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Quantitative concentration of stationary measures ☆

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HIGHLIGHTS

- A new method of constructing sequences of Lyapunov functions and anti-Lyapunov functions is developed.
- Gaussian-like tails estimates and quantitative concentration of stationary measures are obtained.
- Quantitative stabilization/de-stabilization of local attractors/repellers are studied.
- Upper bounds for the stationary differential entropy and entropy-dimension inequalities are established.

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ABSTRACT

We consider an Itô stochastic differential equation and study the asymptotic behaviors of stationary measures of the corresponding Fokker–Planck equation in the vicinity of a deterministic attractor or a repeller. By establishing global measure estimates in domains of interest, we show Gaussian-like behaviors of these measures in the basins of the attractor and the repeller. Not only do our results quantify the concentration results of stationary measures obtained in Huang et al. (2018), but also they are valid near a usual attractor (resp. repeller) instead of a strong attractor (resp. repeller) assumed in Huang et al. (2018). Our approach in conducting the measure estimates is based on a new idea of constructing a sequence of Lyapunov functions (anti-Lyapunov functions) in the basin of attraction (resp. expansion) of the attractor (resp. repeller). As applications of these measure estimates, we also derive upper bounds for the differential entropy and establish certain entropy-dimension inequalities.

1. Introduction

Realistic physical systems are often subject to Itô white noise perturbations due to the existence of internal or external uncertainties. In the case of ordinary differential equations (ODEs), such noise perturbed systems become Itô stochastic differential equations (SDEs). This paper aims at making quantitative understanding of the asymptotic behaviors and concentration of stationary distributions of a SDE in the vicinity of an attractor or a repeller of the unperturbed ODE. The theory of large deviations has been substantially developed for studying these problems (see e.g. [1,2]). In particular, it is shown in [1, Theorem 4.3 in Chapter 4] that stationary measures concentrate on a globally attracting equilibrium of the ODE with Gaussian-like tails, resembling the case of a Gibbs measure. With respect to the measure distribution of stationary measures near a general attractor of the ODE which is not necessary global, several natural questions then arise: (a) Whether the measure distribution in the basin of attraction behaves like that of Gaussian; (b) If the attractor is global, then whether stationary measures concentrate on the global attractor under a suitable Lyapunov condition; (c) Whether and under what mechanism a local attractor can be stabilized via certain multiplicative noises. Answers to these questions are of fundamental importance to the understanding of stochastic stability of a compact invariant set (see [3] for details) and they are closely related to the issue of thermodynamic limit in mesoscopic systems (see e.g. [4]).

The present paper aims at investigating these fundamental questions. We will give an affirmative answer to question (a) and provide quantitative results with respect to questions (b) and (c) which improve relevant results contained in [3]. We note that not only do results contained in [3] work with so-called strong attractors and repellers, but also they are not quantified on the

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domain of interest. Like in [3], we also provide similar results for the case of a repeller. In addition, as applications of these results, we derive upper bounds for the stationary differential entropy and establish entropy-dimension inequalities.

To be more precise, we consider a system of ordinary differential equations (ODEs)

$$\dot{x} = V(x), \quad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set, and $V \in C^2(\mathcal{U}, \mathbb{R}^n)$. Adding general (multiplicative) Itô white noises, the system gives rise to the following family of Itô SDEs

$$dx = V(x)dt + \epsilon G(x)dW_t, \quad x \in \mathcal{U}, \tag{1.2}$$

where W_t is the standard m-dimensional Brownian motion for some $m \geq n$, $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ for some fixed $p > n$, $\mathrm{Rank}(G) \equiv n$, and $\epsilon > 0$ is a small parameter. The density of probability distribution of solutions of (1.2), if exists, satisfies the Fokker–Planck equation

$$\begin{cases} \partial_t u = L_\epsilon u := \epsilon^2 \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad t > 0,\ x \in \mathcal{U}, \\ u \geq 0,\ \displaystyle\int_{\mathcal{U}} u dx = 1, \end{cases} \tag{1.3}$$

where $A = (a^{ij}) = \frac{GG^\top}{2} \in W^{1,p}_{loc}(\mathcal{U}, GL(n,\mathbb{R}))$ is the diffusion matrix, $\partial_i = \partial_{x_i}$, $\partial^2_{ij} = \partial^2_{x_i x_j}$ with the usual summation convention used, and L_ϵ is the Fokker–Planck operator.

A *stationary measure* associated to (1.2) or corresponding to the adjoint Fokker–Planck operator $\mathcal{L}_\epsilon = \epsilon^2 a^{ij}\partial^2_{ij} + V^i\partial_i$ is a measure valued stationary solution of (1.3), namely, a Borel probability measure on $\mathcal{U}$ solving the stationary Fokker–Planck equation in the weak sense (see Definition 2.5). The existence of stationary measures corresponding to $\mathcal{L}_\epsilon$ are guaranteed by various Lyapunov-type conditions (see e.g., [5–7]).

Our main results concerning asymptotic behaviors and quantitative concentration of stationary measures are as follows. We refer to Definition 2.1 for the definition of attractors and repellers.

Theorem A. *Let $\mathcal{E}_0$ be either a maximal attractor $\mathcal{J}_0$ or a maximal repeller $\mathcal{R}_0$ of (1.1) with the basin of attraction or basin of expansion $B(\mathcal{E}_0)$, respectively. Then the following hold for stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_\epsilon\}$:*

(1) (Concentration) *Let $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ with $A = \frac{GG^\top}{2} \in W^{1,p}_{loc}(\mathcal{U}, GL(n,\mathbb{R}))$. Then for any Borel set $\mathcal{W}_0 \subset\subset B(\mathcal{E}_0) \setminus \mathcal{E}_0$, there are constants $w_0, \epsilon_0 > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{W}_0) \leq e^{-\frac{w_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_0].$$

Consequently, any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies

$$\mu(B(\mathcal{E}_0) \setminus \mathcal{E}_0) = 0. \tag{1.4}$$

In particular, if (1.1) is dissipative and $\mathcal{E}_0$ is the global attractor, then $B(\mathcal{E}_0) = \mathcal{U}$ and μ is concentrated on the global attractor $\mathcal{E}_0$.

(2) (Pointwise estimate) *Let $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ with $A = \frac{GG^\top}{2} \in W^{1,\infty}_{loc}(\mathcal{U}, GL(n,\mathbb{R}))$. Then for any Borel set $\mathcal{W}_0 \subset\subset B(\mathcal{E}_0) \setminus \mathcal{E}_0$, there are constants $w_0, \epsilon_0 > 0$ independent of ϵ such that*

$$u_\epsilon(x) \leq e^{-\frac{w_0}{\epsilon^2}}, \quad \forall x \in \mathcal{W}_0, \quad \epsilon \in (0, \epsilon_0],$$

where $\{u_\epsilon\}$ are the densities of $\{\mu_\epsilon\}$.

(3) (Stabilization) *Suppose (1.1) is dissipative and let $\mathcal{J}_0$ be a local attractor of (1.1). Then, there exists a $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ such that for any Borel set $\mathcal{U}_0 \subset\subset \mathcal{U} \setminus \mathcal{J}_0$, there are constants $u_0, \epsilon_0 > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{U}_0) \leq e^{-\frac{u_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_0].$$

Consequently, any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies

$$\mu(\mathcal{U} \setminus \mathcal{J}_0) = 0, \tag{1.5}$$

that is, μ is concentrated on the local attractor $\mathcal{J}_0$.

(4) (De-stabilization) *Suppose $\mathcal{R}_0$ is a local repeller. Then, there exists a $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ such that for any Borel set $\mathcal{V}_0 \subset\subset B(\mathcal{R}_0)$, there are constants $v_0, \epsilon_0 > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{V}_0) \leq e^{-\frac{v_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_0].$$

Consequently, any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies

$$\mu(B(\mathcal{R}_0)) = 0, \tag{1.6}$$

that is, μ is concentrated away from the local repeller $\mathcal{R}_0$.

The concentration results (1.4)–(1.6) are established in [3] with respect to so-called strong attractors (resp. strong repellers) which are roughly the ones admitting strict Lyapunov functions (resp. strict anti-Lyapunov functions) near the boundaries of the basins (see Definition 2.3 for details). It is also shown in [3] that if $\mathcal{R}_0$ in Theorem A(3) is a strongly repelling equilibrium then (1.6) holds with respect to any $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$. These results are proven using the level-sets method and measure estimates established in [8] based on Lyapunov or anti-Lyapunov functions and dynamical properties of an attractor or a repeller of (1.1). The level-sets method introduced in [3] compensates the classical large derivation approach which, even for SDEs on a compact manifold with Lipschitz drift and noise terms, involves non-explicit (and usually less regular) quasi-potential functions that are hard to use in measure estimates if a complicated attractor is involved.

Theorem A shows that not only do the concentration results established in [3] hold for attractors and repellers in the usual sense but also any possible concentrations of μ_ϵ, as $\epsilon \to 0$, on an attractor (either local or global) must do so with Gaussian-like tails. The key ingredient of our approach is to make use of dynamical properties of an attractor or a repeller $\mathcal{E}_0$ in the basin $B(\mathcal{E}_0)$ to construct a sequence of nested domains $\{\Omega_i\}$ in the basin with $\cup_i \Omega_i = \mathcal{B}(\mathcal{E}_0)$ and $\cap_i \Omega_i = \mathcal{E}_0$ so that the vicinity of the boundary of each Ω_i admits a strict Lyapunov or strict anti-Lyapunov function. As a result, the usual attractors and repellers are actually strong attractors and repellers in the sense of Definition 2.3. Using measure estimates established in [3], the concentration estimates in Theorem A then follow from those on the nested domains exhausting the region of interest. In fact, with the construction of these nested domains, we are able to provide more concrete information on concentration rates than the ones stated in Theorem A (see Theorems 4.1, 5.1, 5.2 and their proofs for more details).

Our idea of constructing a sequence of Lyapunov functions (resp. anti-Lyapunov functions) of (1.1) in the basin of an attractor (resp. a repeller) is motivated by the converse Lyapunov theorem (see e.g. [9, Theorem 5.7.24] and [10, Theorem 4.3.1]) for a system of ODEs in the basin of an attracting equilibrium which says that a smooth, strict, entire Lyapunov function exists in the basin of such an equilibrium. We remark that in the presence of a general attractor or repeller, the construction of a smooth entire Lyapunov or anti-Lyapunov function in the basin is in general impossible. Instead, we will show that one can in fact construct a sequence of local Lyapunov or anti-Lyapunov functions in the basin whose domains exhaust the entire basin. Such a result is new and interesting in its own right even for studying the dynamics of (1.1) alone.

Our measure estimates are also useful in studying the (stationary) differential entropy

$$\mathcal{H}[u_\epsilon] = -\int_{\mathcal{U}} u_\epsilon \ln u_\epsilon dx, \tag{1.7}$$

where u_ϵ is the density of a stationary measure μ_ϵ corresponding to $\mathcal{L}_\epsilon$. While the differential entropy $\mathcal{H}[u_\epsilon]$ captures the global information of u_ϵ, its properties rely heavily on local concentration of u_ϵ, which is greatly affected by the dynamical structure of (1.1). We remark that without additional assumptions on the coefficients V and A, the differential entropy $\mathcal{H}[u_\epsilon]$ could take any number between $-\infty$ and ∞ (including $-\infty$ and ∞). In fact, let us take $\epsilon = 1$ for clarity and consider the following SDE

$$dx = \frac{f'(x)}{f(x)}dt + \sqrt{2}dW_t, \quad x \in (a,b),$$

where $-\infty \le a < b \le \infty$, f is a positive and continuously differentiable probability density function on (a,b) and W_t is the standard one-dimensional Brownian motion. It is straightforward to check that f is the density of a stationary measure associated to the above SDE. Upon the choice of f, the differential entropy $\mathcal{H}[f] = -\int_a^b f \ln f dx$ can take any number between $-\infty$ and ∞. In particular, if $f(x) = \frac{1}{x\ln^2 x}$ for $x \in (e,\infty)$, then $\mathcal{H}[f] = \infty$, and if $f(x) = \frac{1}{x(1-\ln x)^2}$ for $x \in (0,1)$, then $\mathcal{H}[f] = -\infty$. We refer the reader to [11,12] for more information about the differential entropy.

In the next theorem, we state some results concerning the differential entropy $\mathcal{H}[u_\epsilon]$ near and away from the global attractor of (1.1) under Lyapunov conditions. We refer to Definition 2.6 for the definition of a uniform Lyapunov function corresponding to $\{\mathcal{L}_\epsilon\}$.

Theorem B. *Suppose there is a uniform Lyapunov function in $\mathcal{U}$, with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant γ, with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ for some $0 < \epsilon_0 \ll 1$. Denote $\mathcal{J}$ as the global attractor of (1.1) and let $\{\mu_\epsilon\}$, with densities $\{u_\epsilon\}$, be the stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Then the following hold.*

(1) *If $|\mathcal{J}| = 0$, then for each $\sigma \in (0,1)$,*

$$\liminf_{\epsilon\to 0} \frac{-\int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln|\mathcal{J}_{h_\sigma(\epsilon)}|} \ge \sigma$$

and

$$\liminf_{\epsilon\to 0} \frac{-\int_{\mathcal{N}} u_\epsilon \ln u_\epsilon dx}{\ln|\mathcal{J}_{h_\sigma(\epsilon)}|} \ge \sigma$$

for all open set $\mathcal{N}$ such that $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$, where $\mathcal{J}_h$ is the h-neighborhood of $\mathcal{J}$ and $h_\sigma(\epsilon) > 0$ is the unique number such that $\mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)}) = \sigma$.

(2) *If $|\mathcal{J}| > 0$, then*

$$\limsup_{\epsilon\to 0}\left[-\int_{\mathcal{N}} u_\epsilon \ln u_\epsilon dx\right] \le \ln|\mathcal{J}|$$

for all open set $\mathcal{N}$ such that $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$.

(3) *Suppose $|\mathcal{U}| < \infty$. Then, for any open set $\mathcal{N}$ such that $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$, there are constants $c > 0$ and $\epsilon_* > 0$ such that*

$$-\int_{\mathcal{U}\setminus\mathcal{N}} u_\epsilon \ln u_\epsilon dx \le \left(\frac{c}{\epsilon^2} + \ln|\mathcal{U}\setminus\mathcal{N}|\right)e^{-\frac{c}{\epsilon^2}}, \quad \forall \epsilon \in (0,\epsilon_*].$$

(4) *Suppose $|\mathcal{U}| = \infty$ and $\rho_M = \infty$. Also suppose that there are positive constants C_1, C_2, ℓ_1 and ℓ_2 such that*

$$\int_{\rho_m}^{\rho} \frac{1}{H(t)}dt \ge C_1\rho^{\ell_1} \quad \text{and} \quad |\Omega_\rho| \le e^{C_2\rho^{\ell_2}}, \quad \forall \rho \in [\rho_m,\infty),$$

where $H(\rho) := \sup_{\partial\Omega_\rho} a^{ij}\partial_i U \partial_j U$ and $\Omega_\rho := \{x \in \mathcal{U} : U(x) < \rho\}$ for $\rho \in [\rho_m,\infty)$. Then, for any $\ell > 0$ and open set $\mathcal{N}$ with $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$, there is $\epsilon_* > 0$ such that

$$-\int_{\mathcal{U}\setminus\mathcal{N}} u_\epsilon \ln u_\epsilon dx \le \epsilon^\ell, \quad \forall \epsilon \in (0,\epsilon_*].$$

(5) *Suppose $|\mathcal{U}| = \infty$ and $\rho_M < \infty$. Also suppose that there are positive constants C_1, C_2, ℓ_1 and ℓ_2 such that*

$$\int_{\rho_m}^{\rho} \frac{1}{H(t)}dt \ge \frac{C_1}{(\rho_M-\rho)^{\ell_1}} \quad \text{and} \quad |\Omega_\rho| \le e^{\frac{C_2}{(\rho_M-\rho)^{\ell_2}}},$$
$$\forall \rho \in [\rho_m,\rho_M).$$

Then, for any $\ell > 0$ and open set $\mathcal{N}$ with $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$, there is $\epsilon_* > 0$ such that

$$-\int_{\mathcal{U}\setminus\mathcal{N}} u_\epsilon \ln u_\epsilon dx \le \epsilon^\ell, \quad \forall \epsilon \in (0,\epsilon_*].$$

Remark 1.1. We make some remarks about the quantity $h_\sigma(\epsilon)$ in the statement of Theorem B(1).

(i) We argue that $h_\sigma(\epsilon)$ is well-defined. By Proposition 2.2(1), u_ϵ is continuous and positive on $\mathcal{U}$. It follows that the function $h \mapsto \mathcal{J}_h$ is continuous and increasing on $(0,h_{\max})$, where $h_{\max}$ is the smallest number such that $\mathcal{U} \subset \mathcal{J}_{h_{\max}}$, and satisfies $\lim_{h\to 0^+}\mu_\epsilon(\mathcal{J}_h) = 0$ and $\lim_{h\to h_{\max}^-}\mu_\epsilon(\mathcal{J}_h) = 1$. Then, for each $\sigma \in (0,1)$, there exists a unique $h_\sigma(\epsilon)$ such that $\mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)}) = \sigma$.

(ii) Suppose there exists a uniform Lyapunov function. Then, for each $\sigma \in (0,1)$, it is not hard to see that $h_\sigma(\epsilon) \to 0$ as $\epsilon \to 0$ (see the proof of Theorem B(1)). However, it is hard to acquire detailed information about the function $\epsilon \mapsto h_\sigma(\epsilon)$. To see this, let us consider the following family of SDEs

$$dx = -f'(x)dt + \epsilon\sqrt{2}dW_t, \quad x \in \mathbb{R},$$

where $0 < \epsilon \ll 1$, f is C^2 and grows such fast as $|x| \to \infty$ that $e^{-\frac{f}{\epsilon^2}}$ is integrable on $\mathbb{R}$, and W_t is the standard one-dimensional Brownian motion. For each ϵ, the unique stationary measure associated to the SDE is given by the Gibbs density $u_\epsilon = \frac{1}{Z_\epsilon}e^{-\frac{f}{\epsilon^2}}$, where $Z_\epsilon = \int_{\mathbb{R}} e^{-\frac{f}{\epsilon^2}}dx$. Then, the asymptotic behaviors of $h_\sigma(\epsilon)$ as $\epsilon \to 0$ depend heavily on the properties of f near its global minimal points. This can be easily seen by examining simple examples $f(x) = x^2$ and x^4. If $f(x) = x^2$, $h_\sigma(\epsilon)$ is proportional to ϵ, and if $f(x) = x^4$, $h_\sigma(\epsilon)$ is proportional to $\sqrt{\epsilon}$.

As an application of entropy estimates stated in Theorem B, we obtain entropy-dimension inequalities. Note that for any bounded set $S \subset \mathcal{U}$, there holds $\liminf_{h\to 0^+}\frac{\ln|S_h|}{\ln h} \in [0,n]$, where S_h is the h-neighborhood of S. The number $n - \liminf_{h\to 0^+}\frac{\ln|S_h|}{\ln h}$ is often referred to as the upper Minkowski dimension of S. If $\lim_{h\to 0^+}\frac{\ln|S_h|}{\ln h}$ exists, then $n - \lim_{h\to 0^+}\frac{\ln|S_h|}{\ln h}$ is the Minkowski dimension of S. In this case, S is called *regular*. The definition can be easily extended to unbounded S by approximation, but it is not needed here. Regular sets form a large class including smooth manifolds and even fractal sets such as cantor sets and Julia sets.

Theorem C. *Suppose there is a uniform Lyapunov function in $\mathcal{U}$, with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant γ, with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ for some $0 < \epsilon_0 \ll 1$. Let $\{\mu_\epsilon\}$, with densities $\{u_\epsilon\}$, be the stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Denote $\mathcal{J}$ as the global attractor of (1.1). Suppose $|\mathcal{J}| = 0$ and assumptions in Theorem B(3)–(5) are satisfied. Then for each $\sigma \in (0,1)$, there holds*

$$\liminf_{\epsilon\to 0}\frac{\mathcal{H}[u_\epsilon]}{\ln h_\sigma(\epsilon)} \ge \sigma(n-d),$$

where $h_\sigma(\epsilon)$ is given in Theorem B(1) and $d := n - \liminf_{h\to 0^+}\frac{\ln|\mathcal{J}_h|}{\ln h} \in [0,n]$ is the upper Minkowski dimension of $\mathcal{J}$.

Measure estimates near the global attractor, and entropy-dimension inequalities for the regular global attractor have been studied in [13] for a class of stochastically perturbed dissipative dynamical systems admitting a strict Lyapunov function U in an isolated neighborhood $\mathcal{N}$ of the global attractor $\mathcal{J}$ satisfying $U(x) \geq Cd^2(x, \mathcal{J})$ for $x \in \mathcal{N}$. Under such a strong Lyapunov condition in $\mathcal{N}$ and additional Lyapunov conditions on an exterior domain, it is shown in [13] that for any $0 < \delta \ll 1$ there are $\epsilon_0 > 0$ and $M > 0$ such that

$$\mu_\epsilon(\mathcal{J}_{M\epsilon}) \geq 1 - \delta, \quad \forall \epsilon \in (0, \epsilon_0], \tag{1.8}$$

which yields the entropy-dimension inequality $\liminf_{\epsilon\to 0} \frac{\mathcal{H}[u_\epsilon]}{\ln \epsilon} \geq n - d$, where d is the Minkowski dimension of $\mathcal{J}$. This is then used in [14] to study the mutual information that quantifies the degeneracy and complexity in a biological network modeled by SDEs. We remark that, though the measure estimate (1.8) is not expected in general unless the strong conditions in [13] are satisfied, the entropy-dimension inequalities as stated in Theorem C remain valid under much weaker conditions assumed in the theorem. In particular, Theorem C does not require the global attractor $\mathcal{J}$ to be regular. But for regular $\mathcal{J}$, it is possible to have the entropy-dimension equality $\lim_{\epsilon\to 0} \frac{\mathcal{H}[u_\epsilon]}{\ln \epsilon} = n - d$ as established in [13, Theorem 4.7] under additional assumptions.

The rest of the paper is organized as follows. In Section 2, we recall basic definitions and preliminary results. In Section 3, we construct uniform Lyapunov functions (resp. uniform anti-Lyapunov functions) near a maximal attractor (resp. maximal repeller) and establish corresponding measure estimates in their basins. In Section 4, we quantify concentrations of stationary measures and prove Theorem A(1), (2). In Section 5, we quantify noise stabilization (resp. de-stabilization) of a local attractor (resp. local repeller) and prove Theorem A(3), (4). In Section 6, we study entropy estimates and prove Theorems B, C.

2. Preliminaries

In the section, we recall some basic definitions and preliminary results from [3,7,8,15]. We refer the reader to [3] for extensive remarks. Throughout the rest of the paper, we denote φ^t as the local flow generated by (1.1) and let $\Omega \subset \mathcal{U}$ be a connected open set.

2.1. Dissipative/anti-dissipative dynamical systems

Definition 2.1.

(1) φ^t is said to be *dissipative* (resp. *anti-dissipative*) in Ω if Ω is positively (resp. negatively) invariant and there is a compact set $B \subset \Omega$ such that for any $\xi \in \Omega$, there is a $t_0(\xi) > 0$ such that $\varphi^t(\xi) \in B$ for all $t \geq t_0(\xi)$ (resp. $t \leq -t_0(\xi)$).

(2) When Ω is positively (resp. negatively) invariant, the *maximal attractor* $\mathcal{J}$ (resp. *maximal repeller* $\mathcal{R}$) in Ω is a compact invariant set satisfying

$$\lim_{t\to\infty} d_H\left(\varphi^t(K), \mathcal{J}\right) = 0 \quad \text{(resp. } \lim_{t\to-\infty} d_H\left(\varphi^t(K), \mathcal{R}\right) = 0)$$

for any $K \subset\subset \Omega$, where $d_H(E, F) = \sup_{x\in E} d(x, F)$ is the Hausdorff semi-distance for any $E, F \subset \mathbb{R}^n$.

(3) Suppose the maximal attractor $\mathcal{J}$ (resp. maximal attractor $\mathcal{R}$) in Ω exists. If $\Omega = \mathcal{U}$, then $\mathcal{J}$ (resp. $\mathcal{R}$) is called the *global attractor* (resp. *global repeller*). Otherwise, it is called the *local attractor* (resp. *local repeller*).

Clearly, if a maximal attractor or a maximal repeller in Ω exists, then it is unique in Ω. In particular, a global attractor or a global repeller is unique. Moreover, a maximal attractor or a maximal repeller is connected as Ω is connected.

For a maximal attractor $\mathcal{J}$ (resp. maximal repeller $\mathcal{R}$), the domain Ω can be extended to its *basin of attraction* $B(\mathcal{J})$ (resp. *basin of expansion* $B(\mathcal{R})$) defined as

$$\begin{aligned} &B(\mathcal{J}) = \left\{x \in \mathcal{U} : \varphi^t(x) \in \mathcal{U},\ \forall t \geq 0 \text{ and} \right.\\ &\qquad \left. \times\ d(\varphi^t(x), \mathcal{J}) \to 0 \text{ as } t \to \infty\right\} \\ &\text{(resp. } B(\mathcal{R}) = \left\{x \in \mathcal{U} : \varphi^t(x) \in \mathcal{U},\ \forall t \leq 0 \text{ and} \right.\\ &\qquad \left. \times\ d(\varphi^t(x), \mathcal{R}) \to 0 \text{ as } t \to -\infty\right\}), \end{aligned} \tag{2.1}$$

which is open, connected and positively (resp. negatively) invariant.

Lyapunov functions (resp. anti-Lyapunov functions) are important in the study of dissipative (resp. anti-dissipative) dynamical systems. But they are usually defined on the entire domain of interest. Weaker notions are introduced in [8] as follows.

Definition 2.2.

(1) A continuous and nonnegative function U defined on Ω is said to be *compact* in Ω if $U(x) < \rho_M$ for all $x \in \Omega$ and $\lim_{x\to\partial\Omega} U(x) = \rho_M$, where $\rho_M = \sup_\Omega U$ is called the *essential upper bound* of U. A compact function U is called *unbounded* if $\rho_M = \infty$.

(2) A C^1 compact function U is called a *Lyapunov function* (resp. *anti-Lyapunov function*) in Ω of (1.1) if there exist $\rho_m \in (0, \rho_M)$, called the *essential lower bound*, and $\gamma > 0$, called the *Lyapunov constant* (resp. *anti-Lyapunov constant*), such that

$$V \cdot \nabla U \leq -\gamma \quad \text{(resp. } \geq \gamma) \quad \text{on} \quad \Omega \setminus \overline{\Omega}_{\rho_m},$$

where $\Omega_{\rho_m} = \{x \in \mathcal{U} : U(x) < \rho_m\}$. The set $\Omega \setminus \overline{\Omega}_{\rho_m}$ is called the *essential domain*.

Clearly, if U is a Lyapunov function or an anti-Lyapunov function in Ω of (1.1), then $\nabla U \neq 0$ on a neighborhood of $\Omega \setminus \overline{\Omega}_{\rho_m}$. For any compact function U in Ω, we define

$$\begin{aligned} \Omega_\rho &= \{x \in \Omega : U(x) < \rho\}, \quad \forall \rho \in (0, \rho_M], \\ \Omega_\rho^* &= \{x \in \Omega : U(x) \leq \rho\}, \quad \forall \rho \in [0, \rho_M). \end{aligned}$$

Note that $\Omega = \Omega_{\rho_M}$. In particular, $\mathcal{U} = \Omega_{\rho_M}$ if $\Omega = \mathcal{U}$.

The following results giving basic relations between dissipative (resp. anti-dissipative) dynamical systems and Lyapunov functions (resp. anti-Lyapunov functions) are proven in [15, Appendix].

Proposition 2.1.

(1) *φ^t is dissipative (resp. anti-dissipative) in Ω if and only if Ω is positively (resp. negatively) invariant and admits a maximal attractor (resp. maximal repeller) in Ω.*

(2) *If (1.1) admits a Lyapunov (resp. an anti-Lyapunov) function in Ω, then φ^t is dissipative (resp. anti-dissipative) in Ω.*

The following notions of a strong local attractor and a strong local repeller are introduced in [3].

Definition 2.3. A compact invariant set $\mathcal{J}_0 \subset \mathcal{U}$ (resp. $\mathcal{R}_0 \subset \mathcal{U}$) is called a *strong local attractor* (resp. *strong local repeller*) if there is a connected and positively (resp. negatively) invariant neighborhood $\mathcal{W}$ of $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) in $\mathcal{U}$, called an *isolating neighborhood*, with oriented C^2 boundary such that $\mathcal{J}_0$ attracts $\mathcal{W}$ (resp. $\mathcal{R}_0$ repels $\mathcal{W}$), and $V \cdot \nu < 0$ (resp. $V \cdot \nu > 0$) on $\partial\mathcal{W}$, where ν denotes the outward unit normal vector field on $\partial\mathcal{W}$.

Clearly, a strong local attractor (resp. strong local repeller) is a local attractor (resp. local repeller). We show in Corollary 3.1 that the converse is also true for C^2 drift fields.

2.2. Lyapunov/anti-Lyapunov functions and stationary measures

Consider the Fokker–Planck equation

$$\partial_t u = Lu := \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad t > 0, \quad x \in \mathcal{U}, \tag{2.2}$$

corresponding to the Itô SDE

$$dx = V(x)dt + G(x)dW_t, \quad x \in \mathcal{U},$$

where $A = (a^{ij}) = \frac{GG^\top}{2}$ is the diffusion matrix and $V = (V^i)$ is the drift field. The assumptions on the regularity of V and G are given in Section 1. Denote by

$$\mathcal{L} = a^{ij}\partial^2_{ij} + V^i\partial_i$$

the L^2-formal adjoint of the Fokker–Planck operator L.

Definition 2.4. A C^2 compact function U in Ω is called a *Lyapunov function* (resp. *anti-Lyapunov function*) in Ω with respect to $\mathcal{L}$ if there are $\rho_m \in (0, \rho_M)$, called the *essential lower bound*, and $\gamma > 0$, called the *Lyapunov constant* (resp. *anti-Lyapunov constant*), such that

$$\mathcal{L}U \le -\gamma \quad (\text{resp. } \ge \gamma) \quad \text{on} \quad \Omega \setminus \overline{\Omega}_{\rho_m}.$$

The set $\Omega \setminus \overline{\Omega}_{\rho_m}$ is called the *essential domain*.

Definition 2.5. Suppose $A \in W^{1,p}_{loc}(\Omega, \mathbb{R}^{n\times n})$ and $V \in L^p_{loc}(\Omega, \mathbb{R}^n)$ for some fixed $p > n$. A Borel probability measure μ on Ω is called a *stationary measure* of the Fokker–Planck equation (2.2) in Ω (or a stationary measure corresponding to $\mathcal{L}$ in Ω) if $L\mu = 0$ in Ω in the sense that

$$V^i \in L^1_{loc}(\Omega, \mu),\ i = 1, \dots, n, \quad \text{and} \quad \int_\Omega \mathcal{L}\phi d\mu = 0, \quad \forall \phi \in C^\infty_0(\Omega).$$

The existence and uniqueness of stationary measures of (2.2) under Lyapunov-type conditions have been extensively studied (see e.g. [5,7,16–22]). We recall the following results concerning the regularity and existence of stationary measures.

Proposition 2.2. *Suppose $A \in W^{1,p}_{loc}(\Omega, GL(n, \mathbb{R}))$ and $V \in L^p_{loc}(\Omega, \mathbb{R}^n)$. Then the following hold.*

(1) *[23] Any stationary measure corresponding to $\mathcal{L}$ in Ω has a positive density in $W^{1,p}_{loc}(\Omega)$.*
(2) *[7] If there is a Lyapunov function in Ω with respect to $\mathcal{L}$, then (2.2) admits a stationary measure in Ω, which is unique if, in addition, the Lyapunov function is unbounded.*

We remark that the existence of an anti-Lyapunov function does not mean the non-existence of stationary measures (see [15, Section 5]). Using Lyapunov or anti-Lyapunov functions, stationary measures can be estimated on essential domains as follows.

Proposition 2.3. *Suppose $A \in W^{1,p}_{loc}(\Omega, GL(n, \mathbb{R}))$ and $V \in L^p_{loc}(\Omega, \mathbb{R}^n)$. Let U be either a Lyapunov function or an anti-Lyapunov function in Ω with respect to $\mathcal{L}$ with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant or anti-Lyapunov constant γ, and satisfies $\nabla U \ne 0$ on $U^{-1}(\rho)$ for a.e. $\rho \in [\rho_m, \rho_M)$, and $a^{ij}\partial_i U \partial_j U \le H(\rho)$ on $\partial\Omega_\rho$ for $\rho \in [\rho_m, \rho_M)$ for some measurable function $H : [\rho_m, \rho_M) \to [0, \infty)$. Let μ be a stationary measure corresponding to $\mathcal{L}$ in Ω.*

(1) *If U is a Lyapunov function, then*

$$\mu(\Omega \setminus \Omega_\rho) \le e^{-\gamma \int_{\rho_m}^{\rho} \frac{1}{H(t)}dt}, \quad \forall \rho \in [\rho_m, \rho_M).$$

(2) *If U is an anti-Lyapunov function, then*

$$\mu(\Omega_\rho \setminus \Omega^*_{\rho_m}) \ge \mu(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m})e^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)}dt}, \quad \forall \rho_m < \rho_0 < \rho < \rho_M.$$

Proof. See [8, Theorem A(b)] for (1) and [8, Theorem B(a)] for (2). □

We note that above results only give estimates of stationary measures in the essential domain of a Lyapunov function or an anti-Lyapunov function, not in the entire domain of interest.

2.3. Uniform Lyapunov/anti-Lyapunov functions

Let $0 < \epsilon_0 \ll 1$. For $\epsilon \in (0, \epsilon_0]$, we recall that

$$\mathcal{L}_\epsilon = \epsilon^2 a^{ij}\partial^2_{ij} + V^i\partial_i,$$

is the L^2-formal adjoint of the Fokker–Planck operator L_ϵ given in (1.3).

Definition 2.6. A C^2 compact function U in Ω is called a *uniform Lyapunov function* (resp. *uniform anti-Lyapunov function*) in Ω with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ if it is a Lyapunov function (resp. anti-Lyapunov function) in Ω with respect to $\mathcal{L}_\epsilon$ for each $\epsilon \in (0, \epsilon_0]$, and the essential lower bound and the Lyapunov constant (resp. anti-Lyapunov constant) are independent of $\epsilon \in (0, \epsilon_0]$, that is, there are $\rho_m \in (0, \rho_M)$ and $\gamma > 0$ such that $\mathcal{L}_\epsilon U \le -\gamma$ (resp. $\ge \gamma$) on $\Omega \setminus \overline{\Omega}_{\rho_m}$ for all $\epsilon \in (0, \epsilon_0]$.

We remark that if U is a uniform Lyapunov function (resp. an anti-Lyapunov function) in Ω with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$, then it is easy to see that U must be a Lyapunov function (resp. anti-Lyapunov function) in Ω of (1.1). The following result, whose proof is trivial, gives a partial converse.

Lemma 2.1. *Suppose that (1.1) admits a C^2 Lyapunov function (resp. anti-Lyapunov function) U in Ω with bounded second-order partial derivatives on Ω. If A is bounded on Ω, then there exists $\epsilon_0 = \epsilon_0(A) > 0$ such that U is a uniform Lyapunov function (resp. uniform anti-Lyapunov function) in Ω with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$.*

2.4. Harnack inequality

Consider the differential operator

$$Lu := \partial_i(a^{ij}(x)\partial_j u + b^i(x)u) \quad \text{on} \quad \Omega,$$

where a^{ij}, b_i, $i, j = 1, \dots, n$, are measurable and bounded functions on Ω. Suppose that there are constants $\lambda > 0$, $\Lambda > 0$ and $\nu \ge 0$ such that for any $x \in \Omega$,

$$a^{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \ \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j}|a^{ij}(x)|^2 \le \Lambda^2, \quad \text{and} \quad \frac{1}{\lambda^2}\sum_i |b^i(x)|^2 \le \nu^2.$$

The following result is a special case of [24, Theorem 8.20].

Proposition 2.4 (*Harnack Inequality*)**.** *Let $u \in W^{1,2}(\Omega)$ be a nonnegative solution of $Lu = 0$ on Ω. Then for any ball $B_{4R}(y) \subset \Omega$,*

$$\sup_{B_R(y)} u \le C_0^{(\frac{\Lambda}{\lambda} + \nu R)} \inf_{B_R(y)} u,$$

where $C_0 = C_0(n)$ depends only on n.

We only use Proposition 2.4 in the case $b^i = \partial_j a^{ij} - V^i$ so that $Lu = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u)$ is in the form of a Fokker–Planck operator.

3. Noise distribution in the basin of maximal attractors /repellers

In this section, we first construct uniform Lyapunov functions (resp. uniform anti-Lyapunov functions) in the basin of maximal attractors (resp. maximal repellers), and then use them to conduct measure estimates of stationary measures.

3.1. Construction of uniform Lyapunov/anti-Lyapunov functions

The existence of a uniform Lyapunov function (resp. uniform anti-Lyapunov functions) in the basin of a maximal attractor (resp. maximal repeller) of (1.1) is stated in the following theorem.

Theorem 3.1. *Let $\mathcal{E}$ be a maximal attractor (resp. maximal repeller) of (1.1) with the basin of attraction (resp. basin of expansion) $B(\mathcal{E})$ given in (2.1). Let $\mathcal{V}_*$ and $\mathcal{V}_{**}$ be any open subsets of $\mathcal{U}$ such that $\mathcal{V}_{**}$ is positively (resp. negatively) invariant and $\mathcal{E}\subset\mathcal{V}_*\subset\mathcal{V}_{**}\subset\subset B(\mathcal{E})$. Then, the following hold.*

(1) *There is an open, connected and positively (resp. negatively) invariant set $\mathcal{U}_*$ satisfying $\mathcal{V}_{**}\subset\subset\mathcal{U}_*\subset\subset B(\mathcal{E})$, and a Lyapunov function (resp. an anti-Lyapunov function) U in $\mathcal{U}_*$ of (1.1) with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant (resp. anti-Lyapunov constant) 2γ. Moreover, the following properties hold:*
   - *for any $\rho\in(0,\rho_M]$, the set $\Omega_\rho:=\{x\in\mathcal{U}_*:U(x)<\rho\}$ is connected;*
   - *$\mathcal{E}\subset\Omega_{\rho_m}\subset\subset\mathcal{V}_*$;*
   - *$U\in C^2(\overline{\mathcal{U}}_*)$, $V\cdot\nabla U<0$ (resp. $V\cdot\nabla U>0$) on $\overline{\mathcal{U}}_*\setminus\Omega_{\rho_m}$; in particular, $\nabla U\neq 0$ on $\overline{\mathcal{U}}_*\setminus\Omega_{\rho_m}$.*

(2) *Let $\epsilon_*:=\max\left\{\epsilon>0:\epsilon^2|A|_{C(\overline{\mathcal{U}}_*)}|U|_{C^2(\overline{\mathcal{U}}_*)}\le\gamma\right\}$. Then, U is a uniform Lyapunov function (resp. uniform anti-Lyapunov function) in $\mathcal{U}_*$ with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_*]}$ with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant (resp. anti-Lyapunov constant) γ.*

Before proving Theorem 3.1, we make several remarks as follows.

Remark 3.1.

(1) From the statement of Theorem 3.1, we see that U is first constructed to be a Lyapunov function (resp. an anti-Lyapunov function) in $\mathcal{U}_*$ of (1.1) with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant 2γ (see Lemma 3.8). As a result, $\mathcal{U}_*$, U, ρ_m, ρ_M and γ are independent of ϵ and A.
(2) The condition V being C^2 ensures that U is a C^2 function. If we only look for Lyapunov functions (resp. anti-Lyapunov functions) of (1.1), then V being C^1 suffices.
(3) We see that $\mathcal{V}_{**}\setminus\mathcal{V}_*\subset\mathcal{U}_*\setminus\overline{\Omega}_{\rho_m}$. Choosing an increasing sequence of $\mathcal{V}_{**}\setminus\mathcal{V}_*$ whose union exhausting $B(\mathcal{E})\setminus\mathcal{E}$, we can construct a family of Lyapunov functions (resp. anti-Lyapunov functions) whose essential domains exhaust $B(\mathcal{E})\setminus\mathcal{E}$. See Section 4 for more details.

Below, for the sake of simplicity, we only prove Theorem 3.1 for the case that $\mathcal{E}=\mathcal{J}$ is a maximal attractor of φ^t. The proof for the case of a maximal repeller simply follows by reversing the time. In all lemmas contained in this subsection, we always assume conditions of Theorem 3.1.

Lemma 3.1. *$\mathcal{V}_{**}$ is connected, and $\overline{\mathcal{V}}_{**}$ is positively invariant and connected.*

Proof. Since $\mathcal{J}$ is compact and $\mathcal{V}_{**}$ is open, the inclusion $\mathcal{J}\subset\mathcal{V}_{**}$ ensures that $\mathcal{J}_\delta\subset\mathcal{V}_{**}$ for some $\delta>0$, where $\mathcal{J}_\delta$ is the δ-neighborhood of $\mathcal{J}$. As $\mathcal{J}$ is connected, so is $\mathcal{J}_\delta$.

Let $x,y\in\mathcal{V}_{**}$ and $x\neq y$. Since $\mathcal{V}_{**}\subset B(\mathcal{J})$, the attracting property of $\mathcal{J}$ ensures the existence of moments $t_x,t_y>0$ such that $\varphi^{t_x}(x),\varphi^{t_y}(y)\in\mathcal{J}_\delta$. The connectedness of $\mathcal{J}_\delta$ ensures the existence of some curve $c_1\subset\mathcal{J}_\delta$ connecting $\varphi^{t_x}(x)$ and $\varphi^{t_y}(y)$. Let

$$c_2:=c_1\cup\left\{\varphi^t(x):t\in[0,t_x]\right\}\cup\left\{\varphi^t(y):t\in[0,t_y]\right\}.$$

As $\mathcal{V}_{**}$ is positively invariant, we have

$$\left\{\varphi^t(x):t\in[0,t_x]\right\}\cup\left\{\varphi^t(y):t\in[0,t_y]\right\}\subset\mathcal{V}_{**}.$$

Thus, $c_2\subset\mathcal{V}_{**}$. Clearly, c_2 connects x and y. Hence, $\mathcal{V}_{**}$ is connected.

The positive invariance of $\overline{\mathcal{V}}_{**}$ follows from that of $\mathcal{V}_{**}$ and the continuity of φ^t. The connectedness of $\overline{\mathcal{V}}_{**}$ follows from that of $\mathcal{V}_{**}$ and the above arguments. □

Lemma 3.2. *There is an open, connected and positively invariant set $\mathcal{V}$ satisfying $\mathcal{V}_{**}\subset\subset\mathcal{V}\subset\subset B(\mathcal{J})$. Moreover, $\overline{\mathcal{V}}$ is positively invariant and connected, and satisfies $\overline{\varphi^t(\mathcal{V})}=\varphi^t(\overline{\mathcal{V}})$ for all $t\ge 0$.*

Proof. Since $\mathcal{V}_{**}\subset\subset B(\mathcal{J})$, there is an open set $\tilde{\mathcal{V}}$ such that $\mathcal{V}_{**}\subset\subset\tilde{\mathcal{V}}\subset\subset B(\mathcal{J})$. Let $\mathcal{V}:=\cup_{t\ge0}\varphi^t(\tilde{\mathcal{V}})$.

Clearly, $\mathcal{V}$ is open, positively invariant and connected. To show the compactness of $\overline{\mathcal{V}}$, we let $\mathcal{N}$ be an open neighborhood of $\mathcal{J}$ and satisfy $\mathcal{N}\subset\subset B(\mathcal{J})$. Since $\tilde{\mathcal{V}}\subset\subset B(\mathcal{J})$, the attracting property of $\mathcal{J}$ ensures the existence of some $T>0$ such that $\varphi^t(\tilde{\mathcal{V}})\subset\mathcal{N}$ for all $t\ge T$. In particular, $\cup_{t\ge T}\varphi^t(\tilde{\mathcal{V}})\subset\mathcal{N}$. From the boundedness of $\tilde{\mathcal{V}}$ and the continuity of the map $(t,x)\mapsto\varphi^t(x):[0,\infty)\times B(\mathcal{J})\to B(\mathcal{J})$, the compactness of $\cup_{t\in[0,T]}\varphi^t(\overline{\tilde{\mathcal{V}}})$ follows. As $\overline{\tilde{\mathcal{V}}}\subset B(\mathcal{J})$, we have

$$\overline{\bigcup_{t\in[0,T]}\varphi^t(\tilde{\mathcal{V}})}\subset\bigcup_{t\in[0,T]}\varphi^t(\overline{\tilde{\mathcal{V}}})\subset B(\mathcal{J}).$$

Hence, $\overline{\mathcal{V}}\subset\overline{\mathcal{N}}\cup\left(\cup_{t\in[0,T]}\varphi^t(\overline{\tilde{\mathcal{V}}})\right)$, which implies the compactness of $\overline{\mathcal{V}}$.

The positive invariance and connectedness of $\overline{\mathcal{V}}$ follow readily. To show $\overline{\varphi^t(\mathcal{V})}=\varphi^t(\overline{\mathcal{V}})$, we first observe that $\overline{\varphi^t(\mathcal{V})}\subset\varphi^t(\overline{\mathcal{V}})$, as $\varphi^t(\overline{\mathcal{V}})$ is compact. The inclusion $\varphi^t(\overline{\mathcal{V}})\subset\overline{\varphi^t(\mathcal{V})}$ follows from the continuity of φ^t. □

Let $\mathcal{V}$ be as in Lemma 3.2. The attracting property of $\mathcal{J}$ ensures the existence of some $T\gg 1$ such that

$$\mathcal{W}:=\varphi^T(\mathcal{V})\subset\subset\mathcal{V}_*.$$

Lemma 3.3. *The set $\mathcal{W}$ is open, positively invariant and connected, and satisfies $\overline{\varphi^t(\mathcal{W})}=\varphi^t(\overline{\mathcal{W}})=\varphi^{t+T}(\overline{\mathcal{V}})$ for all $t\ge0$. The set $\overline{\mathcal{W}}$ is positively invariant and connected.*

Proof. It follows from Lemma 3.2 and the definition of $\mathcal{W}$. □

We construct a nonnegative and smooth function vanishing exactly on $\overline{\mathcal{W}}$. Since the topology of $\mathcal{U}$ is countably generated, there exist countably many open balls $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ in $\mathcal{U}\setminus\overline{\mathcal{W}}$ such that $\mathcal{U}\setminus\overline{\mathcal{W}}=\bigcup_{i\in\mathbb{N}}B_{r_i}(x_i)$. Let us consider the following bump functions

$$\zeta_i(x)=\begin{cases}e^{-\frac{r_i^2}{r_i^2-|x-x_i|^2}}, & x\in B_{r_i}(x_i),\\ 0, & x\in\mathcal{U}\setminus B_{r_i}(x_i),\end{cases}\qquad i\in\mathbb{N},$$

and define

$$\zeta(x):=\sum_{i\in\mathbb{N}}\frac{2^{-i}}{C_i}\zeta_i(x),\quad x\in\mathcal{U},$$

where $C_i = \sup_{|\alpha|\leq i}\sup_{\mathcal{U}}|\partial^\alpha \zeta_i|$, which controls ζ_i up to its ith order partial derivatives. The proof of the next lemma is straightforward.

Lemma 3.4. *The function $\zeta : \mathcal{U} \to \mathbb{R}$ is nonnegative and smooth, and vanishes exactly on $\overline{\mathcal{W}}$.*

As in Lemma 3.4, we can construct a nonnegative and smooth function $\eta : \mathcal{U} \to \mathbb{R}$ vanishing exactly on $\overline{\mathcal{V}}_{**}$. Consider the function

$$\tilde{\zeta} = \zeta(1+\eta)^N,$$

where $N \geq 0$ is to be chosen, and define

$$U(x) = \int_0^T \tilde{\zeta}(\varphi^t(x))dt, \quad x \in \overline{\mathcal{V}}.$$

Since N is eventually fixed, we used N-independent notations here and we do so in the sequel for notational simplicity. We refer the read to Remark 3.2 for the reason why the term $(1+\eta)^N$ is introduced.

For $\rho > 0$, let $\Omega_\rho = \{x \in \mathcal{V} : U(x) < \rho\}$.

Lemma 3.5. *The following hold for any $N \geq 0$.*

(1) *$U \in C^2(\overline{\mathcal{V}})$;*
(2) *$U \geq 0$ on $\overline{\mathcal{V}}$, and $U(x) = 0$ if and only if $x \in \overline{\mathcal{W}}$;*
(3) *$\frac{d}{dt}U(\varphi^t(x))\big|_{t=0} = \nabla U(x)\cdot V(x) = -\tilde{\zeta}(x)$ for $x \in \overline{\mathcal{V}}$; in particular, $\nabla U \neq 0$ on $\overline{\mathcal{V}} \setminus \overline{\mathcal{W}}$;*
(4) *for each $\rho > 0$, Ω_ρ is connected;*
(5) *if $\rho < \inf_{\partial\mathcal{V}} U$, then $\partial\Omega_\rho = U^{-1}(\rho)$ is C^2.*

Proof. (1) It follows from the facts that $\tilde{\zeta} \in C^\infty(\mathcal{U})$ and $\varphi^t \in C^2(B(\mathcal{J}), B(\mathcal{J}))$ for any $t \geq 0$.

(2) It is an immediate consequence of the properties of $\tilde{\zeta}$ and the positive invariance of $\overline{\mathcal{W}}$.

(3) Let $x \in \overline{\mathcal{V}}$. Since $\varphi^T(x) \subset \varphi^T(\overline{\mathcal{V}}) = \overline{\mathcal{W}}$, we have $\tilde{\zeta}(\varphi^T(x)) = 0$. It follows that

$$\begin{aligned}\nabla U(x)\cdot V(x) = \frac{d}{dt}U(\varphi^t(x))\bigg|_{t=0} &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^T \tilde{\zeta}(\varphi^{t+h}(x))dt - \int_0^T \tilde{\zeta}(\varphi^t(x))dt\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_T^{T+h} \tilde{\zeta}(\varphi^t(x))dt - \int_0^h \tilde{\zeta}(\varphi^t(x))dt\right]\\ &= \tilde{\zeta}(\varphi^T(x)) - \tilde{\zeta}(x) = -\tilde{\zeta}(x).\end{aligned}$$

(4) Let $x, y \in \Omega_\rho$. Note that (3) says that the function U is decreasing along trajectories of φ^t as long as the trajectories have not reached $\overline{\mathcal{W}}$. This in particular implies that $\varphi^t(x), \varphi^t(y) \in \Omega_\rho$ for all $t \geq 0$. By the attracting property of $\mathcal{J}$, there are $t_x, t_y > 0$ such that $\varphi^{t_x}(x), \varphi^{t_y}(y) \in \mathcal{W}$. Since $\mathcal{W}$ is connected, there is a curve in $\mathcal{W}$ connecting $\varphi^{t_x}(x)$ and $\varphi^{t_y}(y)$. Catenating this curve with $\{\varphi^t(x) : t \in [0, t_x]\}$ and $\{\varphi^t : t \in [0, t_y]\}$, we find a curve in Ω_ρ connecting x and y. This proves the connectedness of Ω_ρ.

(5) It follows from (1), (3) and the implicit function theorem. □

In the next two lemmas, we consider some special level sets of U in order to determine the essential lower bound and the essential upper bound in the definition of a Lyapunov function.

Lemma 3.6. *For any $N \geq 0$, there exists $\rho_m > 0$ such that $\mathcal{W} \subset\subset \Omega_{\rho_m} \subset\subset \mathcal{V}_*$. Moreover, Ω_{ρ_m} is connected and its boundary $\partial\Omega_{\rho_m} = U^{-1}(\rho_m)$ is C^2.*

Proof. Since $\mathcal{W} \subset\subset \mathcal{V}_* \subset\subset \mathcal{V}$, we have $\partial\mathcal{V}_* \subset \mathcal{V} \setminus \overline{\mathcal{W}}$. Thus, U is positive on $\partial\mathcal{V}_*$. The compactness of $\partial\mathcal{V}_*$ yields $\rho_m := \frac{1}{2}\min_{\partial\mathcal{V}_*} U > 0$. The inclusion $\overline{\mathcal{W}} \subset \Omega_{\rho_m}$ then follows. By Lemma 3.5(4), (5), $\partial\Omega_{\rho_m} = U^{-1}(\rho_m)$ is C^2.

It remains to show that $\Omega_{\rho_m} \subset\subset \mathcal{V}_*$. Suppose $\overline{\Omega}_{\rho_m} \subset \mathcal{V}_*$ fails. Let $x_* \in \overline{\Omega}_{\rho_m} \setminus \mathcal{V}_*$. Then, we can find $t_* \geq 0$ and $y_* \in \partial\mathcal{V}_*$ such that $y_* = \varphi^{t_*}(x_*)$. It follows from Lemma 3.5(3) that $U(x_*) \geq U(y_*) > \rho_m$. However, $U(x_*) \leq \rho_m$, which leads to a contradiction. □

Remark 3.2. Note that if $N = 0$, then the inequality

$$\sup_{\mathcal{V}_{**}} U < \inf_{\partial\mathcal{V}} U \tag{3.1}$$

needs NOT be true. As a result, we would not be able to define the essential upper bound. In order for the validity of (3.1) (at least for sufficient large N), we introduced the term $(1+\eta)^N$ to modify the function ζ near $\partial\mathcal{V}$.

Lemma 3.7. *If N is sufficiently large, then there exists $\rho_M > 0$ such that $\mathcal{V}_{**} \subset\subset \Omega_{\rho_M} \subset \mathcal{V}$. Moreover, Ω_{ρ_M} is connected and its boundary $\partial\Omega_{\rho_M} = U^{-1}(\rho_M)$ is C^2.*

Proof. We point out that the inclusion $\mathcal{V}_{**} \subset\subset \mathcal{V}$ yields the existence of some $t^* > 0$ such that

$$\mathcal{V}_{**} \subset\subset \varphi^{t^*}(\mathcal{V}) \subset \mathcal{V}. \tag{3.2}$$

In fact, if $\overline{\mathcal{V}}_{**}$ does not belong to $\varphi^t(\mathcal{V})$ for any $t > 0$, then the continuity yields that $\overline{\mathcal{V}}_{**}$ does not belong to $\mathcal{V}$. The inclusion $\varphi^{t^*}(\mathcal{V}) \subset \mathcal{V}$ is trivial as $\mathcal{V}$ is positively invariant.

Due to the positive invariance of $\overline{\mathcal{V}}_{**}$ by Lemma 3.1 and the fact that $\tilde{\zeta} = \zeta$ on $\overline{\mathcal{V}}_{**}$ for all $N \geq 0$, the value $\sup_{\mathcal{V}_{**}} U$ is independent of N. But, the value $\inf_{\partial\mathcal{V}} U$ increases greatly with the increase of N. In fact, for any $x \in \partial\mathcal{V}$, we have $\varphi^t(x) \subset \overline{\mathcal{V}} \setminus \varphi^{t^*}(\mathcal{V})$ for all $t \in [0, t^*]$. For otherwise, $\varphi^{t_0}(x) \in \varphi^{t^*}(\mathcal{V})$ for some $t_0 \in (0, t^*]$. Then $\varphi^{t^*}(x) = \varphi^{t^*-t_0}(\varphi^{t_0}(x)) \in \varphi^{t^*}(\mathcal{V})$ due to the positive invariance of $\varphi^{t^*}(\mathcal{V})$, and hence, $x \in \varphi^{-t^*}(\varphi^{t^*}(\mathcal{V})) = \mathcal{V}$, which leads to a contradiction. Since $\varphi^t(x) \in \overline{\mathcal{W}}$, we have $\tilde{\zeta}(\varphi^t(x)) = 0$ for all $t \geq T$. It follows that

$$\begin{aligned}U(x) \geq \int_0^{t^*}\tilde{\zeta}(\varphi^t(x))dt &\geq t^* \inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\tilde{\zeta} \geq t^*\\ &\times\left[\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\zeta\right]\left[1+\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\eta\right]^N.\end{aligned}$$

Due to (3.2) and the construction of ζ and η, both $\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\zeta$ and $\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\eta$ are positive. As a result, the estimate

$$t^*\left[\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\zeta\right]\left[1+\inf_{\overline{\mathcal{V}}\setminus\varphi^{t^*}(\mathcal{V})}\eta\right]^N \geq 1+\sup_{\mathcal{V}_{**}} U$$

holds for all $N \gg 1$. As the above is independent of $x \in \partial\mathcal{V}$, we conclude that $U(x) \geq 1+\sup_{\mathcal{V}_{**}} U$ for any $x \in \partial\mathcal{V}$. In particular, (3.1) holds for $N \gg 1$.

Set $\rho_M := \frac{1}{2}\left(\sup_{\mathcal{V}_{**}} U + \inf_{\partial\mathcal{V}} U\right)$ for $N \gg 1$. The result then follows from (3.1) and Lemma 3.5. □

In what follows, we fix some $N \gg 1$ so that Lemma 3.7 holds. Lemmas 3.6 and 3.7 together give a C^2 Lyapunov function in Ω_{ρ_M} of (1.1).

Lemma 3.8. *U is a C^2 Lyapunov function in $\mathcal{U}_* := \Omega_{\rho_M}$ of (1.1) with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant $r := \inf_{\Omega_{\rho_M}\setminus\overline{\Omega}_{\rho_m}}\tilde{\zeta} > 0$.*

Proof. Recall that U is C^2 and Ω_{ρ_M} is a connected open set. Since $U(x) < \rho_M$ for $x \in \Omega_{\rho_M}$ and $U = \rho_M$ on $\partial\Omega_{\rho_M}$, U is a compact function. By Lemmas 3.5(3) and 3.6, we have that $\nabla U\cdot V\le -r$ on $\Omega_{\rho_M}\setminus\overline{\Omega}_{\rho_m}$, that is, U is a Lyapunov function in Ω_{ρ_M} of (1.1) with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant r. □

Remark 3.3. If we assume V is C^1, then U is C^1, and therefore, U is a Lyapunov function in Ω_{ρ_M} of (1.1).

Proof of Theorem 3.1. (1) Lemma 3.8 gives the existence part of (1) with $r=2\gamma$. Additional properties of U and Ω_{ρ_m} stated in (1) follow from Lemmas 3.5 and 3.6.

(2) We see that on $\mathcal{U}_*\setminus\overline{\Omega}_{\rho_m}$,

$$\mathcal{L}_\epsilon U=\epsilon^2a^{ij}\partial^2_{ij}U+V^i\partial_iU\le\epsilon^2|A|_{C(\overline{\mathcal{U}}_*)}|U|_{C^2(\overline{\mathcal{U}}_*)}-2\gamma.$$

With ϵ_* being as in the statement, we have $\mathcal{L}_\epsilon U\le-\gamma$ on $\mathcal{U}_*\setminus\overline{\Omega}_{\rho_m}$ for all $\epsilon\in(0,\epsilon_*]$. □

We see from Lemma 3.8 and the proof of Theorem 3.1 that

$$\gamma=\frac{1}{2}\inf_{\Omega_{\rho_M}\setminus\overline{\Omega}_{\rho_m}}\tilde{\zeta}.$$

By the definitions of $\tilde{\zeta}$ and ρ_m (see the proof of Lemma 3.6), if $\mathcal{V}_*$ is a small neighborhood of $\mathcal{J}$, so is Ω_{ρ_m}. Then, ρ_m, and hence, γ is small. Otherwise, Ω_{ρ_m} is not necessarily a small neighborhood, and thus, both ρ_m and γ could be large.

Corollary 3.1. *A local attractor (resp. local repeller) is a strong local attractor (resp. strong local repeller).*

Proof. We prove the result for a local attractor $\mathcal{J}$; the result for a local repeller follows similarly. Let $\mathcal{V}_{**}$ be an open and positively invariant neighborhood of $\mathcal{J}$ such that $\mathcal{V}_{**}\subset\subset B(\mathcal{J})$. Such a set does exist, as we can first choose an open set $\mathcal{N}$ satisfying $\mathcal{J}\subset\mathcal{N}\subset\subset B(\mathcal{J})$, and then define $\mathcal{V}_{**}$ to be $\cup_{t\ge0}\varphi^t(\mathcal{N})$.

Choosing an open neighborhood $\mathcal{V}_*$ of $\mathcal{J}$ satisfying $\mathcal{V}_*\subset\mathcal{V}_{**}$. Then assumptions in Theorem 3.1 are satisfied. It follows that $\mathcal{U}_*$ is an isolating neighborhood (see Definition 2.3) satisfying all desired properties. □

3.2. Measure estimates

We derive measure estimates for stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_\epsilon\}$ in the basin of a maximal attractors or maximal repeller $\mathcal{E}$.

Theorem 3.2. *Let $\mathcal{V}_*$, $\mathcal{V}_{**}$, $\mathcal{U}_*$, U, ρ_m, ρ_M, γ and ϵ_* be as in Theorem 3.1. Then the following hold for any stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$.*

(1) *If $\mathcal{E}$ is a maximal attractor, then*

$$\mu_\epsilon^{\mathcal{U}_*}(\mathcal{U}_*\setminus\Omega_\rho)\le e^{-\frac{\gamma(\rho-\rho_m)}{\epsilon^2H(\rho_m,\rho)}},\quad\forall\rho\in[\rho_m,\rho_M)$$

for all $\epsilon\in(0,\epsilon_*]$, where $\mu_\epsilon^{\mathcal{U}_*}:=\frac{\mu_\epsilon|_{\mathcal{U}_*}}{\mu_\epsilon(\mathcal{U}_*)}$ and

$$H(\rho_1,\rho_2)=|A|_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})}|\nabla U|^2_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})},\quad\forall\rho_m\le\rho_1<\rho_2<\rho_M.$$

(2) *If $\mathcal{E}$ is a maximal repeller, then*

$$\mu_\epsilon^{\mathcal{U}_*}(\Omega_\rho\setminus\Omega^*_{\rho_m})\ge\mu_\epsilon^{\mathcal{U}_*}(\Omega_{\rho_0}\setminus\Omega^*_{\rho_m})e^{\frac{\gamma(\rho-\rho_0)}{\epsilon^2H(\rho_0,\rho)}},\quad\forall\rho_m<\rho_0<\rho<\rho_M$$

for all $\epsilon\in(0,\epsilon_*]$.

Proof. (1) By Theorem 3.1, U is a uniform Lyapunov function in $\mathcal{U}_*$ with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_*]}$ with essential lower bound ρ_m, essential upper bound ρ_M and Lyapunov constant γ, and satisfies $\nabla U\ne0$ on $\overline{\mathcal{U}}_*\setminus\Omega_{\rho_m}$, and $a^{ij}\partial_iU\partial_jU\le H(\rho)$ on $\partial\Omega_\rho$ for any $\rho\in[\rho_m,\rho_M)$, where $H(\rho):=|A|_{C(\partial\Omega_\rho)}|\nabla U|^2_{C(\partial\Omega_\rho)}$. Therefore, conditions of Proposition 2.3(1) are satisfied. As $\mu_\epsilon^{\mathcal{U}_*}$ is a stationary measure corresponding to $\mathcal{L}_\epsilon$ in $\mathcal{U}_*$, we apply Proposition 2.3(1) to obtain

$$\mu_\epsilon^{\mathcal{U}_*}(\mathcal{U}_*\setminus\Omega_\rho)\le e^{-\gamma\int_{\rho_m}^{\rho}\frac{1}{\epsilon^2H(t)}dt}\le e^{-\frac{\gamma(\rho-\rho_m)}{\epsilon^2\sup_{t\in[\rho_m,\rho]}H(t)}},\quad\rho\in[\rho_m,\rho_M)$$

for all $\epsilon\in(0,\epsilon_*]$. The result then follows from the fact that $\sup_{t\in[\rho_m,\rho]}H(t)=H(\rho_m,\rho)$ for all $\rho\in[\rho_m,\rho_M)$.

(2) It follows from Proposition 2.3(2). □

4. Quantitative concentration of stationary measures

Let $\mathcal{E}$ be a maximal attractor (resp. maximal repeller). To apply Theorem 3.2, we need to construct the open sets $\mathcal{V}_*$ and $\mathcal{V}_{**}$ satisfying conditions of Theorem 3.1. We first construct a family of such sets in the basin of attraction (resp. basin of expansion) $B(\mathcal{E})$ of $\mathcal{E}$.

Lemma 4.1. *There is a family of open sets $\mathcal{V}_*^\delta$ and $\mathcal{V}_{**}^\delta$, indexed by $\delta\in(0,\delta_*]$ for some $\delta_*>0$, such that for each $\delta\in(0,\delta_*]$, the sets $\mathcal{V}_*^\delta$ and $\mathcal{V}_{**}^\delta$ satisfy conditions of Theorem 3.1. Moreover, the following properties hold.*

(1) *$\mathcal{E}\subset\mathcal{V}_*^{\delta_1}\subset\mathcal{V}_*^{\delta_2}$ if $0<\delta_1<\delta_2\le\delta_*$, and $\bigcap_{\delta\in(0,\delta_*]}\mathcal{V}_*^\delta=\mathcal{E}$;*

(2) *$\mathcal{V}_{**}^{\delta_2}\subset\mathcal{V}_{**}^{\delta_1}\subset\subset B(\mathcal{E})$ if $0<\delta_1<\delta_2\le\delta_*$, and $\bigcup_{\delta\in(0,\delta_*]}\mathcal{V}_{**}^\delta=B(\mathcal{E})$.*

Proof. For each $\delta>0$, define

$$\tilde{\mathcal{V}}_{**}^\delta:=\left\{x\in B(\mathcal{E})\cap B_{\frac{1}{\delta}}:d(x,\partial(B(\mathcal{E})\cap B_{\frac{1}{\delta}}))>\delta\right\},$$

$$\mathcal{V}_{**}^\delta:=\begin{cases}\bigcup_{t\ge0}\varphi^t(\tilde{\mathcal{V}}_{**}^\delta), & \mathcal{E}\text{ maximal attractor},\\ \bigcup_{t\le0}\varphi^t(\tilde{\mathcal{V}}_{**}^\delta), & \mathcal{E}\text{ maximal repeller},\end{cases}$$

where B_r is an open ball with radius r and center $0\in\mathbb{R}^n$. We fix a $\delta_*>0$ sufficiently small such that $\mathcal{E}_{\delta_*}\subset\subset\mathcal{V}_{**}^{\delta_*}$, where $\mathcal{E}_{\delta_*}$ is the δ_*-neighborhood of $\mathcal{E}_{\delta^*}$. Then, arguing as in the proof of Lemma 3.2, it is not hard to see that $\mathcal{V}_{**}^\delta$ satisfies desired properties.

We can simply take $\mathcal{V}_*^\delta$ to be the δ-neighborhood of $\mathcal{E}$. □

For each $\delta\in(0,\delta_*]$, let $\mathcal{U}_*^\delta$, U^δ, ρ_m^δ, ρ_M^δ, γ^δ and ϵ_*^δ be as in Theorem 3.1 associated to the sets $\mathcal{V}_*^\delta$ and $\mathcal{V}_{**}^\delta$ constructed in Lemma 4.1. Let

$$\Omega_\rho^\delta:=\left\{x\in\mathcal{U}_*^\delta:U^\delta(x)<\rho\right\},\quad\forall\rho\in(\rho_m^\delta,\rho_M^\delta],$$
$$(\Omega_\rho^\delta)^*:=\left\{x\in\mathcal{U}_*^\delta:U^\delta(x)\le\rho\right\},\quad\forall\rho\in[\rho_m^\delta,\rho_M^\delta).$$

Recall that

$$\mathcal{E}\subset\Omega^\delta_{\rho_m^\delta}\subset\subset\mathcal{V}_*^\delta\subset\mathcal{V}_{**}^\delta\subset\subset\mathcal{U}_*^\delta=\Omega^\delta_{\rho_M^\delta}\subset\subset B(\mathcal{E}).$$

The next lemma follows directly from Theorem 3.2.

Lemma 4.2. *Let $\{\mu_\epsilon\}$ be the stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Then the following hold for each $\delta\in(0,\delta_*]$.*

(1) *If $\mathcal{E}$ is a maximal attractor, then*

$$\mu_\epsilon(\mathcal{U}_*^\delta\setminus\Omega_\rho^\delta)\le e^{-\frac{\gamma^\delta(\rho-\rho_m^\delta)}{\epsilon^2H^\delta(\rho_m^\delta,\rho)}},\quad\forall\rho\in[\rho_m^\delta,\rho_M^\delta)$$

for all $\epsilon\in(0,\epsilon_*^\delta]$, where

$$H^\delta(\rho_1,\rho_2)=|A|_{C(\overline{\Omega^\delta_{\rho_2}}\setminus\Omega^\delta_{\rho_1})}|\nabla U^\delta|^2_{C(\overline{\Omega^\delta_{\rho_2}}\setminus\Omega^\delta_{\rho_1})}$$

for all $\rho_m^\delta\le\rho_1<\rho_2<\rho_M^\delta$.

(2) *If $\mathcal{E}$ is a maximal repeller, then*

$$\mu_\epsilon(\Omega^\delta_{\rho_0} \setminus (\Omega^\delta_{\rho^\delta_m})^*) \le e^{-\frac{\gamma^\delta(\rho-\rho_0)}{\epsilon^2 H^\delta(\rho_0,\rho)}}, \quad \forall \rho^\delta_m < \rho_0 < \rho < \rho^\delta_M \tag{4.1}$$

for all $\epsilon \in (0, \epsilon^\delta_*]$.

We now establish concentration estimates in $B(\mathcal{E}) \setminus \mathcal{E}$ for stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Part (1) in Theorem A follows from the following result.

Theorem 4.1. *Let $\{\mu_\epsilon\}$ be the family of stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Then, for each $\delta \in (0, \delta^*]$, there exists a constant $c^\delta > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{V}^\delta_{**} \setminus \mathcal{V}^\delta_*) \le e^{-\frac{c^\delta}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon^\delta_*]. \tag{4.2}$$

In particular, the following hold.

(1) *For any Borel set $\mathcal{W} \subset\subset B(\mathcal{E})\setminus\mathcal{E}$, there are constants $w, \epsilon_\mathcal{W} > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{W}) \le e^{-\frac{w}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\mathcal{W}].$$

(2) *Any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies $\mu(B(\mathcal{E})\setminus\mathcal{E}) = 0$. In particular, if (1.1) is dissipative and $\mathcal{E}$ is the global attractor, then $B(\mathcal{E}) = \mathcal{U}$ and μ is concentrated on $\mathcal{E}$.*

Proof. Fix $\delta \in (0, \delta_*]$. First suppose $\mathcal{E}$ is a maximal attractor. Since $\Omega^\delta_{\rho^\delta_m} \subset\subset \mathcal{V}^\delta_*$, we can find some $\rho_0 \in (\rho^\delta_m, \rho^\delta_M)$ such that $\Omega^\delta_{\rho^\delta_m} \subset\subset \Omega^\delta_{\rho_0} \subset\subset \mathcal{V}^\delta_*$, which together with Lemma 4.2(1) lead to

$$\mu_\epsilon(\mathcal{V}^\delta_{**} \setminus \mathcal{V}^\delta_*) \le \mu_\epsilon(\mathcal{U}^\delta_* \setminus \Omega^\delta_{\rho_0}) \le e^{-\frac{\gamma^\delta(\rho_0-\rho^\delta_m)}{\epsilon^2 H^\delta(\rho^\delta_m,\rho_0)}}, \quad \forall \epsilon \in (0, \epsilon^\delta_*].$$

Thus, (4.2) holds by setting $c^\delta := \frac{\gamma^\delta(\rho_0-\rho^\delta_m)}{H^\delta(\rho^\delta_m,\rho_0)}$.

Now, suppose $\mathcal{E}$ is a maximal repeller. Since $\mathcal{V}^\delta_{**} \subset\subset \mathcal{U}^\delta_* = \Omega^\delta_{\rho^\delta_M}$, we can find $\rho^\delta_0 \in (\rho^\delta_m, \rho^\delta_M)$ such that $\mathcal{V}^\delta_{**} \subset\subset \Omega^\delta_{\rho^\delta_0} \subset\subset \mathcal{U}^\delta_*$. Setting $\rho_0 = \rho^\delta_0$ and $\rho = \frac{\rho^\delta_0+\rho^\delta_M}{2}$ in (4.1), we have

$$\begin{aligned}\mu_\epsilon(\mathcal{V}^\delta_{**} \setminus \mathcal{V}^\delta_*) &\le \mu_\epsilon(\mathcal{V}^\delta_{**} \setminus (\Omega^\delta_{\rho^\delta_m})^*) \le \mu_\epsilon(\Omega^\delta_{\rho^\delta_0} \setminus (\Omega^\delta_{\rho^\delta_m})^*) \\ &\le e^{-\frac{\gamma^\delta(\rho^\delta_M-\rho^\delta_0)}{2\epsilon^2 H^\delta(\rho^\delta_0, \frac{\rho^\delta_0+\rho^\delta_M}{2})}}, \quad \forall \epsilon \in (0, \epsilon^\delta_*].\end{aligned}$$

Thus, (4.2) also holds in this case by setting $c^\delta = \frac{\gamma^\delta(\rho^\delta_M-\rho^\delta_0)}{2H^\delta(\rho^\delta_0, \frac{\rho^\delta_0+\rho^\delta_M}{2})}$.

(1) Lemma 4.1 gives $\mathcal{V}^{\delta_2}_{**} \setminus \mathcal{V}^{\delta_2}_* \subset \mathcal{V}^{\delta_1}_{**} \setminus \mathcal{V}^{\delta_1}_*$ if $0 < \delta_1 < \delta_2 \le \delta_*$ and $\cup_{\delta\in(0,\delta_*]}\left(\mathcal{V}^\delta_{**} \setminus \mathcal{V}^\delta_*\right) = B(\mathcal{E}) \setminus \mathcal{E}$. Therefore, for each $\mathcal{W} \subset\subset B(\mathcal{E}) \setminus \mathcal{E}$, there is a $\delta_\mathcal{W} \in (0, \delta_*]$ such that $\mathcal{W} \subset \mathcal{V}^{\delta_\mathcal{W}}_{**} \setminus \mathcal{V}^{\delta_\mathcal{W}}_*$. The result follows.

(2) It is a simple consequence of (1). □

We end this section by establishing pointwise estimates. Part (2) in Theorem A follows from the following result.

Theorem 4.2. *Suppose ∂a^{ij} is locally bounded. Let $\{\mu_\epsilon\}$, with densities $\{u_\epsilon\}$, be the family of stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$. Then, for any $\mathcal{W} \subset\subset B(\mathcal{E}) \setminus \mathcal{E}$, there are constants $w, \epsilon_\mathcal{W} > 0$ independent of ϵ such that*

$$u_\epsilon(x) \le e^{-\frac{w}{\epsilon^2}}, \quad \forall x \in \mathcal{W},\ \epsilon \in (0, \epsilon_\mathcal{W}]. \tag{4.3}$$

Proof. Let $r_* > 0$ be such that $\Omega := \mathcal{W}_{r_*} \subset B(\mathcal{E})\setminus\mathcal{E}$, where $\mathcal{W}_{r_*}$ is the r_*-neighborhood of $\mathcal{W}$. By Theorem 4.1(1), there are constants $c_\Omega, \epsilon_\Omega > 0$ such that

$$\mu_\epsilon(\Omega) \le e^{-\frac{c_\Omega}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\Omega]. \tag{4.4}$$

Suppose (4.3) is not true. Then there are sequences $\{\epsilon_k\}$ and $\{x_k\} \subset \mathcal{W}$ with $\epsilon_k \to 0$ as $k \to \infty$, and $x_k \to x_*$ as $k \to \infty$ for some $x_* \in \overline{\mathcal{W}}$ such that

$$u_{\epsilon_k}(x_k) > e^{-\frac{c_\Omega}{2\epsilon_k^2}}, \quad \forall k.$$

Applying Proposition 2.4 in Ω, we have

$$\sup_{B_R(x)} u_\epsilon \le C_0^{\frac{\Lambda(\Omega)}{\lambda(\Omega)}+\frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}} \inf_{B_R(x)} u_\epsilon, \quad \forall x \in \mathcal{W},\ R < \frac{r_*}{4},\ \epsilon \in (0, \epsilon_*],$$

where $C_0 = C_0(\Omega) \ge 1$ and $C_1(\epsilon, \Omega) = 2n\left[\epsilon^2 \sup_\Omega |\partial A| + \sup_\Omega |V|\right]$. Setting $x = x_k$, $R = \epsilon_k^2$ and $\epsilon = \epsilon_k$ in the above inequality, we have

$$\sup_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k} \le C_0^{\frac{\Lambda(\Omega)}{\lambda(\Omega)}+\frac{C_1(\epsilon_k,\Omega)}{\lambda(\Omega)}} \inf_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k}, \quad k \gg 1.$$

Since ∂a^{ij} is locally bounded, there is a constant $C > 0$ independent of k, such that

$$\sup_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k} \le C \inf_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k}, \quad k \gg 1.$$

It follows that

$$\begin{aligned}\mu_{\epsilon_k}(B_{\epsilon_k^2}(x_k)) &= \int_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k}(x)dx \ge \frac{|B_{\epsilon_k^2}(x_k)|}{C} \sup_{B_{\epsilon_k^2}(x_k)} u_{\epsilon_k} \\ &\ge C_1 \epsilon_k^{2n} e^{-\frac{c_\Omega}{2\epsilon_k^2}}, \quad \forall k \gg 1,\end{aligned}$$

where C_1 is independent of k.

Since $B_{\epsilon_k^2}(x_k) \subset \Omega$ for all $k \gg 1$, we have

$$\mu_{\epsilon_k}(B_{\epsilon_k^2}(x_k)) \le \mu_{\epsilon_k}(\Omega) \le e^{-\frac{c_\Omega}{\epsilon_k^2}}, \quad \forall k \gg 1.$$

As $C_1\epsilon_k^{2n} e^{-\frac{c_\Omega}{2\epsilon_k^2}} > e^{-\frac{c_\Omega}{\epsilon_k^2}}$ for $k \gg 1$, we deduce a contradiction. This completes the proof. □

5. Quantitative stabilization/de-stabilization of local attractors/repellers

Let $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) be a local attractor (resp. local repeller) of (1.1). The purpose of this section is to quantify the stochastic stabilization of $\mathcal{J}_0$ (resp. stochastic de-stabilization of $\mathcal{R}_0$). More precisely, we characterize multiplicative noise coefficients G or $A = \frac{GG^\top}{2}$ so that stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$ tend to concentrate on $\mathcal{J}_0$ (resp. off $B(\mathcal{R}_0)$) as $\epsilon \to 0$.

We prove the following two theorems from which parts (3), (4) of Theorem A follow.

Theorem 5.1. *Suppose (1.1) is dissipative and let $\mathcal{J}_0$ be a local attractor of (1.1). Then, there exists a $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ such that the following holds for the corresponding stationary measures $\{\mu_\epsilon\}$: For any Borel set $\mathcal{U}_0 \subset\subset \mathcal{U} \setminus \mathcal{J}_0$, there are constants $u_0, \epsilon_0 > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{U}_0) \le e^{-\frac{u_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_0].$$

Consequently, any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies $\mu(\mathcal{U} \setminus \mathcal{J}_0) = 0$, that is, μ is concentrated on the local attractor $\mathcal{J}_0$.

Theorem 5.2. *Let $\mathcal{R}_0$ be a local repeller. Then, there exists a $G \in W^{1,p}_{loc}(\mathcal{U}, \mathbb{R}^{n\times m})$ such that the following holds for the corresponding stationary measures $\{\mu_\epsilon\}$: For any given Borel set $\mathcal{U}_0 \subset\subset B(\mathcal{R}_0)$, there are constants $u_0, \epsilon_0 > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{U}_0) \le e^{-\frac{u_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_0].$$

Consequently, any limit measure μ of $\{\mu_\epsilon\}$ as $\epsilon \to 0$ satisfies $\mu(B(\mathcal{R}_0)) = 0$, that is, μ is concentrated away from the local repeller $\mathcal{R}_0$.

For each $x \in \mathcal{U}$, denote

$$\lambda(x) = \inf_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\xi^\top A(x)\xi}{|\xi|^2} \quad \text{and} \quad \Lambda(x) = \sqrt{\sum_{i,j} |a^{ij}(x)|^2}.$$

For any bounded open set $\Omega \subset \mathcal{U}$, we let $\lambda(\Omega) = \inf_\Omega \lambda$ and $\Lambda(\Omega) = \sup_\Omega \Lambda$.

Proof of Theorem 5.1. The proof consists of four steps. In **Step 1**, **Step 2** and **Step 3** below, we establish estimates for $\{\mu_\epsilon\}$ on $\mathcal{U} \setminus \mathcal{J}_0$ for any given G or $A = \frac{GG^\top}{2} \in W^{1,p}_{loc}(\mathcal{U}, GL(n,\mathbb{R}))$. Based on these estimates, we are able to characterize a special G or A in **Step 4** to satisfy desired properties stated in the theorem. Let $\mathcal{J}$ be the global attractor.

Step 1 Applying Theorem 4.1 to $\mathcal{J}$, we find that for any Borel set $\mathcal{W} \subset\subset B(\mathcal{J}) \setminus \mathcal{J}$, there are constants $w, \epsilon_{\mathcal{W}} > 0$ independent of ϵ such that

$$\mu_\epsilon(\mathcal{W}) \le e^{-\frac{w}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_{\mathcal{W}}]. \tag{5.1}$$

Similarly, applying Theorem 4.1 to $\mathcal{J}_0$, we find that for any Borel set $\mathcal{W}_0 \subset\subset B(\mathcal{J}_0) \setminus \mathcal{J}_0$, there are constants $w_0, \epsilon_{\mathcal{W}_0} > 0$ independent of ϵ such that

$$\mu_\epsilon(\mathcal{W}_0) \le e^{-\frac{w_0}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_{\mathcal{W}_0}]. \tag{5.2}$$

Step 2 We derive appropriate estimates for $\{\mu_\epsilon\}$ on a neighborhood of $\mathcal{J} \setminus B(\mathcal{J}_0)$. Firstly, for some fixed $\delta \in (0, \delta_*]$, we apply Lemmas 4.1 and 4.2(1) to $\mathcal{J}_0$ to obtain the sets

$$\mathcal{J}_0 \subset \Omega_{\rho_m} \subset\subset \mathcal{V}_* \subset \mathcal{V}_{**} \subset\subset \mathcal{U}_* = \Omega_{\rho_M} \subset\subset B(\mathcal{J}_0),$$

and the estimates

$$\mu_\epsilon(\mathcal{U}_* \setminus \Omega_\rho) \le e^{-\frac{\gamma(\rho-\rho_m)}{\epsilon^2 H(\rho_m,\rho)}}, \quad \forall \rho \in [\rho_m, \rho_M)$$

for all $\epsilon \in (0, \epsilon_*]$, where $H(\rho_1, \rho_2) = |A|_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})} |\nabla U|^2_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})}$ for all $\rho_m \le \rho_1 < \rho_2 < \rho_M$. Fix some $\rho_0 \in (\rho_m, \rho_M)$ such that $\Omega_{\rho_m} \subset\subset \Omega_{\rho_0} \subset\subset \mathcal{V}_{**}$ and

$$\mu_\epsilon(\mathcal{U}_* \setminus \Omega_{\rho_0}) \le e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}}, \quad \forall \epsilon \in (0, \epsilon_*]. \tag{5.3}$$

Next, we fix an open, bounded and connected neighborhood $\mathcal{N}$ of $\mathcal{J}$, and set $\tilde{\mathcal{N}} := \mathcal{N} \cup \mathcal{V}_{**}$, which is open, bounded and connected. The connectedness follows from the facts that both $\mathcal{N}$ and $\mathcal{V}_{**}$ are connected and their intersection is nonempty. Clearly, $\mathcal{J} \setminus B(\mathcal{J}_0) \subset \tilde{\mathcal{N}} \setminus \mathcal{V}_{**}$. We derive estimates for $\{\mu_\epsilon\}$ on $\tilde{\mathcal{N}} \setminus \mathcal{V}_{**}$. As $\Omega_{\rho_0} \subset\subset \mathcal{V}_{**}$, there is a neighborhood Ω of $\tilde{\mathcal{N}} \setminus \mathcal{V}_{**}$ such that

$$\overline{\Omega}_{\rho_0} \cap \overline{\Omega} = \emptyset. \tag{5.4}$$

Due to the compactness of $\overline{\tilde{\mathcal{N}}} \setminus \mathcal{V}_{**}$, we can find N small balls $\{B_R(x_k)\}_{k=1}^N$ with radius $R > 0$ such that

$$x_k \in \tilde{\mathcal{N}} \setminus \mathcal{V}_{**} \subset \overline{\tilde{\mathcal{N}}} \setminus \mathcal{V}_{**} \subset \bigcup_{k=1}^N B_R(x_k) \subset \bigcup_{k=1}^N B_{4R}(x_k) \subset \Omega.$$

Denoted by, D_j, $j = 1, 2, \ldots, J$, where J is an integer between 1 and N, the components of $\cup_{k=1}^N B_R(x_k)$. Since $\Omega_{\rho_0} \subset\subset \mathcal{V}_{**} \subset\subset \mathcal{U}_*$, we see $D_j \cap (\mathcal{U}_* \setminus \overline{\Omega}_{\rho_0}) \neq \emptyset$, and hence, $|D_j \cap (\mathcal{U}_* \setminus \Omega_{\rho_0})| = |D_j \cap (\mathcal{U}_* \setminus \overline{\Omega}_{\rho_0})| > 0$, for each j. In fact, if there is j_0 such that $D_{j_0} \cap (\mathcal{U}_* \setminus \overline{\Omega}_{\rho_0}) = \emptyset$, then D_{j_0} stays away from the boundary of $\mathcal{V}_{**}$, and hence, the set $(\tilde{\mathcal{N}} \setminus \mathcal{V}_{**}) \cap D_{j_0}$ contains no portion of $\partial\mathcal{V}_{**}$. Now, let $x \in (\tilde{\mathcal{N}} \setminus \mathcal{V}_{**}) \cap D_{j_0}$. Since $\tilde{\mathcal{N}}$ is connected, there are point $y \in \partial\mathcal{V}_{**}$ and curve $c_{xy} \subset \tilde{\mathcal{N}} \setminus \mathcal{V}_{**}$ connecting x and y. Note that there must be a point $z \in c_{xy} \cap \partial D_{j_0}$. However, $z \notin \cup_j D_j$, which implies $z \notin \tilde{\mathcal{N}} \setminus \mathcal{V}_{**}$. This leads to a contradiction.

Let $u_\epsilon \in W^{1,p}_{loc}(\mathcal{U})$ be the probability density of μ_ϵ. Applying Proposition 2.4 with $b^i = \epsilon^2 \partial_j a^{ij} - V^i$ in each component of Ω containing some D_j, we have

$$\sup_{B_R(x_k)} u_\epsilon \le C_0^{\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}} \inf_{B_R(x_k)} u_\epsilon, \quad k = 1, 2, \ldots, N,$$

where $C_0 = C_0(\Omega) \ge 1$ and $C_1(\epsilon, \Omega) = 2n\left[\epsilon^2 \sup_\Omega |\partial A| + \sup_\Omega |V|\right]$. It follows that for each j,

$$\sup_{D_j} u_\epsilon \le C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}\right]} \inf_{D_j} u_\epsilon.$$

For each j, using (5.3), we have

$$\inf_{D_j} u_\epsilon \le \inf_{D_j\cap(\mathcal{U}_*\setminus\Omega_{\rho_0})} u_\alpha \le \frac{\mu_\alpha(\mathcal{U}_* \setminus \Omega_{\rho_0})}{|D_j \cap (\mathcal{U}_* \setminus \Omega_{\rho_0})|} \le \frac{e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}}}{|D_j \cap (\mathcal{U}_* \setminus \Omega_{\rho_0})|},$$
$$\forall \epsilon \in (0, \epsilon_*].$$

Hence, for any $\epsilon \in (0, \epsilon_*]$,

$$\mu_\epsilon(\tilde{\mathcal{N}} \setminus \mathcal{V}_{**}) \le \sum_{j=1}^J \mu_\epsilon(D_j) \le \sum_{j=1}^J |D_j| \sup_{D_j} u_\epsilon \le C_2 C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}\right]} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}}, \tag{5.5}$$

where $C_2 = \sum_{j=1}^J \frac{|D_j|}{|D_j\cap(\mathcal{U}_*\setminus\Omega_{\rho_0})|}$.

Step 3 For any Borel set $\mathcal{U}_0 \subset\subset \mathcal{U} \setminus \mathcal{J}_0$, we can find connected open sets $\mathcal{W} \subset\subset \mathcal{U} \setminus \mathcal{J}$ and $\mathcal{W}_0 \subset\subset B(\mathcal{J}_0) \setminus \mathcal{J}_0$ such that $\mathcal{U}_0 \subset \mathcal{W} \cup \mathcal{W}_0 \cup (\tilde{\mathcal{N}} \setminus \mathcal{V}_{**})$. Applying (5.1), (5.2) and (5.5), we have

$$\mu_\epsilon(\mathcal{U}_0) \le e^{-\frac{w}{\epsilon^2}} + e^{-\frac{w_0}{\epsilon^2}} + C_2 C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}\right]} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}},$$
$$\forall \epsilon \in (0, \epsilon_0], \tag{5.6}$$

where $\epsilon_0 = \min\{\epsilon_{\mathcal{W}}, \epsilon_{\mathcal{W}_0}, \epsilon_*, 1\}$.

Step 4 We note that the term $C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}\right]}$ depends on A in Ω, while the term $e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}}$ depends on A in $\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m}$. Moreover, Ω and $\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m}$ are disjoint due to (5.4).

We now specify particular A's to finish the proof. We take $A : \mathcal{U} \to \mathbb{R}^{n\times n}$ to be C^1 and pointwise positive definite. To control $C_1(\epsilon, \Omega)$, we require A to satisfy

$$\sup_\Omega |\partial A| \le 1 \tag{5.7}$$

so that $C_1(\epsilon, \Omega) \le 2n(1 + \sup_\Omega |V|)$ for all $0 < \epsilon \le 1$.

It remains to control the term

$$C_0^{\frac{NC_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}} = e^{\frac{NC_1(\epsilon,\Omega)R\ln C_0}{\epsilon^2\lambda(\Omega)}} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}}.$$

The equivalence of matrix norms ensures the existence of a constant $C_3 > 0$ such that

$$H(\rho_m, \rho_0) = |A|_{C(\overline{\Omega}_{\rho_0}\setminus\Omega_{\rho_m})} |\nabla U|^2_{C(\overline{\Omega}_{\rho_0}\setminus\Omega_{\rho_m})} \le C_3 \Lambda(\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m}) |\nabla U|^2_{C(\overline{\Omega}_{\rho_0}\setminus\Omega_{\rho_m})}.$$

Setting $C_4 = NC_1(\epsilon, \Omega)R\ln C_0 \le NR\ln C_0$ and $C_5 = \frac{\gamma(\rho_0-\rho_m)}{C_3|\nabla U|^2_{C(\overline{\Omega}_{\rho_0}\setminus\Omega_{\rho_m})}}$, we have

$$C_0^{\frac{NC_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}} \le \exp\left\{\left[\frac{C_4}{\lambda(\Omega)} - \frac{C_5}{\Lambda(\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m})}\right]\frac{1}{\epsilon^2}\right\}.$$

Now, we further require A to satisfy

$$\frac{C_4}{\lambda(\Omega)} - \frac{C_5}{\Lambda(\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m})} \le -\frac{1}{\lambda(\Omega)} \quad \text{or} \quad \frac{\lambda(\Omega)}{\Lambda(\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m})} \ge \frac{C_4+1}{C_5}. \tag{5.8}$$

Note that conditions (5.7) and (5.8) can be satisfied simultaneously as $\overline{\Omega}$ and $\overline{\Omega}_{\rho_0} \setminus \Omega_{\rho_m}$ are disjoint. Thus, $C_0^{\frac{NC_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}} e^{-\frac{\gamma(\rho_0-\rho_m)}{\epsilon^2 H(\rho_m,\rho_0)}} \le e^{-\frac{1}{\epsilon^2\lambda(\Omega)}}$.

To finish the proof, we set $C_6 := C_2 C_0^{N\frac{\Lambda(\Omega)}{\lambda(\Omega)}}$. Then (5.6) reads

$$\mu_\epsilon(\mathcal{U}_0) \le e^{-\frac{w}{\epsilon^2}} + e^{-\frac{w_0}{\epsilon^2}} + C_6 e^{-\frac{1}{\epsilon^2\lambda(\Omega)}}, \quad \forall \epsilon \in (0, \epsilon_0],$$

This completes the proof. □

Proof of Theorem 5.2. The proof also consists of four steps similar to that of Theorem 5.1.

Step 1 Applying Theorem 4.1 to $\mathcal{R}_0$, we find that for any Borel set $\mathcal{W} \subset\subset B(\mathcal{R}_0) \setminus \mathcal{R}_0$, there are constants $w > 0$ and $\epsilon_{\mathcal{W}} > 0$ such that

$$\mu_\epsilon(\mathcal{W}) \le e^{-\frac{w}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_{\mathcal{W}}]. \tag{5.9}$$

Step 2 To estimates $\{\mu_\epsilon\}$ in a neighborhood of $\mathcal{R}_0$, we first apply Lemmas 4.1 and 4.2(2) to $\mathcal{R}_0$ with respect to some fixed $\delta \in (0, \delta_*]$ to obtain the estimates

$$\mu_\epsilon(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m}) \le e^{-\frac{\gamma(\rho-\rho_0)}{\epsilon^2 H(\rho_0,\rho)}}, \quad \forall \rho_m < \rho_0 < \rho < \rho_M$$

for all $\epsilon \in (0, \epsilon_*]$, where $H(\rho_1, \rho_2) = |A|_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})} |\nabla U|^2_{C(\overline{\Omega}_{\rho_2}\setminus\Omega_{\rho_1})}$ for all $\rho_m \le \rho_1 < \rho_2 < \rho_M$. Fix some $\rho_0 \in (\rho_m, \rho_M)$ and set $\rho = \frac{\rho_0+\rho_M}{2}$. Then $\Omega_{\rho_m} \subset\subset \Omega_{\rho_0}$ and

$$\mu_\epsilon(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m}) \le e^{-\frac{\gamma(\rho_M-\rho_0)}{2\epsilon^2 H(\rho_0, \frac{\rho_0+\rho_M}{2})}}, \quad \forall \epsilon \in (0, \epsilon_*]. \tag{5.10}$$

Note that Ω_{ρ_m} is open, bounded and connected, and $\mathcal{R}_0 \subset \overline{\Omega}_{\rho_m}$. Let $\Omega \subset \Omega_{\rho_0}$ be an open and connected neighborhood of $\overline{\Omega}_{\rho_m}$. Then

$$\overline{\Omega} \cap (\mathcal{U} \setminus \Omega_{\rho_0}) = \emptyset. \tag{5.11}$$

Let $\{B_R(x_k)\}_{k=1}^N$ be N small balls with radius $R > 0$ such that

$$x_k \in \Omega_{\rho_m} \subset \overline{\Omega}_{\rho_m} \subset \bigcup_{k=1}^N B_R(x_k) \subset \bigcup_{k=1}^N B_{4R}(x_k) \subset \Omega.$$

Clearly, $D := \cup_{k=1}^N B_R(x_k)$ is connected and $|D \cap (\Omega_{\rho_0} \setminus \Omega^*_{\rho_m})| > 0$.

Let $u_\epsilon \in W^{1,p}_{loc}(\mathcal{U})$ be the probability density of μ_ϵ. Arguing as in **Step 2** in the proof of Theorem 5.1 and using (5.11), we arrive at

$$\mu_\epsilon(\Omega_{\rho_m}) \le \mu_\epsilon(D) \le C_2 C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon,\Omega)R}{\epsilon^2\lambda(\Omega)}\right]} e^{-\frac{\gamma(\rho_M-\rho_0)}{2\epsilon^2 H(\rho_0, \frac{\rho_0+\rho_M}{2})}}, \quad \forall \epsilon \in (0, \epsilon_*], \tag{5.12}$$

where $C_0 = C_0(\Omega) \ge 1$, $C_1(\epsilon, \Omega) = 2n\left[\epsilon^2 \sup_\Omega |\partial A| + \sup_\Omega |V|\right]$ and $C_2 = \frac{|D|}{|D\cap(\Omega_{\rho_0}\setminus\Omega^*_{\rho_m})|}$.

Step 3 For any Borel set $\mathcal{U}_0 \subset\subset B(\mathcal{R}_0)$, we can find connected open sets $\mathcal{W} \subset\subset B(\mathcal{R}_0) \setminus \mathcal{R}_0$ such that $\mathcal{U}_0 \subset \mathcal{W} \cup \Omega_{\rho_m}$. Applying (5.9) and (5.12), we have

$$\mu_\epsilon(\mathcal{U}_0) \le e^{-\frac{w}{\epsilon^2}} + C_2 C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon)R}{\epsilon^2\lambda(\Omega)}\right]} e^{-\frac{\gamma(\rho_M-\rho_0)}{2\epsilon^2 H(\rho_0, \frac{\rho_0+\rho_M}{2})}}, \quad \forall \epsilon \in (0, \epsilon_0],$$

where $\epsilon_0 = \min\{\epsilon_{\mathcal{W}}, \epsilon_*, 1\}$.

Step 4 Note that $C_0^{N\left[\frac{\Lambda(\Omega)}{\lambda(\Omega)} + \frac{C_1(\epsilon)R}{\epsilon^2\lambda(\Omega)}\right]}$ depends on A in Ω, while $e^{-\frac{\gamma(\rho_M-\rho_0)}{2\epsilon^2 H(\rho_0, \frac{\rho_0+\rho_M}{2})}}$ depends on A in $\overline{\Omega}_{\frac{\rho_0+\rho_M}{2}} \setminus \Omega_{\rho_0}$. Moreover, Ω and $\overline{\Omega}_{\frac{\rho_0+\rho_M}{2}} \setminus \Omega_{\rho_0}$ are disjoint due to (5.11). We can now characterize special A's as in **Step 4** of the proof of Theorem 5.1 to satisfy the desired properties stated in the theorem. □

6. Entropy estimates and entropy-dimension inequalities

Throughout this section, we assume that there is a uniform Lyapunov function in $\mathcal{U}$, with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant γ, with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ for some $0 < \epsilon_0 \ll 1$. Let $\mathcal{J}$ be the global attractor of (1.1), and $\{\mu_\epsilon\}$, with densities $\{u_\epsilon\}$, be the stationary measures corresponding to $\{\mathcal{L}_\epsilon\}$ in $\mathcal{U}$.

The purpose of this section is to apply measure estimates established in Section 4 to study the differential entropy defined in (1.7). In particular, we prove Theorems B and C. Before doing so, we prove several lemmas.

Lemma 6.1. *Let $\Omega \subset \mathcal{U}$ be such that $|\Omega| \in (0, \infty)$. Then any probability density function $v : \mathcal{U} \to [0, \infty)$ satisfies*

$$\int_\Omega v \ln v dx \ge \int_\Omega v dx \left[\ln \int_\Omega v dx - \ln |\Omega|\right].$$

Proof. Set $\delta := \int_\Omega v dx$ and define the functional $F(u) = \int_\Omega u \ln u dx$. Minimizing F over all probability density functions u on $\mathcal{U}$ subject to $\int_\Omega u dx = \delta$, we find from the Lagrange multiplier that F is minimized at any u_0 satisfying $u_0 \equiv \frac{\delta}{|\Omega|}$ on Ω. Hence, $\int_\Omega v \ln v dx \ge F(u_0) = \delta \ln \frac{\delta}{|\Omega|}$. This completes the proof. □

Lemma 6.2. *For any $\mathcal{W} \subset\subset \mathcal{U} \setminus \mathcal{J}$, there are constants $w, \epsilon_* > 0$ independent of ϵ such that*

$$-\int_{\mathcal{W}} u_\epsilon \ln u_\epsilon dx \le \left(\frac{w}{\epsilon^2} + \ln |\mathcal{W}|\right) e^{-\frac{w}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_*].$$

Proof. Applying Lemma 6.1, we have $\int_{\mathcal{W}} u_\epsilon \ln u_\epsilon dx \ge |\mathcal{W}| \frac{\mu_\epsilon(\mathcal{W})}{|\mathcal{W}|} \ln \frac{\mu_\epsilon(\mathcal{W})}{|\mathcal{W}|}$. By Theorem A(1), there are constants $w, \epsilon_* > 0$ independent of ϵ such that

$$\mu_\epsilon(\mathcal{W}) \le e^{-\frac{w}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_*].$$

Making ϵ_* smaller if necessary, we may assume $\frac{1}{|\mathcal{W}|} e^{-\frac{w}{\epsilon_*^2}} < \frac{1}{e}$. Since the function $x \mapsto x \ln x$ is decreasing on $(0, \frac{1}{e})$, we have

$$\frac{\mu_\epsilon(\mathcal{W})}{|\mathcal{W}|} \ln \frac{\mu_\epsilon(\mathcal{W})}{|\mathcal{W}|} \ge \frac{1}{|\mathcal{W}|} e^{-\frac{w}{\epsilon^2}} \left(-\frac{w}{\epsilon^2} - \ln |\mathcal{W}|\right).$$

The result follows readily. □

The following result gives measure estimates on exterior domains.

Lemma 6.3. *For any open set $\mathcal{N}$ such that $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$, there are constants $c, \epsilon_* > 0$ independent of ϵ such that*

$$\mu_\epsilon(\mathcal{U} \setminus \mathcal{N}) \le e^{-\frac{c}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_*].$$

Proof. In the presence of the uniform Lyapunov function, we apply Proposition 2.3(1) to find in particular $\mu_\epsilon(\mathcal{U} \setminus \Omega_{\rho_0}) \le e^{-\frac{r}{\epsilon^2}\int_{\rho_m}^{\rho_0} \frac{1}{H(t)} dt}$ for some fixed $\rho_0 \in (\rho_m, \rho_M)$ and all sufficiently small ϵ.

Let $\mathcal{W} \subset\subset \mathcal{U} \setminus \mathcal{J}$ be an open set such that $\mathcal{U} \setminus \mathcal{N} \subset \mathcal{W} \cup (\mathcal{U} \setminus \Omega_{\rho_0})$. The lemma then follows from a simple application of Theorem A(1) to $\mathcal{W}$. □

We are ready to prove Theorem B.

Proof of Theorem B. (1) We first claim that for each $\sigma \in (0,1)$,

$h_\sigma(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.$

In fact, if there is a sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that $\liminf_{n\to\infty} h_\sigma(\epsilon_n) > h$ for some $h > 0$, then $\mu_{\epsilon_n}(\mathcal{U}\setminus\mathcal{J}_h) \geq \mu_{\epsilon_n}(\mathcal{U}\setminus\mathcal{J}_{h_\sigma(\epsilon_n)}) = 1-\sigma$ for all $n \gg 1$. However, Lemma 6.3 says that $\mu_{\epsilon_n}(\mathcal{U}\setminus\mathcal{J}_h) \to 0$ as $n \to \infty$, which leads to a contradiction.

By Lemma 6.1, we have

$$\int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx \geq \mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)})\left[\ln \mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)}) - \ln|\mathcal{J}_{h_\sigma(\epsilon)}|\right] = \sigma\left[\ln\sigma - \ln|\mathcal{J}_{h_\sigma(\epsilon)}|\right]. \tag{6.1}$$

Since $|\mathcal{J}_{h_\sigma(\epsilon)}| \to 0$ as $\epsilon \to 0$, the first inequality in the statement follows.

Let $\mathcal{N}$ be as in the statement. For $0 < \epsilon \ll 1$, we have from (6.1) and Lemma 6.1 that

$$\begin{aligned}\int_{\mathcal{N}} u_\epsilon \ln u_\epsilon dx &= \int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx + \int_{\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx\\ &\geq \sigma\left[\ln\sigma - \ln|\mathcal{J}_{h_\sigma(\epsilon)}|\right] + \mu_\epsilon(\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)})\\ &\quad\times\left[\ln\mu_\epsilon(\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}) - \ln|\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}|\right].\end{aligned}$$

Since $|\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}| \to |\mathcal{N}|$ as $\epsilon \to 0$ and

$$\begin{aligned}\mu_\epsilon(\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}) &= \mu_\epsilon(\mathcal{N}) - \mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)})\\ &= \mu_\epsilon(\mathcal{N}) - \sigma \to 1-\sigma \quad \text{as} \quad \epsilon \to 0,\end{aligned}$$

we deduce the second inequality in the statement.

(2) Let $\mathcal{N}$ be as in the statement. For $0 < h \ll 1$, we write $\mathcal{N} = \mathcal{J}_h \cup (\mathcal{N}\setminus\mathcal{J}_h)$. By Lemmas 6.1 and 6.2, we find that for each $0 < h \ll 1$, there are $c_h > 0$ and $\epsilon_h > 0$ such that

$$\begin{aligned}\int_{\mathcal{N}} u_\epsilon \ln u_\epsilon dx &= \int_{\mathcal{J}_h} u_\epsilon \ln u_\epsilon dx + \int_{\mathcal{N}\setminus\mathcal{J}_h} u_\epsilon \ln u_\epsilon dx\\ &\geq \mu_\epsilon(\mathcal{J}_h)[\ln\mu_\epsilon(\mathcal{J}_h)\\ &\quad - \ln|\mathcal{J}_h|] - \left(\frac{c_h}{\epsilon^2} + \ln|\mathcal{N}\setminus\mathcal{J}_h|\right)e^{-\frac{c_h}{\epsilon^2}}\end{aligned}$$

for all $\epsilon \in (0, \epsilon_h]$.

Due to Lemma 6.3, there holds $\mu_\epsilon(\mathcal{J}_h) \to 1$ as $\epsilon \to 0$, which implies that

$$\liminf_{\epsilon\to 0}\int_{\mathcal{N}} u_\epsilon \ln u_\epsilon dx \geq -\ln|\mathcal{J}_h|, \quad \forall 0 < h \ll 1.$$

The result follows by setting $h \to 0$.

(3) It follows from Lemmas 6.1, 6.3 and arguments as in the proof of Lemma 6.2.

(4) We first prove the following claim: for any increasing sequence $\{\rho_n\}_{n\in\mathbb{N}_0}$ satisfying

$$\rho_0 = \rho_m, \quad \lim_{n\to\infty}\rho_n = \rho_M = \infty, \quad \liminf_{n\in\mathbb{N}_0}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt > 0 \quad \text{and}$$
$$\inf_{n\in\mathbb{N}_0}|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}| \geq 1,$$

there is $\epsilon_* \in (0, \epsilon_0)$, depending on the sequence $\{\rho_n\}_{n\in\mathbb{N}_0}$, such that

$$\int_{\mathcal{U}\setminus\Omega_{\rho_1}} u_\epsilon \ln u_\epsilon dx \geq \sum_{n\in\mathbb{N}}\left\{\left[-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt - \ln|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}|\right]e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}\right\} \tag{6.2}$$

for all $\epsilon \in (0, \epsilon_*]$.

Clearly, U is a Lyapunov function in $\mathcal{U}$ of (1.1). In particular, $\nabla U \neq 0$ on $\mathcal{U}\setminus\Omega_{\rho_m}$. Let $\{\rho_n\}_{n\in\mathbb{N}_0}$ be an increasing sequence as in the statement of the claim. It is easy to see that for each $n \in \mathbb{N}$, the uniform Lyapunov function U, restricted to $\Omega_{\rho_{n+1}}$, can be considered as a uniform Lyapunov function in $\Omega_{\rho_{n+1}}$ with respect to $\{\mathcal{L}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ with essential domain $\Omega_{\rho_{n+1}}\setminus\overline{\Omega}_{\rho_{n-1}}$, essential upper bound ρ_{n+1}, essential lower bound ρ_{n-1} and Lyapunov constant γ. Since $\frac{\mu_\epsilon|_{\Omega_{\rho_{n+1}}}}{\mu_\epsilon(\Omega_{\rho_{n+1}})}$ is a stationary measure corresponding to $\mathcal{L}_\epsilon$ in $\Omega_{\rho_{n+1}}$, we apply Proposition 2.3(1) to find in particular

$$\mu_\epsilon(\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}) \leq e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}\mu_\epsilon(\Omega_{\rho_{n+1}}) \leq e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}.$$

By the choice of the sequence $\{\rho_n\}_{n\in\mathbb{N}_0}$, we can ensure that

$$\sup_{n\in\mathbb{N}}\left[\frac{1}{|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}|}e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}\right] < \frac{1}{e}$$

for all sufficiently small ϵ. Applications of Lemma 6.1 and arguments as in the proof of Lemma 6.2 then yield that

$$\int_{\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}} u_\epsilon \ln u_\epsilon dx \geq \left[-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt - \ln|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}|\right] \times e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}$$

for all $n \in \mathbb{N}$ and all sufficiently small ϵ. Summarizing the above inequalities over $n \in \mathbb{N}$ leads to the claim.

Fix $\ell > 0$. We choose a special sequence $\{\rho_n\}_{n\in\mathbb{N}_0}$ as follows: let $\rho_0 = \rho_m$ and $\rho_1 = \rho_0 + 1$. For $n = 1, 2, \ldots$, we recursively define ρ_{n+1} to be the smallest number such that

$$\begin{aligned}\rho_{n+1} \geq \rho_n + 1, \quad &\int_{\rho_n}^{\rho_{n+1}}\frac{1}{H(t)}dt \geq \alpha_{n+1}\\ &:= \max\left\{\frac{C_1}{2}\rho_{n+1}^{\ell_1}, n+1\right\} \quad \text{and} \quad |\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}| \geq 1.\end{aligned}$$

Since the function $x \mapsto -xe^{-x}$ is increasing on $(1,\infty)$, we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\left\{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt \times e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}\right\}\\ &\geq \sum_{n\in\mathbb{N}}\left[-\frac{\gamma\alpha_{n+1}}{\epsilon^2}e^{-\frac{\gamma\alpha_{n+1}}{\epsilon^2}}\right] \geq -\epsilon^\ell\end{aligned}$$

for all small ϵ. Since $\ln|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}| \leq \ln|\Omega_{\rho_{n+1}}| \leq C_2\rho_{n+1}^{\ell_2}$, we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\left\{-\ln|\Omega_{\rho_{n+1}}\setminus\Omega_{\rho_n}| \times e^{-\frac{\gamma}{\epsilon^2}\int_{\rho_{n-1}}^{\rho_n}\frac{1}{H(t)}dt}\right\}\\ &\geq \sum_{n\in\mathbb{N}}\left[-C_2\rho_{n+1}^{\ell_2}e^{-\frac{\gamma\alpha_{n+1}}{\epsilon^2}}\right] \geq -\epsilon^\ell\end{aligned}$$

for all small ϵ. Applying (6.2), we arrive at

$$\int_{\mathcal{U}\setminus\Omega_{\rho_m+1}} u_\epsilon \ln u_\epsilon dx \geq -\epsilon^\ell \tag{6.3}$$

for all small ϵ.

To finish the proof, let $\mathcal{N}$ be as in the statement and choose an open set $\mathcal{W} \subset\subset \mathcal{U}\setminus\mathcal{J}$ such that $\mathcal{N} \subset \mathcal{W}\cup\Omega_{\rho_m+1}$. Applying Lemma 6.2 to $\int_{\mathcal{W}} u_\epsilon \ln u_\epsilon dx$, we conclude the result from (6.3).

(5) It follows from arguments as in the proof of (4). □

It remains to prove Theorem C.

Proof of Theorem C. Let $\mathcal{N}$ be an open set such that $\mathcal{J} \subset \mathcal{N} \subset\subset \mathcal{U}$. For each $\sigma \in (0, 1)$, we have

$$\frac{\mathcal{H}[u_\epsilon]}{\ln h_\sigma(\epsilon)} = \frac{-\int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)} + \frac{-\int_{\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)} + \frac{-\int_{\mathcal{U}\setminus\mathcal{N}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)}$$

for all small $\epsilon > 0$.

Theorem B(1) and $\liminf_{h\to 0^+} \frac{\ln|\mathcal{J}_h|}{\ln h} = n - d$ ensure that

$$\liminf_{\epsilon\to 0} \frac{-\int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)} = \liminf_{\epsilon\to 0} \left[\frac{-\int_{\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln|\mathcal{J}_{h_\sigma(\epsilon)}|} \frac{\ln|\mathcal{J}_{h_\sigma(\epsilon)}|}{\ln h_\sigma(\epsilon)} \right] \ge \sigma(n-d).$$

By Lemma 6.1, we have

$$\begin{aligned}\int_{\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx &\ge \mu_\epsilon(\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}) \\ &\quad \times \left[\ln \mu_\epsilon(\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}) - \ln|\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}|\right] \\ &\ge \left[\mu_\epsilon(\mathcal{N}) - \mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)})\right] \\ &\quad \times \left\{\ln\left[\mu_\epsilon(\mathcal{N}) - \mu_\epsilon(\mathcal{J}_{h_\sigma(\epsilon)})\right] - \ln|\mathcal{N}|\right\} \\ &= \left[\mu_\epsilon(\mathcal{N}) - \sigma\right]\left\{\ln\left[\mu_\epsilon(\mathcal{N}) - \sigma\right] - \ln|\mathcal{N}|\right\}.\end{aligned}$$

As $\lim_{\epsilon\to 0}\mu_\epsilon(\mathcal{N}) = 1$ due (3) to Lemma 6.3, we conclude that

$$\liminf_{\epsilon\to 0} \frac{-\int_{\mathcal{N}\setminus\mathcal{J}_{h_\sigma(\epsilon)}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)} = 0.$$

Whenever the assumptions in Theorem B(3), (4) or (5) are satisfied, the results of Theorem B(3), (4) or (5) yield

$$\liminf_{\epsilon\to 0} \frac{-\int_{\mathcal{U}\setminus\mathcal{N}} u_\epsilon \ln u_\epsilon dx}{\ln h_\sigma(\epsilon)} = 0.$$

This completes the proof. □

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