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ASYMPTOTIC DYNAMICS OF A CLASS OF COUPLED OSCILLATORS DRIVEN BY WHITE NOISES

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This paper is devoted to the study of the asymptotic dynamics of a class of coupled second order oscillators driven by white noises. It is shown that any system of such coupled oscillators with positive damping and coupling coefficients possesses a global random attractor. Moreover, when the damping and the coupling coefficients are sufficiently large, the global random attractor is a one-dimensional random horizontal curve regardless of the strength of the noises, and the system has a rotation number, which implies that the oscillators in the system tend to oscillate with the same frequency eventually and therefore the so-called frequency locking is successful. The results obtained in this paper generalize many existing results on the asymptotic dynamics for a single second order noisy oscillator to systems of coupled second order noisy oscillators. They show that coupled damped second order oscillators with large damping have similar asymptotic dynamics as the limiting coupled first order oscillators as the damping goes to infinite and also that coupled damped second order oscillators have similar asymptotic dynamics as their proper space continuous counterparts, which are of great practical importance.

Keywords: Coupled second order oscillators; white noises; random attractor; random horizontal curve; rotation number; frequency locking.

AMS Subject Classification: 60H10, 34F05, 37H10

1. Introduction

This paper is devoted to the study of the asymptotic dynamics of the following system of second order oscillators driven by additive noises:

$$d\dot{u}_j + \alpha du_j + K(Au)_j dt + \beta g(u_j)dt = f_j dt + \epsilon_j dW_j, \tag{1.1}$$

where $j \in \mathbb{Z}_N^d := \{j = (j_1, \ldots, j_d) \in \mathbb{Z}^d : 1 \le j_1, \ldots, j_d \le N\}$, u_j is a scalar unknown function of t and $u = \{u_j\}_{j \in \mathbb{Z}_N^d}$, α and K are positive constants, A is an $N^d \times N^d$ matrix and $(Au)_j$ stands for the jth component of the vector Au, $\beta \in \mathbb{R}$, g is a periodic function, f_j and ϵ_j are constants, and $\{W_j(t)\}_{j \in \mathbb{Z}_N^d}$ are independent two-sided real-valued Wiener processes. Moreover, A and g satisfy

(HA) *A is an $N^d \times N^d$ non-negative definite symmetric matrix with eigenvalues denoted by λ_i, $i = 0, 1, \ldots, N^d - 1$ satisfying that*

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{N^d - 1},$$

λ_0 is algebraically simple and $(1, \ldots, 1)^\top \in \mathbb{R}^{N^d}$ is an eigenvector corresponding to $\lambda_0 = 0$.

(HG) *$g \in C^1(\mathbb{R}, \mathbb{R})$ has the following properties*

$$g(x + \kappa) = g(x), \quad |g(x)| \le c_1, \quad |g'(x)| \le c_2, \ \forall\, x \in \mathbb{R},$$

where $c_1 > 0$, $c_2 > 0$ and $\kappa > 0$ is the smallest positive period of g.

System (1.1) appears in many applied problems including Josephson junction arrays and coupled pendula (see [11, 13, 22, 27], etc.). Physically, α in (1.1) represents the damping of the system and K is the coupling coefficient of the system. (1.1) then represents a system of N^d coupled damped oscillators independently driven by white noises.

System (1.1) also arises from various spatial discretizations of certain damped hyperbolic partial differential equations. For example, the $N^d \times N^d$ matrix A in (1.1) includes the discretization of negative Laplace operator $-\Delta$ with Neumann or periodic boundary conditions defined as follows:

$$\begin{aligned}(Au)_j &= (Au)_{(j_1, j_2, \ldots, j_d)} \\ &= \frac{1}{h^2}[2d u_j - u_{(j_1+1, j_2, \ldots, j_d)} - u_{(j_1, j_2+1, \ldots, j_d)} - \cdots - u_{(j_1, j_2, \ldots, j_d+1)} \\ &\qquad - u_{(j_1-1, j_2, \ldots, j_d)} - u_{(j_1, j_2-1, \ldots, j_d)} - \cdots - u_{(j_1, j_2, \ldots, j_d-1)}],\end{aligned}$$

with Neumann boundary condition

$$\begin{aligned} u_{(j_1, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_d)} &= u_{(j_1, \ldots, j_{i-1}, 1, j_{i+1}, \ldots, j_d)}, \\ u_{(j_1, \ldots, j_{i-1}, N+1, j_{i+1}, \ldots, j_d)} &= u_{(j_1, \ldots, j_{i-1}, N, j_{i+1}, \ldots, j_d)} \end{aligned}$$

or periodic boundary condition

$$\begin{aligned} u_{(j_1, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_d)} &= u_{(j_1, \ldots, j_{i-1}, N, j_{i+1}, \ldots, j_d)}, \\ u_{(j_1, \ldots, j_{i-1}, N+1, j_{i+1}, \ldots, j_d)} &= u_{(j_1, \ldots, j_{i-1}, 1, j_{i+1}, \ldots, j_d)} \end{aligned}$$

for $j = (j_1, \ldots, j_d) \in \mathbb{Z}_N^d$ and $i = 1, \ldots, d$. Thus, (1.1) with $u_j = u(j_1 h, \ldots, j_i h, \ldots, j_d h)$ $(h = \frac{L}{N})$, A being as above, $f_j = f$, $\epsilon_j = \epsilon$, and $W_j = W$ is a spatial discretization of the following problem

$$d\dot{u} + \alpha du - K\Delta u dt + \beta g(u)dt = fdt + \epsilon dW, \quad \text{in } U \times \mathbb{R}^+ \tag{1.2}$$

with Neumann boundary condition or periodic boundary condition, i.e.

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial U \times \mathbb{R}^+$$

or

$$u|_{\Gamma_j} = u|_{\Gamma_{j+d}}, \quad \left.\frac{\partial u}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial u}{\partial x_j}\right|_{\Gamma_{j+d}}, \quad j = 1, \dots, d,$$

where $\Gamma_j = \partial U \cap \{x_j = 0\}$, $\Gamma_{j+d} = \partial U \cap \{x_j = L\}$, $j = 1, \dots, d$ and $U = \prod_{i=1}^{d}(0, L)$. Note that if $g(u) = \sin u$, (1.2) is the so-called damped sine–Gordon equation, which is used to model, for instance, the dynamics of a continuous family of junctions (see [25]).

Two of the main dynamical aspects about coupled oscillators and damped wave equations considered in the literature are the existence and structure of global attractors and the phenomenon of frequency locking. A large amount of research has been carried out toward these two aspects for a variety of systems related to (1.1). See for example, [17, 18, 20, 22, 28] for the study of coupled oscillators with constant or periodic external forces; [12, 19, 21, 25, 26] for the study of the deterministic damped sine–Gordon equation; [8, 16, 23] for the study of coupled oscillators driven by white noises; and [10, 24, 29] for the study of stochastic damped sine–Gordon equation. Many of the existing works focus on the existence of global attractors and the estimate of the dimension of the global attractors. In [8, 23, 24], the existence and structure of random attractors of stochastic oscillators and stochastic damped wave equations are studied. In particular, the asymptotic dynamics of a single second order noisy oscillator, i.e. (1.1) with $N = 1$, is studied in [23]. The author of [23] proved the existence of a random attractor which is a family of horizontal curves and the existence of a rotation number which implies the frequency locking. In [8], the authors considered a class of coupled first order oscillators driven by white noises. Among those, the existence of a one-dimensional random attractor and the existence of a rotation number are proved in [8]. The system of coupled first order oscillators considered in [8] is of the form

$$du_j + K(Au)_j dt + \beta g(u_j)dt = f_j dt + \epsilon dW_j, \quad j \in \mathbb{Z}_N^d. \tag{1.3}$$

Note that, by re-scaling the time variable by $t \to \frac{t}{\alpha}$, (1.1) becomes

$$\frac{1}{\alpha} d\dot{u}_j + du_j + K(Au)_j dt + \beta g(u_j)dt = f_j dt + \epsilon dW_j, \quad j \in \mathbb{Z}_N^d. \tag{1.4}$$

Hence, (1.3) can be formally viewed as the limiting system of (1.1) as the damping coefficient α goes to infinite. In [24], the authors investigated the existence and structure of random attractors of damped sine–Gordon equations of the form (1.2) with Neumann boundary condition, which is a space continuous counterpart of (1.1) as mentioned above.

However, many important dynamical aspects including the existence of global attractor and the occurrence of frequency locking have been hardly studied for coupled second order oscillators of the form (1.1) driven by white noises. It is of great interest to investigate the extent to which the existing results on asymptotic dynamics of a single second order noisy oscillator may be generalized to systems of coupled second order noisy oscillators. Thanks to the relations between (1.1) and (1.3) and between (1.1) and (1.2), it is also of great interest to explore the similarity and difference between the dynamics of coupled damped second order oscillators and its limiting coupled first order oscillators as the damping coefficient goes to infinite and between the dynamics of coupled damped second order oscillators and their proper space continuous counterparts. The objective of this paper is to carry out a study along this line. In particular, we study the asymptotic or global dynamics of (1.1), including the existence and structure of global attractor in proper phase space and the success of frequency locking.

In order to do so, as usual, we first change (1.1) to some system of coupled first order random equations. Assume $N \geq 2$ and $d \geq 1$ ($N = 1$ reduces to the single noisy oscillator case considered in [23]). Let $u = (u_j)_{j\in\mathbb{Z}_N^d}$, $g(u) = (g(u_j))_{j\in\mathbb{Z}_N^d}$, $f = (f_j)_{j\in\mathbb{Z}_N^d}$, $W(t) = (\epsilon_j W_j(t))_{j\in\mathbb{Z}_N^d}$. Then, (1.1) can be written as the following matrix form,

$$d\dot{u} + \alpha du + KAudt + \beta g(u)dt = fdt + dW(t). \tag{1.5}$$

Let

$$\Omega_j = \Omega_0 = \{\omega_0 \in C(\mathbb{R}, \mathbb{R}) : \omega_0(0) = 0\}$$

be equipped with the compact open topology, $\mathcal{F}_j = \mathcal{B}(\Omega_0)$ be the Borel σ-algebra of Ω_0 and $\mathbb{P}_j$ be the corresponding Wiener measure for $j \in \mathbb{Z}_N^d$. Let $\Omega = \prod_{j\in\mathbb{Z}_N^d} \Omega_j$, $\mathcal{F}$ be the product σ-algebra on Ω and $\mathbb{P}$ be the induced product Wiener measure. Define $(\theta_t)_{t\in\mathbb{R}}$ on Ω via

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$

Then, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system (see [1]). Consider the Ornstein–Uhlenbeck equation,

$$dz + zdt = dW(t), \quad z \in \mathbb{R}^{N^d}. \tag{1.6}$$

Let $z(\theta_t\omega) = (z_j(\theta_t\omega))_{j\in\mathbb{Z}_N^d}$ be the unique stationary solution of (1.6) (see [1, 2, 9] for the existence and various properties of $z(\cdot)$). Let $v = \dot{u} - z(\theta_t\omega)$. We obtain the following equivalent system of (1.5),

$$\begin{cases} \dot{u} = v + z(\theta_t\omega), \\ \dot{v} = -KAu - \alpha v + f - \beta g(u) + (1 - \alpha)z(\theta_t\omega). \end{cases} \tag{1.7}$$

To study the global dynamics of (1.1), it is therefore equivalent to study the global dynamics of (1.7). Observe that the natural phase space for (1.7) is $E := \mathbb{R}^{N^d} \times \mathbb{R}^{N^d}$ with the standard Euclidean norm. Thanks to the presence of the damping, it is expected that (1.7) possesses a global attractor in certain sense. However, due to the uncontrolled component of the solutions along the direction of the eigenvectors of the linear operator in the right of (1.7) corresponding to the zero eigenvalue, there is no bounded attracting sets in E with the standard Euclidean norm, which will lead to nontrivial dynamics. There is also some additional difficulty if one studies (1.7) in E with the standard Euclidean norm due to the zero limit of some eigenvalues of the linear operator in the right of (1.7) as $\alpha \to \infty$. The later difficulty does not appear for coupled first order oscillators studied in [8] and for a single noisy oscillator considered in [23]. We will overcome the difficulty by using some equivalent norm on E and considering (1.7) in some proper quotient space of E and prove the existence of a global random attractor as well as the existence of a rotation number of (1.7).

To be more precise, let

$$C = \begin{pmatrix} 0 & I \\ -KA & -\alpha I \end{pmatrix}. \tag{1.8}$$

By simple matrix analysis, the eigenvalues of C are given by (see [14, 20] for example)

$$\mu_i^{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4K\lambda_i}}{2}, \quad i = 0, 1, \ldots, N^d - 1. \tag{1.9}$$

Note that $\mu_0^+ = 0$, which requires some special consideration for the solutions along the direction of the eigenvector $\eta_0 = (1, \ldots, 1, 0, \ldots, 0)^\top$ corresponding to μ_0^+. We overcome this difficulty by considering (1.7) in the cylindrical space $E_1/\kappa\eta_0\mathbb{Z} \times E_2$, where $E_1 = \mathrm{span}\{\eta_0\}$, E_2 is the space spanned by all the eigenvectors corresponding to nonzero eigenvalues of C (see Sec. 4 for details). We then prove

(1) *For any* $\alpha > 0$ *and* $K > 0$, *system* (1.1) *possesses a global random attractor* (*which is unbounded along the one-dimensional space* E_1 *and bounded along the one-codimensional space* E_2) (*see Theorem* 4.2, *Corollary* 4.3 *and Remark* 4.4).

It is expected physically that when the damping coefficient $\alpha \to \infty$, the dynamics of (1.7) becomes simpler or the structure of the global attractor of (1.7) becomes simpler. However, $\mu_i^+ \to 0$ as $\alpha \to \infty$ for $i = 1, 2, \ldots, N^d - 1$, which gives rise to some difficulty for studying the structure of the global attractor in E with the standard Euclidean norm. We introduce an equivalent norm on E to overcome this difficulty (see Sec. 3 for the introduction of the equivalent norm, the choice of such equivalent norm was first discovered in [15]) and prove

(2) *When* α *and* K *are sufficiently large*, *the global random attractor of* (1.1) *is a one-dimensional random horizontal curve* (*see Theorem* 5.3 *and Corollary* 5.4),

and the rotation number (*see Definition* 6.1) *of* (1.1) *exists* (*see Theorem* 6.3 *and Corollary* 6.4).

Note that roughly a real number $\rho \in \mathbb{R}$ is called the *rotation number* of (1.1) or (1.7) if for any solution $\{u_j(t)\}_{j\in\mathbb{Z}_N^d}$ of (1.1), the limit $\lim_{t\to\infty} \frac{u_j(t)}{t}$ exists almost surely for any $j \in \mathbb{Z}_N^d$ and

$$\lim_{t\to\infty} \frac{u_j(t)}{t} = \rho \quad \text{for a.e. } \omega \in \Omega \text{ and } j = 1, 2, \ldots, N^d$$

(see Definition 6.1 and the remark after Definition 6.1). Hence if (1.1) has a rotation number, then the oscillators in the system tend to oscillate with the same frequency eventually and therefore the so-called frequency locking is successful.

(1) and (2) above are the main results of the paper. They make an important contribution to the understanding of coupled second order oscillators driven by noises. Property (1) shows that system (1.1) is dissipative along the one-codimensional space E_2. By property (2), the asymptotic dynamics of (1.1) with sufficiently large α and K is one-dimensional regardless of the strength of noise. Property (2) also shows that all the solutions of (1.1) tend to oscillate with the same frequency eventually almost surely and hence frequency locking is successful in (1.1) provided that α and K are sufficiently large.

The results obtained in this paper generalize many existing results on the asymptotic dynamics for a single damped noisy oscillator to systems of coupled damped noisy oscillators. They show that coupled damped second order oscillators with large damping have similar asymptotic dynamics as the limiting coupled first order oscillators as the damping goes to infinite and hence one may use coupled first order oscillators to analyze qualitative properties of coupled second order oscillators with large damping, which is of great practical importance. They also show that coupled damped second order oscillators have similar asymptotic dynamics as their proper space continuous counterparts and hence one may use finitely many coupled oscillators to study qualitative properties of damped wave equations, which is of great practical importance too.

The rest of the paper is organized as follows. In Sec. 2, we present some basic concepts and properties for general random dynamical systems. In Sec. 3, we provide some basic settings about (1.1) and show that it generates a random dynamical system. We prove in Sec. 4 the existence of a global random attractor of the random dynamical system ϕ generated by (1.1) for any $\alpha > 0$ and $K > 0$. We show in Sec. 5 that the global random attractor of ϕ is a random horizontal curve and in Sec. 6 that (1.1) has a rotation number, respectively, provided that α and K are sufficiently large.

2. Random Dynamical Systems

In this section, we collect some basic knowledge about general random dynamical system (see [1, 4] for details). Let (X, d) be a complete and separable metric space with Borel σ-algebra $\mathcal{B}(X)$.

Definition 2.1. A *continuous random dynamical system over* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

such that the following properties hold:

(1) $\varphi(0, \omega, x) = x$ for all $\omega \in \Omega$;
(2) $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$ for all $s, t \geq 0$ and $\omega \in \Omega$;
(3) $\varphi(t, \omega, x)$ is continuous in x for every $t \geq 0$ and $\omega \in \Omega$.

For given $x \in X$ and $E, F \subset X$, we define

$$d(x, F) = \inf_{y\in F} d(x, y)$$

and

$$d_H(E, F) = \sup_{x\in E} d(x, F).$$

$d_H(E, F)$ is called the *Hausdorff semi-distance* from E to F.

Definition 2.2. (1) A set-valued mapping $\omega \mapsto D(\omega) : \Omega \to 2^X$ is said to be a *random set* if the mapping $\omega \mapsto d(x, D(\omega))$ is measurable for any $x \in X$. If $\omega \mapsto d(x, D(\omega))$ is measurable for any $x \in X$ and $D(\omega)$ is closed (compact) for each $\omega \in \Omega$, then $\omega \mapsto D(\omega)$ is called a *random closed* (*compact*) *set*. A random set $\omega \mapsto D(\omega)$ is said to be *bounded* if there exist $x_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$D(\omega) \subset \{x \in X : d(x, x_0) \leq R(\omega)\} \quad \text{for all } \omega \in \Omega.$$

(2) A random set $\omega \mapsto D(\omega)$ is called *tempered* provided that for some $x_0 \in X$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup\{d(b, x_0) : b \in D(\theta_{-t}\omega)\} = 0 \quad \text{for all } \beta > 0.$$

(3) A random set $\omega \mapsto B(\omega)$ is said to be a *random absorbing set* if for any tempered random set $\omega \mapsto D(\omega)$, there exists $t_0(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for all } t \geq t_0(\omega),\ \omega \in \Omega.$$

(4) A random set $\omega \mapsto B_1(\omega)$ is said to be a *random attracting set* if for any tempered random set $\omega \mapsto D(\omega)$, we have

$$\lim_{t\to\infty} d_H(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega), B_1(\omega))) = 0 \quad \text{for all } \omega \in \Omega.$$

(5) A random compact set $\omega \mapsto A(\omega)$ is said to be a *global random attractor* if it is a random attracting set and $\varphi(t, \omega, A(\omega)) = A(\theta_t\omega)$ for all $\omega \in \Omega$ and $t \geq 0$.

Theorem 2.3. *Let φ be a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. If there is a random compact attracting set $\omega \mapsto B(\omega)$ of φ, then $\omega \mapsto A(\omega)$ is a global random attractor of φ, where*

$$A(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega, B(\theta_{-\tau}\omega))}, \quad \omega \in \Omega.$$

Proof. See [1, 4]. □

3. Basic Settings

In this section, we give some basic settings about (1.1) and show that it generates a random dynamical system.

First, let $Y = (u, v)^\top$ and $F(\theta_t\omega, Y) = (z(\theta_t\omega), f - \beta g(u) + (1-\alpha)z(\theta_t\omega))^\top$. System (1.7) can then be written as

$$\dot{Y} = CY + F(\theta_t\omega, Y), \tag{3.1}$$

where C is as in (1.8).

Recall that $z(\theta_t\omega) = (z_j(\theta_t\omega))_{j\in\mathbb{Z}_N^d}$ is the unique stationary solution of (1.6). Note that the random variable $|z_j(\omega)|$ is tempered and the mapping $t \mapsto z_j(\theta_t\omega)$ is $\mathbb{P}$-a.s. continuous (see [1, 2]). More precisely, there is a θ_t-invariant $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that $t \mapsto z_j(\theta_t\omega)$ is continuous for $\omega \in \tilde{\Omega}$ and $j \in \mathbb{Z}_N^d$. We will consider (1.7) or (3.1) for $\omega \in \tilde{\Omega}$ and write $\tilde{\Omega}$ as Ω from now on.

Let $E = \mathbb{R}^{N^d} \times \mathbb{R}^{N^d}$ and $F^\omega(t, Y) := F(\theta_t\omega, Y)$, then $F^\omega(\cdot,\cdot) : \mathbb{R} \times E \to E$ is continuous in t and globally Lipschitz continuous in Y for each $\omega \in \Omega$. By classical theory of ordinary differential equations concerning existence and uniqueness of solutions, for each $\omega \in \Omega$ and any $Y_0 \in E$, (3.1) has a uniqueness solution $Y(t, \omega, Y_0)$, $t \geq 0$, satisfying

$$Y(t, \omega, Y_0) = e^{Ct}Y_0 + \int_0^t e^{C(t-s)}F(\theta_s\omega, Y(s, \omega, Y_0))ds, \quad t \geq 0 \tag{3.2}$$

and $Y(0, \omega, Y_0) = Y_0$. Moreover, it follows from [1] that $Y(t, \omega, Y_0)$ is measurable in (t, ω, Y_0). Hence (3.1) generates a continuous random dynamical system on E,

$$Y : \mathbb{R}^+ \times \Omega \times E \to E, \quad (t, \omega, Y_0) \mapsto Y(t, \omega, Y_0). \tag{3.3}$$

Define a mapping $\phi : \mathbb{R}^+ \times \Omega \times E \to E$ by

$$\phi(t, \omega, \phi_0) = Y(t, \omega, Y_0(\omega)) + (0, z(\theta_t\omega))^\top, \tag{3.4}$$

where $\phi_0 = (u_0, u_1)^\top \in E$ and $Y_0(\omega) = (u_0, u_1 - z(\omega))^\top$. Then ϕ is a continuous random dynamical system associated with the problem (1.1) on E.

Recall that the eigenvalues of C are given by (see [14, 20] for example)

$$\mu_i^\pm = \frac{-\alpha \pm \sqrt{\alpha^2 - 4K\lambda_i}}{2}, \quad i = 0, 1, \ldots, N^d - 1. \tag{3.5}$$

By (3.5), C has at least two real eigenvectors 0 and $-\alpha$ with eigenvalues $\eta_0 = (1,\ldots,1,0,\ldots,0)^\top$, $\eta_{-1} = (1,\ldots,1,-\alpha,\ldots,-\alpha)^\top \in E$, respectively. Let $E_1 = \mathrm{span}\{\eta_0\}$, $E_{-1} = \mathrm{span}\{\eta_{-1}\}$, $E_{11} = E_1 + E_{-1}$ and $E_{22} = E_{11}^\perp$, the orthogonal complement space of E_{11} in E, then $E = E_{11} \oplus E_{22}$. To control the unboundedness of solutions in the direction of η_0, we will study (3.1) in the cylindrical space $E_1/\kappa\eta_0\mathbb{Z} \times E_2$, where $E_2 = E_{-1} \oplus E_{22}$ (see Sec. 4 for details).

Observe that the Lipschitz constant of F with respect to Y in E with the standard Euclidean norm is independent of $\alpha > 0$. But $\mu_i^+ \to 0$ as $\alpha \to \infty$ for $i \geq 1$, which gives rise to some difficulty for the investigation of (3.1) in E with the standard Euclidean norm. To overcome the difficulty, we introduce a new norm which is equivalent to the standard Euclidean norm on E. Here, we only collect some results about the new norm (see [15, 20] for details).

Define two bilinear forms on E_{11} and E_{22}, respectively. For $Y_i = (u_i, v_i)^\top \in E_{11}$, $i = 1, 2$, let

$$\langle Y_1, Y_2\rangle_{E_{11}} = \frac{\alpha^2}{4}\langle u_1, u_2\rangle + \left\langle \frac{\alpha}{2}u_1 + v_1, \frac{\alpha}{2}u_2 + v_2 \right\rangle, \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^{N^d}$, and for $Y_i = (u_i, v_i)^\top \in E_{22}$, $i = 1, 2$, let

$$\langle Y_1, Y_2\rangle_{E_{22}} = \langle KAu_1, u_2\rangle + \left(\frac{\alpha^2}{4} - \delta K\lambda_1\right)\langle u_1, u_2\rangle + \left\langle \frac{\alpha}{2}u_1 + v_1, \frac{\alpha}{2}u_2 + v_2 \right\rangle, \tag{3.7}$$

where $\delta \in (0, 1]$. It is easy to check that the Poincaré-type inequality

$$\langle Au, u\rangle \geq \lambda_1\|u\|^2, \quad \forall\, Y = (u, v)^\top \in E_{22}$$

holds (see [20] for example), where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^{N^d}$. Thus (3.7) is positive-definite. For any $Y_i = Y_i^{(1)} + Y_i^{(2)} \in E$, $i = 1, 2$, where $Y_1^{(1)}, Y_2^{(1)} \in E_{11}$, $Y_1^{(2)}, Y_2^{(2)} \in E_{22}$, we define

$$\langle Y_1, Y_2\rangle_E = \langle Y_1^{(1)}, Y_2^{(1)}\rangle_{E_{11}} + \langle Y_1^{(2)}, Y_2^{(2)}\rangle_{E_{22}}. \tag{3.8}$$

Lemma 3.1. ([20]) (1) (3.6) *and* (3.7) *define inner products on* E_{11} *and* E_{22}, *respectively.*

(2) (3.8) *defines an inner product on* E, *and the corresponding norm* $\|\cdot\|_E$ *is equivalent to the standard Euclidean norm on* E.

(3) *In terms of the inner product* $\langle\cdot,\cdot\rangle_E$, E_1 *and* E_{11} *are orthogonal to* E_{-1} *and* E_{22}, *respectively.*

(4) *In terms of the norm* $\|\cdot\|_E$, *the Lipschitz constant* L_F *of* F *with respect to* Y *satisfies*

$$L_F = \frac{2c_2|\beta|}{\alpha}, \tag{3.9}$$

where c_2 *is as in* **(HG)**.

Note that E_2 is orthogonal to E_1 and $E = E_1 \oplus E_2$. Denote by P and $Q\ (= I - P)$ the projections from E into E_1 and E_2, respectively. Set

$$a = \frac{\alpha}{2} - \left|\frac{\alpha}{2} - \frac{\delta K \lambda_1}{\alpha}\right|. \tag{3.10}$$

Lemma 3.2. (1) *For any* $Y \in E_2$, $\langle CY, Y\rangle_E \le -a\|Y\|_E^2$.
(2) $\|e^{Ct}Q\|_E \le e^{-at}$ *for* $t \ge 0$.
(3) $e^{Ct}PY = PY$ *for* $Y \in E$, $t \ge 0$.

Proof. (1) and (2) follow from similar arguments as in Lemma 2.3 and Corollary 2.4 in [19]. Let us show (3). For $Y \in E$, since $PY \in E_1$ and $\frac{d}{dt}e^{Ct}PY = e^{Ct}CPY = 0$, we have $e^{Ct}PY = e^{C0}PY = PY$. □

By Lemma 3.2(2), the constant a in (3.10) describes the exponential decay rate of $e^{Ct}|_{QE}$ in the new norm. By Lemma 3.1(4), L_F tends to 0 as $\alpha \to \infty$ with respect to the new norm, which essentially helps to overcome the difficulty induced from the fact that $\mu_i^+ \to 0$ as $\alpha \to \infty$ for $i \ge 1$.

The following lemma will be needed to take care of the unboundedness of the solutions along the direction of the eigenvectors corresponding to μ_0^+.

Lemma 3.3. *Let* $p_0 = \kappa\eta_0 \in E$ *(*κ *is the smallest positive period of* g*). The random dynamical system* Y *defined in* (3.3) *is* p_0*-translation invariant in the sense that*

$$Y(t, \omega, Y_0 + p_0) = Y(t, \omega, Y_0) + p_0, \quad t \ge 0,\ \omega \in \Omega,\ Y_0 \in E.$$

Proof. Since $Cp_0 = 0$ and $F(t, \omega, Y)$ is p_0-periodic in Y, $Y(t, \omega, Y_0) + p_0$ is a solution of (3.1) with initial data $Y_0 + p_0$. Thus, $Y(t, \omega, Y_0) + p_0 = Y(t, \omega, Y_0 + p_0)$. □

By (3.3) and Lemma 3.3, ϕ is also p_0-translation invariant.

Definition 3.4. A random variable $\omega \mapsto r(\omega) : \Omega \to (0, \infty)$ is called *tempered* with respect to the dynamical system $(\theta_t)_{t\in\mathbb{R}}$ on Ω if for $\mathbb{P}$-a.e. $\omega \in \Omega$, there holds

$$\lim_{t\to\pm\infty} \frac{\ln r(\theta_t\omega)}{t} = 0.$$

Lemma 3.5. *For any* $\epsilon > 0$, *there is a tempered random variable* $\tilde{r}(\omega) > 0$ *such that*

$$\|z(\theta_t\omega)\| \le e^{\epsilon|t|}\tilde{r}(\omega) \quad \text{for all } t \in \mathbb{R},\ \omega \in \Omega, \tag{3.11}$$

where $\tilde{r}(\omega)$ *satisfies*

$$e^{-\epsilon|t|}\tilde{r}(\omega) \le \tilde{r}(\theta_t\omega) \le e^{\epsilon|t|}\tilde{r}(\omega), \quad t \in \mathbb{R},\ \omega \in \Omega. \tag{3.12}$$

Proof. For $j \in \mathbb{Z}_N^d$, since $|z_j(\omega)|$ is a tempered random variable and the mapping $t \mapsto \ln|z_j(\theta_t\omega)|$ is $\mathbb{P}$-a.s. continuous, it follow from Proposition 4.3.3 in [1] that for any $\epsilon_j > 0$ there is tempered random variable $r_j(\omega) > 0$ such that

$$\frac{1}{r_j(\omega)} \leq |z_j(\omega)| \leq r_j(\omega),$$

where $r_j(\omega)$ satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$e^{-\epsilon_j|t|}r_j(\omega) \leq r_j(\theta_t\omega) \leq e^{\epsilon_j|t|}r_j(\omega), \quad t \in \mathbb{R}. \tag{3.13}$$

Let $r(\omega) = (r_j(\omega))_{j\in\mathbb{Z}_N^d}$, $\omega \in \Omega$ and take $\epsilon_j = \epsilon$, $j \in \mathbb{Z}_N^d$, then we have

$$\|z(\theta_t\omega)\| \leq \left(\sum_{j\in\mathbb{Z}_N^d} e^{2\epsilon|t|}r_j^2(\omega)\right)^{\frac{1}{2}} = e^{\epsilon|t|}\|r(\omega)\|, \quad t \in \mathbb{R},\ \omega \in \Omega.$$

Let $\tilde{r}(\omega) = \|r(\omega)\|$, $\omega \in \Omega$. Then (3.11) is satisfied and (3.12) is trivial from (3.13). □

4. Existence of Random Attractor

In this section, we study the existence of a random attractor. We assume that $p_0 = \kappa\eta_0 \in E_1$ and $\delta \in (0, 1]$ is such that $a > 0$, where a is as in (3.10). We remark in the end of this section that such δ always exists.

By Lemma 3.3 and the fact that C has a zero eigenvalue, we will define a random dynamical system $\mathbf{Y}$ on some cylindrical space induced from the random dynamical system Y on E. Then by properties of Y restricted on E_2, we can prove the existence of a global random attractor of $\mathbf{Y}$. Thus, we can say that Y has a global random attractor which is unbounded along E_1 and bounded along E_2. Now, we define $\mathbf{Y}$.

Let $\mathbb{T}^1 = E_1/p_0\mathbb{Z}$ and $\mathbf{E} = \mathbb{T}^1 \times E_2$, where $p_0\mathbb{Z} = \{kp_0 : k \in \mathbb{Z}\}$. For $Y_0 \in E$, let $\mathbf{Y}_0 := Y_0 \pmod{p_0}$, which is an element of $\mathbf{E}$. Note that, by Lemma 3.3, $Y(t, \omega, Y_0 + kp_0) = Y(t, \omega, Y_0) + kp_0$, $\forall\, k \in \mathbb{Z}$ for $t \geq 0$, $\omega \in \Omega$ and $Y_0 \in E$. With this, we define $\mathbf{Y} : \mathbb{R}^+ \times \Omega \times \mathbf{E} \to \mathbf{E}$ by setting

$$\mathbf{Y}(t, \omega, \mathbf{Y}_0) = Y(t, \omega, Y_0) \pmod{p_0}, \tag{4.1}$$

where $\mathbf{Y}_0 = Y_0 \pmod{p_0}$. Then $\mathbf{Y} : \mathbb{R}^+ \times \Omega \times \mathbf{E} \to \mathbf{E}$ is a random dynamical system. Similarly, the random dynamical system ϕ defined in (3.4) also induces a random dynamical system Φ on $\mathbf{E}$. By (3.3), (3.4) and (4.1), Φ is defined by

$$\Phi(t, \omega, \Phi_0) = \mathbf{Y}(t, \omega, \mathbf{Y}_0) + \tilde{z}(\theta_t\omega) \pmod{p_0}, \quad t \geq 0,\ \omega \in \Omega, \tag{4.2}$$

where $\Phi_{\mathbf{0}} = \phi_0 \pmod{p_0}$, $\tilde{z}(\theta_t\omega) = (0, z(\theta_t\omega))^\top$ and $\mathbf{Y}_0 = \Phi_0 - \tilde{z}(\omega) \pmod{p_0}$.

Recall that P and $Q\,(= I - P)$ are the projections from E into E_1 and E_2, respectively.

Definition 4.1. Let $\omega \in \Omega$ and $R : \Omega \to \mathbb{R}^+$ be a random variable. A *random pseudo-ball* $\omega \mapsto B(\omega)$ *in* E *with random radius* $\omega \mapsto R(\omega)$ is a set of the form

$$\omega \mapsto B(\omega) = \{b \in E : \|Qb\|_E \le R(\omega)\}.$$

Furthermore, a random set $\omega \mapsto B(\omega) \in E$ is called *pseudo-tempered* provided that $\omega \mapsto QB(\omega)$ is a tempered random set in E, i.e. for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup\{\|Qb\|_E : b \in B(\theta_{-t}\omega)\} = 0 \quad \text{for all } \beta > 0.$$

Clearly, any random pseudo-ball $\omega \mapsto B(\omega)$ in E has the form $\omega \mapsto E_1 \times QB(\omega)$, where $\omega \mapsto QB(\omega)$ is a random ball in E_2. Then the measurability of $\omega \mapsto B(\omega)$ is trivial. By Definition 4.1, if $\omega \mapsto B(\omega)$ is a random pseudo-ball in E, then $\omega \mapsto B(\omega) \pmod{p_0}$ is random bounded set in $\mathbf{E}$. And if $\omega \mapsto B(\omega)$ is a pseudo-tempered random set in E, then $\omega \mapsto B(\omega) \pmod{p_0}$ is a tempered random set in $\mathbf{E}$.

We next show the existence of a global random attractor of the induced random dynamical system $\mathbf{Y}$ defined in (4.1).

Theorem 4.2. *Let* $\alpha > 0$ *and* $K > 0$. *Then the induced random dynamical system* $\mathbf{Y}$ *defined in* (4.1) *has a global random attractor* $\omega \mapsto \mathbf{A}_0(\omega)$. *Moreover,* $\omega \mapsto \mathbf{A}_0(\omega)$ *is tempered.*

Proof. For $\omega \in \Omega$, we obtain from (3.2) that

$$Y(t,\omega,Y_0(\omega)) = e^{Ct}Y_0(\omega) + \int_0^t e^{C(t-s)}F(\theta_s\omega, Y(s,\omega,Y_0(\omega)))ds. \tag{4.3}$$

The projection of (4.3) on E_2 is

$$QY(t,\omega,Y_0(\omega)) = e^{Ct}QY_0(\omega) + \int_0^t e^{C(t-s)}QF(\theta_s\omega, Y(s,\omega,Y_0(\omega)))ds. \tag{4.4}$$

By replacing ω by $\theta_{-t}\omega$, it follows from (4.4) that

$$\begin{aligned} QY(t,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)) &= e^{Ct}QY_0(\theta_{-t}\omega) \\ &\quad + \int_0^t e^{C(t-s)}QF(\theta_{s-t}\omega, Y(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)))ds. \end{aligned}$$

It then follows from Lemma 3.2 and $Q^2 = Q$ that

$$\begin{aligned} &\|QY(t,\theta_{-t}\omega, Y_0(\theta_{-t}\omega))\|_E \\ &\quad \le e^{-at}\|QY_0(\theta_{-t}\omega)\|_E + \int_0^t e^{-a(t-s)}\|F(\theta_{s-t}\omega, Y(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)))\|_E ds. \end{aligned} \tag{4.5}$$

By Lemma 3.5 with $\epsilon = \frac{a}{2}$ and the equivalence of $\|\cdot\|_E$ and $\|\cdot\|$ on E, there is an $M_1 > 0$ such that

$$\begin{aligned}
&\|F(\theta_{s-t}\omega, Y(s, \theta_{-t}\omega, Y_0(\theta_{-t}\omega)))\|_E \\
&\quad \le M_1 \|F(\theta_{s-t}\omega, Y(s, \theta_{-t}\omega, Y_0(\theta_{-t}\omega)))\| \\
&\quad = M_1(\|z(\theta_{s-t}\omega)\|^2 + \|f - \beta g(Y_u(s, \theta_{-t}\omega)) + (1-\alpha)z(\theta_{s-t}\omega)\|^2)^{\frac{1}{2}} \\
&\quad \le M_1((3\alpha^2 - 6\alpha + 4)\|z(\theta_{s-t}\omega)\|^2 + 3\|f\|^2 + 3\beta^2 c_1^2 N^d)^{\frac{1}{2}} \\
&\quad \le a_1 e^{\frac{a}{2}(t-s)} \tilde{r}(\omega) + a_2,
\end{aligned}$$

where Y_u satisfies $Y(s, \theta_{-t}\omega, Y_0(\theta_{-t}\omega)) = (Y_u(s, \theta_{-t}\omega), Y_v(s, \theta_{-t}\omega))^\top$, $a_1 = M_1\sqrt{3\alpha^2 - 6\alpha + 4}$ and $a_2 = M_1\sqrt{3\|f\|^2 + 3\beta^2 c_1^2 N^d}$. We then find from (4.5) that

$$\begin{aligned}
&\|QY(t, \theta_{-t}\omega, Y_0(\theta_{-t}\omega))\|_E \\
&\quad \le e^{-at}\|QY_0(\theta_{-t}\omega)\|_E + \frac{2a_1}{a}(1 - e^{-\frac{a}{2}t})\tilde{r}(\omega) + \frac{a_2}{a}(1 - e^{-at}).
\end{aligned}$$

Now for $\omega \in \Omega$, define

$$R_0(\omega) = \frac{4a_1}{a}\tilde{r}(\omega) + \frac{2a_2}{a}.$$

Then, for any pseudo-tempered random set $\omega \mapsto B(\omega)$ in E and any $Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there is a $T_B(\omega) > 0$ such that for $t \ge T_B(\omega)$,

$$\|QY(t, \theta_{-t}\omega, Y_0(\theta_{-t}\omega))\|_E \le R_0(\omega), \quad \omega \in \Omega,$$

which implies

$$Y(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset B_0(\omega) \quad \text{for all } t \ge T_B(\omega),\ \omega \in \Omega,$$

where $\omega \mapsto B_0(\omega)$ is the random pseudo-ball centered at origin with random radius $\omega \mapsto R_0(\omega)$. Note that $\omega \mapsto R_0(\omega)$ is a random variable since $\omega \mapsto \tilde{r}(\omega)$ is a random variable, then the measurability of random pseudo-ball $\omega \mapsto B_0(\omega)$ is trivial from Definition 4.1.

For $\omega \in \Omega$, let $\mathbf{B}(\omega) = B(\omega) \pmod{p_0}$ and $\mathbf{B_0}(\omega) = B_0(\omega) \pmod{p_0}$, we then have

$$\mathbf{Y}(t, \theta_{-t}\omega, \mathbf{B}(\theta_{-t}\omega)) \subset \mathbf{B_0}(\omega) \quad \text{for all } t \ge T_{\mathbf{B}}(\omega),\ \omega \in \Omega,$$

where $T_{\mathbf{B}}(\omega) = T_B(\omega)$ for $\omega \in \Omega$, i.e. $\omega \mapsto \mathbf{B}_0(\omega)$ is the random absorbing set of $\mathbf{Y}$. Moreover, $\omega \mapsto \mathbf{B}_0(\omega)$ is bounded and closed, hence compact in $\mathbf{E}$, it then follows from Theorem 2.3 that $\mathbf{Y}$ has a global random attractor $\omega \mapsto \mathbf{A}_0(\omega)$, where

$$\mathbf{A}_0(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau \ge t} \mathbf{Y}(\tau, \theta_{-\tau}\omega, \mathbf{B}_0(\theta_{-\tau}\omega))}, \quad \omega \in \Omega.$$

Moreover, $\omega \mapsto \mathbf{A}_0(\omega)$ is tempered. Indeed, the random variable $\omega \mapsto R_0(\omega)$ is tempered since $\omega \mapsto \tilde{r}(\omega)$ is tempered. It then follows that the random pseudo-ball $\omega \mapsto B_0(\omega)$ is pseudo-tempered, and therefore, $\omega \mapsto \mathbf{B}_0(\omega)$ is tempered, which leads to the temperedness of $\omega \mapsto \mathbf{A}_0(\omega)$, since $\mathbf{A}_0(\omega) \subset \mathbf{B}_0(\omega)$ for all $\omega \in \Omega$. This completes the proof. □

Corollary 4.3. *Let* $\alpha > 0$ *and* $K > 0$. *Then the induced random dynamical system* Φ *defined in* (4.2) *has a global random attractor* $\omega \mapsto \mathbf{A}(\omega)$, *where* $\mathbf{A}(\omega) = \mathbf{A_0}(\omega) + \tilde{z}(\omega) \pmod{p_0}$ *for all* $\omega \in \Omega$. *Moreover,* $\omega \mapsto \mathbf{A}(\omega)$ *is tempered.*

Proof. It follows from (4.2) and Theorem 4.2. □

Remark 4.4. (1) For any $\alpha > 0$ and $K > 0$, there is a $\delta \in (0, 1]$ such that $\alpha^2 > 2\delta K \lambda_1$ which implies $a > 0$, where a is as in (3.10) and λ_1 is the smallest positive eigenvalue of A.

(2) We say that the random dynamical system Y (or ϕ) has a global random attractor in the sense that the induced random dynamical system $\mathbf{Y}$ (or Φ) has a global random attractor, and we will say that Y (or ϕ) has a global random attractor directly in the sequel. We denote the global random attractor of Y and ϕ by $\omega \mapsto A_0(\omega)$ and $\omega \mapsto A(\omega)$ respectively. Indeed, $\omega \mapsto A_0(\omega)$ and $\omega \mapsto A(\omega)$ satisfy

$$\mathbf{A}_0(\omega) = A_0(\omega) \pmod{p_0}, \quad \mathbf{A}(\omega) = A(\omega) \pmod{p_0}, \quad \omega \in \Omega.$$

Hence a global random attractor of Y (or ϕ) is unbounded along the one-dimensional space E_1 and bounded along the one-codimensional space E_2.

(3) Observe that the global attractors of many dissipative systems related to (1.1) are one-dimensional (see [8, 17–21, 23, 24, 26]). Similarly, we expect that the random attractor $\omega \mapsto A(\omega)$ of ϕ is one-dimensional for each $\omega \in \Omega$ provided that α is sufficiently large. We prove that this is true in the next section.

(4) By (2), the system (1.1) is dissipative along E_2 (i.e. it possesses a global random attractor which is bounded along E_2). In Sec. 6, we will show that (1.1) with sufficiently large α and K also has a rotation number and hence all the solutions tend to oscillate with the same frequency eventually.

5. One-Dimensional Random Attractor

In this section, we apply the invariant and inertial manifold theory, in particular, the theory established in [7] to show that the random attractor of the random dynamical system ϕ generated by (1.1) is one-dimensional (more precisely, is a horizontal curve) provided that α and K are sufficiently large (see Remark 4.4(2) for the random attractor). This method has been applied by Chow, Shen and Zhou [8] to systems of coupled first order noisy oscillators and by Shen, Zhou and Shen [24] to the stochastic damped sine–Gordon equation. The reader is referred to [3, 5] for the theory and application of inertial manifold theory for stochastic evolution equations.

Assume that $p_0 = \kappa\eta_0$ and $a > 4L_F$ (see (3.10) for a and (3.9) for L_F). Note that the condition $a > 4L_F$ indicates that the exponential decay rate of $e^{Ct}|_{QE}$ in the norm $\|\cdot\|_E$ is larger than four times the Lipschitz constant of F in the norm $\|\cdot\|_E$. It will be seen at the end of this section that the condition $a > 4L_F$ can be satisfied provided that α and K are sufficiently large.

Definition 5.1. Suppose $\{\Phi^\omega\}_{\omega\in\Omega}$ is a family of maps from E_1 to E_2 and $n \in \mathbb{N}$. A family of graphs $\omega \mapsto \ell(\omega) \equiv \{(p, \Phi^\omega(p)) : p \in E_1\}$ is said to be a *random np_0-period horizontal curve* if $\omega \mapsto \ell(\omega)$ is a random set and $\{\Phi^\omega\}_{\omega\in\Omega}$ satisfy the Lipshitz condition

$$\|\Phi^\omega(p_1) - \Phi^\omega(p_2)\|_E \le \|p_1 - p_2\|_E \quad \text{for all } p_1, p_2 \in E_1,\ \omega \in \Omega$$

and the periodic condition

$$\Phi^\omega(p + np_0) = \Phi^\omega(p) \quad \text{for all } p \in E_1,\ \omega \in \Omega.$$

Note that for any $\omega \in \Omega$, $\ell(\omega)$ is a deterministic np_0-period horizontal curve. When $n = 1$, we simply call it a horizontal curve.

Lemma 5.2. *Let $a > 4L_F$. Suppose that $\omega \mapsto \ell(\omega)$ is a random np_0-period horizontal curve in E. Then, $\omega \mapsto Y(t, \omega, \ell(\omega))$ is also a random np_0-period horizontal curve in E for all $t > 0$. Moreover, $\omega \mapsto Y(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega))$ is a random np_0-period horizontal curve for all $t > 0$.*

The proof of Lemma 5.2 is similar to that of Lemma 4.2 in [24]. We hence omit it here.

Choose $\gamma \in (0, \frac{a}{2})$ such that

$$\frac{2c_2|\beta|}{\alpha}\left(\frac{1}{\gamma} + \frac{1}{a - 2\gamma}\right) < 1, \tag{5.1}$$

where $\frac{2c_2|\beta|}{\alpha}$ is the Lipschitz constant of F (see (3.9)). We remark at the end of this section that such a γ exists provided that α and K are sufficiently large. The main result in this section is as follows.

Theorem 5.3. *Assume that $a > 4L_F$ and there is $\gamma \in (0, \frac{a}{2})$ such that* (5.1) *holds. Then the global random attractor $\omega \mapsto A_0(\omega)$ of the random dynamical system Y is a random horizontal curve.*

Proof. Since Eq. (3.1) can be viewed as a deterministic system with a random parameter $\omega \in \Omega$, we write it here as $(3.1)_\omega$ for $\omega \in \Omega$. We first show that for any fixed $\omega \in \Omega$, $(3.1)_\omega$ has a one-dimensional attracting invariant manifold $W(\omega)$.

In order to do so, for fixed $\omega \in \Omega$, let

$$W(\omega) = \left\{Y_0 \in E \,|\, Y(t, \omega, Y_0) \text{ exists for } t \le 0 \text{ and } \sup_{t\le 0} \|e^{\gamma t} Y(t, \omega, Y_0)\|_E < \infty\right\}.$$

We prove that $W(\omega)$ is a one-dimensional attracting invariant manifold of $(3.1)_\omega$.

First of all, by the definition of $W(\omega)$, it is clear that for any $t \in \mathbb{R}$,

$$Y(t, \omega, W(\omega)) = W(\theta_t \omega),$$

that is, $\{W(\omega)\}_{\omega \in \Omega}$ is invariant. By the variation of constant formula, $Y_0 \in W(\omega)$ if and only if there is $\tilde{Y}(t)$ with $\tilde{Y}(0) = Y_0$, $\sup_{t \le 0} \|e^{\gamma t}\tilde{Y}(t)\|_E < \infty$,

$$\begin{aligned}\tilde{Y}(t) = e^{Ct}\xi + \int_0^t e^{C(t-s)} P F^{\omega}(s, \tilde{Y}(s)) ds \\ + \int_{-\infty}^t e^{C(t-s)} Q F^{\omega}(s, \tilde{Y}(s)) ds, \quad t \le 0,\end{aligned} \tag{5.2}$$

and $Y(t, \omega, Y_0) = \tilde{Y}(t)$, where $F^{\omega}(t, Y) = F(\theta_t \omega, Y)$ and $\xi = P\tilde{Y}(0) \in E_1$. For $H : (-\infty, 0] \to E$ such that $\sup_{t \le 0} \|e^{\gamma t} H(t)\|_E < \infty$, define

$$(LH)(t) = \int_0^t e^{C(t-s)} P H(s) ds + \int_{-\infty}^t e^{C(t-s)} Q H(s) ds, \quad t \le 0.$$

Then

$$\begin{aligned}\sup_{t \le 0} \|e^{\gamma t}(LH)(t)\|_E &\le \left(\frac{1}{\gamma} + \frac{1}{a - \gamma}\right) \sup_{t \le 0} \|e^{\gamma t} H(t)\|_E \\ &\le \left(\frac{1}{\gamma} + \frac{1}{a - 2\gamma}\right) \sup_{t \le 0} \|e^{\gamma t} H(t)\|_E,\end{aligned}$$

which means that $\|L\| \le \frac{1}{\gamma} + \frac{1}{a-2\gamma}$. Thus, Theorem 3.3 in [7] shows that for any $\xi \in E_1$, Eq. (5.2) has a unique solution $\tilde{Y}^{\omega}(t, \xi)$ satisfying $\sup_{t \le 0} \|e^{\gamma t}\tilde{Y}^{\omega}(t, \xi)\|_E < \infty$. Let

$$h(\omega, \xi) = Q\tilde{Y}^{\omega}(0, \xi) = \int_{-\infty}^0 e^{-Cs} Q F^{\omega}(s, \tilde{Y}^{\omega}(s, \xi)) ds, \quad \omega \in \Omega.$$

Then,

$$W(\omega) = \{\xi + h(\omega, \xi) : \xi \in E_1\}$$

and $W(\omega)$ is a one-dimensional invariant manifold of $(3.1)_{\omega}$. Furthermore, for any $\epsilon \in (0, \gamma)$, by Lemma 3.5, we have

$$\|h(\theta_{-t}\omega, \xi)\|_E \le \frac{a_1}{a - \epsilon}\tilde{r}(\omega) e^{\epsilon t} + \frac{a_2}{a}, \quad t \ge 0, \tag{5.3}$$

where a_1, a_2 are the same as in the proof of Theorem 4.2.

To show the attracting property of $W(\omega)$, we prove that for each given $\omega \in \Omega$ there exists a stable foliation $\{W_s(Y_0, \omega) : Y_0 \in W(\omega)\}$ of the invariant manifold

$W(\omega)$ of $(3.1)_\omega$. Consider the following integral equation

$$\hat{Y}(t) = e^{Ct}\eta + \int_0^t e^{C(t-s)} Q(F^\omega(s, \hat{Y}(s) + Y^\omega(s, \xi + h(\omega, \xi))) - F^\omega(s, Y^\omega(s, \xi + h(\omega, \xi))))ds + \int_\infty^t e^{C(t-s)} P(F^\omega(s, \hat{Y}(s) + Y^\omega(s, \xi + h(\omega, \xi))) - F^\omega(s, Y^\omega(s, \xi + h(\omega, \xi))))ds, \quad t \geq 0, \tag{5.4}$$

where $\xi + h(\omega, \xi) \in W(\omega)$, $\eta = Q\hat{Y}(0) \in E_2$ and $Y^\omega(t, \xi + h(\omega, \xi)) := Y(t, \omega, \xi + h(\omega, \xi))$, $t \geq 0$ is the solution of (3.1) with initial data $\xi + h(\omega, \xi)$ for fixed $\omega \in \Omega$. Theorem 3.4 in [7] shows that for any $\xi \in E_1$ and $\eta \in E_2$, Eq. (5.4) has a unique solution $\hat{Y}^\omega(t, \xi, \eta)$ satisfying $\sup_{t\geq 0} \|e^{\gamma t}\hat{Y}^\omega(t, \xi, \eta)\|_E < \infty$ and for any $\xi \in E_1$, η_1, $\eta_2 \in E_2$,

$$\sup_{t\geq 0} e^{\gamma t} \|\hat{Y}^\omega(t, \xi, \eta_1) - \hat{Y}^\omega(t, \xi, \eta_2)\|_E \leq M_2 \|\eta_1 - \eta_2\|_E, \tag{5.5}$$

where $M_2 = \frac{1}{1 - \frac{2c_2|\beta|}{\alpha}(\frac{1}{\gamma} + \frac{1}{a-2\gamma})}$. Let

$$\begin{aligned} \hat{h}(\omega, \xi, \eta) &= \xi + P\hat{Y}^\omega(0, \xi, \eta) \\ &= \xi + \int_\infty^0 e^{-Cs} P(F^\omega(s, \hat{Y}^\omega(s, \xi, \eta) + Y^\omega(s, \xi + h(\omega, \xi))) \\ &\quad - F^\omega(s, Y^\omega(s, \xi + h(\omega, \xi))))ds. \end{aligned}$$

Then, $W_s(\omega, \xi + h(\omega, \xi)) = \{\eta + h(\omega, \xi) + \hat{h}(\omega, \xi, \eta) : \eta \in E_2\}$ is a foliation of $W(\omega)$ at $\xi + h(\omega, \xi)$.

Observe that

$$\begin{aligned} &\hat{Y}^\omega(t, \xi, \eta) + Y^\omega(t, \xi + h(\omega, \xi)) - Y^\omega(t, \xi + h(\omega, \xi)) \\ &\quad = \hat{Y}^\omega(t, \xi, \eta) \\ &\quad = e^{Ct}(\eta + h(\omega, \xi) + \hat{h}(\omega, \xi, \eta) - \xi - h(\omega, \xi)) \\ &\qquad + \int_0^t e^{C(t-s)}(F^\omega(s, \hat{Y}^\omega(s, \xi, \eta) + Y^\omega(s, \xi + h(\omega, \xi))) \\ &\qquad - F^\omega(s, Y^\omega(s, \xi + h(\omega, \xi))))ds \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &Y^\omega(t, \eta + h(\omega, \xi) + \hat{h}(\omega, \xi, \eta)) - Y^\omega(t, \xi + h(\omega, \xi)) \\ &\quad = e^{Ct}(\eta + h(\omega, \xi) + \hat{h}(\omega, \xi, \eta) - \xi - h(\omega, \xi)) \\ &\qquad + \int_0^t e^{C(t-s)}(F^\omega(s, Y^\omega(s, \eta + h(\omega, \xi) + \hat{h}(\omega, \xi, \eta))) \\ &\qquad - F^\omega(s, Y^\omega(s, \xi + h(\omega, \xi))))ds. \end{aligned} \tag{5.7}$$

Comparing (5.6) with (5.7), we find that

$$\hat{Y}^\omega(t,\xi,\eta) = Y^\omega(t,\eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)) - Y^\omega(t,\xi + h(\omega,\xi)), \quad t \geq 0. \tag{5.8}$$

In addition, if $\eta = 0$, then by the uniqueness of solution of (5.4), $\hat{Y}^\omega(t,\xi,0) \equiv 0$ for $t \geq 0$, which together with (5.5) and (5.8) shows that

$$\sup_{t\geq 0} e^{\gamma t}\|Y^\omega(t,\eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)) - Y^\omega(t,\xi + h(\omega,\xi))\|_E \leq M_2\|\eta\|_E \tag{5.9}$$

for any $\xi \in E_1$ and $\eta \in E_2$. Therefore, $\{W_s(Y_0,\omega) : Y_0 \in W(\omega)\}$ is a stable foliation of the invariant manifold $W(\omega)$ of $(3.1)_\omega$ and then $W(\omega)$ is a one-dimensional attracting invariant manifold of $(3.1)_\omega$.

Next we show that $A_0(\omega) = W(\omega)$ and $A_0(\omega)$ is a random horizontal curve. Let $\omega \mapsto B(\omega)$ be any pseudo-tempered random set in E. For any $t > 0$ and $Y_0 \in B(\theta_{-t}\omega)$, there is $\xi(\theta_{-t}\omega, Y_0) \in E_1$ such that

$$Y_0 \in W_s(\theta_{-t}\omega, \xi(\theta_{-t}\omega, Y_0) + h(\theta_{-t}\omega, \xi(\theta_{-t}\omega, Y_0))).$$

Let $\eta(\theta_{-t}\omega) = \sup_{Y_0\in B(\theta_{-t}\omega)} \|QY_0 - h(\theta_{-t}\omega, \xi(\theta_{-t}\omega, Y_0))\|_E$. Then by (5.3) and (5.9),

$$\begin{aligned}&\|Y(t,\theta_{-t}\omega, Y_0) - Y(t,\theta_{-t}\omega, \xi(\theta_{-t}\omega, Y_0) + h(\theta_{-t}\omega, \xi(\theta_{-t}\omega, Y_0)))\|_E \\ &\quad \leq M_2 e^{-\gamma t}\eta(\theta_{-t}\omega) \to 0 \quad \text{as} \quad t \to \infty,\end{aligned}$$

which implies that for $\omega \in \Omega$,

$$d_H(Y(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), W(\omega)) \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore,

$$A_0(\omega) = W(\omega) \quad \text{for} \quad \omega \in \Omega.$$

Moreover, for any random horizontal curve $\omega \mapsto \ell(\omega)$ in E contained in some pseudo-tempered random set,

$$d_H(Y(t,\theta_{-t}\omega, \ell(\theta_{-t}\omega)), A_0(\omega)) \to 0 \quad \text{as} \quad t \to \infty$$

for every $\omega \in \Omega$, which means that $\lim_{t\to\infty} Y(t,\theta_{-t}\omega, \ell(\theta_{-t}\omega)) \subset A_0(\omega)$. Since $A_0(\omega)$ is one-dimensional, we have for $\omega \in \Omega$,

$$A_0(\omega) = \lim_{t\to\infty} Y(t,\theta_{-t}\omega, \ell(\theta_{-t}\omega)).$$

It then follows from Lemma 5.2 that $\omega \mapsto A_0(\omega)$ is a random horizontal curve. □

Corollary 5.4. *Assume that $a > 4L_F$ and there is $\gamma \in (0, \frac{a}{2})$ such that* (5.1) *holds. Then the random attractor $\omega \mapsto A(\omega)$ of the random dynamical system ϕ is a random horizontal curve.*

Proof. It follows from Corollary 4.3, Remark 4.4 and Theorem 5.3. □

Remark 5.5. At the beginning of this section, we assume that $a > 4L_F$. Since $a = \frac{\alpha}{2} - |\frac{\alpha}{2} - \frac{\delta K\lambda_1}{\alpha}|$ and $L_F = \frac{2c_2|\beta|}{\alpha}$, we can take α, K satisfying $\frac{\alpha}{2} - |\frac{\alpha}{2} - \frac{\delta K\lambda_1}{\alpha}| > \frac{8c_2|\beta|}{\alpha}$, where λ_1 is the smallest positive eigenvalue of A. On the other hand, we need some $\gamma \in (0, \frac{a}{2})$ such that (5.1) holds. Note that

$$\min_{\gamma\in(0,\frac{a}{2})}\left(\frac{1}{\gamma} + \frac{1}{a-2\gamma}\right) = \left(\frac{1}{\gamma} + \frac{1}{a-2\gamma}\right)\Bigg|_{\gamma=\frac{a}{2+\sqrt{2}}} = \frac{(\sqrt{2}+1)^2}{a},$$

which implies that there exist α, K satisfying

$$\frac{\alpha}{2} - \left|\frac{\alpha}{2} - \frac{\delta K\lambda_1}{\alpha}\right| > \frac{2c_2|\beta|(\sqrt{2}+1)^2}{\alpha} > \frac{8c_2|\beta|}{\alpha}. \tag{5.10}$$

Indeed, let $c = 2c_2|\beta|(\sqrt{2}+1)^2$, then for any $\alpha > \sqrt{2c}$ and $K > \frac{c}{\lambda_1}$, there is a $\delta > 0$ satisfying

$$\frac{c}{K\lambda_1} < \delta < \min\left\{\frac{\alpha^2 - c}{K\lambda_1}, 1\right\}$$

such that (5.10) holds.

6. Rotation Number

In this section, we study the phenomenon of frequency locking, i.e. the existence of a rotation number of the coupled second order oscillators with white noises (1.1).

Definition 6.1. The coupled second order system with white noises (1.1) is said to have a *rotation number* $\rho \in \mathbb{R}$ if, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and each $\phi_0 = (u_0, u_1)^\top \in E$, the limit $\lim_{t\to\infty} \frac{P\phi(t,\omega,\phi_0)}{t}$ exists and

$$\lim_{t\to\infty} \frac{P\phi(t,\omega,\phi_0)}{t} = \rho\eta_0,$$

where η_0 is the basis of E_1.

Note that the rotation number is considered here by restricting ϕ on E_1, since ϕ is dissipative on E_2 and limits likewise in Definition 6.1 vanish. From (3.4), we have

$$\frac{P\phi(t,\omega,\phi_0)}{t} = \frac{PY(t,\omega,Y_0(\omega))}{t} + \frac{P(0,z(\theta_t\omega))^\top}{t}, \tag{6.1}$$

where $\phi_0 = (u_0, u_1)^\top$ and $Y_0(\omega) = (u_0, u_1 - z(\omega))^\top$. By Lemma 2.1 in [9], it is easy to prove that $\lim_{t\to\infty} \frac{P(0,z(\theta_t\omega))^\top}{t} = (0,0)^\top$. Thus, it is sufficient to prove the existence of the rotation number of the random system (3.1).

Let us show a simple lemma which will be used. For any $p_i = s_i\eta_0 \in E_1$, $i = 1, 2$, we define

$$p_1 \le p_2 \quad \text{if } s_1 \le s_2.$$

Then we have

Lemma 6.2. *Suppose that $a > 4L_F$. Let ℓ be any deterministic np_0-periodic horizontal curve (ℓ satisfies the Lipschitz and periodic condition in Definition 5.1). For any $Y_1,\ Y_2 \in \ell$ with $PY_1 \le PY_2$, there holds*

$$PY(t,\omega,Y_1) \le PY(t,\omega,Y_2) \quad \text{for } t > 0,\ \omega \in \Omega. \tag{6.2}$$

The proof of this lemma is similar to that of Lemma 6.3 in [24]. We then omit it here. We now have the main result in this section.

Theorem 6.3. *Let $a > 4L_F$. Then the rotation number of* (3.1) *exists.*

Proof. By the random dynamical system $\mathbf{Y}$ defined in (4.1), we define the corresponding skew-product semiflow $\Theta_t : \Omega \times \mathbf{E} \to \Omega \times \mathbf{E}$ for $t \ge 0$ by setting

$$\Theta_t(\omega, \mathbf{Y}_0) = (\theta_t\omega, \mathbf{Y}(t,\omega,\mathbf{Y}_0)).$$

Obviously, $(\Omega \times \mathbf{E},\ \mathcal{F} \times \mathcal{B},\ (\Theta_t)_{t\ge 0})$ is a measurable dynamical system, where $\mathcal{B} = \mathcal{B}(\mathbf{E})$ is the Borel σ-algebra of $\mathbf{E}$. It can also be verified that there is a measure μ on $\Omega \times \mathbf{E}$ such that $(\Omega \times \mathbf{E},\ \mathcal{F} \times \mathcal{B},\ \mu,\ (\Theta_t)_{t\ge 0})$ becomes an ergodic metric dynamical system (see [6]). Note that

$$\frac{PY(t,\omega,Y_0)}{t} = \frac{PY_0}{t} + \frac{1}{t}\int_0^t PF(\theta_s\omega, Y(s,\omega,Y_0))ds.$$

Since $F(\theta_s\omega, Y(s,\omega,Y_0) + kp_0) = F(\theta_s\omega, Y(s,\omega,Y_0))$, $\forall\, k \in \mathbb{Z}$, we can identify $F(\theta_s\omega, \mathbf{Y}(s,\omega,\mathbf{Y_0}))$ with $F(\theta_s\omega, Y(s,\omega,Y_0))$ and write

$$F(\theta_s\omega, Y(s,\omega,Y_0)) = F(\theta_s\omega, \mathbf{Y}(s,\omega,\mathbf{Y}_0)).$$

Thus,

$$\begin{aligned}\frac{PY(t,\omega,Y_0)}{t} &= \frac{PY_0}{t} + \frac{1}{t}\int_0^t PF(\theta_s\omega, \mathbf{Y}(s,\omega,\mathbf{Y}_0))ds \\ &= \frac{PY_0}{t} + \frac{1}{t}\int_0^t \mathbf{F}(\Theta_s(\omega,\mathbf{Y}_0))ds,\end{aligned} \tag{6.3}$$

where $\mathbf{F} = P \circ F \in L^1(\Omega \times \mathbf{E},\ \mathcal{F} \times \mathcal{B},\ \mu)$. Let $t \to \infty$ in (6.3), $\lim_{t\to\infty} \frac{PY_0}{t} = (0,0)^\top$ and by Ergodic Theorems in [1], there exists a constant $\rho \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{F}(\Theta_s(\omega,\mathbf{Y}_0))ds = \rho\eta_0,$$

which means

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y_0)}{t} = \rho\eta_0$$

for μ-a.e. $(\omega, Y_0) \in \Omega \times E$. Thus, there is $\Omega^* \subset \Omega$ with $\mathbb{P}(\Omega^*) = 1$ such that for any $\omega \in \Omega^*$, there is $Y_0^*(\omega) \in E$ such that

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y_0^*(\omega))}{t} = \rho\eta_0.$$

By Lemma 3.3, we have that for any $n \in \mathbb{N}$ and $\omega \in \Omega^*$,

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y_0^*(\omega) \pm np_0)}{t} = \lim_{t\to\infty} \frac{PY(t,\omega,Y_0^*(\omega)) \pm np_0}{t} = \rho\eta_0. \tag{6.4}$$

Now for any $\omega \in \Omega^*$ and any $Y \in E$, there is $n_0(\omega) \in \mathbb{N}$ such that

$$PY_0^*(\omega) - n_0(\omega)p_0 \le PY \le PY_0^*(\omega) + n_0(\omega)p_0$$

and there is a $n_0(\omega)p_0$-periodic horizontal curve $l_0(\omega)$ such that $Y_0^*(\omega) - n_0(\omega)p_0$, Y, $Y_0^*(\omega) + n_0(\omega)p_0 \in l_0(\omega)$. Then by Lemma 6.2, we have

$$PY(t,\omega,Y_0^*(\omega) - n_0(\omega)p_0) \le PY(t,\omega,Y) \le PY(t,\omega,Y_0^*(\omega) + n_0(\omega)p_0),$$

which together with (6.4) implies that for any $\omega \in \Omega^*$ and any $Y \in E$,

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y)}{t} = \rho\eta_0.$$

Consequently, for any a.e. $\omega \in \Omega$ and any $Y \in E$,

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y)}{t} = \rho\eta_0.$$

The theorem is thus proved. □

Corollary 6.4. *Assume that $a > 4L_F$. Then the rotation number of the coupled second order system with white noises* (1.1) *exists.*

Proof. It follows from (6.1) and Theorem 6.3. □

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