SIAM J. MATH. ANAL.
Vol. 53, No. 2, pp. 1958–1992

CONVERGENCE TO PERIODIC PROBABILITY SOLUTIONS IN FOKKER–PLANCK EQUATIONS*

MIN JI†, WEIWEI QI‡, ZHONGWEI SHEN§, AND YINGFEI YI¶

Abstract. The present paper is devoted to the study of convergence of solutions of a Fokker–Planck equation (FPE) associated to a periodic stochastic differential equation with less regular coefficients under various Lyapunov conditions. In the case of nondegenerate noises, we prove two types of convergence of solutions to the unique periodic probability solution, namely, convergence in mean and exponential convergence. In the case of degenerate noises, we show the convergence of solutions in mean to the set of periodic probability solutions. New results on the uniqueness of periodic probability solutions and global probability solutions of the FPE are also obtained. As applications, we study the long-time behaviors of the FPEs associated to stochastic damping Hamiltonian systems and stochastic slow-fast systems, and of weak solutions of periodic stochastic differential equations with less regular coefficients.

Key words. Fokker–Planck equation, periodic probability solution, uniqueness, convergence

AMS subject classifications. Primary, 35Q84; Secondary, 35J25, 37B25, 60J60

DOI. 10.1137/20M1319127

1. Introduction.

Consider ordinary differential equations (ODEs) of the form

$$\dot{x} = V(x,t), \quad x \in \mathcal{U}, \tag{1.1}$$

where t is the time variable, $\dot{x}$ stands for the time derivative of $x = x(t)$, $\mathcal{U} \subset \mathbb{R}^d$ is an open connected domain, and $V = (V^i) : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^d$ is a time T-periodic vector field, called the *drift field*, for some $T > 0$. The periodic time dependence in (1.1) is frequently used in applications, for instance in biology, ecology, physics, and engineering, to model time recurrence and seasonal variations in the vector field. As real world problems are often subject to noise perturbations from either surrounding environments or intrinsic uncertainties [20], more realistic models should often take the fluctuations or noises into consideration. This motivates us to consider noise perturbations to the ODE (1.1) that result in the following stochastic differential

*Received by the editors February 18, 2020; accepted for publication (in revised form) January 12, 2021; published electronically April 8, 2021.
https://doi.org/10.1137/20M1319127
Funding: The work of the first author was partially supported by National Natural Science Foundation of China grant 11571344. The work of the second author was partially supported by a postdoctoral fellowship from the University of Alberta. The work of the third author was partially supported by a start-up grant from the University of Alberta and NSERC grants RGPIN-2018-04371 and NSERC DGECR-2018-00353. The work of the fourth author was partially supported by NSERC grant RGPIN-2020-04451, the PIMS CRG grant, a faculty development grant from the University of Alberta, and a scholarship from Jilin University.
†Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Beijing 100190, People's Republic of China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China (jimin@math.ac.cn).
‡University of Chinese Academy of Sciences, Beijing 100049, China, and Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (wqi2@ualberta.ca).
§Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (zhongwei@ualberta.ca).
¶Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada, and School of Mathematics, Jilin University, Changchun 130012, People's Republic of China (yingfei@ualberta.ca).

equation (SDE):

$$\mathrm{d}x = V(x,t)\mathrm{d}t + G(x,t)\mathrm{d}W_t, \quad x \in \mathcal{U}, \tag{1.2}$$

where $G : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^{d\times m}$ is a time T-periodic *noise coefficient matrix* with $m \geq d$, and $W = (W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process. The SDE (1.2) is naturally connected to the following Fokker–Planck equation (FPE):

$$\partial_t u = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad x \in \mathcal{U}, \tag{1.3}$$

where $A := (a^{ij}) = \frac{1}{2}GG^\top$ is the *diffusion matrix*, $\partial_i = \partial_{x_i}$, $\partial^2_{ij} = \partial^2_{x_i x_j}$, and the summation convention is used in the right-hand side of (1.3). Not only does the FPE (1.3) govern the distributions of the solutions of (1.2), but it has also been directly used to model the evolution of the distributions for many stochastic processes [33]. We emphasize that a general domain $\mathcal{U}$ rather than the whole space $\mathbb{R}^d$ expands the scope of applications. For instance, mathematical models arising in population biology often have the first quadrant of $\mathbb{R}^d$ as the phase space.

Two fundamental problems concerning the long-time dynamics of the SDE (1.2) and the FPE (1.3) are the existence and uniqueness of *steady states* and the convergence of their solutions to the steady states. These problems have been extensively studied when $V(x,t) = V(x)$ and $G(x,t) = G(x)$ are autonomous in both regular or less regular cases, in which steady states are often defined to be the *stationary measures*, or *stationary distributions*. We refer the reader to [9, 5, 7, 6, 18, 19] and references therein for the existence and uniqueness of stationary measures, and [25, 32, 5, 29, 26, 3, 6, 23] and references therein for the convergence of solutions of (1.2) and (1.3) to stationary measures. Many different approaches have been taken and developed to study these problems. For instance, ergodic properties of Markov processes and stochastic analytical techniques are adopted in [25, 32, 29, 26, 3], theories of Dirichlet forms and semigroups are used in [9, 5, 6], and PDE techniques are developed in [9, 5, 7, 6, 18, 19, 23]. We emphasize that (1.2) and (1.3) with less regular coefficients arise naturally in applications, for instance in modeling complex fluid flows [31], and their study gives rise to challenging mathematical problems.

When V and G are T-periodic in t and admit at least Lipschitz regularity in x, steady states of (1.2) and (1.3) are characterized by the periodic analogues of stationary measures, called *periodic solutions*, that appeared in the literature under different names and definitions. The investigation of these fundamental problems for (1.2) and (1.3) with locally Lipschitz coefficients has attracted much attention, especially in recent years. In [26], Khasminskii defined periodic solutions for the SDE (1.2) in the sense of periodic Markov processes and proved the existence under periodic Lyapunov conditions. In [11], Chen et al. studied the existence of classical periodic solutions of the FPE (1.3), assuming the existence of an uncommon Lyapunov function. The existence of periodic solutions of semilinear SDEs has been established in [30, 21, 13, 10] and references therein. Zhao and Zheng [35] and Feng, Zhao, and Zhou [16] studied the existence of the so-called random periodic solutions of (1.2) in the framework of random dynamical systems. As for the convergence, Feng, Zhao, and Zhong investigated in [15] the ergodic property of (1.2) that generalizes the classical ergodic theory of (1.2) in the autonomous case.

For (1.2) and (1.3) with less regular coefficients, the authors of the present paper adopted PDE techniques in [22] to show the existence of periodic probability solutions (see Definition 1.1) of (1.3) under a Lyapunov condition. The uniqueness of periodic probability solutions and the convergence of solutions of (1.3) remained open.

The main purpose of the present paper is to investigate the uniqueness of periodic probability solutions of (1.3) as well as the convergence of the solutions of (1.2) and (1.3) when V and G are less regular. Our study of the convergence issue also gives an alternative approach for the existence of periodic probability solutions of (1.3) but under stronger conditions than those required in [22]. We recall from [22] the definition of periodic probability solutions of (1.3). Denote by

$$\mathcal{L} := \partial_t + a^{ij}\partial_{ij}^2 + V^i\partial_i$$

the parabolic operator associated to the dual equation of (1.3).

DEFINITION 1.1 (periodic probability solution). *A Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is called a* periodic probability solution *of* (1.3) *if there exists a family of Borel probability measures $(\mu_t)_{t\in\mathbb{R}}$ on $\mathcal{U}$ satisfying $\mu_t = \mu_{t+T}$ for all $t\in\mathbb{R}$, $a^{ij}, V^i \in L^1_{loc}(\mathcal{U}\times\mathbb{R}, \mathrm{d}\mu_t\mathrm{d}t)$ for all $i,j\in\{1,\dots,d\}$, and $\int_{\mathbb{R}}\int_{\mathcal{U}}\mathcal{L}\phi\,\mathrm{d}\mu_t\mathrm{d}t = 0$ for all $\phi\in C_0^{2,1}(\mathcal{U}\times\mathbb{R})$, such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$, writing $\mu = (\mu_t)_{t\in\mathbb{R}}$ in short.*

To proceed, dissipative conditions in terms of Lyapunov functions are needed. For a nonnegative function $U\in C_T(\mathcal{U}\times\mathbb{R})$ (see Table 1 for the definition), we define for each $\rho>0$, the *ρ-sublevel set* $\Omega_\rho = \{(x,t)\in\mathcal{U}\times\mathbb{R} : U(x,t)<\rho\}$, and its *$t$-sections* $\Omega_\rho^t = \{x\in\mathcal{U} : U(x,t)<\rho\}$ for $t\in\mathbb{R}$. From now on, we begin to use some function spaces, which, except for the usual ones, are collected in Table 1 at the end of this section. We define four types of Lyapunov functions as follows.

DEFINITION 1.2. *A function $U\in C_T(\mathcal{U}\times\mathbb{R})$ is called an* unbounded compact function *if $U\geq 0$ and there is a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open sets in $\mathcal{U}$ satisfying $\mathcal{U}_n\subset\mathcal{U}_{n+1}\subset\subset\mathcal{U}$ for all $n\in\mathbb{N}$ and $\mathcal{U}=\bigcup_{n=1}^\infty\mathcal{U}_n$ such that*

$$\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U \to \infty \quad \text{as} \quad n\to\infty. \tag{1.4}$$

An unbounded compact function $U\in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ is called a Lyapunov function *(with respect to $\mathcal{L}$) of the following types:*

(L1) *There are positive constants ρ_m, C_1, and C_2 such that $\mathcal{L}U\leq C_1U+C_2$ in $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$.*
(L2) *There are positive constants ρ_m and γ such that $\mathcal{L}U\leq-\gamma$ in $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$.*
(L3) $\lim_{n\to\infty}\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\mathcal{L}U=-\infty$.
(L4) *There are positive constants ρ_m, C_1, and C_2 such that $\mathcal{L}U\leq-C_1U+C_2$ in $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$.*

Note that in the definition of Lyapunov functions of types (L1) and (L4), the domain $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$ cannot be replaced by $\mathcal{U}\times\mathbb{R}$, as the drift field V is not necessarily continuous or bounded. As the definitions of these Lyapunov functions are based on *unbounded* compact functions, they are necessarily unbounded. The word "unbounded" often appears in front of these functions in literature just to highlight their unboundedness. In this paper, we choose to suppress the word "unbounded" in front of these functions for the sake of simplicity. Whenever no confusion is caused, we also suppress the phrase "with respect to $\mathcal{L}$".

We point out that the unboundedness of a Lyapunov function of type (L1), the weakest among all types, ensures the conservation of mass (see Theorem 2.3). In the case of smooth coefficients, it simply says that trajectories of the solution process of (1.2) do not reach the boundary of $\mathcal{U}$ in finite time. Thus, the system on $\mathcal{U}$ can be treated in the same way as it is on $\mathbb{R}^d$.

To study the uniqueness of periodic probability solutions of (1.3), we make the following assumption.

(H1) For fixed $p > d+2$, $a^{ij} \in L^\infty(\mathbb{R}; W^{1,p}_{loc}(\mathcal{U}))$ and $V^i \in L^p_{loc}(\mathcal{U}\times\mathbb{R})$ for each $i,j = 1,\dots,d$. The diffusion matrix $A = (a^{ij})$ is *locally uniformly positive definite*; that is, for each open set $\mathcal{V} \subset\subset \mathcal{U}$, there are positive constants $\lambda_\mathcal{V}$ and $\Lambda_\mathcal{V}$ such that $\lambda_\mathcal{V}|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \Lambda_\mathcal{V}|\xi|^2$ for all $(x,t)\in\mathcal{V}\times\mathbb{R}$ and $\xi\in\mathbb{R}^d$.

Our result on uniqueness is as follows.

THEOREM A (uniqueness). *Assume* **(H1)**. *If $\mathcal{L}$ admits a Lyapunov function of type* (L1)*, then there exists at most one periodic probability solution of* (1.3).

We remark that the existence of periodic probability solutions of (1.3) is established in [22, Theorem A] under **(H1)** and a Lyapunov function of type (L2). This together with Theorem A gives the following corollary.

COROLLARY A. *Assume* **(H1)**. *If $\mathcal{L}$ admits a Lyapunov function of type* (L2)*, then there exists a unique periodic probability solution of* (1.3).

Let $\mathcal{M}_p(\mathcal{U})$ be the space of all Borel probability measures on $\mathcal{U}$. We recall the definition of global probability solutions, whose existence and uniqueness are investigated in subsection 2.2.

DEFINITION 1.3. *Let $\mathcal{I}\subset\mathbb{R}$ be an open interval, and let $s\in\mathbb{R}$:*

(1) *A Borel measure μ on $\mathcal{U}\times\mathcal{I}$ is called a* measure solution *of* (1.3) *(in $\mathcal{U}\times\mathcal{I}$) if there exists a family of Borel measures $(\mu_t)_{t\in\mathcal{I}}$ on $\mathcal{U}$ satisfying $a^{ij}, V^i \in L^1_{loc}(\mathcal{U}\times\mathcal{I}, \mathrm{d}\mu_t\mathrm{d}t)$ for all $i,j\in\{1,\dots,d\}$ and*

$$\iint_{\mathcal{U}\times\mathcal{I}} \mathcal{L}\phi \mathrm{d}\mu_t\mathrm{d}t = 0 \quad \forall \phi\in C^{2,1}_0(\mathcal{U}\times\mathcal{I}), \tag{1.5}$$

such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$. In this case, we write $\mu = (\mu_t)_{t\in\mathcal{I}}$. If, in addition, $\mathcal{I} = (s,\infty)$ and $\mu_t(\mathcal{U})\le 1$ (resp., $\mu_t(\mathcal{U}) = 1$) for a.e. $t\in\mathcal{I}$, then μ is called a global subprobability solution *(resp.,* global probability solution*) of* (1.3).

(2) *Let $\mathcal{I} = (s, t_0)$ for some $t_0\in(s,\infty]$. A measure solution $\mu = (\mu_t)_{t\in\mathcal{I}}$ of* (1.3) *is said to satisfy the initial condition*

$$\mu_s = \nu \in \mathcal{M}_p(\mathcal{U}) \tag{1.6}$$

if for each $\phi\in C^\infty_c(\mathcal{U})$, there is a set $J_\phi\subset\mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi| = 0$ such that

$$\lim_{J_\phi\ni t\to s}\int_\mathcal{U}\phi\mathrm{d}\mu_t = \int_\mathcal{U}\phi\mathrm{d}\nu. \tag{1.7}$$

In this case, $\mu = (\mu_t)_{t\in\mathcal{I}}$ is simply called a measure solution *of the Cauchy problem* (1.3) *and* (1.6)*. If, in addition, $\mathcal{I} = (s,\infty)$ and $\mu = (\mu_t)_{t\in(s,\infty)}$ is a global subprobability solution (resp., global probability solution) of* (1.3)*, then μ is called a* global subprobability solution *(resp.,* global probability solution*) of the Cauchy problem* (1.3) *and* (1.6).

We prove three results on the convergence of global probability solutions of the Cauchy problem (1.3) and (1.6) to periodic probability solutions. To state the first one, we make the following assumptions on A and V.

(H2) a^{ij}, $V^i\in C(\mathcal{U}\times\mathbb{R})$ for each $i,j = 1,\dots,d$.

THEOREM B (convergence in mean). *Assume* **(H2)** *and that $\mathcal{L}$ admits a Lyapunov function of type* (L3) *U. Let $\mu = (\mu_t)_{t\in(s,\infty)}$ be a global probability solution of the Cauchy problem* (1.3) *and* (1.6) *with $\int_\mathcal{U} U(\cdot,s)\mathrm{d}\nu < \infty$. Then for any sequence* *of positive integers* $\{n_j\}_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} n_j = \infty$, *there exist a subsequence, still denoted by* $\{n_j\}_{j\in\mathbb{N}}$, *and a periodic probability solution* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *of* (1.3) *such that the following hold:*

(1) *For each bounded* $\phi \in C_T(\mathcal{U}\times\mathbb{R})$, *there holds that*

$$\lim_{j\to\infty} \frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau = \frac{1}{T}\int_0^T\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall t \ge s.$$

(2) *For each* $\psi \in C_c^2(\mathcal{U})$, *there holds that*

$$\lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+kT} = \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t \quad \textit{for a.e. } t > s.$$

In particular, if (1.3) *admits a unique periodic probability solution* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$, *then the convergence in* (1) *and* (2) *holds for the whole sequence* $\mathbb{N}$.

Under the conditions of Theorem B, the diffusion matrix A is allowed to be degenerate in $\mathcal{U}$, in which case the FPE (1.3) can admit multiple periodic probability solutions. This is why the main part in the statement of Theorem B only asserts the average attractiveness of global probability solutions of the Cauchy problem (1.3) and (1.6) by the set of periodic probability solutions of (1.3). If we assume, in addition, that A is locally uniformly positive definite as in **(H1)**, then Theorem A guarantees the validity of the "In particular" part in the statement of Theorem B. The same results can be established under slightly weaker conditions on the coefficients. As the proof is almost the same, we state the results in the next corollary, which is our second result on the convergence.

COROLLARY B. *Assume* **(H1)** *and that* $\mathcal{L}$ *admits a Lyapunov function of type* (L3) U. *Then for any global probability solution* $\mu = (\mu_t)_{t\in(s,\infty)}$ *of the Cauchy problem* (1.3) *and* (1.6) *with* $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$, *there holds that for any* $\psi \in C_b(\mathcal{U})$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+kT} = \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t \quad \forall t \in (s, s+T],$$

where $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *is the unique periodic probability solution of* (1.3).

Compared to the convergence in Theorem B (2), the convergence in Corollary B holds for a larger class of test functions. This is because the assumption **(H1)** (more precisely, the locally uniform positive definiteness of A in **(H1)**) ensures the existence of the continuous density of a global probability solution $\mu = (\mu_t)_{t\in(s,\infty)}$, which guarantees the continuity of the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\mu_t$ on (s,∞) for each $\phi \in C_b(\mathcal{U})$, while under the conditions in the statement of Theorem B, the continuity of the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\mu_t$ on (s,∞) is only obtained when $\phi \in C_c^2(\mathcal{U})$.

Conclusions in Theorem B and Corollary B can be regarded as weak forms of Birkhoff's ergodic theorem. Moreover, their proofs do not require the standard semiflow property that plays essential roles in the proof of the classical ergodic theorem for measure-preserving dynamical systems and Markov processes. Indeed, under the assumption **(H1)** or **(H2)**, the uniqueness of solutions of the Cauchy problem (1.3) and (1.6) is unknown. Even if we assume the uniqueness, they are only known to generate a semiflow under the weak*-topology. Such weak ergodic theorems without semiflow property can potentially serve as theoretical foundations for the evolution of practical systems that are often too complicated to admit the standard semiflow property or get it tested.

Our third convergence result concerns the exponential convergence of global probability solutions of the Cauchy problem (1.3) and (1.6) to periodic probability solutions under Lyapunov functions of type (L4). This requires Lipschitz conditions on $A = (a^{ij})$ as follows.

(H3) For each $i, j = 1, \ldots, d$, the entry a^{ij} is *locally Lipschitz* in x; that is, for each open set $\mathcal{V} \subset\subset \mathcal{U}$, there is an $L_{\mathcal{V}} > 0$ such that $\left|a^{ij}(x_1, t) - a^{ij}(x_2, t)\right| \leq L_{\mathcal{V}}|x_1 - x_2|$ for all $x_1, x_2 \in \overline{\mathcal{V}}$ and a.e. $t \in \mathbb{R}$.

Theorem C (exponential convergence). *Assume* **(H1)** *and* **(H3)**. *Suppose $\mathcal{L}$ admits a Lyapunov function of type* (L4) *U. Then there exist positive constants C_1 and C_2 such that for any global probability solution $\mu = (\mu_t)_{t\in(s,\infty)}$ of the Cauchy problem* (1.3) *and* (1.6) *with $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$, there holds that*

$$\|\mu_t - \tilde{\mu}_t\|_{TV} \leq C_1 e^{-C_2(t-s)} \quad \forall t > s,$$

where $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (1.3) *and $\|\cdot\|_{TV}$ denotes the total variation norm.*

An important piece in the proof of Theorem C is the construction of the transition probability densities $p(s, x, t, y)$ for $s < t$ and $x, y \in \mathcal{U}$ associated to the global probability solutions of the Cauchy problem (1.3) and (1.6) (see subsection 5.1). This benefits from the assumption **(H3)**, which, together with the assumption **(H1)**, ensures the existence, regularity, and uniqueness of global probability solutions of the Cauchy problem (1.3) and (1.6) (see Theorem 2.3). Consequently, the transition probability densities can be defined and shown to satisfy expected properties resulting in the applicability of classical arguments leading to the exponential convergence.

It is worthwhile to point out that our approaches for the convergence are different from those in [23] for the autonomous case based on the sophisticated theory of *generalized Markov semigroups* associated to stationary measures developed in [9]. In fact, it is unclear whether there is an analogous theory of generalized Markov semigroups associated to periodic probability solutions. Any progress along this direction would be helpful for improving the convergence results in Theorem B and Corollary B.

In this paper, we also consider three applications of Theorem B, Corollary B, and Theorem C as follows. (i) For a class of stochastic damping Hamiltonian systems, Lyapunov functions of type (L3) are constructed to ensure the convergence of global probability solutions of the associated FPEs as stated in Theorem B. (ii) For a class of stochastic slow-fast systems with very strong dissipative properties along the fast directions and nondegenerate noises only along the slow directions, we show the existence and uniqueness of periodic probability solutions as well as the convergence of global probability solutions of the associated FPEs under Lyapunov conditions along the slow directions. (iii) For an SDE with less regular coefficients, we show that the distributions of their globally defined weak solutions are global probability solutions of the associated FPE, and hence, under appropriate Lyapunov conditions, the convergence of globally defined weak solutions is established as simple consequences of our convergence results. The details of these applications are given in section 6.

The rest of the paper is organized as follows. In section 2, we recall some basic facts, including, in particular, equivalent formalisms of global probability solutions of the Cauchy problem (1.3) and (1.6) and the regularity theory of measure solutions of (1.3), and we prove the global well-posedness of the Cauchy problem (1.3) and

Table 1
Notations.

$\mathcal{I} \subset \mathbb{R}$	An interval
$C_c(\mathcal{U})$	Space of compactly supported continuous functions on $\mathcal{U}$
$C_b(\mathcal{U})$	Space of bounded continuous functions on $\mathcal{U}$
$C_c^2(\mathcal{U})/C_c^\infty(\mathcal{U})$	$C_c(\mathcal{U}) \cap C^2(\mathcal{U})/C_c(\mathcal{U}) \cap C^\infty(\mathcal{U})$
$C_0(\mathcal{U} \times \mathcal{I})$	Space of compactly supported continuous functions on $\mathcal{U} \times \mathcal{I}$
$C_c(\mathcal{U} \times \mathcal{I})$	Space of continuous functions $u : \mathcal{U} \times \mathcal{I} \to \mathbb{R}$ such that $u(t, \cdot) \in C_c(\mathcal{U})$ for each $t \in \mathcal{I}$
$C_T(\mathcal{U} \times \mathbb{R})$	Space of T-periodic and continuous functions on $\mathcal{U} \times \mathbb{R}$
$C^{2,1}(\mathcal{U} \times \mathcal{I})$	Space of continuous functions that are twice continuously differentiable in x and continuously differentiable in t
$C_c^{2,1}(\mathcal{U} \times \mathcal{I})$	$C^{2,1}(\mathcal{U} \times \mathcal{I}) \cap C_c(\mathcal{U} \times \mathcal{I})$
$C_T^{2,1}(\mathcal{U} \times \mathbb{R})$	$C^{2,1}(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$
$C_0^{2,1}(\mathcal{U} \times \mathcal{I})$	Space of functions in $C^{2,1}(\mathcal{U} \times \mathcal{I})$ with compact support
$L^\infty(\mathbb{R}; W_{loc}^{1,p}(\mathcal{U}))$	Space of measurable functions $u : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ such that $u(t, \cdot) \in W_{loc}^{1,p}(\mathcal{U})$ for a.e. $t \in \mathbb{R}$ and for each subdomain $\Omega \subset\subset \mathcal{U}$, the function $t \mapsto \|u(t, \cdot)\|_{W^{1,p}(\Omega)}$ is essentially bounded

(1.6). In section 3, we study the uniqueness of periodic probability solutions of (1.3) with nondegenerate noises and prove Theorem A. We study the convergence of global probability solutions of the Cauchy problem (1.3) and (1.6) in sections 4 and 5. In particular, Theorems B and C are proven, respectively, in sections 4 and 5. The proof of Corollary B is sketched out at the end of subsection 4.2. Some applications of our convergence results, namely, Theorem B, Corollary B, and Theorem C, are presented in section 6.

2. Preliminaries. In subsection 2.1, we present some equivalent formalisms of measure solutions, given in Definition 1.3, of (1.3) or the Cauchy problem (1.3) and (1.6), and we recall the regularity theory. In subsection 2.2, we present some results on the global well-posedness of the Cauchy problem (1.3) and (1.6).

2.1. Measure solutions and regularity. Arguing as in [6, Proposition 6.1.2] and [8, Lemma 1.1], the following equivalent formalisms hold for (1.5) or (1.5) and (1.7) in $\mathcal{U} \times \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}$ is an open interval.

Lemma 2.1. *Let $(\mu_t)_{t\in\mathcal{I}}$ be Borel measures such that a^{ij}, $V^i \in L^1_{loc}(\mathcal{U} \times \mathcal{I}, \mathrm{d}\mu_t \mathrm{d}t)$ for all $i, j = 1, \dots, d$:*

(1) *The following statements are equivalent to* (1.5)*:*
   (a) *For each $\phi \in C_c^2(\mathcal{U})$, there exists $J_\phi \subset \mathcal{I}$ satisfying $|\mathcal{I} \setminus J_\phi| = 0$ such that*

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall r, t \in J_\phi \text{ with } r < t.$$

   (b) *For each $\phi \in C_c^{2,1}(\mathcal{U} \times \mathcal{I})$, there exists $J_\phi \subset \mathcal{I}$ satisfying $|\mathcal{I} \setminus J_\phi| = 0$ such that*

$$\int_{\mathcal{U}} \phi(\cdot, t) \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi(\cdot, r) \mathrm{d}\mu_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall r, t \in J_\phi \text{ with } r < t.$$

(2) *Let $\mathcal{I} = (s, t_0)$ for some $-\infty < s < t_0 \le \infty$ The following statements are equivalent to* (1.5) *and* (1.7)*:*
   (a) *For each $\phi \in C_c^2(\mathcal{U})$, there exists $J_\phi \subset \mathcal{I}$ satisfying $|\mathcal{I} \setminus J_\phi| = 0$ such that*

$$(2.1) \qquad \int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\nu + \lim_{J_\phi \ni r \to s} \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall t \in J_\phi.$$

(b) *For each* $\phi \in C_c^{2,1}(\mathcal{U}\times[s,t_0))$, *there exists* $J_\phi \subset \mathcal{I}$ *satisfying* $|\mathcal{I}\setminus J_\phi| = 0$ *such that*

$$\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t = \int_{\mathcal{U}} \phi(\cdot,s)\mathrm{d}\nu + \lim_{J_\phi \ni r\to s}\int_r^t\int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall t\in J_\phi.$$

LEMMA 2.2. *Let* $(\mu_t)_{t\in\mathcal{I}}$ *be as in Lemma* 2.1. *If* $t\mapsto \int_{\mathcal{U}}\phi\mathrm{d}\mu_t$ *is continuous on* $\mathcal{I}$ *for any* $\phi\in C_c^2(\mathcal{U})$, *then* J_ϕ *can be taken to be* $\mathcal{I}$ *in each case of Lemma* 2.1.

Proof. We only show the case in Lemma 2.1 (2)(a); the other cases can be proven in the same manner.

For fixed $\phi \in C_c^2(\mathcal{U})$, it is clear that the function $(r,t)\mapsto \int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ is continuous on $\{(r,t)\in\mathcal{I}^2 : r<t\}$. Fix $t_*\in\mathcal{I}\setminus J_\phi$. There exists a sequence $\{t_n\}_{n\in\mathbb{N}}\subset J_\phi$ such that $t_n\to t_*$ as $n\to\infty$. Setting $t=t_n$ in (2.1) and letting $n\to\infty$, we see that (2.1) holds for $t=t_*$.

It remains to show that for each $t\in\mathcal{I}$, there holds that $\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau = \int_{\mathcal{U}}\phi\mathrm{d}\mu_t - \int_{\mathcal{U}}\phi\mathrm{d}\nu =: A_t$. Clearly, the above limit is the case if r takes values in J_ϕ. If the above limit is not the case, then there exist an $\epsilon_0>0$ and a sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}\setminus J_\phi$ satisfying $r_n\to s$ as $n\to\infty$ such that $|\int_{r_n}^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau - A_t| > \epsilon_0$ for all $n\gg 1$. By the continuity of $r\mapsto\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ and the density of J_ϕ in $\mathcal{I}$, we find a sequence $\{\tilde r_n\}_{n\in\mathbb{N}}\subset J_\phi$ satisfying $\tilde r_n\to s$ as $n\to\infty$ such that $|\int_{\tilde r_n}^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau - A_t| > \frac{\epsilon_0}{2}$ for all $n\gg1$, which leads to a contradiction. This proves (2)(a). ∎

Now we recall the regularity theory of measure solutions of (1.3) in $\mathcal{U}\times\mathcal{I}$. Recall $p>d+2$. Let $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$ be the space of measurable functions u on $\mathcal{U}\times\mathcal{I}$ such that $u(\cdot,t)\in W_0^{1,p}(\mathcal{U})$ for a.e. $t\in\mathcal{I}$ and the function $t\mapsto \|u(t,\cdot)\|_{W_0^{1,p}(\mathcal{U})}$ lies in $L^p(\mathcal{I})$. Let $\mathbb{H}^{-1,p'}(\mathcal{U}\times\mathcal{I})$ be the dual space of $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$, where $p'>1$ is such that $\frac{1}{p}+\frac{1}{p'}=1$.

Let $\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$ be the space of measurable functions u on $\mathcal{U}\times\mathcal{I}$ such that $\eta u\in\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$ and $\partial_t(\eta u)\in\mathbb{H}^{-1,p}(\mathcal{U}\times\mathcal{I})$ for each $\eta\in C_0^\infty(\mathbb{R}^{d+1})$. By [6, Theorem 6.2.2], there exist $\alpha>\frac{1}{p}$ and $\gamma>0$, depending only on d and p, such that $\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$ is continuously embedded into $C^{\alpha-\frac{1}{p}}(\mathcal{I},C^\gamma(\mathcal{U}))$. Here, $C^\alpha(\mathcal{I},C^\gamma(\mathcal{U}))$ denotes the space of all continuous functions $u:\mathcal{U}\times\mathcal{I}\to\mathbb{R}$ such that $u(t,\cdot)\in C^\gamma(\mathcal{U})$ for all $t\in\mathcal{I}$ and for each subdomain $\Omega\subset\subset\mathcal{U}$, the function $t\mapsto |u(t,\cdot)|_{C^\gamma(\overline{\Omega})}$ lies in $C^\alpha(\mathcal{I})$.

THEOREM 2.1 (see [4, 6]). *Assume* **(H1)**. *Let* $\mu=(\mu_t)_{t\in\mathcal{I}}$ *be a measure solution of* (1.3). *Then* μ *admits a positive density* $u\in\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$. *Moreover, for closed intervals* $[s_1,t_1]\subset\subset[s_2,t_2]\subset\mathcal{I}$ *and open subsets* $\mathcal{W}\subset\subset\mathcal{W}_1\subset\subset\mathcal{U}$, *there exist* $\alpha>\frac{1}{p}$, $\gamma>0$, *and* $N>0$, *independent of* μ *or* u, *such that*

$$\|u\|_{C^{\alpha-\frac{1}{p}}((s_1,t_1),C^\gamma(\mathcal{W}))} \le N\int_{s_2}^{t_2}\mu_\tau(\mathcal{W}_1)\mathrm{d}\tau. \tag{2.2}$$

2.2. Global well-posedness. The following result on the existence of global probability solutions of the Cauchy problem (1.3) and (1.6) is taken from [28].

THEOREM 2.2 (see [28]). *Assume* **(H1)** *or* **(H2)**. *Suppose* $\mathcal{L}$ *admits a Lyapunov function of type* (L1). *Then the Cauchy problem* (1.3) *and* (1.6) *admits a global probability solution* $\mu=(\mu_t)_{t\in(s,\infty)}$. *Moreover, under* **(H1)**, μ *admits a density in* $C^\alpha((s,\infty),C^\gamma(\mathcal{U}))$ *for some* $\alpha>\frac{1}{p}$ *and* $\gamma>0$.

We prove a uniqueness result.

THEOREM 2.3. *Assume* **(H1)** *and* **(H3)**. *Suppose* $\mathcal{L}$ *admits a Lyapunov function of type* (L1). *Then the Cauchy problem* (1.3) *and* (1.6) *admits a unique (in the class of global subprobability solutions) global probability solution.*

Proof. Let $\mu^1 = (\mu_t^1)_{t\in(s,\infty)}$ and $\mu^2 = (\mu_t^2)_{t\in(s,\infty)}$ be, respectively, a global probability solution and a global subprobability solution of the Cauchy problem (1.3) and (1.6). Applying Theorem 2.1, we may assume that for each $i = 1, 2$, μ^i admits a positive density $\rho_i \in C(\mathcal{U} \times (s,\infty))$. We show that $\rho_1 = \rho_2$. Setting $w := \frac{\rho_2}{\rho_1}$, it is equivalent to prove $w \equiv 1$ on $\mathcal{U} \times (s,\infty)$.

Define $f_\lambda(t) := e^{\lambda(1-t)} - e^\lambda$ for $t \geq 0$, where $\lambda > 0$ is a parameter. Following the main ideas of [34, Lemma 2.2], we deduce that for any nonnegative function $\phi \in C_c^{2,1}(\mathcal{U} \times \mathbb{R})$, there holds that

$$\int_{\mathcal{U}} f_\lambda(w)\phi \mathrm{d}\mu_t^1 \leq f_\lambda(1) \int_{\mathcal{U}} \phi \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\mathcal{L}\phi \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \quad \forall t > s. \tag{2.3}$$

Since the proof of (2.3), using the main ideas in [34, Lemma 2.2], is relatively independent and long, we include it in the accompanying supplemental file (supplement.pdf [local/web 279KB]) for the sake of readability and completeness.

Let U be the Lyapunov function of type (L1). Fix $\rho_0 > \rho_m$. We introduce a smooth and nondecreasing function θ satisfying $\theta = 0$ on $[0, \rho_m]$ and $\theta(t) = t$ for $t \in [\rho_0, \infty)$. There exists $\bar{C} > 0$ such that $t\theta'(t) \leq \bar{C}\theta(t)$ for $t \geq 0$. Since $\theta'' \not\equiv 0$ on $[\rho_m, \rho_0]$ and $\theta'' = 0$ otherwise, **(H1)** ensures the existence of $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\begin{aligned}\mathcal{L}\theta(U) &= \theta'(U)\mathcal{L}U + \theta''(U)a^{ij}\partial_i U \partial_j U \\ &\leq \theta'(U)(C_1 U + C_2) + \Lambda_{\Omega_{\rho_0}} |\theta''|_\infty \max_{\Omega_{\rho_0}} |\nabla U|^2 \leq \tilde{C}_1\theta(U) + \tilde{C}_2 \quad \text{in} \quad \mathcal{U} \times \mathbb{R}.\end{aligned}$$

Hence, we find a new Lyapunov function of type (L1) $\tilde{U} := \theta(U)$ whose Lyapunov condition holds on $\mathcal{U} \times \mathbb{R}$. This allows us to proceed as in [28, Theorem 3.5].

Let $\zeta \in C_c^\infty([0,\infty))$ satisfy $\zeta(0) = 1$, $\zeta = 0$ on $[1,\infty)$, $\zeta' \leq 0$, and $\zeta'' \geq 0$. It is clear that $\zeta(\frac{\tilde{U}}{N}) \in C_c^{2,1}(\mathcal{U} \times (s,\infty))$ for $N \gg 1$. Setting $\phi = \zeta(\frac{\tilde{U}}{N})$ in (2.3), we find

$$\begin{aligned}\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 &\leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\mathcal{L}\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \\ &= f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N}\zeta'\left(\frac{\tilde{U}}{N}\right) \mathcal{L}\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \\ &\quad + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N^2}\zeta''\left(\frac{\tilde{U}}{N}\right) a^{ij}\partial_i\tilde{U}\partial_j\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \quad \forall t > s.\end{aligned} \tag{2.4}$$

Since $f_\lambda \leq 0$, $\zeta'' \geq 0$, and (a^{ij}) is positive definite, the last term in (2.4) is nonpositive. Thus,

$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N}\zeta'\left(\frac{\tilde{U}}{N}\right) \mathcal{L}\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau.$$

Since $\mathcal{L}\tilde{U} \leq \tilde{C}_1\tilde{U} + \tilde{C}_2$, $\zeta' \leq 0$, and $f_\lambda \leq 0$, we find

$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\zeta'\left(\frac{\tilde{U}}{N}\right) \frac{\tilde{C}_1\tilde{U} + \tilde{C}_2}{N} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau.$$

It follows from $\zeta'(t)=0$ for $t\geq 1$ and $|f_\lambda|\leq e^\lambda$ that

(2.5)
$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right)\mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}}\zeta\left(\frac{\tilde{U}}{N}\right)\mathrm{d}\nu + \frac{C_3}{N}\int_s^t\int_{\{\tilde{U}\leq N\}}\left[\tilde{C_1}\tilde{U}+\tilde{C_2}\right]\mathrm{d}\mu_\tau^1\mathrm{d}\tau,$$

where $C_3 = e^\lambda|\zeta'|_\infty$.

Applying the dominated convergence theorem, we find

$$\lim_{N\to+\infty}\frac{1}{N}\int_s^t\int_{\{\tilde{U}\leq N\}}\left[\tilde{C_1}\tilde{U}+\tilde{C_2}\right]\mathrm{d}\mu_\tau^1\mathrm{d}\tau = 0.$$

Since $\lim_{N\to\infty}\zeta(\frac{t}{N})=1$ for each $t\in[0,\infty)$, we pass to the limit $N\to\infty$ in (2.5) to find from the dominated convergence theorem that $\int_{\mathcal{U}} f_\lambda(w)\mathrm{d}\mu_t^1\leq f_\lambda(1)$ for all $t>s$, namely, $\int_{\mathcal{U}}\left[e^{\lambda(1-w)}-e^\lambda\right]\mathrm{d}\mu_t^1\leq(1-e^\lambda)$ for all $t>s$.

Since μ^1 is a global probability solution of the Cauchy problem (1.3) and (1.6) so that $\mu_t^1(\mathcal{U})=1$ for all $t>s$, we deduce $\int_{\mathcal{U}} e^{\lambda(1-w)}\mathrm{d}\mu_t^1\leq 1$ for all $t>s$. For $t\in\mathbb{R}$, if there is $a\in(0,1)$ such that $\mu_t^1(\{x\in\mathcal{U}:0<w(x,t)<a\})>0$, then

$$\int_{\{x\in\mathcal{U}:0<w(x,t)<a\}} e^{\lambda(1-a)}\mathrm{d}\mu_t^1 \leq \int_{\mathcal{U}} e^{\lambda(1-w)}\mathrm{d}\mu_t^1\leq 1.$$

Letting $\lambda\to\infty$, the left-hand side in the above inequality approaches ∞, giving rise to a contradiction. Thus, $w(x,t)\geq 1$ for μ_t^1-a.e. $x\in\mathcal{U}$. As μ^1 has a pointwise positive density ρ_1 and w is continuous on $\mathcal{U}\times(s,\infty)$, then $w(x,t)\geq 1$ for all $(x,t)\in\mathcal{U}\times(s,\infty)$. If $w\not\equiv 1$, we integrate the equality $w\rho_1=\rho_2$ to find $T<\int_0^T\int_{\mathcal{U}} w(x,t)\rho_1(x,t)dxdt = \int_0^T\int_{\mathcal{U}}\rho_2(x,t)dxdt\leq T$, which leads to a contradiction. ∎

3. Proof of Theorem A. Throughout this section, we assume **(H1)**. Let μ^1 and μ^2 be two periodic probability solutions of (1.3). By Theorem 2.1, for each $i=1,2$, μ^i admits a positive and T-periodic density $\rho_i\in\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathbb{R})\cap C^{\alpha-\frac{1}{p}}(\mathbb{R},C^\gamma(\mathcal{U}))$ for some $\alpha>\frac{1}{p}$ and $\gamma>0$. To show $\mu_1=\mu_2$, it suffices to prove that $w:=\frac{\rho_2}{\rho_1}\equiv 1$. This is done within two steps. We first establish an important inequality for w. Using this inequality for a family of carefully chosen test functions, we conclude that $w\equiv 1$.

The inequality for w is contained in the following result. Set $C_{c,T}^{2,1}(\mathcal{U}\times\mathbb{R}) := C_c^{2,1}(\mathcal{U}\times\mathbb{R})\cap C_T^{2,1}(\mathcal{U}\times\mathbb{R})$.

LEMMA 3.1. *Let $f(t):=e^{1-t}-e$ for $t\in[0,\infty)$. Then there exists a $C>0$ such that for each nonnegative function $\phi\in C_{c,T}^{2,1}(\mathcal{U}\times\mathbb{R})$, there holds that*

$$\int_t^{t+T}\int_{\mathcal{U}}\phi f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\leq C\int_t^{t+T}\int_{\mathcal{U}} f(w)\mathcal{L}\phi\mathrm{d}\mu_s^1\mathrm{d}s\quad\forall t\in\mathbb{R}.$$

As the proof of Lemma 3.1 follows from similar arguments leading to (2.3), we sketch it in the supplementary material file (supplement.pdf [local/web 279KB]).

Now we prove Theorem A.

Proof of Theorem A. Let $\zeta\in C^\infty([0,+\infty))$ be a nonnegative function satisfying $\zeta(0)=1$, $\zeta=0$ on $[1,\infty)$, $\zeta'\leq 0$, and $\zeta''\geq 0$. Let θ be as in the proof of Theorem 2.3. Clearly, $\tilde{U}:=\theta(U)$ satisfies $\mathcal{L}\tilde{U}\leq\tilde{C_1}\tilde{U}+\tilde{C_2}$ for some $\tilde{C_1},\tilde{C_2}>0$. It is easy to see that $\zeta(\frac{\tilde{U}}{N})\in C_{c,T}^{2,1}(\mathcal{U}\times\mathbb{R})$ for $N\gg 1$. Applying Lemma 3.1 with $\phi:=\zeta(\frac{\tilde{U}}{N})$, we find

$$\begin{aligned}
&\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde U}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\\
&\quad\le C\int_t^{t+T}\int_{\mathcal{U}}f(w)\mathcal{L}\zeta\left(\frac{\tilde U}{N}\right)\mathrm{d}\mu_s^1\mathrm{d}s\\
&\quad= C\int_t^{t+T}\int_{\mathcal{U}}f(w)\left[\zeta''\left(\frac{\tilde U}{N}\right)\frac{1}{N^2}a^{ij}\partial_i\tilde U\partial_j\tilde U+\zeta'\left(\frac{\tilde U}{N}\right)\frac{1}{N}\mathcal{L}\tilde U\right]\mathrm{d}\mu_s^1\mathrm{d}s\quad\forall t\in\mathbb{R}.
\end{aligned}$$

Note that $\int_t^{t+T}\int_{\mathcal{U}}f(w)\zeta''(\frac{\tilde U}{N})\frac{1}{N^2}a^{ij}\partial_i\tilde U\partial_j\tilde U\mathrm{d}\mu_s^1\mathrm{d}s\le 0$ due to the positive definiteness of (a^{ij}), $f<0$, and $\zeta''\ge 0$. As a result,

$$\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde U}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\le\frac{C}{N}\int_t^{t+T}\int_{\mathcal{U}}f(w)\zeta'\left(\frac{\tilde U}{N}\right)\mathcal{L}\tilde U\mathrm{d}\mu_s^1\mathrm{d}s.$$

Using $\mathcal{L}\tilde U\le\tilde C\tilde U+\tilde C_2$ and $|f|\le e$, we find

$$\begin{aligned}
0&\le\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde U}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\\
&\le\frac{Ce}{N}|\zeta'|_\infty\int_t^{t+T}\int_{\{(x,s):\tilde U(x,s))\le N\}}\left(\tilde C_1\tilde U+\tilde C_2\right)\mathrm{d}\mu_s^1\mathrm{d}s\quad\forall t\in\mathbb{R}.
\end{aligned}$$

Letting $N\to\infty$ in the above inequality, we conclude from $f''(t)=e^{1-t}$ and the dominated convergence theorem that $\int_t^{t+T}\int_{\mathcal{U}}e^{1-w}a^{ij}\partial_i w\partial_j w\rho_1\mathrm{d}x\mathrm{d}t=0$ for all $t\in\mathbb{R}$. Since (a^{ij}) is locally uniformly positive definite, we conclude that $\nabla w=0$ a.e. on $\mathcal{U}\times\mathbb{R}$, which together with the continuity of w implies that $w(\cdot,t)\equiv$ const for $t\in\mathbb{R}$. As $w=\frac{\rho_2}{\rho_1}$ and both $\rho_1(\cdot,t)$ and $\rho_2(\cdot,t)$ are continuous probability densities for each $t\in\mathbb{R}$, there must hold that $w\equiv 1$. This completes the proof. ◻

4. Proof of Theorem B.

In subsection 4.1, we establish an estimate for global probability solutions of the Cauchy problem (1.3) and (1.6). It is then applied to prove Theorem B in subsection 4.2.

4.1. An estimate.

We first prove a result on the time regularity of global sub-probability solutions of (1.3). It is indeed a variation of a classical result (see [2, Lemma 8.1.2]). However, there is no global integrability of a^{ij} and V^i in our case.

DEFINITION 4.1. *Let $\mathcal{I}$ be an interval and $\mu=(\mu_t)_{t\in\mathcal{I}}$ be a family of Borel measures on $\mathcal{U}$. A* continuous modification *of μ is a family of Borel measures $\tilde\mu=(\tilde\mu_t)_{t\in\mathcal{I}}$ on $\mathcal{U}$ satisfying the following property:*

$$\forall\phi\in C_c^{2,1}(\mathcal{U}\times\mathcal{I}),\ \textit{the function}\ t\mapsto\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde\mu_t\ \textit{is continuous on}\ \mathcal{I},$$

such that $\mu_t=\tilde\mu_t$ for a.e. $t\in\mathcal{I}$.

LEMMA 4.1. *Let $\mathcal{I}$ be an interval and $\mu=(\mu_t)_{t\in\mathcal{I}}$ be a family of Borel measures on $\mathcal{U}$. Then there exists at most one continuous modification of μ.*

Proof. Suppose both $\tilde\mu^1=(\tilde\mu_t^1)_{t\in\mathcal{I}}$ and $\tilde\mu^2=(\tilde\mu_t^2)_{t\in\mathcal{I}}$ are continuous modifications of $\mu=(\mu_t)_{t\in\mathcal{I}}$. By Definition 4.1, $\tilde\mu_t^1=\tilde\mu_t^2$ for a.e. $t\in\mathbb{R}$, and for each $\phi\in C_c^2(\mathcal{U})$, the functions $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^1$ and $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^2$ are continuous on $\mathcal{I}$. It follows that for each $\phi \in C_c^2(\mathcal{U})$, there holds that $\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^1 = \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^2$ for all $t \in \mathcal{I}$. This implies that $\tilde{\mu}_t^1 = \tilde{\mu}_t^2$ for all $t \in \mathcal{I}$. ◻

LEMMA 4.2. *Assume* **(H2)**. *Let $\mu = (\mu_t)_{t\in(s,\infty)}$ be a global subprobability solution of* (1.3). *Then μ admits a unique continuous modification.*

Proof. By Lemma 4.1, we only need to show the existence. We first show that there is a family of subprobability measures $(\tilde{\mu}_t)_{t\in(s,\infty)}$ on $\mathcal{U}$ satisfying the following property:

$$\forall \phi \in C_c^2(\mathcal{U}), \text{ the function } t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t \text{ is continuous on } (s,\infty),$$

such that $\tilde{\mu}_t = \mu_t$ for a.e. $t \in (s,\infty)$.

As $\mu = (\mu_t)_{t\in(s,\infty)}$ is a subprobability solution of (1.3), Lemma 2.1 (1)(a) implies that for any $\phi \in C_c^2(\mathcal{U})$, there exists $J_\phi \subset (s,\infty)$ satisfying $|(s,\infty) \setminus J_\phi| = 0$ such that

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall r,t \in J_\phi \text{ with } r < t.$$

For each $\phi \in C_c^2(\mathcal{U})$, we define a function f_ϕ on J_ϕ by setting $f_\phi(t) := \int_{\mathcal{U}} \phi \mathrm{d}\mu_t$ for $t \in J_\phi$. Since a^{ij} and V^i are locally bounded and T-periodic for each $i,j = 1,\dots,d$, and ϕ is compactly supported in $\mathcal{U}$, the boundedness of $\mathcal{L}\phi$ follows. As a result,

$$\begin{aligned} |f_\phi(t) - f_\phi(r)| &= \left| \int_{\mathcal{U}} \phi \mathrm{d}\mu_t - \int_{\mathcal{U}} \phi \mathrm{d}\mu_r \right| \\ &= \left| \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \right| \le \max_{\mathcal{U}\times[r,t]} |\mathcal{L}\phi| \times (t-r) \quad \forall r,t \in J_\phi \text{ with } r < t. \end{aligned}$$

It follows that there exists a locally Lipschitz continuous function $\tilde{f}_\phi$ on (s,∞) such that $\tilde{f}_\phi(t) = f_\phi(t)$ for $t \in J_\phi$. Obviously, $\tilde{f}_\phi \ge 0$ if $\phi \ge 0$ and $|\tilde{f}_\phi(t)| \le |\phi|_\infty$ for $\phi \in C_c^2(\mathcal{U})$ and $t \in (s,\infty)$.

For each $t \in [s,\infty)$, we define a functional: $K_t : C_c^2(\mathcal{U}) \to \mathbb{R}$, $\phi \mapsto \tilde{f}_\phi(t)$. Obviously, K_t is linear and positive, and $|K_t\phi| \le |\phi|_\infty$. As $C_c^2(\mathcal{U})$ is dense in $C_c(\mathcal{U})$ under the topology of uniform convergence on $\mathcal{U}$, K_t has a unique linear continuous extension $\overline{K}_t$ onto $C_c(\mathcal{U})$ satisfying $\overline{K}_t\phi = K_t\phi$ for all $\phi \in C_c^2(\mathcal{U})$. We see that $\overline{K}_t$ is positive. In fact, for any nonnegative function $\phi \in C_c(\mathcal{U})$, there exists a sequence of nonnegative functions $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^2(\mathcal{U})$ that converges uniformly to ϕ on $\mathcal{U}$. Therefore,

$$\overline{K}_t\phi = \lim_{n\to\infty} K_t\phi_n = \lim_{n\to\infty} \tilde{f}_{\phi_n}(t) \ge 0.$$

Applying the Riesz representation theorem, we find a Borel measure $\tilde{\mu}_t$ on $\mathcal{U}$ such that $\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \overline{K}_t\phi$ for all $\phi \in C_c(\mathcal{U})$. As a consequence, we obtain a family of Borel measures $(\tilde{\mu}_t)_{t\in(s,\infty)}$ on $\mathcal{U}$ satisfying $\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \tilde{f}_\phi(t)$ for all $t \in (s,\infty)$ and $\phi \in C_c^2(\mathcal{U})$. In particular, the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t$ is continuous on (s,∞) for any $\phi \in C_c^2(\mathcal{U})$.

Let $\mathcal{D}$ be a countable basis of $C_c^2(\mathcal{U})$ under the topology of uniform convergence on $\mathcal{U}$, and set $J := \cap_{\phi\in\mathcal{D}} J_\phi$. Clearly, $|(s,\infty) \setminus J| = 0$ and

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \tilde{f}_\phi(t) = f_\phi(t) = \int_{\mathcal{U}} \phi \mathrm{d}\mu_t \quad \forall \phi \in \mathcal{D} \text{ and } t \in J. \tag{4.1}$$

As $C_c^2(\mathcal{U})$ is dense in $C_c(\mathcal{U})$ and $\mathcal{D}$ is dense in $C_c^2(\mathcal{U})$, (4.1) holds for all $\phi \in C_c(\mathcal{U})$ and $t \in J$. Hence, $\tilde{\mu}_t = \mu_t$ and $\tilde{\mu}_t(\mathcal{U}) = \mu_t(\mathcal{U}) \le 1$ for all $t \in J$. From the continuity of the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t$ on (s,∞) for each $\phi \in C_c^2(\mathcal{U})$, we conclude that $\tilde{\mu}_t(\mathcal{U}) \le 1$ for all $t \in (s,\infty)$.

It remains to show that for each $\phi \in C_c^{2,1}(\mathcal{U}\times(s,\infty))$, the function $t \mapsto \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t$ is continuous on (s,∞). For fixed $t \in (s,\infty)$ and $\phi \in C_c^{2,1}(\mathcal{U}\times(s,\infty))$, the local Lipschitz continuity of $\tilde{f}_{\phi(\cdot,t)}$ on (s,∞) implies that

$$\left|\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_r - \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right| = \left|\tilde{f}_{\phi(\cdot,t)}(r) - \tilde{f}_{\phi(\cdot,t)}(t)\right| \to 0 \quad \text{as} \quad r \to t.$$

It follows that

$$\begin{aligned}&\left|\int_{\mathcal{U}} \phi(\cdot,r)\mathrm{d}\tilde{\mu}_r - \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right| \\ &\qquad \le \max_{x\in\mathcal{U}} |\phi(x,r)-\phi(x,t)| + \left|\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_r - \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right| \to 0 \quad \text{as} \quad r \to t.\end{aligned}$$

This proves the required continuity and hence completes the proof. ∎

A similar result can be proven for global subprobability solutions of the Cauchy problem (1.3) and (1.6).

LEMMA 4.3. *Assume* **(H2)**. *Let* $\mu = (\mu_t)_{t\in(s,\infty)}$ *be a global subprobability solution of the Cauchy problem* (1.3) *and* (1.6). *Then* μ *admits a continuous modification* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *satisfying* $\lim_{t\to s^+}\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\nu$.

Proof. The proof follows from arguments as in the proof of Lemma 4.1. The differences are that we use Lemma 2.1 (2)(a) instead of Lemma 2.1 (1)(a) and define $\tilde{f}_\phi$ on $[s,\infty)$ for $\phi \in C_c^2(\mathcal{U})$ with $\mu_s = \nu$. ∎

Remark 4.1. If μ is a global subprobability solution of (1.3) or a global subprobability solution of the Cauchy problem (1.3) and (1.6), so is its continuous modification $\tilde{\mu}$. Moreover, Lemma 2.2 applies in particular to $\tilde{\mu}$. This would allow us to get rid of J_ϕ in many situations in what follows.

The expected estimate is stated in the next result.

PROPOSITION 4.1. *Assume* **(H2)** *and that* $\mathcal{L}$ *admits a Lyapunov function of type* (L3) U. *Let* $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ *be a sequence of open sets in* $\mathcal{U}$ *as in Definition* 1.2 *and* $\mu = (\mu_t)_{t\in(s,\infty)}$ *be a global subprobability solution of the Cauchy problem* (1.3) *and* (1.6) *with* $\int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu < \infty$. *Let* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in(s,\infty)}$ *be the continuous modification of* $\mu = (\mu_t)_{t\in(s,\infty)}$ *given in Lemma* 4.3. *Then* $\tilde{\mu}_t(\mathcal{U}) = 1$ *for all* $t > s$, *and there exists some* $C > 0$, *independent of* s, ν, *and* μ, *such that*

$$C_n \int_s^t \tilde{\mu}_\tau(\mathcal{U}\setminus\mathcal{U}_n)\mathrm{d}\tau + D_n\tilde{\mu}_t(\mathcal{U}\setminus\mathcal{U}_n) \le \int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + C(t-s) \tag{4.2}$$

for all $t > s$ *and* $n \in \mathbb{N}$, *where* $C_n := -\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} \mathcal{L}U > 0$ *and* $D_n := \inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U$.

Proof. For notational simplicity, we write $\tilde{\mu} = (\tilde{\mu}_t)_{t\in(s,\infty)}$ as $\mu = (\mu_t)_{t\in(s,\infty)}$.

We see from Lemma 2.1 (2)(b) and Lemma 2.2 that for each $\phi \in C_c^{2,1}(\mathcal{U}\times[s,\infty))$,

$$\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t = \int_{\mathcal{U}} \phi(\cdot,s)\mathrm{d}\mu_s + \lim_{r\to s}\int_r^t\int_{\mathcal{U}} \mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau \quad \forall t > s. \tag{4.3}$$

Since U is a Lyapunov function of type (L3), there is a $\rho_m > 0$ such that $\mathcal{L}U \le 0$ on $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$. Fix $\rho_0 > \rho_m$, and let $\{\zeta_\rho\}_{\rho>\rho_0}$ be a family of smooth and nondecreasing functions on $\mathbb{R}$ satisfying

$$\zeta_\rho(t) = \begin{cases} 0, & t \in [0, \rho_m], \\ t, & t \in [\rho_0, \rho], \\ \rho + 1, & t \in [\rho + 2, \infty), \end{cases} \qquad \zeta_\rho(t) \le t, \ t \in [\rho_m, \rho_0], \quad \text{and} \quad \zeta_\rho''(t) \le 0, \ t \in [\rho, \rho{+}2].$$

In addition, we let the functions $\{\zeta_\rho\}_{\rho \ge \rho_0}$ coincide on $[0, \rho_0]$.

Clearly, $\zeta_\rho(U) - (\rho+1) \in C_c^{2,1}(\mathcal{U} \times \mathbb{R})$. Setting $\phi = \zeta_\rho(U) - (\rho+1)$ in (4.3) gives

$$\int_{\mathcal{U}} (\zeta_\rho(U) - (\rho+1)) \mathrm{d}\mu_t = \int_{\mathcal{U}} (\zeta_\rho(U) - (\rho+1)) \mathrm{d}\nu + \lim_{r \to s} \int_r^t \int_{\mathcal{U}} \mathcal{L}(\zeta_\rho(U)) \mathrm{d}\mu_\tau \mathrm{d}\tau.$$

It follows from $\mathcal{L}(\zeta_\rho(U)) = \zeta_\rho'(U)\mathcal{L}U + \zeta_\rho''(U) a^{ij}\partial_i U \partial_j U$ that

$$\begin{aligned} \int_{\mathcal{U}} \zeta_\rho(U) \mathrm{d}\mu_t = & \int_{\mathcal{U}} \zeta_\rho(U) \mathrm{d}\nu + (\rho+1) \times [\mu_t(\mathcal{U}) - \nu(\mathcal{U})] \\ & + \lim_{r \to s} \int_r^t \int_{\mathcal{U}} \zeta_\rho'(U) \mathcal{L}U \mathrm{d}\mu_\tau \mathrm{d}\tau + \lim_{r \to s} \int_r^t \int_{\mathcal{U}} \zeta_\rho''(U) a^{ij} \partial_i U \partial_j U \mathrm{d}\mu_\tau \mathrm{d}\tau. \end{aligned} \tag{4.4}$$

Due to (1.4), there exists an $n_0 \in \mathbb{N}$ such that $\Omega_{\rho_0} \subset\subset \mathcal{U}_n \times \mathbb{R}$ for all $n > n_0$. Since $\zeta_\rho' = 0$ on $[0, \rho_m]$, $\zeta_\rho' = 1$ on $[\rho_0, \rho]$, and $\zeta_\rho' \ge 0$ otherwise, we see from $\mathcal{L}U \le 0$ in $(\mathcal{U} \times \mathbb{R}) \setminus \Omega_{\rho_m}$ that $\zeta_\rho'(U)\mathcal{L}U \le [\sup_{(\mathcal{U} \setminus \mathcal{U}_n) \times \mathbb{R}} \mathcal{L}U] \mathbf{1}_{\Omega_\rho \setminus (\mathcal{U}_n \times \mathbb{R})}$. Thus,

$$\begin{aligned} \lim_{r \to s} \int_r^t \int_{\mathcal{U}} \zeta_\rho'(U) \mathcal{L}U \mathrm{d}\mu_\tau \mathrm{d}\tau & \le \sup_{(\mathcal{U} \setminus \mathcal{U}_n) \times \mathbb{R}} \mathcal{L}U \times \lim_{r \to s} \int_r^t \mu_\tau(\Omega_\rho^\tau \setminus \mathcal{U}_n) \mathrm{d}\tau \\ & = -C_n \int_s^t \mu_\tau(\Omega_\rho^\tau \setminus \mathcal{U}_n) \mathrm{d}\tau, \quad n > n_0, \end{aligned} \tag{4.5}$$

where $C_n := -\sup_{(\mathcal{U} \setminus \mathcal{U}_n) \times \mathbb{R}} \mathcal{L}U > 0$ and the monotone convergence theorem is used in the above equality.

As $\zeta_\rho'' \ne 0$ on $[\rho_m, \rho_0]$, $\zeta'' \le 0$ on $[\rho, \rho+2]$, and $\zeta'' = 0$ otherwise, the nonnegative definiteness of (a^{ij}) gives $\zeta_\rho''(U) a^{ij} \partial_i U \partial_j U \le [C_* \max_{\overline{\Omega}_{\rho_0}} a^{ij} \partial_i U \partial_j U] \mathbf{1}_{\Omega_{\rho_0} \setminus \Omega_{\rho_m}}$, where $C_* := \max_{t \in [\rho_m, \rho_0]} \zeta_\rho''(t)$ is independent of ρ due to the construction of $\{\zeta_\rho\}_{\rho > \rho_0}$. Hence, setting $C := C_* \big(\max_{\overline{\Omega}_{\rho_0}} a^{ij} \partial_i U \partial_j U \big)$, we find

$$\int_r^t \int_{\mathcal{U}} \zeta_\rho''(U) a^{ij} \partial_i U \partial_j U \mathrm{d}\mu_\tau \mathrm{d}\tau \le C_* \left(\max_{\overline{\Omega}_{\rho_0}} a^{ij} \partial_i U \partial_j U \right) \times (t - r) = C(t - r). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) yields

$$\begin{aligned} \int_{\mathcal{U}} \zeta_\rho(U) \mathrm{d}\mu_t \le & \int_{\mathcal{U}} \zeta_\rho(U) \mathrm{d}\nu + (\rho+1) \times [\mu_t(\mathcal{U}) - \nu(\mathcal{U})] \\ & - C_n \int_s^t \mu_\tau(\Omega_\rho^\tau \setminus \mathcal{U}_n) \mathrm{d}\tau + C(t - s) \quad \forall t > s. \end{aligned} \tag{4.7}$$

As $\zeta_\rho \ge 0$ and $\zeta_\rho(t) = t$ for $t \in [\rho_0, \rho]$, we derive from $\Omega_{\rho_0} \subset \mathcal{U}_n \times \mathbb{R}$ that

$$\int_{\mathcal{U}} \zeta_\rho(U) \mathrm{d}\mu_t \ge \int_{\Omega_\rho^t \setminus \Omega_{\rho_0}^t} U \mathrm{d}\mu_t \ge \int_{\Omega_\rho^t \setminus \mathcal{U}_n} U \mathrm{d}\mu_t \ge D_n \mu_t(\Omega_\rho^t \setminus \mathcal{U}_n) \quad \forall n > n_0, \tag{4.8}$$

where $D_n := \inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U$. As $\zeta_\rho(t) \le t$ for $t \ge 0$ and $\rho > \rho_0$, we find from (4.7) and (4.8) that

$$
\begin{aligned}
D_n\mu_t(\Omega_\rho^t \setminus \mathcal{U}_n) \le & \int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu + (\rho+1) \times [\mu_t(\mathcal{U}) - \nu(\mathcal{U})] \\
& - C_n \int_s^t \mu_\tau(\Omega_\rho^\tau \setminus \mathcal{U}_n)\mathrm{d}\tau + C(t-s) \quad \forall t > s.
\end{aligned} \tag{4.9}
$$

Note that the ν-integrability of $U(\cdot, s)$ ensures the nontriviality of the above inequalities. If $\mu_t(\mathcal{U}) < \nu(\mathcal{U}) = 1$ for some $t > s$, we deduce from (4.9) that

$$
0 \le \int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu + (\rho+1) \times (\mu_t(\mathcal{U}) - \nu(\mathcal{U})) + C(t-s) \to -\infty \quad \text{as} \quad \rho \to \infty,
$$

which leads to a contradiction. Therefore, $\mu_t(\mathcal{U}) = \nu(\mathcal{U}) = 1$ for all $t > s$. Consequently, letting $\rho \to \infty$ in (4.9) leads to

$$
C_n \int_s^t \mu_\tau(\mathcal{U} \setminus \mathcal{U}_n)\mathrm{d}\tau + D_n\mu_t(\mathcal{U} \setminus \mathcal{U}_n) \le \int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu + C(t-s) \quad \forall t > s.
$$

This completes the proof. ∎

Remark 4.2. A priori estimates as in Proposition 4.1 have been established in [26, Chapter 3, Theorem 3.8] in the case of smooth coefficients. The proof in [26] follows from a direct application of Dynkin's/Itô's formula to the Lyapunov function of type (L3) U. However, the situation in the setting of weak coefficients is much worse, as only the definition of global subprobability solutions and their equivalent formalisms can be used. As a result, we have to truncate the Lyapunov function U in order to use the definition and establish estimates in order to pass to the limit.

We point out that a similar situation is encountered in the proof of Lemma 5.3.

4.2. Proof of Theorem B. We recall the definition of the weak*-topology for Borel measures on $\mathcal{U} \times \mathbb{R}$.

DEFINITION 4.2. *A sequence of σ-finite Borel measures $\{\mu^n, n \in \mathbb{N}\}$ on $\mathcal{U} \times \mathbb{R}$ is said to converge to a σ-finite Borel measure μ on $\mathcal{U} \times \mathbb{R}$ under the* weak*-topology *as $n \to \infty$ if*

$$
\lim_{n\to\infty} \iint_{\mathcal{U}\times\mathbb{R}} \phi \mathrm{d}\mu^n = \iint_{\mathcal{U}\times\mathbb{R}} \phi \mathrm{d}\mu \quad \forall \phi \in C_0(\mathcal{U} \times \mathbb{R}).
$$

Set $C_{c,T}(\mathcal{U} \times \mathbb{R}) := C_c(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$.

Proof of Theorem B. For clarity, we assume $s = 0$. Applying Lemma 4.3, we may replace $\mu = (\mu_t)_{t\in(0,\infty)}$ by its continuous modification, still denoted by $\mu = (\mu_t)_{t\in(0,\infty)}$. Since $U(\cdot, 0)$ is ν-integrable, Proposition 4.1 yields the existence of some $C > 0$ such that

$$
C_n \int_0^t \mu_\tau(\mathcal{U} \setminus \mathcal{U}_n)\mathrm{d}\tau + D_n\mu_t(\mathcal{U} \setminus \mathcal{U}_n) \le \int_{\mathcal{U}} U(\cdot, 0)\mathrm{d}\nu + Ct, \quad t > 0, \tag{4.10}
$$

where $\mathcal{U}_n$, C_n, and D_n are as in the statement of Proposition 4.1.

For each $n \in \mathbb{N}$, we define $\mu_t^n := \frac{1}{n}\sum_{k=0}^{n-1} \mu_{t+kT}$ for $t > 0$ and $\mathrm{d}\mu^n := \mathrm{d}\mu_t^n \mathrm{d}t$ on $\mathcal{U} \times (0, \infty)$. It is easy to see that for any bounded $\phi \in C_T(\mathcal{U} \times \mathbb{R})$,

$$
\int_t^{t+T} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau^n \mathrm{d}\tau = \frac{1}{n}\int_t^{t+nT} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \quad \forall t > 0. \tag{4.11}
$$

Let $\{n_j\}_{j\in\mathbb{N}}\subset\mathbb{N}$ be fixed. The proof is finished in seven steps. In Steps 1–5, we construct a periodic probability solution of (1.3). The convergence results are proven in Steps 6 and 7.

Step 1. We show the existence of a subsequence of $\{n_j\}_{j\in\mathbb{N}}$, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, such that μ^{n_j} converges under the weak*-topology to some Borel measure $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$, and for each $t>0$, there holds that

$$\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}\phi \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=\iint_{\mathcal{U}\times[t,t+T]}\phi \mathrm{d}\tilde{\mu}\quad\forall\text{ bounded }\phi\in C(\mathcal{U}\times[t,t+T]). \tag{4.12}$$

For any compact set $K\subset\mathcal{U}\times(0,\infty)$, there holds that $\sup_{j\in\mathbb{N}}\mu^{n_j}(K)<\infty$. Applying [14, Corollary A2.6.V], we conclude the existence of a subsequence of $\{n_j\}_{j\in\mathbb{N}}$, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, such that μ^{n_j} converges under the weak*-topology to some Borel measure $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$.

To show (4.12), we may apply [12, Theorem 4.4.2], which says, in particular, that it is equivalent to showing the following:

(i) $\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}f\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=\iint_{\mathcal{U}\times[t,t+T]}f\mathrm{d}\tilde{\mu}$ for each $f\in C_0(\mathcal{U}\times[t,t+T])$;

(ii) $\tilde{\mu}(\mathcal{U}\times[t,t+T])=T$.

We prove (i) and (ii) in the rest of Step 1.

(i) Note that for any $f\in C_0(\mathcal{U}\times[t,t+T])$, there exist an $\epsilon_0\in(0,1)$ and a family of functions $\{f_\epsilon\}_{\epsilon\in(0,\epsilon_0)}\subset C_0(\mathcal{U}\times(0,\infty))$ satisfying the following:

- $|f_\epsilon|_\infty\le|f|_\infty$ for all $\epsilon\in(0,\epsilon_0)$;
- for each $\epsilon\in(0,\epsilon_0)$, $f_\epsilon=\begin{cases}f & \text{on}\quad \mathcal{U}\times[t,t+T],\\ 0 & \text{on}\quad \mathcal{U}\times(0,t-\epsilon]\cup[t+\epsilon+T,\infty).\end{cases}$

Clearly, $\lim_{\epsilon\to0}f_\epsilon(x,\tau)=f(x,\tau)\mathbb{1}_{[t,t+T]}(\tau)$ for $(x,\tau)\in\mathcal{U}\times(0,\infty)$. As μ^{n_j} converges to $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$ under the weak*-topology, there holds that

$$\lim_{j\to\infty}\int_0^\infty\int_{\mathcal{U}}f_\epsilon\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=\iint_{\mathcal{U}\times(0,\infty)}f_\epsilon\mathrm{d}\tilde{\mu}\quad\forall\epsilon\in(0,\epsilon_0). \tag{4.13}$$

The construction of $\{f_\epsilon\}_{\epsilon\in(0,\epsilon_0)}$ ensures that

$$\int_0^\infty\int_{\mathcal{U}}f_\epsilon\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau-2\epsilon|f|_\infty\le\int_t^{t+T}\int_{\mathcal{U}}f\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\le\int_0^\infty\int_{\mathcal{U}}f_\epsilon\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau+2\epsilon|f|_\infty$$

for all $\epsilon\in(0,\epsilon_0)$. Letting $j\to\infty$ in the above inequality, we find from (4.13)

$$\begin{aligned}\iint_{\mathcal{U}\times(0,\infty)}f_\epsilon\mathrm{d}\tilde{\mu}-2\epsilon|f|_\infty&\le\liminf_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}f\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\\&\le\limsup_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}f\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\le\iint_{\mathcal{U}\times(0,\infty)}f_\epsilon\mathrm{d}\tilde{\mu}+2\epsilon|f|_\infty.\end{aligned}$$

Since $\lim_{\epsilon\to0}\iint_{\mathcal{U}\times(0,\infty)}f_\epsilon\mathrm{d}\tilde{\mu}=\iint_{\mathcal{U}\times[t,t+T]}f\mathrm{d}\tilde{\mu}$ thanks to the dominated convergence theorem, passing to the limit $\epsilon\to0$ in the above inequalities yields (i).

(ii) It follows from the definition of $\{\mu^{n_j}\}_{j\in\mathbb{N}}$ that

$$\mu^{n_j}((\mathcal{U}\setminus\mathcal{U}_m)\times[t,t+T])=\frac{1}{n_j}\int_t^{t+n_jT}\mu_\tau(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau\quad\forall m\in\mathbb{N}\text{ and }j\in\mathbb{N}.$$

By (4.10), $\frac{1}{n_j}\int_0^{t+n_jT}\mu_\tau(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau\le\frac{1}{n_jC_m}\left(\int_{\mathcal{U}}U(\cdot,0)\mathrm{d}\nu+C\times(t+n_jT)\right)$ holds for all $m\in\mathbb{N}$ and $j\in\mathbb{N}$, where we recall that $C_m=-\sup_{(\mathcal{U}\setminus\mathcal{U}_m)\times\mathbb{R}}\mathcal{L}U\to\infty$. As a result, for any $0<\epsilon\ll 1$, there exists an $m_0=m_0(\epsilon)\in\mathbb{N}$ such that $\mu^{n_j}((\mathcal{U}\setminus\mathcal{U}_m)\times[t,t+T])\leq\epsilon$ for all $m\geq m_0$ and $j\in\mathbb{N}$. Equivalently, $\mu^{n_j}(\mathcal{U}_m\times[t,t+T])\geq T-\epsilon$ for all $m\geq m_0$ and $j\in\mathbb{N}$. This means, $\{\mu^{n_j}\}$, when restricted on $\mathcal{U}\times[t,t+T]$, is tight. Hence, we apply the Portmanteau theorem to find

$$\tilde{\mu}(\overline{\mathcal{U}}_m\times[t,t+T])\geq\limsup_{j\to\infty}\mu^{n_j}(\overline{\mathcal{U}}_m\times[t,t+T])\geq T-\epsilon\quad\forall m\geq m_0.$$

Letting $\epsilon\to 0$, we conclude that $\tilde{\mu}(\mathcal{U}\times[t,t+T])\geq T$.

By (i), we deduce $\tilde{\mu}(\mathcal{U}_m\times[t,t+T])\leq\liminf_{j\to\infty}\mu^{n_j}(\mathcal{U}_m\times[t,t+T])\leq T$, which implies $\tilde{\mu}(\mathcal{U}\times[t,t+T])\leq T$. Hence, $\tilde{\mu}(\mathcal{U}\times[t,t+T])=T$, and (ii) follows.

Step 2. We show that the measure $\tilde{\mu}$ obtained in Step 1 admits t-sections. More precisely, we show the existence of a family of Borel measures $\{\tilde{\mu}_t\}_{t\in(0,\infty)}$ on $\mathcal{U}$ satisfying $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ and $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$ such that $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$.

Let $\eta\in C_c(0,\infty)$ and $|\mathrm{supp}(\eta)|\leq T$. Setting $\phi=\eta$ in (4.12) gives

$$\iint_{\mathcal{U}\times(0,\infty)}\eta\mathrm{d}\tilde{\mu}=\lim_{j\to\infty}\int_0^\infty\int_{\mathcal{U}}\eta\mathrm{d}\mu_t^{n_j}\mathrm{d}t=\int_0^\infty\eta\mathrm{d}t. \tag{4.14}$$

Arguing as in the proof of [22, Lemma 4.2], we derive from (4.14) the existence of a family of Borel measures $\{\tilde{\mu}_t\}_{t\in(0,\infty)}$ satisfying $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$ such that $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$.

It remains to show that $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for a.e. $t>0$. It follows from (4.11) that for any $\phi\in C_{c,T}(\mathcal{U}\times\mathbb{R})$ and $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, there holds that

$$\begin{aligned}\int_{t_1}^{t_1+T}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau&=\frac{1}{n_j}\int_{t_1}^{t_1+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\\&=\frac{1}{n_j}\left(\int_{t_1}^{t_2}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau-\int_{t_1+n_jT}^{t_2+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau+\int_{t_2}^{t_2+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\right)\\&=\frac{1}{n_j}\left(\int_{t_1}^{t_2}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau-\int_{t_1+n_jT}^{t_2+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\right)+\int_{t_2}^{t_2+T}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau.\end{aligned}$$

Letting $j\to\infty$ in the above equality, we find from (4.12) that $\iint_{\mathcal{U}\times[t_1,t_1+T]}\phi\mathrm{d}\tilde{\mu}=\iint_{\mathcal{U}\times[t_2,t_2+T]}\phi\mathrm{d}\tilde{\mu}$ for all $0<t_1<t_2$. We then argue as in the proof of [22, Lemma 4.1] to find $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for a.e. $t>0$.

Step 3. We show that $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3).

We claim that for each $t>0$, there holds that

$$\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\tilde{\mu}_\tau\mathrm{d}\tau=0\quad\forall\phi\in C_0^{2,1}(\mathcal{U}\times(t,t+T)). \tag{4.15}$$

Fix $t\in\mathbb{R}$. For any $\phi\in C_0^{2,1}(\mathcal{U}\times(t,t+T))$ and $k\in\mathbb{N}\cup\{0\}$, we define

$$\phi_k(x,\tau):=\phi(x,\tau-kT)\mathbf{1}_{\mathcal{U}\times(t+kT,t+(k+1)T)}(x,\tau),\quad(x,\tau)\in\mathcal{U}\times(0,\infty).$$

Obviously, $\phi_k\in C_0^{2,1}(\mathcal{U}\times(0,\infty))$ for each $k\in\mathbb{N}\cup\{0\}$. As $\mu=(\mu_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3), $\int_{t+kT}^{t+(k+1)T}\int_{\mathcal{U}}\mathcal{L}\phi_k\mathrm{d}\mu_\tau\mathrm{d}\tau=\int_0^\infty\int_{\mathcal{U}}\mathcal{L}\phi_k\mathrm{d}\mu_\tau\mathrm{d}\tau=0$. This together with the T-periodicity of (a^{ij}) and (V^i) gives $\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_{\tau+kT}\mathrm{d}\tau=$

$\int_{t+kT}^{t+(k+1)T}\int_{\mathcal{U}}\mathcal{L}\phi_k \mathrm{d}\mu_\tau \mathrm{d}\tau=0$, which yields

$$\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_\tau^n \mathrm{d}\tau=\frac{1}{n}\sum_{k=0}^{n-1}\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_{\tau+kT}\mathrm{d}\tau=0 \quad \forall n\in\mathbb{N}. \tag{4.16}$$

As $\mathcal{L}\phi\in C_0(\mathcal{U}\times(t,t+T))$, we deduce from (4.16) and (4.12) that $\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau=\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=0$. This proves (4.15).

From (4.15) and Lemma 2.1 (1)(a), we find that for each $t>0$ and $\phi\in C_c^2(\mathcal{U})$, there exists a subset $J_\phi^t\subset(t,t+T)$ satisfying $|(t,t+T)\setminus J_\phi^t|=0$ such that

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall t_1,t_2\in J_\phi^t \text{ with } t_1<t_2.$$

As t is arbitrary in $(0,\infty)$, there exists $J_\phi\subset(0,\infty)$ with $|(0,\infty)\setminus J_\phi|=0$ such that

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall t_1,t_2\in J_\phi \text{ with } t_1<t_2.$$

That is, $\tilde{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times(0,\infty)$. As $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$ by Step 2, $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3).

Step 4. We show that $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$ admits a continuous modification, still denoted by $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, such that $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ and $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$.

By Lemma 4.2, there is a modification of $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, still denoted by $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, satisfying the following property: the function $t\mapsto\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_t$ is continuous on $(0,\infty)$ for all $\phi\in C_c^2(\mathcal{U})$. This together with the fact that $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for a.e. $t>0$ (from Step 2) yields $\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_t=\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t+T}$ for all $\phi\in C_c^2(\mathcal{U})$ and $t>0$. Hence, $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for all $t>0$.

It remains to show that $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$. Note that up to now we only know $\tilde{\mu}_t(\mathcal{U})\le 1$ for all $t>0$ and $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$. Fix $t_0>0$. Since U is a Lyapunov function of type (L3), we can follow the arguments as in the proof of [28, Proposition 2.8] (see the proof of [22, Theorem A] for more details) to find a nonnegative function $\tilde{U}\in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ satisfying the following properties:

(1) $\int_{\mathcal{U}}\tilde{U}(\cdot,t_0)\mathrm{d}\tilde{\mu}_{t_0}<\infty$.

(2) $\lim_{n\to\infty}\inf_{x\in\mathcal{U}\setminus\mathcal{U}_n}\tilde{U}(x,t)=\infty$ for all $t\in\mathbb{R}$.

(3) There is a $\tilde{\rho}_m>0$ such that $\mathcal{L}\tilde{U}\le 0$ on $(\mathcal{U}\times\mathbb{R})\setminus\tilde{\Omega}_{\rho_m}$, where $\tilde{\Omega}_{\rho_m}:=\{(x,t)\in\mathcal{U}\times\mathbb{R}:\tilde{U}(x,t)<\tilde{\rho}_m\}$.

We see from Lemma 2.1 (1)(b) and Lemma 2.2 that for each $\phi\in C_c^{2,1}(\mathcal{U}\times(0,\infty))$,

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall 0<t_1<t_2. \tag{4.17}$$

Note that $\zeta_\rho(\tilde{U})-(\rho+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$ thanks to property (2), where $\{\zeta_\rho\}$ is defined as in the proof of Proposition 4.1. Setting $\phi=\zeta_\rho(\tilde{U})-(\rho+1)$ in (4.17), we find

$$\int_{\mathcal{U}}(\zeta_\rho(\tilde{U})-(\rho+1))\mathrm{d}\tilde{\mu}_t=\int_{\mathcal{U}}(\zeta_\rho(\tilde{U})-(\rho+1))\mathrm{d}\tilde{\mu}_{t_0} + \int_{t_0}^t\int_{\mathcal{U}}\left[\zeta_\rho'(\tilde{U})\mathcal{L}\tilde{U}+\zeta_\rho''(\tilde{U})a^{ij}\partial_i\tilde{U}\partial_j\tilde{U}\right]\mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall t>t_0.$$

Arguing as in the proof of Proposition 4.1 yields the existence of $C>0$ such that

$$0 \le \int_{\mathcal{U}} \tilde{U}(\cdot,t_0)\mathrm{d}\tilde{\mu}_{t_0} + (\rho+1)\times[\tilde{\mu}_t(\mathcal{U})-\tilde{\mu}_{t_0}(\mathcal{U})] + C(t-t_0) \quad \forall t>t_0.$$

Since $\tilde{U}(\cdot,t_0)$ is $\tilde{\mu}_{t_0}$-integrable, if $\tilde{\mu}_t(\mathcal{U})<\tilde{\mu}_{t_0}(\mathcal{U})$ for some $t>t_0$, a contradiction is readily derived by letting $\rho\to\infty$ in the above inequality. As a result, $\tilde{\mu}_t(\mathcal{U})\ge\tilde{\mu}_{t_0}(\mathcal{U})$ for all $t>t_0$. Since $t_0>0$ is arbitrary and $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$, we conclude that $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$.

Step 5. We extend $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, the continuous modification obtained in Step 4, to obtain a periodic probability solution $\hat{\mu}=(\hat{\mu}_t)_{t\in\mathbb{R}}$ of (1.3).

To do so, we define

$$\hat{\mu}_t = \begin{cases} \tilde{\mu}_t, & t>0,\\ \tilde{\mu}_{t+kT}, & t\in(-kT,-(k-1)T] \text{ and } k\in\mathbb{N}. \end{cases}$$

Obviously, $\hat{\mu}_t(\mathcal{U})=1$ and $\hat{\mu}_t=\hat{\mu}_{t+T}$ for all $t\in\mathbb{R}$. Thus, $\hat{\mu}:=(\hat{\mu}_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (1.3) if we can show that $\hat{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times\mathbb{R}$.

As $\tilde{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times(0,\infty)$, the definition of $\hat{\mu}$ implies that for any $\phi\in C_c^2(\mathcal{U})$ and $k\in\mathbb{N}$, there holds that

(4.18)
$$\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_{t_1}-\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_{t_2}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau \ \forall t_1,t_2\in(-kT,-(k-1)T] \text{ with } t_1<t_2.$$

As $k\in\mathbb{N}$ is arbitrary and $\hat{\mu}=\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$, we see that (4.18) holds for all $t_1,t_2\in\mathbb{R}$ with $t_1<t_2$. That is, $\hat{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times\mathbb{R}$.

Step 6. We show that for any bounded $\phi\in C_T(\mathcal{U}\times\mathbb{R})$, there holds that

$$\lim_{j\to\infty}\frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau=\frac{1}{T}\int_0^T\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau \quad \forall t\ge 0. \tag{4.19}$$

It follows from (4.11), (4.12), and the definition of $\hat{\mu}=(\hat{\mu}_t)_{t\in\mathbb{R}}$ that for each bounded $\phi\in C_T(\mathcal{U}\times\mathbb{R})$, $t\ge0$, and $\epsilon>0$,

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau &= \lim_{j\to\infty}\frac{1}{n_jT}\left[\int_t^{t+\epsilon}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau-\int_{t+n_jT}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\right]\\ &\quad+\lim_{j\to\infty}\frac{1}{n_jT}\int_{t+\epsilon}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\\ &=\lim_{j\to\infty}\frac{1}{n_jT}\int_{t+\epsilon}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\\ &=\lim_{j\to\infty}\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau.\end{aligned}$$

The T-periodicity of ϕ and $\hat{\mu}=(\hat{\mu}_t)_{t\in\mathbb{R}}$ then ensures that $\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau=\frac{1}{T}\int_0^T\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau$ for all $t\ge0$ and $\epsilon>0$. Hence, (4.19) follows.

Step 7. We show that for any $\psi\in C_c^2(\mathcal{U})$, there holds that

$$\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi\mathrm{d}\mu_{t+kT}=\int_{\mathcal{U}}\psi\mathrm{d}\hat{\mu}_t \quad \forall t>0. \tag{4.20}$$

Fix $\psi \in C_c^2(\mathcal{U})$. Clearly, $\mathcal{L}\psi$ is bounded on $\mathcal{U} \times \mathbb{R}$. Since $\mu = (\mu_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3), Lemma 2.1 (1)(a) and Lemma 2.2 imply that

$$\begin{aligned}\left|\int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_1} - \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_2}\right| &\le \int_{t_1}^{t_2} \int_{\mathcal{U}} |\mathcal{L}\psi| \mathrm{d}\mu_\tau \mathrm{d}\tau \\ &\le \max_{\mathrm{supp}(\psi)\times\mathbb{R}} |\mathcal{L}\psi| \times (t_2 - t_1) \quad \forall 0 < t_1 < t_2.\end{aligned} \tag{4.21}$$

Fix $t_0 \in (0, T)$, and let $\eta \in C_c^\infty(\mathbb{R})$ be nonnegative and satisfy $\mathrm{supp}(\eta) \subset [-1, 1]$ and $\int_{\mathbb{R}} \eta \mathrm{d}t = 1$. Define $\eta_\epsilon(t) := \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})$ for $t \in \mathbb{R}$ and $0 < \epsilon \ll 1$. Clearly, $\int_{\mathbb{R}} \eta_\epsilon \mathrm{d}t = 1$ for $0 < \epsilon \ll 1$. It follows that for all $t > 0$ and $0 < \epsilon \ll 1$,

$$\begin{aligned}&\left|\int_t^{t+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \eta_\epsilon(\tau - (t + t_0))\mathrm{d}\tau - \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+t_0}\right| \\ &\quad\le \int_t^{t+T} \left|\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau - \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+t_0}\right| \eta_\epsilon(\tau - (t + t_0))\mathrm{d}\tau \le \epsilon \times \max_{\mathrm{supp}(\psi)\times\mathbb{R}} |\mathcal{L}\psi|,\end{aligned}$$

where we used (4.21). Equivalently, for all $t > 0$ and $0 < \epsilon \ll 1$, there holds that

$$\begin{aligned}&\int_t^{t+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \eta_\epsilon(\tau - (t + t_0))\mathrm{d}\tau - C\epsilon \\ &\quad\le \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+t_0} \le \int_t^{t+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \eta_\epsilon(\tau - (t + t_0))\mathrm{d}\tau + C\epsilon,\end{aligned} \tag{4.22}$$

where $C = \max_{\mathrm{supp}(\psi)\times\mathbb{R}} |\mathcal{L}\psi|$.

Fix $t_1 > 0$. For each fixed n_j, setting $t = t_1 + kT$ for $k = 0, \ldots, n_j - 1$ in (4.22) and then summarizing the resulting inequalities, we arrive at

$$\begin{aligned}&\sum_{k=0}^{n_j-1} \int_{t_1+kT}^{t_1+(k+1)T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \eta_\epsilon\left(\tau - (t_0 + t_1 + kT)\right)\mathrm{d}\tau - n_j C\epsilon \\ &\quad\le \sum_{k=0}^{n_j-1} \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_0+t_1+kT} \\ &\quad\le \sum_{k=0}^{n_j-1} \int_{t_1+kT}^{t_1+(k+1)T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \eta_\epsilon(\tau - (t_0 + t_1 + kT))\mathrm{d}\tau + n_j C\epsilon, \quad 0 < \epsilon \ll 1.\end{aligned} \tag{4.23}$$

For each $\epsilon > 0$, we define a function $\tilde{\eta}_\epsilon$ on $\mathbb{R}$ by setting

$$\tilde{\eta}_\epsilon(t) = \eta_\epsilon(t - (t_0 + t_1 + kT)), \quad t \in [t_1 + kT, t_1 + (k + 1)T) \text{ and } k \in \mathbb{Z}.$$

Obviously, $\tilde{\eta}_\epsilon$ is T-periodic and $\tilde{\eta}_\epsilon \in C_c^\infty(\mathbb{R})$ for each $0 < \epsilon \ll 1$. Setting $\phi(x, t) := \psi(x)\tilde{\eta}_\epsilon(t)$ for $(x, t) \in \mathcal{U} \times \mathbb{R}$ in (4.19) gives

$$\lim_{j\to\infty} \frac{1}{n_j T} \int_{t_1}^{t_1+n_jT} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau\right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau = \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau\right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau. \tag{4.24}$$

It follows that

$$\begin{aligned}
&\lim_{j\to\infty}\frac{1}{n_jT}\sum_{k=0}^{n_j-1}\int_{t_1+kT}^{t_1+(k+1)T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon(\tau-(t_0+t_1+kT))\mathrm{d}\tau\\
&\quad=\lim_{j\to\infty}\frac{1}{n_jT}\sum_{k=0}^{n_j-1}\int_{t_1+kT}^{t_1+(k+1)T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau\\
&\quad=\lim_{j\to\infty}\frac{1}{n_jT}\int_{t_1}^{t_1+n_jT}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau=\frac{1}{T}\int_{t_1}^{t_1+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau.
\end{aligned}$$

Dividing (4.23) by n_jT and then letting $j\to\infty$, we derive from the above limit that

$$\begin{aligned}
\frac{1}{T}\int_{t_1}^{t_1+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau-\frac{C\epsilon}{T}&\le\liminf_{j\to\infty}\frac{1}{n_jT}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_0+t_1+kT}\\
&\le\limsup_{j\to\infty}\frac{1}{n_jT}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_0+t_1+kT}\le\frac{1}{T}\int_{t_1}^{t_1+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau+\frac{C\epsilon}{T}
\end{aligned}$$

for all $0<\epsilon\ll 1$. Since the continuity of the function $t\mapsto\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_t$ on $\mathbb{R}$ ensures $\lim_{\epsilon\to 0}\frac{1}{T}\int_{t_1}^{t_1+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_\tau\right)\tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau=\frac{1}{T}\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_{t_0+t_1}$, we pass to the limit $\epsilon\to 0$ in the above inequalities to find $\lim_{j\to\infty}\frac{1}{n_jT}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_0+t_1+kT}=\frac{1}{T}\int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_{t_0+t_1}$ for all $t_0\in(0,T)$ and $t_1>0$. This proves (4.20).

If the periodic probability solution of (1.3) is unique, then it is clear that (4.19) and (4.20) hold for the whole sequence $\{\mu^n\}_{n\in\mathbb{N}}$. This completes the proof. ∎

We sketch the proof of Corollary B.

Proof of Corollary B. The proof is almost the same as that of Theorem B. The main difference lies in Step 7. More precisely, in this situation, we apply Theorem A to find that $\tilde{\mu}=(\tilde{\mu})_{t\in\mathbb{R}}$ is the unique periodic probability solution of (1.3) and admits a continuous density. Therefore, the function $t\mapsto\int_{\mathcal{U}}\psi \mathrm{d}\tilde{\mu}_t$ is continuous on $\mathbb{R}$ for any $\psi\in C_b(\mathcal{U})$. Hence, the result follows from arguments as in Step 7. ∎

5. Proof of Theorem C.

The main idea of proving Theorem C is to apply Harris's theorem (see, e.g., [17, Theorem 3.6]) to a map induced by transition probability densities associated to global probability solutions of the Cauchy problem (1.3) and (1.6). Such transition probability densities are constructed and studied in subsection 5.1. We prove Theorem C in subsection 5.2.

Throughout this section, we assume **(H1)** and **(H3)** and that $\mathcal{L}$ admits a Lyapunov function of type (L4) U. Hence, Theorems 2.3 and A hold. Moreover, we denote $M_b(\mathcal{U})$ as the collection of all bounded measurable functions on $\mathcal{U}$ and write $\langle\mu,\phi\rangle:=\int_{\mathcal{U}}\phi \mathrm{d}\mu$ for $\mu\in\mathcal{M}_p(\mathcal{U})$ and $\phi\in M_b(\mathcal{U})$, where we recall that $\mathcal{M}_p(\mathcal{U})$ is the set of all Borel probability measures on $\mathcal{U}$.

5.1. Transition probability densities.

For fixed $s\in\mathbb{R}$ and $x\in\mathcal{U}$, let $\mu^{s,x}$ be the unique global probability solution of the Cauchy problem (1.3) and (1.6) with $\nu=\delta_x$ given in Theorem 2.3. Following Theorem 2.1, $\mu^{s,x}$ admits a Hölder continuous density $(y,t)\mapsto p(s,x,t,y)$ on $\mathcal{U}\times(s,\infty)$. We prove some properties of $p(s,x,t,y)$ in the rest of this subsection.

LEMMA 5.1. *The following hold:*

(1) *For $s<t$ and $y\in\mathcal{U}$, the function $x\mapsto p(s,x,t,y)$ is continuous on $\mathcal{U}$.*

(2) *For $s\in\mathbb{R}$, the function $(x,t,y)\mapsto p(s,x,t,y)$ is measurable on $\mathcal{U}\times(s,\infty)\times\mathcal{U}$.*

Proof. (1) Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{U}$ converge to some $x_*\in\mathcal{U}$ as $n\to\infty$. We show that

$$\lim_{n\to\infty}p(s,x_n,t,y)=p(s,x_*,t,y)\quad\forall(y,t)\in\mathcal{U}\times(s,\infty). \tag{5.1}$$

Set $u_n(y,t):=p(s,x_n,t,y)$, $(y,t)\in\mathcal{U}\times(s,\infty)$ and $\mathrm{d}\mu^n=\mathrm{d}\mu_t^n\mathrm{d}t:=u_n(y,t)\mathrm{d}y\mathrm{d}t$. Note that μ^n is nothing but μ^{s,x_n}.

By Theorem 2.1, we see that for any $\mathcal{V}\subset\subset\mathcal{U}$ and $t_1,t_2\in(s,\infty)$ with $t_1<t_2$, there exists a $C>0$, independent of n, such that $|u_n|_{C^{\alpha-\frac{1}{p}}([t_1,t_2],C^\gamma(\overline{\mathcal{V}}))}\leq C$ for all $n\in\mathbb{N}$. Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is precompact under the topology of locally uniform convergence on $\mathcal{U}\times(s,\infty)$ thanks to the Arzelà–Ascoli theorem and the standard diagonal argument. In particular, any subsequence of $\{u_n\}_{n\in\mathbb{N}}$ has a further subsequence that is locally uniformly convergent on $\mathcal{U}\times(s,\infty)$.

Let us fix a subsequence $\{u_{n_j}\}$ that converges locally uniformly to some nonnegative continuous function u on $\mathcal{U}\times(s,\infty)$. We show that the Borel measure μ defined by $\mathrm{d}\mu=\mathrm{d}\mu_t\mathrm{d}t:=u(y,t)\mathrm{d}y\mathrm{d}t$ coincides with μ^{s,x_*}. That is, for any $\phi\in C_c^2(\mathcal{U})$,

$$\int_{\mathcal{U}}\phi\mathrm{d}\mu_t=\phi(x_*)+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\quad\forall t>s. \tag{5.2}$$

As μ^{n_j} is the global probability solution of the Cauchy problem (1.3) and (1.6) with $\nu=\delta_{x_{n_j}}$, we apply Lemma 2.1 (1)(a) and Lemma 2.2 to find for any $\phi\in C_c^2(\mathcal{U})$ and $t>r>s$, $\int_{\mathcal{U}}\phi\mathrm{d}\mu_t^{n_j}=\int_{\mathcal{U}}\phi\mathrm{d}\mu_r^{n_j}+\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau$, which is rewritten as

$$\int_{\mathcal{U}}\phi\mathrm{d}\mu_t^{n_j}-\phi(x_{n_j})-\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=\int_{\mathcal{U}}\phi\mathrm{d}\mu_r^{n_j}-\phi(x_{n_j}).$$

Following the proof of [28, Theorem 2.3], we see that for fixed $t_0>s$ and $\phi\in C_c^2(\mathcal{U})$ there exist $C_1>0$ and $\alpha>0$, independent of n, such that $\left|\int_{\mathcal{U}}\phi\mathrm{d}\mu_t^n-\phi(x_n)\right|\leq C_1|t-s|^\alpha$ for all $t\in(s,t_0)$ and $n\in\mathbb{N}$. Hence, for any $t>s$, there holds that

$$\begin{aligned}&\left|\int_{\mathcal{U}}\phi\mathrm{d}\mu_t^{n_j}-\phi(x_{n_j})-\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\right|\\&\qquad\leq\left|\int_{\mathcal{U}}\phi\mathrm{d}\mu_r^{n_j}-\phi(x_{n_j})\right|\leq C_1|r-s|^\alpha\quad\forall s<r<\min\{t,t_0\}\text{ and }j\in\mathbb{N}.\end{aligned}$$

Letting $j\to\infty$, we find $\left|\int_{\mathcal{U}}\phi\mathrm{d}\mu_t-\phi(x_*)-\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\right|\leq C|r-s|^\alpha$ for all $s<r<\min\{t,t_0\}$. Letting $r\to s$ in the above inequality yields (5.2).

Since the above result holds for any locally uniformly convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$, the sequence u_n converges locally uniformly to $p(s,x_*,\cdot,\cdot)$ as $n\to\infty$. In particular, (5.1) follows.

(2) In addition to (1), we know that for each $s\in\mathbb{R}$ and $x\in\mathcal{U}$, the function $(y,t)\mapsto p(s,x,t,y)$ is continuous on $\mathcal{U}\times(s,\infty)$. Hence, the function $(x,t,y)\mapsto p(s,x,t,y)$ is a Carathéodory function on $\mathcal{U}\times(s,\infty)\times\mathcal{U}$ and its measurability follows from [1, Lemma 4.51]. ∎

LEMMA 5.2. *Let $\mu=(\mu_t)_{t\in(s,\infty)}$ be the unique global probability solution of the Cauchy problem* (1.3) *and* (1.6). *Then $\mathrm{d}\mu=\mathrm{d}\mu_t\mathrm{d}t=\int_{\mathcal{U}}p(s,x,t,y)\mathrm{d}\nu(x)\mathrm{d}y\mathrm{d}t$. In particular, for any $\phi\in M_b(\mathcal{U})$, $\langle\mu_t,\phi\rangle=\langle\nu,\langle\mu_t^{s,\bullet},\phi\rangle\rangle=\int_{\mathcal{U}}\langle\mu_t^{s,x},\phi\rangle\mathrm{d}\nu(x)$.*

Proof. Define $\mathrm{d}\tilde{\mu} = \mathrm{d}\tilde{\mu}_t \mathrm{d}t := \int_{\mathcal{U}} p(s,x,t,y)\mathrm{d}\nu(x)\mathrm{d}y\mathrm{d}t$ on $\mathcal{U} \times (s,\infty)$. By the definition of $\mu^{s,x}$, there holds that for any $\phi \in C_c^2(\mathcal{U})$,

$$\lim_{t\to s}\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} = \phi(x), \tag{5.3}$$

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r^{s,x} + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau^{s,x} \mathrm{d}\tau \quad \forall s < r < t. \tag{5.4}$$

It follows from Lemma 5.1 that each term in (5.4) is measurable with respect to x. Integrating (5.4) with respect to ν and applying Fubini's theorem, we find

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall s < r < t. \tag{5.5}$$

That is, $\tilde{\mu}$ is a global probability solution of (1.3) in $\mathcal{U} \times (s,\infty)$.

For $\phi \in C_c^2(\mathcal{U})$, we deduce from $\left|\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x}\right| \le |\phi|_\infty$, (5.3), and the dominated convergence theorem that $\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}}\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} \mathrm{d}\nu(x) \to \int_{\mathcal{U}} \phi \mathrm{d}\nu$ as $t \to s$. Hence, $\tilde{\mu}$ is a global probability solution of the Cauchy problem (1.3) and (1.6). The uniqueness result in Theorem 2.3 ensures that $\tilde{\mu} = \mu$. The "In particular" part follows readily. ◻

COROLLARY 5.1. *For each $x, y \in \mathcal{U}$ and $t_2 > t_1 > s$, there holds that*

$$p(s,x,t_2,y) = \int_{\mathcal{U}} p(s,x,t_1,z)p(t_1,z,t_2,y)\mathrm{d}z.$$

Proof. Fix $s \in \mathbb{R}$ and $x \in \mathcal{U}$. Lemma 5.2 ensures that the measure

$$\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t := \left[\int_{\mathcal{U}} p(t_1,z,t,y)\mathrm{d}\mu_{t_1}^{s,x}(z)\right]\mathrm{d}y\mathrm{d}t = \left[\int_{\mathcal{U}} p(t_1,z,t,y)p(s,x,t_1,z)\mathrm{d}z\right]\mathrm{d}y\mathrm{d}t$$

on $\mathcal{U} \times (t_1,\infty)$ is the unique global probability solution of the Cauchy problem (1.3) and (1.6) (with $s = t_1$ and $\nu = \mu_{t_1}^{s,x}$). So is the measure $\mu^{s,x}$ restricted on $\mathcal{U}\times(t_1,\infty)$. Theorem 2.3 then yields $\mu^{s,x} = \mu$ on $\mathcal{U}\times(t_1,\infty)$. Hence, they have the same densities, that is, $p(s,x,t,y) = \int_{\mathcal{U}} p(s,x,t_1,z)p(t_1,z,t,y)\mathrm{d}z$ for all $t > t_1$ and $y \in \mathcal{U}$. The corollary follows. ◻

5.2. Proof of Theorem C. We prove two lemmas before proving Theorem C. Recall that U is a Lyapunov function of type (L4).

The first lemma gives evolutionary estimates of a global probability solution of the Cauchy problem (1.3) and (1.6) against U.

LEMMA 5.3. *There are positive constants C_1 and C_2 such that for any global probability solution $\mu = (\mu_t)_{t\in(s,\infty)}$ of the Cauchy problem* (1.3) *and* (1.6) *with $\int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu < \infty$, there holds that*

$$\int_{\mathcal{U}} U(\cdot,t)\mathrm{d}\mu_t \le e^{-C_1(t-s)}\int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + C_2 \quad \forall t > s.$$

Proof. For simplicity, integrals of the forms $\int_{\mathcal{U}} g(\cdot,t)\mathrm{d}\mu_t$ and $\int_{\mathcal{U}} g(\cdot,s)\mathrm{d}\nu$ are respectively written as $\int_{\mathcal{U}} g\mathrm{d}\mu_t$ and $\int_{\mathcal{U}} g\mathrm{d}\nu$ in the rest of the proof.

By Theorem 2.1, μ admits a density $u \in C(\mathcal{U}\times(s,\infty))$, namely, $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t = u(x,t)\mathrm{d}x\mathrm{d}t$. By Lemma 2.1 (1)(b) and Lemma 2.2, there holds that for each $\phi \in C_c^{2,1}(\mathcal{U}\times(s,\infty))$, $\int_{\mathcal{U}} \phi\mathrm{d}\mu_t = \int_{\mathcal{U}} \phi\mathrm{d}\mu_r + \int_r^t\int_{\mathcal{U}} \mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ for all $t > r > s$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \phi\mathrm{d}\mu_t = \int_{\mathcal{U}} \mathcal{L}\phi\mathrm{d}\mu_t \quad \forall t > s. \tag{5.6}$$

As U is a Lyapunov function of type (L4), there are positive constants C_1, C_2, and ρ_m such that $\mathcal{L}U \le -C_1U + C_2 < 0$ in $(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m}$. Fix $\rho_0 > \rho_m$, and set $N_0 = [\rho_0]+1$, where $[\rho_0]$ is the integer part of ρ_0. Let $\{\zeta_N\}_{N\ge N_0}$ be a sequence of smooth and nondecreasing functions on $\mathbb{R}$ satisfying

$$\zeta_N(t) = \begin{cases} 0, & t\in[0,\rho_m],\\ t, & t\in[\rho_0,N],\\ N+1, & t\ge N+2,\end{cases}\ \ \zeta_N(t)\le t,\ \ t\in[\rho_m,\rho_0],\ \text{ and }\ \zeta_N''(t)\le 0,\ \ t\in[N,N{+}2].$$

In addition, let the functions $\{\zeta_N\}_{N\ge N_0}$ coincide on $[0,\rho_0]$.

We claim that there exists $\tilde{C}_1 > 0$ such that

$$\begin{aligned}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t \le e^{-C_1(t-s)}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\nu + \frac{\tilde{C}_1}{C_1}&\\ + C_1(N+1)\int_s^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau \ \ \forall t>s \text{ and } N\ge N_0.&\end{aligned}\tag{5.7}$$

Note that $\zeta_N(U)-(N+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$. Applying Lemma 2.1 (2)(b) and Lemma 2.2 with $\phi=\zeta_N(U)-(N+1)$, we find $\lim_{r\to s}\int_{\mathcal{U}}[\zeta_N(U)-(N+1)]\mathrm{d}\mu_r = \int_{\mathcal{U}}[\zeta_N(U)-(N+1)]\mathrm{d}\nu$. It follows from $\mu_r(\mathcal{U})=\nu(\mathcal{U})=1$ for all $r>s$ that

$$\lim_{r\to s}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_r = \int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\nu.\tag{5.8}$$

Setting $\phi=\zeta_N(U)-(N+1)$ in (5.6), we find $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}[\zeta_N(U)-(N+1)]\,\mathrm{d}\mu_t = \int_{\mathcal{U}}\mathcal{L}\left(\zeta_N(U)-(N+1)\right)\mathrm{d}\mu_t$. Since $\mu_t(\mathcal{U})=1$ for all $t>0$ and $\mathcal{L}(\zeta_N(U)-(N+1)) = \zeta_N'(U)\mathcal{L}U+\zeta_N''(U)a^{ij}\partial_iU\partial_jU$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t = \int_{\mathcal{U}}\zeta_N'(U)\mathcal{L}U\mathrm{d}\mu_t + \int_{\mathcal{U}}\zeta_N''(U)a^{ij}\partial_iU\partial_jU\mathrm{d}\mu_t \quad \forall t>s.\tag{5.9}$$

As $\zeta_N'=0$ on $[0,\rho_m]$, $\zeta_N'(t)=1$ on $[\rho_0,N]$, and $\zeta_N'\ge 0$ otherwise, we find $\zeta_N'(U)\mathcal{L}U\le(-C_1U+C_2)\mathbf{1}_{\Omega_N\setminus\Omega_{\rho_0}}$, which implies that

$$\int_{\mathcal{U}}\zeta_N'(U)\mathcal{L}U\mathrm{d}\mu_t \le \int_{\Omega_N^t\setminus\Omega_{\rho_0}^t}(-C_1U+C_2)\mathrm{d}\mu_t.\tag{5.10}$$

Since $\zeta_N''\ne 0$ on $[\rho_m,\rho_0]$, $\zeta_N''\le 0$ on $[N,N+2]$, and $\zeta_N''=0$ otherwise, the nonnegative definiteness of (a^{ij}) gives $\zeta_N''a^{ij}\partial_iU\partial_jU \le (C_*\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU)\mathbf{1}_{\Omega_{\rho_0}\setminus\Omega_{\rho_m}}$, where $C_*:=\max_{t\in[\rho_m,\rho_0]}\zeta_N''(t)$, which is independent of N due to the coincidence of $\{\zeta_N\}_{N\ge N_0}$ on $[0,\rho_0]$. Thus, $\int_{\mathcal{U}}\zeta_N''(U)a^{ij}\partial_iU\partial_jU\mathrm{d}\mu_t \le C_*\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU$, which together with (5.9) and (5.10) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t + C_1\int_{\Omega_N^t\setminus\Omega_{\rho_0}^t}U\mathrm{d}\mu_t \le C_2 + C_*\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU.\tag{5.11}$$

As $\zeta_N\le\rho_0$ on $[0,\rho_0]$, $\zeta_N(t)=t$ on $[\rho_0,N]$, and $\zeta_N\le N+1$ on $[N,\infty)$, we see that

$$\begin{aligned}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t &= \left(\int_{\Omega_{\rho_0}^t}+\int_{\Omega_N^t\setminus\Omega_{\rho_0}^t}+\int_{\mathcal{U}\setminus\Omega_N^t}\right)\zeta_N(U)\mathrm{d}\mu_t\\ &\le \rho_0 + \int_{\Omega_N^t\setminus\Omega_{\rho_0}^t}U\mathrm{d}\mu_t + (N+1)\mu_t(\mathcal{U}\setminus\Omega_N^t).\end{aligned}\tag{5.12}$$

Setting $\tilde{C}_1 := C_1\rho_0 + C_2 + C_* \max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U$, we find from (5.11) and (5.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t + C_1\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t \leq \tilde{C}_1 + C_1(N+1)\mu_t(\mathcal{U}\setminus\Omega_N^t) \quad \forall t>s.$$

Applying Gronwall's inequality yields

$$\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t \leq e^{-C_1(t-r)}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_r + \frac{\tilde{C}_1}{C_1} + C_1(N+1)\int_r^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau \quad \forall t>r>s.$$

Letting $r\to s$ in the above inequality, we conclude (5.7) from (5.8) and the monotone convergence theorem.

Note that if there holds that

$$(N+1)\int_s^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau \to 0 \quad \text{as} \quad N\to\infty, \tag{5.13}$$

then we can pass to the limit $N\to\infty$ in (5.7) to find from $\zeta_N(t)\leq t$ for $N\geq N_0$ and $t\geq 0$ that $\int_{\mathcal{U}\setminus\Omega^t_{\rho_0}} U\mathrm{d}\mu_t \leq e^{-C_1(t-s)}\int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + \frac{\tilde{C}_1}{C_1}$ for all $t>s$, which readily leads to the lemma.

To finish the proof, we show (5.13). Fix $t>s$. We define

$$\begin{aligned} f &:= (1+U)u\mathbf{1}_{\bigcup_{\tau\in[s,t]}\left((\mathcal{U}\setminus\Omega_{\rho_0})\times\{\tau\}\right)},\\ f_N(x,\tau) &:= (N+1)u(x,\tau)e^{-C_1(t-\tau)}\mathbf{1}_{\bigcup_{\tau\in[s,t]}\left((\mathcal{U}\setminus\Omega_{\rho_N})\times\{\tau\}\right)}(x,\tau), \quad N\gg 1. \end{aligned}$$

Obviously, $f_N\leq f$ for $N\gg1$ and $f_N\to 0$ as $N\to\infty$. As

$$\int_s^t\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_0}} f(x,\tau)\mathrm{d}x\mathrm{d}\tau = \int_s^t\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_0}} U\mathrm{d}\mu_\tau\mathrm{d}\tau,$$

$$\int_s^t\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_0}} f_N(x,\tau)\mathrm{d}x\mathrm{d}\tau = (N+1)\int_s^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau,$$

the limit (5.13) follows from the dominated convergence theorem if there holds that

$$\int_s^t\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_0}} U\mathrm{d}\mu_\tau\mathrm{d}\tau < \infty. \tag{5.14}$$

It remains to show (5.14). Set $\tilde{C}_2 := C_2 + C_*\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U$. For any fixed $r\in(s,t)$, integrating (5.11) over $[r,t]$ gives

$$\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t + C_1\int_r^t\int_{\Omega^\tau_N\setminus\Omega^\tau_{\rho_0}} U\mathrm{d}\mu_\tau\mathrm{d}\tau \leq \int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_r + \tilde{C}_2(t-r).$$

Letting $r\to s$ in the above inequality, we deduce from (5.8) that

$$\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t + C_1\int_s^t\int_{\Omega^\tau_N\setminus\Omega^\tau_{\rho_0}} U\mathrm{d}\mu_\tau\mathrm{d}\tau \leq \int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\nu + \tilde{C}_2(t-s).$$

Since $\zeta_N(t)\leq t$ for all $t>0$ and $N\geq N_0$, and $U(\cdot,s)$ is ν-integrable, passing to the limit $N\to\infty$ in the above inequality yields $\int_s^t\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_0}} U\mathrm{d}\mu_\tau\mathrm{d}\tau \leq \int_{\mathcal{U}} U\mathrm{d}\nu + \tilde{C}_2(t-s) < \infty$. Hence, (5.14) holds. This completes the proof. ∎

The second lemma is a version of the minorization condition of the measures $\{\mu^{s,x}\}$, where $\mu^{s,x}=(\mu^{s,x}_t)_{t\in(0,\infty)}$ is defined at the beginning of subsection 5.1.

LEMMA 5.4. *Let $s,t\in\mathbb{R}$ with $s<t$. For each $R>0$, there exists $\alpha>0$ such that $\|\mu^{s,x_1}_t-\mu^{s,x_2}_t\|_{TV}\le 2(1-\alpha)$ for all $x_1,x_2\in\mathcal{U}$ satisfying $U(x_1,s)+U(x_2,s)\le R$, where $\|\cdot\|_{TV}$ denotes the total variation norm.*

Proof. Fix $s,t\in\mathbb{R}$ with $s<t$ and $R>0$. Note that

$$\{(x_1,x_2)\in\mathcal{U}\times\mathcal{U}:U(x_1,s)+U(x_2,s)\le R\}\subset\overline{\Omega}^s_R\times\overline{\Omega}^s_R,$$

where we recall that $\Omega^\tau_\rho:=\{x\in\mathcal{U}:U(x,\tau)<\rho\}$ for $\tau\in\mathbb{R}$ and $\overline{\Omega}^\tau_R$ denotes the closure of Ω^τ_R.

We first claim that there exist positive constants ρ_1 and M such that

$$\inf_{y\in\Omega^t_{\rho_1}}p(s,x,t,y)\ge M\quad\forall x\in\overline{\Omega}^s_R. \tag{5.15}$$

For $x\in\overline{\Omega}^s_R$, we denote $\mu=(\mu_t)_{t\in(s,\infty)}:=\mu^{s,x}$ and $u(y,t):=p(s,x,t,y)$ for $y\in\mathcal{U}$ for notational simplicity. Applying Lemma 5.3, we find

$$\int_{\mathcal{U}}U(y,\tau)u(y,\tau)\mathrm{d}y=\int_{\mathcal{U}}U(\cdot,\tau)\mathrm{d}\mu_\tau\le e^{-C_1(\tau-s)}U(x,s)+C_2\quad\forall\tau>s.$$

Set $\Delta:=\frac{t-s}{4}$. Integrating the above inequality over $[s+\Delta,s+2\Delta]$ gives

$$\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}}U(y,\tau)u(t,\tau)\mathrm{d}y\mathrm{d}\tau\le\frac{U(x,s)}{C_1}\left(e^{-C_1\Delta}-e^{-2C_1\Delta}\right)+C_2\Delta.$$

Setting $C_3:=\frac{1}{C_1}\left(e^{-C_1\Delta}-e^{-2C_1\Delta}\right)\max_{x\in\overline{\Omega}^s_R}U(x,s)+C_2\Delta$, we arrive at

$$\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_\rho}U(y,\tau)u(t,\tau)\mathrm{d}y\mathrm{d}\tau\le C_3\quad\forall\rho\ge\min_{\mathcal{U}\times\mathbb{R}}U. \tag{5.16}$$

As U satisfies (1.4), there holds that $\lim_{\rho\to\infty}\inf_{(\mathcal{U}\times\mathbb{R})\setminus\Omega_\rho}U=\infty$. This together with (5.16) yields the existence of some $\rho_1>\min_{\mathcal{U}\times\mathbb{R}}U$ such that $\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_1}}u(y,\tau)\mathrm{d}y\mathrm{d}\tau\le\frac{\Delta}{2}$, which implies that

$$\frac{\Delta}{2}\le\Delta-\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_1}}u(y,\tau)\mathrm{d}y\mathrm{d}\tau=\int_{s+\Delta}^{s+2\Delta}\int_{\Omega^\tau_{\rho_1}}u(y,\tau)\mathrm{d}y\mathrm{d}\tau\le|Q^1_{\rho_1}|\sup_{Q^1_{\rho_1}}u,$$

where $Q^1_{\rho_1}:=\cup_{\tau\in[s+\Delta,s+2\Delta]}\left(\Omega^\tau_{\rho_1}\times\{\tau\}\right)$. By Harnack's inequality (see, e.g., [27, Theorem 10.1]) to u, there is $C>0$, independent of u, such that $\frac{\Delta}{2|Q^1_{\rho_1}|}\le\sup_{Q^1_{\rho_1}}u\le C\inf_{Q^2_{\rho_1}}u\le C\inf_{y\in\Omega^t_{\rho_1}}u(y,t)$, where $Q^2_{\rho_1}:=\cup_{\tau\in[s+3\Delta,t]}\left(\Omega^\tau_{\rho_1}\times\{\tau\}\right)$. Setting $M:=\frac{\Delta}{2C|Q^1_{\rho_1}|}$, (5.15) follows.

We prove the lemma. For $x_1,x_2\in\overline{\Omega}^s_R$, set $\mu^i=(\mu^i_t)_{t\in(0,\infty)}:=\mu^{s,x_i}$ and $u_i(y,t):=p(s,x_i,t,y)$ for $y\in\mathcal{U}$ and $i=1,2$. It follows from (5.15) that for $i,j=1,2$ with $i\ne j$,

$$\begin{aligned}\int_{\Omega^t_{\rho_1}}(u_i(y,t)-u_j(y,t))^+\mathrm{d}y&\le\int_{\Omega^t_{\rho_1}}(u_i(y,t)-M)\mathbb{1}_{\{u_i\ge u_j\}}(y,t)\mathrm{d}y\\&\le\mu^i_t(\Omega^t_{\rho_1})-M\int_{\Omega^t_{\rho_1}}\mathbb{1}_{\{u_i\ge u_j\}}(y,t)\mathrm{d}y.\end{aligned}$$

As a result, for i, $j = 1, 2$ with $i \neq j$,

$$\begin{aligned}\int_{\mathcal{U}} (u_i(y,t) - u_j(y,t))^+ \mathrm{d}y &= \left(\int_{\mathcal{U} \setminus \Omega^t_{\rho_1}} + \int_{\Omega^t_{\rho_1}} \right) (u_i(y,t) - u_j(y,t))^+ \mathrm{d}y \\ &\leq \mu^i_t(\mathcal{U} \setminus \Omega^t_{\rho_1}) + \mu^i_t(\Omega^t_{\rho_1}) - M \int_{\Omega^t_{\rho_1}} \mathbb{1}_{\{u_i \geq u_j\}}(y,t) \mathrm{d}y \\ &= 1 - M \int_{\Omega^t_{\rho_1}} \mathbb{1}_{\{u_i \geq u_j\}}(y,t) \mathrm{d}y.\end{aligned}$$

As $|u_1 - u_2| = (u_1 - u_2)^+ + (u_2 - u_1)^+$, we arrive at $\int_{\mathcal{U}} |u_1(y,t) - u_2(y,t)|\, \mathrm{d}y \leq 2 - M|\Omega^t_{\rho_1}|$. It follows that

$$\|\mu^1_t - \mu^2_t\|_{TV} \leq \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \int_A |u_1(x,t) - u_2(x,t)| \mathrm{d}x \leq 2 \left(1 - \frac{1}{2} M |\Omega^t_{\rho_1}| \right).$$

As $0 < M|\Omega^t_{\rho_1}| \leq \int_{\Omega^t_{\rho_1}} u_1(y,t) \mathrm{d}x \leq 1$, we find $\alpha := \frac{1}{2} M |\Omega^t_{\rho_1}| \in (0,1)$. This completes the proof. ∎

We are ready to prove Theorem C.

Proof of Theorem C. We assume, without loss of generality, that $s = 0$. For each $x \in \mathcal{U}$, we denote $\mu^x = (\mu^x_t)_{t \in (0,\infty)} := \mu^{0,x}$, where we recall $\mu^{0,x}$ is the unique global probability solution of the Cauchy problem (1.3) and (1.6) with $s = 0$ and $\nu = \delta_x$. Then $\mathrm{d}\mu^x := \mathrm{d}\mu^x_t \mathrm{d}t = p(0,x,t,y) \mathrm{d}y \mathrm{d}t$. The proof is divided into four steps.

Step 1. We show that there exist a unique measure $\mu_* \in \mathcal{M}_p(\mathcal{U})$ and positive constants C and $\varrho \in (0,1)$ such that $\|\mu^x_{nT} - \mu_*\|_{TV} \leq C \varrho^n (1 + U(x,0))$ for all $x \in \mathcal{U}$ and $n \in \mathbb{N}$.

Let $\mathcal{P} : M_b(\mathcal{U}) \to M_b(\mathcal{U})$ be defined by $\mathcal{P}\phi(x) := \langle \mu^x_T, \phi \rangle = \int_{\mathcal{U}} p(0,x,T,y) \phi(y) \mathrm{d}y$ for $x \in \mathcal{U}$ and $\phi \in M_b(\mathcal{U})$. By Lemma 5.1, $\mathcal{P}$ is well-defined. In particular, $\mathcal{P}\mathbf{1} = \mathbf{1}$, where $\mathbf{1} \equiv 1$. Let $\mathcal{P}^* : \mathcal{M}_p(\mathcal{U}) \to \mathcal{M}_p(\mathcal{U})$ be the adjoint operator of $\mathcal{P}$ defined by

$$\langle \mathcal{P}^* \mu, \phi \rangle = \langle \mu, \mathcal{P}\phi \rangle \quad \forall \mu \in \mathcal{M}_p(\mathcal{U}) \text{ and } \phi \in M_b(\mathcal{U}).$$

Since a^{ij} and V^i are T-periodic for $i, j = 1, \ldots, d$, we find from Theorem 2.3 that $p(nT, x, (n+1)T, y) = p(0,x,T,y)$ for all $n \in \mathbb{N}$ and $x, y \in \mathcal{U}$, which implies that $\mathcal{P}\phi(x) = \int_{\mathcal{U}} p(nT, x, (n+1)T, y) \phi(y) \mathrm{d}y$ for $x \in \mathcal{U}$. It follows from Corollary 5.1 that

$$\mathcal{P}^n \phi(x) = \langle \mu^x_{nT}, \phi \rangle, \quad x \in \mathcal{U}. \tag{5.17}$$

Define a weighted supremum norm: $\|\phi\|_* := \operatorname{ess\,sup}_{x \in \mathcal{U}} \left| \frac{\phi(x)}{1 + U(x,0)} \right|$ for $\phi \in M_b(\mathcal{U})$. Thanks to Lemmas 5.3 and 5.4, we apply Harris's theorem (see, e.g., [17, Theorem 3.6]) to find that $\mathcal{P}$ admits a unique invariant measure μ_* of $\mathcal{P}$, namely, $\mathcal{P}^* \mu_* = \mu_*$, and there exist constants $C > 0$ and $\varrho \in (0,1)$ such that

$$\|\langle \mu^\bullet_{nT}, \phi \rangle - \langle \mu_*, \phi \rangle\|_* = \|\mathcal{P}^n \phi - \langle \mu_*, \phi \rangle\|_* \leq C \varrho^n \|\phi - \langle \mu_*, \phi \rangle\|_* \leq 2C \varrho^n \|\phi\|_\infty$$

holds for all $\phi \in M_b(\mathcal{U})$, where we used (5.17) in the equality. Consequently,

$$\begin{aligned}\|\mu^x_{nT} - \mu_*\|_{TV} &= \sup_{|\phi| \leq 1} \frac{|\langle \mu^x_{nT}, \phi \rangle - \langle \mu_*, \phi \rangle|}{1 + U(x,0)} [1 + U(x,0)] \\ &\leq \sup_{|\phi| \leq 1} \|\langle \mu^\bullet_{nT}, \phi \rangle - \langle \mu_*, \phi \rangle\|_* [1 + U(x,0)] \leq 2C \varrho^n [1 + U(x,0)].\end{aligned}$$

Step 2. We show that (1.3) admits a unique periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$. Moreover, there holds that $\tilde{\mu}_{nT} = \mu_*$ for all $n \in \mathbb{N}$.

Denote $\mathrm{d}\tilde{\mu} := \mathrm{d}\tilde{\mu}_t \mathrm{d}t$ as the unique global probability solution of the Cauchy problem (1.3) and (1.6) with $s = 0$ and $\nu = \mu_*$. It follows from Lemma 5.2 that $\langle\tilde{\mu}_t, \phi\rangle = \langle\mu_*, \langle\mu_t^\bullet, \phi\rangle\rangle$ for all $t > 0$ and $\phi \in M_b(\mathcal{U})$, which together with (5.17) and the fact that μ_* is invariant under $\mathcal{P}$ implies that for each $n \in \mathbb{N}$ and $\phi \in M_b(\mathcal{U})$,

$$\langle\tilde{\mu}_{nT}, \phi\rangle = \langle\mu_*, \langle\mu_{nT}^\bullet, \phi\rangle\rangle = \langle\mu_*, \mathcal{P}^n\phi\rangle = \langle\mathcal{P}^{*n}\mu_*, \phi\rangle = \langle\mu_*, \phi\rangle .$$

That is, $\tilde{\mu}_{nT} = \mu_*$ for all $n \in \mathbb{N}$. Therefore, $\tilde{\mu}_{t+T} = \tilde{\mu}_t$ for all $t > 0$. We then extend $\tilde{\mu}$ to $\mathcal{U} \times (-\infty, 0)$ by defining $\tilde{\mu}_t := \tilde{\mu}_{t+nT}$ for $t \in (-nT, (n-1)T]$ and $n \in \mathbb{N}$. It is not hard to check that $\tilde{\mu} := (\tilde{\mu}_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (1.3). The uniqueness follows from Theorem A.

Step 3. We prove there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that $\|\mu_t^x - \tilde{\mu}_t\|_{TV} \leq \tilde{C}_1 e^{-\tilde{C}_2 t}[1 + U(x, 0)]$ for all $x \in \mathcal{U}$ and $t > 0$.

For $t > 0$, there exist unique $n_t \in \mathbb{N}_0$ and $0 \leq r_t < T$ such that $t = n_tT + r_t$. For $\phi \in M_b(\mathcal{U})$, we set $\phi_{r_t}(x) := \langle\mu_{r_t}^x, \phi\rangle$ for $x \in \mathcal{U}$. Clearly, $\|\phi_{r_t}\| \leq \|\phi\|_\infty$ if ϕ is bounded. As $p(0, x, r, y) = p(nT, x, r+nT, y)$ for all $n \in \mathbb{N}$, $x, y \in \mathcal{U}$, and $r > 0$, we find $\phi_{r_t}(x) := \int_{\mathcal{U}} p(n_tT, x, t, y)\phi(y)\mathrm{d}y$ for $x \in \mathcal{U}$. It follows from Lemma 5.2 and Corollary 5.1 that for each $\phi \in M_b(\mathcal{U})$, there hold that $\langle\tilde{\mu}_t, \phi\rangle = \langle\tilde{\mu}_{r_t}, \phi\rangle = \big\langle\mu_*, \langle\mu_{r_t}^\bullet, \phi\rangle\big\rangle = \langle\mu_*, \phi_{r_t}\rangle$, and for each $x \in \mathcal{U}$,

$$\langle\mu_t^x, \phi\rangle = \int_{\mathcal{U}} p(0, x, nT, z)\left[\int_{\mathcal{U}} p(nT, z, t, y)\phi(y)\mathrm{d}y\right]\mathrm{d}z = \mathcal{P}^n\phi_{r_t}(x) = \langle\mu_{nT}^x, \phi_{r_t}\rangle.$$

Consequently, we derive from Step 1 that

$$\begin{aligned}\|\mu_t^x - \tilde{\mu}_t\|_{TV} &= \sup_{|\phi|\leq 1} |\langle\mu_t^x, \phi\rangle - \langle\tilde{\mu}_t, \phi\rangle| \\ &= \sup_{|\phi|\leq 1} |\langle\mu_{nT}^x, \phi_{r_t}\rangle - \langle\mu_*, \phi_{r_t}\rangle| \leq C\varrho^n[1 + U(x, 0)] \leq \tilde{C}_1 e^{-\tilde{C}_2 t}[1 + U(x, 0)],\end{aligned}$$

where $\tilde{C}_1 = C\varrho^{-1}$ and $\tilde{C}_2 = \frac{1}{T}\ln\varrho$.

Step 4. Applying Lemma 5.2, we find $\langle\mu_t, \phi\rangle = \langle\nu, \langle\mu_t^\bullet, \phi\rangle\rangle = \int_{\mathcal{U}}\langle\mu_t^x, \phi\rangle\mathrm{d}\nu(x)$ for all $\phi \in M_b(\mathcal{U})$. It follows from Step 3 that

$$\|\mu_t - \tilde{\mu}_t\|_{TV} \leq \int_{\mathcal{U}} \sup_{|\phi|\leq 1} |\langle\mu_t^x, \phi\rangle - \langle\tilde{\mu}_t, \phi\rangle|\,\mathrm{d}\nu(x) \leq \tilde{C}_1 e^{-\tilde{C}_2 t}\int_{\mathcal{U}}[1 + U(\cdot, 0)]\,\mathrm{d}\nu$$

for all $t \geq 0$. This completes the proof. □

6. Applications.

In this section, we discuss some applications of Theorem B, Corollary B, and Theorem C. Applications to stochastic damping Hamiltonian systems and stochastic slow-fast systems are discussed, respectively, in subsections 6.1 and 6.2. In subsection 6.3, we investigate the convergence of weak solutions of an SDE with less regular coefficients.

6.1. Stochastic damping Hamiltonian systems.

Consider the following stochastic damping Hamiltonian system:

$$(6.1)\qquad \begin{cases}\dot{x} = y, \\ \mathrm{d}y = -[b(x, y)y + \nabla V(x, t)]\,\mathrm{d}t + F(x, y, t)\mathrm{d}t + \sigma(x, y, t)\mathrm{d}W_t,\end{cases}$$

where $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, the damping $b=(b^{ij}):\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}^{d\times d}$ is continuous, the potential $V:\mathbb{R}^d\times\mathbb{R}\mapsto(0,\infty)$ is twice continuously differentiable in x and continuously differentiable in t, the external force $F:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\mapsto\mathbb{R}^d$ is continuous, the noise coefficient matrix $\sigma:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\mapsto\mathbb{R}^{d\times m}$ belongs to $C(\mathbb{R},W^{1,p}_{loc}(\mathbb{R}^d\times\mathbb{R}^d))$, where $p>d+2$ and $m\geq d$ are fixed, and $(W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process. We assume V, F, and σ are all T-periodic in t for some $T>0$.

The FPE associated to (6.1) reads as

$$\partial_t u=\partial^2_{y_iy_j}(a^{ij}u)-\partial_{x_i}(y_iu)+\partial_{y_i}\big((b^{ij}y_j+\partial_{x_i}V)u\big)-\partial_{y_i}(F^iu),\tag{6.2}$$

where $(a^{ij}):=\frac{\sigma\sigma^\top}{2}$ is the diffusion matrix. Denote

$$\mathcal{L}_H:=\partial_t+a^{ij}\partial^2_{y_iy_j}+y^i\partial_{x_i}-(b^{ij}y_j+\partial_{x_i}V)\partial_{y_i}+F^i\partial_{y_i}.$$

We make the following additional assumptions on the coefficients:

(A1) There exists $b_0>0$ such that $b^{ij}y_iy_j\geq b_0|y|^2$ for all $y\in\mathbb{R}^d$.

(A2) The functions F and σ, and ∂_tV, are uniformly bounded on $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$ and $\mathbb{R}^d\times\mathbb{R}$, respectively.

(A3) $\Phi\in C^2(\mathbb{R}^d)$ is lower bounded and satisfies

$$\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}\sum_{i,j=1}^d\left|-b^{ji}(x,y)\frac{x_j}{|x|}+\partial_{x_i}\Phi(x)\right|<\infty.$$

(A4) $\nabla_xV\cdot\frac{x}{|x|}\to\infty$ as $|x|\to\infty$.

Note that **(A1)** says that the system (6.1) is damped. When $b(x,y)$ is bounded, the function Φ in **(A3)** can be taken to be 0.

Following the arguments as in the proof of [22, Theorem 5.1], we can construct a Lyapunov function of type (L3) with respect to $\mathcal{L}_H$. Hence, Theorem B is applied to give the following result.

THEOREM 6.1. *Assume* **(A1)**–**(A4)**. *Let $\mu=(\mu_t)_{t\in(s,\infty)}$ be a global probability solution of the Cauchy problem associated to* (6.2) *with initial condition $\mu_s=\nu$, where $\nu\in\mathcal{M}_p(\mathbb{R}^d\times\mathbb{R}^d)$ is compactly supported. Then for any sequence of positive integers $\{n_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}n_j=\infty$, there exists a subsequence, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, and a periodic probability solution $\tilde\mu=(\tilde\mu_t)_{t\in\mathbb{R}}$ of* (6.2) *such that the following hold:*

(1) *for each bounded $\phi\in C_T(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R})$, there holds that*

$$\lim_{j\to\infty}\frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi\,\mathrm{d}\mu_\tau\mathrm{d}\tau=\frac{1}{T}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi\,\mathrm{d}\tilde\mu_\tau\mathrm{d}\tau\quad\forall t\geq s;$$

(2) *for each $\psi\in C^2_c(\mathbb{R}^d\times\mathbb{R}^d)$, there holds that*

$$\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi\,\mathrm{d}\mu_{t+kT}=\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi\,\mathrm{d}\tilde\mu_t\quad\textit{for a.e. } t\in(s,s+T].$$

We point out that the uniqueness of periodic probability solutions of (6.2) (with nonsmooth coefficients) remains an interesting open question.

6.2. Stochastic slow-fast systems. Consider the following SDE:

$$\begin{cases}\epsilon\dot x=f(x,y,t),\\ \mathrm{d}y=g(x,y,t)\mathrm{d}t+\sigma(x,y,t)\mathrm{d}W_t,\end{cases}\qquad(x,y)\in\mathbb{R}^m\times\mathbb{R}^n,\tag{6.3}$$

where $0 < \epsilon \ll 1$, $f = (f^k) : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^m$, $g = (g^i) : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$, $\sigma = (\sigma^{ij}) : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^{n\times\ell}$ is the *noise coefficient matrix* with $\ell \geq n$, and $W = (W_t)_{t\in\mathbb{R}}$ is a standard ℓ-dimensional Wiener process. We assume f, g, and σ are T-periodic in t for some $T > 0$.

As here we are only interested in the dynamics of (6.3) for each fixed $0 < \epsilon \ll 1$, we set $\epsilon = 1$ in (6.3) and consider the following system for clarity:

$$\begin{cases} \dot{x} = f(x, y, t), \\ \mathrm{d}y = g(x, y, t)\mathrm{d}t + \sigma(x, y, t)\mathrm{d}W_t, \end{cases} \qquad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n.$$

The associated FPE reads as

$$\partial_t u = \partial^2_{y_i y_j}(a^{ij}u) - \partial_{x_k}(f^k u) - \partial_{y_j}(g^i u), \quad (x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}, \tag{6.4}$$

where $A := (a^{ij}) = \frac{1}{2}\sigma\sigma^\top$. Denote $\mathcal{L}_{SF} := \partial_t + a^{ij}\partial_{y_i y_j} + f^k\partial_{x_k} + g^i\partial_{y_i}$.

We make the following assumptions on the coefficients:

- **(B1)** Let $p > m+n+2$. $A(x, y, t)$ is positive definite for each $(x, y, t) \in \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$, and $a^{ij} \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathbb{R}^m \times \mathbb{R}^n))$ and $g^i \in C_T(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ for each $i, j = 1, \ldots, n$. Moreover, for each $a > 0$, $\sup_{\{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|\leq a\}} |A| < \infty$,
- **(B2)** There exists a positive T-periodic function $U \in C^{1,1}(\mathbb{R}^m \times \mathbb{R})$ satisfying $\lim_{|x|\to\infty} U(x, t) = \infty$ for all $t \in \mathbb{R}$ such that $\sup_{|y|\leq a} \sup_{t\in\mathbb{R}} \mathcal{L}_{SF}U \to -\infty$ as $|x| \to \infty$ for each $a > 0$, and

$$\begin{aligned} \mathcal{L}_{SF}U = 0 \quad &\text{on} \quad \{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |x| = 0\}, \\ \mathcal{L}_{SF}U < 0 \quad &\text{on} \quad \{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |x| \neq 0\}. \end{aligned}$$

To proceed, we need dissipativity along the slow direction, namely, the y-direction.

DEFINITION 6.1. *Let $V \in C^{2,1}_T(\mathbb{R}^n \times \mathbb{R})$ be a nonnegative function, and let it satisfy $\lim_{|y|\to\infty} \inf_{t\in\mathbb{R}} V(y, t) = \infty$. It is called a* semi-Lyapunov function *(with respect to $\mathcal{L}_{SF}$) of*

1. *type* (L2) *if there exist positive constants γ and a such that $\mathcal{L}_{SF}V \leq -\gamma$ on $\{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |y| > a\}$ and*
2. *type* (L3) *if $\lim_{|y|\to\infty} \sup_{(x,t)\in\mathbb{R}^m\times\mathbb{R}} \mathcal{L}_{SF}V(x, y, t) = -\infty$.*

THEOREM 6.2. *Assume* **(B1)** *and* **(B2)**. *If $\mathcal{L}_{SF}$ admits a semi-Lyapunov function of type* (L2)*, then there exists a unique periodic probability solution $\mu = (\mu_t)_{t\in\mathbb{R}}$ of* (6.4)*. Moreover,* $\mathrm{supp}(\mu) = \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$*, where 0_m denotes the origin in $\mathbb{R}^m$.*

Proof. We write $\mathcal{L}_{SF}$ as $\mathcal{L}$ for simplicity. The proof is divided into three steps.

Step 1. We show that (6.4) admits a periodic probability solution $\mu = (\mu_t)_{t\in\mathbb{R}}$.

Let V be a semi-Lyapunov function of type (L2) and $\gamma, a > 0$ be as in Definition 6.1 (1). Define $W(x, y, t) := U(x, t) + V(y, t)$ for $(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$. Obviously, W is nonnegative and T-periodic, and satisfies $\inf_{t\in\mathbb{R}} W(x, y, t) \to \infty$ as $|x|+|y| \to \infty$ and $\mathcal{L}W = \mathcal{L}U + \mathcal{L}V \leq -\gamma$ on $\{(x, y, t) \in \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R} : |y| > a\}$. Moreover, it follows from **(B1)** that $\mathcal{L}V$ is bounded on $\{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |y| \leq a\}$ and from **(B2)** that $\lim_{|x|\to\infty} \sup_{|y|\leq a} \sup_{t\in\mathbb{R}} \mathcal{L}U = -\infty$. Hence, there exists a constant $b > 0$ such that

$$\mathcal{L}W = \mathcal{L}U + \mathcal{L}V \leq -\gamma \quad \text{on} \quad \{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |y| \leq a,\ |x| > b\}.$$

As a result, $\mathcal{L}W \leq -\gamma$ on $\{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : |y| > a \text{ or } |x| > b\}$. That is, W is a Lyapunov function of type (L2). Hence, we apply [22, Theorem B] to find a periodic probability solution $\mu = (\mu_t)_{t\in\mathbb{R}}$ of (6.4).

By virtue of Lemma 4.2, we may assume, without loss of generality, that for any $\phi \in C_c^{2,1}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, the function $t \mapsto \iint_{\mathbb{R}^m\times\mathbb{R}^n} \phi(\cdot,t)\mathrm{d}\mu_t$ is continuous on $\mathbb{R}$.

Step 2. We show that μ is supported on $\{0_m\} \times \mathbb{R}^n \times \mathbb{R}$. By Lemma 2.1 (1)(b) and Lemma 2.2, there holds that for any T-periodic $\phi \in C_c^{2,1}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$,

$$\int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau = 0 \quad \forall t \in \mathbb{R}. \tag{6.5}$$

For each $\alpha > 1$, set $W_\alpha(x, y, t) := \alpha U(x, t) + V(y, t)$ for $(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$. Obviously, $W_\alpha(x, y, t) \to \infty$ as $|x| + |y| \to \infty$ for each $\alpha > 1$.

Let $\{\zeta_\rho\}_{\rho>0}$ be smooth and nondecreasing functions on $\mathbb{R}$ satisfying $\zeta_\rho(t) = t$ for $t \in [0, \rho]$, $\zeta_\rho = \rho + 1$ on $[\rho + 2, \infty)$, and $\zeta_\rho'' \le 0$ on $[\rho, \rho + 2]$. Clearly, for each $\rho > 0$, the function $\zeta_\rho(W_\alpha) - (\rho+1)$ is T-periodic and belongs to $C_c^{2,1}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$. Setting $\phi := \zeta_\rho(W_\alpha) - (\rho+1)$ in (6.5), we find from $\mathcal{L}\zeta_\rho(W_\alpha) = \zeta_\rho'(W_\alpha)(\alpha\mathcal{L}U + \mathcal{L}V) + \zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V$ that

$$\begin{aligned} 0 = \alpha \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \zeta_\rho'(W_\alpha)\mathcal{L}U \mathrm{d}\mu_\tau\mathrm{d}\tau + \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \zeta_\rho'(W_\alpha)\mathcal{L}V \mathrm{d}\mu_\tau\mathrm{d}\tau \\ + \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V \mathrm{d}\mu_\tau\mathrm{d}\tau. \end{aligned} \tag{6.6}$$

As $\zeta_\rho' \ge 0$ on $[0, \infty)$, we see from Definition 6.1 (1) that

$$\zeta_\rho'(W_\alpha)\mathcal{L}V \le \begin{cases} \max_{\Omega\times\mathbb{R}} |\mathcal{L}U|\zeta_\rho'(W_\alpha), & (x, y, t) \in \Omega \times \mathbb{R}, \\ -\gamma\zeta_\rho'(W_\alpha), & (x, y, t) \in \Omega^c \times \mathbb{R}, \end{cases} \tag{6.7}$$

where $\Omega := \{(x, y) : |y| \le a\}$, $\Omega^c := \{(x, y) : |y| > a\}$, and $a > 0$ is as Definition 6.1 (1). Since $\zeta_\rho'' \le 0$ on $[\rho, \rho + 2]$ and $\zeta_\rho'' = 0$ otherwise, we derive from the nonnegative definiteness of $A = (a^{ij})$ that $\zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V \le 0$ on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$, which together with (6.6) and (6.7) gives

$$\begin{aligned} &- \alpha \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \zeta_\rho'(W_\alpha)\mathcal{L}U \mathrm{d}\mu_\tau\mathrm{d}\tau + \gamma \int_t^{t+T} \iint_{\Omega^c} \zeta_\rho'(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau \\ &\quad \le \max_{\Omega\times\mathbb{R}} |\mathcal{L}V| \int_t^{t+T} \iint_{\Omega} \zeta_\rho'(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau \le \max_{\Omega\times\mathbb{R}} |\mathcal{L}V| \times T. \end{aligned}$$

In particular, $-\alpha \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \zeta_\rho'(W_\alpha)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau \le \max_{\Omega\times\mathbb{R}} |\mathcal{L}V| \times T$. Note that $\lim_{\rho\to\infty} \zeta_\rho'(W_\alpha) = 1$. Letting $\rho \to \infty$ in the last inequality, we find

$$-\alpha \int_t^{t+T} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \mathcal{L}U \mathrm{d}\mu_\tau\mathrm{d}\tau \le \max_{\Omega\times\mathbb{R}} |\mathcal{L}V| \times T.$$

To see $\mathrm{supp}(\mu) \subset \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$, we suppose on the contrary that there exists a closed set $B \subset \mathbb{R}^m$ satisfying $0_m \notin B$ such that $\int_t^{t+T} \mu_\tau(\{(x, y) : x \in B\})\mathrm{d}\tau > 0$ for all $t \in \mathbb{R}$. Note that $\sup_{B\times\mathbb{R}} \mathcal{L}U < 0$ by **(B2)**. Hence,

$$-\alpha \left(\sup_{B\times\mathbb{R}} \mathcal{L}U\right) \int_t^{t+T} \mu_\tau(\{(x, y) : x \in B\})\mathrm{d}\tau \le \max_{\Omega\times\mathbb{R}} |\mathcal{L}V| \times T,$$

which leads to a contradiction when letting $\alpha \to \infty$.

Step 3. We claim that $\mathrm{supp}(\mu) = \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$ and $\mu = (\mu_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (6.4).

Define $\mu_t^*(B) := \mu_t(\{0_m\} \times B)$ for $B \in \mathcal{B}(\mathbb{R}^n)$ and $t \in \mathbb{R}$, and $\mu^* := (\mu_t^*)_{t\in\mathbb{R}}$, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra of $\mathbb{R}^n$. We further define $\mathcal{L}_0 := \partial_t + \alpha^{ij}\partial^2_{y_iy_j} + \beta^i\partial_{y_i}$, where $\alpha^{ij}(y,t) = a^{ij}(0_m,y,t)$ and $\beta^i(y,t) := g^i(0_m,y,t)$ for $(y,t) \in \mathbb{R}^n \times \mathbb{R}$ and $i,j = 1,\dots,n$.

As $\mu = (\mu_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (6.4) and is supported on $\{0_m\} \times \mathbb{R}^n \times \mathbb{R}$, we find $\mu_t^*(\mathbb{R}^n) = 1$ and $\mu_t^* = \mu_{t+T}^*$ for $t \in \mathbb{R}$ and $\int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathcal{L}_0\phi \mathrm{d}\mu_\tau^* \mathrm{d}\tau = 0$ for all $\phi \in C_0^{2,1}(\mathbb{R}^n \times \mathbb{R})$. That is, $\mu^* = (\mu_t^*)_{t\in\mathbb{R}}$ is a periodic probability solution of the following FPE:

$$\partial_t u = \partial^2_{y_iy_j}(\alpha^{ij}u) - \partial_{y_i}(\beta^i u), \quad (y,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{6.8}$$

By Theorem 2.1, μ^* admits a positive density on $\mathbb{R}^n \times \mathbb{R}$. Hence, $\mathrm{supp}(\mu^*) = \mathbb{R}^n \times \mathbb{R}$, or equivalently, $\mathrm{supp}(\mu) = \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$. Note that V is a Lyapunov function of type (L2) with respect to $\mathcal{L}_0$. Hence, we apply Theorem A to conclude that (6.4) as well as (6.8) admit a unique periodic probability solution. ∎

When the semi-Lyapunov function of type (L2) in Theorem 6.2 is of type (L3), we can apply Theorem B to deduce a convergence result.

THEOREM 6.3. *Assume* **(B1)**, **(B2)** *and that* $\mathcal{L}_{SF}$ *admits a semi-Lyapunov function of type* (L3). *Then, for any global probability solution* $\mu = (\mu_t)_{t\in(s,\infty)}$ *of the Cauchy problem associated to* (6.4) *with initial condition* $\mu_s = \nu$, *where* $\nu \in \mathcal{M}_p(\mathbb{R}^m \times \mathbb{R}^n)$ *is compactly supported, there holds that for each* $t \in (s, s+T]$,

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^K \iint_{\mathbb{R}^m\times\mathbb{R}^n} \phi \mathrm{d}\mu_{t+kT} = \iint_{\mathbb{R}^m\times\mathbb{R}^n} \phi \mathrm{d}\tilde{\mu}_t \quad \forall \phi \in C_c^2(\mathbb{R}^m \times \mathbb{R}^n),$$

where $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *is the unique periodic probability solution of* (6.4)

Proof. Let $V \in C_T^{2,1}(\mathbb{R}^n \times \mathbb{R})$ be the strong semi-Lyapunov function with respect to $\mathcal{L}_{SF}$. Arguing as in the proof of Theorem 6.2, we show that the function $W(x,y,t) := U(x,t) + V(y,t)$ for all $(x,y,t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ is a Lyapunov function of type (L3). The conclusion follows from Theorems B and 6.2. ∎

6.3. Convergence of weak solutions of an SDE. Fix $s \in \mathbb{R}$. Consider the following initial value problem associated to the SDE (1.2):

$$\begin{cases} \mathrm{d}x = V(x,t)\mathrm{d}t + G(x,t)\mathrm{d}W_t, & x \in \mathcal{U}, \\ x_s \sim \nu, \end{cases} \tag{6.9}$$

where ν is a given Borel probability measure on $\mathcal{U}$. We assume V and G are continuous on $\mathcal{U} \times \mathbb{R}$ and T-periodic in t for some $T > 0$.

Recall that a *(globally defined) weak solution* of (6.9) is a triple of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq s}, \mathbb{P})$, an adapted Wiener process $(W_t)_{t\geq s}$, and an adapted stochastic process $(X_t)_{t\geq s}$ such that

$$X_s \sim \nu, \quad X_t = X_s + \int_s^t V(X_\tau,\tau)\mathrm{d}\tau + \int_s^t G(X_\tau,\tau)\mathrm{d}W_\tau \quad \forall t > s.$$

In what follows, we simply call $(X_t)_{t\geq s}$ a weak solution of (6.9) without mentioning the underlying probability space and Wiener process.

Let $(a^{ij}) = \frac{1}{2}GG^\top$, and set $\mathcal{L} := \partial_t + a^{ij}\partial^2_{ij} + V^i\partial_i$.

LEMMA 6.1. *Let* $(X_t)_{t\geq s}$ *be a weak solution of* (6.9) *and* μ_t *be the distribution of* X_t *for* $t \geq s$. *Then* $(\mu_t)_{t\in(s,\infty)}$ *is a global probability solution of the Cauchy problem* (1.3) *and* (1.6).

Proof. It is well known [24] that under the current assumptions on the coefficients, $(X_t)_{t\geq s}$ induces a solution of the associated martingale problem. Hence, for each $\phi \in C_c^2(\mathcal{U})$, there holds that $\mathbb{E}\phi(X_t) - \mathbb{E}\phi(X_s) - \int_s^t \mathbb{E}\left[\mathcal{L}\phi(X_\tau)\right]\mathrm{d}\tau = 0$ for all $t > s$, that is, $\int_{\mathcal{U}} \phi \mathrm{d}\mu_t - \int_{\mathcal{U}} \phi \mathrm{d}\nu - \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau = 0$ for all $t > s$. The conclusion then follows from Lemma 2.1 (1) (b). ∎

In the presence of Lemma 6.1, we can apply Theorems B and C to derive the following convergence results of weak solutions.

THEOREM 6.4. *Suppose* $\mathcal{L}$ *admits a Lyapunov function of type* (L3) U. *Let* $(X_t)_{t\geq s}$ *be a weak solution of* (6.9) *with* $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$. *Then for any sequence of positive integers* $\{n_j\}_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} n_j = \infty$, *there exists a subsequence, still denoted by* $\{n_j\}_{j\in\mathbb{N}}$, *and a periodic probability solution* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *of* (1.3) *such that the following hold:*

(1) *for each bounded* $\phi \in C_T(\mathcal{U} \times \mathbb{R})$, *there holds that*

$$\lim_{j\to\infty} \frac{1}{n_j T} \int_t^{t+n_j T} \mathbb{E}\phi(X_\tau)\mathrm{d}\tau = \frac{1}{T}\int_0^T \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau \quad \forall t \geq s;$$

(2) *for each* $\psi \in C_c^2(\mathcal{U})$, *there holds that*

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \mathbb{E}\psi(X_{t+kT}) = \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t \quad \textit{for a.e. } t > s.$$

In particular, if (1.3) *admits a unique periodic probability solution* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$, *then the convergence in* (1) *and* (2) *holds for the whole sequence* $\mathbb{N}$.

THEOREM 6.5. *Assume* $GG^\top$ *is locally Lipschitz continuous in* x *and pointwise positive definite. Suppose* $\mathcal{L}$ *admits a Lyapunov function of type* (L4) U. *Then there exist positive constants* C_1 *and* C_2 *such that any weak solution* $(X_t)_{t\geq s}$ *of* (6.9) *with* $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$ *satisfies*

$$\left|\mathbb{E}\phi(X_t) - \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t\right| \leq C_1 e^{-C_2(t-s)} \quad \forall t > s$$

for all bounded measurable function ϕ *on* $\mathcal{U}$, *where* $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ *is the unique probability solution of* (1.3).

Acknowledgment. We would like to thank the anonymous referees for carefully reading the manuscript and providing invaluable suggestions.

REFERENCES

[1] C. D. ALIPRANTIS AND K. C. BORDER, *Infinite Dimensional Analysis. A Hitchhiker's Guide*, 3rd ed., Springer, Berlin, 2006.

[2] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures Math. ETH Zürich, Birkhäuser Verlag, Basel, 2005.

[3] A. ARAPOSTATHIS, V. S. BORKAR, AND M. K. GHOSH, *Ergodic Control of Diffusion Processes*, Encyclopedia Math. Appl. 143, Cambridge University Press, Cambridge, UK, 2012.

[4] V. I. BOGACHEV, N. V. KRYLOV, AND M. RÖCKNER, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations, 26 (2001), pp. 2037–2080.

[5] V. I. Bogachev, N. V. Krylov, and M. Rökner, *Elliptic and parabolic equations for measures*, Uspekhi Mat. Nauk, 64 (2009), pp. 5–116 (in Russian); Russian Math. Surveys, 64 (2009), pp. 973–1078 (in English).

[6] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov, *Fokker-Planck-Kolmogorov Equations*, Math. Surveys Monogr. 207, American Mathematical Society, Providence, RI, 2015.

[7] V. I. Bogachev, M. Röckner, and S. V. Shaposhnikov, *On uniqueness problems related to elliptic equations for measures*, J. Math. Sci. (N.Y.), 176 (2011), pp. 759–773.

[8] V. I. Bogachev, M. Röckner, and S. V. Shaposhnikov, *On uniqueness problems related to the Fokker-Planck-Kolmogorov equation for measures,* J. Math. Sci. (N.Y.), 179 (2011), pp. 7–47.

[9] V. I Bogachev, M. Rökner, and V. Stannat, *Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions*, Mat. Sb., 193 (2002), pp. 3–36 (in Russian); Sb. Math., 193 (2002), pp. 945–976 (in English).

[10] C. Cheban and Z. Liu, *Periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent and Poisson stable solutions for stochastic differential equations*, J. Differential Equations, 269 (2020), pp. 3652–3685.

[11] F. Chen, Y. Han, Y. Li, and X. Yang, *Periodic solutions of Fokker-Planck equations*, J. Differential Equations, 263 (2017), pp. 285–298.

[12] K. Chung, *A Course in Probability Theory*, 3rd ed., Academic Press, San Diego, CA, 2001.

[13] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Math. Soc. Lecture Note Ser. 229, Cambridge University Press, Cambridge, UK, 1996.

[14] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes Volume* I: *Elementary Theory and Methods*, 2nd ed., Springer-Verlag, New York, Berlin, Heidelberg, 2003.

[15] C. Feng, H. Zhao, and J. Zhong, *Existence of Geometric Ergodic Periodic Measures of Stochastic Differential Equations*, preprint, https://arxiv.org/abs/1904.08091.

[16] C. Feng, H. Zhao, and B. Zhou, *Pathwise random periodic solutions of stochastic differential equations*, J. Differential Equations, 251 (2011), pp. 119–149.

[17] M. Hairer, *Convergence of Markov Processes*, http://www.hairer.org/notes/Convergence.pdf.

[18] W. Huang, M. Ji, Z. Liu, and Y. Yi, *Steady states of Fokker-Planck equations:* I. *Existence*, J. Dynam. Differential Equations, 27 (2015), pp. 721–742.

[19] W. Huang, M. Ji, Z. Liu, and Y. Yi, *Steady states of Fokker-Planck equations:* III. *Degenerate diffusion*, J. Dynam. Differential Equations, 28 (2016), pp. 127–141.

[20] W. Huang, M. Ji, Z. Liu, and Y. Yi, *Concentration and limit behaviors of stationary measures*, Phys. D, 369 (2018), pp. 1–17.

[21] A. Ichikawa, *Bounded solutions and periodic solutions of a linear stochastic evolution equation*, in Probability Theory and Mathematical Statistics (Kyoto, 1986), Lecture Notes in Math. 1299, Springer, Berlin, 1988, pp. 124–130.

[22] M. Ji, W. Qi, Z. Shen, and Y. Yi, *Existence of periodic probability solutions to Fokker-Planck equations with applications*, J. Funct. Anal., 277 (2019), 108281.

[23] M. Ji, Z. Shen, and Y. Yi, *Convergence to equilibrium in Fokker-Planck equations*, J. Dynam. Differential Equations, 31 (2019), pp. 1591–1615.

[24] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Grad. Texts Math. 113, Springer-Verlag, New York, 1991.

[25] R. Z. Khasminskii, *Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations*, Teor. Verojatnost. i Primenen., 5 (1960), pp. 196–214 (in Russian).

[26] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*, Stoch. Model. Appl. Probab. 66, Springer, Heidelberg, 2012.

[27] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type,* Transl. Math. Monogr., American Mathematical Society, Providence, RI, 1968.

[28] O. A. Manita and V. S. Shaposhnikov, *On the Cauchy problem for Fokker-Planck-Kolmogorov equations with potential terms on arbitrary domains*, J. Dynam. Differential Equations, 28 (2016), pp. 493–518.

[29] S. Meyn and R. L Tweedie, *Markov Chains and Stochastic Stability*, 2nd ed., Cambridge University Press, Cambridge, UK, 2009.

[30] T. Morozan, *Periodic solutions of affine stochastic differential equations*, Stochastic Anal. Appl., 4 (1986), pp. 87–110.

[31] H. C. Öttinger, *Stochastic Processes in Polymeric Fluids. Tools and Examples for Developing Simulation Algorithms*, Springer-Verlag, Berlin, 1996.

[32] L. Rey-Bellet, *Ergodic properties of Markov processes*, in Open Quantum Systems. II, Lecture Notes in Math. 1881, Springer, Berlin, 2006, pp. 1–39.
[33] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications*, 2nd ed., Springer Ser. Synergetics 18, Springer-Verlag, Berlin, 1989.
[34] S. V. Shaposhnikov, *On the uniqueness of a probabilistic solution of the Cauchy problem for the Fokker-Planck-Kolmogorov equation*, Teor. Veroyatn. Primen., 56 (2011), pp. 77–99 (in Russian); Theory Probab. Appl., 56 (2012), pp. 96–115 (in English), https://doi.org/10.1137/S0040585X97985212.
[35] H. Zhao and Z.-H. Zheng, *Random periodic solutions of random dynamical systems*, J. Differential Equations, 246 (2009), pp. 2020–2038.