SUPPLEMENTARY MATERIALS: CONVERGENCE TO PERIODIC PROBABILITY SOLUTIONS IN FOKKER-PLANCK EQUATIONS*

MIN JI[†], WEIWEI QI[‡], ZHONGWEI SHEN[§], AND YINGFEI YI[¶]

In this supplement, we provide the proofs of (2.3) and Lemma 3.1.

SM1. Proof of (2.3). Let $\rho_1, \rho_2 \in C(\mathcal{U}\times(s,\infty))$ be respectively a global probability solution and a global sub-probability solution of the Cauchy problem (1.3) and (1.6). Define $w := \frac{\rho_2}{\rho_1}$ and $f_\lambda(t) := e^{\lambda(1-t)} - e^{\lambda}$ for $t \geq 0$, where $\lambda > 0$ is a parameter. Then for any non-negative function $\phi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, there holds

$$\int_{\mathcal{U}} f_\lambda(w)\phi \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}}\phi \mathrm{d}\nu + \int_s^t\int_{\mathcal{U}} f(w)\mathcal{L}\phi \mathrm{d}\mu_\tau^1 \mathrm{d}\tau, \quad \forall t > s. \tag{SM1.1}$$

Note that the above inequality is just (2.3).

The rest of this section is devoted to the proof of (SM1.1).

Proof of (SM1.1)**.** Define

$$\eta(x) = \begin{cases} c_d e^{-\frac{1}{1-|x|^2}}, & \text{if} \quad |x| \leq 1,\\ 0, & \text{if} \quad |x| > 1,\end{cases}$$

where $c_d > 0$ is such that $\int_{\mathbb{R}^d}\eta dx = 1$. It is well-known that $\eta \in C_c^\infty(\mathbb{R}^d)$. Let $\eta_\epsilon(x) := \frac{1}{\epsilon^d}\eta(\frac{x}{\epsilon})$ for $x \in \mathbb{R}^d$ and $0 < \epsilon \ll 1$.

For a measurable function $g : \mathcal{U}\times(s,\infty) \to \mathbb{R}$, we define

$$g_\epsilon(x,t) := \int_{\{y\in\mathbb{R}^d : x-y\in\mathcal{U}\}} g(x-y,t)\eta_\epsilon(y)\mathrm{d}y, \quad (x,t) \in \mathcal{U}\times(s,\infty).$$

In particular, for each $i = 1, 2$ and $(x,t) \in \mathcal{U}\times(s,\infty)$,

$$\rho_{i,\epsilon}(x,t) = \int_{\{y\in\mathbb{R}^d : x-y\in\mathcal{U}\}} \rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y = \int_{\mathcal{U}}\rho_i(y,t)\eta_\epsilon(x-y)\mathrm{d}y.$$

It is not hard to check that

$$\lim_{\epsilon\to 0}\rho_{i,\epsilon} = \rho_i \quad \text{locally uniformly in} \quad \mathcal{U}\times(s,\infty), \tag{SM1.2}$$

and that for each $0 < \epsilon \ll 1$,

$$\lim_{t\to s}\rho_{i,\epsilon}(x,t) = \nu_\epsilon(x), \quad x \in \mathcal{U}, \tag{SM1.3}$$

*Supplementary material for SIMA#M131912
https://doi.org/10.1137/20M1319127

[†]Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Beijing 100190, People's Republic of China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China (jimin@math.ac.cn).

[‡]University of Chinese Academy of Sciences, Beijing 100049, China, and Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (wqi2@ualberta.ca).

[§]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (zhongwei@ualberta.ca).

[¶]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada, and School of Mathematics, Jilin University, Changchun 130012, People's Republic of China (yingfei@ualberta.ca).

where $\nu_\epsilon(x) := \int_{\mathcal{U}} \eta_\epsilon(x-y)\mathrm{d}\nu(y)$ for $x \in \mathcal{U}$. Note that for each $0 < \epsilon \ll 1$ and $i = 1, 2$, there holds $\rho_{i,\epsilon}(\cdot, t) \le |\eta_\epsilon|_\infty$ on $\mathcal{U}$ for each $t \in (s, \infty)$, which together with (SM1.3) and the dominated convergence theorem implies that

$$\lim_{t\to s^+} \rho_{i,\epsilon}(\cdot, t) = \nu_\epsilon \quad \text{in} \quad L^1(\mathcal{U}). \tag{SM1.4}$$

It is straightforward to check that for each $i = 1, 2$, $\rho_{i,\epsilon}$ satisfies

$$\partial_t \rho_{i,\epsilon} = \partial_{kl}(a^{kl}\rho_{i,\epsilon}) - \partial_k((V^k\rho_i)_\epsilon - R^k_{\rho_i,\epsilon}),$$

where $R^k_{\rho_i,\epsilon} := \partial_l(a^{kl}\rho_i)_\epsilon - \partial_l(a^{kl}\rho_{i,\epsilon})$. Set $w_\epsilon := \frac{\rho_{2,\epsilon}}{\rho_{1,\epsilon}}$. Multiplying by $\phi \in C^{2,1}_c(\mathcal{U}\times\mathbb{R})$ the equation satisfied by $\rho_{2,\epsilon}$ and integrating by parts, we arrive at

$$\int_{t_1}^{t_2}\int_{\mathcal{U}} \partial_t(w_\epsilon\rho_{1,\epsilon})\phi \mathrm{d}x\mathrm{d}\tau = \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[w_\epsilon\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + \left((V^k\rho_2)_\epsilon - R^k_{\rho_2,\epsilon}\right)\partial_k\phi\right]\mathrm{d}x\mathrm{d}\tau$$

for all $t_2 > t_1 > s$. Setting $\phi = f'_\lambda(w_\epsilon)\psi$ in the above equality, where $\psi \in C^{2,1}_c(\mathcal{U}\times\mathbb{R})$ is non-negative, we find

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathcal{U}} \partial_t(w_\epsilon\rho_{1,\epsilon})f'_\lambda(w_\epsilon)\psi \mathrm{d}x\mathrm{d}\tau \\
&\quad= \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[w_\epsilon\rho_{1,\epsilon}a^{kl}\partial_{kl}(f'_\lambda(w_\epsilon)\psi) + \left((V^k\rho_2)_\epsilon - R^k_{\rho_2,\epsilon}\right)\partial_k(f'_\lambda(w_\epsilon)\psi)\right]\mathrm{d}x\mathrm{d}\tau
\end{aligned} \tag{SM1.5}$$

for all $t_2 > t_1 > s$. Note that there holds the equality

$$\partial_t(w_\epsilon\rho_{1,\epsilon})f'_\lambda(w_\epsilon) = \partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon}) - \left(f_\lambda(w_\epsilon) - f'_\lambda(w_\epsilon)w_\epsilon\right)\partial_t\rho_{1,\epsilon}.$$

Inserting the above equality into the left-hand side of (SM1.5) and then utilizing the equation satisfied by $\rho_{1,\epsilon}$, the following equality follows from straightforward calculations.

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathcal{U}} \partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon})\psi \mathrm{d}x\mathrm{d}\tau + \int_{t_1}^{t_2}\int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau \\
&\quad= \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[f_\lambda(w_\epsilon)\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau \\
&\qquad+ \int_{t_1}^{t_2}\int_{\mathcal{U}} (W_\epsilon\cdot\nabla\psi)f'_\lambda(w_\epsilon)\mathrm{d}x\mathrm{d}\tau \\
&\qquad+ \int_{t_1}^{t_2}\int_{\mathcal{U}} (W_\epsilon\cdot\nabla w_\epsilon)f''_\lambda(w_\epsilon)\psi\mathrm{d}x\mathrm{d}\tau \\
&\qquad- \int_{t_1}^{t_2}\int_{\mathcal{U}} f'_\lambda(w_\epsilon)R^k_{\rho_2,\epsilon}\partial_k\psi\mathrm{d}x\mathrm{d}\tau \\
&\qquad+ \int_{t_1}^{t_2}\int_{\mathcal{U}} (f_\lambda(w_\epsilon) - w_\epsilon f'_\lambda(w_\epsilon))R^k_{\rho_1,\epsilon}\partial_k\psi\mathrm{d}x\mathrm{d}\tau \\
&\qquad+ \int_{t_1}^{t_2}\int_{\mathcal{U}} f''_\lambda(w_\epsilon)(R^k_{\rho_2,\epsilon} - w_\epsilon R^k_{\rho_1,\epsilon})\partial_k w_\epsilon\psi\mathrm{d}x\mathrm{d}\tau \\
&\quad= \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[f_\lambda(w_\epsilon)\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau + \sum_{j=1}^{5} I_j,
\end{aligned} \tag{SM1.6}$$

where $W_\epsilon := (V\rho_2)_\epsilon - (V\rho_1)_\epsilon w_\epsilon$.

We estimate the terms I_j, $j = 1, \dots, 5$. Note that $f'_\lambda(x) = -\lambda e^{\lambda(1-x)}$ and $f''_\lambda(x) = \lambda^2 e^{\lambda(1-x)}$. Obviously,

$$|I_1| \le \lambda e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |W_\epsilon| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau.$$

By Young's inequality, there holds for any $\delta > 0$,

$$\begin{aligned} |I_2| &\le \delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{1}{4\delta} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_{t_1}^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|W_\epsilon|^2}{\rho_{1,\epsilon}} f''_\lambda(w_\epsilon) \mathrm{d}x\mathrm{d}\tau \\ &\le \delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|W_\epsilon|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

For I_3 and I_4, we have

$$\begin{aligned} |I_3| &\le \lambda e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |R_{\rho_2,\epsilon}| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau, \\ |I_4| &\le \lambda e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |R_{\rho_1,\epsilon}| (1+\lambda w_\epsilon) e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau, \end{aligned}$$

where $R_{\rho_i,\epsilon} = (R^k_{\rho_i,\epsilon})$, $i = 1, 2$. For I_5, we find from Young's inequality and $f''_\lambda(x) = \lambda^2 e^{\lambda(1-x)}$ that

$$\begin{aligned} |I_5| &\le 2\delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}} |w_\epsilon|^2 e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau, \quad \delta > 0, \end{aligned}$$

It follows from (SM1.6) that

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathcal{U}} \partial_t (f_\lambda(w_\epsilon)\rho_{1,\epsilon}) \psi \mathrm{d}x\mathrm{d}\tau + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \psi f''_\lambda(w_\epsilon) a^{kl} \partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad \le \int_{t_1}^{t_2} \int_{\mathcal{U}} [f_\lambda(w_\epsilon)\rho_{1,\epsilon} a^{kl} \partial_{kl}\psi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon \partial_k\psi] \mathrm{d}x\mathrm{d}\tau \\ &\qquad + 3\delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon,\delta), \quad \forall t_2 > t_1 > s, \end{aligned} \tag{SM1.7}$$

where

$$
\begin{aligned}
\Omega(\epsilon,\delta) &= \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))} |W_\epsilon| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad + \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|W_\epsilon|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))} \left[|R_{\rho_2,\epsilon}| + |R_{\rho_1,\epsilon}|(1+\lambda w_\epsilon)\right] e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad + \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad + \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}} |w_\epsilon|^2 e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

Arguing as in the proof of [SM1, Lemma 3.1 and Lemma 3.2], we find $\lim_{\epsilon\to 0}\Omega(\epsilon,\delta) = 0$ for all $\delta > 0$. It follows from the Newton-Leibniz formula that

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathcal{U}} \partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon})\psi \mathrm{d}x\mathrm{d}\tau &= \int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x \\
&\quad - \int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x \\
&\quad - \int_{t_1}^{t_2}\int_{\mathcal{U}} f_\lambda(w_\epsilon)\rho_{1,\epsilon}\partial_t\psi \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{SM1.8}
$$

As $\psi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, when restricted on $\mathcal{U}\times[s,t_2]$, is compactly supported, and (a^{ij}) is locally uniform positive definite, there is a positive number m such that

$$
\left(a^{ij}\partial_i w_\epsilon \partial_j w_\epsilon\right)(x,t) \geq m|\nabla w_\epsilon(x,t)|^2, \quad \forall (x,t) \in \mathrm{supp}(\psi)\cap(\mathcal{U}\times[s,t_2]).
$$

This together with $f_\lambda'' \geq 0$ and $\psi \geq 0$ yields that

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathcal{U}} \psi f_\lambda''(w_\epsilon) a^{kl}\partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau &\geq \frac{3}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}} \psi f_\lambda''(w_\epsilon) a^{kl}\partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad + \frac{m}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}} \psi f_\lambda''(w_\epsilon)|\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{SM1.9}
$$

Set $\delta = \frac{m}{12}$. We find from (SM1.7), (SM1.8) and (SM1.9) that

$$
\begin{aligned}
&\int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x \\
&\quad \leq \int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x - \frac{3}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}} \psi f_\lambda''(w_\epsilon) a^{kl}\partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[\rho_{1,\epsilon} f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi) + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon \partial_k\psi\right] \mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon, \frac{m}{12}) \\
&\quad \leq \int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x \\
&\qquad + \int_{t_1}^{t_2}\int_{\mathcal{U}} \left[\rho_{1,\epsilon} f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi) + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon \partial_k\psi\right] \mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon, \frac{m}{12}).
\end{aligned} \tag{SM1.10}
$$

Since $|f_\lambda(t)-f_\lambda(1)|\leq \lambda e^\lambda|t-1|$ holds for all $t\geq 0$, we apply the dominated convergence theorem to find for each $0<\epsilon\ll 1$,

$$\begin{aligned}&\int_{\mathcal{U}}|f_\lambda(w_\epsilon(x,t_1))-f_\lambda(1)|\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x\\&\qquad\leq \lambda e^\lambda\int_{\mathcal{U}}|\rho_{2,\epsilon}(x,t_1)-\rho_{1,\epsilon}(x,t_1)|\psi(x,t_1)\mathrm{d}x\to 0\quad\text{as}\quad t_1\to s^+,\end{aligned}$$

By (SM1.4) and the dominated convergence theorem, we deduce

$$\begin{aligned}&\int_{\mathcal{U}}|\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)-\nu_\epsilon(x)\psi(x,s)|\mathrm{d}x\\&\quad\leq\int_{\mathcal{U}}|\rho_{1,\epsilon}(x,t_1)-\nu_\epsilon(x)|\psi(x,t_1)\mathrm{d}x+\int_{\mathcal{U}}\nu_\epsilon(x)|\psi(x,t_1)-\psi(x,s)|\mathrm{d}x\\&\quad\leq\max_{\mathcal{U}\times[s,t_2]}|\psi|\cdot\|\rho_{1,\epsilon}(\cdot,t_1)-\nu_\epsilon(\cdot)\|_{L^1(\mathcal{U})}+\int_{\mathcal{U}}\nu_\epsilon(x)|\psi(x,t_1)-\psi(x,s)|\mathrm{d}x\\&\quad\to 0\quad\text{as}\quad t_1\to s^+.\end{aligned}$$

Thus,$\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t)\psi(x,t_1)\mathrm{d}x\to f_\lambda(1)\int_{\mathcal{U}}\nu_\epsilon(x)\psi(x,s)\mathrm{d}x$ as $t_1\to s$. Note that $\rho_{1,\epsilon}f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi$ is integrable on $\mathcal{U}\times[s,t_2)$. It follows that

$$\begin{aligned}&\lim_{t_1\to s^+}\int_{t_1}^{t_2}\int_{\mathcal{U}}\rho_{1,\epsilon}\left[f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau\\&\qquad=\int_s^{t_2}\int_{\mathcal{U}}\left[\rho_{1,\epsilon}f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Passing to the limit $t_1\to s^+$ in the inequality (SM1.10) yields

$$\begin{aligned}&\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x\\&\quad\leq f_\lambda(1)\int_{\mathcal{U}}(\nu*\eta_\epsilon)\psi\mathrm{d}x+\int_s^{t_2}\int_{\mathcal{U}}\left[f_\lambda(w_\epsilon)\rho_{1,\epsilon}f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)\right.\\&\qquad\left.+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\frac{m}{12}),\quad\forall t_2>s.\end{aligned}$$

As $\lim_{\epsilon\to 0}\Omega(\epsilon,\frac{m}{12})=0$, we let $\epsilon\to 0$ in the above inequality to find from (SM1.2), (SM1.3) and the dominated convergence theorem that (SM1.1), with t_2 and ψ replaced by t and ϕ, respectively, holds. This completes the proof. ∎

SM2. Proof of Lemma 3.1. The proof follows from similar arguments leading to (2.3). To proceed, we need a lemma concerning the convergence of mollified functions.

Let $\{\eta_\epsilon\}$ be as defined in the proof of (SM1.1). For a T-periodic measurable function g on $\mathcal{U}\times\mathbb{R}$, we define

$$g_\epsilon(x,t):=\int_{\{y\in\mathbb{R}^d:x-y\in\mathcal{U}\}}g(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad(x,t)\in\mathcal{U}\times\mathbb{R}.\tag{SM2.1}$$

In particular, for each $i=1,2$,

$$\rho_{i,\epsilon}(x,t)=\int_{\{y\in\mathbb{R}^d:x-y\in\mathcal{U}\}}\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad(x,t)\in\mathcal{U}\times\mathbb{R}.\tag{SM2.2}$$

Set $w_\epsilon := \frac{\rho_{2,\epsilon}}{\rho_{1,\epsilon}}$, $W_\epsilon := (V\rho_2)_\epsilon - (V\rho_1)_\epsilon w_\epsilon$ and $R^k_{\rho_i,\epsilon} := \partial_l(a^{kl}\rho_i)_\epsilon - \partial_l(a^{kl}\rho_{i,\epsilon})$ for $i=1,2$ and $k=1,\dots,d$.

LEMMA SM2.1. *Assume* **(H1)**. *Then, both W_ϵ and $R^k_{\rho_i,\epsilon}$ (for each $i=1,2$ and $k=1,\dots,d$) converge to 0 in $L^p_{loc}(\mathcal{U}\times\mathbb{R})$ as $\epsilon\to 0$.*

Proof. Let $\mathcal{K}\subset\subset\mathcal{U}$ and $t>s$. We see from the formula (SM2.1) that there is an $\epsilon_{\mathcal{K}}>0$ such that for each T-periodic measurable function g on $\mathcal{U}\times\mathbb{R}$, there holds

$$g_\epsilon(x,t) = \int_{\mathbb{R}^d} g(x-y,t)\eta_\epsilon(y)\mathrm{d}y, \quad (x,t)\in\mathcal{K}\times\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_{\mathcal{K}}).$$

Then, it follows from definitions of W_ϵ and w that for each $\epsilon\in(0,\epsilon_{\mathcal{K}})$, there holds

$$\begin{aligned} W_\epsilon(x,t) &= \int_{\mathbb{R}^d} V(x-y,t)\rho_2(x-y,t)\eta_\epsilon(y)\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^d} V(x-y,t)\rho_1(x-y,t)\eta_\epsilon(y)\mathrm{d}y w_\epsilon(x,t) \\ &= \int_{\mathbb{R}^d} V(x-y,t)\rho_1(x-y,t)\eta_\epsilon(y)\left[w(x-y,t)-w_\epsilon(x,t)\right]\mathrm{d}y \end{aligned}$$

for all $(x,t)\in\mathcal{K}\times\mathbb{R}$. Since $w\in C_T(\mathcal{U}\times\mathbb{R})$, we see that for any $0<\delta\ll 1$, there is an $\epsilon_0=\epsilon_0(\delta)\in(0,\epsilon_{\mathcal{K}})$ such that $\mathcal{K}_{\epsilon_0} := \{x\in\mathcal{U}:\mathrm{dist}(x,\mathcal{K})<\epsilon_0\}\subset\subset\mathcal{U}$,

$$\begin{aligned} &\sup_{|y|\le\epsilon_0}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}} |w(x-y,t)-w(x,t)| < \frac{\delta}{2}, \quad \text{and} \\ &\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}} |w(x,t)-w_\epsilon(x,t)| < \frac{\delta}{2}, \quad \forall\epsilon\in(0,\epsilon_0). \end{aligned}$$

It follows that $\sup_{|y|\le\epsilon_0}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}|w(x-y,t)-w_\epsilon(x,t)|<\delta$ for all $\epsilon\in(0,\epsilon_0)$. This together with Hölder's inequality yields

$$\begin{aligned} \int_s^t\int_{\mathcal{K}} |W_\epsilon|^p \mathrm{d}x\mathrm{d}\tau &\le \delta^p \int_s^t\int_{\mathcal{K}} \left|\int_{\mathbb{R}^d} V(x-y,\tau)\rho_1(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right|^p \mathrm{d}x\mathrm{d}\tau \\ &\le \delta^p \int_s^t\int_{\mathcal{K}} \left[\left(\int_{\mathbb{R}^d} |V(x-y,\tau)|^p\rho_1^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right)\left(\int_{\mathbb{R}^d} 1^{p'}\eta_\epsilon(y)\mathrm{d}y\right)^{\frac{p}{p'}}\right]\mathrm{d}x\mathrm{d}\tau \\ &= \delta^p \int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d} |V(x-y,\tau)|^p\rho_1^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau, \quad \forall\epsilon\in(0,\epsilon_0). \end{aligned}$$

A simple change of variable gives

$$\begin{aligned} \int_s^t\int_{\mathcal{K}} |W_\epsilon|^p\mathrm{d}x\mathrm{d}\tau &\le \delta^p\int_s^t\int_{\mathcal{K}}\int_{\mathcal{K}_{\epsilon_0}} |V(z,\tau)|^p\rho_1^p(z,\tau)\eta_\epsilon(x-z)\mathrm{d}z\mathrm{d}x\mathrm{d}\tau \\ &\le \delta^p\int_s^t\int_{\mathcal{K}_{\epsilon_0}} |V(z,\tau)|^p\rho_1^p(z,\tau)\left(\int_{\mathbb{R}^d}\eta_\epsilon(x-z)\mathrm{d}x\right)\mathrm{d}z\mathrm{d}\tau \\ &\le \delta^p\left(\sup_{\mathcal{K}_{\epsilon_0}\times\mathbb{R}}\rho_1^p\right)\int_s^t\int_{\mathcal{K}_{\epsilon_0}} |V|^p\mathrm{d}z\mathrm{d}\tau, \quad \forall\epsilon\in(0,\epsilon_0), \end{aligned}$$

where we used Fubini's theorem in the second inequality. Thus,

$$\lim_{\epsilon\to 0}\|W_\epsilon\|_{L^p(\mathcal{K}\times[s,t])} = 0.$$

Now, we deal with $R^k_{\rho_i,\epsilon}$. Note that for $\epsilon\in(0,\epsilon_{\mathcal{K}})$,

$$
\begin{aligned}
R^k_{\rho_i,\epsilon}(x,t) &= \partial_l\int_{\mathbb{R}^d} a^{kl}(x-y,t)\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y - \partial_l\int_{\mathbb{R}^d} a^{kl}(x,t)\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y\\
&= \int_{\mathbb{R}^d}\left[\partial_l a^{kl}(x-y,t)-\partial_l a^{kl}(x,t)\right]\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y\\
&\quad -\int_{\mathbb{R}^d}\left(a^{kl}(x-y,t)-a^{kl}(x,t)\right)\partial_l\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad (x,t)\in\mathcal{K}\times\mathbb{R}.
\end{aligned}
$$

Since $a^{kl}\in L^\infty(\mathbb{R},W^{1,p}_{loc}(\mathcal{U}))$ is T-periodic for each $k,l=1,\dots,d$, we find

$$
\text{(SM2.3)}\qquad \sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\to 0\quad\text{as}\quad\epsilon\to0,\qquad\text{and}
$$

$$
\text{(SM2.4)}\qquad
\begin{aligned}
&\sup_{|y|\le\epsilon}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}\left|a^{kl}(x-y,t)-a^{kl}(x,t)\right|\\
&\quad\le\epsilon^{1-\frac{d}{p}}\operatorname*{ess\,sup}_{t\in\mathbb{R}}\|a^{kl}(\cdot,t)\|_{W^{1,p}(\mathcal{K}_\epsilon\times\mathbb{R})}\to0\quad\text{as}\quad\epsilon\to0,
\end{aligned}
$$

where we used the Sobolev embedding theorem.

Applying Hölder's inequality and (SM2.3), we find

(SM2.5)
$$
\begin{aligned}
&\int_s^t\int_{\mathcal{K}}\left|\int_{\mathbb{R}^d}\left(\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right)\rho_i(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right|^p\mathrm{d}x\mathrm{d}\tau\\
&\quad\le\int_s^t\int_{\mathcal{K}}\left(\int_{\mathbb{R}^d}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\rho_i^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right)\\
&\qquad\qquad\times\left(\int_{\mathbb{R}^d}1^{p'}\eta_\epsilon(y)\mathrm{d}y\right)^{\frac{p}{p'}}\mathrm{d}x\mathrm{d}\tau\\
&\quad=\int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\rho_i^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\\
&\quad\le\left(\sup_{\mathcal{K}_\epsilon\times\mathbb{R}}\rho_i^p\right)\left(\sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\right)\int_{\mathbb{R}^d}\eta_\epsilon(y)\mathrm{d}y\\
&\quad=\left(\sup_{\mathcal{K}_\epsilon\times\mathbb{R}}\rho_i^p\right)\sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\\
&\quad\to0\quad\text{as}\quad\epsilon\to0.
\end{aligned}
$$

Applying Hölder's inequality, a change of variable and (SM2.4), we find

$$
\begin{aligned}
&\int_s^t \int_{\mathcal{K}} \left| \int_{\mathbb{R}^d} \left(a^{kl}(x-y,\tau) - a^{kl}(x,\tau)\right) \partial_l \rho_i(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y \right|^p \mathrm{d}x\mathrm{d}\tau \\
&\quad \le \int_s^t \int_{\mathcal{K}} \int_{\mathbb{R}^d} \left|a^{kl}(x-y,\tau) - a^{kl}(x,\tau)\right|^p |\partial_l \rho_i(x-y,\tau)|^p \eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau \\
&\quad \le \sup_{|y|\le\epsilon} \sup_{(x,\tau)\in\mathcal{K}\times\mathbb{R}} \left|a^{kl}(x-y,\tau) - a^{kl}(x,\tau)\right|^p \\
&\qquad\qquad\qquad \times \int_s^t \int_{\mathcal{K}} \int_{\mathbb{R}^d} |\partial_l \rho_i(x-y,t)|^p \eta_\epsilon(y)\mathrm{d}x\mathrm{d}\tau \\
&\quad \le \sup_{|y|\le\epsilon} \sup_{(x,\tau)\in\mathcal{K}\times\mathbb{R}} \left|a^{kl}(x-y,\tau) - a^{kl}(x,\tau)\right|^p \\
&\qquad\qquad\qquad \times \int_s^t \int_{\mathcal{K}_\epsilon} |\partial_l \rho_i(z,t)|^p \left(\int_{\mathbb{R}^d} \eta_\epsilon(x-z)\mathrm{d}x\right)\mathrm{d}z\mathrm{d}\tau \\
&\quad \le \epsilon^{p-d} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|a^{kl}(\cdot,t)\|^p_{W^{1,p}(\mathcal{K}_\epsilon\times\mathbb{R})} \times \int_s^t \int_{\mathcal{K}_\epsilon} |\partial_l \rho_i|^p \mathrm{d}x\mathrm{d}\tau \\
&\quad \to 0 \quad \text{as} \quad \epsilon \to 0,
\end{aligned} \tag{SM2.6}
$$

where the L^p-integrability of $\partial_l \rho_i$ (for each $l = 1, \dots, d$ and $i = 1, 2$) on $\mathcal{K}_\epsilon \times [s, t]$ follows from $\rho_i \in \mathcal{H}^{1,p}_{loc}(\mathcal{U} \times \mathbb{R})$ for $i = 1, 2$.

It follows from (SM2.5) and (SM2.6) that $\lim_{\epsilon\to 0} \|R^k_{\rho_i,\epsilon}\|_{L^p(\mathcal{K}\times[s,t])} = 0$. This completes the proof. ∎

Proof of Lemma 3.1. We only point out the differences from the arguments leading to (2.3). Fix a non-negative function $\phi \in C^{2,1}_{c,T}(\mathcal{U} \times \mathbb{R})$. We see that the inequality (SM1.7) holds with t_1, t_2, ψ and f_λ replaced by t, $t+T$, ϕ and f, respectively. That is,

$$
\begin{aligned}
&\int_t^{t+T} \int_{\mathcal{U}} \partial_t (f(w_\epsilon)\rho_{1,\epsilon})\phi \mathrm{d}x\mathrm{d}\tau + \int_t^{t+T} \int_{\mathbb{R}^d} \phi f''(w_\epsilon) a^{kl} \partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\
&\quad \le \int_t^{t+T} \int_{\mathcal{U}} [f(w_\epsilon)\rho_{1,\epsilon} a^{kl} \partial_{kl}\phi + f(w_\epsilon)(V^k\rho_1)_\epsilon \partial_k \phi] \mathrm{d}x\mathrm{d}\tau \\
&\qquad + 3\delta \int_t^{t+T} \int_{\mathcal{U}} \phi f''(w_\epsilon)|\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon,\delta), \quad \forall t \in \mathbb{R},
\end{aligned} \tag{SM2.7}
$$

where $\delta > 0$ is to be determined and

$$
\begin{aligned}
\Omega(\epsilon,\delta) = {} & \lambda e^\lambda \sup_{\mathcal{U}\times[t,t+T]} |\nabla\phi| \int_t^{t+T} \int_{\mathrm{supp}(\phi(\cdot,\tau))} |W_\epsilon| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
& + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[t,t+T]} |\phi| \int_t^{t+T} \int_{\mathrm{supp}(\phi(\cdot,\tau))} \frac{|W_\epsilon|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
& + \lambda e^\lambda \sup_{\mathcal{U}\times[t,t+T]} |\nabla\phi| \int_t^{t+T} \int_{\mathrm{supp}(\phi(\cdot,\tau))} \left[|R_{\rho_2,\epsilon}| + |R_{\rho_1,\epsilon}|(1+\lambda w_\epsilon)\right] e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
& + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[t,t+T]} |\phi| \int_t^{t+T} \int_{\mathrm{supp}(\phi(\cdot,\tau))} \frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\
& + \frac{\lambda^2}{4\delta} e^\lambda \sup_{\mathcal{U}\times[t,t+T]} |\phi| \int_t^{t+T} \int_{\mathrm{supp}(\phi(\cdot,\tau))} \frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}} |w_\epsilon|^2 e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

It follows from Lemma SM2.1 that $\lim_{\epsilon\to 0}\Omega(\epsilon,\delta)=0$ for any $\delta>0$.

From the T-periodicity of $w,\rho_{1,\epsilon}$ and w_ϵ, and the Newton-Leibniz formula, we find $\int_t^{t+T}\int_{\mathbb{R}^d}\partial_t(f(w_\epsilon)\rho_{1,\epsilon})\phi \mathrm{d}x\mathrm{d}\tau = -\int_t^{t+T}\int_{\mathbb{R}^d} f(w_\epsilon)\rho_{1,\epsilon}\partial_t\phi \mathrm{d}x\mathrm{d}\tau$. It follows from (SM2.7) that

$$
\begin{aligned}
&\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\
&\quad\le\int_t^{t+T}\int_{\mathcal{U}}[f(w_\epsilon)\rho_{1,\epsilon}(\partial_t\phi+a^{kl}\partial_{kl}\phi)+f(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\phi]\mathrm{d}x\mathrm{d}\tau\\
&\qquad+3\delta\int_t^{t+T}\int_{\mathcal{U}}\phi f''(w)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\delta),\quad\forall t\in\mathbb{R},
\end{aligned}
\tag{SM2.8}
$$

Since ϕ, when restricted on $\mathcal{U}\times[t,t+T]$, is compactly supported, and (a^{ij}) is locally uniform positive definite, there exists $\lambda>0$ such that

$$\left(a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\right)(x,\tau)\ge\lambda|\nabla w_\epsilon(x,\tau)|^2,\quad\forall(x,\tau)\in\operatorname{supp}(\phi)\cap(\mathcal{U}\times[t,t+T]),$$

which together with the positiveness of f'' and ϕ gives

$$\frac{\lambda}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\le\frac{1}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau.$$

Setting $\delta=\frac{\lambda}{6}$ in (SM2.8), we use the above inequality to find

$$
\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\
&\quad\le\int_t^{t+T}\int_{\mathbb{R}^d}[\rho_{1,\epsilon}f(w_\epsilon)(\partial_t\phi+a^{kl}\partial_{kl}\phi)+f(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\phi]\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\frac{\lambda}{6})
\end{aligned}
$$

for all $t\in\mathbb{R}$. The result follows from letting $\epsilon\to 0$. ∎

REFERENCES

[SM1] V. S. Shaposhnikov, *On the uniqueness of the probabilistic solution of the Cauchy problem for the Fokker-Planck-Kolmogorov equation*, (Russian. Russian summary), Teor. Veroyatn. Primen., 56 (2011), pp. 77–99; translation in Theory Probab. Appl., 56 (2012), pp. 96–115.