DYNAMICS OF ABSORBED SINGULAR DIFFUSIONS, PART I: LARGE DEVIATION PRINCIPLE OF QUASI-STATIONARY DISTRIBUTIONS

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Abstract. The present paper is Part I of a series of two papers devoted to the investigation of a family of one-dimensional absorbed singular diffusion processes exhibiting transient dynamics, namely, interesting dynamical behaviours over fine time scales. In this part, we establish the powerful sub-exponential large deviation principle (LDP) of quasi-stationary distributions (QSDs), which determines the quasi-potential function and prefactor in the WKB expansion, and therefore, provides very fine asymptotic properties of QSDs. As consequences, new results about estimations of QSDs near the extinction state and infinity, the sub-exponential asymptotic of the principal eigenvalue, and the asymptotic of the principal eigenfunction are obtained. Applications to diffusion models arising from chemical reactions and population dynamics are discussed.

In Part II [47], we use the sub-exponential LDP of QSDs and its consequences to explore the multiscale dynamics in detail and prove the asymptotic reciprocal relationship between the mean extinction time and the principal eigenvalue of the diffusion operator. While the former characterizes the dynamics over different time scales, the latter is a fundamental principle quantifying in particular the lifespan of interesting dynamical behaviours combined and its natural connection with the principal eigenvalue.

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1. Introduction

A large number of experimental and numerical evidences show that complex processes in biology, chemistry, climate dynamics, fluids, neuroscience, etc. often exhibit transient dynamics, namely, intriguing or important dynamical behaviours over a relatively long but finite time period. For instance, species in a community usually coexist for a long period that may span dozens or even hundreds of generations before the extinction of at least one species (see e.g. [26, 27, 46]). In a closed chemical reaction system, chemical oscillations could last for a long period before the system eventually relaxes to the thermal equilibrium due to the inevitable heat dissipation (see e.g. [48, 55]). In an open flow, transiently chaotic advective dynamics can be generated to impact the spreading of pollutants, the population dynamics of plankton and larvae, biological and chemical reactions and so on (see e.g. [35, 52]). The treatment of such dynamical behaviours is out of the scope of traditional dynamical system theories focusing on long-term dynamics. Addressing long but finite-time dynamical behaviors, transient dynamics has demonstrated its significance in many scientific areas and been attracting an increasing amount of attention. Given more and more results from experiments and numerical studies (see e.g. [35, 44, 27]), rigorous mathematical frameworks are expected to classify transient dynamics of different mechanisms and further stimulate the research to have a better understanding of transient dynamics.

In this series of two papers, we continue to study the transient dynamics and related properties of a class of absorbed singular diffusion processes arising from chemical reactions and population dynamics initiated in the works [49, 32]. More precisely, we consider the following randomly perturbed dynamical systems:

\[ dx = b(x)dt + \epsilon \sqrt{a(x)}dW_t, \quad x \in [0, \infty), \]

where \( 0 < \epsilon \ll 1 \) is a parameter, \( b : [0, \infty) \to \mathbb{R}, \ a : [0, \infty) \to [0, \infty) \) and \( W_t \) is the standard one-dimensional Wiener process on some probability space. The equation (1.1) can be derived as a diffusion approximation of rescaled birth-death processes modelling the evolution of some species in a community or some type of molecules in a chemical reaction system. We make the following standard assumptions on the coefficients \( a \) and \( b \) throughout this paper.

**(H)** The functions \( b : [0, \infty) \to \mathbb{R} \) and \( a : [0, \infty) \to [0, \infty) \) satisfy the following conditions:

1. \( b \in C^1((0, \infty)) \cap C^2((0, \infty)), \ b(0) = 0, \ b'(0) > 0, \) and \( \limsup_{x \to \infty} b(x) < 0; \)
(2) \( a \in C^2((0, \infty)) \cap C^0((0, \infty)) \), \( a(0) = 0 \), \( a'(0) > 0 \), and \( a > 0 \) on \((0, \infty)\);

(3) \( \lim_{x \to \infty} \frac{b^2(x)}{a(x)} = \infty \), \( \limsup_{x \to \infty} \frac{\max\{a(x), |a'(x)|, |a''(x)|, |b'(x)|\}}{b(x)} < \infty \), and there is \( m > 0 \) such that \( \frac{b(x)}{a(x)} \leq \int_0^x \frac{b(s)}{a(s)} \, ds \) for \( x \gg 1 \).

\((H)(1)\) says that \( b \) is a logistic-type growth rate function that plays important roles in especially biological and ecological applications. \((H)(2)\) assumes that \( a \) is degenerate at 0 and behaves like \( a'(0)x \) near 0. In particular, \( \sqrt{a} \) vanishes and is singular at 0, causing the non-integrability of the Gibbs density near 0 that leads to substantial difficulties in the analysis of (1.1). The assumptions \( \limsup_{x \to \infty} b(x) < 0 \) in \((H)(1)\) and \( \lim_{x \to \infty} \frac{b^2(x)}{a(x)} = \infty \) in \((H)(3)\) ensure the dissipativity of (1.1). Other conditions in \((H)(3)\) restricting the behaviours of \( a, b \) and the ratio \( \frac{b}{a} \) near \( \infty \) are mild technical assumptions, and they are sufficiently general for applications (See Section 5).

Let \( X_t^\epsilon \) be the diffusion process on \([0, \infty)\) generated by solutions of (1.1). Clearly, 0 is an absorbing state of \( X_t^\epsilon \), and is often called an extinction state in applications. Under \((H)\), \( X_t^\epsilon \) reaches the extinction state 0 in finite time almost surely \([6, 31]\). Therefore, the long-term behavior of \( X_t^\epsilon \) tells nothing interesting and thus drives us to look at the dynamics of \( X_t^\epsilon \) before hitting 0. Since the ordinary differential equation (ODE) \( \dot{x} = b(x) \) may contain multiple attractors in \((0, \infty)\), the sample path large deviation principle (LDP) \([23]\) indicates with probability almost one that trajectories of \( X_t^\epsilon \) sojourn around these attractors for a long time before going to extinction, demonstrating fascinating transient dynamics. As in \([49, 32]\), we adopt a distribution/observable-based viewpoint and use quasi-stationary distributions (QSDs), being essentially stationary distributions of \( X_t^\epsilon \) conditioned on not hitting 0, to capture the transient dynamics of \( X_t^\epsilon \).

Denote by \( T_0^\epsilon \) the extinction time of \( X_t^\epsilon \), namely, the first time that \( X_t^\epsilon \) hits 0. More precisely, \( T_0^\epsilon = \inf\{t \geq 0 : X_t^\epsilon = 0\} \). Then, \( \mathbb{P}_\mu[T_0^\epsilon < \infty] = 1 \), where \( \mathbb{P}_\mu \) denotes the law of \( X_t^\epsilon \) with initial distribution \( \mu \). The expectation associated with \( \mathbb{P}_\mu \) is written as \( \mathbb{E}_\mu \). When \( \mu = \delta_x \) is the Dirac measure at \( x \), we simply write \( \mathbb{P}_x^\epsilon \) and \( \mathbb{E}_x^\epsilon \) as \( \mathbb{P}_x^\epsilon \) and \( \mathbb{E}_x^\epsilon \), respectively.

**Definition 1.1** (Quasi-stationary distribution). A Borel probability measure \( \mu_\epsilon \) on \((0, \infty)\) is called a quasi-stationary distribution (QSD) of \( X_t^\epsilon \) if

\[
\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in B | t < T_0^\epsilon] = \mu_\epsilon(B), \quad \forall t \geq 0, \quad B \in \mathcal{B}((0, \infty)),
\]

where \( \mathcal{B}((0, \infty)) \) is the Borel \( \sigma \)-algebra of \((0, \infty)\).

It is known from the general theory of QSDs \([41, 14]\) that if \( \mu_\epsilon \) is a QSD of \( X_t^\epsilon \), then there is a unique positive number \( \lambda_{\epsilon,1} \), often referred to as the extinction rate, such that \( T_0^\epsilon \sim \exp(\lambda_{\epsilon,1}) \) provided \( X_0^\epsilon \sim \mu_\epsilon \).

We state in Proposition 2.1 the existence of a unique QSD \( \mu_\epsilon \) of \( X_t^\epsilon \) with a continuously differentiable density \( u_\epsilon \). Moreover, the associated extinction rate \( \lambda_{\epsilon,1} \) is exactly the first eigenvalue or principal eigenvalue of \( -\mathcal{L}_\epsilon \), where \( \mathcal{L}_\epsilon \) denotes an appropriate closed extension of the diffusion operator of (1.1) (see Subsection 2.1 for details).

In previous works \([49, 32]\), the authors study the tightness and rough concentration estimates of \( \{\mu_\epsilon\}_\epsilon \), as well as the asymptotic of the first two eigenvalue values of \( -\mathcal{L}_\epsilon \) in order
to characterize the transient dynamics of $X^\epsilon_t$. The main purpose of the present paper is to establish the powerful sub-exponential LDP of the QSD $\mu^\epsilon$ or its density $u^\epsilon$, which captures very fine asymptotic properties of $\mu^\epsilon$ as $\epsilon \to 0$ and is of independent interest. That is, we intend to rigorously justify the WKB expansion

$$u^\epsilon(x) = \frac{1}{\epsilon a(x)} e^{-\frac{2}{\epsilon^2} v(x)} \left[ R_0(x) + \epsilon^2 R_1(x) + \cdots + \epsilon^{2n} R_n(x) + o(\epsilon^{2(n+1)}) \right], \quad x \in (0, \infty)$$

in the case $n = 0$, so that

$$u^\epsilon(x) = \frac{R^\epsilon(x)}{\epsilon a(x)} e^{-\frac{2}{\epsilon^2} v(x)} \quad \text{and} \quad R^\epsilon(x) = R_0(x) + o(\epsilon^2), \quad x \in (0, \infty),$$

where $v(x)$ is the quasi-potential function and the sub-exponential term $\frac{R^\epsilon(x)}{\epsilon a(x)}$ is often called the prefactor in physical literature. The sub-exponential LDP and its consequences are then used in Part II [47] to investigate the multiscale dynamics of $X^\epsilon_t$ in detail and establish the asymptotic reciprocal relationship between the mean extinction time $\mathbb{E}^\epsilon[T^\epsilon_0]$ and the principal eigenvalue $\lambda_{\epsilon,1}$. While the former characterizes the global dynamics including both transient dynamics and long-term dynamics, the latter is a fundamental principle quantifying in particular the lifespan of transient dynamics and its natural connection with the principal eigenvalue. Not only do results proven in the present paper and Part II [47] greatly improve many of those contained in [49, 32], but also they widely broaden the scope of the study.

To state our main results, we consider the potential function:

$$V(x) = -\int_0^x \frac{b}{a} ds, \quad x \in (0, \infty). \quad (1.2)$$

We follow [39] to define valleys of $V$.

**Definition 1.2.** An open interval $I \subset (0, \infty)$ is called a valley (of $V$) if it is one of the connected components of the sublevel set $\{x \in (0, \infty) : V(x) < \rho\}$ and satisfies $V(\partial I) = \rho$ for some $\rho \in \mathbb{R}$. We say $I \subset (0, \infty)$ a $d$-valley if it is a valley of depth $d$, namely, $\sup_I V - \inf_I V = d$.

Set

$$d_1 := \sup_{x \in (0, \infty)} \left[ \sup_{(0,x)} V - V(x) \right] > 0,$$

which is the depth of the deepest valleys of $V$. Since $V(0+) = 0$ and $V(\infty) = \infty$ by (H), there exist finitely many $d_1$-valleys and no $d$-valley with $d > d_1$. We point out that in literature $d_1$ is often defined by

$$d_1 = \sup_{x \in (0, \infty)} \left[ \inf_{\xi \in C_x} \sup_{t \in [0,1]} V(\xi(t)) - V(x) \right],$$

where for each $x \in (0, \infty)$,

$$C_x = \{ \xi : [0, 1] \to [0, \infty) : \xi \text{ is continuous and satisfies } \xi(0) = x \text{ and } \xi(1) = 0 \}.$$

But it is easy to check that they coincide.
The LDP is proven when there exists a unique $d_1$-valley, which is a generic case. Recall that $u_\epsilon$ is the density of the QSD $\mu_\epsilon$.

**Theorem A.** Assume (H) and the existence of a unique $d_1$-valley $(\alpha, \beta)$. Then,

$$\lim_{\epsilon \to 0} \frac{\epsilon^2}{2} \ln u_\epsilon = -v \quad \text{locally uniformly in } (0, \infty),$$

where $v$ is a locally Lipschitz viscosity solution of $(v')^2 + \frac{b}{a}v' = 0$ in $(0, \infty)$, and is given as follows:

- if $\alpha = 0$, then $v = d_1 + V$;
- if $\alpha > 0$, then $v(x) = \begin{cases} d_1 + V(x) - \sup_{(0,x)} V, & x \in (0, \alpha], \\ d_1 + V(x) - V(\alpha), & x \in (\alpha, \infty). \end{cases}$

Obviously, in the case of a unique $d_1$-valley $(\alpha, \beta)$ with $\alpha > 0$, the quasi-potential function $v$ obtained in Theorem A is not continuously differentiable everywhere and could be non-differentiable at many points depending on the geometry of $V$ on $(0, \alpha)$. See Figure 1 for an illustration of $V$ and $v$ in the case $\alpha > 0$.

![Figure 1. Illustration of $V$ and $v$ in the case $\alpha > 0$.](image)

In what follows, besides assuming the existence of a unique $d_1$-valley $(\alpha, \beta)$, we often impose the following assumptions on $V$ in $(\alpha, \beta)$ in order to establish finer results.

**(H$_V$)** $\{x \in (\alpha, \beta) : V(x) = \min_{(\alpha,\beta)} V\} = \{x_1, \ldots, x_N\}$ for some $N \in \mathbb{N}$ and $b'(x_i) < 0$ for each $i \in \{1, \ldots, N\}$.

That is, $V|_{(\alpha,\beta)}$ attains its minimal value at only finitely many points and they are non-degenerate. We point out that $x_1, \ldots, x_N$ are the only global minima of $V$ when $\alpha = 0$, and they may not be when $\alpha > 0$. Whenever (H$_V$) is assumed, we denote

$$M_0 := \left( \sum_{i=1}^N \frac{1}{a(x_i)} \sqrt{\frac{\pi}{V''(x_i)}} \right)^{-1}.$$
Let $v$ be the quasi-potential function as in Theorem A. To establish the sub-exponential asymptotic of $u_\epsilon$ as $\epsilon \to 0$ (or to determine the prefactor in the WKB expansion of $u_\epsilon$), we set

$$R_\epsilon := \epsilon au_\epsilon e^{\frac{2}{\epsilon}v}$$

and study the asymptotic of $R_\epsilon$ as $\epsilon \to 0$.

Throughout this paper, for positive numbers $A_\epsilon$ and $B_\epsilon$ indexed by $\epsilon$, we write $A_\epsilon \approx_\epsilon B_\epsilon$ if $\lim_{\epsilon \to 0} \frac{A_\epsilon}{B_\epsilon} = 1$.

**Theorem B.** Assume (H), the existence of a unique $d_1$-valley $(\alpha, \beta)$ and (H$_V$).

1. If $\alpha = 0$, then $\lim_{\epsilon \to 0} R_\epsilon = M_0$ locally uniformly in $(0, \infty)$.
2. If $\alpha > 0$, $V(\alpha) > V(x)$ for all $x \in (0, \alpha)$ and $b'(\alpha) > 0$, then

$$\lim_{\epsilon \to 0} \frac{R_\epsilon}{\epsilon} = -\frac{M_0}{2V'(0^+)} \sqrt{-V''(\alpha)} \pi \text{ locally uniformly in } (0, x_0),$$

$$R_\epsilon(x) \approx_\epsilon \frac{M_0}{\epsilon} \sqrt{-V''(\alpha)} \int_0^x e^{\frac{2}{\epsilon}V - \sup_{[0,x]}V} dz, \quad x \in [x_0, \alpha),$$

$$\lim_{\epsilon \to 0} R_\epsilon(x) = \begin{cases} M_0 \sqrt{\alpha} \frac{2}{\epsilon}, & x = \alpha, \\ M_0 \text{ locally uniformly in } x \in (\alpha, \infty), & \end{cases}$$

where $x_0 \in (0, \alpha]$ is the smallest positive zero of $V$.

**Remark 1.1.** We comment on Theorem B (2).

1. Note that $V(\alpha) \geq V(x)$ for all $x \in (0, \alpha)$ with strict inequality for $x \in (0, \delta)$ for some $\delta \in (0, \alpha)$. If there is $x \in (0, \alpha)$ such that $V(\alpha) = V(x)$, then we need to impose conditions on $V$ at such an $x$ in order to determine the asymptotic. While it is certainly doable, the statement would be messy.

2. $b'(\alpha) > 0$ is not a strong restriction, and can be replaced by a higher order derivative condition at $\alpha$ if $a$ and $b$, so $V$, have enough differentiability near $\alpha$.

3. Note that $x_0 = \alpha$ if $V(\alpha) = 0$. When $x_0 < \alpha$ (if and only if $V(\alpha) > 0$), it is theoretically possible to establish a similar result for $x \in [x_0, \alpha)$ by means of Laplace’s method. But, it is hard to state the result explicitly because the asymptotic of $\int_0^x e^{\frac{2}{\epsilon}V - \sup_{[0,x]}V} dz$ as $\epsilon \to 0$ for $x \in [x_0, \alpha)$ depends heavily on the geometry of $V$ on $[x_0, \alpha)$.

At this moment, we mention relevant works on the LDP of stationary distributions and QSDs, and compare our approach with those contained in literature. For stationary distributions of randomly perturbed dynamical systems of the form $dx = f(x) dt + \epsilon \sigma(x) dW_t$ in any dimension with the vector field $f$ having a non-degenerate globally asymptotically stable equilibrium and the diffusion matrix $\sigma \sigma^\top$ being uniformly positive definite, the LDP as in Theorem A has been studied in [23, 51], the sub-exponential LDP as in Theorem B has been established in [50, 16, 4], and the WKB asymptotic expansion in a small neighbourhood of the asymptotically stable equilibrium has been justified in [42, 43]. All of them build on
the sample path LDP due to Freidlin and Wentzell [23], except the work [4] in which the authors tackle the problem from a control theoretic viewpoint and are able to treat vector fields admitting finitely many asymptotically stable equilibria and no other \( \omega \)-limit sets. In [51], the author replaces the positive definiteness of \( \sigma \sigma^\top \) by some conditions on the controlled trajectories (see the condition (A4) in [51, Theorem 1] for details), and therefore, some degenerate cases can be treated. In [38], the authors study a family of continuous time symmetric random walks on the unit circle and establish the LDP of stationary distributions by means of the Aubry-Mather theory (see e.g. [3, 22]).

As for the LDP of QSDs, there exist a few results [39, 8, 5, 13]. In [39], the author considers regular reversible diffusion processes restricted on a bounded domain and killed on its boundary, and investigate by a functional analytic approach the asymptotic of the principal eigenfunction of the generator in the deepest valley (assuming the uniqueness of such a valley), leading to the sub-exponential LDP in that valley. More precise asymptotic in a neighbourhood of a non-degenerate local minimal point of the potential function is obtained in [5] by a potential theoretic approach. One-dimensional absorbed birth-and-death processes whose mean field ODE has a single asymptotically stable equilibrium are investigated in [8, 13]. The LDP and sub-exponential LDP are respectively established in [8] and [13]. Both of them heavily use the recursive formula satisfied by the QSD. The work [8] also treats processes whose mean field ODE has multiple stable equilibria.

Our two-step approach is different from those contained in literature. The first step studying the vanishing viscosity limit of the logarithmic transform \( v_\epsilon = -\frac{\epsilon^2}{2} \ln(a u_\epsilon) \) is somewhat standard. Establishing the local uniform boundedness of \( \{v_\epsilon\} \) and \( \{v'_\epsilon\} \), we find candidates for the quasi-potential function who are viscosity solutions of the Hamilton-Jacobi equation \( (v')^2 + \frac{h}{a} v' = 0 \) in \((0, \infty)\). Previous studies on the tightness and rough concentration estimates of QSDs [39] give basic properties of the candidates (see Section 3 for details). Due to the non-uniqueness of viscosity solutions of the Hamilton-Jacobi equation (although some properties of the candidates have been established), an approach to the determination of the quasi-potential function is needed. This is the purpose of the second step. In literature, methods based on the Freidlin-Wentzell theory, control theory, Aubry-Mather theory, etc. have been used to achieve this goal as mentioned earlier. Here, we tackle the problem from a completely different perspective that takes full advantage of the one-dimensional structure. More precisely, exploring the properties of \( u_\epsilon \) near 0 and \( \infty \), we are able to establish integral identities for \( u_\epsilon \), \( v_\epsilon \) and \( v'_\epsilon \) (see Proposition 4.1 for details). Elementary analysis based on these identities and Laplace’s method then allows us to establish the LDP as stated in Theorems A and B.

As byproducts of the proof and consequences of Theorems A and B, we obtain new results about uniform-in-\( \epsilon \) estimates of \( \mu_\epsilon \) or \( u_\epsilon \) near 0 and \( \infty \), sub-exponential asymptotic of the principal eigenvalue \( \lambda_{\epsilon,1} \), and asymptotic of the eigenfunction \( \phi_{\epsilon,1} \) of \(-L_\epsilon\) corresponding to \( \lambda_{\epsilon,1} \) and satisfying the normalization \( \|\phi_{\epsilon,1}\|_{L^2(u^G_\epsilon)} = 1 \), where \( u^G_\epsilon := \frac{1}{a} e^{-\frac{\epsilon}{a} V} \) is the non-integrable Gibbs density. Note from Proposition 2.1 that \( u_\epsilon \) and \( \phi_{\epsilon,1} \) are related by \( u_\epsilon = \frac{\phi_{\epsilon,1} u^G_\epsilon}{\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}} \).
Theorem C. Assume \((H)\). The following hold.

(1) There exist \(L \gg 1\), \(C > 0\) and \(0 < \epsilon_* \ll 1\) such that
\[
u_{\epsilon} \leq \frac{C}{\epsilon^4} e^{\frac{1}{\epsilon^2} \int_{L}^{\infty} b\,ds} \quad \text{in} \quad [L, \infty), \quad \forall \epsilon \in (0, \epsilon_*).
\]

(2) For each \(0 < \delta \ll 1\), there are \(0 < x_\delta \ll 1\) and \(0 < \epsilon_\delta \ll 1\) such that
\[
e^{-\frac{2}{\epsilon^2}(d_1+\delta)} \leq \nu_{\epsilon} \leq e^{-\frac{2}{\epsilon^2}(d_1-\delta)} \quad \text{in} \quad (0, x_\delta), \quad \forall \epsilon \in (0, \epsilon_\delta).
\]

(3) Suppose in addition the existence of a unique \(d_1\)-valley \((\alpha, \beta)\) and \((H_V)\).

(i) If \(\alpha = 0\), then
\[
\lim_{\epsilon \to 0} \epsilon \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} d_1} = \frac{b'(0)}{a'(0)} M_0
\]
and
\[
\lim_{\epsilon \to 0} \|\phi_{\epsilon,1}\|_{L^1(u_{\epsilon}^C)} \phi_{\epsilon,1} = 1 \quad \text{locally uniformly in} \quad (0, \infty).
\]

(ii) If \(\alpha > 0\), \(V(\alpha) > V(x)\) for \(x \in (0, \alpha)\) and \(b'(\alpha) > 0\), then
\[
\lim_{\epsilon \to 0} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} d_1} = \frac{M_0}{2} \sqrt{-\frac{V''(\alpha)}{\pi}}
\]
and
\[
\lim_{\epsilon \to 0} \|\phi_{\epsilon,1}\|_{L^1(u_{\epsilon}^C)} \phi_{\epsilon,1}(x) = \begin{cases}
0, & \text{uniformly in } x \in (0, \tilde{\alpha}) \text{ for each } \tilde{\alpha} \in (0, \alpha), \\
\frac{1}{2}, & x = \alpha, \\
1, & \text{locally uniformly in } x \in (\alpha, \infty).
\end{cases}
\]

Conclusions like those in Theorems A, B and C (3) have fruitful and far-reaching consequences. For instance, in [16], the author used the sub-exponential LDP of stationary measures to rigorously justify a formula concerning the asymptotic exit distribution originally derived in [40]. In the works [17, 18, 19] (see [36] for an exposition) studying exit events and the Eyring-Kramers formula on the basis of QSDs for the overdamped Langevin equation, the sub-exponential asymptotic of the principal eigenvalue plays an significant role in computing the asymptotic of transition rates and determining the asymptotic exit distribution.

In Part II [47], we use them to establish the multiscale dynamics of \(X_t^\epsilon\) and the asymptotic reciprocal relationship between \(E_{\epsilon}[T_0^\epsilon]\) and \(\lambda_{\epsilon,1}\).

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results, including spectral theory of \(L_{\epsilon}\), Liouville-type transform of \(L_{\epsilon}\) and the resulting semiclassical Schrödinger operators, and concentration estimates for QSDs. As mentioned earlier, our approach to establishing the sub-exponential LDP of \(\{\mu_{\epsilon}\}\) consists of two steps. The first step addressing the vanishing viscosity limits of \(v_{\epsilon} = -\frac{\epsilon^2}{2} \ln(au_{\epsilon})\) is contained in Section 3. The second step including proving the crucial integral identities for \(u_{\epsilon}, v_{\epsilon}\) and \(v'_\epsilon\) and completing the proof of Theorems A, B and C is presented in Section 4. Applications to logistic diffusion processes are discussed in Section 5.
2. Preliminary

In this section, we recall and establish some preliminary results for later purposes. We assume \((H)\) throughout this section.

2.1. Generator and spectral theory. For each \(0 < \epsilon \ll 1\), we consider the symmetric quadratic form \(\mathcal{E}_\epsilon : C_0^\infty((0, \infty)) \times C_0^\infty((0, \infty)) \to \mathbb{R}\) defined by

\[
\mathcal{E}_\epsilon(\phi, \psi) = \frac{\epsilon^2}{2} \int_0^\infty a\phi'\psi' u_\epsilon^G dx, \quad \forall \phi, \psi \in C_0^\infty((0, \infty)),
\]

where \(u_\epsilon^G := \frac{1}{\epsilon} e^{-\frac{\epsilon^2}{2}V}\) is the non-integrable Gibbs density. That is, \(u_\epsilon^G\) is the unique (up to constant multiplication) solution to \(\frac{\epsilon^2}{2}(au)' - bu = 0\) in \((0, \infty)\). In particular, it solves the stationary Fokker-Planck equation

\[
\frac{\epsilon^2}{2}(au)'' - (bu)' = 0 \quad \text{in} \quad (0, \infty).
\]

The form \(\mathcal{E}_\epsilon\) is Markovian and closable [24]. Its smallest closed extension, again denoted by \(\mathcal{E}_\epsilon\), is a Dirichlet form with domain \(D(\mathcal{E}_\epsilon)\) being the closure of \(C_0^\infty((0, \infty))\) under the norm \(\|\phi\|^2_D(\mathcal{E}_\epsilon) := \|\phi\|^2_{L^2(u_\epsilon^G)} + \mathcal{E}_\epsilon(\phi, \phi)\), where \(L^2(u_\epsilon^G) := L^2((0, \infty), u_\epsilon^G dx)\). Denote by \(\mathcal{L}_\epsilon\) the non-positive self-adjoint operator in the weighted space \(L^2(u_\epsilon^G)\) associated with \(\mathcal{E}_\epsilon\) such that

\[
\mathcal{E}_\epsilon(\phi, \psi) = \langle -\mathcal{L}_\epsilon \phi, \psi \rangle_{L^2(u_\epsilon^G)}, \quad \forall \phi \in D(\mathcal{L}_\epsilon), \psi \in D(\mathcal{E}_\epsilon),
\]

where

\[
D(\mathcal{L}_\epsilon) := \left\{ \phi \in D(\mathcal{E}_\epsilon) : \exists f \in L^2(u_\epsilon^G) \text{ s.t. } \mathcal{E}_\epsilon(\phi, \psi) = \langle f, \psi \rangle_{L^2(u_\epsilon^G)}, \forall \psi \in D(\mathcal{E}_\epsilon) \right\}
\]

is the domain of \(\mathcal{L}_\epsilon\) and contained in particular in \(L^2(u_\epsilon^G)\). Note that \(\mathcal{L}_\epsilon \phi = \frac{\epsilon^2}{2} a\phi'' + b\phi'\) for \(\phi \in C_0^\infty((0, \infty))\), that is, \(\mathcal{L}_\epsilon\) is a self-adjoint extension of the generator of (1.1).

We present the following results about the spectrum of \(-\mathcal{L}_\epsilon\) and the semigroup generated by \(\mathcal{L}_\epsilon\).

**Lemma 2.1** ([6, 32]). For each \(0 < \epsilon \ll 1\), the following hold.

1. \(-\mathcal{L}_\epsilon\) has purely discrete spectrum contained in \((0, \infty)\) and listed as follows:
   \[
   \lambda_{\epsilon,1} < \lambda_{\epsilon,2} < \lambda_{\epsilon,3} < \cdots \to \infty.
   \]

2. Each \(\lambda_{\epsilon,i}\) is associated with a unique eigenfunction \(\phi_{\epsilon,i} \in D(\mathcal{L}_\epsilon) \cap L^1(u_\epsilon^G) \cap C^2((0, \infty))\) subject to the normalization \(\|\phi_{\epsilon,i}\|_{L^2(u_\epsilon^G)} = 1\). Moreover, \(\phi_{\epsilon,1}\) is positive on \((0, \infty)\).

3. The set \(\{\phi_{\epsilon,i}, i \in \mathbb{N}\}\) is an orthonormal basis of \(L^2(u_\epsilon^G)\).

The following result is proven in [32].

**Lemma 2.2** ([32, Theorem A]). For each \(i \in \mathbb{N}\), \(\lim_{\epsilon \to 0^+} \frac{\epsilon^2}{2} \ln \lambda_{\epsilon,1} = -d_1\).

We point out that the notations \(r_1\) used in [32] correspond to \(2d_1\) used in the present paper. Since \(d_1 > 0\), Lemma 2.2 says that \(\lambda_{\epsilon,1}\) is exponentially small in \(\epsilon\).
2.2. **Semi-classical Schrödinger operators.** Consider the transform

\[ y = \xi(x) = \int_0^x \frac{1}{\sqrt{a(z)}} \, dz, \quad x \in (0, \infty). \]

Assumptions on \( a \) ensure that \( \xi' > 0 \) on \((0, \infty)\) and \( \xi(0^+) = 0 \). Set \( y_\infty := \xi(\infty) \in (0, \infty] \). In particular, \( \xi : (0, \infty) \to (0, y_\infty) \) is invertible. Let \( v_\epsilon^G(y) := \frac{u_\epsilon^G(x)}{\xi'(x)} = \sqrt{a(x)}u_\epsilon^G(x) \) and set \( L^2(v_\epsilon^G) := L^2((0, y_\infty), v_\epsilon^G \, dy) \). Define

\[ \mathcal{L}_\epsilon^Y := \frac{\epsilon^2}{2} \frac{d^2}{dy^2} - q_\epsilon(y) \frac{d}{dy} \quad \text{in} \quad L^2(v_\epsilon^G), \]

where \( q_\epsilon = -(\mathcal{L}_\epsilon \xi) \circ \xi^{-1} \). It is not hard to check that \( U_\epsilon \mathcal{L}_\epsilon = \mathcal{L}_\epsilon^Y U_\epsilon \), where \( U_\epsilon : L^2(u_\epsilon^G) \to L^2(v_\epsilon^G) \), \( f \mapsto f \circ \xi^{-1} \) is a unitary transform.

Consider the semi-classical Schrödinger operator

\[ \mathcal{L}_\epsilon^S := \frac{\epsilon^2}{2} \frac{d^2}{dy^2} - \frac{1}{2} \left[ \frac{q_\epsilon^2(y)}{\epsilon^2} - q_\epsilon'(y) \right] \quad \text{in} \quad L^2((0, y_\infty)). \]

It is easy to verify that \( \tilde{U}_\epsilon \mathcal{L}_\epsilon^Y = \mathcal{L}_\epsilon^S \tilde{U}_\epsilon \), where \( \tilde{U}_\epsilon : L^2(v_\epsilon) \to L^2((0, y_\infty)) \), \( f \mapsto f \sqrt{v_\epsilon^G} \) is a unitary transform. Hence, \( \tilde{U}_\epsilon \mathcal{L}_\epsilon \tilde{U}_\epsilon = \mathcal{L}_\epsilon^S \tilde{U}_\epsilon \), that is, \( \mathcal{L}_\epsilon \) is unitarily equivalent to \( \mathcal{L}_\epsilon^S \).

Denote by \( V_\epsilon \) the potential of the Schrödinger operator \( \mathcal{L}_\epsilon^S \), namely, \( V_\epsilon = \frac{1}{2} \left( \frac{q_\epsilon^2}{\epsilon^2} - q_\epsilon' \right) \).

**Lemma 2.3.** The following hold.

1. There exist \( C_1 > 0 \) and \( y_1 \in (0, y_\infty) \) such that

\[ V_\epsilon \geq \frac{C_1 \epsilon^2}{\xi^{-1}} \quad \text{in} \quad (0, y_1), \quad \forall 0 < \epsilon \ll 1 \quad \text{and} \quad \inf_{0 < \epsilon \ll 1} \inf_{(0, y_1)} V_\epsilon > 0; \]

2. For each \( y_2 \in (0, y_\infty) \) with \( \xi^{-1}(y_2) \gg 1 \), there exists \( C_2 = C_2(y_2) > 0 \) such that

\[ V_\epsilon \geq \frac{C_2 b_\epsilon^2 \circ \xi^{-1}}{\epsilon^2 a \circ \xi^{-1}} \quad \text{in} \quad [y_2, y_\infty), \quad \forall 0 < \epsilon \ll 1. \]

3. The family \( \{V_\epsilon\}_\epsilon \) is uniformly lower bounded, that is, \( \inf_{0 < \epsilon \ll 1} \min_{(0, y_\infty)} V_\epsilon > -\infty. \)

**Proof.** The proof of this lemma is given in [32, Lemma 2.2]. The only difference is that in (2), we fixed a \( y_2 \in (0, y_\infty) \) there, while we do not fix it here. \( \square \)

2.3. **Concentration estimates and tightness of QSDs.** For \( 0 < \epsilon \ll 1 \), denote

\[ u_\epsilon := \frac{\phi_{\epsilon,1} u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}} \quad \text{and} \quad d\mu_\epsilon := u_\epsilon \, dx. \]

By Lemma 2.1 (2), \( \mu_\epsilon \) is a Borel probability measure on \((0, \infty)\).

**Proposition 2.1 ([6]).** For each \( 0 < \epsilon \ll 1 \), \( \mu_\epsilon \) is the unique QSD of \( X^\epsilon_t \) with extinction rate \( \lambda_{\epsilon,1} \).
We point out that \( \mu \), being a QSD of \( X_t \), follows directly from Lemma 2.1. Proving the uniqueness is however much more involved. In [6], the authors achieve this by exploring the so-called “coming down from infinity”. As a result, they show that for any \( \mu \in \mathcal{P}((0, \infty)) \) the conditioned dynamics \( \mathbb{P}_\mu[X_t \in \bullet t < T_\mu^\bullet] \) converges to the QSD \( \mu \) as \( t \to \infty \). This can be improved to exponential convergence with rate \( \lambda_{\epsilon,2} - \lambda_{\epsilon,1} \) if \( \mu \) is compactly supported in \((0, \infty)\).

Since the breakthrough [6], the study of QSDs for singular diffusion processes has been attracting a lot of attention. We refer the reader to [6, 7, 37, 45, 10, 11, 28, 12, 29] and references therein for the developments focusing on the existence and uniqueness of QSDs and the (exponential) convergence to QSDs.

Let \( A \) be the global attractor of \( \dot{x} = b(x) \) on \((0, \infty)\). Under the assumptions on \( b \), \( A \) is a closed interval with its left endpoint and right endpoint being respectively the smallest positive zero and largest zero of \( b \).

We recall from [49] concentration estimates of \( \{u_\epsilon\}_\epsilon \) away from \( A \) and \( \infty \).

**Lemma 2.4** ([49]). The following hold.

1. For each \( \mathcal{O} \subset (0, \infty) \setminus A \), there are \( \gamma_\mathcal{O} > 0 \) and \( 0 < \epsilon_\mathcal{O} \ll 1 \) such that
   \[
   \sup_{\mathcal{O}} u_\epsilon \leq e^{-\frac{2\epsilon}{\epsilon_\mathcal{O}}}, \quad \forall \epsilon \in (0, \epsilon_\mathcal{O}).
   \]

2. For each \( \kappa \in (0, 1) \), there are \( x_\kappa \in (0, 1) \) and \( 0 < \epsilon_\kappa \ll 1 \) such that
   \[
   u_\epsilon(x) \leq \frac{1}{\epsilon_\kappa}, \quad \forall x \in (0, x_\kappa), \quad \epsilon \in (0, \epsilon_\kappa).
   \]

The proof of Lemma 2.4 (1) in [49] is based on the sub-level set approach developed in [30] and the construction of uniform-in-noise Lyapunov functions. Lemma 2.4 (2) is the most important result in [49]. It addresses the tightness of \( \{u_\epsilon\}_\epsilon \) near 0 by circumventing the difficulties caused by the degeneracy and singularity of the noise at 0.

In the rest of this subsection, we establish concentration estimates of \( \{u_\epsilon\}_\epsilon \) near \( \infty \) that turn out to be very useful in the sequel. Recall from Subsection 2.2 that \( y = \xi(x) \) and \( y_\infty = \xi(\infty) \).

**Lemma 2.5.** Let \( L \gg 1 \). The following hold for each \( 0 < \epsilon \ll 1 \).

1. If \( y_\infty = \infty \), then
   \[
   u_\epsilon \leq u_\epsilon(L) \left[ \frac{a(L)}{a} \right]^{\frac{3}{2}} e^{-\gamma_{\epsilon,L}\epsilon[L-\xi(L)]} \frac{1}{\epsilon_{\epsilon,L}} \int_L^a \frac{L}{\epsilon_{\epsilon,L}} ds \quad \text{in} \quad [L, \infty),
   \]
   where \( \gamma_{\epsilon,L} = \sqrt{\frac{2}{c^2}(C_L^2 - \lambda_{\epsilon,1})} \). In which, \( C_L := C_2 \inf_{\epsilon[L, y_\infty]} \frac{\beta_0\epsilon^{-1}}{\alpha\epsilon^{-1}} \), where \( C_2 = C_2(\xi(L)) \) is given in Lemma 2.3 (2).

2. If \( y_\infty < \infty \), then
   \[
   u_\epsilon \leq u_\epsilon(L) \left[ \frac{a(L)}{a} \right]^{\frac{3}{2}} e^{\gamma_{\epsilon,L}[y_\infty - \xi]} \frac{e^{\gamma_{\epsilon,L}[y_\infty - \xi]} - e^{\gamma_{\epsilon,L}[y_\infty - \xi]}}{e^{\gamma_{\epsilon,L}[\xi(L) - y_\infty]} - e^{\gamma_{\epsilon,L}[y_\infty - \xi(L)]}} \frac{1}{\epsilon_{\epsilon,L}} \int_L^a \frac{L}{\epsilon_{\epsilon,L}} ds \quad \text{in} \quad [L, \infty),
   \]
where $\gamma_{\epsilon,L}$ is as in (1).

In particular, if $L \gg \sup A$, then $u_\epsilon \leq \left[\frac{a(L)}{a}\right]^\frac{3}{2} e^{\frac{1}{2\lambda_1} \int_0^L \frac{1}{x} \, ds}$ in $[L, \infty)$.

**Proof.** The “In particular” part follows directly from (1), (2) and Lemma 2.4 (1). We prove (1) and (2).

Note that $w_\epsilon := \frac{u_\epsilon}{w_\epsilon}$ satisfies $\int_0^\infty w_\epsilon^2 u_\epsilon^G dx < \infty$ and $L_{\epsilon,1} w_\epsilon = -\lambda_{\epsilon,1} w_\epsilon$, and $\bar{w}_\epsilon := \bar{U}_\epsilon U_\epsilon w_\epsilon$ satisfies $\int_0^\infty (V_\epsilon + M) \bar{w}_\epsilon^2 dy < \infty$ and $L_{\epsilon,1} \bar{w}_\epsilon = -\lambda_{\epsilon,1} \bar{w}_\epsilon$, where $M = \inf_{0 \leq \epsilon \leq 1} \inf V_\epsilon < \infty$ due to Lemma 2.3 (3). We readily check that

$$\bar{w}_\epsilon(y) = w_\epsilon(x) \sqrt{a_G^{\epsilon}(y)} = \sqrt{a(x)} w_\epsilon(x) \sqrt{u_\epsilon^G(x)}.$$

Thus,

$$\frac{\bar{w}_\epsilon(y) \sqrt{u_\epsilon^G(x)}}{\sqrt{a(x)}} = w_\epsilon(x) u_\epsilon^G(x) = u_\epsilon(y). \quad (2.2)$$

Fix $L \gg 1$ and set $y_L := \xi(L)$. We distinguish between the cases $y_\infty = \infty$ and $y_\infty < \infty$.

**Case** $y_\infty = \infty$. Consider the following problem:

$$\begin{cases}
\frac{\epsilon^2}{2} \bar{W}_\epsilon'' - \frac{C_L}{\epsilon} \bar{W}_\epsilon = -\lambda_{\epsilon,1} \bar{W}_\epsilon & \text{in } (y_L, \infty), \\
\bar{W}_\epsilon(y_L) = \bar{w}_\epsilon(y_L), & \bar{W}_\epsilon(\infty) = 0,
\end{cases}$$

where $C_L$ is as in the statement. The unique solution is given by

$$\bar{W}_\epsilon(y) = \bar{w}_\epsilon(y_L) e^{-\gamma_{\epsilon,L}(y-y_L)}, \quad y \in [y_L, \infty),$$

where $\gamma_{\epsilon,L}$ is given in the statement. Since $V_\epsilon \geq C_L \geq \lambda_{\epsilon,1}$ on $[y_L, \infty)$ ensured by Lemma 2.3 (2), we find from the comparison principle (see e.g. [2, Chapter 2, Section 2.3]) that $\bar{w}_\epsilon \leq \bar{W}_\epsilon$ in $[y_L, \infty)$. This together with (2.2) implies that for $x \in [L, \infty)$,

$$u_\epsilon(x) \leq \frac{\bar{W}_\epsilon(y) \sqrt{u_\epsilon^G(x)}}{\sqrt{a(x)}} = \frac{\sqrt{u_\epsilon^G(x)}}{\sqrt{a(x)}} \frac{\sqrt{a(L)}}{\sqrt{u_\epsilon^G(L)}} u_\epsilon(L) e^{-\gamma_{\epsilon,L}(y-y_L)} = u_\epsilon(L) \frac{[a(L)]^\frac{3}{2}}{[a(x)]^\frac{3}{2}} e^{-\gamma_{\epsilon,L}[\xi(x)-\xi(L)]} e^{\frac{1}{2\lambda_1} \int_0^L \frac{1}{x} \, ds}.$$

**Case** $y_\infty < \infty$. Consider the following problem:

$$\begin{cases}
\frac{\epsilon^2}{2} \bar{W}_\epsilon'' - \frac{C_L}{\epsilon} \bar{W}_\epsilon = -\lambda_{\epsilon,1} \bar{W}_\epsilon & \text{in } (y_L, y_\infty), \\
\bar{W}_\epsilon(y_L) = \bar{w}_\epsilon(y_L), & \bar{W}_\epsilon(y_\infty) = 0.
\end{cases}$$

The unique solution is given by

$$\bar{W}_\epsilon(y) = \bar{w}_\epsilon(y_L) \frac{e^{\gamma_{\epsilon,L}(y-y_\infty)} - e^{\gamma_{\epsilon,L}(y_\infty-y)}}{e^{\gamma_{\epsilon,L}(y-y_\infty)} - e^{\gamma_{\epsilon,L}(y_\infty-y_L)}}, \quad y \in [y_L, y_\infty).$$
To apply the comparison principle, we verify \( \bar{w}_\epsilon(y_\infty) := \lim_{y \to y_\infty} \bar{w}_\epsilon(y) = 0 \). To see this, we first claim for fixed \( K \gg 1 \),

\[
\int_{\xi(K)}^{y_\infty} V_\epsilon dy = \infty. \tag{2.3}
\]

Indeed, we see from Lemma 2.3 (2) that

\[
\int_{\xi(K)}^{y_\infty} V_\epsilon dy \geq \frac{C_2}{\epsilon^2} \int_{\xi(K)}^{y_\infty} b^2 \circ \xi^{-1} dy = \frac{C_2}{\epsilon^2} \int_K^{\infty} \frac{b^2}{a} dx,
\]

where \( C_2 = C_2(\xi(K)) \). Since \( \lim_{x \to \infty} \frac{b^2(x)}{a(x)} = \infty \) by (H)(3), there is \( c_1 > 0 \) such that

\[
\int_{\{x \in (K, \infty) : a(x) \leq 1\}} \frac{b^2}{a} dx \geq c_1 \left| \{x \in (K, \infty) : a(x) \leq 1\} \right|. \tag{2.4}
\]

As \( \limsup_{x \to \infty} \frac{a(x)}{b(x)} < \infty \) by (H)(3), there is \( c_2 > 0 \) such that \( \frac{a(x)}{b(x)} \leq \frac{1}{c_2} \) for all \( x > K \) (making \( K \) larger if necessary). It follows that

\[
\int_{\{x \in (K, \infty) : a(x) > 1\}} \frac{b^2}{a} dx \geq \int_{\{x \in (K, \infty) : a(x) > 1\}} \sqrt{a} \left( \frac{b}{a} \right)^2 dx \geq c_2^2 \left| \{x \in (K, \infty) : a(x) > 1\} \right|,
\]

which together with (2.4) yields \( \int_K^{\infty} \frac{b^2}{a} dx = \infty \) and thus, (2.3) holds.

Now, we show that

\[
\bar{w}_\epsilon(y_\infty) = 0. \tag{2.5}
\]

It follows from \( \int_0^{y_\infty} (V_\epsilon + M) \bar{w}_\epsilon dy < \infty, \) (2.3) and the positivity of \( \bar{w}_\epsilon \) that \( \liminf_{y \to y_\infty} \bar{w}_\epsilon(y) = 0 \). Suppose for contradiction that \( \limsup_{y \to y_\infty} \bar{w}_\epsilon(y) > 0 \). Then, there exists \( y_\ast \) (which can be chosen to be arbitrary close to \( y_\infty \)) such that \( \bar{w}_\epsilon \) has a local maximum at \( y_\ast \). In particular, \( \bar{w}_\epsilon''(y_\ast) \leq 0 \). This together with \( \frac{c_1^2}{\epsilon^2} \bar{w}_\epsilon''(y_\ast) - V_\epsilon(y_\ast) \bar{w}_\epsilon(y_\ast) = -\lambda_{\epsilon,1} \bar{w}_\epsilon(y_\ast) \) implies that \( V_\epsilon(y_\ast) \leq \lambda_{\epsilon,1} \). Since \( V_\epsilon(y) \to \infty \) as \( y \to y_\infty \), we arrive at a contradiction. Hence, (2.5) is true.

Due to (2.5) and \( V_\epsilon \geq \frac{C_4}{\epsilon^2} \geq \lambda_{\epsilon,1} \) on \([y_L, \infty)\), we apply the comparison principle to conclude that \( \bar{w}_\epsilon(y) \leq \bar{W}_\epsilon(y) \) for all \( y \in [y_L, \infty) \). This together with (2.2) implies

\[
\frac{u_\epsilon(x)}{\sqrt{a(x)}} \leq \frac{\bar{W}_\epsilon(y) \sqrt{u_\epsilon^2(x)} \sqrt{a(L)}}{\sqrt{a(x)}} = \frac{\sqrt{u_\epsilon^2(x)} \sqrt{a(L)}}{\sqrt{a(x)}} u_\epsilon(L) e^{\gamma_{c,L}(y-y_\infty)} - e^{\gamma_{c,L}y_\infty} - e^{\gamma_{c,L}y_{L-y_\infty}} e^{\gamma_{c,L}y_{L-y_\infty}} - e^{\gamma_{c,L}y_{L-y_\infty}}
\]

\[
= u_\epsilon(L) \frac{[a(L)]^{\frac{1}{2}} e^{\gamma_{c,L}[\xi(x)-y_\infty]} - e^{\gamma_{c,L}|y_\infty-\xi(x)|}}{[a(x)]^{\frac{1}{2}} e^{\gamma_{c,L}|\xi(L)-y_\infty|} - e^{\gamma_{c,L}|\xi(L)-\xi(x)|}} e^{\frac{1}{\epsilon^2} \int_L^x 
\]

for all \( x \in [L, \infty) \). This completes the proof.

The following result is a direct consequence of Lemma 2.4 and Lemma 2.5.

**Corollary 2.1.** For any open set \( O \) containing \( A \), there holds \( \lim_{\epsilon \to 0} \mu_\epsilon(O) = 1 \). In particular, the family of QSDs \( \{\mu_\epsilon\} \) is tight.
We end this section by pointing out the difference between [49] and the present paper in treating the tightness of \( \mu_\epsilon \) near infinity. Assuming the existence of a uniform-in-noise Lyapunov function near \( \infty \), the authors proved in [49] the exponential smallness in \( \epsilon \) of the tail estimate appealing to the sub-level set approach put forward in [30]. Here, explicit assumptions on \( a \) and \( b \) allow us to use decaying properties of eigenfunctions of the Schrödinger operator \( L_\epsilon^S \) (which is unitarily equivalent to \( L_\epsilon \)) to establish exponential decaying estimates for the density \( u_\epsilon \). Corresponding results, presented in Lemma 2.5, turn out to be crucial in applying the identities in Proposition 4.1 to derive sharp asymptotic of \( \{u_\epsilon\}_\epsilon \).

3. Vanishing viscosity limits

To study the exponential asymptotic of \( \mu_\epsilon \) or its density \( u_\epsilon \) as \( \epsilon \to 0 \), we introduce for each \( 0 < \epsilon \ll 1 \) the logarithmic transform:

\[
v_\epsilon = -\frac{\epsilon^2}{2} \ln(au_\epsilon) \quad \text{in} \quad (0, \infty).
\]

(3.1)

It is well-defined as both \( a \) and \( u_\epsilon \) are positive on \((0, \infty)\). Moreover, since \( a, u_\epsilon \in C^3((0, \infty)) \), there holds \( v_\epsilon \in C^3((0, \infty)) \). Clearly, the local uniform convergence of \(-\frac{\epsilon^2}{2} \ln u_\epsilon \) to some \( v \in C((0, \infty)) \) as \( \epsilon \to 0 \) is equivalent to the local uniform convergence of \( v_\epsilon \) to \( v \) as \( \epsilon \to 0 \).

It is straightforward to check that \( v_\epsilon \) satisfies the following singularly perturbed equation:

\[
-\frac{\epsilon^2}{2} v_\epsilon'' + \left(\frac{b}{a} v_\epsilon'\right)^2 + \frac{b}{a} v_\epsilon' = \frac{\epsilon^2}{2} \left[ \frac{b}{a} \right]' - \frac{\lambda_{\epsilon,1}}{a} \quad \text{in} \quad (0, \infty).
\]

(3.2)

The next result addresses the local uniform boundedness of \( \{v_\epsilon\}_\epsilon \) and \( \{v'_\epsilon\}_\epsilon \). Its proof is postponed to the end of this section.

**Lemma 3.1.** The following hold.

1. For each \( O \subset (0, \infty) \), there exist \( \gamma^1_O \in \mathbb{R}, \gamma^2_O > 0 \) and \( 0 < \epsilon_O \ll 1 \) such that

\[
\gamma^1_O \leq \inf_O v_\epsilon \leq \sup_O v_\epsilon \leq \gamma^2_O, \quad \forall \epsilon \in (0, \epsilon_O).
\]

Moreover, if \( O \subset (0, \infty) \setminus \mathcal{A} = \emptyset \), then \( \gamma^1_O > 0 \).

2. For each \( O \subset (0, \infty) \), there exist \( \Gamma_O > 0 \) and \( 0 < \epsilon_O \ll 1 \) such that

\[
\sup_O |v'_\epsilon| \leq \Gamma_O, \quad \forall \epsilon \in (0, \epsilon_O).
\]

Denote by \( V \) the set of limit points of \( \{v_\epsilon\}_\epsilon \) under the topology of locally uniform convergence in \((0, \infty)\) as \( \epsilon \to 0 \). By Lemma 3.1, we apply the Arzelá-Ascoli theorem and standard diagonal argument to conclude \( V \neq \emptyset \) and \( V \subset C((0, \infty)) \). Moreover, the well-known result on the stability of viscosity solutions (see e.g. [15]) ensures that each element of \( V \) is a viscosity solution of the following Hamilton-Jacobi equation:

\[
\left(\frac{b}{a} v'\right)^2 + \frac{b}{a} v' = 0 \quad \text{in} \quad (0, \infty).
\]

(3.3)

It is unfortunate that (3.3) admits infinitely many viscosity solutions.
We prove some properties of functions in $\mathcal{V}$, which will be used in Section 4. Recall that $\mathcal{A}$ is the global attractor of $\dot{x} = b(x)$ in $(0, \infty)$.

**Proposition 3.1.** Each $v \in \mathcal{V}$ is locally Lipschitz continuous and satisfies

$$(v')^2 + \frac{b}{a}v' = 0 \text{ a.e. in } (0, \infty).$$

Moreover, $v > 0$ on $(0, \infty) \setminus \mathcal{A}$, $v(0+) \in (0, \infty)$, $v(\infty) = \infty$ and $\min_{\mathcal{A}} v = 0$.

**Proof.** Let $v \in \mathcal{V}$. By Lemma 3.1 (2), $v$ is locally Lipschitz continuous. Since $v$ is a viscosity solution of (3.3), it is well-known (see e.g. [15]) that if $v$ is differentiable at $x_0 \in (0, \infty)$, then $v(x_0) = 0$ holds at $x_0$. Hence, $v(x)$ satisfies $(v')^2 + \frac{b}{a}v' = 0$ a.e. in $(0, \infty)$.

Lemma 3.1 (1) ensures that $v > 0$ on $(0, \infty) \setminus \mathcal{A}$. Since $b > 0$ in $(0, \inf \mathcal{A})$, we see from the equation that $v' \leq 0$ a.e. in $(0, \inf \mathcal{A})$, and thus, $v$ is non-increasing on $(0, \inf \mathcal{A})$. It follows that $v(0+) \in (0, \infty]$. Since $v' \geq -\frac{b}{a}$ a.e. in $(0, \inf \mathcal{A})$, $v(0+)$ must be finite, and hence, $v(0+) \in (0, \infty)$.

We see from Lemma 2.5 that $v(x) \geq -\frac{1}{2} \int_0^x \frac{b(s)}{a(s)} ds$ for $x \geq L$. Since $\int_0^x \frac{b(s)}{a(s)} ds \to -\infty$ as $x \to \infty$, we conclude $v(\infty) = \infty$.

It remains to show $\min_{\mathcal{A}} v = 0$. Let $I$ be an open interval such that $\mathcal{A} \subset I \subset (0, \infty)$. Corollary 2.1 ensures that $\lim_{\epsilon \to 0} \int_I u_\epsilon dx = 1$, or $\lim_{\epsilon \to 0} \int_I \frac{1}{a} e^{-\frac{2}{\gamma} \nu \epsilon} dx = 1$. This together with the uniform convergence of $v_\epsilon$ (up to a subsequence) to $v$ on $I$ as $\epsilon \to 0$ implies that $\inf_I v = 0$, and hence, $\min_{\mathcal{A}} v = 0$. □

The rest of this section is devoted to the proof of Lemma 3.1.

**Proof of Lemma 3.1.** (1) Let $\mathcal{O} \subset (0, \infty)$ be open. It follows from the classical interior estimates for elliptic equations (see e.g. [25]) that there exists $\gamma_\mathcal{O} > 0$ such that $\sup_{\mathcal{O}} u_\epsilon \leq e^{-\frac{\gamma_\mathcal{O}}{\epsilon}}$ for $0 < \epsilon \ll 1$, which together with (3.1) leads to $\inf_{\mathcal{O}} v_\epsilon \geq -\frac{\gamma_\mathcal{O}}{2}$ for $0 < \epsilon \ll 1$.

To see the upper bound of $v_\epsilon$ on $\mathcal{O}$, we let $\mathcal{O}_1$ be an open interval satisfying $\mathcal{A} \cup \mathcal{O} \subset \mathcal{O}_1 \subset (0, \infty)$. Fix $0 < \delta \ll 1$. Since Corollary 2.1 ensures

$$|\mathcal{O}_1| \sup_{\mathcal{O}_1} u_\epsilon \geq \int_{\mathcal{O}_1} u_\epsilon dx \geq 1 - \delta, \quad \forall 0 < \epsilon \ll 1,$$

we see from Harnack’s inequality that there exists $\gamma_{\mathcal{O}_1} > 0$ such that $\inf_{\mathcal{O}_1} u_\epsilon \geq e^{-\frac{\gamma_{\mathcal{O}_1}}{\epsilon^2}}$ for $0 < \epsilon \ll 1$, which together with (3.1) yields $\sup_{\mathcal{O}_1} v_\epsilon \leq \sup_{\mathcal{O}_1} u_\epsilon \leq \frac{\gamma_{\mathcal{O}_1}}{2}$ for $0 < \epsilon \ll 1$.

For the “Moreover” part, we let $\mathcal{O} \subset (0, \infty) \setminus \mathcal{A}$. Then, Lemma 2.4 (1) yields the existence of $\gamma_{\mathcal{O}} > 0$ such that $\sup_{\mathcal{O}} u_\epsilon \leq e^{-\frac{\gamma_{\mathcal{O}}}{\epsilon^2}}$ for $0 < \epsilon \ll 1$, leading to $\inf_{\mathcal{O}} v_\epsilon \geq \frac{\gamma_{\mathcal{O}}}{2} > 0$ for $0 < \epsilon \ll 1$. This completes the proof of (1).

(2) The proof is inspired by the Bernstein-type estimate in [21, Lemma 2.2]. The key point here lies in the non-negativeness of the term $(v'_\epsilon)^2$ in (3.2). Let $I_1, I_2$ be open intervals and satisfy $0 \neq I_1 \subset \subset I_2 \subset \subset (0, \infty)$. Let $\eta : (0, \infty) \to [0, 1]$ be smooth and satisfy $\eta = 1$ on $I_1$ and $\eta = 0$ on $(0, \infty) \setminus I_2$. 


Fix $0 < \epsilon_* \ll 1$. For each $\epsilon \in (0, \epsilon_*)$, we define the auxiliary function $z_\epsilon = \eta^4(v'_\epsilon)^2$. We claim that
\[
\sup_{\epsilon \in (0, \epsilon_*)} \max z_\epsilon < \infty. \tag{3.4}
\]
If this is the case, then $\sup_{\epsilon \in (0, \epsilon_*)} \sup I_1 |v'_\epsilon| < \infty$, leading to the conclusion.

It remains to show (3.4). Since $z_\epsilon$ is continuous and compactly supported in $I_2$, there exists $x_\epsilon \in I_2$ such that $z_\epsilon(x_\epsilon) = \max z_\epsilon$. We may assume, without loss of generality, that $\max z_\epsilon > 0$. Then, $\eta(x_\epsilon) > 0$ and $v'_\epsilon(x_\epsilon) \neq 0$.

We calculate
\[
z'_\epsilon = 4\eta^3 \eta'(v'_\epsilon)^2 + 2\eta^4 v'_\epsilon v''_\epsilon, \\
z''_\epsilon = (\eta^4)'(v'_\epsilon)^2 + 16\eta^3 \eta' v'_\epsilon v''_\epsilon + 2\eta^4 (v''_\epsilon)^2 + 2\eta^4 v'_\epsilon v''''_\epsilon.
\]
Multiplying the expression of $z''_\epsilon$ by $-\frac{\epsilon^2}{2}$ and setting $c_\epsilon := \frac{\epsilon^2}{2} \left( \left( \frac{b}{a} \right)' - \frac{\lambda_1}{a} \right)$ (i.e., the right hand side of (3.2)), we find from (3.2) that
\[
-\frac{\epsilon^2}{2} z''_\epsilon = -\frac{\epsilon^2}{2} (\eta^4)'(v'_\epsilon)^2 - 8\epsilon^2 \eta^3 \eta' v'_\epsilon v''_\epsilon - \epsilon^2 \eta^4 (v''_\epsilon)^2 + 2\eta^4 v'_\epsilon \left( \frac{-\epsilon^2}{2} v''_\epsilon \right)' = -\frac{\epsilon^2}{2} (\eta^4)'(v'_\epsilon)^2 - 8\epsilon^2 \eta^3 \eta' v'_\epsilon v''_\epsilon - \epsilon^2 \eta^4 (v''_\epsilon)^2 + 2\eta^4 v'_\epsilon \left[ c_\epsilon - (v'_\epsilon)^2 - \frac{b}{a} v''_\epsilon \right]' = -\frac{\epsilon^2}{2} (\eta^4)'(v'_\epsilon)^2 - 8\epsilon^2 \eta^3 \eta' v'_\epsilon v''_\epsilon - \epsilon^2 \eta^4 (v''_\epsilon)^2 + 2\eta^4 v'_\epsilon c'_\epsilon - 4\eta^4 (v'_\epsilon)^2 v''_\epsilon - 2\eta^4 \left( \frac{b}{a} \right)' (v'_\epsilon)^2 - 2\eta^4 \frac{b}{a} v'_\epsilon v''_\epsilon. 
\tag{3.5}
\]
At the point $x_\epsilon$, there holds $z'_\epsilon = 0$, namely, $4\eta^3 \eta'(v'_\epsilon)^2 + 2\eta^4 v'_\epsilon v''_\epsilon = 0$. Since $\eta(x_\epsilon) > 0$ and $v'_\epsilon(x_\epsilon) \neq 0$, we find
\[
\eta v''_\epsilon = -2\eta v'_\epsilon \quad \text{at} \quad x_\epsilon. \tag{3.6}
\]
As $z''_\epsilon(x_\epsilon) \leq 0$, we find from (3.5) and (3.6) that at the point $x_\epsilon$ there holds
\[
e^2 \eta^4 (v''_\epsilon)'^2 \leq -\frac{\epsilon^2}{2} (\eta^4)'(v'_\epsilon)^2 - 8\epsilon^2 \eta^3 \eta' v'_\epsilon v''_\epsilon + 2\eta^4 v'_\epsilon c'_\epsilon - 4\eta^4 (v'_\epsilon)^2 v''_\epsilon - 2\eta^4 \left( \frac{b}{a} \right)' (v'_\epsilon)^2 - 2\eta^4 \frac{b}{a} v'_\epsilon v''_\epsilon = -\frac{\epsilon^2}{2} (\eta^4)'(v'_\epsilon)^2 - 8\epsilon^2 \eta^3 \eta' v'_\epsilon v''_\epsilon - \eta^4 v'_\epsilon v''_\epsilon + \eta^4 (c'_\epsilon)^2 - 4\eta^3 (v'_\epsilon)^2 (2\eta v'_\epsilon + \eta v''_\epsilon) - 2\eta^3 \frac{b}{a} v'_\epsilon (2\eta v'_\epsilon) = 6\eta^3 (v'_\epsilon)^3 + \left[ -\frac{\epsilon^2}{2} (12(\eta')^2 + 4\eta \eta') + 16\epsilon^2 (\eta')^2 + \eta^2 - 2\eta^2 \left( \frac{b}{a} \right)' + 4\eta \frac{b}{a} \right] \eta^2 (v'_\epsilon)^2 + \eta^4 (c'_\epsilon)^2.
\]
Thus, $\epsilon^2 \eta^4 (v''_\epsilon) \leq C_1 \eta^3 |v'|^3 + C_2 \eta^2 (v'_\epsilon)^2 + C_3$ at $x_\epsilon$, where

$$C_1 = 8 \max_{\epsilon \in (0, \epsilon_*)} |\eta'|, \quad C_3 = \sup_{\epsilon \in (0, \epsilon_*)} \max \left[ \epsilon^2 \eta^4 (v'_\epsilon)^2 \right],$$

$$C_2 = \sup_{\epsilon \in (0, \epsilon_*)} \max \left[ -\frac{\epsilon^2}{2} \left( 12 (\eta^4)'' + 4 \eta^3 \eta'' \right) + 16 \epsilon^2 (\eta')^2 + \eta^2 - 2 \eta^2 \left( \frac{b}{a} \right)' + 4 \eta' \eta \frac{b}{a} \right].$$

Since $C_2 \eta^2 (v'_\epsilon)^2 \leq \frac{C^2}{3} + \frac{2}{3} \eta^3 |v'|^3$ by Young’s inequality, we arrive at

$$\epsilon^2 \eta^4 (v''_\epsilon)^2 \leq C_4 \eta^3 |v'|^3 + C_5 \quad \text{at} \quad x_\epsilon,$$

where $C_4 = C_1 + \frac{2}{3}$ and $C_5 = \frac{C^2}{3} + C_3$.

Because of $(v'_\epsilon)^2 = c_\epsilon - \frac{b}{a} v'_\epsilon + \frac{2}{2} v''_\epsilon$ by (3.2) and $|\frac{b}{a} v'_\epsilon| \leq \frac{1}{2} \left( \frac{b}{a} \right)^2 + \frac{1}{2} (v'_\epsilon)^2$ by Hölder’s inequality, we find

$$\frac{1}{2} (v'_\epsilon)^2 \leq \frac{2}{2} v''_\epsilon + c_\epsilon + \frac{1}{2} \left( \frac{b}{a} \right)^2.$$

Thus,

$$\eta^4 (v'_\epsilon)^4 \leq \eta^4 \left[ \epsilon^2 v''_\epsilon + 2 c_\epsilon + \left( \frac{b}{a} \right)^2 \right]^2 \leq 2 \epsilon^4 \eta^4 (v''_\epsilon)^2 + 2 \eta^4 \left[ 2 c_\epsilon + \left( \frac{b}{a} \right)^2 \right]^2.$$

This together with (3.7) implies that $\eta^4 (v'_\epsilon)^4 \leq 2 \epsilon^2 C_4 \eta^3 |v'|^3 + C_6$ at $x_\epsilon$, where

$$C_6 = \sup_{\epsilon \in (0, \epsilon_*)} \left\{ 2 \epsilon^2 C_5 + \max \left[ 2 \epsilon^2 C_5 + \max \left( \frac{b}{a} \right)^2 \right] \right\}.$$

Let $\kappa > 0$ be such that $\frac{3}{2} \kappa^4 = \frac{1}{2}$. Applying Young’s inequality, we find

$$\eta^4 (v'_\epsilon)^4 \leq \frac{2 \kappa^2 C_4}{\kappa} \eta^3 |v'|^3 + C_6 \leq \frac{116 \kappa^2 C_4^4}{\kappa^4} + \frac{1}{2} \eta^4 (v'_\epsilon)^4 + C_6 \quad \text{at} \quad x_\epsilon,$$

which gives $\eta^4 (v'_\epsilon)^4 \leq C_7 := \frac{8 \kappa^2 C_4^3}{\kappa^4} + 2 C_6$ at $x_\epsilon$. It follows that

$$\max z_\epsilon = \eta^4 (x_\epsilon) (v'_\epsilon)^2 (x_\epsilon) \leq \sqrt{C_7} \max \eta^2.$$

Since this estimate holds for all $\epsilon \in (0, \epsilon_*)$, we conclude (3.4), and hence, complete the proof. \qed

### 4. Large deviation principle of QSDs

In this section, we study the LDP for QSDs $\{\mu_\epsilon\}_\epsilon$ and prove in particular Theorems A, B and C.
4.1. **Identities.** Recall $u_\epsilon$ and $v_\epsilon$ from (2.1) and (3.1), respectively. We derive identities for $u_\epsilon$, $v_\epsilon$ and $v'_\epsilon$ that play crucial roles in proving the LDP for \( \{\mu_\epsilon\}_\epsilon \).

**Proposition 4.1.** Assume (H). For each $0 < \epsilon \ll 1$,
\[
\begin{align*}
 u_\epsilon &= \frac{2\lambda_{\epsilon,1}}{e^2a} e^{-\frac{2}{\epsilon^2}V} \int_0^\epsilon e^{-\frac{2}{\epsilon^2}V(z)} \left( \int_z^\infty u_\epsilon \, dz \right) \, dz, \\
 v_\epsilon &= -\frac{\epsilon^2}{2} \ln \frac{2}{\epsilon^2} - \frac{\epsilon^2}{2} \ln \lambda_{\epsilon,1} - \frac{\epsilon^2}{2} \ln \int_0^\epsilon e^{-\frac{2}{\epsilon^2}V(z)} \left( \int_z^\infty u_\epsilon \, dz \right) \, dz + V, \\
 v'_\epsilon &= -\frac{b}{a} - \lambda_{\epsilon,1} e^{-\frac{2}{\epsilon^2}V} \int_0^\epsilon \frac{1}{a} e^{-\frac{2}{\epsilon^2}V} \, dz.
\end{align*}
\]

We establish several lemmas before proving Proposition 4.1. Recall from the proof of Lemma 2.5 that $w_\epsilon = \frac{u_\epsilon}{v_\epsilon}$.

**Lemma 4.1.** For each $0 < \epsilon \ll 1$, $\lim_{x \to \infty} \frac{w_\epsilon(x)}{\sqrt{a(x)}} e^{-\frac{V(x)}{\epsilon^2}} = 0$.

**Proof.** It is a byproduct of the proof of Lemma 2.5. Indeed, since
\[
\frac{w_\epsilon(x)}{\sqrt{a(x)}} e^{-\frac{V(x)}{\epsilon^2}} = \sqrt{a(x)} w_\epsilon(x) \sqrt{u_\epsilon'(x)} = \tilde{w}_\epsilon(\xi(x)) \leq W_\epsilon(\xi(x)), \quad \forall x \gg 1,
\]
the lemma follows immediately from $\lim_{y \to y_0} W_\epsilon(y) = 0$ and $y = \xi(x)$.

**Lemma 4.2.** For each $0 < \epsilon \ll 1$,
\[
\frac{\epsilon^2}{2} \left( w_\epsilon'(e^{-\frac{2}{\epsilon^2}V})' \right) = -\lambda_{\epsilon,1} u_\epsilon \quad \text{in } (0, \infty).
\]

In particular, $w'_\epsilon > 0$ in $(0, \infty)$ and $w_\epsilon(0+) = 0$.

**Proof.** Note that $w_\epsilon$ satisfies $L_\epsilon w_\epsilon = -\lambda_{\epsilon,1} w_\epsilon$, namely, $\frac{\epsilon^2}{2} a w_\epsilon'' + b w_\epsilon' = -\lambda_{\epsilon,1} w_\epsilon$. Multiplying this equation by $w_\epsilon'^2$, we readily derive the identity as in the statement.

We show $w'_\epsilon > 0$. Suppose for contradiction that there is $x_\ast \in (0, \infty)$ such that $w'_\epsilon(x_\ast) \leq 0$. Fix $x_{**} > x_\ast$. Integrating the identity over $[x_\ast, x_{**}]$ yields $w'_\epsilon(x_{**}) < 0$. We then integrate the identity over $[x_{**}, x]$ to find
\[
w'_\epsilon(x) e^{-\frac{2}{\epsilon^2}V(x)} < -C_1 := w'_\epsilon(x_{**}) e^{-\frac{2}{\epsilon^2}V(x_{**})} < 0 \quad \text{for } x > x_{**}.
\]
It follows that $w'_\epsilon(x) < -C_1 e^{-\frac{2}{\epsilon^2}V(x)}$ for $x > x_{**}$. Since $V(x) \to \infty$ as $x \to \infty$, there exists $C_2 > 0$ such that $w'_\epsilon(x) \leq -C_2$ for all $x \gg 1$, which implies that $w_\epsilon < 0$ for all $x \gg 1$, leading to a contradiction.

It remains to show $w_\epsilon(0+) = 0$. Since $w_\epsilon u_\epsilon' = u_\epsilon \in L^1((0, \infty))$, we conclude from the behavior of $u_\epsilon'(x)$ near $x = 0$ and the monotonicity of $u_\epsilon$ that $w_\epsilon(0+) = 0$.

**Lemma 4.3.** For each $0 < \epsilon \ll 1$, $\lim_{x \to \infty} w_\epsilon(x) e^{-\frac{2}{\epsilon^2}V(x)} = 0$. 

Proof. By Lemma 4.2, the function $w'_e e^{-\frac{2}{e}V}$ is positive and decreasing. So,

$$C := \lim_{x \to \infty} w'_e(x) e^{-\frac{2}{e}V(x)} \geq 0.$$ 

It suffices to show $C = 0$.

Suppose on the contrary that $C > 0$. Then, there exists $x_0 > 0$ such that $w'_e e^{-\frac{2}{e}V} \geq \frac{C}{2}$ in $(x_0, \infty)$, and hence,

$$w_e(x) = w_e(x_0) + \int_{x_0}^x w'_e(s)ds \geq w_e(x_0) + \frac{C}{2} \int_{x_0}^x e^{-\frac{2}{e}V(s)}ds, \quad \forall x > x_0. \tag{4.1}$$

Since (H)(3) ensures $V'(x) \leq V^m(x)$ for $x \gg 1$, we derive

$$\frac{\frac{d}{dx} \int_{x_0}^x e^{\frac{2}{e}V(s)}ds}{\frac{d}{dx} e^{\frac{2}{e}V(x)}} = \frac{e^{\frac{2}{e}V(x)}}{3V'(x)e^{\frac{2}{e}V(x)}} = \frac{2\epsilon^2 e^{\frac{1}{2}\epsilon} V(x)}{3V'(x)} \geq \frac{2\epsilon^2 e^{\frac{1}{2}\epsilon} V(x)}{3V^m(x)} \to \infty \quad \text{as} \quad x \to \infty,$$

where we used the fact $\lim_{x \to \infty} V(x) = \infty$ to claim the last limit. From which, it follows that

$$\lim_{x \to \infty} \int_{x_0}^x e^{\frac{2}{e}V(s)}ds = \infty. \quad \text{This together with (4.1) yields}$$

$$w_e(x) \geq w_e(x_0) + \frac{C}{2} e^{\frac{3}{2}\epsilon V(x)}, \quad \forall x \gg L. \tag{4.2}$$

Thanks to Lemma 2.5 and $w_e = au_e e^{\frac{2}{e}V}$, we find $C_1 > 0$ such that $w_e(x) \leq C_1 a^\frac{1}{2} e^{\frac{1}{2}[V(x)+V(L)]}$ for $x \gg 1$. By (H)(3), there is $C_2 > 0$ such that $a^\frac{1}{4}(x) \leq e^{C_2x}$ and $V(x) \geq C_2x$ for $x \gg 1$. As a result,

$$w_e(x) \leq C_1 e^{C_2x} e^{\frac{1}{2}[V(x)+V(L)]} \leq C_1 e^{\frac{1}{2\epsilon} V(x)}, \quad \forall x \gg L.$$

This contradicts (4.2) due to $\lim_{x \to \infty} V(x) = \infty$. Hence, $C = 0$. \hfill \Box

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Integrating the identity in Lemma 4.2 over $[x, \tilde{x}] \subset (0, \infty)$ yields

$$\frac{\epsilon^2}{2} w'_e(x) e^{-\frac{2}{e}V(x)} - \frac{\epsilon^2}{2} w'_e(\tilde{x}) e^{-\frac{2}{e}V(\tilde{x})} = -\lambda_{e,1} \int_x^{\tilde{x}} u_e dz.$$ 

Passing to the limit $\tilde{x} \to \infty$, we deduce from Lemma 4.3 that $\frac{\epsilon^2}{2} w'_e e^{-\frac{2}{e}V} = \lambda_{e,1} \int_0^\infty u_e dz$, which together with $w_e(0+) = 0$ (by Lemma 4.2) gives

$$w_e = \frac{2\lambda_{e,1}}{\epsilon^2} \int_0^\infty e^{\frac{2}{e}V(z)} \left( \int_z^\infty u_e dz \right) dz.$$ 

As $u_e = u_e u_G^G$, we derive the formula for $u_e$. The formula for $v_e$ then is a direct consequence of its definition.

Note that $v'_e = -\frac{\epsilon^2}{2} \frac{(au_e)'}{au_e}$. Integrating $\frac{\epsilon^2}{2} (au_e)^{''} - (bu_e)' = -\lambda_{e,1} u_e$, we find $\frac{\epsilon^2}{2} (au_e)^{''} - bu_e = \lambda_{e,1} \int_0^\infty u_e dz$, leading to $v'_e = -\frac{b}{a} + \frac{\lambda_{e,1}}{au_e} \int_0^\infty u_e dz$. The conclusion follows readily from $u_e = \frac{1}{a} e^{-\frac{2}{e}v_e}$. \hfill \Box
The formula for $u_\epsilon$ in Proposition 4.1 leads to refined estimates of $\{u_\epsilon\}_\epsilon$ near 0 in comparison to those given in Lemma 2.4 (2).

**Lemma 4.4.** Assume (H). For each $0 < \delta \ll 1$, there are $0 < x_\delta \ll 1$ and $0 < \epsilon_\delta \ll 1$ such that
\[ e^{-\frac{2}{\epsilon^2} (d_1 + \delta)} \leq u_\epsilon(x) \leq e^{-\frac{2}{\epsilon^2} (d_1 - \delta)}, \quad \forall x \in (0, x_\delta), \quad \epsilon \in (0, \epsilon_\delta). \]

**Proof.** We establish the lower bound; the upper bound follows similarly. Consider the formula for $u_\epsilon$ in Proposition 4.1. Note that for each $0 < \delta \ll 1$, there is $0 < x_\delta \ll 1$ such that $|V(x) - V(y)| \leq \frac{\delta}{2}$ for all $x, y \in (0, x_\delta)$. By Corollary 2.1, there exists $0 < \epsilon_\delta \ll 1$ such that $\int_{x_\delta}^\infty u_\epsilon d\tilde{z} \geq 1 - \delta$ for all $\epsilon \in (0, \epsilon_\delta)$. Then, for each $x \in (0, x_\delta)$ and $\epsilon \in (0, \epsilon_\delta),$
\[ u_\epsilon(x) \geq \frac{2(1 - \delta)\lambda_{\epsilon, 1}}{\epsilon^2a(x)} \int_0^x e^{-\frac{2}{\epsilon^2} (V(x) - V(z))} dz = \frac{2(1 - \delta)\lambda_{\epsilon, 1}}{\epsilon^2a(x)} xe^{-\frac{2}{\epsilon^2} (V(x) - V(\xi))}, \]
where we used the mean value theorem in the equality and $\xi \in (0, x)$. The desired inequality then follows from Lemma 2.2 and the facts that $a(0) = 0$ and $a'(0) > 0$. \qed

The next result, improving Corollary 2.1, is a simple consequence of Lemma 2.4 (1), Lemma 2.5 and Lemma 4.4.

**Corollary 4.1.** Assume (H). For each open set $O$ satisfying $A \subset O \subset (0, \infty)$, there exist $\gamma_O > 0$ and $0 < \epsilon_O \ll 1$ such that
\[ \mu_{\epsilon}((0, \infty) \setminus O) \leq e^{-\frac{\gamma_O}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_O). \]

4.2. **Proof of Theorem A.** Let $(\alpha, \beta) \subset (0, \infty)$ be the unique $d_1$-valley. We focus on the case $\alpha > 0$; the case $\alpha = 0$ can be treated in the same way and is easier.

Up to a subsequence, we may assume without loss of generality that $\lim_{\epsilon \to 0} v_\epsilon = v$ locally uniformly in $(0, \infty)$. We determine $v$ within three steps.

**Step 1.** Let $x_0$ be the smallest zero of $v$. By Proposition 3.1, $x_0$ exists and belongs to $[\inf A, \sup A]$. We show
\[ v(x) = d_1 + V(x) - \sup_{(0, x)} V = \begin{cases} d_1 + V(x) - \sup_{(0, x)} V, & x \in (0, \alpha], \\ d_1 + V(x) - V(\alpha), & x \in (\alpha, x_0], \end{cases} \quad (4.3) \]
and
\[ x_0 = \min \left\{ x \in (\alpha, \beta) : V(x) = \min_{(\alpha, \beta)} V \right\}. \quad (4.4) \]

Fix $x \in (0, x_0)$. Note that Proposition 3.1 and the definition of $x_0$ ensure $\min_{(0, x]} v > 0$. The locally uniform convergence of $v_\epsilon$ to $v$ as $\epsilon \to 0$ and Lemma 4.4 then imply $\inf_{z \in (0, x]} \int_z^\infty u_\epsilon d\tilde{z} \to 1$ as $\epsilon \to \infty$, and hence,
\[ \lim_{\epsilon \to 0} \frac{e^2}{2} \ln \int_0^x e^{\frac{2}{\epsilon^2} V(z)} \left( \int_z^\infty u_\epsilon d\tilde{z} \right) dz = \sup_{(0, x]} V. \]
This together with the formula for $v_{\epsilon}$ in Proposition 4.1 and Lemma 2.2 yields $\lim_{\epsilon \to 0} v_{\epsilon}(x) = d_1 + V(x) - \sup_{(0,x)} V$. From which and the continuity of $v$, the first equality in (4.3) follows readily.

Since $v(x_0) = 0$ by the definition of $x_0$, we see from the first equality in (4.3) that $\sup_{(0,x_0)} V - V(x_0) = d_1$. As $(\alpha, \beta)$ is the unique $d_1$-valley, there must hold

$$x_0 \in \left\{ x \in (\alpha, \beta) : V(x) = \min_{(\alpha, \beta)} V \right\}$$

and (4.4), otherwise $v$ attains 0 in $(0, x_0)$.

Observing that $V(\alpha) = \max_{(0, \beta)} V$, we deduce the second equality in (4.3).

**Step 2.** We prove that for any $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$, there holds

$$\lim_{\epsilon \to 0} \frac{\epsilon^2}{2} \ln \int_{x_1}^{x_2} \lambda_{\epsilon, 1} e^{\frac{z}{\epsilon^2} v_{\epsilon}(z)} \left( \int_{z}^{\infty} \frac{1}{a} e^{-\frac{z}{\epsilon^2} v_{\epsilon}} d\tilde{z} \right) dz = -d_1 + \gamma(x_1, x_2), \quad (4.5)$$

where

$$\gamma(x_1, x_2) = \sup_{z \in [x_1, x_2]} \sup_{\tilde{z} \in (z, \infty)} [v(z) - v(\tilde{z})]. \quad (4.6)$$

To see this, we fix $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$ and split the integral

$$\int_{x_1}^{x_2} \lambda_{\epsilon, 1} e^{\frac{z}{\epsilon^2} v_{\epsilon}(z)} \left( \int_{z}^{\infty} \frac{1}{a} e^{-\frac{z}{\epsilon^2} v_{\epsilon}} d\tilde{z} \right) dz = \int_{x_1}^{x_2} \lambda_{\epsilon, 1} e^{\frac{z}{\epsilon^2} v_{\epsilon}(z)} \left( \int_{z}^{z_1} \frac{1}{a} e^{-\frac{z}{\epsilon^2} v_{\epsilon}} d\tilde{z} + \int_{z_1}^{\infty} \frac{1}{a} e^{-\frac{z}{\epsilon^2} v_{\epsilon}} d\tilde{z} \right) dz,$$

where $z_1 \gg x_2$ is such that

$$\inf_{0 < \epsilon \ll 1} v_{\epsilon} > \sup_{0 < \epsilon \ll 1} v_{\epsilon}.$$
Thanks to the locally uniform convergence of \( v_\epsilon \) to \( v \) as \( \epsilon \to 0 \), there exists \( 0 < \epsilon_0 < 1 \) such that
\[
\lambda_{\epsilon,1} e^{-\frac{\delta}{2}} \int_{x_1}^{x_2} \int_{z}^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon} \left[ v(z) - v(\tilde{z}) \right]} d\tilde{z} dz \\
\leq \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} v_\epsilon(z)} \left( \int_{z}^{z_1} \frac{1}{a} e^{-\frac{2}{\epsilon} v_\epsilon} d\tilde{z} \right) dz \\
\leq \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} \int_{x_1}^{x_2} \int_{z}^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon} \left[ v(z) - v(\tilde{z}) \right]} d\tilde{z} dz}, \quad \forall 0 < \epsilon < \epsilon_0.
\]

Applying Laplace’s method, we find
\[
\lim_{\epsilon \to 0} \frac{\epsilon^2}{2} \ln \int_{x_1}^{x_2} \int_{z}^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon} \left[ v(z) - v(\tilde{z}) \right]} d\tilde{z} dz = \gamma(x_1, x_2),
\]
which together with the above two-sided inequalities, Lemma 2.2 and the arbitrariness of \( \delta > 0 \) leads to (4.5).

**Step 3.** We finish the proof by showing
\[
v = d_1 + V - V(\alpha) \text{ in } (x_0, \infty).
\]

Integrating the formula for \( v_\epsilon' \) in Proposition 4.1 over \((x_1, x_2) \subset (\alpha, \infty)\) yields
\[
v_\epsilon(x_2) - v_\epsilon(x_1) = -\int_{x_1}^{x_2} \frac{b}{a} ds - \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} v_\epsilon(z)} \left( \int_{z}^{z_1} \frac{1}{a} e^{-\frac{2}{\epsilon} v_\epsilon} d\tilde{z} \right) d\tilde{z} \\
= V(x_2) - V(x_1) - \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} v_\epsilon(z)} \left( \int_{z}^{z_1} \frac{1}{a} e^{-\frac{2}{\epsilon} v_\epsilon} d\tilde{z} \right) d\tilde{z}.
\]

Since the above identity holds for any \( x_1, x_2 \in (\alpha, \infty) \) with \( x_1 < x_2 \), we see from the definition of \( \gamma \) in Definition 4.6 that \( \gamma(x_1, x_2) < d_1 \). As a consequence, we let \( \epsilon \to 0 \) in (4.8) and apply (4.5) to conclude that
\[
v(x_2) = v(x_1) + V(x_2) - V(x_1).
\]

Letting \( x_1 \to \alpha^+ \) and setting \( x_2 = x \in (x_0, \infty) \), we conclude (4.7) from (4.3) and the continuity of \( v \).

### 4.3. Proof of Theorem B. (1) The formula for \( u_\epsilon \) in Proposition 4.1 and the definition of \( R_\epsilon \) (see (1.3)) give
\[
R_\epsilon = \epsilon \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} d_1} \frac{2}{\epsilon^2} \int_{0}^{\infty} e^{\frac{2}{\epsilon} V(z)} \left( \int_{z}^{\infty} u_\epsilon d\tilde{z} \right) dz.
\]

We claim
\[
\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_{0}^{\infty} e^{\frac{2}{\epsilon} V(z)} \left( \int_{z}^{\infty} u_\epsilon d\tilde{z} \right) dz = -\frac{1}{V(0^+)} = \frac{a'(0)}{b'(0)} \text{ locally uniformly in } (0, \infty),
\]
and
\[
\lim_{\epsilon \to 0} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon} d_1} = \frac{b'(0)}{a'(0)} M_0.
\]

These together with (4.9) lead to the conclusion.
We prove (4.10). Let \([\ell_1, \ell_2] \subset (0, \infty)\) and fix \(x_* \in (0, \min\{A, \ell_1\})\). For each \(0 < \epsilon \ll 1\) and \(x \in [\ell_1, \ell_2]\), there holds
\[
C_1(\epsilon) \frac{2}{\epsilon^2} \int_0^{x_*} e^{\frac{z}{\epsilon}} V(z) dz \leq \frac{2}{\epsilon^2} \int_0^{x_*} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz \\
\leq \frac{2}{\epsilon^2} \int_0^{x_*} e^{\frac{z}{\epsilon}} V(z) dz + \frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz,
\]
where \(C_1(\epsilon) := \inf_{z \in (0, x_*)} \int_z^\infty u_* dz \to 1\) as \(\epsilon \to 0\) thanks to Corollary 4.1. Since \(V\) is decreasing on \((0, x_*)\) and \(V'(0+) = -\frac{b(0)}{a(0)} < 0\), we apply Laplace’s method to find \(\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{x_*} e^{\frac{z}{\epsilon}} V(z) dz = -\frac{1}{V'(0+)}\). Therefore, (4.10) follows immediately if we show
\[
\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz = 0. \tag{4.12}
\]
Since the integral is increasing in \(\ell_2\), we assume without loss of generality that \(\ell_2 > \beta\). Take \(x^* \in (x_N, \beta)\), where \(x_N \in (0, \beta)\) is given in (H\(_V\)). Then,
\[
\frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz \leq \frac{2}{\epsilon^2} \int_0^{x^*} e^{\frac{z}{\epsilon}} V(z) dz + \frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz. \tag{4.13}
\]
Since \(\sup_{[x_*, x^*]} V(x) < 0\), we find \(\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{x^*} e^{\frac{z}{\epsilon}} V(z) dz \to 0\) as \(\epsilon \to 0\). Note that Lemma 2.5 ensures the existence of some \(z_* \gg \ell_2\) such that
\[
\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^{z_*} u_* dz \right) dz = 0.
\]
Moreover, Theorem A and the fact that \(V(z) - \min_{[z, z_*]} V - d_1 < 0\) for \(z \in [x_N, z_*]\) (otherwise, there are more than one \(d_1\)-valleys) yield
\[
\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{\ell_2} e^{\frac{z}{\epsilon}} V(z) \left( \int_z^{z_*} u_* dz \right) dz = \lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_0^{\ell_2} \int_z^{z_*} e^{\frac{z}{\epsilon}} V(z) dz dz dz = 0.
\]
Then, (4.12) follows from (4.13). This proves (4.10).

Now, we show (4.11). Fix \(0 < \delta_0 \ll 1\). Note that
\[
C_2(\epsilon) := \int_{(0, \infty) \setminus \bigcup_{i=1}^N (x_i - \delta_0, x_i + \delta_0)} u_* dx \to 0 \quad \text{as} \quad \epsilon \to 0
\]
thanks to Theorem A and Corollary 4.1. Set
\[
C_3(x, \epsilon) := \frac{2}{\epsilon^2} \int_0^x e^{\frac{z}{\epsilon}} V(z) \left( \int_z^\infty u_* dz \right) dz.
\]
The formula for \(u_*\) in Proposition 4.1 yields
\[
1 - C_2(\epsilon) = \sum_{i=1}^N \int_{x_i - \delta_0}^{x_i + \delta_0} u_* dx = \lambda_{i,1} e^{\frac{1}{\epsilon^2} d_1} \sum_{i=1}^N \int_{x_i - \delta_0}^{x_i + \delta_0} e^{\frac{z}{\epsilon} [d_1 + V(x)]} dx.
\tag{4.14}
\]
Step 2. Let \( \epsilon \) due to Laplace’s method, we pass to the limit \( \epsilon \to 0 \) in (4.14) to find (4.11).

(2) The proof follows from similar arguments, but the mechanism is slightly different. We break the proof into three steps.

**Step 1.** We prove

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_i - \delta_0}^{x_i + \delta_0} \frac{1}{a(x_i)} e^{-\frac{2}{a(x_i)} [d_1 + V]} dx = \frac{1}{a(x_i)} \sqrt{\frac{\pi}{V''(x_i)}}, \quad \forall i \in \{1, \ldots, N\}
\]
due to Laplace’s method, we pass to the limit \( \epsilon \to 0 \) in (4.14) to find (4.11).

Let \([\ell_1, \ell_2] \subset (\alpha, \infty)\) satisfy \( \{x_1, \ldots, x_N\} \subset (\ell_1, \ell_2) \) and \( \ell_2 > \beta \). Fix \( x_* \in (\alpha, \ell_1) \). Then, for each \( x \in [\ell_1, \ell_2] \),

\[
\frac{C_4(\epsilon)}{\epsilon} \int_{x_*}^{x} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz \\
\leq \frac{1}{\epsilon} \int_{0}^{x} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} \left( \int_{z}^{\infty} u_{\epsilon} d\tilde{z} \right) dz \\
\leq \frac{1}{\epsilon} \int_{0}^{x_*} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz + \frac{1}{\epsilon} \int_{x_*}^{\ell_2} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} \left( \int_{z}^{\infty} u_{\epsilon} d\tilde{z} \right) dz,
\]

where \( C_4(\epsilon) := \inf_{x \in (0, x_*)} \int_{z}^{\infty} u_{\epsilon} d\tilde{z} \to 1 \) as \( \epsilon \to 0 \) thanks to Theorem A and Corollary 4.1. Recall the assumption \( V(\alpha) > V(x) \) for all \( x \in (0, \alpha) \). In particular, \( V(\alpha) \geq 0 \).

- If \( V(\alpha) > 0 \), then the function \( z \mapsto V(z) - V(\alpha) \) on \([0, x_*] \) has the maximum value \( 0 \) attained only at \( z = \alpha \). Moreover, \( V''(\alpha) = -\frac{b'(\alpha)}{a(\alpha)} < 0 \) by assumption. It follows from Laplace’s method that \( \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{0}^{x_*} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz = \sqrt{\frac{\pi}{-V''(\alpha)}} \).

- If \( V(\alpha) = 0 \), then the function \( z \mapsto V(z) - V(\alpha) \) on \([0, x_*] \) has the maximum value \( 0 \) attained only at \( z = 0 \) and \( z = \alpha \). Since \( V'(0) < 0 \), we find from Laplace’s method that the integral \( \int_{0}^{x_*} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz \) is dominated by \( \int_{0}^{\alpha + \delta} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz \) for any fixed \( 0 < \delta \ll 1 \). Hence, \( \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{0}^{x_*} e^{\frac{2}{a(x)} [V(z) - V(\alpha)]} dz = \sqrt{\frac{\pi}{-V''(\alpha)}} \).

Arguing as in (1), we deduce that the second integral in the last line of (4.16) tends to 0 as \( \epsilon \to 0 \). Then, (4.15) follows readily.

**Step 2.** We show

\[
\lim_{\epsilon \to 0} 2\lambda e^{\frac{2}{a(x)} [d_1]} = \sqrt{\frac{-V''(\alpha)}{\pi}} M_0. \quad (4.17)
\]

Fix \( 0 < \delta_0 \ll 1 \). Note that

\[
C_5(\epsilon) := \int_{(0, \infty) \setminus \bigcup_{i=1}^{N} (x_i - \delta_0, x_i + \delta_0)} u_{\epsilon} dx \to 0 \quad \text{as} \quad \epsilon \to 0
\]
due to Corollary 4.1 and Theorem A. Set
\[ C_6(x, \epsilon) := \frac{1}{\epsilon} \int_0^x e^{\frac{2}{\epsilon} [V(\alpha) - V(\alpha)]} \left( \int_z^\infty u_\epsilon d\tilde{z} \right) dz. \]

It follows from the formula for \( u_\epsilon \) in Proposition 4.1 that
\[ 1 - C_5(\epsilon) = \sum_{i=1}^N \int_{x_i - \delta_0}^{x_i + \delta_0} u_\epsilon dx = \frac{2\lambda_i}{\epsilon} e^{\frac{2}{\epsilon} d_1} \sum_{i=1}^N \int_{x_i - \delta_0}^{x_i + \delta_0} \frac{C_6(x, \epsilon)}{a(x)} e^{-\frac{2}{\epsilon} [d_1 + V(x) - V(\alpha)]} dx. \] (4.18)

By (4.15) and Laplace’s method,
\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_i - \delta_0}^{x_i + \delta_0} C_6(x, \epsilon) e^{-\frac{2}{\epsilon} [d_1 + V(x) - V(\alpha)]} dx = \frac{\pi}{-V''(\alpha)} \frac{1}{a(x_i)} \sqrt{\frac{\pi}{V''(\alpha)}}, \quad \forall i \in \{1, \ldots, N\}. \]

This together with (4.18) yields (4.17).

**Step 3.** We prove the limit for \( R_\epsilon \). By the formula for \( u_\epsilon \) in Proposition 4.1 and the definition of \( R_\epsilon \),
\[ R_\epsilon(x) = \begin{cases} \frac{2\lambda_i}{\epsilon} e^{\frac{2}{\epsilon} d_1} \int_0^x e^{\frac{2}{\epsilon} [V(z) - V(\sup(0,\alpha))]} \left( \int_z^\infty u_\epsilon d\tilde{z} \right) dz, & x \in (0, \alpha), \\ \frac{2\lambda_i}{\epsilon} e^{\frac{2}{\epsilon} d_1} \int_x^{\infty} e^{\frac{2}{\epsilon} [V(z) - V(\alpha)]} \left( \int_z^\infty u_\epsilon d\tilde{z} \right) dz, & x \in [\alpha, \infty). \end{cases} \] (4.19)

It follows from (4.15) and (4.17) that \( \lim_{\epsilon \to 0} R_\epsilon = M_0 \) locally uniformly in \((\alpha, \infty)\). We claim
\[ \lim_{\epsilon \to 0} R_\epsilon(\alpha) = \frac{M_0}{2}. \] (4.20)

Clearly, \( \lim_{\epsilon \to 0} \int_z^\infty u_\epsilon d\tilde{z} = 1 \) uniformly in \( z \in (0, \alpha] \). Arguing as in **Step 1**, we find
\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\alpha e^{\frac{2}{\epsilon} [V(z) - V(\alpha)]} \left( \int_z^\infty u_\epsilon d\tilde{z} \right) dz = \frac{1}{2} \frac{\pi}{V''(\alpha)}. \]

This together with (4.17) leads to (4.20).

Let \( x_0 \) be as in the statement. Clearly, \( x_0 \in (0, \alpha] \). We claim that
\[ \lim_{\epsilon \to 0} \frac{R_\epsilon}{\epsilon} = - \frac{M_0}{2V'(0+)} \sqrt{\frac{-V''(\alpha)}{\pi}} \quad \text{locally uniformly in} \quad (0, x_0). \]

Indeed, given (4.17) and the fact that \( \lim_{\epsilon \to 0} \int_z^\infty u_\epsilon d\tilde{z} = 1 \) uniformly in \( z \in (0, \alpha) \), it suffices to study the asymptotic of the integral \( \int_0^x e^{\frac{2}{\epsilon} [V - \sup(0,\alpha)]} dz \) as \( \epsilon \to 0 \). Clearly, \( \sup(0,\alpha) \leq V = 0 \).

Since \( V(0+) > V(z) \) for all \( z \in (0, x] \) and \( V'(0+) = -\frac{K'(0)}{\alpha'(0)} < 0 \), Laplace’s method yields
\[ \lim_{\epsilon \to 0} \frac{2}{\pi} \int_0^x e^{\frac{2}{\epsilon} V} dz = -\frac{1}{V'(0+)}, \quad \text{which is locally uniform in} \quad (0, x_0). \]

The limit follows.

In consideration of (4.17) and \( \lim_{\epsilon \to 0} \int_z^\infty u_\epsilon d\tilde{z} = 1 \) uniformly in \( z \in (0, \alpha) \), the result about \( R_\epsilon \) in \([x_0, \alpha)\) follows readily. This proves the conclusions in (2).
4.4. **Proof of Theorem C.**  
(1) It follows from Lemma 2.5.
(2) It follows from Lemma 4.4.
(3) The limits concerning \( \lambda_{e,} \) in (i) and (ii) follow from (4.11) and (4.17), respectively. It remains to show the limit of \( \| \phi_{e,1} \|_{L^1(u_0^e)} \phi_{e,1} \). Recall that \( u_e = \frac{\phi_{e,1} u_0^e}{\| \phi_{e,1} \|_{L^1(u_0^e)}} \) and \( u_0^G = \frac{1}{ae} e^{-\frac{2}{a}v} \).

(i) By the definition of \( R_e \) and Theorem A, we find
\[
\frac{\phi_{e,1} u_0^G}{\| \phi_{e,1} \|_{L^1(u_0^e)}} = u_e = \frac{R_e}{\epsilon a} e^{-\frac{2}{a}v} = \frac{R_e}{\epsilon} e^{-\frac{2}{\epsilon^2}d_1} u_0^G,
\]
giving \( R_e = \frac{\epsilon}{\epsilon} e^{-\frac{2}{\epsilon^2}d_1} \| \phi_{e,1} \|_{L^1(u_0^e)} \). It follows that
\[
1 = \int_0^\infty \phi_{e,1} u_0^G \, dx = \| \phi_{e,1} \|_{L^1(u_0^e)} \int_0^\infty \phi_{e,1} u_e \, dx = \frac{1}{\epsilon} e^{-\frac{2}{\epsilon^2}d_1} \| \phi_{e,1} \|_{L^1(u_0^e)} \int_0^\infty R_e u_e \, dx,
\]
and therefore, \( \| \phi_{e,1} \|_{L^1(u_0^e)} \phi_{e,1} = \frac{R_e}{\epsilon} \int_0^\infty R_e u_e \, dx \). Since \( \lim_{\epsilon \to 0} R_e = M_0 \) locally uniformly in \((0, \infty)\) by Theorem B (1), the result follows if we can show
\[
\lim_{\epsilon \to 0} \int_0^\infty R_e u_e \, dx = M_0. \quad (4.21)
\]

Note that \( R_e u_e = eau_0^2 e^{-\frac{2}{a}v} \). Fix \( K \gg 1 \). Since \( \lim_{\epsilon \to 0} \int_0^K u_e \, dx = 1 \) due to Corollary 4.1, we find from Theorem B (1) that \( \lim_{\epsilon \to 0} \int_0^K R_e u_e \, dx = M_0 \). An application of Lemma 4.4 leads to \( \lim_{\epsilon \to 0} \int_0^\infty R_e u_e \, dx = 0 \).

It remains to treat \( \int_0^\infty R_e u_e \, dx \). Fix \( 1 \ll L < K \). We distinguish between \( y_\infty = \infty \) and \( y_\infty < \infty \).

- If \( y_\infty = \infty \), then Lemma 2.5 and \( u_e(L) = \frac{R_e(L)}{\epsilon a(L)} e^{-\frac{2}{\epsilon^2}v(L)} \) give
  \[
  u_e \leq \frac{R_e(L)}{\epsilon a(L)} e^{-\frac{2}{\epsilon^2}v(L)} e^{-\gamma_{e,L} [\xi(L) - \xi]} \int_K^L \frac{b}{a} \, ds \quad \text{in} \quad [L, \infty)
  \]
  for all \( 0 < \epsilon \ll 1 \), where \( \gamma_{e,L} = \sqrt{2 \frac{C_2}{e} - \lambda_{e,L}} \). In which, \( C_2 := C_2(\xi(L), y_\infty) \) \( \frac{b\xi_{\infty} - 1}{a\xi_{\infty} - 1} \), where \( C_2 = C_2(\xi(L)) \) is given in Lemma 2.5 (2). It follows that
  \[
  \int_K^\infty R_e u_e \, dx \leq \int_K^\infty \frac{|R_e(L)|^2}{\epsilon a(L)} e^{-\frac{2}{\epsilon^2}v(L)} e^{-2\gamma_{e,L} [\xi(L) - \xi]} \int_K^\infty \frac{1}{\sqrt{a}} e^{-2\gamma_{e,L} [\xi(L)]} \, dx
  \]
  \[
  = \frac{|R_e(L)|^2}{\epsilon a(L)} e^{-2\gamma_{e,L} [\xi(L)]} \int_K^\infty \frac{1}{\sqrt{a}} e^{-2\gamma_{e,L} [\xi(L)]} \, dx
  \]
  \[
  = \frac{|R_e(L)|^2}{\epsilon a(L)} e^{-2\gamma_{e,L} [\xi(L)]} \int_K^\infty \frac{1}{\sqrt{a}} e^{-2\gamma_{e,L} [\xi(L)]} \, ds = 0 \quad \text{as} \quad \epsilon \to 0.
  \]

- If \( y_\infty < \infty \), then Lemma 2.5 and \( u_e(L) = \frac{R_e(L)}{\epsilon a(L)} e^{-\frac{2}{\epsilon^2}v(L)} \) give
  \[
  u_e \leq \frac{R_e(L)}{\epsilon a(L)} e^{-\frac{2}{\epsilon^2}v(L)} e^{\gamma_{e,L} [\xi(y_\infty)] - \gamma_{e,L} [\xi(L)]} \int_K^L \frac{b}{a} \, ds \quad \text{in} \quad [L, \infty)
  \]
for all $0 < \epsilon \ll 1$, where $\gamma_{\epsilon,L}$ is as above. It follows that

$$\int_{K}^{\infty} R_{\epsilon} u_{\epsilon} dx \leq \frac{[R_{\epsilon}(L)]^{2}}{\epsilon \sqrt{a(L)}} e^{-\frac{2}{\epsilon} v(L)} \int_{\xi(K)}^{\infty} \left( \frac{e^{\gamma_{\epsilon,L}[y-y_{0}]} - e^{\gamma_{\epsilon,L}[y_{0}-y]}}{e^{\gamma_{\epsilon,L}[y-y_{0}]} - e^{\gamma_{\epsilon,L}[y_{0}-\xi(L)]}} \right)^{2} dy$$

$$= \frac{[R_{\epsilon}(L)]^{2}}{\epsilon \sqrt{a(L)}} e^{-\frac{2}{\epsilon} v(L)} \int_{\xi(K)}^{\infty} \left( \frac{e^{2\gamma_{\epsilon,L}[y-y_{0}]} - 1}{e^{2\gamma_{\epsilon,L}[y-y_{0}]} - 1} \right)^{2} dy$$

$$\leq \frac{[R_{\epsilon}(L)]^{2}}{\epsilon \sqrt{a(L)}} e^{-\frac{2}{\epsilon} v(L)} \int_{\xi(K)}^{\infty} e^{-2\gamma_{\epsilon,L}[y-\xi(L)]} dy \to 0 \quad \text{as} \quad \epsilon \to 0.$$  

Hence, $\lim_{\epsilon \to 0} \int_{K}^{\infty} R_{\epsilon} u_{\epsilon} dx = 0$. This proves (4.21), and thus, the result.

(ii) The definition of $R_{\epsilon}$ and Theorem A give

$$\frac{\phi_{\epsilon,1} u_{\epsilon}^{G}}{\| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})}} = R_{\epsilon} e^{-\frac{2}{\epsilon (V-V)} u_{\epsilon}^{G}},$$

leading to

$$\phi_{\epsilon,1}(x) = \frac{R_{\epsilon}(x)}{\epsilon} e^{-\frac{2}{\epsilon} [v(x) - V(x)] \| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})}}$$

$$= \left\{ \begin{array}{ll} \frac{R_{\epsilon}(x)}{\epsilon} e^{-\frac{2}{\epsilon} [d_{1} - \sup_{(0,\alpha)} V] \| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})}}, & x \in (0, \alpha], \\
\frac{R_{\epsilon}(x)}{\epsilon} e^{-\frac{2}{\epsilon} [d_{1} - V(\alpha)] \| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})}}, & x \in (\alpha, \infty). \end{array} \right.$$

It follows that

$$1 = \int_{0}^{\infty} \phi_{\epsilon,1}^{2} u_{\epsilon}^{G} dx = \| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})} \int_{0}^{\infty} \phi_{\epsilon,1} u_{\epsilon} dx$$

$$= \frac{1}{\epsilon} e^{-\frac{2}{\epsilon} [d_{1} - V(\alpha)] \| \phi_{\epsilon,1} \|_{L^{1}(u_{\epsilon}^{G})}} \left( \int_{0}^{\alpha} R_{\epsilon}(x) e^{2 \sup_{(0,\alpha)} V-V(\alpha)} u_{\epsilon}(x) dx + \int_{\alpha}^{\infty} R_{\epsilon} u_{\epsilon} dx \right).$$

Fix $K \gg 1$. As $\sup_{(0,\alpha)} V \leq V(\alpha)$, we find

$$\int_{0}^{\alpha} R_{\epsilon}(x) e^{2 \sup_{(0,\alpha)} V-V(\alpha)} u_{\epsilon}(x) dx \leq \int_{0}^{\alpha} R_{\epsilon} u_{\epsilon} dx.$$

We see from (4.19) and (4.17) that $\sup_{(0,\alpha)} R_{\epsilon} \leq C_{\epsilon}$ for some $C > 0$ and all $0 < \epsilon \ll 1$. Since $\int_{0}^{\alpha} u_{\epsilon} dx$ is exponentially small in $\epsilon$ due to Theorem A and Corollary 4.1, we conclude $\lim_{\epsilon \to 0} \int_{0}^{\alpha} R_{\epsilon} u_{\epsilon} dx = 0$, and thus, $\lim_{\epsilon \to 0} \int_{0}^{\alpha} R_{\epsilon}(x) e^{2 \sup_{(0,\alpha)} V-V(\alpha)} u_{\epsilon}(x) dx = 0$. Write

$$\int_{\alpha}^{\infty} R_{\epsilon} u_{\epsilon} dx = \left( \int_{\alpha}^{\alpha + \frac{1}{\pi}} R_{\epsilon} u_{\epsilon} dx + \int_{\alpha + \frac{1}{\pi}}^{\alpha + \frac{1}{\pi} K} R_{\epsilon} u_{\epsilon} dx + \int_{\alpha + \frac{1}{\pi} K}^{\infty} R_{\epsilon} u_{\epsilon} dx \right).$$

Arguing as above yields $\lim_{\epsilon \to 0} \int_{\alpha + \frac{1}{\pi} K}^{K} R_{\epsilon} u_{\epsilon} dx = 0$. Arguments as in (i) yield

$$\lim_{\epsilon \to 0} \int_{\alpha + \frac{1}{\pi} K}^{K} R_{\epsilon} u_{\epsilon} dx = M_{0} \quad \text{and} \quad \lim_{\epsilon \to 0} \int_{\alpha + \frac{1}{\pi} K}^{\alpha + \frac{1}{\pi} K} R_{\epsilon} u_{\epsilon} dx = 0.$$
Hence, letting $\epsilon \to 0$ in (4.23), we deduce
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-\frac{2}{\epsilon^2} [d_1 - V(\alpha)]} \| \phi_{\epsilon,1} \|_{L^1(uG)}^2 = \frac{1}{M_0} \tag{4.24}
\]
It follows from (4.22) and Theorem B (2) that
\[
\lim_{\epsilon \to 0} \| \phi_{\epsilon,1} \|_{L^1(uG)} \phi_{\epsilon,1} (x) = \begin{cases} 
\frac{1}{2}, & x = \alpha, \\
1, & \text{locally uniformly in } x \in (\alpha, \infty) \end{cases}
\]
It remains to treat $\lim_{\epsilon \to 0} \| \phi_{\epsilon,1} \|_{L^1(uG)} \phi_{\epsilon,1}$ in $(0, \alpha)$. We find from (4.22), (4.24) and the fact $\sup_{(0,\alpha)} R_\epsilon \leq \frac{C}{\epsilon}$ for all $0 < \epsilon \ll 1$ that for each $x \in (0, \alpha)$,
\[
\| \phi_{\epsilon,1} \|_{L^1(uG)} \phi_{\epsilon,1} (x) = \frac{R_\epsilon}{\epsilon} e^{-\frac{2}{\epsilon^2} [d_1 - \sup_{(0,x)} V]} \| \phi_{\epsilon,1} \|_{L^1(uG)}^2 \to 0 \quad \text{as } \epsilon \to 0.
\]
This completes the proof.

5. Applications

We discuss some applications in chemical reactions and population dynamics.

5.1. Logistic diffusion processes. Consider the following family of SDEs:
\[
dx = (b_1 x - b_2 x^2) dt + \epsilon \sqrt{a_1 x + a_2 x^2} dW_t, \quad x \in [0, \infty),
\]
where $0 < \epsilon \ll 1$ is a parameter, $b_1$, $b_2$ and $a_1$ are positive constants, $a_2 \geq 0$, and $W_t$ is a standard one-dimensional Wiener process.

We roughly describe two typical situations giving rise to (5.1).

**Chemical reactions.** Consider the following chemical reactions:
\[
A + X \xrightarrow{k_1} 2X, \quad X \xrightarrow{k_2} C,
\]
where $k_1$, $k_{-1}$ and $k_2$ are reaction rates. The concentration of $A$ molecules, denoted by $x_A$, is assumed to remain constant. We assume $k_1 x_A > k_2$.

Let $V \gg 1$ be the generalized volume of the system and $X^V_t$ be the continuous-time Markov jump process counting the number of $X$ molecules. The law of large numbers [20, 1] ensures that as the volume $V$ grows to infinity, the rescaled process $\frac{X^V_t}{V}$ converges to the solutions of the following mean field ODE for the concentration of $X$ molecules:
\[
\dot{x} = -k_{-1} x^2 + k_1 x_A x - k_2 x, \quad x \in [0, \infty).
\]

The fluctuation of $\frac{X^V_t}{V}$ around solutions of (5.3) is captured by the central limit theorem [20, 1], leading to the diffusion approximation of $\frac{X^V_t}{V}$:
\[
dx = (-k_{-1} x^2 + k_1 x_A x - k_2 x) dt + \epsilon \sqrt{k_{-1} x^2 + k_1 x_A x + k_2 x} dW_t, \quad x \in [0, \infty),
\]
where $\epsilon = \frac{1}{\sqrt{V}}$ and $W_t$ is a standard one-dimensional Wiener process.

It is not hard to check that solutions of (5.4) almost surely reach the extinction state 0 in finite time, while solutions of (5.3) with positive initial data converge to the unique positive
equilibrium. Such a dynamical disagreement between deterministic and stochastic models is often referred to as Keizer’s paradox [33], which is often formulated in terms of the chemical master equation satisfied by the distributions of $X^Y_t$ or $\frac{X^Y_t}{K}$ (see e.g. [34, 53, 9]).

**Logistic BDPs.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda > \mu > 0$. Consider a continuous-time birth-and-death process (BDP) $Y^K_t$ on the state space $\mathbb{N}_0$ with birth rates $\lambda^K_n = \lambda_n$, $n \in \mathbb{N}_0$, and death rates $\mu^K_n = n (\mu + n)$, $n \in \mathbb{N}$, where $K \gg 1$ is the scaling parameter, often called the carrying capacity. By the law of large numbers and central limit theorem, for sufficiently large $K$, the process $\frac{Y^K_t}{K}$ stays close to solutions of the following SDE:

$$dx = (\lambda x - \mu x - x^2)dt + \epsilon \sqrt{\lambda x + \mu x + x^2}dW_t, \quad x \in [0, \infty),$$

where $\epsilon = \frac{1}{\sqrt{K}}$. The SDE (5.5) is the diffusion approximation of $\frac{Y^K_t}{K}$, and is in the form of (5.1).

Let $a(x) = a_1 x + a_2 x^2$ and $b(x) = b_1 x - b_2 x^2$. Clearly, the assumption (H) is satisfied. Let $V$ be as in (1.2). Denote by $X^Y_t$ the solution processes of (5.1) and by $T_0^\epsilon$ the associated extinction time. Set $x_* := \frac{b_1}{b_2}$.

**Theorem 5.1.** Consider (5.1).

1. For each $0 < \epsilon \ll 1$, (5.1) admits a unique QSD $\mu_\epsilon$ with a density $u_\epsilon$.
2. $u_\epsilon = \frac{R_\epsilon}{a(x_*)} e^{\frac{b_1}{b_2} x_* \frac{1}{\epsilon} x} dt$, where $\lim_{\epsilon \to 0} R_\epsilon = a(x_*) \sqrt{-\frac{V'(x_*)}{\pi a(x_*)}}$ locally uniformly in $(0, \infty)$.
3. $\lim_{\epsilon \to 0} \|\mu_\epsilon - G_\epsilon\|_{TV} = 0$, where $G_\epsilon$ is a probability measure on $(0, \infty)$ whose density is proportional to $exp \left\{ \frac{V'(x_*)}{a(x_*)} \left( x - x_* \right)^2 \right\}$.
4. For any $p \in [1, \infty)$, $\lim_{\epsilon \to 0} W_p(\mu_\epsilon, G_\epsilon) = 0$, where $W_p$ is the $p$-Wasserstein distance.

**Proof.** (1) See [6].

(2) It follows directly from Theorem A and Theorem B (1).

(3) Denote by $G_\epsilon$ the density of $G_\epsilon$, namely, $G_\epsilon(x) = \frac{1}{Z_\epsilon} e^{-\frac{V'(x_*)}{2 \epsilon^2} (x - x_*)^2}$, where $Z_\epsilon = \epsilon \int_{\mathbb{R}} \pi(x_*)^{-\frac{1}{2}} e^{-\frac{V''(x_*)}{2 \epsilon^2} (x - x_*)^2} dx$. Fix $0 < \delta_0 \ll 1$ and $\kappa \in (\frac{2}{3}, 1)$. Set $I_\epsilon := (x_* - \epsilon^\kappa, x_* + \epsilon^\kappa)$ and $I_{\delta_0} := (x_* - \delta_0, x_* + \delta_0)$, and write

$$2 \text{dist}_{TV}(\mu_\epsilon, G_\epsilon) = \left( \int_{(0, \infty) \setminus I_{\delta_0}} + \int_{I_{\delta_0} \setminus I_\epsilon} + \int_{I_\epsilon} \right) |u_\epsilon - G_\epsilon| dx.$$

It remains to treat the integrals.

By Corollary 4.1 and the tail of $G_\epsilon$, there exists $\gamma_1 > 0$ such that

$$\int_{(0, \infty) \setminus I_{\delta_0}} \left( u_\epsilon - G_\epsilon \right) dx \leq e^{-\frac{\gamma_1}{\epsilon^2}}, \quad \forall 0 < \epsilon \ll 1. \quad (5.6)$$

Clearly, there is $0 < \eta \ll 1$ such that

$$V(x) - V(x_*) \geq \left[ \frac{V''(x_*)}{2} - \eta \right] (x - x_*)^2, \quad \forall x \in I_{\delta_0}.$$
It follows from (2) that for each $0 < \epsilon \ll 1$,
\[
\int_{I_{\delta_0} \setminus I_\epsilon} |u_\epsilon - G_\epsilon| \, dx \leq \int_{I_{\delta_0} \setminus I_\epsilon} \frac{R_\epsilon}{a \epsilon} e^{-\frac{2}{\epsilon^2} |V - V(x_\star)|} \, dx + \int_{I_{\delta_0} \setminus I_\epsilon} G_\epsilon \, dx 
\leq C_4 \int_{I_{\delta_0} \setminus I_\epsilon} \frac{1}{\epsilon} e^{-\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{2} - \eta\right] (x-x_\star)^2} \, dx + \int_{I_{\delta_0} \setminus I_\epsilon} G_\epsilon \, dx,
\]
where $C_4 = 1 + \sup_{I_{\delta_0}} \frac{a(x_\star)}{a} \sqrt{\frac{\nabla^m(x_\star)}{\pi}}$. Note that there is $\gamma_2 > 0$ (independent of $\kappa$) such that
\[
\max \left\{ \int_{I_{\delta_0} \setminus I_\epsilon} \frac{1}{\epsilon} e^{-\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{2} - \eta\right] (x-x_\star)^2} \, dx, \int_{I_{\delta_0} \setminus I_\epsilon} G_\epsilon \, dx \right\} \leq e^{-\frac{\gamma_2}{\epsilon^{2(1-\kappa)}}}, \quad \forall 0 < \epsilon \ll 1.
\]
Hence,
\[
\int_{I_{\delta_0} \setminus I_\epsilon} |u_\epsilon - G_\epsilon| \, dx \leq 2C_4 e^{-\frac{\gamma_2}{\epsilon^{2(1-\kappa)}}}, \quad \forall 0 < \epsilon \ll 1. \tag{5.7}
\]
Since $V \in C^3((0, \infty))$, there holds
\[
V(x) = V(x_\star) + \frac{V''(x_\star)}{2} (x-x_\star)^2 + \frac{V'''(x_\star)}{6} (x-x_\star)^3 + o(|x-x_\star|^3), \quad \forall x \in I_{\delta_0}.
\]
Then,
\[
\int_{I_\epsilon} |u_\epsilon - G_\epsilon| \, dx = \int_{I_\epsilon} \left| \frac{R_\epsilon(x)}{a(x)} - \frac{1}{Z_\epsilon} e^{-\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{6} (x-x_\star)^3 + o(|x-x_\star|^3)\right]} \right| \right| e^{-\frac{2}{\epsilon^2} |V(x) - V(x_\star)|} \, dx 
\leq \int_{I_\epsilon} C_5(x, \epsilon) \, dx \leq \sqrt{\frac{2\pi}{V''(x_\star)}} \sup_{x \in I_\epsilon} C_5(x, \epsilon),
\]
where
\[
C_5(x, \epsilon) = \left| \frac{R_\epsilon(x)}{a(x)} - \frac{1}{Z_\epsilon} e^{-\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{6} (x-x_\star)^3 + o(|x-x_\star|^3)\right]} \right| \right| e^{-\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{6} (x-x_\star)^3 + o(|x-x_\star|^3)\right]}.
\]
Note that $\kappa > \frac{2}{3}$ gives
\[
\lim_{\epsilon \to 0} \sup_{x \in I_\epsilon} e^{\frac{2}{\epsilon^2} \left[\frac{\nabla^m(x_\star)}{6} (x-x_\star)^3 + o(|x-x_\star|^3)\right]} = 1.
\]
This together with (2) and the fact that $\frac{\epsilon}{Z_\epsilon} = \sqrt{\frac{\nabla^m(x_\star)}{\pi}}$ implies that $\sup_{x \in I_\epsilon} C_5(x, \epsilon) \to 0$ as $\epsilon \to 0$. Hence, $\int_{I_\epsilon} |u_\epsilon - G_\epsilon| \, dx \to 0$ as $\epsilon \to 0$, which together with (5.6) and (5.7) yields $\|\mu_\epsilon - G_\epsilon\|_{TV} \to 0$ as $\epsilon \to 0$. This completes the proof.

(4) Let $p \in [1, \infty)$. By [54, Theorem 6.15],
\[
W_p(\mu_\epsilon, G_\epsilon) \leq 2 \frac{1}{p'} \left( \int_0^\infty |x-x_\star|^p |u_\epsilon(x) - G_\epsilon(x)| \, dx \right)^{\frac{1}{p'}},
\]
where $p'$ is the dual exponent of $p$. It follows from (3) and the tails of $u_\epsilon$ (see Lemma 2.5) and $G_\epsilon$ that $W_p(\mu_\epsilon, G_\epsilon) \to 0$ as $\epsilon \to 0$. \qed
5.2. Distance between discrete and continuous QSDs. We further examine logistic BDPs with the focus on the distance between QSDs of $X^K_t := \frac{Y^K_t}{K}$ and that of (5.5). This concerns the compatibility of the birth-death process $X^K_t$ and the diffusion process (5.5) as models for the evolution of the same species.

It is well-established (see e.g. [41, 14]) that for each $K \gg 1$, $X^K_t$ admits a unique QSD $\mu^K$ on $\mathbb{N}$. In [13], the authors proved the following asymptotic of $\mu^K$ as $K \to \infty$.

**Proposition 5.1** ([13]). The following hold.

1. There is $C_0 > 0$ such that $\|\mu^K - G^K\|_{TV} \leq \frac{C_0}{\sqrt{K}}$ for all $K \gg 1$, where $G^K = \{G^K(\frac{n}{K})\}_{n \in \mathbb{N}}$ is a probability measure on $\frac{\mathbb{N}}{K}$ given by

$$G^K(\frac{n}{K}) = \frac{1}{Z^K} e^{-\frac{K}{2\lambda} (\frac{n}{K} - (\lambda - \mu)K)^2}$$

with $Z^K$ being the normalization constant.

2. For any $p \in [1, \infty)$, there exists $C_p > 0$ such that $W_p(\mu^K, G^K) \leq \frac{C_p}{\sqrt{K}}$ for all $K \gg 1$.

Proposition 5.1 (2) is not stated in [13]. But, it is a simple consequence of Proposition 5.1 (1), the tails of $\mu^K$ given in the proof of [13, Theorem 3.7], and the control of Wasserstein distance by weighted total variation distance (see [54, Theorem 6.15]).

We identify $\mu^K$ and $G^K$ with their natural extensions to probability measures on $(0, \infty)$. In particular, they are singular with respect to the Lebesgue measure on $(0, \infty)$.

Recall that $\epsilon = \frac{1}{\sqrt{K}}$. Denote by $\mu^K := \mu_\epsilon$ the unique QSD of (5.5). As the total variation distance between $\mu^K$ and $\mu_\epsilon$ is 1, we use somewhat weaker distances.

**Theorem 5.2.** The following hold.

1. For any $p \in [1, \infty)$, $\lim_{K \to \infty} W_p(\mu^K, \mu_\epsilon) = 0$.
2. $\lim_{K \to \infty} \text{dist}_{Kol}(\mu^K, \mu_\epsilon) = 0$, where $\text{dist}_{Kol}$ denotes the Kolmogorov metric.

**Proof.** Let $G_K$ be a probability measure on $(0, \infty)$ with density

$$G_K(x) = \frac{1}{Z_K} \exp\left\{-\frac{K}{2\lambda} (x - (\lambda - \mu))^2\right\},$$

where $Z_K$ is the normalization constant. Theorem 5.1 says that

$$\lim_{K \to \infty} \|\mu^K - G_K\|_{TV} = 0 \quad \text{and} \quad \lim_{K \to \infty} W_p(\mu^K, G_K) = 0, \quad \forall p \in [1, \infty). \quad (5.8)$$

(1) Let $p \in [1, \infty)$. Note that

$$W_p(\mu^K, \mu_\epsilon) \leq W_p(\mu^K, G^K) + W_p(G^K, G_K) + W_p(G_K, \mu_\epsilon).$$

It follows from Proposition 5.1 (2) and (5.8) that the limit $\lim_{K \to \infty} W_p(\mu^K, \mu_\epsilon) = 0$ holds if we show

$$\lim_{K \to \infty} W_p(G^K, G_K) = 0. \quad (5.9)$$
To show (5.9), we note that
\[
W_p(G^K, \delta_{\lambda-\mu}) \leq W_p(G^K, \delta_{\lambda-\mu}) + W_p(\delta_{\lambda-\mu}, G_K)
\]
\[
= \left( \int_0^\infty |x - (\lambda - \mu)|^p dG^K(x) \right)^{\frac{1}{p}} + \left( \int_0^\infty |x - (\lambda - \mu)|^p dG_K(x) \right)^{\frac{1}{p}}
\]
\[
= \left( \sum_{n \in \mathbb{N}} \left| \frac{n}{K} - (\lambda - \mu) \right|^p G^K \left( \frac{n}{K} \right) \right)^{\frac{1}{p}} + \left( \int_0^\infty |x - (\lambda - \mu)|^p G_K(x) dx \right)^{\frac{1}{p}}.
\]

By Laplace’s method, \( \int_0^\infty |x - (\lambda - \mu)|^p G_K(x) dx \to 0 \) as \( K \to \infty \). To show that the sum vanishes as \( K \to \infty \), we write
\[
\sum_{n \in \mathbb{N}} \left| \frac{n}{K} - (\lambda - \mu) \right|^p G^K \left( \frac{n}{K} \right) = \left( \sum_{n \in \mathbb{N}_K} + \sum_{n \in \mathbb{N}_K^c} \right) \left| \frac{n}{K} - (\lambda - \mu) \right|^p G^K \left( \frac{n}{K} \right),
\]
where
\[
\mathbb{N}_K = \left\{ n \in \mathbb{N} : \left| \frac{n}{K} - \frac{(\lambda - \mu)K}{K} \right| < \frac{1}{K} \right\} \quad \text{and} \quad \mathbb{N}_K^c = \mathbb{N} \setminus \mathbb{N}_K. \tag{5.10}
\]
It is then straightforward to check that the sums over \( \mathbb{N}_K \) and \( \mathbb{N}_K^c \) vanish as \( K \to \infty \). Hence, \( \sum_{n \in \mathbb{N}} \left| \frac{n}{K} - (\lambda - \mu) \right|^p G^K \left( \frac{n}{K} \right) \to 0 \) as \( K \to \infty \). This proves (5.9).

(2) Due to Proposition 5.1 (1), (5.8) and
\[
dist_{Kol}(\mu^K, \mu_K) \leq dist_{Kol}(\mu^K, G^K) + dist_{Kol}(G^K, G_K) + dist_{Kol}(G_K, \mu_K)
\]
\[
\leq \| \mu^K - G^K \|_{TV} + dist_{Kol}(G^K, G_K) + \| G_K - \mu_K \|_{TV},
\]
the limit \( \lim_{K \to \infty} \text{dist}_{Kol}(\mu^K, \mu_K) = 0 \) follows if we show
\[
\lim_{K \to \infty} \text{dist}_{Kol}(G^K, G_K) = 0. \tag{5.11}
\]

Denote by \( F_{G^K} \) and \( F_{G_K} \) distribution functions of \( G^K \) and \( G_K \), respectively. Clearly,
\[
F_{G^K}(t) = \begin{cases} 0, & t \in (0, \frac{1}{K}), \\ \sum_{m=1}^{n} G^K \left( \frac{m}{K} \right), & t = \left[ \frac{n}{K}, \frac{n+1}{K} \right), \quad n \in \mathbb{N}, \end{cases}
\]
By the definition,
\[
\text{dist}_{Kol}(G^K, G_K) = \sup_{t \in (0, \frac{1}{K})} |F_{G^K}(t) - F_{G_K}(t)|
\]
\[
\leq \sup \left\{ \sup_{t \in (0, \frac{1}{K})} |F_{G^K}(t) - F_{G_K}(t)|, \sup_{t \in \left[ \frac{n}{K}, \frac{n+1}{K} \right)} |F_{G^K}(t) - F_{G_K}(t)|, \quad n \in \mathbb{N} \right\}.
\]
Obviously, \( \sup_{t \in (0, \frac{1}{K})} \left| F_{G_K}(t) - F_{G_K}(t) \right| \to 0 \) as \( K \to \infty \). Note that

\[
\sup_{n \in \mathbb{N}} \sup_{t \in \left( \frac{n}{K}, \frac{n+1}{K} \right)} \left| F_{G_K}(t) - F_{G_K}(t) \right|
\leq \left( \sum_{n \in \mathbb{N}_K} + \sum_{n \in \mathbb{N}_K^c} \right) \left| G_K \left( \frac{n}{K} \right) - \int_{\frac{n}{K}}^{\frac{n+1}{K}} G_K dx \right| + \sup_{n \in \mathbb{N}} \int_{\frac{n}{K}}^{\frac{n+1}{K}} G_K dx,
\]

where \( \mathbb{N}_K \) and \( \mathbb{N}_K^c \) are given in (5.10). It is easy to see \( \lim_{K \to \infty} \sup_{n \in \mathbb{N}} \int_{\frac{n}{K}}^{\frac{n+1}{K}} G_K dx = 0 \). By the exponential tails of \( G^K \) and \( G_K \), the sum over \( \mathbb{N}_K^c \) vanishes as \( K \to \infty \). For the sum over \( \mathbb{N}_K \), we set \( x_K := \frac{\left( (\lambda - \mu)K \right)}{K} \) and estimate for \( n \in \mathbb{N}_K \)

\[
\left| G_K \left( \frac{n}{K} \right) - \int_{\frac{n}{K}}^{\frac{n+1}{K}} G_K dx \right| \leq \frac{1}{ZK} e^{-\frac{n}{2K} \cdot (\frac{n}{K} - x_K)^2} \left| 1 - Z^K e^{-\frac{x_K}{2K} \cdot (\frac{n}{K} - x_K)^2} \right| e^{-\frac{x_K}{2K} \cdot (\frac{n}{K} - x_K)^2} dx.
\]

Note that

\[
\sup_{n \in \mathbb{N}_K} \sup_{x \in \left( \frac{n}{K}, \frac{n+1}{K} \right)} \left| \frac{K}{2\lambda} \left[ (x - (\lambda - \mu))^2 - \left( \frac{n}{K} - x_K \right)^2 \right] \right|
= \frac{K}{2\lambda} \sup_{n \in \mathbb{N}_K} \sup_{x \in \left( \frac{n}{K}, \frac{n+1}{K} \right)} \left| x - (\lambda - \mu) - \frac{n}{K} + x_K \right| \cdot \left| x - (\lambda - \mu) + \frac{n}{K} - x_K \right|
\leq \frac{K}{2\lambda} \sup_{n \in \mathbb{N}_K} \sup_{x \in \left( \frac{n}{K}, \frac{n+1}{K} \right)} \left| x - \frac{n}{K} + x_K - (\lambda - \mu) \right| \cdot \left| x - \frac{n}{K} + x_K - (\lambda - \mu) + \frac{n}{K} - x_K \right|
\leq \frac{K}{2\lambda K} \left( \frac{2}{K} + \frac{2}{K^2} \right) \to 0 \quad \text{as} \quad K \to \infty.
\]

This together with the fact that \( \frac{Z^K}{Z_K} \to 1 \) as \( K \to \infty \) implies that

\[
\sup_{n \in \mathbb{N}_K} \left| 1 - \frac{Z^K}{Z_K} \int_{\frac{n}{K}}^{\frac{n+1}{K}} e^{-\frac{x}{2K} \cdot (\frac{n}{K} - x_K)^2} dx \right| \to 0 \quad \text{as} \quad K \to \infty.
\]

Hence, \( \sum_{n \in \mathbb{N}_K} \left| G_K \left( \frac{n}{K} \right) - \int_{\frac{n}{K}}^{\frac{n+1}{K}} G_K(x) dx \right| \to 0 \) as \( K \to \infty \). This proves (5.11), and completes the proof of the theorem.

\[\square\]

**References**


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