SHAPE OPTIMIZATION OF PRINCIPAL EIGENVALUES AND THE GEOMETRY OF CLIMATE ENVELOPES

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Abstract. Integrodifference equations (IDEs) have been widely used to model the evolution of species under the influence of climate change. In the absence of an Allee effect, the principal eigenvalue of the linear integral operator associated with the linearization of the IDE at the extinction state is well-known to play a fundamental role in characterizing extinction/persistence dynamics. Thus, it becomes imperative to investigate the impact of climate change on this principal eigenvalue. Particularly crucial are considerations regarding the shifting velocity (i.e., speed and direction), size, and geometry of the climate envelope incorporated in the IDE to represent climate change. While effects of shifting velocity and size have attracted some attention, the influence of the climate envelope geometry remains largely unexplored.

The present paper is devoted to studying how the geometry of the climate envelope affects the principal eigenvalue, leading to a two-dimensional shape optimization problem. Assuming radial symmetry and analyticity of the dispersal kernel, we demonstrate that among rectangles of equal area, squares maximize the principal eigenvalue for all small shifting speeds. Our approach explores fine properties of the principal eigenfunction in response to the fact that the presence of the shifting speed breaks the symmetry of the linear integral operator, and takes advantage of the analyticity of the principal eigenvalue with respect to the shifting velocity and the smallness of the shifting speed.

1. Introduction

Climate change is a serious threat to the survival of species. Altering and shifting the favorable habitat, climate change forces species to shift or expand their range accordingly in order to remain persistence. Therefore, it is of fundamental significance to understand whether a species can keep up with the shifting habitat and the dependence of the answer to this question on the species and shifting habitat. These problems have been attracting a lot of attention in biological and ecological literature (see [12, 14, 17, 18, 23] and references therein). In mathematical literature, models of different types such as reaction-diffusion equations (see e.g. [3, 4, 5, 8, 20, 21, 22]) and integrodifference equations (IDEs) (see e.g. [25, 26]) are widely used to study these problems. IDEs are advantageous in the sense that they offer effective models for both local and nonlocal dispersal.

Consider the following IDE:

$$u_{n+1}(x) = \int_{\mathbb{R}^d} k(x-y)1_{\Omega}(y-ncv)f(u_n(y))dy, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where the space dimension $d = 1$ or 2, $u_n : \mathbb{R}^d \to [0,\infty)$ is the distribution of a species in the n-th generation, $k : \mathbb{R}^d \to [0,\infty)$ is a dispersal kernel (i.e., a continuous probability density function on $\mathbb{R}^d$), the climate envelope Ω is compact and has positive measure, c is the shifting speed, $v \in \mathbb{S}^{d-1}$

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stands for the direction that the favorable habitat shifts towards and $f : [0, \infty) \to \mathbb{R}$ is the growth rate function without Allee effect. We have chosen the index function 1_Ω as the climate envelope function for clarity; in general, it can be any non-negative, bounded, and compactly supported function. A typical example of the growth rate function f is the Beverton-Holt function [6]

$$f(u) = \frac{Ru}{1 + \frac{(R-1)u}{K^*}},$$

where $R > 0$ is the growth rate, and $K^* > 0$ is the carrying capacity. The IDE (1.1) prescribes the distribution u_{n+1} in the $(n+1)$-th generation given the distribution in the n-th generation through two stages: a sedentary stage and a dispersal stage.

In terms of (1.1), studying the effects of climate change on population persistence/extinction is equivalent to studying the long-term behavior of solutions of (1.1), and its dependence on the shifting speed c and the climate envelope function 1_Ω. Many results have been established in this regard (see e.g. [15, 19, 25, 26]). Introducing the moving frame $w_n(x) := u_n(x + ncv)$, we can rewrite (1.1) as

$$w_{n+1}(x) = \int_\Omega k(x - y + cv) f(w_n(y)) dy, \quad x \in \Omega. \tag{1.2}$$

Note that we have restricted w_n to the bounded domain Ω in (1.2). This is important for the purpose of the present paper. It should be pointed out that a reaction-diffusion equation model can not be converted to an equivalent problem on a bounded domain.

Let $\mathcal{F}_{\Omega,c,v} : C(\Omega) \to C(\Omega)$ be the linear operator associated with the linearization of (1.2) about 0, that is,

$$\mathcal{F}_{\Omega,c,v}[w] = r_0 \int_\Omega k(\cdot - y + cv) w(y) dy, \quad w \in C(\Omega),$$

where $r_0 = f'(0)$. Under mild assumptions on k ensuring the applicability of the Kreĭn-Rutman theorem (see e.g. [13]), the principal eigenvalue $\lambda(\Omega, c, v)$ of $\mathcal{F}_{\Omega,c,v}$ exists and is given by its spectral radius. Moreover, the sign of $\lambda(\Omega, c, v) - 1$ determines the persistence/extinction dynamics of (1.2) or equivalently (1.1), that is, $\lambda(\Omega, c, v) - 1 > 0$ ensures persistence while $\lambda(\Omega, c, v) - 1 \le 0$ yields extinction (see e.g. [15, 24]).

The dependence of $\lambda(\Omega, c, v)$ on the shifting speed c is known in some situations. For instance, if k is radially symmetric and decreasing, then $\lambda(\Omega, c, v)$ is decreasing in c. In dimension $d = 1$, the monotone dependence of $\lambda([-\frac{L}{2}, \frac{L}{2}], c, v)$ on L − the size of the climate envelope − is known as well. See [15, 25, 26] and reference therein.

The main purpose of the present paper is to investigate the dependence of $\lambda(\Omega, c, v)$ on the geometry of the domain Ω in dimension $d = 2$. To be more specific, assuming that k is radially symmetric and non-increasing, we would like to shed some light on the following shape optimization problem

$$\arg\max_{|\Omega|=1} \lambda(\Omega, c, v). \tag{1.3}$$

The area being fixed to be 1 does not lose the generality. When $c = 0$, it is well-known that $\lambda(\Omega, 0, v)$, which no longer depends on v, is maximized only at Ω = disks. This, providing an answer to the problem (1.3), is a simple consequence of the variational formula of $\lambda(\Omega, 0, v)$ and Riesz's rearrangement inequality (see e.g. [16, Theorem 3.7]). When $c \neq 0$, the principal eigenvalue $\lambda(\Omega, c, v)$ does not admit a variational formula that is as simple as that in the case $c = 0$. Therefore, the classical rearrangement inequality does not apply and this fact increases the difficulty of the problem (1.3) significantly.

We deviate to mention that the shape optimization problem has been especially investigated for the principal eigenvalue of elliptic operators on bounded domains with homogeneous Dirichlet boundary condition. More precisely, one is interested in the following problem

$$\arg\min_{|\Omega|=1} \Lambda(\Omega), \tag{1.4}$$

where $\Lambda(\Omega) > 0$ is the principal eigenvalue of the elliptic operator $\mathcal{L} := -\nabla \cdot (A\nabla) + b \cdot \nabla + V$ on $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary condition. Here, we save the energy but refer the reader to [10] for the introduction of general conditions on Ω, A, b and V that are required to investigate the problem (1.4). In the case of the negative Laplace operator (namely, $A = I_{d\times d}$, $b = 0$ and $V = 0$), the problem (1.4) is solved by the classical Rayleigh-Faber-Krahn inequality saying that $\arg\min_{|\Omega|=1} \Lambda(\Omega) = \Omega^*$, where Ω^* denotes an arbitrary open ball with $|\Omega^*| = 1$. Generalization to more general cases (even for $-\nabla \cdot (A\nabla)$ or $-\Delta + b \cdot \nabla$) turns out to be highly nontrivial. In [9], the authors considered the operator $-\Delta+b\cdot\nabla$ and extended the classical Rayleigh-Faber-Krahn inequality (see [9, Corollary 1.2]), answering the problem (1.4). They further considered in [10] the operator $\mathcal{L}$ in its full generality. By introducing a new symmetric arrangement associated with the principal eigenfunction of $\mathcal{L}$, they established a series of results, leading in particular to a further extension of the classical Rayleigh-Faber-Krahn inequality (see [10, Corollary 2.8]).

In the present paper, our focus is not on solving the problem (1.3), but rather on a restricted version thereof. Specifically, recognizing the ecological significance of rectangular domains (see e.g. [19]), we aim to maximize $\lambda(\Omega, c, v)$ over rectangular domains with fixed area. That is, for each $|c| \ll 1$ and $v \in \mathbb{S}$ we consider the following problem

$$\arg\max_{\substack{\Omega \text{ rectangle} \\ |\Omega|=1}} \lambda(\Omega, c, v). \tag{1.5}$$

Considering small shifting speed c is biologically meaningful.

We introduce assumptions and notations before stating our results.

(H) $k : \mathbb{R}^2 \to [0, \infty)$ is real analytic, radially symmetric, radially decreasing in $B_\eta(0)$ and non-increasing in $\mathbb{R}^2 \backslash B_\eta(0)$ for some $\eta > 0$, and satisfies $\int_{\mathbb{R}^2} k = 1$, where $B_\eta(0) \subset \mathbb{R}^2$ is the open ball centered at the origin 0 with radius η.

Typical examples of k satisfying conditions in **(H)** are Gaussian probability densities functions centred at the origin.

From now on, we only consider Ω as rectangles with area 1. Under **(H)**, the principal eigenvalue $\lambda(\Omega, c, v)$ of $\mathcal{F}_{\Omega,c,v}$ exists and is given by its spectral radius. As $\lambda(T\Omega, c, v) = \lambda(\Omega, c, v)$ for any translation T on $\mathbb{R}^2$, we only need to consider rectangles centered at the origin, that is,

$$\left\{ \mathcal{R}\Omega_L : \mathcal{R} \text{ is a rotation on } \mathbb{R}^2, \quad \Omega_L = \left[-\frac{L}{2}, \frac{L}{2}\right] \times \left[-\frac{1}{2L}, \frac{1}{2L}\right], \quad L > 0 \right\}.$$

Since the radial symmetry of k asserts that $\lambda(\mathcal{R}\Omega_L, c, v) = \lambda(\Omega_L, c, \mathcal{R}^{-1}v)$, considering (1.5) for each fixed $v \in \mathbb{S}$ is equivalent to studying the following problem

$$\arg\max_{L>0, v\in\mathbb{S}} \lambda(\Omega_L, c, v). \tag{1.6}$$

For convenience, we write

$$\mathcal{F}_{L,c,v} = \mathcal{F}_{\Omega_L,c,v} \quad \text{and} \quad \lambda(L, c, v) = \lambda(\Omega_L, c, v).$$

Note that $\mathcal{F}_{L,0,v}$ and $\lambda(L,0,v)$ are independent of v.

We take a two-step approach to solve the problem (1.6). First, we solve for each $v \in \mathbb{S}$ the following problem

$$\arg\max_{L>0} \lambda(L,c,v). \tag{1.7}$$

It is shown in Theorem A that (1.7) admits a unique solution L_v (actually, $L_v = 1$). It is followed by maximizing the function $v \mapsto \lambda(L_v,c,v)$, that is, to study

$$\arg\max_{v\in\mathbb{S}} \lambda(L_v,c,v). \tag{1.8}$$

In Theorem B, we show that $\lambda(L_v,c,v) = \lambda(1,c,v)$ is independent of $v \in \mathbb{S}$.

Now, we present our findings in detail. Under **(H)**, classical perturbation theories [1, 2] ensure that $\lambda(L,c,v)$ is analytic in (L,c,v). Therefore, for each $v \in \mathbb{S}$ and $L_0 > 0$, there holds the second-order Taylor expansion of $\lambda(L,c,v)$ about $(L,c) = (L_0,0)$:

$$\begin{aligned}\lambda(L,c,v) = {} & \lambda(L_0,0,v) + \lambda'_L(L_0,0,v)(L-L_0) + \lambda'_c(L_0,0,v)c \\ & + \frac{1}{2}\lambda''_{LL}(L_0,0,v)(L-L_0)^2 + \lambda''_{Lc}(L_0,0,v)(L-L_0)c + \frac{1}{2}\lambda''_{cc}(L_0,0,v)c^2 \\ & + h_1(L,c,v)(L-L_0)^2 + h_2(L,c,v)(L-L_0)c + h_3(L,c,v)c^2,\end{aligned} \tag{1.9}$$

where $\lim_{(L,c)\to(L_0,0)} h_i(L,c,v) = 0$ for $i = 1,2,3$. Note that the coefficients $\lambda(L_0,0,v)$, $\lambda'_L(L_0,0,v)$ and $\lambda''_{LL}(L_0,0,v)$ are independent of v.

The following theorem addresses coefficients of the first- and second-order terms in (1.9).

Theorem A. *Assume* **(H)**. *For each $v \in \mathbb{S}$, the following statements hold.*

(i) $\lambda'_L(L_0,0,v) > 0$ *for* $0 < L_0 < 1$,
$\lambda'_L(1,0,v) = 0$ *and* $\lambda''_{LL}(1,0,v) < 0$,
$\lambda'_L(L_0,0,v) < 0$ *for* $L_0 > 1$.
(ii) $\lambda'_c(L_0,0,v) = 0$ *and* $\lambda''_{cc}(L_0,0,v) < 0$ *for all* $L_0 > 0$.
(iii) $\lambda''_{Lc}(L_0,0,v) = 0$ *for all* $L_0 > 0$.

Therefore, the Taylor expansion (1.9) is simplified to the following:

- if $L_0 = 1$,
$$\lambda(L,c,v) = \lambda(1,0,v) + \frac{1}{2}\lambda''_{LL}(1,0,v)(L-1)^2 + \frac{1}{2}\lambda''_{cc}(1,0,v)c^2 + o((L-1)^2) + o(c^2),$$
- if $L_0 \neq 1$,
$$\lambda(L,c,v) = \lambda(L_0,0,v) + \lambda'_L(L_0,0,v)(L-L_0) + \frac{1}{2}\lambda''_{cc}(L_0,0,v)c^2 + o((L-L_0)^2) + o(c^2).$$

An immediate consequence is that for each $|c| \ll 1$ and $v \in \mathbb{S}$, the function $L \mapsto \lambda(L,c,v)$ is unimodal and achieves its largest value at $L = 1$. In particular, the solution to the problem (1.7) is $L = 1$.

It remains to consider the problem (1.8) with $L_v = 1$, that is,

$$\arg\max_{v\in\mathbb{S}} \lambda(1,c,v). \tag{1.10}$$

We prove the following result.

Theorem B. *Assume* **(H)**. *There exists a* $\delta > 0$ *such that*

$$\lambda(1,c,v) = \sum_{i=0}^{\infty} \frac{c^{2i}}{(2i)!} \frac{\partial^{2i}\lambda}{\partial c^{2i}}(1,0,v_0), \quad \forall (c,v) \in (-\delta,\delta) \times \mathbb{S},$$

where $v_0 = (1,0)$. *In particular, for any* $c \in (-\delta,\delta)$, *the function* $v \mapsto \lambda(1,c,v)$ *is constant on* $\mathbb{S}$, *and therefore, the solution of* (1.10) *is given by the whole unit circle* $\mathbb{S}$.

The conclusion of Theorem B is somewhat counter-intuitive, as Ω_1 is not radially symmetric. Perhaps, this is a result of the analyticity and symmetry, and is supported by the following important example.

Example 1.1. *Let* k *be a Gaussian probability density function with zero mean, namely,*

$$k(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad x \in \mathbb{R}^2$$

for some $\sigma > 0$. *Denote by* $\varphi_{\Omega,c,v}$ *the positive eigenfunction associated with* $\lambda(\Omega,c,v)$. *Then,*

$$\lambda(\Omega,c,v)\varphi_{\Omega,c,v}(x) = \frac{r_0}{2\pi\sigma^2}\int_\Omega e^{-\frac{|x-y+cv|^2}{2\sigma^2}}\varphi_{\Omega,c,v}(y)dy = \frac{r_0}{2\pi\sigma^2} e^{-\frac{c^2}{2\sigma^2}} e^{-\frac{cv\cdot x}{\sigma^2}} \int_\Omega e^{-\frac{|x-y|^2}{2\sigma^2}} e^{\frac{cv\cdot y}{\sigma^2}} \varphi_{\Omega,c,v}(y)dy.$$

Setting $\phi(x) = e^{\frac{cv\cdot x}{\sigma^2}}\varphi_{\Omega,c,v}(x)$ *yields*

$$\lambda(\Omega,c,v)e^{\frac{c^2}{2\sigma^2}}\phi(x) = r_0\int_\Omega k(x-y)\phi(y)dy,$$

resulting in

$$\lambda(\Omega,c,v) = e^{-\frac{c^2}{2\sigma^2}}\lambda(\Omega,0,v).$$

Since $\lambda(\Omega,0,v)$ *is independent of* $v \in \mathbb{S}$, *so is* $\lambda(\Omega,c,v)$.

The solution of (1.6) follows immediately from Theorems A and B.

Corollary 1.2. *Assume* **(H)**. *Then, there exists a* $\delta > 0$ *such that for any* $c \in (-\delta,\delta)$, *the solution of* (1.6) *is given by the set* $\{1\} \times \mathbb{S}$. *Equivalently, for any* $c \in (-\delta,\delta)$ *and* $v \in \mathbb{S}$, *the solution of* (1.5) *is given by unit squares.*

Our approach is different from those used in [9, 10]. The proof of Theorem A is based on the analysis of the eigenfunction $\varphi_{L,c,v}$ of $\mathcal{F}_{L,c,v}$ associated with the principal eigenvalue $\lambda(L,c,v)$ under the normalization (fixed throughout the paper)

$$\max_{\Omega_L} \varphi_{L,c,v} = 1. \tag{1.11}$$

From the second-order Taylor expansion (1.9) of $\lambda(L,c,v)$, one finds identities connecting coefficients of the first- and second-order terms in (1.9) with $\varphi_{L,c,v}$ and its partial derivatives calculated at $(L_0,0)$. Crucial ingredients in our proof are the establishment of properties of $\varphi_{L,c,v}$ at $(L_0,0)$ in response to the symmetry breaking, namely, $c \neq 0$. Some of them are collected in Section 2. Others are hidden in Section 3, where the proof of Theorem A is given.

The proof of Theorem B digs into the analyticity and symmetry of $(c,\theta) \mapsto \lambda(1,c,v(\theta))$, where $v(\theta) := (\cos\theta,\sin\theta)$. A crucial step in our proof is to show that the disk of convergence of the simplified series of $\lambda(c,\theta) := \lambda(1,c,v(\theta))$ at $(c,\theta) = (0,0)$ actually contains $(-R_0,R_0) \times [0,\pi/4]$ for some small $R_0 > 0$ (considering $\theta \in [0,\pi/4]$ suffices due to the symmetry), that is,

$$\lambda(c,\theta) = \lambda(0,0) + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k}\binom{2i+2k}{2k}\frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!}, \quad \forall c \in (-R_0,R_0), \quad \theta \in [0,\pi/4],$$

where $\lambda_{c^{2i}\theta^{2k}}(0,0)=\partial^{2(i+k)}_{c^{2i}\theta^{2k}}\lambda(c,\theta)|_{(0,0)}$. This together with the symmetry of $\lambda(c,\theta)$ at $(0,\pi/4)$ helps to draw the conclusion of Theorem B.

The rest of the paper is organized as follows. In Section 2, we establish some properties of the principal eigenfunction $\varphi_{L,c}$ that play significant roles in the proof of Theorem A, whose proof is given in Section 3. Section 4 is devoted to the proof of Theorem B.

2. Properties of the principal eigenfunction

Recall that $\varphi_{L,0,v}$ is the principal eigenfunction of $\mathcal{F}_{L,0,v}$ associated with the principal eigenvalue $\lambda(L,0,v)$ and satisfies the normalization (1.11). In this section, we establish some properties of $\varphi_{L,0,v}$ that play crucial roles in the sequel. Due to the independence of $\varphi_{L,0,v}$ on $v\in\mathbb{S}$, we write $\varphi_{L,0}$ for $\varphi_{L,0,v}$ in the rest of this section.

Given the radial symmetry of k and the geometry of the domain Ω_L, the proof of the following result addressing the symmetry of $\varphi_{L,0}$ is straightforward, and therefore, omitted.

Lemma 2.1. *Assume* **(H)**. *Then, $\varphi_{L,0}$ is symmetric with respect to both $x_1=0$ and $x_2=0$. When $L=1$, there holds in addition*

$$\varphi_{1,0}(x_1,x_2)=\varphi_{1,0}(x_2,x_1)=\varphi_{1,0}(-x_2,-x_1),\quad \forall(x_1,x_2)\in\Omega_1. \tag{2.1}$$

Now, we explore finer properties of the eigenfunction $\varphi_{L,0}$. For $L\geq 1$, straight lines $\ell_L: x_2=x_1-\frac{L^2-1}{2L}$ and $x_1=\frac{L^2-2}{2L}$ divide Ω_L into three subdomains as follows (see Figure 1)

$$\begin{aligned}
A_L&=\left\{(x_1,x_2)\in\mathbb{R}^2:\frac{L^2-2}{2L}\leq x_1\leq\frac{L}{2},-\frac{1}{2L}\leq x_2<x_1-\frac{L^2-1}{2L}\right\},\\
B_L&=\left\{(x_1,x_2)\in\mathbb{R}^2:\frac{L^2-2}{2L}\leq x_1\leq\frac{L}{2},x_1-\frac{L^2-1}{2L}\leq x_2\leq\frac{1}{2L}\right\},\\
C_L&=\Omega_L\setminus(A_L\cup B_L).
\end{aligned}$$

Obviously, $C_L=\emptyset$ if $L=1$.

Similarly, for $0<L<1$, straight lines $\ell_L: x_2=x_1-\frac{L^2-1}{2L}$ and $x_2=\frac{1-2L^2}{2L}$ divide Ω_L into three subdomains as follows (see Figure 2)

$$\begin{aligned}
A_L&=\left\{(x_1,x_2)\in\mathbb{R}^2:-\frac{L}{2}\leq x_1\leq\frac{L}{2},x_1-\frac{L^2-1}{2L}<x_2\leq\frac{1}{2L}\right\},\\
B_L&=\left\{(x_1,x_2)\in\mathbb{R}^2:-\frac{L}{2}\leq x_1\leq\frac{L}{2},\frac{1-2L^2}{2L}\leq x_2\leq x_1-\frac{L^2-1}{2L}\right\},\\
C_L&=\Omega_L\setminus(A_L\cup B_L).
\end{aligned}$$

For $L>0$, we define $\rho_L:A_L\to B_L\setminus\ell_L$ by mapping $x\in A_L$ to $\rho_L(x)\in B_L$ such that x and $\rho_L(x)$ are symmetric with respect to the line ℓ_L, that is,

$$\rho_L(x)=\left(x_2+\frac{L^2-1}{2L},x_1-\frac{L^2-1}{2L}\right),\quad x=(x_1,x_2)\in A_L.$$

Clearly, ρ_L is smooth and has a smooth inverse.

We prove the following result about $\varphi_{L,0}$.

Lemma 2.2. *Assume* **(H)**. *Then,*

$$\varphi_{L,0}\leq\varphi_{L,0}\circ\rho_L\quad in\quad A_L.$$

If $L=1$, then the equality holds.

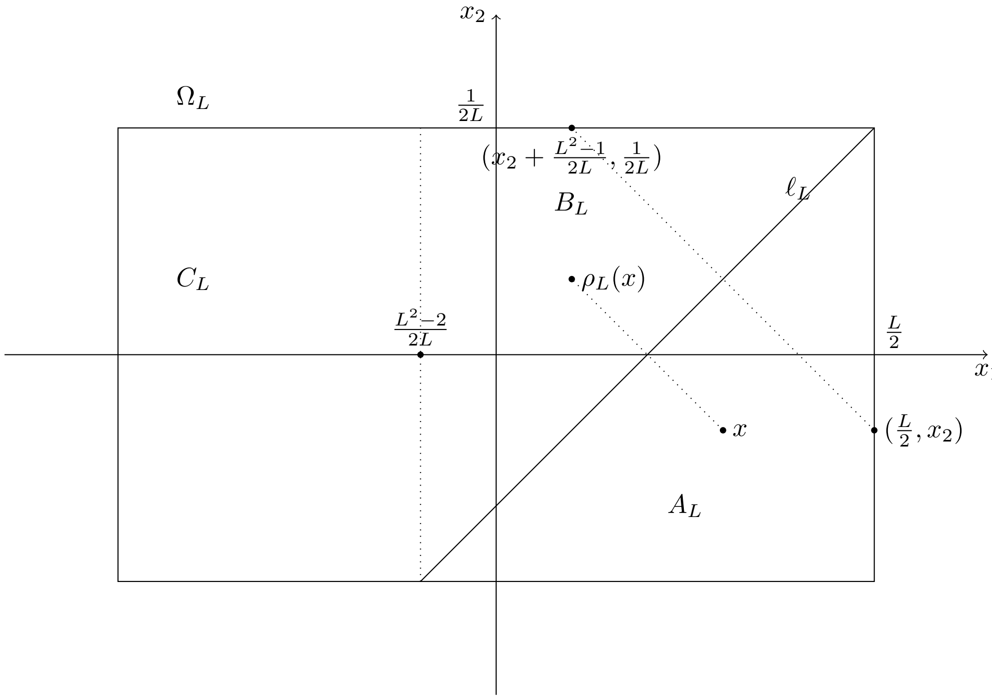

FIGURE 1. Decomposition of Ω_L for $L > 1$

Proof. If $L = 1$, the result follows immediately from Lemma 2.1. Now, we fix $L \neq 1$ and write $\lambda = \lambda(L, 0)$ and $\varphi = \varphi_{L,0}$ for simplicity. Clearly, it is equivalent to prove

$$A_L^+ := \{x \in A_L : \varphi(x) - \varphi(\rho_L(x)) > 0\} = \emptyset. \tag{2.2}$$

Suppose on the contrary that (2.2) fails, that is, $A_L^+ \neq \emptyset$. The continuity of φ and ρ_L ensures that A_L^+ contains an open set, and therefore, has positive Lebesgue measure.

For $x \in A_L$, we deduce from the eigen-equation $\mathcal{F}_{L,0}\varphi = \lambda\varphi$ that

$$\begin{aligned}
\lambda\varphi(x) &= r_0 \int_{A_L \cup C_L} k(x-y)\varphi(y)dy + r_0 \int_{B_L \setminus \ell_L} k(x-y)\varphi(y)dy \\
&= r_0 \int_{A_L \cup C_L} k(x-y)\varphi(y)dy + r_0 \int_{A_L} k(x-\rho_L(y))\varphi(\rho_L(y))dy \\
&= r_0 \int_{A_L \cup C_L} k(x-y)\varphi(y)dy + r_0 \int_{A_L} k(\rho_L(x)-y)\varphi(\rho_L(y))dy,
\end{aligned}$$

since we used the definition of ρ_L in the second equality, and the fact $|x - \rho_L(y)| = |\rho_L(x) - y|$ for $x, y \in A_L$ and the radial symmetry of k in the third equality.

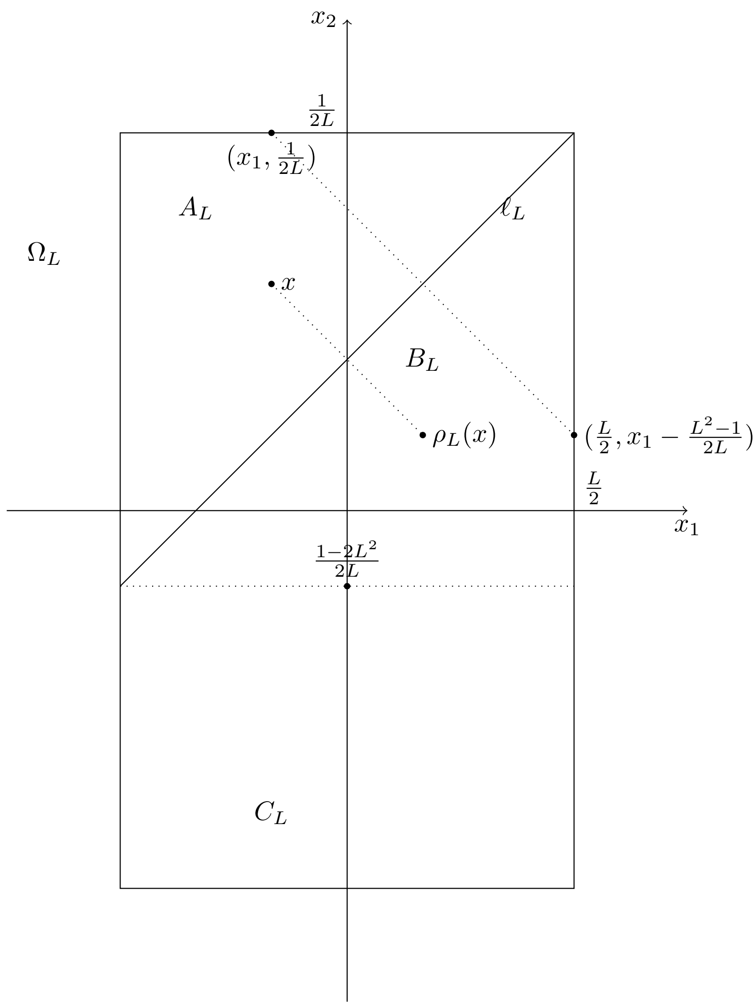

FIGURE 2. Decomposition of Ω_L for $0 < L < 1$

Similarly, for $x \in A_L$,

$$\begin{aligned}
\lambda\varphi(\rho_L(x)) &= r_0 \int_{A_L \cup C_L} k(\rho_L(x) - y)\varphi(y)dy + r_0 \int_{B_L \setminus \ell_L} k(\rho_L(x) - y)\varphi(y)dy \\
&= r_0 \int_{A_L \cup C_L} k(\rho_L(x) - y)\varphi(y)dy + r_0 \int_{A_L} k(\rho_L(x) - \rho_L(y))\varphi(\rho_L(y))dy \\
&= r_0 \int_{A_L \cup C_L} k(\rho_L(x) - y)\varphi(y)dy + r_0 \int_{A_L} k(x - y)\varphi(\rho_L(y))dy,
\end{aligned}$$

where we used the fact that $|\rho_L(x) - \rho_L(y)| = |x - y|$ for $x, y \in A_L$ and the radial symmetry of k in the third equality. Thus, for $x \in A_L$,

$$\lambda(\varphi(x) - \varphi(\rho_L(x))) = r_0 \int_{A_L} (k(x-y) - k(\rho_L(x) - y))(\varphi(y) - \varphi(\rho_L(y)))dy \\ + r_0 \int_{C_L} (k(x-y) - k(\rho_L(x) - y))\varphi(y)dy.$$

Since $|x - y| > |\rho_L(x) - y|$ for $x \in A_L$ and $y \in C_L$, we deduce from k being radially nonincreasing that

$$\int_{C_L} (k(x-y) - k(\rho_L(x) - y))\varphi(y)dy \le 0, \quad \forall x \in A_L.$$

As $|x - y| < |\rho_L(x) - y|$ for $x, y \in A_L$, $\varphi(y) - \varphi(\rho_L(y)) \le 0$ for $y \in A_L \setminus A_L^+$ and $\varphi(y) - \varphi(\rho_L(y)) > 0$ for $y \in A_L^+$, we find for $x \in A_L$,

$$\begin{aligned} &\int_{A_L} (k(x-y) - k(\rho_L(x) - y))(\varphi(y) - \varphi(\rho_L(y)))dy \\ &\quad \le \int_{A_L^+} (k(x-y) - k(\rho_L(x) - y))(\varphi(y) - \varphi(\rho_L(y)))dy \\ &\quad \le \int_{A_L^+} k(x-y)(\varphi(y) - \varphi(\rho_L(y)))dy. \end{aligned}$$

It follows that

$$\lambda(\varphi(x) - \varphi(\rho_L(x))) \le r_0 \int_{A_L^+} k(x-y)(\varphi(y) - \varphi(\rho_L(y)))dy, \quad \forall x \in A_L.$$

Multiplying the above inequality by $\varphi - \varphi \circ \rho_L$ and integrating over A_L^+ yield

$$\begin{aligned} \lambda \int_{A_L^+} (\varphi - \varphi \circ \rho_L)^2 &\le r_0 \int_{A_L^+} \int_{A_L^+} k(x-y)(\varphi(y) - \varphi(\rho_L(y)))(\varphi(x) - \varphi(\rho_L(x)))dxdy \\ &\le \lambda^+ \int_{A_L^+} (\varphi - \varphi \circ \rho_L)^2, \end{aligned}$$

where λ^+ is the principal eigenvalue of $\mathcal{F}_{L,0}$ restricted on A_L^+. It follows that $\lambda \le \lambda^+$. However, [11, Theorem 1.1] asserts that $\lambda > \lambda^+$, leading to a contradiction. Hence, (2.2) holds. □

Then, we have the following lemma.

Lemma 2.3. *Assume* **(H)**. *For any* $L > 0$,

$$\begin{aligned} \partial_{x_1}\varphi_{L,0} \le 0 \quad &in \quad \left[0, \frac{L}{2}\right] \times \left[-\frac{1}{2L}, \frac{1}{2L}\right], \\ \partial_{x_2}\varphi_{L,0} \le 0 \quad &in \quad \left[-\frac{L}{2}, \frac{L}{2}\right] \times \left[0, \frac{1}{2L}\right]. \end{aligned} \tag{2.3}$$

Moreover, there exists $\delta = \delta(L) > 0$ *such that*

$$\begin{aligned} \partial_{x_1}\varphi_{L,0} < 0 \quad &in \quad \left(\frac{L}{2} - \delta, \frac{L}{2}\right] \times \left[-\frac{1}{2L}, \frac{1}{2L}\right], \\ \partial_{x_2}\varphi_{L,0} < 0 \quad &in \quad \left[-\frac{L}{2}, \frac{L}{2}\right] \times \left(\frac{1}{2L} - \delta, \frac{1}{2L}\right]. \end{aligned}$$

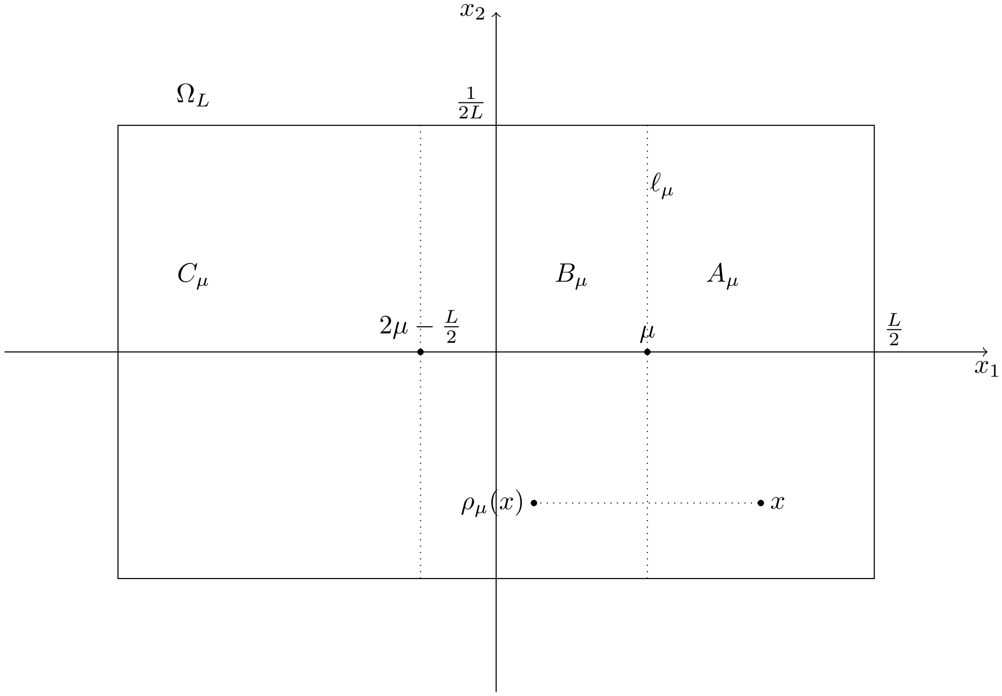

FIGURE 3. Decomposition of Ω_L for $L > 0$

Proof. We only prove the results for $\partial_{x_1}\varphi_{L,0}$; the results for $\partial_{x_2}\varphi_{L,0}$ can be proven in the same way.

For fixed $L > 0$, we write φ for $\varphi_{L,0}$ for simplicity. Since k is radially symmetric, there is $K : [0,\infty) \to [0,\infty)$ such that $k(x) = K(|x|)$ for all $x \in \mathbb{R}^2$. The differentiability of K is ensured by that of k. Differentiating the eigen-equation $\mathcal{F}_{L,0}\varphi = \lambda(L,0)\varphi$ with respect to the first variable x_1, we find

$$\lambda(L,0)\partial_{x_1}\varphi(x) = r_0 \int_{\Omega_L} K'(|x-y|)\frac{x_1 - y_1}{|x-y|}\varphi(y)dy, \quad \forall x \in \Omega_L. \tag{2.4}$$

To proceed, for any $0 < \mu < \frac{L}{2}$, we decompose Ω_L into

$$\Omega_L = A_\mu \cup B_\mu \cup C_\mu,$$

where $A_\mu = \left(\mu, \frac{L}{2}\right] \times \left[-\frac{1}{2L}, \frac{1}{2L}\right]$, $B_\mu = \left[2\mu - \frac{L}{2}, \mu\right] \times \left[-\frac{1}{2L}, \frac{1}{2L}\right]$ and $C_\mu = \Omega_L \setminus (A_\mu \cup B_\mu)$, and define $\rho_\mu : A_\mu \to B_\mu \setminus \ell_\mu$, where $\ell_\mu : x_1 = \mu$, by setting

$$\rho_\mu(x) = (2\mu - x_1, x_2), \quad x = (x_1, x_2) \in A_\mu.$$

See Figure 3 for an illustration. Clearly, for $x \in A_\mu$, x and $\rho_\mu(x)$ are symmetric with respect to the line ℓ_μ. The map ρ_μ is smooth and has a smooth inverse.

We claim that for any $\mu \in (0, \frac{L}{2})$,

$$\varphi \le \varphi \circ \rho_\mu \quad \text{in} \quad A_\mu. \tag{2.5}$$

We continue to prove the result assuming (2.5) whose proof is postponed to the end.

We treat the integral on the right hand side of (2.4). For $x_1 \in (0, \frac{L}{2})$,

$$\begin{aligned}
&\int_{\Omega_L} K'(|x-y|)\frac{x_1-y_1}{|x-y|}\varphi(y)dy \\
&\quad= \int_{A_{x_1}\cup C_{x_1}} K'(|x-y|)\frac{x_1-y_1}{|x-y|}\varphi(y)dy + \int_{B_{x_1}\setminus \ell_{x_1}} K'(|x-y|)\frac{x_1-y_1}{|x-y|}\varphi(y)dy \\
&\quad= \int_{A_{x_1}\cup C_{x_1}} K'(|x-y|)\frac{x_1-y_1}{|x-y|}\varphi(y)dy + \int_{A_{x_1}} K'(|x-y|)\frac{y_1-x_1}{|x-y|}\varphi(\rho_{x_1}(y))dy \\
&\quad= \int_{A_{x_1}} K'(|x-y|)\frac{y_1-x_1}{|x-y|}(\varphi(\rho_{x_1}(y))-\varphi(y))dy + \int_{C_{x_1}} K'(|x-y|)\frac{x_1-y_1}{|x-y|}\varphi(y)dy \\
&\quad\le \int_{A_{x_1}} K'(|x-y|)\frac{y_1-x_1}{|x-y|}(\varphi(\rho_{x_1}(y))-\varphi(y))dy \le 0,
\end{aligned}$$

where we used the symmetry of y and $\rho_{x_1}(y)$ with respect to ℓ_{x_1} in the second equality, k being radially non-increasing (so $K' \le 0$) and $x_1 - y_1 \ge 0$ for $y \in C_{x_1}$ in the first inequality, and (2.5) and $y_1 - x_1 > 0$ for $y \in A_{x_1}$ in the last inequality. Hence, (2.3) is true for $x \in \left[0, \frac{L}{2}\right) \times \left[-\frac{1}{2L}, \frac{1}{2L}\right]$ (note that the correctness of (2.3) for $x_1 = 0$ is trivial due to the symmetry of φ with respect to $x_1 = 0$).

We now show the strict inequality near the edge $x_1 = \frac{L}{2}$. Note that

$$\lambda(L,0)\partial_{x_1}\varphi(L/2,x_2) = r_0\int_{\Omega_L} K'(|\,(L/2,x_2)-y|)\frac{L/2-y_1}{|(L/2,x_2)-y|}\varphi(y)dy, \quad \forall x_2 \in \left[-\frac{1}{2L},\frac{1}{2L}\right]. \tag{2.6}$$

It follows from the assumption that k is radially decreasing on $B_\eta(0)$ (or equivalently, K is decreasing on $[0,\eta)$) that the integral on the right hand side of the above equality is negative. Hence, $\partial_{x_1}\varphi(L/2,x_2) < 0$ for all $x_2 \in \left[-\frac{1}{2L}, \frac{1}{2L}\right]$. As $\varphi \in C^2(\Omega_L)$, there must exist some $\delta > 0$ such that $\partial_{x_1}\varphi < 0$ in $\left(\frac{L}{2}-\delta, \frac{L}{2}\right] \times \left[-\frac{L}{2}, \frac{L}{2}\right]$.

It remains to prove (2.5), which actually follows from arguments as in the proof of Lemma 2.2. We include the details for completeness. It is seen from the eigen-equation $\mathcal{F}_{L,0}\varphi = \lambda(L,0)\varphi$ that for any $x \in A_\mu$,

$$\begin{aligned}
\lambda(L,0)\varphi(x) &= r_0\int_{A_\mu\cup C_\mu} k(x-y)\varphi(y)dy + r_0\int_{B_\mu\setminus\ell_\mu} k(x-y)\varphi(y)dy \\
&= r_0\int_{A_\mu\cup C_\mu} k(x-y)\varphi(y)dy + r_0\int_{A_\mu} k(x-\rho_\mu(y))\varphi(\rho_\mu(y))dy \\
&= r_0\int_{A_\mu\cup C_\mu} k(x-y)\varphi(y)dy + r_0\int_{A_\mu} k(\rho_\mu(x)-y)\varphi(\rho_\mu(y))dy,
\end{aligned}$$

where we used the definition of ρ_μ in the second equality, and the fact that $|x-\rho_\mu(y)| = |\rho_\mu(x)-y|$ for any $x, y \in A_\mu$ and the radial symmetry of k in the last equality. Similarly, for $x \in A_\mu$,

$$\begin{aligned}
\lambda(L,0)\varphi(\rho_\mu(x)) &= r_0\int_{A_\mu\cup C_\mu} k(\rho_\mu(x)-y)\varphi(y)dy + r_0\int_{B_\mu\setminus\ell_\mu} k(\rho_\mu(x)-y)\varphi(y)dy \\
&= r_0\int_{A_\mu\cup C_\mu} k(\rho_\mu(x)-y)\varphi(y)dy + r_0\int_{A_\mu} k(\rho_\mu(x)-\rho_\mu(y))\varphi(\rho_\mu(y))dy \\
&= r_0\int_{A_\mu\cup C_\mu} k(\rho_\mu(x)-y)\varphi(y)dy + r_0\int_{A_\mu} k(x-y)\varphi(\rho_\mu(y))dy,
\end{aligned}$$

where we used the fact that $|\rho_\mu(x)-\rho_\mu(y)|=|x-y|$ for any $x,y\in A_\mu$ and the radial symmetry of k in the last equality. Thus, for $x\in A_\mu$,

$$\begin{aligned}\lambda(L,0)(\varphi(x)-\varphi(\rho_\mu(x)))=r_0\int_{A_\mu}(k(x-y)-k(\rho_\mu(x)-y))(\varphi(y)-\varphi(\rho_\mu(y)))dy\\ +r_0\int_{C_\mu}(k(x-y)-k(\rho_\mu(x)-y))\varphi(y)dy.\end{aligned}$$

Since $|x-y|>|\rho_\mu(x)-y|$ for $x\in A_\mu$ and $y\in C_\mu$, k being radially non-increasing ensures

$$\int_{C_\mu}(k(x-y)-k(\rho_\mu(x)-y))\varphi(y)dy\leq 0,\quad \forall x\in A_\mu.$$

Then,

$$\lambda(L,0)(\varphi(x)-\varphi(\rho_\mu(x)))\leq r_0\int_{A_\mu}(k(x-y)-k(\rho_\mu(x)-y))(\varphi(y)-\varphi(\rho_\mu(y)))dy,\quad \forall x\in A_\mu.$$

Obviously, (2.5) is equivalent to

$$A_\mu^+:=\{x\in A_\mu:\varphi(x)-\varphi(\rho_\mu(x))>0\}=\emptyset.$$

Suppose on the contrary that $A_\mu^+\neq\emptyset$. The continuity of φ guarantees that A_μ^+ has positive Lebesgue measure. By the facts that $|x-y|<|\rho_\mu(x)-y|$ for $x,y\in A_\mu$, $\varphi(y)-\varphi(\rho_\mu(y))\leq 0$ for $y\in A_\mu\setminus A_\mu^+$ and $\varphi(y)-\varphi(\rho_L(y))>0$ for $y\in A_\mu^+$, we have for $x\in A_L$,

$$\begin{aligned}&\int_{A_\mu}(k(x-y)-k(\rho_\mu(x)-y))(\varphi(y)-\varphi(\rho_\mu(y)))dy\\ &\quad\leq\int_{A_\mu^+}(k(x-y)-k(\rho_\mu(x)-y))(\varphi(y)-\varphi(\rho_\mu(y)))dy\\ &\quad\leq\int_{A_\mu^+}k(x-y)(\varphi(y)-\varphi(\rho_\mu(y)))dy.\end{aligned}$$

It follows that

$$\lambda(L,0)(\varphi(x)-\varphi(\rho_\mu(x)))\leq r_0\int_{A_\mu^+}k(x-y)(\varphi(y)-\varphi(\rho_\mu(y)))dy,\quad \forall x\in A_\mu.$$

Multiplying the above inequality by $\varphi-\varphi\circ\rho_\mu$ and integrating over A_μ^+ yield

$$\begin{aligned}\lambda(L,0)\int_{A_\mu^+}(\varphi-\varphi\circ\rho_\mu)^2&\leq r_0\int_{A_\mu^+}\int_{A_\mu^+}k(x-y)(\varphi(y)-\varphi(\rho_\mu(y)))(\varphi(x)-\varphi(\rho_\mu(x)))dxdy\\ &\leq\lambda^+(L,0)\int_{A_\mu^+}(\varphi-\varphi\circ\rho_\mu)^2,\end{aligned}$$

where $\lambda^+(L,0)$ is the principal eigenvalue of $\mathcal{F}_{L,0}$ restricted on A_μ^+. We deduce that $\lambda(L,0)\leq\lambda^+(L,0)$. However, [11, Theorem 1.1] asserts that $\lambda(L,0)>\lambda^+(L,0)$, leading to a contradiction. Hence, (2.5) holds. This completes the proof. □

3. Proof of Theorem A

This section is devoted to the proof of Theorem A.

3.1. **Proof of Theorem A (i).** Observe that it suffices to study $\lambda(L,0,v)$, which is actually independent of $v\in\mathbb{S}$, in order to determine the signs of $\lambda'_L(L_0,0,v)$ and $\lambda''_{LL}(L_0,0,v)$.

Fix $L_0>0$. The second-order Taylor expansion of $\lambda(L,0,v)$ at $L=L_0$ reads

$$\lambda(L,0,v)=\lambda(L_0,0,v)+\lambda'_L(L_0,0,v)(L-L_0)+\frac{1}{2}\lambda''_{LL}(L_0,0,v)(L-L_0)^2+o((L-L_0)^2).$$

Let $-1\ll\epsilon\ll 1$. Note that

$$\Omega_{L_0+\epsilon}=T(L_0,\epsilon)\Omega_{L_0}, \tag{3.1}$$

where

$$T(L_0,\epsilon)=\begin{pmatrix}\frac{L_0+\epsilon}{L_0} & 0\\ 0 & \frac{L_0}{L_0+\epsilon}\end{pmatrix}. \tag{3.2}$$

Set

$$\lambda(\epsilon):=\lambda(L_0+\epsilon,0,v)\quad\text{and}\quad\varphi_\epsilon:=\varphi_{L_0+\epsilon,0,v}.$$

Clearly, $\lambda'(0)=\lambda'_L(L_0,0,v)$ and $\lambda''(0)=\lambda''_{LL}(L_0,0,v)$.

We break the proof into two steps.

Step 1. We determine the sign of $\lambda'(0)$.

In consideration of the symmetry, it is easy to see that $\lambda'(0)=0$ if $L_0=1$, and the treatment for the case $0<L_0<1$ is analogous to the case $L_0>1$. Hence, we focus on the case $L_0>1$ in the rest of this step.

The eigen-equation associated with $\lambda(\epsilon)$ and φ_ϵ reads

$$\lambda(\epsilon)\varphi_\epsilon(x)=r_0\int_{\Omega_{L_0+\epsilon}}k(x-y)\varphi_\epsilon(y)dy,\quad x\in\Omega_{L_0+\epsilon}. \tag{3.3}$$

It follows that $\phi_\epsilon:=\varphi_\epsilon\circ T(L_0,\epsilon)$ satisfies

$$\lambda(\epsilon)\phi_\epsilon(x)=r_0\int_{\Omega_{L_0}}k(T(L_0,\epsilon)(x-y))\phi_\epsilon(y)dy,\quad x\in\Omega_{L_0}. \tag{3.4}$$

Therefore, $(\lambda(\epsilon),\phi_\epsilon)$ is a principal eigenpair of the integral operator on Ω_{L_0} defined by the kernel $r_0k\circ T(L_0,\epsilon)$. As $\max_{\Omega_{L_0}}\phi_\epsilon=1$ due to the normalization of φ_ϵ, we readily see that ϕ_ϵ is twice continuously differentiable with respect to ϵ. In particular, letting $\epsilon=0$ in (3.4) results in

$$\lambda(0)\phi_0(x)=r_0\int_{\Omega_{L_0}}k(x-y)\phi_0(y)dy,\quad x\in\Omega_{L_0}. \tag{3.5}$$

Of course, $\phi_0=\varphi_0$.

Differentiating (3.4) with respect to ϵ leads to

$$\begin{aligned}&\lambda'(\epsilon)\phi_\epsilon(x)+\lambda(\epsilon)\partial_\epsilon\phi_\epsilon(x)\\&\quad=r_0\int_{\Omega_{L_0}}\left\{\langle\nabla k(T(L_0,\epsilon)(x-y)),T'_\epsilon(L_0,\epsilon)(x-y)\rangle\,\phi_\epsilon(y)+k(T(L_0,\epsilon)(x-y))\partial_\epsilon\phi_\epsilon(y)\right\}dy.\end{aligned} \tag{3.6}$$

Setting $\epsilon=0$, we find

$$\begin{aligned}&\lambda'(0)\phi_0(x)+\lambda(0)\partial_\epsilon\phi_\epsilon(x)\big|_{\epsilon=0}\\&\quad=r_0\int_{\Omega_{L_0}}\left\{\langle\nabla k(x-y),T'_\epsilon(L_0,0)(x-y)\rangle\phi_0(y)+k(x-y)\partial_\epsilon\phi_\epsilon(y)\big|_{\epsilon=0}\right\}dy.\end{aligned} \tag{3.7}$$

Multiplying (3.5) by $\partial_\epsilon \phi_\epsilon\big|_{\epsilon=0}$ and integrating over Ω_{L_0} yield

$$\begin{aligned}\lambda(0)\int_{\Omega_{L_0}} \partial_\epsilon \phi_\epsilon\big|_{\epsilon=0}\phi_0 &= r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} k(x-y)\partial_\epsilon\phi_\epsilon(x)\big|_{\epsilon=0}\phi_0(y)dydx\\ &= r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} k(x-y)\partial_\epsilon\phi_\epsilon(y)\big|_{\epsilon=0}\phi_0(x)dydx,\end{aligned}$$

where we used the radially symmetry in the second equality. Now, multiplying (3.7) by ϕ_0, integrating over Ω_{L_0} and applying the above equality, we arrive at

$$\lambda'(0)\int_{\Omega_{L_0}}\phi_0^2 = r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle\phi_0(x)\phi_0(y)dxdy. \tag{3.8}$$

Differentiating (3.5) with respect to x and multiplying by $T'_\epsilon(L_0,0)x$ gives

$$\lambda(0)\langle\nabla\phi_0(x), T'_\epsilon(L_0,0)x\rangle = r_0\int_{\Omega_{L_0}}\langle\nabla k(x-y), T'_\epsilon(L_0,0)x\rangle\phi_0(y)dy.$$

Multiplying the above equality by ϕ_0 and integrating over Ω_{L_0} yields

$$\begin{aligned}&\lambda(0)\int_{\Omega_{L_0}}\langle\nabla\phi_0(x), T'_\epsilon(L_0,0)x\rangle\phi_0(x)dx\\ &\quad= r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y), T'_\epsilon(L_0,0)x\rangle\phi_0(y)\phi_0(x)dxdy\\ &\quad= r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} -\langle\nabla k(x-y), T'_\epsilon(L_0,0)y\rangle\phi_0(y)\phi_0(x)dxdy,\end{aligned}$$

where we switched x and y in the integral and used the radial symmetry of k in the second equality. In particular,

$$2\lambda(0)\int_{\Omega_{L_0}}\langle\nabla\phi_0(x), T'_\epsilon(L_0,0)x\rangle\phi_0(x)dx = r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle\phi_0(x)\phi_0(y)dxdy.$$

It follows from (3.8) that

$$\begin{aligned}\lambda'(0)\int_{\Omega_{L_0}}\phi_0^2 &= 2\lambda(0)\int_{\Omega_{L_0}}\langle\nabla\phi_0(x), T'_\epsilon(L_0,0)x\rangle\phi_0(x)dx\\ &= \frac{\lambda(0)}{L_0}\left(2L_0\int_0^{\frac{1}{2L_0}}\phi_0^2(L_0/2,x_2)dx_2 - \frac{2}{L_0}\int_0^{\frac{L_0}{2}}\phi_0^2(x_1,\frac{1}{2L_0})dx_1\right),\end{aligned} \tag{3.9}$$

where the second equality follows from straightforward calculations of the integral and the symmetry of ϕ_0 with respect to $x_1=0$ and $x_2=0$ (see Lemma 2.1).

Now, we treat the integrals on the last line of (3.9) to determine the sign of $\lambda'(0)$. Note that for any $x_2\in\left[-\frac{1}{2L_0},\frac{1}{2L_0}\right]$, the points $\left(\frac{L_0}{2},x_2\right)$ and $\left(x_2+\frac{L_0^2-1}{2L_0},\frac{1}{2L_0}\right)$ are symmetric with respect to the line ℓ_{L_0} (see Figure 1 for an illustration), that is, $\rho_{L_0}(L_0/2,x_2)=\left(x_2+\frac{L_0^2-1}{2L_0},\frac{1}{2L_0}\right)$. Thus, we can apply Lemma 2.2 to find

$$\int_0^{\frac{1}{2L_0}}\phi_0^2(L_0/2,x_2)dx_2 \le \int_0^{\frac{1}{2L_0}}\phi_0^2\left(x_2+\frac{L_0^2-1}{2L_0},\frac{1}{2L_0}\right)dx_2 = \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}}\Phi^2, \tag{3.10}$$

where $\Phi:=\phi_0(\cdot,\frac{1}{2L_0})$ and a simple change of variable is used in the equality.

By Lemma 2.3, there exists a $\delta > 0$ such that

$$\Phi(\hat{x}_1) < \Phi(L_0/2 - \delta) \le \Phi(\tilde{x}_1), \quad \forall \tilde{x}_1 \in [0, L_0/2 - \delta]\,,\ \hat{x}_1 \in (L_0/2 - \delta, L_0/2]\,.$$

It follows from the mean value theorem that

$$\frac{1}{\delta} \int_{\frac{L_0}{2} - \delta}^{\frac{L_0}{2}} \Phi^2 < \frac{2L_0}{1 - 2\delta L_0} \int_{\frac{L_0^2 - 1}{2L_0}}^{\frac{L_0}{2} - \delta} \Phi^2.$$

Thus,

$$\int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}} \Phi^2 = \left(\int_{\frac{L_0}{2} - \delta}^{\frac{L_0}{2}} + \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2} - \delta} \right) \Phi^2 < \frac{1}{1 - 2\delta L_0} \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2} - \delta} \Phi^2 \le \frac{1}{L_0^2 - 1} \int_0^{\frac{L_0^2-1}{2L_0}} \Phi^2,$$

where we used Lemma 2.3 and the mean value theorem in the last inequality. This implies that

$$2L_0 \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}} \Phi^2 = \frac{2}{L_0} \left[(L_0^2 - 1) \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}} \Phi^2 + \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}} \Phi^2 \right] < \frac{2}{L_0} \left(\int_0^{\frac{L_0^2-1}{2L_0}} + \int_{\frac{L_0^2-1}{2L_0}}^{\frac{L_0}{2}} \right) \Phi^2 = \frac{2}{L_0} \int_0^{\frac{L_0}{2}} \Phi^2,$$

which together with (3.10) leads to $2L_0 \int_0^{\frac{1}{2L_0}} \phi_0^2(\frac{L_0}{2}, x_2) dx_2 < \frac{2}{L_0} \int_0^{\frac{L_0}{2}} \Phi^2$. It then follows from (3.9) that $\lambda'(0) < 0$.

Step 2. We show $\lambda''(0) < 0$ when $L_0 = 1$.

Denote by k'' the Hessian of k. Differentiating (3.6) with respect to ϵ yields

$$\begin{aligned}
&\lambda''(\epsilon)\phi_\epsilon(x) + 2\lambda'(\epsilon)\partial_\epsilon \phi_\epsilon(x) + \lambda(\epsilon)\partial^2_{\epsilon\epsilon}\phi_\epsilon(x) \\
&\quad = r_0 \int_{\Omega_1} \Big\{ (T'_\epsilon(1,\epsilon)(x-y))^\top k''(T(1,\epsilon)(x-y)) T'_\epsilon(1,\epsilon)(x-y)\phi_\epsilon(y) \\
&\qquad\qquad + \langle \nabla k(T(1,\epsilon)(x-y)), T''_\epsilon(1,\epsilon)(x-y)\rangle \phi_\epsilon(y) \\
&\qquad\qquad + 2\langle \nabla k(T(1,\epsilon)(x-y)), T'_\epsilon(1,\epsilon)(x-y)\rangle \partial_\epsilon \phi_\epsilon(y) + k(T(1,\epsilon)(x-y))\partial^2_{\epsilon\epsilon}\phi_\epsilon(y) \Big\} dy.
\end{aligned}$$

As $\lambda'(0) = 0$, setting $\epsilon = 0$ results in

$$\begin{aligned}
&\lambda''(0)\phi_0(x) + \lambda(0)\partial^2_{\epsilon\epsilon}\phi_\epsilon(x)\big|_{\epsilon=0} \\
&\quad = r_0 \int_{\Omega_1} \Big\{ (T'_\epsilon(1,0)(x-y))^\top k''(x-y) T'_\epsilon(1,0)(x-y)\phi_0(y) \\
&\qquad\qquad + \langle \nabla k(x-y), T''_\epsilon(1,0)(x-y)\rangle \phi_0(y) \\
&\qquad\qquad + 2\langle \nabla k(x-y), T'_\epsilon(1,0)(x-y)\rangle \psi(y) + k(x-y)\partial^2_{\epsilon\epsilon}\phi_\epsilon(y)\big|_{\epsilon=0} \Big\} dy,
\end{aligned} \tag{3.11}$$

where $\psi = \partial_\epsilon \phi_\epsilon\big|_{\epsilon=0}$. Thanks to the radial symmetry of k and the fact that Ω_1 is a square, it is easy to verify that ψ is symmetric with respect to $x_1 = 0$ and $x_2 = 0$, that is,

$$\psi(x_1, x_2) = \psi(-x_1, -x_2) = \psi(x_1, -x_2), \quad \forall (x_1, x_2) \in \Omega_1. \tag{3.12}$$

This property of ψ is needed later.

Multiplying (3.11) by ϕ_0 and integrating over Ω_1, we find

$$\begin{aligned}\lambda''(0)\int_{\Omega_1}\phi_0^2 &= r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)(x-y))^\top k''(x-y)T'_\epsilon(1,0)(x-y)\phi_0(x)\phi_0(y)dxdy\\ &\quad + r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(x-y), T''_\epsilon(1,0)(x-y)\rangle\phi_0(x)\phi_0(y)dxdy\\ &\quad + 2r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(x-y), T'_\epsilon(1,0)x\rangle\psi(y)\phi_0(x)dxdy\\ &\quad - 2r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(x-y), T'_\epsilon(1,0)y\rangle\psi(y)\phi_0(x)dxdy\\ &= \text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)},\end{aligned}\tag{3.13}$$

where we used the fact that

$$\begin{aligned}\lambda(0)\int_{\Omega_1}\partial^2_{\epsilon\epsilon}\phi_\epsilon\big|_{\epsilon=0}\phi_0 &= r_0\int_{\Omega_1}\int_{\Omega_1}k(x-y)\partial^2_{\epsilon\epsilon}\phi_\epsilon(x)\big|_{\epsilon=0}\phi_0(y)dydx\\ &= r_0\int_{\Omega_1}\int_{\Omega_1}k(x-y)\partial^2_{\epsilon\epsilon}\phi_\epsilon(y)\big|_{\epsilon=0}\phi_0(x)dydx,\end{aligned}$$

thanks to (3.5) and the radial symmetry of k.

Now, we treat the terms (I)-(IV) on the right hand side of (3.13). For (I), we see from the radial symmetry of k that

$$\begin{aligned}\text{(I)} &= 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top k''(x-y)T'_\epsilon(1,0)x\phi_0(x)\phi_0(y)dxdy\\ &\quad - 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top k''(x-y)T'_\epsilon(1,0)y\phi_0(x)\phi_0(y)dxdy\\ &= 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top k''(x-y)T'_\epsilon(1,0)(x-y)\phi_0(y)\phi_0(x)dydx.\end{aligned}\tag{3.14}$$

For (II), we see from the radial symmetry of k that

$$\text{(II)} = 2r_0\int_{\Omega_1}\int_{\Omega_1}\partial_{x_2}k(x-y)(x_2-y_2)\phi_0(x)\phi_0(y)dxdy.$$

Differentiating the eigen-equation $\mathcal{F}_{1,0}[\phi_0](x)=\lambda(0)\phi_0(x)$ with respect to the second variable x_2, multiplying the resulting equality by x_2 and $\phi_0(x)$, and integrating over Ω_1 yield

$$r_0\int_{\Omega_1}\int_{\Omega_1}x_2\partial_{x_2}k(x-y)\phi_0(x)\phi_0(y)dxdy = \lambda(0)\int_{\Omega_1}x_2\partial_{x_2}\phi_0(x)\phi_0(x)dx.$$

Similarly,

$$r_0\int_{\Omega_1}\int_{\Omega_1}y_2\partial_{y_2}k(y-x)\phi_0(x)\phi_0(y)dxdy = \lambda(0)\int_{\Omega_1}y_2\partial_{y_2}\phi_0(y)\phi_0(y)dy.$$

Since the radial symmetry of k ensures that

$$-r_0\int_{\Omega_1}\int_{\Omega_1}y_2\partial_{x_2}k(x-y)\phi_0(x)\phi_0(y)dxdy = r_0\int_{\Omega_1}\int_{\Omega_1}y_2\partial_{y_2}k(y-x)\phi_0(x)\phi_0(y)dxdy,$$

we find

$$
\begin{aligned}
\text{(II)} &= 2\lambda(0)\int_{\Omega_1} x_2\partial_{x_2}\phi_0(x)\phi_0(x)dx + 2\lambda(0)\int_{\Omega_1} y_2\partial_{y_2}\phi_0(y)\phi_0(y)dy \\
&= 2\lambda(0)\int_{\Omega_1} x_1\partial_{x_1}\phi_0(x)\phi_0(x)dx + 2\lambda(0)\int_{\Omega_1} x_2\partial_{x_2}\phi_0(x)\phi_0(x)dx \\
&= 2r_0\int_{\Omega_1}\int_{\Omega_1}(x_1\partial_{x_1}k(x-y)+x_2\partial_{x_2}k(x-y))\phi_0(x)\phi_0(y)dxdy \\
&= 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top T'_\epsilon(1,0)\nabla k(x-y)\phi_0(x)\phi_0(y)dxdy,
\end{aligned}
\tag{3.15}
$$

where we used the fact that $\phi_0(x_1,x_2)=\phi_0(x_2,x_1)$ for all $(x_1,x_2)\in\Omega_1$ (see (2.1)) in the second equality.

Now, we treat the term (III). By (3.7) and $\lambda'(0)=0$, we have

$$
\lambda(0)\psi(x) = r_0\int_{\Omega_1}\Big\{\langle\nabla k(x-y), T'_\epsilon(1,0)(x-y)\rangle\phi_0(y) + k(x-y)\psi(y)\Big\}dy.
$$

Differentiating the above equality with respect to x, multiplying the resulting equality by $T'_\epsilon(1,0)x$ and $2\phi_0(x)$, and integrating over Ω_1 yield

$$
\begin{aligned}
&2\lambda(0)\int_{\Omega_1}\langle\nabla\psi(x), T'_\epsilon(1,0)x\rangle\phi_0(x)dx \\
&\quad= 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top k''(x-y), T'_\epsilon(1,0)(x-y)\phi_0(y)\phi_0(x)dydx \\
&\qquad+ 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top T'_\epsilon(1,0)\nabla k(x-y)\phi_0(y)\phi_0(x)dydx \\
&\qquad+ 2r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(x-y), T'_\epsilon(1,0)x\rangle\psi(y)\phi_0(x)dydx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{(III)} &= 2\lambda(0)\int_{\Omega_1}\langle\nabla\psi(x), T'_\epsilon(1,0)x\rangle\phi_0(x)dx \\
&\quad- 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top k''(x-y), T'_\epsilon(1,0)(x-y)\phi_0(y)\phi_0(x)dydx \\
&\quad- 2r_0\int_{\Omega_1}\int_{\Omega_1}(T'_\epsilon(1,0)x)^\top T'_\epsilon(1,0)\nabla k(x-y)\phi_0(y)\phi_0(x)dydx.
\end{aligned}
\tag{3.16}
$$

For (IV), we differentiate the eigen-equation $\lambda(0)\phi_0(y)=\mathcal{F}_{1,0}[\phi_0](y)$ with respect to y, multiply the resulting equality by $T'_\epsilon(1,0)y$ and $2\psi(y)$, and integrate over Ω_1 to arrive at

$$
2\lambda(0)\int_{\Omega_1}\langle\nabla\phi_0(y), T'_\epsilon(1,0)y\rangle\psi(y)dy = 2r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(y-x), T'_\epsilon(1,0)y\rangle\psi(y)\phi_0(x)dxdy.
$$

It follows that

$$
\begin{aligned}
\text{(IV)} &= 2r_0\int_{\Omega_1}\int_{\Omega_1}\langle\nabla k(y-x), T'_\epsilon(1,0)y\rangle\psi(y)\phi_0(x)dxdy \\
&= 2\lambda(0)\int_{\Omega_1}\langle\nabla\phi_0(y), T'_\epsilon(1,0)y\rangle\psi(y)dy,
\end{aligned}
\tag{3.17}
$$

where we used the radial symmetry of k in first equality.

Inserting (3.14), (3.15), (3.16) and (3.17) into (3.13) results in

$$\begin{aligned}\lambda''(0)\int_{\Omega_1}\phi_0^2 &= 2\lambda(0)\int_{\Omega_1}\langle\nabla\phi_0(x),T'_\epsilon(1,0)x)\rangle\psi(x)dx+2\lambda(0)\int_{\Omega_1}\langle\nabla\psi(x),T'_\epsilon(1,0)x\rangle\phi_0(x)dx\\ &= 2\lambda(0)\int_{-\frac12}^{\frac12}[x_1\psi(x)\phi_0(x)]\Big|_{x_1=-\frac12}^{x_1=\frac12}dx_2-2\lambda(0)\int_{-\frac12}^{\frac12}[x_2\psi(x)\phi_0(x)]\Big|_{x_2=-\frac12}^{x_2=\frac12}dx_1\\ &= 2\lambda(0)\int_{-\frac12}^{\frac12}\psi(1/2,x_2)\phi_0(1/2,x_2)dx_2-2\lambda(0)\int_{-\frac12}^{\frac12}\psi(x_1,1/2)\phi_0(x_1,1/2)dx_1 \qquad (3.18)\\ &= 2\lambda(0)\int_{-\frac12}^{\frac12}(\psi(1/2,x_2)-\psi(x_2,1/2))\phi_0(x_2,1/2)dx_2\\ &= 4\lambda(0)\int_0^{\frac12}(\psi(1/2,x_2)-\psi(x_2,1/2))\phi_0(x_2,1/2)dx_2,\end{aligned}$$

where the second equality follows from straightforward calculations, the third equality follows from the symmetry of ψ and ϕ_0 with respect to $x_1=0$ and $x_2=0$ (see (3.12) and Lemma 2.1), the fourth equality follows from $\phi_0(x_1,x_2)=\phi_0(x_2,x_1)$ for all $(x_1,x_2)\in\Omega_1$ (see (2.1)), and the last equality follows again from (3.12) and Lemma 2.1.

We claim that

$$\begin{aligned}&\psi(1/2,x_2)\le\psi(x_2,1/2),\quad \forall x_2\in[0,1/2]\,,\\ &\exists\delta\in(0,1/2)\quad \text{s.t.}\quad \psi(1/2,x_2)<\psi(x_2,1/2),\quad \forall x_2\in(1/2-\delta,1/2)\,.\end{aligned}\tag{3.19}$$

Assuming (3.19), we conclude from (3.18) that $\lambda''(0)<0$, completing the proof.

To finish the proof, it remains to show (3.19). Recall that $\psi=\partial_\epsilon\phi_\epsilon\big|_{\epsilon=0}$ and $\phi_\epsilon=\varphi_\epsilon\circ T(1,\epsilon)$. We may assume without loss of generality that $0\le\epsilon\ll1$; the case $-1\ll\epsilon\le0$ can be treated in the same manner.

Note that for $\epsilon>0$, the width of $\Omega_{1+\epsilon}$ is greater than its height as shown in Figure 1. By Lemma 2.2, we have $\varphi_\epsilon\le\varphi_\epsilon\circ\rho_{1+\epsilon}$ in $A_{1+\epsilon}$. In particular,

$$\varphi_\epsilon\left(\frac{1+\epsilon}{2},\tilde x_2\right)\le\varphi_\epsilon\left(\tilde x_1,\frac{1}{2(1+\epsilon)}\right),\quad \tilde x_2\in\left[0,\frac{1}{2(1+\epsilon)}\right],$$

where $\tilde x_1=\tilde x_2+\frac{(1+\epsilon)^2-1}{2(1+\epsilon)}$. Obviously, $(1+\epsilon)^2\tilde x_2\le\tilde x_1$ for $0<\epsilon\ll1$. By the mean value theorem,

$$\varphi_\epsilon\left(\tilde x_1,\frac{1}{2(1+\epsilon)}\right)=\varphi_\epsilon\left((1+\epsilon)^2\tilde x_2,\frac{1}{2(1+\epsilon)}\right)+(\tilde x_1-(1+\epsilon)^2\tilde x_2)\partial_{x_1}\varphi_\epsilon\left(\xi,\frac{1}{2(1+\epsilon)}\right),$$

where $\xi\in((1+\epsilon)^2\tilde x_2,\tilde x_1)$. It follows that

$$\begin{aligned}\varphi_\epsilon\left(\frac{1+\epsilon}{2},\tilde x_2\right)&\le\varphi_\epsilon\left((1+\epsilon)^2\tilde x_2,\frac{1}{2(1+\epsilon)}\right)+(\tilde x_1-(1+\epsilon)^2\tilde x_2)\partial_{x_1}\varphi_\epsilon\left(\xi,\frac{1}{2(1+\epsilon)}\right)\\ &=\varphi_\epsilon\left((1+\epsilon)^2\tilde x_2,\frac{1}{2(1+\epsilon)}\right)+\frac{\epsilon^2+2\epsilon}{1+\epsilon}\left(\frac12-(1+\epsilon)\tilde x_2\right)\partial_{x_1}\varphi_\epsilon\left(\xi,\frac{1}{2(1+\epsilon)}\right).\end{aligned}$$

Setting $x_2:=(1+\epsilon)\tilde x_2$, we find from $\phi_\epsilon=\varphi_\epsilon\circ T(1,\epsilon)$ that

$$\phi_\epsilon(1/2,x_2)\le\phi_\epsilon(x_2,1/2)+\frac{\epsilon^2+2\epsilon}{1+\epsilon}\left(\frac12-x_2\right)\partial_{x_1}\varphi_\epsilon\left(\xi,\frac{1}{2(1+\epsilon)}\right),\tag{3.20}$$

where $\xi \in \left((1+\epsilon)x_2, \frac{x_2}{1+\epsilon} + \frac{\epsilon^2+2\epsilon}{2(1+\epsilon)}\right)$. Since $\phi_\epsilon(x)$ is continuously differentiable in ϵ,

$$\begin{aligned}\partial_\epsilon \phi_\epsilon(1/2, x_2)\big|_{\epsilon=0} &= \lim_{\epsilon\to 0^+} \frac{\phi_\epsilon(1/2,x_2) - \phi_0(1/2,x_2)}{\epsilon} \\ &\le \lim_{\epsilon\to 0^+} \frac{\phi_\epsilon(x_2,1/2) - \phi_0(1/2,x_2)}{\epsilon} + \lim_{\epsilon\to 0^+}\left[\frac{\epsilon+2}{1+\epsilon}\left(\frac{1}{2} - x_2\right)\partial_{x_1}\varphi_\epsilon\left(\xi, \frac{1}{2(1+\epsilon)}\right)\right] \\ &= \lim_{\epsilon\to 0^+} \frac{\phi_\epsilon(x_2,1/2) - \phi_0(x_2,1/2)}{\epsilon} + (1-2x_2)\partial_{x_1}\varphi_0(x_2,1/2) \\ &= \partial_\epsilon\phi_\epsilon(x_2,1/2)\big|_{\epsilon=0} + (1-2x_2)\partial_{x_1}\varphi_0(x_2,1/2),\end{aligned}$$

we used (3.20) in the inequality, and (2.1) and $\lim_{\epsilon\to 0^+}\xi = x_2$ in the second equality. Note that Lemma 2.3 ensures that

$$\begin{aligned}&(1-2x_2)\partial_{x_1}\varphi_0(x_2,1/2) \le 0, \quad \forall x_2 \in (0,1/2]\,, \\ &\exists \delta \in (0,1/2) \quad \text{s.t.} \quad (1-2x_2)\partial_{x_1}\varphi_0(x_2,1/2) < 0, \quad \forall x_2 \in (1/2-\delta, 1/2)\,.\end{aligned}$$

Hence, (3.19) follows. This completes the proof.

3.2. **Proof of Theorem A (ii).** First, recall that the principal eigenvalue problem

$$\lambda(L_0,c,v)\varphi_{L_0,c,v}(x) = r_0 \int_{\Omega_{L_0}} k(x-y+cv)\varphi_{L_0,c,v}(y)dy, \quad x \in \Omega_{L_0}. \tag{3.21}$$

The second-order Taylor expansions of $\lambda(L_0,c,v)$, $\varphi_{L_0,c,v}(x)$ and $k(x-y+cv)$ about $c=0$ read

$$\begin{aligned}\lambda(L_0,c,v) &= \lambda(L_0,0,v) + c\lambda'_c(L_0,0,v) + \frac{c^2}{2}\lambda''_{cc}(L_0,0,v) + o(c^2), \\ \varphi_{L_0,c,v} &= \varphi_{L_0,0,v} + c\partial_c\varphi_{L_0,c,v}\big|_{c=0} + \frac{c^2}{2}\partial^2_{cc}\varphi_{L_0,c,v}\big|_{c=0} + o(c^2), \\ k(x+cv) &= k(x) + c\langle\nabla k(x), v\rangle + \frac{c^2}{2}v^\top k''(x)v + o(c^2).\end{aligned} \tag{3.22}$$

Set

$$\lambda(c) := \lambda(L_0,c,v), \quad \phi_0 := \varphi_{L_0,0}, \quad \phi_1 := \partial_c\varphi_{L_0,c,v}\big|_{c=0} \quad \text{and} \quad \phi_2 := \partial^2_{cc}\varphi_{L_0,c,v}\big|_{c=0}.$$

Then, $\lambda(0) = \lambda(L_0,0,v)$, $\lambda'(0) = \lambda'_c(L_0,0,v)$ and $\lambda''(0) = \lambda''_{cc}(L_0,0,v)$.

Inserting (3.22) into (3.21) and matching terms haing the same order in c lead to

$$\lambda(0)\phi_0(x) = r_0\int_{\Omega_{L_0}} k(x-y)\phi_0(y)dy, \tag{3.23}$$

$$\lambda'(0)\phi_0(x) + \lambda(0)\phi_1(x) = r_0\int_{\Omega_{L_0}}\Big\{k(x-y)\phi_1(y) + \langle\nabla k(x-y), v\rangle\phi_0(y)\Big\}dy, \tag{3.24}$$

and

$$\begin{aligned}&\frac{1}{2}\lambda''(0)\phi_0(x) + \lambda'(0)\phi_1(x) + \frac{1}{2}\lambda(0)\phi_2(x) \\ &\quad = r_0\int_{\Omega_{L_0}}\left\{\frac{1}{2}k(x-y)\phi_2(y) + \langle\nabla k(x-y), v\rangle\phi_1(y) + \frac{1}{2}v^\top k''(x-y)v\phi_0(y)\right\}dy.\end{aligned} \tag{3.25}$$

We show $\lambda'(0) = 0$. Multiplying (3.24) by ϕ_0 and integrating over Ω_{L_0} yield

$$\lambda'(0)\int_{\Omega_{L_0}}\phi_0^2 = r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y), v\rangle\phi_0(y)\phi_0(x)dxdy, \tag{3.26}$$

where we used

$$\lambda(0)\int_{\Omega_{L_0}}\phi_0\phi_1=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi_0(y)\phi_1(x)dxdy=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi_0(x)\phi_1(y)dxdy.$$

Differentiating (3.23) with respect to x, multiplying the resulting equality by v and ϕ_0, and integrating over Ω_{L_0}, we find

$$r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),v\rangle\phi_0(y)\phi_0(x)dxdy=\lambda(0)\int_{\Omega_{L_0}}\langle\nabla\phi_0,v\rangle\phi_0=0,$$

where we used Lemma 2.1 implying $\int_{\Omega_{L_0}}\langle\nabla\phi_0,v\rangle\phi_0=0$. It then follows from (3.26) that $\lambda'(0)=0$.

We show $\lambda''(0)<0$ in the rest. Thanks to $\lambda'(0)=0$, the equalities (3.24) and (3.25) are respectively reduced to

$$\lambda(0)\phi_1(x)=r_0\int_{\Omega_{L_0}}\Big\{k(x-y)\phi_1(y)+\langle\nabla k(x-y),v\rangle\phi_0(y)\Big\}dy, \tag{3.27}$$

and

$$\begin{aligned}&\frac{1}{2}\lambda''(0)\phi_0(x)+\frac{1}{2}\lambda(0)\phi_2(x)\\&\quad=r_0\int_{\Omega_{L_0}}\left\{\frac{1}{2}k(x-y)\phi_2(y)+\langle\nabla k(x-y),v\rangle\phi_1(y)+\frac{1}{2}v^Tk''(x-y)v\phi_0(y)\right\}dy.\end{aligned} \tag{3.28}$$

Note from (3.23) that

$$\lambda(0)=r_0\sup_{\|\phi\|_{L^2(\Omega_{L_0})}=1}\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi(y)\phi(x)dydx.$$

Thus,

$$r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi_1(x)\phi_1(y)dxdy\le\lambda(0)\int_{\Omega_{L_0}}\phi_1^2,$$

which together with (3.27) yields

$$r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),v\rangle\phi_1(x)\phi_0(y)dydx\ge 0. \tag{3.29}$$

We multiply (3.28) by ϕ_0 and integrate over Ω_{L_0} to obtain

$$\frac{1}{2}\lambda''(0)\int_{\Omega_{L_0}}\phi_0^2=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\left\{\langle\nabla k(x-y),v\rangle\phi_1(y)\phi_0(x)+\frac{1}{2}v^\top k''(x-y)v\phi_0(y)\phi_0(x)\right\}dxdy, \tag{3.30}$$

where we used

$$\lambda(0)\int_{\Omega_{L_0}}\phi_0\phi_2=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi_0(y)\phi_2(x)dxdy=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}k(x-y)\phi_0(x)\phi_2(y)dxdy.$$

Since

$$r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}v^\top k''(x-y)v\phi_0(y)\phi_0(x)dxdy=\lambda(0)\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0,$$

where ϕ_0'' denotes the Hessian of ϕ_0, and

$$r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),v\rangle\phi_1(y)\phi_0(x)dxdy=-r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),v\rangle\phi_1(x)\phi_0(y)dxdy,$$

we find from (3.30) that

$$\begin{aligned}\frac{1}{2}\lambda''(0)\int_{\Omega_{L_0}}\phi_0^2 &= -r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),v\rangle\phi_1(x)\phi_0(y)dxdy+\frac{\lambda(0)}{2}\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0\\ &\le\frac{\lambda(0)}{2}\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0,\end{aligned}\tag{3.31}$$

where we used (3.29) in the inequality.

To finish the proof, it remains to show

$$\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0<0.\tag{3.32}$$

Let $v=(v_1,v_2)^\top$. Then,

$$\begin{aligned}\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0 &= v_1^2\int_{\Omega_{L_0}}\partial^2_{x_1x_1}\phi_0\phi_0+2v_1v_2\int_{\Omega_{L_0}}\partial^2_{x_1x_2}\phi_0\phi_0+v_2^2\int_{\Omega_{L_0}}\partial^2_{x_2x_2}\phi_0\phi_0\\ &= v_1^2(\mathrm{I})+2v_1v_2(\mathrm{II})+v_2^2(\mathrm{III}).\end{aligned}$$

We now treat (I), (II) and (III). Integration by parts yields

$$\begin{aligned}(\mathrm{I}) &= \int_{-\frac{1}{2L_0}}^{\frac{1}{2L_0}}\partial_{x_1}\phi_0(L_0/2,x_2)\phi_0(L_0/2,x_2)dx_2-\int_{-\frac{1}{2L_0}}^{\frac{1}{2L_0}}\partial_{x_1}\phi_0(-L_0/2,x_2)\phi_0(-L_0/2,x_2)dx_2-\int_{\Omega_{L_0}}(\partial_{x_1}\phi_0)^2\\ &= 2\int_{-\frac{1}{2L_0}}^{\frac{1}{2L_0}}\partial_{x_1}\phi_0(L_0/2,x_2)\phi_0(L_0/2,x_2)dx_2-\int_{\Omega_{L_0}}(\partial_{x_1}\phi_0)^2\end{aligned}$$

where we used the fact that the symmetry of ϕ_0 with respect to $x_1=0$ in the second equality. Similarly, integration by parts and the symmetry of ϕ_0 with respect to $x_2=0$ imply

$$(\mathrm{III})=2\int_{-\frac{L_0}{2}}^{\frac{L_0}{2}}\partial_{x_2}\phi_0(x_1,\frac{1}{2L_0})\phi_0(x_1,\frac{1}{2L_0})dx_1-\int_{\Omega_{L_0}}(\partial_{x_2}\phi_0)^2.$$

Again, by integration by parts and the symmetry of ϕ_0 with respect to $x_2=0$, we find

$$\begin{aligned}(\mathrm{II}) &= \int_{-\frac{L_0}{2}}^{\frac{L_0}{2}}\partial_{x_1}\phi_0(x_1,\frac{1}{2L_0})\phi_0(x_1,\frac{1}{2L_0})dx_1-\int_{-\frac{L_0}{2}}^{\frac{L_0}{2}}\partial_{x_1}\phi_0(x_1,-\frac{1}{2L_0})\phi_0(x_1,-\frac{1}{2L_0})dx_1-\int_{\Omega_{L_0}}\partial_{x_1}\phi_0\partial_{x_2}\phi_0\\ &= -\int_{\Omega_{L_0}}\partial_{x_1}\phi_0\partial_{x_2}\phi_0.\end{aligned}$$

Since

$$v_1^2\int_{\Omega_{L_0}}(\partial_{x_1}\phi_0)^2+2v_1v_2\int_{\Omega_{L_0}}\partial_{x_1}\phi_0\partial_{x_2}\phi_0+v_2^2\int_{\Omega_{L_0}}(\partial_{x_2}\phi_0)^2=\int_{\Omega_{L_0}}(v_1\partial_{x_1}\phi_0+v_2\partial_{x_2}\phi_0)^2\ge0,$$

we arrive at

$$\int_{\Omega_{L_0}}v^\top\phi_0''v\phi_0\le2v_1^2\int_{-\frac{1}{2L_0}}^{\frac{1}{2L_0}}\partial_{x_1}\phi_0(L_0/2,x_2)\phi_0(L_0/2,x_2)dx_2+2v_2^2\int_{-\frac{L_0}{2}}^{\frac{L_0}{2}}\partial_{x_2}\phi_0(x_1,\frac{1}{2L_0})\phi_0(x_1,\frac{1}{2L_0})dx_1.$$

Since Lemma 2.3 implies

$$\int_{-\frac{1}{2L_0}}^{\frac{1}{2L_0}}\partial_{x_1}\phi_0(L_0/2,x_2)\phi_0(L_0/2,x_2)dx_2<0\quad\text{and}\quad\int_{-\frac{L_0}{2}}^{\frac{L_0}{2}}\partial_{x_2}\phi_0(x_1,\frac{1}{2L_0})\phi_0(x_1,\frac{1}{2L_0})dx_1<0,$$

we conclude $\int_{\Omega_{L_0}} v^\top \phi_0'' v\phi_0 < 0$, that is, (3.32) holds. This completes the proof.

3.3. **Proof of Theorem A (iii).** Set $\lambda(\epsilon, c) := \lambda(L_0+\epsilon, c, v)$ and $\varphi_{\epsilon,c} := \varphi_{L_0+\epsilon,c,v}$. The conclusion follows if we show $\lambda''_{\epsilon c}(0,0,v) = 0$.

Recall $T(L_0,\epsilon)$ from (3.2). Setting $\phi_{\epsilon,c} := \varphi_{\epsilon,c} \circ T(L_0,\epsilon)$, we find from $\mathcal{F}_{L_0+\epsilon,c,v}\varphi_{\epsilon,c} = \lambda(\epsilon,c)\varphi_{\epsilon,c}$ that

$$\lambda(\epsilon,c,v)\phi_{\epsilon,c}(x) = r_0 \int_{\Omega_{L_0}} k(T(L_0,\epsilon)(x-y)+cv)\phi_{\epsilon,c}(y)dy, \quad x \in \Omega_{L_0}. \tag{3.33}$$

Second-order Taylor expansions of $\lambda(\epsilon,c)$, $\phi_{\epsilon,c}$ and $k(T(L_0,\epsilon)(x-y)+cv)$ about $(\epsilon,c) = (0,0)$ read respectively

$$\begin{aligned}
\lambda(\epsilon,c) &= \lambda(0,0) + \epsilon\lambda'_\epsilon(0,0) + \frac{\epsilon^2}{2}\lambda''_{\epsilon\epsilon}(0,0) + \epsilon c\lambda''_{\epsilon c}(0,0) + \frac{c^2}{2}\lambda''_{cc}(0,0)\\
&\quad + \epsilon^2 h_1(\epsilon,c) + \epsilon c h_2(\epsilon,c) + c^2 h_3(\epsilon,c),\\
\phi_{\epsilon,c} &= \phi_0 + \epsilon\phi_1 + c\tilde{\phi}_1 + \frac{\epsilon^2}{2}\phi_2 + \epsilon c\bar{\phi}_2 + \frac{c^2}{2}\tilde{\phi}_2\\
&\quad + \epsilon^2\hat{h}_1(\epsilon,c) + \epsilon c\hat{h}_2(\epsilon,c) + c^2\hat{h}_3(\epsilon,c),\\
k(T(L_0,\epsilon)(x-y)+cv) &= k(x-y) + \epsilon\langle\nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle + c\langle\nabla k(x-y), v\rangle\\
&\quad + \frac{\epsilon^2}{2}\big(\langle\nabla k(x-y), T''_{\epsilon\epsilon}(L_0,0)(x-y)\rangle + (T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)T'_\epsilon(L_0,0)(x-y)\big)\\
&\quad + \epsilon c(T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v + \frac{c^2}{2}v^\top k''(x-y)v\\
&\quad + \epsilon^2\bar{h}_1(\epsilon,c) + \epsilon c\bar{h}_2(\epsilon,c) + c^2\bar{h}_3(\epsilon,c),
\end{aligned}$$

where we used $\lambda'_c(0,0) = 0$ (by Theorem A (ii)) in the expansion of $\lambda(\epsilon,c)$, $\phi_1 := \partial_\epsilon\phi_{\epsilon,c}\big|_{(0,0)}$, $\tilde{\phi}_1 := \partial_c\phi_{\epsilon,c}\big|_{(0,0)}$, $\phi_2 := \partial_{\epsilon\epsilon}\phi_{\epsilon,c}\big|_{(0,0)}$, $\bar{\phi}_2 := \partial_{\epsilon c}\phi_{\epsilon,c}\big|_{(0,0)}$ and $\tilde{\phi}_2 := \partial_{cc}\phi_{\epsilon,c}\big|_{(0,0)}$, and $h_i(\epsilon,c)\to 0$, $\hat{h}_i(\epsilon,c)\to 0$ and $\bar{h}_i(\epsilon,c)\to 0$ as $(\epsilon,c)\to(0,0)$ for $i=1,2,3$.

Inserting them into (3.33) and collecting terms of the same order, we find the following:

- order 1:

$$\lambda(0,0)\phi_0(x) = r_0\int_{\Omega_{L_0}} k(x-y)\phi_0(y)dy, \tag{3.34}$$

- order ϵ:

$$\lambda'_\epsilon(0,0)\phi_0(x) + \lambda(0,0)\phi_1(x) = r_0\int_{\Omega_{L_0}}\Big\{k(x-y)\phi_1(y) + \langle\nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle\phi_0(y)\Big\}dy, \tag{3.35}$$

- order c:

$$\lambda(0,0)\tilde{\phi}_1(x) = r_0\int_{\Omega_{L_0}}\Big\{k(x-y)\tilde{\phi}_1(y) + \langle\nabla k(x-y), v\rangle\phi_0(y)\Big\}dy, \tag{3.36}$$

- order ϵc:

$$\begin{aligned}
&\lambda''_{\epsilon c}(0,0)\phi_0(x) + \lambda'_\epsilon(0,0)\tilde{\phi}_1(x) + \lambda(0,0)\bar{\phi}_2(x) \\
&\quad = r_0 \int_{\Omega_{L_0}} k(x-y)\bar{\phi}_2(y)dy + r_0 \int_{\Omega_{L_0}} \langle \nabla k(x-y), v\rangle \phi_1(y)dy \\
&\qquad + r_0 \int_{\Omega_{L_0}} \langle \nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle \tilde{\phi}_1(y)dy \\
&\qquad + r_0 \int_{\Omega_{L_0}} (T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v\phi_0(y)dy.
\end{aligned} \tag{3.37}$$

Multiplying (3.35) and (3.36) respectively by $\tilde{\phi}_1$ and ϕ_1 and integrating over Ω_{L_0} lead to

$$\begin{aligned}
&\lambda'_\epsilon(0,0)\int_{\Omega_{L_0}} \phi_0\tilde{\phi}_1 + \lambda(0,0)\int_{\Omega_{L_0}} \phi_1\tilde{\phi}_1 \\
&\quad = r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \Big\{ k(x-y)\phi_1(y)\tilde{\phi}_1(x) + \langle \nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle \phi_0(y)\tilde{\phi}_1(x)\Big\} dxdy
\end{aligned}$$

and

$$\lambda(0,0)\int_{\Omega_{L_0}} \tilde{\phi}_1\phi_1 = r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \Big\{ k(x-y)\tilde{\phi}_1(y)\phi_1(x) + \langle \nabla k(x-y), v\rangle \phi_0(y)\phi_1(x)\Big\} dxdy.$$

Inserting the second equality into the first one, we conclude from the fact

$$\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} k(x-y)\Big(\phi_1(y)\tilde{\phi}_1(x) - \tilde{\phi}_1(y)\phi_1(x)\Big)\, dxdy = 0,$$

thanks to the radial symmetry of k that

$$\begin{aligned}
&\lambda'_\epsilon(0,0)\int_{\Omega_{L_0}} \phi_0\tilde{\phi}_1 \\
&\quad = r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \Big\{ \langle \nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle \phi_0(y)\tilde{\phi}_1(x) - \langle \nabla k(x-y), v\rangle \phi_0(y)\phi_1(x)\Big\} dxdy,
\end{aligned} \tag{3.38}$$

It follows from (3.34) and the radial symmetry of k that

$$\lambda(0,0)\int_{\Omega_{L_0}} \bar{\phi}_2\phi_0 = r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} k(x-y)\bar{\phi}_2(y)\phi_0(x)dxdy.$$

Multiplying (3.37) by ϕ_0 and integrating over Ω_{L_0} then yield

$$\begin{aligned}
&\lambda''_{\epsilon c}(0,0)\int_{\Omega_{L_0}} \phi_0^2 + \lambda'_\epsilon(0,0)\int_{\Omega_{L_0}} \tilde{\phi}_1\phi_0 \\
&\quad = r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \Big\{ \langle \nabla k(x-y), v\rangle \phi_1(y)\phi_0(x) + \langle \nabla k(x-y), T'_\epsilon(L_0,0)(x-y)\rangle \tilde{\phi}_1(y)\phi_0(x)\Big\} dxdy \\
&\qquad + r_0 \int_{\Omega_{L_0}}\int_{\Omega_{L_0}} (T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v\phi_0(y)\phi_0(x)dxdy.
\end{aligned} \tag{3.39}$$

We see from the radial symmetry of k that

$$\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \langle \nabla k(x-y), v\rangle \phi_1(y)\phi_0(x)dxdy = -\int_{\Omega_{L_0}}\int_{\Omega_{L_0}} \langle \nabla k(x-y), v\rangle \phi_1(x)\phi_0(y)dxdy$$

and

$$\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),T'_\epsilon(L_0,0)(x-y)\rangle\tilde{\phi}_1(y)\phi_0(x)dxdy=\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),T'_\epsilon(L_0,0)(x-y)\rangle\tilde{\phi}_1(x)\phi_0(y)dxdy.$$

For the second term on the right hand side of (3.39), we claim

$$\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v\phi_0(y)\phi_0(x)dxdy=0. \tag{3.40}$$

Indeed, setting $h(x):=\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v\phi_0(y)\phi_0(x)dy$, we have

$$\begin{aligned}h(-x)&=\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(-x-y))^\top k''(-x-y)v\phi_0(y)\phi_0(-x)dy\\&=-\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(x+y))^\top k''(-x-y)v\phi_0(y)\phi_0(x)dy\\&=-\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(x-y))^\top k''(-x+y)v\phi_0(-y)\phi_0(x)dy\\&=-\int_{\Omega_{L_0}}(T'_\epsilon(L_0,0)(x-y))^\top k''(x-y)v\phi_0(y)\phi_0(x)dy,\end{aligned}$$

we used the fact that k is radially symmetric and $\phi_0(x)=\phi_0(-x)$ for any $x\in\Omega_{L_0}$. Thus, $h(-x)=-h(x)$ for $x\in\Omega_{L_0}$, implying $\int_{\Omega_{L_0}}hdx=0$, namely, (3.40).

Hence, (3.39) becomes

$$\begin{aligned}&\lambda''_{\epsilon c}(0,0)\int_{\Omega_{L_0}}\phi_0^2+\lambda'_\epsilon(0,0)\int_{\Omega_{L_0}}\phi_0\tilde{\phi}_1\\&\quad=r_0\int_{\Omega_{L_0}}\int_{\Omega_{L_0}}\langle\nabla k(x-y),T'_\epsilon(L_0,0)(x-y)\rangle\tilde{\phi}_1(x)\phi_0(y)-\langle\nabla k(x-y),v\rangle\phi_1(x)\phi_0(y)dxdy.\end{aligned}$$

This together with (3.38) yields

$$\lambda''_{\epsilon c}(0,0)\int_{\Omega_{L_0}}\phi_0^2+\lambda'_\epsilon(0,0)\int_{\Omega_{L_0}}\phi_0\tilde{\phi}_1=\lambda'_\epsilon(0,0)\int_{\Omega_{L_0}}\phi_0\tilde{\phi}_1,$$

leading to $\lambda''_{\epsilon c}(0,0)=0$. This completes the proof.

4. Proof of Theorem B

This section is devoted to the proof of Theorem B. Set $v(\theta):=(\cos\theta,\sin\theta)$ and $\lambda(c,\theta):=\lambda(1,c,v(\theta))$ for $\theta\in[0,2\pi]$. Recall that $\varphi_{1,c,v}$ is the principal eigenfunction of $\mathcal{F}_{1,c,v}$ associated with the principal eigenvalue $\lambda(1,c,v)$ and satisfies the normalization (1.11). Setting $\phi_{c,\theta}:=\varphi_{1,c,v(\theta)}$, we see from the eigen-equation that

$$\lambda(c,\theta)\phi_{c,\theta}(x)=r_0\int_{\Omega_1}k(x-y+cv(\theta))\phi_{c,\theta}(y)dy,\quad x\in\Omega_1. \tag{4.1}$$

This together with the radial symmetry of k and the symmetry of Ω_1 ensures that

$$\lambda(c,\theta)=\lambda(-c,\theta)=\lambda(c,-\theta), \tag{4.2}$$

and

$$\lambda(c,\pi/4-\theta)=\lambda(c,\pi/4+\theta). \tag{4.3}$$

Hence, we only need to consider $\lambda(c,\theta)$ for $\theta\in[0,\frac{\pi}{4}]$.

For $\hat{\theta} \in [0, \pi/4]$, Taylor series of $\lambda(c, \theta)$ at $(c, \theta) = (0, \hat{\theta})$ reads

$$\lambda(c,\theta) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{m}{n} \frac{c^{m-n}(\theta - \hat{\theta})^n}{m!} \lambda_{c^{m-n}\theta^n}(0, \hat{\theta}),$$

where $\lambda_{c^{m-n}\theta^n}(0, \hat{\theta}) := \partial^m_{c^{m-n}\theta^n} \lambda(c, \theta)|_{(0,\hat{\theta})}$.

Note from (4.1) with $c = 0$ that $\lambda_0 := \lambda(0, \hat{\theta})$ is independent of $\hat{\theta}$, and hence,

$$\lambda_{\theta^m}(0, \hat{\theta}) = 0, \quad \forall m \in \mathbb{N}^+,$$

where $\mathbb{N}^+$ denotes the set of positive natural numbers. Moreover, the symmetry $\lambda(c, \theta) = \lambda(-c, \theta)$ from (4.2) yields that

$$\lambda_{c^{2i+1}\theta^j}(0, \hat{\theta}) = 0, \quad \forall i, j \in \mathbb{N}$$

where $\mathbb{N} = \{0\} \cup \mathbb{N}^+$. Hence, the Taylor series of $\lambda(c, \theta)$ at $(c, \theta) = (0, \hat{\theta})$ becomes

$$\lambda(c,\theta) = \lambda_0 + \sum_{m=2}^{\infty} \sum_{\substack{n \in \{0,1,\ldots,m-2\} \\ m-n \in 2\mathbb{N}^+}} \binom{m}{n} \frac{c^{m-n}(\theta - \hat{\theta})^n}{m!} \lambda_{c^{m-n}\theta^n}(0, \hat{\theta}).$$

Setting $2i = m - n \in 2\mathbb{N}^+$, we can rewrite the above series of $\lambda(c, \theta)$ as

$$\begin{aligned}
\lambda(c,\theta) &= \lambda_0 + \sum_{m=2}^{\infty} \sum_{i=1}^{[\frac{m}{2}]} \binom{m}{m-2i} \frac{c^{2i}(\theta - \hat{\theta})^{m-2i}}{m!} \lambda_{c^{2i}\theta^{m-2i}}(0, \hat{\theta}) \\
&= \lambda_0 + \sum_{j=1}^{\infty} \sum_{i=1}^{j} \binom{2j}{2j-2i} \frac{c^{2i}(\theta - \hat{\theta})^{2j-2i}}{(2j)!} \lambda_{c^{2i}\theta^{2j-2i}}(0, \hat{\theta}) \\
&\quad + \sum_{j=1}^{\infty} \sum_{i=1}^{j} \binom{2j+1}{2j-2i+1} \frac{c^{2i}(\theta - \hat{\theta})^{2j-2i+1}}{(2j+1)!} \lambda_{c^{2i}\theta^{2j-2i+1}}(0, \hat{\theta}),
\end{aligned}$$

where $[\frac{m}{2}]$ denotes the greatest integer less than or equal to $\frac{m}{2}$. Rearrangement yields

$$\begin{aligned}
\lambda(c,\theta) = \lambda_0 &+ \sum_{i=1}^{\infty} c^{2i} \sum_{j=i}^{\infty} \binom{2j}{2j-2i} \frac{(\theta - \hat{\theta})^{2j-2i}}{(2j)!} \lambda_{c^{2i}\theta^{2j-2i}}(0, \hat{\theta}) \\
&+ \sum_{i=1}^{\infty} c^{2i} \sum_{j=i}^{\infty} \binom{2j+1}{2j-2i+1} \frac{(\theta - \hat{\theta})^{2j-2i+1}}{(2j+1)!} \lambda_{c^{2i}\theta^{2j-2i+1}}(0, \hat{\theta}).
\end{aligned} \tag{4.4}$$

Applying the binomial expansion to $(\theta - \hat{\theta})^{2j-2i}$ and $(\theta - \hat{\theta})^{2j-2i+1}$ and separating even and odd powers of θ result in

$$\begin{aligned}
(\theta - \hat{\theta})^{2j-2i} &= \sum_{n=0}^{2j-2i} \binom{2j-2i}{n} \theta^n (-\hat{\theta})^{2j-2i-n} \\
&= \sum_{k=0}^{j-i} \binom{2j-2i}{2k} \theta^{2k} \hat{\theta}^{2j-2i-2k} - \sum_{k=1}^{j-i} \binom{2j-2i}{2k-1} \theta^{2k-1} \hat{\theta}^{2j-2i-2k+1}
\end{aligned}$$

and

$$
\begin{aligned}
(\theta-\hat{\theta})^{2j-2i+1} &= \sum_{n=0}^{2j-2i+1}\binom{2j-2i+1}{n}\theta^n(-\hat{\theta})^{2j-2i+1-n} \\
&= \sum_{k=1}^{j-i+1}\binom{2j-2i+1}{2k-1}\theta^{2k-1}\hat{\theta}^{2j-2i+2-2k} - \sum_{k=0}^{j-i}\binom{2j-2i+1}{2k}\theta^{2k}\hat{\theta}^{2j-2i+1-2k}.
\end{aligned}
$$

Inserting them into (4.4), collecting even and odd powers of θ, and rearranging, we arrive at

$$
\lambda(c,\theta)=\lambda_0+\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}c^{2i}\theta^{2k}g_{2k}^{(i)}(\hat{\theta})+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}c^{2i}\theta^{2k-1}g_{2k-1}^{(i)}(\hat{\theta}), \tag{4.5}
$$

where

$$
\begin{aligned}
g_{2k}^{(i)}(\hat{\theta}) &= \sum_{j=i+k}^{\infty}\binom{2j}{2j-2i}\binom{2j-2i}{2k}\frac{\hat{\theta}^{2j-2i-2k}}{(2j)!}\lambda_{c^{2i}\theta^{2j-2i}}(0,\hat{\theta}) \\
&\quad - \sum_{j=i+k}^{\infty}\binom{2j+1}{2j-2i+1}\binom{2j-2i+1}{2k}\frac{\hat{\theta}^{2j-2i+1-2k}}{(2j+1)!}\lambda_{c^{2i}\theta^{2j-2i+1}}(0,\hat{\theta}), \\
g_{2k-1}^{(i)}(\hat{\theta}) &= \sum_{j=i+k-1}^{\infty}\binom{2j+1}{2j-2i+1}\binom{2j-2i+1}{2k-1}\frac{\hat{\theta}^{2j-2i+2-2k}}{(2j+1)!}\lambda_{c^{2i}\theta^{2j-2i+1}}(0,\hat{\theta}) \\
&\quad - \sum_{j=i+k}^{\infty}\binom{2j}{2j-2i}\binom{2j-2i}{2k-1}\frac{\hat{\theta}^{2j-2i-2k+1}}{(2j)!}\lambda_{c^{2i}\theta^{2j-2i}}(0,\hat{\theta}).
\end{aligned}
$$

Note that when $\hat{\theta}=0$, the symmetry $\lambda(c,\theta)=\lambda(c,-\theta)$ from (4.2) implies that

$$
\lambda_{c^{2i}\theta^{m-2i}}(0,0)=0,\quad \forall m\geq 3\text{ odd},\quad i=1,2,\dots,[m/2].
$$

It then follows from (4.4) with $\hat{\theta}=0$ that

$$
\begin{aligned}
\lambda(c,\theta) &= \lambda_0+\sum_{i=1}^{\infty}c^{2i}\sum_{j=i}^{\infty}\binom{2j}{2j-2i}\frac{\theta^{2j-2i}}{(2j)!}\lambda_{c^{2i}\theta^{2j-2i}}(0,0) \\
&= \lambda_0+\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}c^{2i}\theta^{2k}\binom{2i+2k}{2k}\frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!}.
\end{aligned} \tag{4.6}
$$

For $\hat{\theta}\in[0,\pi/4]$, let $R_{\hat{\theta}}>0$ be such that the rectangle $(-R_{\hat{\theta}},R_{\hat{\theta}})\times(\hat{\theta}-R_{\hat{\theta}},\hat{\theta}+R_{\hat{\theta}})$ is contained in the disk of convergence of the series of $\lambda(c,\theta)$ at $(c,\theta)=(0,\hat{\theta})$. Set $I_{\hat{\theta}}:=(\hat{\theta}-R_{\hat{\theta}},\hat{\theta}+R_{\hat{\theta}})$. Obviously, $\{I_{\hat{\theta}},\hat{\theta}\in[0,\pi/4]\}$ is an open cover of $\in[0,\pi/4]$. By the Heine-Borel Theorem, there are $\theta_1,\theta_2,\dots,\theta_M\in[0,\pi/4]$ such that $\{I_{\theta_m},m=1,\dots,M\}$ covers $[0,\pi/4]$. We may assume, without loss of generality, that

$$
\theta_m<\theta_{m+1}\quad\text{and}\quad I_{\theta_m}\cap I_{\theta_{m+1}}\neq\emptyset,\quad \forall m=1,2,\dots,M-1.
$$

Moreover, we may assume $\theta_1=0$. Otherwise, we just add I_0 to the above open cover of $[0,\pi/4]$ and then relabel the intervals.

Set $R_0:=\min\{R_{\theta_1},\dots,R_{\theta_M}\}$. We finish the proof of Theorem B within three steps.

Step 1. We show that

$$\lambda(c,\theta) = \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} \binom{2i+2k}{2k} \frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!}, \quad \forall c \in (-R_0, R_0) \tag{4.7}$$

holds for all $\theta \in I_{\theta_1} \cup I_{\theta_2}$.

Recall that $\theta_1 = 0$. Note that (4.6) ensures that

$$\lambda(c,\theta) = \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} \binom{2i+2k}{2k} \frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!}, \quad \forall c \in (-R_0, R_0), \quad \theta \in I_{\theta_1}. \tag{4.8}$$

That is, (4.7) holds for all $\theta \in I_{\theta_1}$. It remains to show that it also holds for all $\theta \in I_{\theta_2}$. We find from (4.5) with $\hat{\theta} = \theta_2$ that

$$\lambda(c,\theta) = \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} g_{2k}^{(i)}(\theta_2) + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} c^{2i}\theta^{2k-1} g_{2k-1}^{(i)}(\theta_2), \quad \forall c \in (-R_0, R_0), \quad \theta \in I_{\theta_2}. \tag{4.9}$$

Since $I_{\theta_1} \cap I_{\theta_2} \neq \emptyset$, we compare (4.8) with (4.9) in $I_{\theta_1} \cap I_{\theta_2}$ to find

$$g_{2k-1}^{(i)}(\theta_2) = 0, \quad \forall i \in \mathbb{N}^+, \quad k \in \mathbb{N}^+,$$
$$g_{2k}^{(i)}(\theta_2) = \binom{2i+2k}{2k} \frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!}, \quad \forall i \in \mathbb{N}^+, \quad k \in \mathbb{N}.$$

Inserting them into (4.9) asserts the correctness of (4.7) for $\theta \in I_{\theta_2}$. Hence, (4.7) holds for all $\theta \in I_{\theta_1} \cup I_{\theta_2}$.

Step 2. We show that (4.7) holds for all $\theta \in I_{\theta_1} \cup I_{\theta_2} \cup \cdots \cup I_{\theta_M}$.

Note that (4.5) with $\hat{\theta} = \theta_3$ reads in particular

$$\lambda(c,\theta) = \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} g_{2k}^{(i)}(\theta_3) + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} c^{2i}\theta^{2k-1} g_{2k-1}^{(i)}(\theta_3), \quad \forall c \in (-R_0, R_0), \quad \theta \in I_{\theta_3}. \tag{4.10}$$

Since $I_{\theta_2} \cap I_{\theta_3} \neq \emptyset$, we compare (4.10) with (4.7) to find

$$g_{2k-1}^{(i)}(\theta_3) = 0, \quad i \in \mathbb{N}^+, \quad k \in \mathbb{N}^+,$$
$$g_{2k}^{(i)}(\theta_3) = \binom{2i+2k}{2k} \frac{1}{(2i+2k)!} \lambda_{c^{2i}\theta^{2k}}(0,0), \quad i \in \mathbb{N}^+, \quad k \in \mathbb{N}.$$

It then follows from (4.10) that (4.7) holds for all $\theta \in I_{\theta_3}$, and hence, it holds for all $\theta \in I_{\theta_1} \cup I_{\theta_2} \cup I_{\theta_3}$.

The claim then follows readily from repeating the above process.

Step 3. We finish the proof of Theorem B.

Since $[0, \pi/4] \subset I_{\theta_1} \cup I_{\theta_2} \cup \cdots \cup I_{\theta_M}$, we have from **Step 2** that

$$\begin{aligned} \lambda(c,\theta) &= \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} \binom{2i+2k}{2k} \frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i+2k)!} \\ &= \lambda_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c^{2i}\theta^{2k} \frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2i)!(2k)!}, \quad \forall c \in (-R_0, R_0), \quad \theta \in [0, \pi/4]. \end{aligned} \tag{4.11}$$

Note that (4.3) ensures that $\lambda_{c^{2i}\theta^{2n-1}}(0,\pi/4)=0$ for all $i,n\in\mathbb{N}^+$. From which, we differentiate (4.11) to arrive at

$$\sum_{k\geq n}^{\infty}\left(\frac{\pi}{4}\right)^{2k-2n+1}\frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2k-2n+1)!}=0,\quad \forall i,n\in\mathbb{N}^+.$$

Multiplying the above equality by $(\frac{\pi}{4})^{2n-1}$ yields

$$\sum_{k\geq n}^{\infty}\left(\frac{\pi}{4}\right)^{2k}\frac{\lambda_{c^{2i}\theta^{2k}}(0,0)}{(2k-2n+1)!}=0,\quad \forall i,n\in\mathbb{N}^+. \tag{4.12}$$

Define the lower triangular matrix

$$A:=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{3!} & 1 & 0 & 0 & 0 & \ddots \\ \frac{1}{5!} & \frac{1}{3!} & 1 & 0 & 0 & \ddots \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & 0 & \ddots \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

For each $i\in\mathbb{N}^+$, we set $a_k^{(i)}:=(\frac{\pi}{4})^{2k}\lambda_{c^{2i}\theta^{2k}}(0,0)$ for $k\in\mathbb{N}^+$, and define $X^{(i)}:=(a_1^{(i)},a_2^{(i)},a_3^{(i)},\cdots)$. It is straightforward to check that (4.12) can be rewritten as

$$X^{(i)}A=0,\quad \forall i\in\mathbb{N}^+.$$

We apply [7, Theorem 2.2.9] to conclude that $X^{(i)}=0$ for all $i\in\mathbb{N}^+$, and hence, $\lambda_{c^{2i}\theta^{2k}}(0,0)=0$ for all $i,k\in\mathbb{N}^+$. It then follows from (4.11) that

$$\lambda(c,\theta)=\lambda_0+\sum_{i=1}^{\infty}\frac{c^{2i}}{(2i)!}\lambda_{c^{2i}}(0,0),\quad \forall c\in(-R_0,R_0),\quad \theta\in[0,\pi/4].$$

This completes the proof.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] H. Baumgärtel, Analytic perturbation theory for linear operators depending on several complex variables. *Mat. Issled.* 9 (1974), no. 1, 17-39.

[2] H. Baumgärtel, *Analytic perturbation theory for matrices and operators.* Birkhäuser Verlag, Basel, 1985.

[3] H. Berestycki, O. Diekmann, C. J. Nagelkerke and P. A. Zegeling, Can a species keep pace with a shifting climate? *Bull. Math. Biol.* 71 (2009), no. 2, 399-429.

[4] H. Berestycki and L. Rossi, Reaction-diffusion equations for population dynamics with forced speed, I – the case of the whole space. *Discrete Contin. Dyn. Syst.* 21 (2008), no. 1, 41–67.

[5] H. Berestycki and L. Rossi, Reaction-diffusion equations for population dynamics with forced speed, II – cylindrical type domains. *Discrete Contin. Dyn. Syst.* 25 (2009), no. 1, 19–61.

[6] R. J. H. Beverton and S. J. Holt, On the dynamics of exploited fish populations, Her Majesty's Stationery Office, London. (1957)

[7] J. Boos, *Classical and modern methods in summability.* Oxford University Press, Oxford, 2000.

[8] J. Bouhours and G. Nadin, A variational approach to reaction-diffusion equations with forced speed in dimension 1. *Discrete Contin. Dyn. Syst.* 35 (2015), no. 5, 1843-1872.

[9] F. Hamel, N. Nadirashvili and E. Russ, A Faber–Krahn inequality with drift. https://arxiv.org/abs/math/0607585

[10] F. Hamel, N. Nadirashvili and E. Russ, Rearrangement inequalities and applications to isoperimetric problems for eigenvalues. *Ann. of Math.* (2) 174 (2011), no. 2, 647-755.

[11] J. Garcia-Melian and J. D. Rossi, On the principal eigenvalue of some nonlocal diffusion problems. *J. Differential Equations* 246 (2009), no. 1, 21–38.

[12] J. A. Hodgson, C. D. Tomas, B. A. Wintle and A. Moilanen, Climate change, connectivity and conservation decision making: back to basics. *J. Appl. Ecol.* 46 (2009), 964–969.

[13] M. G. Kreĭn and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space. *Uspehi Matem. Nauk (N. S.)* 3, (1948). no. 1(23), 3-95.

[14] J. Lenoir, J. C. Gegout, P. A. Marquet, P. de Ruffray and H. Brisse, A significant upward shift in plant species optimum elevation during the 20th century. *Science* 320 (2008), 1768-1771.

[15] M. A. Lewis, N. G. Marculis and Z. Shen, Integrodifference equations in the presence of climate change: persistence criterion, travelling waves and inside dynamics. *J. Math. Biol.* 77 (2018), no. 6-7, 1649–1687.

[16] E. H. Lieb and M. Loss, *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[17] R. Menendez, A. Gonzalez-Megias, P. Jay-Robert and R. Marquez-Ferrando, Climate change and elevational range shifts: evidence from dung beetles in two European mountain ranges. *Glob Ecol. Biogeogr.* 23 (2014), no. 6, 646-657.

[18] C. Parmesan, Ecological and evolutionary responses to recent climate change. *Annu. Rev. Ecol. Evol. Syst.* 37 (2006), 637-669.

[19] A. Phillips and M. Kot, Persistence in a two-dimensional moving-habitat model. *Bull. Math. Biol.* 77 (2015), no. 11, 2125–2159.

[20] A. B. Potapov and M. A. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility. *Bull. Math. Biol.* 66 (2004), no. 5, 975–1008.

[21] W. Shen, Z. Shen, S. Xue and D. Zhou, Population dynamics under climate change: persistence criterion and effects of fluctuations. *J. Math. Biol.* 84 (2022), no. 4, Paper No. 30, 42 pp.

[22] H.-H. Vo, Persistence versus extinction under a climate change in mixed environments. *J. Differential Equations* 259 (2015), no. 10, 4947–4988.

[23] G. R. Walther, Community and ecosystem responses to recent climate change. *Phil. Trans. R. Soc. B.* 365 (2010), 2019-2024.

[24] X.-Q. Zhao, Global attractivity and stability in some monotone discrete dynamical systems. *Bull. Austral. Math. Soc.* 53 (1996), no. 2, 305-324.

[25] Y. Zhou and M. Kot, Discrete-time growth-dispersal models with shifting species ranges. *Theor. Ecol.* 4 (2011), 13-25.

[26] Y. Zhou and M. Kot, Life on the move: modeling the effects of climate-driven range shifts with integrodifference equations. Dispersal, individual movement and spatial ecology. 263-292, *Lecture Notes in Math.*, 2071, Springer, Heidelberg, 2013.

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