

SYNCHRONIZATION OF COUPLED STOCHASTIC SYSTEMS WITH MULTIPLICATIVE NOISE

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We consider the synchronization of solutions to coupled systems of the conjugate random ordinary differential equations (RODEs) for the N -Stratonovich stochastic ordinary differential equations (SODEs) with linear multiplicative noise ($N \in \mathbb{N}$). We consider the synchronization between two solutions and among different components of solutions under one-sided dissipative Lipschitz conditions. We first show that the random dynamical system generated by the solution of the coupled RODEs has a singleton sets random attractor which implies the synchronization of any two solutions. Moreover, the singleton sets random attractor determines a stationary stochastic solution of the equivalently coupled SODEs. Then we show that any solution of the RODEs converge to a solution of the averaged RODE within any finite time interval as the coupled coefficient tends to infinity. Our results generalize the work of two Stratonovich SODEs in [9].

Keywords: Synchronization; random dynamical systems; multiplicative noise.

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1. Introduction

The synchronization of coupled systems is a well-known phenomenon in both biology and physics. It is also known to occur in many other different fields. Descriptions of its diversity of occurrence can be found in [1–4, 7, 12–15, 17, 20]. Synchronization of deterministic coupled systems has been investigated mathematically in [5, 11, 19] for autonomous systems and in [16] for nonautonomous systems. For coupled systems of Itô stochastic ordinary differential equations with additive noise, Caraballo and Kloeden proved its synchronization of solutions in [8].

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} = C_0(\mathbb{R}, \mathbb{R}),$$

the Borel σ -algebra \mathcal{F} on Ω is generated by the compact open topology (see [6, 14]), and \mathbb{P} is the corresponding Wiener measure on (Ω, \mathcal{F}) . Define $(\theta_t)_{t \in \mathbb{R}}$ on Ω via

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ becomes an ergodic metric dynamical system.

Consider the following N -Stratonovich stochastic ordinary differential equations (SODEs) in \mathbb{R}^d ($d \in \mathbb{N}$):

$$dX_t^{(j)} = f^{(j)}(X_t^{(j)})dt + \sum_{i=1}^m c_i^{(j)} X_t^{(j)} \circ dW_t^{(i)}, \quad j = 1, \dots, N, \tag{1.1}$$

where $c_i^{(j)} \in \mathbb{R}$, $W_t^{(i)}$ are independent two-sided scalar Wiener processes on $(\Omega, \mathcal{F}, \mathbb{P})$ for $i = 1, \dots, m$, and $f^{(j)}$, $j = 1, \dots, N$, are regular enough to ensure the existence and uniqueness of solutions and satisfy the one-sided dissipative Lipschitz conditions

$$\langle x_1 - x_2, f^{(j)}(x_1) - f^{(j)}(x_2) \rangle \leq -L\|x_1 - x_2\|^2, \quad j = 1, \dots, N \tag{1.2}$$

on \mathbb{R}^d for some $L > 0$.

Set

$$x^{(j)}(t, \omega) = e^{-O_t^{(j)}(\omega)} X_t^{(j)}(\omega), \quad t \in \mathbb{R}, \quad \omega \in \Omega, \quad j = 1, \dots, N,$$

where

$$O_t^{(j)} = \sum_{i=1}^m c_i^{(j)} e^{-t} \int_{-\infty}^t e^\tau dW_\tau^{(i)}, \quad j = 1, \dots, N,$$

are N stationary Ornstein–Uhlenbeck processes which solve the following Ornstein–Uhlenbeck stochastic differential equations, respectively,

$$dO_t^{(j)} = -O_t^{(j)} dt + \sum_{i=1}^m c_i^{(j)} dW_t^{(i)}, \quad j = 1, \dots, N.$$

Then SODEs (1.1) can be transformed into the following conjugate pathwise random ordinary differential equations (RODEs)

$$\begin{aligned} \frac{dx^{(j)}}{dt} &= F^{(j)}(x^{(j)}, O_t^{(j)}(\omega)) \\ &:= e^{-O_t^{(j)}(\omega)} f^{(j)}(e^{O_t^{(j)}(\omega)} x^{(j)}) + O_t^{(j)}(\omega) x^{(j)}, \quad j = 1, \dots, N \end{aligned} \tag{1.3}$$

(see [18] for the conjugate theory of SODE and RODE).

Now we consider the linear coupled RODEs of (1.3)

$$\frac{dx^{(j)}}{dt} = F^{(j)}(x^{(j)}, O_t^{(j)}(\omega)) + \nu(x^{(j-1)} - 2x^{(j)} + x^{(j+1)}), \quad j = 1, \dots, N \tag{1.4}$$

with coupling coefficient $\nu > 0$, where $x^{(0)} = x^{(N)}$ and $x^{(N+1)} = x^{(1)}$. Now (1.4) can be written as the following equivalent SODEs

$$dX_t^{(j)} = (f^{(j)}(X_t^{(j)}) + \nu(e^{\rho_t^{(j)}} X_t^{(j-1)} - 2X_t^{(j)} + e^{\varrho_t^{(j)}} X_t^{(j+1)}))dt + \sum_{i=1}^m c_i^{(j)} X_t^{(j)} \circ dW_t^{(i)}, \quad j = 1, \dots, N, \tag{1.5}$$

where $\rho_t^{(j)} = O_t^{(j)} - O_t^{(j-1)}$, $\varrho_t^{(j)} = O_t^{(j)} - O_t^{(j+1)}$, $O_t^{(0)} = O_t^{(N)}$ and $O_t^{(N+1)} = O_t^{(1)}$. For synchronization of solutions to coupled RODEs (1.4), there are two cases: one for any two solutions and the other for components of solutions. When $N = 2$, i.e. for two Stratonovich SODEs, Caraballo *et al.* [9] considered both types of synchronization. Under the assumption of one-sided dissipative Lipschitz conditions (1.2), they first proved that synchronization of any two solutions occurs and the random dynamical system generated by the solution of (1.4) _{$N=2$} has a singleton sets random attractor; then they proved that the synchronization between any two components of solutions occurs as the coupled coefficient ν tends to infinity. Moreover, when the driving noise is the same in each system, exact synchronization occurs no matter how large the intensity coefficients of noise are. Based on the work of [9], in this paper, we consider the above two types of synchronization of solutions of (1.4) in the case of $N \geq 3$ and obtain similar results. Explicitly, we show that the random dynamical system generated by the solution of the coupled RODEs (1.4) has a singleton sets random attractor which implies the synchronization of any two solutions of (1.4). Moreover, the singleton sets random attractor determines a stationary stochastic solution of the equivalently coupled SODEs (1.5). We also show that any solutions of RODEs (1.4) converge to a solution $\mathbf{z}(t, \omega)$ of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{N} \sum_{j=1}^N e^{-O_t^{(j)}} f^{(j)}(e^{O_t^{(j)}} z) + \frac{1}{N} \sum_{j=1}^N O_t^{(j)} z \tag{1.6}$$

as the coupling coefficient $\nu \rightarrow \infty$. Here it is worth mentioning that this generalization is not trivial since some new techniques are used especially in Sec. 4.

The rest of this paper is organized as follows. In Sec. 2, we introduce two lemmas which will be used frequently. In Sec. 3, we show synchronization of two solutions to the coupled RODEs and obtain the stationary stochastic solution to the equivalent SODEs. In Sec. 4, we study synchronization of components of solutions to the coupled RODEs and obtain the exact synchronization of the equivalent SODEs provided by same driving noise.

2. Two Lemmas

We will frequently use the following two lemmas.

Lemma 2.1. *There exists a $(\theta_t)_{t \in \mathbb{R}}$ invariant subset $\bar{\Omega} \in \mathcal{F}$ of $\Omega = C_0(\mathbb{R}, \mathbb{R})$ of full measure such that for $\omega \in \bar{\Omega}$,*

$$\lim_{t \rightarrow \pm\infty} \frac{|\omega(t)|}{t} = 0, \tag{2.1}$$

and for $j = 1, \dots, N$, there exist random variables $\bar{O}^{(j)} = O_t^{(j)}$ and $T_\omega > 0$ such that

$$\bar{O}^{(j)}(\theta_t \omega) = O_t^{(j)}(\omega), \quad \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t \bar{O}^{(j)}(\theta_\tau \omega) d\tau = 0, \quad \omega \in \bar{\Omega} \tag{2.2}$$

and

$$e^{2 \int_s^t O_\tau^{(j)} d\tau} \leq e^{\frac{L}{2}(t-s)} \quad \text{for } -s, t > T_\omega.$$

Proof. The equalities (2.1) and (2.2) can be found in [9, 10]. By (2.2), $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t O_\tau^{(j)} d\tau = 0$, thus, there exists $T_\omega(1) > 0$ such that $\int_0^t O_\tau^{(j)} d\tau \leq \frac{L}{4}t$ for $t > T_\omega(1)$. Similarly, $\lim_{s \rightarrow -\infty} \frac{1}{s} \int_s^0 O_\tau^{(j)} d\tau = 0$ implies that there exists $T_\omega(2) > 0$ such that $\int_s^0 O_\tau^{(j)} d\tau \leq -\frac{L}{4}s$ for $-s > T_\omega(2)$. Taking $T_\omega = \max\{T_\omega(1), T_\omega(2)\}$, we have $2 \int_s^t O_\tau^{(j)} d\tau \leq \frac{L}{2}(t-s)$ for $-s, t > T_\omega$, which yields the assertion. \square

We remark that the proof of (2.1) and (2.2) requires the ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. In the following sections, since $\bar{\Omega}$ is an $(\theta_t)_{t \in \mathbb{R}}$ invariant set with full measure, we consider $(\theta_t)_{t \in \mathbb{R}}$ defined on $\bar{\Omega}$ instead of Ω . This mapping has the same properties as the original one if we choose for \mathcal{F} the trace σ -algebra with respect to $\bar{\Omega}$.

Lemma 2.2. *Suppose that $A(t)$ is a $p \times p$ matrix and $\varphi(t), \psi(t)$ are p -dimensional vectors on $[t_0, t]$ ($t \geq t_0, t, t_0 \in \mathbb{R}$) which are sufficiently regular. If the following inequality holds in the componentwise sense*

$$\frac{d}{dt} \varphi(t) \leq A(t)\varphi(t) + \psi(t), \quad t \geq t_0, \tag{2.3}$$

then

$$\varphi(t) \leq \exp\left(\int_{t_0}^t A(\tau) d\tau\right) \varphi(t_0) + \int_{t_0}^t \exp\left(\int_u^t A(\tau) d\tau\right) \psi(u) du, \quad t \geq t_0. \tag{2.4}$$

Proof. It follows from (2.3) that

$$\begin{aligned} \frac{d}{dt} \left(\exp\left(-\int_{t_0}^t A(\tau) d\tau\right) \varphi(t) \right) &= \exp\left(-\int_{t_0}^t A(\tau) d\tau\right) \left(\frac{d}{dt} \varphi(t) - A(t)\varphi(t) \right) \\ &\leq \exp\left(-\int_{t_0}^t A(\tau) d\tau\right) \psi(t), \end{aligned}$$

then

$$\exp\left(-\int_{t_0}^t A(\tau) d\tau\right) \varphi(t) - \varphi(t_0) \leq \int_{t_0}^t \exp\left(-\int_{t_0}^u A(\tau) d\tau\right) \psi(u) du,$$

which implies inequality (2.4). \square

3. Synchronization of Two Solutions

Consider the coupled RODEs (1.4)

$$\frac{dx^{(j)}}{dt} = F^{(j)}(x^{(j)}, O_t^{(j)}(\omega)) + \nu(x^{(j-1)} - 2x^{(j)} + x^{(j+1)}), \quad j = 1, \dots, N \quad (3.1)$$

with initial data

$$x^{(j)}(0, \omega) = x_0^{(j)}(\omega) \in \mathbb{R}^d, \quad \omega \in \Omega, \quad j = 1, \dots, N, \quad (3.2)$$

where $\nu > 0$, and

$$F^{(j)}(x^{(j)}, O_t^{(j)}(\omega)) = e^{-O_t^{(j)}(\omega)} f^{(j)}(e^{O_t^{(j)}(\omega)} x^{(j)}) + O_t^{(j)}(\omega) x^{(j)}, \quad j = 1, \dots, N. \quad (3.3)$$

Here $f^{(j)}$ are regular enough to ensure the existence and uniqueness of global solutions on \mathbb{R} and satisfy the one-sided dissipative Lipschitz conditions (1.2) for $j = 1, \dots, N$.

For asymptotic behavior of the difference between two solutions of RODEs (3.1) and (3.2), we have

Lemma 3.1. *For any two solutions $(x_1^{(1)}(t), x_1^{(2)}(t), \dots, x_1^{(N)}(t))^\top$ and $(x_2^{(1)}(t), x_2^{(2)}(t), \dots, x_2^{(N)}(t))^\top$ of RODEs (3.1) and (3.2) (omitting ω for brevity),*

$$\lim_{t \rightarrow \infty} \|x_1^{(j)}(t) - x_2^{(j)}(t)\| = 0, \quad j = 1, \dots, N,$$

that is, all solutions of the coupled RODEs (3.1) and (3.2) converge pathwise to each other as time t goes to infinity.

Proof. By the one-sided dissipative Lipschitz conditions (1.2), we obtain for $j = 1, \dots, N$,

$$\begin{aligned} \frac{d}{dt} \|x_1^{(j)}(t) - x_2^{(j)}(t)\|^2 &= 2 \left\langle x_1^{(j)}(t) - x_2^{(j)}(t), \frac{d}{dt} x_1^{(j)}(t) - \frac{d}{dt} x_2^{(j)}(t) \right\rangle \\ &= 2e^{-O_t^{(j)}} \langle x_1^{(j)}(t) - x_2^{(j)}(t), f^{(j)}(e^{O_t^{(j)}} x_1^{(j)}(t)) \\ &\quad - f^{(j)}(e^{O_t^{(j)}} x_2^{(j)}(t)) \rangle \\ &\quad + (2O_t^{(j)} - 4\nu) \|x_1^{(j)}(t) - x_2^{(j)}(t)\|^2 \\ &\quad + 2\nu \langle x_1^{(j)}(t) - x_2^{(j)}(t), x_1^{(j-1)}(t) \\ &\quad - x_2^{(j-1)}(t) + x_1^{(j+1)}(t) - x_2^{(j+1)}(t) \rangle \\ &\leq (2O_t^{(j)} - 2L - 2\nu) \|x_1^{(j)}(t) - x_2^{(j)}(t)\|^2 \\ &\quad + \nu \|x_1^{(j-1)}(t) - x_2^{(j-1)}(t)\|^2 + \nu \|x_1^{(j+1)}(t) - x_2^{(j+1)}(t)\|^2. \end{aligned}$$

Define

$$\mathbf{x}(t) = (\|x_1^{(1)}(t) - x_2^{(1)}(t)\|^2, \|x_1^{(2)}(t) - x_2^{(2)}(t)\|^2, \dots, \|x_1^{(N)}(t) - x_2^{(N)}(t)\|^2)^\top, \quad t \in \mathbb{R}$$

and

$$A_\nu(t) = \begin{pmatrix} a_\nu^{(1)}(t) & \nu & 0 & \cdots & 0 & \nu \\ \nu & a_\nu^{(2)}(t) & \nu & 0 & \cdots & 0 \\ 0 & \nu & a_\nu^{(3)}(t) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \nu & 0 \\ 0 & \cdots & 0 & \nu & a_\nu^{(N-1)}(t) & \nu \\ \nu & 0 & \cdots & 0 & \nu & a_\nu^{(N)}(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

where diagonal entries $a_\nu^{(j)}(t) = 2O_t^{(j)} - 2L - 2\nu$, $j = 1, \dots, N$. Thus the above differential inequalities can be written as a simple form

$$\dot{\mathbf{x}}(t) \leq A_\nu(t)\mathbf{x}(t), \tag{3.4}$$

componentwisely. By Lemma 2.2 and (3.4) we obtain

$$\mathbf{x}(t) \leq \exp\left(\int_0^t A_\nu(\tau)d\tau\right)\mathbf{x}(0)$$

componentwisely.

The proof of this lemma will be completed in the following Lemma 3.2. □

Lemma 3.2. For $t \geq T_\omega$ and $\nu > 0$,

$$\left\| \exp\left(\int_0^t A_\nu(\tau)d\tau\right)\mathbf{x}(0) \right\| \leq e^{-Lt}\|\mathbf{x}(0)\|,$$

where T_ω is defined as in Lemma 2.1.

Proof. Matrix $\int_0^t A_\nu(\tau)d\tau$ is real symmetric, which implies that there exists an orthonormal basis consisting of eigenvectors $\eta_{\nu,t}^{(1)}, \eta_{\nu,t}^{(2)}, \dots, \eta_{\nu,t}^{(N)}$ of \mathbb{R}^N with eigenvalues $\lambda_{\nu,t}^{(1)}, \lambda_{\nu,t}^{(2)}, \dots, \lambda_{\nu,t}^{(N)}$, and therefore there exist $c_{\mathbf{x}(0),\nu,t}^{(1)}, c_{\mathbf{x}(0),\nu,t}^{(2)}, \dots, c_{\mathbf{x}(0),\nu,t}^{(N)}$ such that

$$\mathbf{x}(0) = \sum_{j=1}^N c_{\mathbf{x},\nu,t}^{(j)}\eta_{\nu,t}^{(j)}.$$

Since $\eta_{\nu,t}^{(1)}, \eta_{\nu,t}^{(2)}, \dots, \eta_{\nu,t}^{(N)}$ are orthogonal and $\exp(\int_0^t A_\nu(\tau) d\tau) \eta_{\nu,t}^{(j)} = e^{\lambda_{\nu,t}^{(j)}} \eta_{\nu,t}^{(j)}$ for $j = 1, \dots, N$, we have

$$\begin{aligned} \left\| \exp\left(\int_0^t A_\nu(\tau) d\tau\right) \mathbf{x}(0) \right\|^2 &= \left\| \sum_{j=1}^N c_{\mathbf{x}(0), \nu, t}^{(j)} \exp\left(\int_0^t A_\nu(\tau) d\tau\right) \eta_{\nu,t}^{(j)} \right\|^2 \\ &= \left\| \sum_{j=1}^N e^{\lambda_{\nu,t}^{(j)}} c_{\mathbf{x}(0), \nu, t}^{(j)} \eta_{\nu,t}^{(j)} \right\|^2 \\ &\leq e^{2 \max\{\lambda_{\nu,t}^{(1)}, \lambda_{\nu,t}^{(2)}, \dots, \lambda_{\nu,t}^{(N)}\}} \|\mathbf{x}(0)\|^2. \end{aligned} \tag{3.5}$$

Next, let us estimate the upper bound of eigenvalues of matrix $\int_0^t A_\nu(\tau) d\tau$. The quadratic form satisfies

$$\begin{aligned} f(\xi_1, \xi_2, \dots, \xi_N) &= \xi^\top \left(\int_0^t A_\nu(\tau) d\tau \right) \xi \\ &= \sum_{j=1}^N \left(2 \int_0^t O_\tau^{(j)} d\tau - 2Lt - 2\nu t \right) \xi_j^2 + 2\nu t \sum_{j=1}^N \xi_j \xi_{j-1} \\ &\leq \sum_{j=1}^N \left(2 \int_0^t O_\tau^{(j)} d\tau - Lt \right) \xi_j^2 - Lt \sum_{j=1}^N \xi_j^2, \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_N)^\top \in \mathbb{R}^N$ and $\xi_0 = \xi_N$. Hence, it follows from Lemma 2.1 that

$$f(\xi_1, \xi_2, \dots, \xi_N) \leq -Lt \sum_{j=1}^N \xi_j^2, \tag{3.6}$$

for $t \geq T_\omega$ and for all $\nu > 0$. Inequality (3.6) implies that the quadratic form is negative definite and eigenvalues of $\int_0^t A_\nu(\tau) d\tau$ satisfy

$$\max\left\{ \lambda_{\nu,t}^{(1)}, \lambda_{\nu,t}^{(2)}, \dots, \lambda_{\nu,t}^{(N)} \right\} \leq -Lt. \tag{3.7}$$

Combining (3.5) and (3.7) yields the assertion. □

Now we use the theory of random dynamical systems to find what the solutions of (3.1) and (3.2) will converge to. It is easy to see from [6] that the solution

$$\phi(t, \omega) = (x^{(1)}(t, \omega), x^{(2)}(t, \omega), \dots, x^{(N)}(t, \omega))^\top, \quad \omega \in \Omega$$

of (3.1) and (3.2) generates a random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with state space \mathbb{R}^{Nd} . For this random dynamical system $\phi(t, \omega)$, we have

Theorem 3.3. $\phi(t, \omega)$, $t \in \mathbb{R}$, $\omega \in \Omega$, has a singleton sets random attractor $\{A_\nu(\omega)\}$ where

$$A_\nu(\omega) = (\bar{x}_\nu^{(1)}(\omega), \bar{x}_\nu^{(2)}(\omega), \dots, \bar{x}_\nu^{(N)}(\omega))^\top,$$

which implies the synchronization of any two solutions of (3.1) and (3.2). Moreover,

$$(\bar{x}_\nu^{(1)}(\theta_t\omega)e^{O_t^{(1)}(\omega)}, \bar{x}_\nu^{(2)}(\theta_t\omega)e^{O_t^{(2)}(\omega)}, \dots, \bar{x}_\nu^{(N)}(\theta_t\omega)e^{O_t^{(N)}(\omega)})^\top$$

is the stationary stochastic solution of the equivalently coupled SODEs (1.5).

Proof. First,

$$\begin{aligned} \frac{d}{dt}\|x^{(j)}(t)\|^2 &= 2\left\langle x^{(j)}(t), \frac{d}{dt}x^{(j)}(t) \right\rangle \\ &= 2\langle x^{(j)}(t), e^{-O_t^{(j)}}f^{(j)}(e^{O_t^{(j)}}x^{(j)}(t)) \rangle + 2\langle x^{(j)}(t), O_t^{(j)}x^{(j)}(t) \rangle \\ &\quad + 2\nu\langle x^{(j)}(t), x^{(j-1)}(t) - 2x^{(j)}(t) + x^{(j+1)}(t) \rangle \\ &\leq 2e^{-2O_t^{(j)}}\langle e^{O_t^{(j)}}x^{(j)}(t) - 0, f^{(j)}(e^{O_t^{(j)}}x^{(j)}(t)) - f^{(j)}(0) \rangle \\ &\quad + 2e^{-O_t^{(j)}}\langle x^{(j)}(t), f^{(j)}(0) \rangle + (2O_t^{(j)} - 4\nu)\|x^{(j)}(t)\|^2 \\ &\quad + 2\nu\langle x^{(j)}(t), x^{(j-1)}(t) + x^{(j+1)}(t) \rangle \\ &\leq (2O_t^{(j)} - 2L - 2\nu)\|x^{(j)}(t)\|^2 + \nu\|x^{(j-1)}(t)\|^2 + \nu\|x^{(j+1)}(t)\|^2 \\ &\quad + 2\|x^{(j)}(t)\|\|f^{(j)}(0)\|e^{-O_t^{(j)}} \\ &\leq (2O_t^{(j)} - L - 2\nu)\|x^{(j)}(t)\|^2 + \nu\|x^{(j-1)}(t)\|^2 + \nu\|x^{(j+1)}(t)\|^2 \\ &\quad + \frac{e^{-2O_t^{(j)}}}{L}\|f^{(j)}(0)\|^2, \end{aligned}$$

for $j = 1, \dots, N$. Analogous to (3.4), we obtain

$$\dot{\tilde{\mathbf{x}}}(t) \leq \tilde{A}_\nu(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{f}}(t)$$

with

$$\begin{aligned} \tilde{\mathbf{x}}(t) &= (\|x^{(1)}(t)\|^2, \|x^{(2)}(t)\|^2, \dots, \|x^{(N)}(t)\|^2)^\top, \quad t \in \mathbb{R}, \\ \tilde{\mathbf{f}}(t) &= \frac{1}{L}(e^{-2O_t^{(1)}}\|f^{(1)}(0)\|^2, e^{-2O_t^{(2)}}\|f^{(2)}(0)\|^2, \dots, e^{-2O_t^{(N)}}\|f^{(N)}(0)\|^2)^\top, \quad t \in \mathbb{R} \end{aligned}$$

and

$$\tilde{A}_\nu(t) = \begin{pmatrix} \tilde{a}_\nu^{(1)}(t) & \nu & 0 & \cdots & 0 & \nu \\ \nu & \tilde{a}_\nu^{(2)}(t) & \nu & 0 & \cdots & 0 \\ 0 & \nu & \tilde{a}_\nu^{(3)}(t) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \nu & 0 \\ 0 & \cdots & 0 & \nu & \tilde{a}_\nu^{(N-1)}(t) & \nu \\ \nu & 0 & \cdots & 0 & \nu & \tilde{a}_\nu^{(N)}(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

where diagonal entries $\tilde{a}_\nu^{(j)}(t) = 2O_t^{(j)} - L - 2\nu$ for $j = 1, \dots, N$. Then by Lemma 2.2,

$$\tilde{\mathbf{x}}(t) \leq \exp\left(\int_{t_0}^t \tilde{A}_\nu(\tau) d\tau\right) \tilde{\mathbf{x}}(t_0) + \int_{t_0}^t \exp\left(\int_u^t \tilde{A}_\nu(\tau) d\tau\right) \tilde{\mathbf{f}}(u) du, \quad t \geq t_0.$$

Analogous to Lemma 3.2, we have

$$\left\| \exp\left(\int_{t_0}^t \tilde{A}_\nu(\tau) d\tau\right) \tilde{\mathbf{x}}(t_0) \right\| \leq e^{-\frac{L}{2}(t-t_0)} \|\tilde{\mathbf{x}}(t_0)\|, \quad -t_0, t \geq T_\omega, \quad \nu > 0.$$

Define

$$\begin{aligned} C_\nu(\omega) &:= \int_{-\infty}^0 \exp\left(\int_u^0 \tilde{A}_\nu(\tau) d\tau\right) \tilde{\mathbf{f}}(t) du, \\ R_\nu^2(\omega) &= 1 + \|C_\nu(\omega)\|^2 \end{aligned} \tag{3.8}$$

and let $B_\nu(\omega)$ be a random ball in \mathbb{R}^{Nd} centered at the origin with radius $R_\nu(\omega)$. Note that the infinite integral on the right-hand side of (3.8) is well defined by Lemma 2.1.

Note that if $\lim_{t \rightarrow \infty} e^{-kt} \|\tilde{\mathbf{x}}(t_0)\| = 0$ for all $k > 0$, then

$$\sum_{j=1}^N \|x^{(j)}(0)\|^2 < R_\nu^2(\omega) \quad \text{as } t_0 \rightarrow -\infty,$$

which implies that the closed random ball $B_\nu(\omega)$ is a pullback absorbing set at $t = 0$ of $\phi(t, \omega)$, that is, for any $\omega \in \Omega$ and any $D \in \mathcal{D}$ (\mathcal{D} is a collection of tempered random bounded sets, i.e. $\lim_{t \rightarrow \infty} e^{-kt} \sup_{u \in D(\theta_{-t}\omega)} \|u\| = 0$), there exists $t_{B_\nu}(\omega)$ such that

$$\phi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) \subset B_\nu(\omega) \quad \text{for all } t \geq t_{B_\nu}(\omega).$$

Hence by Theorem 4.1 in [14], the random dynamical system $\phi(t, \omega)$ generated by the coupled RODEs (3.1) and (3.2) has a random attractor $A_\nu(\omega)$ in $B_\nu(\omega)$ for each ω with the properties that $A_\nu(\omega)$ is compact, ϕ -invariant ($\phi(t, \omega)A_\nu(\omega) = A_\nu(\theta_t\omega)$ for all $t \geq 0$ and $\omega \in \Omega$) and attracting in \mathcal{D} , i.e. for all $D \in \mathcal{D}$,

$$\lim_{t \rightarrow +\infty} H_d^*(\phi(t, \theta_{-t}\omega) D(\theta_{-t}\omega), A_\nu(\omega)) = 0, \quad \omega \in \Omega,$$

where H_d^* is the Hausdorff semi-distance on \mathbb{R}^{Nd} . By Lemma 3.1, all solutions of (3.1) and (3.2) converge pathwise to each other, therefore, $A_\nu(\omega)$ consists of singleton sets, i.e.

$$A_\nu(\omega) = (\bar{x}_\nu^{(1)}(\omega), \bar{x}_\nu^{(2)}(\omega), \dots, \bar{x}_\nu^{(N)}(\omega))^\top.$$

When we transfer the coupled RODEs (3.1) back to the coupled SODEs (1.5), the corresponding pathwise singleton sets attractor is then

$$(\bar{x}_\nu^{(1)}(\theta_t\omega)e^{O_t^{(1)}(\omega)}, \bar{x}_\nu^{(2)}(\theta_t\omega)e^{O_t^{(2)}(\omega)}, \dots, \bar{x}_\nu^{(N)}(\theta_t\omega)e^{O_t^{(N)}(\omega)})^\top,$$

which is exactly a stationary stochastic solution of the coupled SODEs (1.5) because the Ornstein–Uhlenbeck process is stationary. □

4. Synchronization of Components of Solutions

It is known in Sec. 2 that all solutions of the coupled RODEs (3.1) and (3.2) converge pathwise to each other in the future for a fixed $\nu(> 0)$. Here, we consider what will happen to solutions of the coupled RODEs (3.1) and (3.2) as the coupling coefficient ν goes to infinity. First, we prove a lemma which plays an important role in this section.

Lemma 4.1. *For fixed $p \in \mathbb{N}$ and any $\alpha \in (0, 2)$, there exist a $\alpha_0(p) \in (0, 2)$ such that the $p \times p$ real symmetric triple diagonal matrix*

$$A = \begin{pmatrix} -\alpha & 1 & 0 & \cdots & 0 \\ 1 & -\alpha & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\alpha & 1 \\ 0 & \cdots & 0 & 1 & -\alpha \end{pmatrix}$$

is negative definite for all $\alpha \geq \alpha_0(p)$.

Proof. Let $\tilde{A} = -A$. We assert that there exists an $\alpha_0(p) \in (0, 2)$ such that \tilde{A} is positive definite for $\alpha \geq \alpha_0(p)$, then A is negative definite for $\alpha \geq \alpha_0(p)$. In fact, let $a + b = \lambda - \alpha$ and $ab = 1$, we have

$$\begin{aligned} |\lambda E - \tilde{A}| &= \begin{vmatrix} \lambda - \alpha & 1 & 0 & \cdots & 0 \\ 1 & \lambda - \alpha & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda - \alpha & 1 \\ 0 & \cdots & 0 & 1 & \lambda - \alpha \end{vmatrix} \\ &= \begin{vmatrix} a + b & ab & 0 & \cdots & 0 \\ 1 & a + b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a + b & ab \\ 0 & \cdots & 0 & 1 & a + b \end{vmatrix} = \frac{a^{p+1} - b^{p+1}}{a - b}. \end{aligned}$$

If $a \neq b$, then $|\lambda E - \tilde{A}| = 0$ implies that $(\frac{a}{b})^{p+1} = 1$ and thus $\frac{a}{b} = e^{i\frac{2k\pi}{p+1}}$ for $k = 1, \dots, p$ ($k \neq 0$ since $\frac{a}{b} \neq 1$). It follows from $ab = 1$ that

$$a_k = \pm \left(\cos \frac{k\pi}{p+1} + i \sin \frac{k\pi}{p+1} \right), \quad b_k = \pm \left(\cos \frac{k\pi}{p+1} - i \sin \frac{k\pi}{p+1} \right),$$

$k = 1, \dots, p$. Then $\lambda = \alpha + (a+b) = \alpha \pm 2 \cos \frac{k\pi}{p+1}$, $k = 1, \dots, p$. Note that $\cos \frac{k\pi}{p+1} = -\cos \frac{(p+1-k)\pi}{p+1}$, thus the p different eigenvalues of \tilde{A} are $\lambda_k = \alpha + 2 \cos \frac{k\pi}{p+1}$, $k = 1, \dots, p$.

Otherwise, $a = b$ implies $a = b = 1$ or $a = b = -1$, then $\lambda = \alpha + 2$ or $\lambda = \alpha - 2$. But $|\lambda E - \tilde{A}| \neq 0$ for these two λ . Hence, all eigenvalues of \tilde{A} are

$$\lambda_k = \alpha + 2 \cos \frac{k\pi}{p+1}, \quad k = 1, \dots, p.$$

It follows that for any $\alpha_0(p) \in (-2 \cos \frac{p\pi}{p+1}, 2) \subset (0, 2)$, for example, $\alpha_0(p) = 1 - \cos \frac{p\pi}{p+1}$, \tilde{A} is positive definite for $\alpha \geq \alpha_0(p)$. □

We also need the following estimations. Suppose that $\phi(t) = (x_\nu^{(1)}(t), x_\nu^{(2)}(t), \dots, x_\nu^{(N)}(t))^\top$ is a solution of the coupled RODEs (3.1) and (3.2). For any two different components $x_\nu^{(k)}(t), x_\nu^{(l)}(t)$ of the solution,

$$\begin{aligned} y_\nu^{k,l}(t) &= 2\langle x_\nu^{(k)}(t) - x_\nu^{(l)}(t), F^{(k)}(x_\nu^{(k)}(t), O_t^{(k)}) - F^{(l)}(x_\nu^{(l)}(t), O_t^{(l)}) \rangle \\ &= 2\langle x_\nu^{(k)}(t) - x_\nu^{(l)}(t), e^{-O_t^{(k)}} f^{(k)}(e^{O_t^{(k)}} x^{(k)}(t)) - e^{-O_t^{(l)}} f^{(l)}(e^{O_t^{(l)}} x^{(l)}(t)) \rangle \\ &\quad + 2\langle x_\nu^{(k)}(t) - x_\nu^{(l)}(t), O_t^{(k)} x^{(k)}(t) - O_t^{(l)} x^{(l)}(t) \rangle \\ &\leq 2\|x_\nu^{(k)}(t) - x_\nu^{(l)}(t)\| (e^{-O_t^{(k)}} \|f^{(k)}(e^{O_t^{(k)}} x^{(k)}(t))\| + |O_t^{(k)}| \cdot \|x^{(k)}(t)\|) \\ &\quad + 2\|x_\nu^{(k)}(t) - x_\nu^{(l)}(t)\| (e^{-O_t^{(l)}} \|f^{(l)}(e^{O_t^{(l)}} x^{(l)}(t))\| + |O_t^{(l)}| \cdot \|x^{(l)}(t)\|), \end{aligned}$$

thus, for fixed $\beta > 0$, we have

$$\begin{aligned} &-\beta\nu \|x_\nu^{(k)}(t) - x_\nu^{(l)}(t)\|^2 + y_\nu^{k,l}(t) \\ &\leq \frac{1}{\nu} \left(\frac{4}{\beta} e^{-2O_t^{(k)}} \|f^{(k)}(e^{O_t^{(k)}} x^{(k)}(t))\|^2 + \frac{4}{\beta} |O_t^{(k)}|^2 \|x^{(k)}(t)\|^2 \right) \\ &\quad + \frac{1}{\nu} \left(\frac{4}{\beta} e^{-2O_t^{(l)}} \|f^{(l)}(e^{O_t^{(l)}} x^{(l)}(t))\|^2 + \frac{4}{\beta} |O_t^{(l)}|^2 \|x^{(l)}(t)\|^2 \right). \end{aligned}$$

Let

$$\begin{aligned} M_{T_1, T_2}^{k,l,\beta}(\nu, \omega) &= \sup_{t \in [T_1, T_2]} \left(\frac{4}{\beta} e^{-2O_t^{(k)}} \|f^{(k)}(e^{O_t^{(k)}} x^{(k)}(t))\|^2 + \frac{4}{\beta} |O_t^{(k)}|^2 \|x^{(k)}(t)\|^2 \right) \\ &\quad + \sup_{t \in [T_1, T_2]} \left(\frac{4}{\beta} e^{-2O_t^{(l)}} \|f^{(l)}(e^{O_t^{(l)}} x^{(l)}(t))\|^2 + \frac{4}{\beta} |O_t^{(l)}|^2 \|x^{(l)}(t)\|^2 \right) \end{aligned}$$

for any bounded interval $[T_1, T_2]$. Note that $C_\nu(\omega)$ in (3.8) satisfies

$$\frac{d}{d\nu} \|C_\nu(\omega)\|^2 = 2 \left\langle C_\nu(\omega), \frac{d}{d\nu} C_\nu(\omega) \right\rangle \leq 0$$

and consequently, $R_\nu(\omega) \leq R_1(\omega)$ for $\nu \geq 1$. Hence, $M_{T_1, T_2}^{k, l, \beta}(\nu, \omega)$ is uniformly bounded in ν and

$$-\beta\nu\|x_\nu^{(k)}(t) - x_\nu^{(l)}(t)\|^2 + y_\nu^{k, l}(t) \leq \frac{1}{\nu}M_{T_1, T_2}^{k, l, \beta}(\omega) \tag{4.1}$$

uniformly for $t \in [T_1, T_2]$ with

$$M_{T_1, T_2}^{k, l, \beta}(\omega) = \sup_{\nu \geq 1} M_{T_1, T_2}^{k, l, \beta}(\nu, \omega).$$

Now let us estimate the difference between any two components of a solution to the coupled RODEs (3.1) and (3.2) as $\nu \rightarrow \infty$.

Lemma 4.2. *The difference between any two components of a solution $(x_\nu^{(1)}(t), x_\nu^{(2)}(t), \dots, x_\nu^{(N)}(t))^\top$ of the coupled RODEs (3.1) and (3.2) vanishes uniformly in any bounded time interval as the coupled coefficient ν goes to infinity, namely, for any bounded interval $[T_1, T_2]$ and any $t \in [T_1, T_2]$,*

$$\lim_{\nu \rightarrow \infty} \|x_\nu^{(j)}(t) - x_\nu^{(k)}(t)\| = 0$$

for all $j, k \in \{1, 2, \dots, N\}$.

Proof. Equivalently, we can estimate the difference between any two adjacent components only, where the first and the last components of the solution are considered to be adjacent. From now on, we call the difference between two components of the solution a term. In the following process of estimations, we note that only one new term will be involved in each step which continues the process, except the last step that ends the process.

Let us begin our estimations with $x_\nu^{(1)}(t), x_\nu^{(2)}(t)$.

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2 &= 2\langle x_\nu^{(1)}(t) - x_\nu^{(2)}(t), F^{(1)}(x_\nu^{(1)}(t), O_t^{(1)}) - F^{(2)}(x_\nu^{(2)}(t), O_t^{(2)}) \rangle \\ &\quad + 2\langle x_\nu^{(1)}(t) - x_\nu^{(2)}(t), -3\nu(x_\nu^{(1)}(t) - x_\nu^{(2)}(t)) \rangle \\ &\quad + 2\langle x_\nu^{(1)}(t) - x_\nu^{(2)}(t), \nu(x_\nu^{(N)}(t) - x_\nu^{(3)}(t)) \rangle \\ &\leq -5\nu\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2 + \nu\|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 + y_\nu^{1, 2}(t) \\ &\leq -\alpha\nu\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2 + \nu\|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 \\ &\quad + \frac{1}{\nu}M_{T_1, T_2}^{1, 2, 5-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$ by (4.1). Here, we take

$$\alpha = \begin{cases} 1 - \cos \frac{N\pi}{N+2}, & N \text{ is even,} \\ 1 - \cos \frac{(N-1)\pi}{N+1}, & N \text{ is odd.} \end{cases}$$

In fact, we can take any $\alpha \in (-2 \cos \frac{N\pi}{N+2}, 2)$ when N is even and any $\alpha \in (-2 \cos \frac{(N-1)\pi}{N+1}, 2)$ when N is odd.

Note that the above estimations generate $x_\nu^{(3)}(t) - x_\nu^{(N)}(t)$.

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 &= 2\langle x_\nu^{(3)}(t) - x_\nu^{(N)}(t), F^{(3)}(x_\nu^{(3)}(t), O_t^{(3)}) \\ &\quad - F^{(N)}(x_\nu^{(N)}(t), O_t^{(N)}) \rangle \\ &\quad + 2\langle x_\nu^{(3)}(t) - x_\nu^{(N)}(t), -2\nu(x_\nu^{(3)}(t) - x_\nu^{(N)}(t)) \rangle \\ &\quad + 2\langle x_\nu^{(3)}(t) - x_\nu^{(N)}(t), \nu(x_\nu^{(2)}(t) - x_\nu^{(1)}(t)) \rangle \\ &\quad + 2\langle x_\nu^{(3)}(t) - x_\nu^{(N)}(t), \nu(x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)) \rangle \\ &\leq -2\nu \|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 + \nu \|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\|^2 + y_\nu^{3,N}(t) \\ &\leq -\alpha\nu \|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 + \nu \|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\|^2 + \frac{1}{\nu} M_{T_1, T_2}^{3,N,2-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$.

Note that $x_\nu^{(1)}(t) - x_\nu^{(2)}(t)$ has been estimated and $x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)$ is generated. Similarly, we have

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\|^2 &\leq -\alpha\nu \|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\|^2 + \nu \|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(5)}(t) - x_\nu^{(N-2)}(t)\|^2 + \frac{1}{\nu} M_{T_1, T_2}^{4,N-1,2-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$.

Continue such estimations, we obtain

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(j+3)}(t) - x_\nu^{(N-j)}(t)\|^2 &\leq -\alpha\nu \|x_\nu^{(j+3)}(t) - x_\nu^{(N-j)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(j+2)}(t) - x_\nu^{(N-j+1)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(j+4)}(t) - x_\nu^{(N-j-1)}(t)\|^2 + \frac{1}{\nu} M_{T_1, T_2}^{j+3,N-j,2-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$, for $j = 2, 3, \dots$

Now there exists a question: when and where does this process end? There are two cases: N is even and N is odd.

Case 1. N is even

Go on the above process with j increasing. When $j = \frac{N}{2} - 3$, we have

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(\frac{N}{2})}(t) - x_\nu^{(\frac{N}{2}+3)}(t)\|^2 &\leq -\alpha\nu \|x_\nu^{(\frac{N}{2})}(t) - x_\nu^{(\frac{N}{2}+3)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(\frac{N}{2}-1)}(t) - x_\nu^{(\frac{N}{2}+4)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(\frac{N}{2}+1)}(t) - x_\nu^{(\frac{N}{2}+2)}(t)\|^2 + \frac{1}{\nu} M_{T_1, T_2}^{\frac{N}{2}, \frac{N}{2}+3, 2-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$.

As j increases to $\frac{N}{2} - 2$, we have

$$\begin{aligned} \frac{d}{dt} \|x_\nu^{(\frac{N}{2}+1)}(t) - x_\nu^{(\frac{N}{2}+2)}(t)\|^2 &\leq -\alpha\nu \|x_\nu^{(\frac{N}{2}+1)}(t) - x_\nu^{(\frac{N}{2}+2)}(t)\|^2 \\ &\quad + \nu \|x_\nu^{(\frac{N}{2})}(t) - x_\nu^{(\frac{N}{2}+3)}(t)\|^2 \\ &\quad + \frac{1}{\nu} M_{T_1, T_2}^{\frac{N}{2}+1, \frac{N}{2}+2, 5-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$, which ends this process.

For ease of notation, we rewrite the above inequalities in the matrix form,

$$\dot{\mathbf{y}}(t) \leq \mathbf{A}_\nu \mathbf{y}(t) + \frac{1}{\nu} \mathbf{M} \tag{4.2}$$

uniformly for $t \in [T_1, T_2]$ with two $\frac{N}{2}$ -dimensional vectors

$$\begin{aligned} \mathbf{y}(t) &= (\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2, \|x_\nu^{(3)}(t) \\ &\quad - x_\nu^{(N)}(t)\|^2, \dots, \|x_\nu^{(\frac{N}{2}+1)}(t) - x_\nu^{(\frac{N}{2}+2)}(t)\|^2)^\top, \quad t \in \mathbb{R}, \\ \mathbf{M} &= (M_{T_1, T_2}^{1, 2, 5-\alpha}(\omega), M_{T_1, T_2}^{3, N, 2-\alpha}(\omega), \dots, M_{T_1, T_2}^{\frac{N}{2}, \frac{N}{2}+3, 2-\alpha}(\omega), M_{T_1, T_2}^{\frac{N}{2}+1, \frac{N}{2}+2, 5-\alpha}(\omega))^\top \end{aligned}$$

and an $\frac{N}{2} \times \frac{N}{2}$ matrix

$$\mathbf{A}_\nu = \begin{pmatrix} -\alpha\nu & \nu & 0 & \cdots & 0 \\ \nu & -\alpha\nu & \nu & \ddots & \vdots \\ 0 & \nu & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\alpha\nu & \nu \\ 0 & \cdots & 0 & \nu & -\alpha\nu \end{pmatrix}.$$

By Lemma 2.2, it follows from (4.2) that

$$\mathbf{y}(t) \leq e^{(t-t_0)\mathbf{A}_\nu} \mathbf{y}(t_0) + \frac{1}{\nu} \int_{t_0}^t e^{(t-u)\mathbf{A}_\nu} \mathbf{M} du. \tag{4.3}$$

By Lemma 4.1, $\frac{1}{\nu}\mathbf{A}_\nu$ is negative definite, then similar to Lemma 3.2,

$$\|e^{\nu(t-t_0)\mathbf{A}_\nu}\mathbf{y}(t_0)\| \leq e^{\nu(t-t_0)\lambda_{\max}}\|\mathbf{y}(t_0)\|,$$

where $\lambda_{\max} = -\alpha - 2\cos\frac{N\pi}{N+2} < 0$ is the maximal eigenvalue of $\frac{1}{\nu}\mathbf{A}_\nu$. Thus, it follows from (4.3) that

$$\mathbf{y}(t) \rightarrow 0 \quad \text{as } \nu \rightarrow \infty$$

uniformly for $t \in [T_1, T_2]$, which implies that $\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2$ and $\|x_\nu^{(\frac{N}{2}+1)}(t) - x_\nu^{(\frac{N}{2}+2)}(t)\|^2$ tend to 0 uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$.

Case 2. N is odd

Similarly, when $j = \frac{N-1}{2} - 3$, we have

$$\begin{aligned} \frac{d}{dt}\|x_\nu^{(\frac{N-1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+3)}(t)\|^2 &\leq -\alpha\nu\|x_\nu^{(\frac{N-1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+3)}(t)\|^2 \\ &\quad + \nu\|x_\nu^{(\frac{N-1}{2}-1)}(t) - x_\nu^{(\frac{N+1}{2}+4)}(t)\|^2 \\ &\quad + \nu\|x_\nu^{(\frac{N+1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+2)}(t)\|^2 \\ &\quad + \frac{1}{\nu}M_{T_1, T_2}^{\frac{N-1}{2}, \frac{N+1}{2}+3, 2-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$.

As j increases to $\frac{N+1}{2} - 3$, we have

$$\begin{aligned} \frac{d}{dt}\|x_\nu^{(\frac{N+1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+2)}(t)\|^2 &\leq -\alpha\nu\|x_\nu^{(\frac{N+1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+2)}(t)\|^2 \\ &\quad + \nu\|x_\nu^{(\frac{N-1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+3)}(t)\|^2 \\ &\quad + \frac{1}{\nu}M_{T_1, T_2}^{\frac{N+1}{2}, \frac{N+1}{2}+2, 5-\alpha}(\omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$, which ends this process.

We can also rewrite the above inequalities in the matrix form

$$\dot{\tilde{\mathbf{y}}}(t) \leq \tilde{\mathbf{A}}_\nu\tilde{\mathbf{y}}(t) + \frac{1}{\nu}\tilde{\mathbf{M}} \tag{4.4}$$

uniformly for $t \in [T_1, T_2]$ with two $\frac{N-1}{2}$ -dimensional vectors

$$\begin{aligned} \tilde{\mathbf{y}}(t) &= (\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2, \|x_\nu^{(3)}(t) \\ &\quad - x_\nu^{(N)}(t)\|^2, \dots, \|x_\nu^{(\frac{N+1}{2})}(t) - x_\nu^{(\frac{N+1}{2}+2)}(t)\|^2)^\top, \quad t \in \mathbb{R}, \\ \tilde{\mathbf{M}} &= (M_{T_1, T_2}^{1, 2, 5-\alpha}(\omega), M_{T_1, T_2}^{3, N, 2-\alpha}(\omega), \dots, M_{T_1, T_2}^{\frac{N-1}{2}, \frac{N+1}{2}+3, 2-\alpha}(\omega), \\ &\quad M_{T_1, T_2}^{\frac{N+1}{2}, \frac{N+1}{2}+2, 5-\alpha}(\omega))^\top \end{aligned}$$

and a $\frac{N-1}{2} \times \frac{N-1}{2}$ matrix

$$\tilde{\mathbf{A}}_\nu = \begin{pmatrix} -\alpha\nu & \nu & 0 & \cdots & 0 \\ \nu & -\alpha\nu & \nu & \ddots & \vdots \\ 0 & \nu & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\alpha\nu & \nu \\ 0 & \cdots & 0 & \nu & -\alpha\nu \end{pmatrix}.$$

By Lemma 2.2, it follows from (4.4) that

$$\tilde{\mathbf{y}}(t) \leq e^{(t-t_0)\tilde{\mathbf{A}}_\nu} \tilde{\mathbf{y}}(t_0) + \frac{1}{\nu} \int_{t_0}^t e^{(t-u)\tilde{\mathbf{A}}_\nu} \tilde{\mathbf{M}} du. \tag{4.5}$$

Similar to the case that N is even, it follows from (4.5) that $\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\|^2$ tends to 0 uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$.

For other adjacent components, the above process can be duplicated. Hence, after we have dealt with any adjacent components, we can conclude that the difference between any two components of a solution of the coupled RODEs (3.1) goes to 0 uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$. In fact, if N is even, another adjacent component will be involved while we focus on current adjacent components. For example, $x_{\nu^{\frac{N}{2}+1}}(t) - x_{\nu^{\frac{N}{2}+2}}(t)$ is involved while we duplicate with $x_\nu^{(1)}(t) - x_\nu^{(2)}(t)$. So the above process can be done for only $\frac{N}{2}$ times if N is even. \square

Remark 4.3. In the case of $N = 3$, the proof of Lemma 4.2 can be simplified since each term is only related to itself.

Lemma 4.2 implies that all components of a solution of (3.1) and (3.2) tend to the same limit uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$. Now, we find what they converge to.

Consider the averaged RODE (1.6)

$$\frac{dz}{dt} = \frac{1}{N} \sum_{j=1}^N e^{-O_t^{(j)}} f^{(j)}(e^{O_t^{(j)}} z) + \frac{1}{N} \sum_{j=1}^N O_t^{(j)} z. \tag{4.6}$$

Lemma 4.4. *The random dynamical system $\varphi(t, \omega)$ generated by the solution of RODE (4.6) has a singleton sets random attractor denoted by $\{\bar{z}(\omega)\}$. Furthermore,*

$$\bar{z}(\theta_t \omega) \exp\left(\frac{1}{N} \sum_{j=1}^N O_t^{(j)}(\omega)\right)$$

is the stationary stochastic solution of equivalently averaged SODE

$$dZ_t = \frac{1}{N} \sum_{j=1}^N e^{-\zeta_t^{(j)}} f^{(j)}(e^{\zeta_t^{(j)}} Z_t) dt + \frac{1}{N} \sum_{i=1}^m \left(\sum_{j=1}^N c_i^{(j)} \right) Z_t \circ dW_t^{(i)}, \tag{4.7}$$

where

$$\zeta_t^{(j)} = \frac{1}{N} \sum_{k=1}^N (O_t^{(j)} - O_t^{(k)}), \quad j = 1, \dots, N.$$

Proof. Assume that $z_1(t), z_2(t)$ are two solutions of (4.6). We have

$$\frac{d}{dt} \|z_1(t) - z_2(t)\|^2 \leq \left(-2L + \frac{2}{N} \sum_{j=1}^N O_t^{(j)} \right) \|z_1(t) - z_2(t)\|^2.$$

It follows from Gronwall’s lemma that

$$\|z_1(t) - z_2(t)\|^2 \leq \exp \left(-2t \left(L - \frac{1}{N} \sum_{j=1}^N \frac{1}{t} \int_0^t O_\tau^{(j)} d\tau \right) \right) \|z_1(0) - z_2(0)\|^2.$$

Hence, by Lemma 2.1, we have

$$\lim_{t \rightarrow \infty} \|z_1(t) - z_2(t)\|^2 = 0,$$

which means all solutions of (4.6) converge pathwise to each other.

Now we use the theory of random dynamical systems to see what they converge to. Suppose $z(t)$ is a solution of (4.6), we have

$$\frac{d}{dt} \|z(t)\|^2 \leq \left(-L + \frac{2}{N} \sum_{j=1}^N O_t^{(j)} \right) \|z(t)\|^2 + \frac{1}{N} \sum_{j=1}^N \frac{e^{-2O_t^{(j)}}}{L} \|f^{(j)}(0)\|^2.$$

It follows from Gronwall’s lemma that

$$\begin{aligned} \|z(t)\|^2 &\leq e^{-L(t-t_0) + \frac{2}{N} \sum_{j=1}^N \int_{t_0}^t O_\tau^{(j)} d\tau} \|z(t_0)\|^2 \\ &\quad + \frac{1}{N} \sum_{j=1}^N \frac{\|f^{(j)}(0)\|^2}{L} \int_{t_0}^t e^{-2O_u^{(j)}} e^{-L(t-u) + \frac{2}{N} \sum_{k=1}^N \int_u^t O_\tau^{(k)} d\tau} du. \end{aligned}$$

Thus, by Lemma 2.1, we obtain

$$\begin{aligned} \|z(t)\|^2 &\leq e^{-\frac{L}{2}(t-t_0)} \|z(t_0)\|^2 \\ &\quad + \frac{1}{N} \sum_{j=1}^N \frac{\|f^{(j)}(0)\|^2}{L} \int_{t_0}^t e^{-2O_u^{(j)}} e^{-L(t-u) + \frac{2}{N} \sum_{k=1}^N \int_u^t O_\tau^{(k)} d\tau} du \end{aligned}$$

for $-t_0, t > T_\omega$.

By pathwise pullback convergence with $t_0 \rightarrow -\infty$, the random closed ball centered at the origin with random radius $R(\omega)$ is a pullback absorbing set of $\varphi(t, \omega)$ in \mathcal{D} for $t > T_\omega$, where

$$R^2(\omega) = 1 + \frac{1}{N} \sum_{j=1}^N \frac{\|f^{(j)}(0)\|^2}{L} \int_{-\infty}^0 e^{Lu-2O_u^{(j)}} e^{\frac{2}{N} \sum_{k=1}^N \int_u^0 O_\tau^{(k)} d\tau} du.$$

Note that the integrals on the right-hand side are well defined by Lemma 2.1.

By Theorem 4.1 in [9], there exists a random attractor $\{\bar{z}(\omega)\}$ for $\varphi(t, \omega)$. Since all solutions of (4.6) converge pathwise to each other, the random attractor $\{\bar{z}(\omega)\}$ are composed of singleton sets.

Note that the averaged RODE (4.6) is transformed from the averaged SODE (4.7) by the transformation

$$z(t, \omega) = \exp\left(-\frac{1}{N} \sum_{j=1}^N O_t^{(j)}(\omega)\right) Z_t(\omega),$$

so the pathwise singleton sets attractor $\bar{z}(\theta_t \omega) \exp(\frac{1}{N} \sum_{j=1}^N O_t^{(j)}(\omega))$ is a stationary solution of the averaged SODE (4.7) since the Ornstein–Uhlenbeck process is stationary. □

We now show another main result of this paper.

Theorem 4.5. *Let*

$$(\bar{x}_{\nu_n}^{(1)}(t, \omega), \bar{x}_{\nu_n}^{(2)}(t, \omega), \dots, \bar{x}_{\nu_n}^{(N)}(t, \omega))^\top = (\bar{x}_{\nu_n}^{(1)}(\theta_t \omega), \bar{x}_{\nu_n}^{(2)}(\theta_t \omega), \dots, \bar{x}_{\nu_n}^{(N)}(\theta_t \omega))^\top$$

be the singleton sets random attractor of the random dynamical system $\phi(t, \omega)$ generated by the solution of RODEs (3.1) and (3.2), then

$$(\bar{x}_{\nu_n}^{(1)}(t, \omega), \bar{x}_{\nu_n}^{(2)}(t, \omega), \dots, \bar{x}_{\nu_n}^{(N)}(t, \omega))^\top \rightarrow (\bar{z}(t, \omega), \bar{z}(t, \omega), \dots, \bar{z}(t, \omega))^\top$$

pathwise uniformly for $t \in [T_1, T_2]$ for any sequence $\nu_n \rightarrow \infty$, where $\bar{z}(t, \omega) = \bar{z}(\theta_t \omega)$ solves the averaged RODE (4.6) and $\bar{z}(\omega)$ is the singleton sets random attractor of the random dynamical system $\varphi(t, \omega)$ generated by the solution of the averaged RODE (4.6).

Proof. Define

$$\bar{z}_\nu(\omega) = \frac{1}{N} \sum_{j=1}^N \bar{x}_\nu^{(j)}(\omega),$$

where $\{(\bar{x}_\nu^{(1)}(\omega), \bar{x}_\nu^{(2)}(\omega), \dots, \bar{x}_\nu^{(N)}(\omega))\}$ is the singleton sets random attractor of the random dynamical system generated by RODEs (3.1) and (3.2). Thus, $\bar{z}_\nu(t, \omega) = \bar{z}_\nu(\theta_t \omega)$ satisfies

$$\frac{d}{dt} \bar{z}_\nu(t, \omega) = \frac{1}{N} \sum_{j=1}^N (e^{-O_t^{(j)}(\omega)} f^{(j)}(e^{O_t^{(j)}(\omega)} \bar{x}_\nu^{(j)}(t, \omega)) + O_t^{(j)}(\omega) \bar{x}_\nu^{(j)}(t, \omega)). \tag{4.8}$$

Note that

$$\begin{aligned} \left\| \frac{d}{dt} \bar{z}_\nu(t, \omega) \right\|^2 &\leq \frac{2}{N} \sum_{j=1}^N (e^{-2O_t^{(j)}(\omega)} \|f^{(j)}(e^{O_t^{(j)}(\omega)} \bar{x}_\nu^{(j)}(t, \omega))\|^2 \\ &\quad + |O_t^{(j)}(\omega)|^2 \|\bar{x}_\nu^{(j)}(t, \omega)\|^2), \end{aligned}$$

by continuity and the fact that these solutions belong to the compact ball $B_1(\omega)$, it follows that

$$\sup_{t \in [T_1, T_2]} \left\| \frac{d}{dt} \bar{z}_\nu(t, \omega) \right\| \leq \left(\frac{2}{N} \sum_{j=1}^N \frac{\beta}{4} M_{T_1, T_2}^{j, \bullet, \beta}(\omega) \right)^{\frac{1}{2}} < \infty.$$

By Ascoli–Arzelà theorem, there exists a subsequence $\nu_{n_k} \rightarrow \infty$ such that $\bar{z}_{\nu_{n_k}}(t, \omega)$ converges to $\bar{z}(t, \omega)$ as $n_k \rightarrow \infty$.

Since difference between any two components of a solution of the coupled RODEs (3.1) tends to 0 uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$, we have

$$\begin{aligned} \bar{x}_{\nu_{n_k}}^{(j)}(t, \omega) &= N \bar{z}_{\nu_{n_k}}(t, \omega) - \sum_{j' \neq j} \bar{x}_{\nu_{n_k}}^{(j')}(t, \omega) \\ &= \bar{z}_{\nu_{n_k}}(t, \omega) + \sum_{j' \neq j} (\bar{z}_{\nu_{n_k}}(t, \omega) - \bar{x}_{\nu_{n_k}}^{(j')}(t, \omega)) \\ &= \bar{z}_{\nu_{n_k}}(t, \omega) + \frac{1}{N} \sum_{j' \neq j} \sum_{j'' \neq j'} (\bar{x}_{\nu_{n_k}}^{(j'')}(t, \omega) - \bar{x}_{\nu_{n_k}}^{(j')}(t, \omega)) \\ &\rightarrow \bar{z}(t, \omega) \end{aligned}$$

uniformly for $t \in [T_1, T_2]$ as $\nu_{n_k} \rightarrow \infty$ for $j = 1, \dots, N$.

Furthermore, it follows from (4.8) that

$$\begin{aligned} \bar{z}_\nu(t, \omega) &= \bar{z}_\nu(T_1, \omega) + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t e^{-O_s^{(j)}(\omega)} f^{(j)}(e^{O_s^{(j)}(\omega)} \bar{x}_\nu^{(j)}(s, \omega)) ds \\ &\quad + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t O_s^{(j)}(\omega) \bar{x}_\nu^{(j)}(s, \omega) ds. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{z}(t, \omega) &= \bar{z}(T_1, \omega) + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t e^{-O_s^{(j)}(\omega)} f^{(j)}(e^{O_s^{(j)}(\omega)} \bar{z}(s, \omega)) ds \\ &\quad + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t O_s^{(j)}(\omega) \bar{z}(s, \omega) ds \end{aligned}$$

uniformly for $t \in [T_1, T_2]$ as $\nu_{n_k} \rightarrow \infty$, which means that $\bar{z}(t, \omega)$ solves RODE (4.6).

Note that any possible subsequences converge to the same limit, so every sequence $\bar{z}_{\nu_n}(t, \omega)$ converges to $\bar{z}(t, \omega)$ uniformly for $t \in [T_1, T_2]$ as $\nu_n \rightarrow \infty$ by Lemma 2.2 in [8].

Finally, since the random dynamical system generated by the solution of RODE (4.6) has a singleton sets random attractor $\{\bar{z}(\omega)\}$, the stationary stochastic process $\bar{z}(\theta_t \omega)$ must be equal to $\bar{z}(t, \omega)$, namely $\bar{z}(t, \omega) = \bar{z}(\theta_t \omega)$. □

As a straightforward consequence of Theorem 4.5, we have

Corollary 4.6.

$$(\bar{x}_\nu^{(1)}(t, \omega), \bar{x}_\nu^{(2)}(t, \omega), \dots, \bar{x}_\nu^{(N)}(t, \omega))^\top \rightarrow (\bar{z}(t, \omega), \bar{z}(t, \omega), \dots, \bar{z}(t, \omega))^\top$$

pathwise uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$.

In terms of the coupled SODEs (1.5), its stationary stochastic solution

$$(\bar{x}_\nu^{(1)}(\theta_t \omega) e^{O_t^{(1)}(\omega)}, \bar{x}_\nu^{(2)}(\theta_t \omega) e^{O_t^{(2)}(\omega)}, \dots, \bar{x}_\nu^{(N)}(\theta_t \omega) e^{O_t^{(N)}(\omega)})^\top$$

tends pathwisely to

$$(\bar{z}(\theta_t \omega) e^{O_t^{(1)}(\omega)}, \bar{z}(\theta_t \omega) e^{O_t^{(2)}(\omega)}, \dots, \bar{z}(\theta_t \omega) e^{O_t^{(N)}(\omega)})^\top$$

uniformly for $t \in [T_1, T_2]$ as $\nu \rightarrow \infty$. Obviously, if $c_i^{(1)} = c_i^{(2)} = \dots = c_i^{(N)} = c_i$ for $i = 1, \dots, m$ in (1.5), i.e. the driving noise is the same, exact synchronization of solutions of the coupled SODEs

$$\begin{aligned} dX_t^{(j)} &= (f^{(j)}(X_t^{(j)}) + \nu(X_t^{(j-1)} - 2X_t^{(j)} + X_t^{(j+1)})) dt \\ &\quad + \sum_{i=1}^m c_i X_t^{(j)} \circ dW_t^{(i)}, \quad j = 1, \dots, N \end{aligned}$$

occurs.

Remark 4.7. The results in this paper hold just in the almost everywhere sense because $\omega \in \bar{\Omega}$ here (see Lemma 2.1 and some interpretations below the lemma).

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