

Completeness for sparse potential scattering

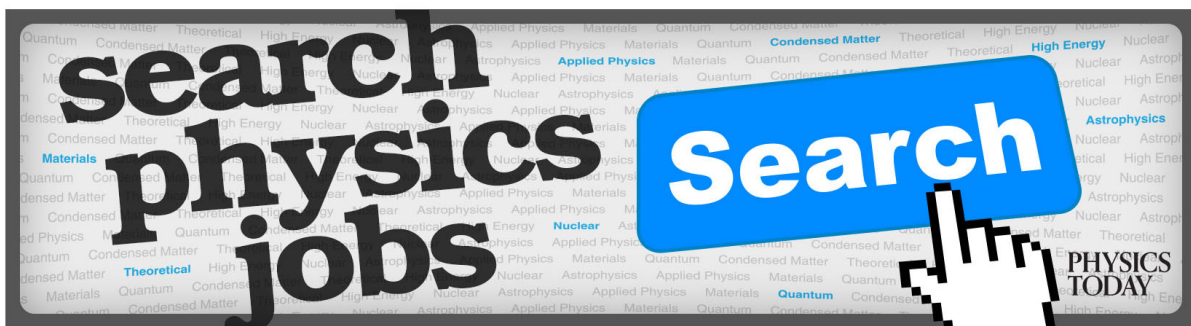
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Completeness for sparse potential scattering

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The present paper is devoted to the scattering theory of a class of continuum Schrödinger operators with deterministic sparse potentials. We first establish the limiting absorption principle for both modified free resolvents and modified perturbed resolvents. This actually is a weak form of the classical limiting absorption principle. We then prove the existence and completeness of local wave operators, which, in particular, imply the existence of wave operators. Under additional assumptions on the sparse potential, we prove the completeness of wave operators. In the context of continuum Schrödinger operators with sparse potentials, this paper gives the first proof of the completeness of wave operators. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4862192>]

I. INTRODUCTION

This paper is concerned with the scattering theory (especially the completeness of wave operators) of a class of Schrödinger operators with sparse potentials, that is,

$$H = H_0 + V \quad \text{on} \quad L^2(\mathbb{R}^d), \quad (1.1)$$

where $H_0 = -\Delta$ is the negative Laplacian and V is the sparse potential of the form $\sum_{n=1}^{\infty} v_n(\cdot - x_n)$, in which, $\{v_n\}_{n \in \mathbb{N}}$ are real-valued, uniformly bounded and uniformly compactly supported functions on \mathbb{R}^d , and $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ is the sequence satisfying certain sparse condition.

As an important ingredient of the spectral theory of Schrödinger operators, both discrete and continuum Schrödinger operators with sparse potentials have attracted a lot of attention (see Refs. 3, 8, 10, 11, 14, 16–18, 20–24, and references therein). In particular, in one dimension, if $v_n = a_n v$ with v being nonzero, nonnegative, continuous, and compactly supported and $\{a_n\}_{n \in \mathbb{N}}$ satisfying $a_n \rightarrow 0$ as $n \rightarrow \infty$, and the sparse sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies $\frac{x_n}{x_{n+1}} \rightarrow 0$ as $n \rightarrow \infty$, then the spectrum is purely absolutely continuous (a.c.) on $\mathbb{R}_+ (= [0, \infty))$ if $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and is purely singularly continuous otherwise (see Ref. 18). In higher dimensions, $d \geq 3$, the situation is a little different. Hundertmark and Kirsch proved in Ref. 8 that the a.c. spectrum contains \mathbb{R}_+ if $\inf_{m \neq n} |x_m - x_n| \gtrsim n^\gamma$ for some $\gamma > \frac{2}{d(d-2)}$. In Ref. 17, Krutikov obtained similar results under slightly different assumptions.

In Refs. 8 and 17, the a.c. spectrum containing \mathbb{R}_+ is established as a result of the existence of wave operators. Their methods are Cook's method in nature. Since Klaus's theorem (see, e.g., Refs. 2, 8, and 15) says that Schrödinger operators with sparse potentials may have essential spectrum below zero, we cannot simply exclude the possible existence of a.c. spectrum below zero, and therefore, we do not expect the completeness of wave operators in general. But, we can still expect the completeness of local wave operators. This is one goal of the present paper. We also know from Klaus's theorem the essential spectrum below zero, if exists, is characterized by the discrete spectrum of $H_0 + v_n, n \in \mathbb{N}$. Therefore, under additional assumptions on $\{v_n\}_{n \in \mathbb{N}}$, the completeness may be obtained. This is the other goal of the paper.

Consider the Schrödinger operator H in (1.1). We attempt to prove the existence and completeness of (local) wave operators by means of the smooth method of Kato (see, e.g., Refs. 12 and 27

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and the Appendix). By writing

$$H - H_0 = |V|^{1/2} V^{1/2},$$

where $V^{1/2} = \text{sgn}(V)|V|^{1/2}$ so that $V = |V|^{1/2} V^{1/2}$, this requires to show the local H -smoothness of $|V|^{1/2}$ and local H_0 -smoothness of $V^{1/2}$, which for short-range potentials comes from the classical limiting absorption principle, which, however, is not expected to be true in our case since the sparse potential considered here may not be decaying. Instead, we want to prove some properties about the resolvents that closely relate to Kato's smooth method. After considering this and the resolvent identity, we define for $\lambda > 0$ and $\epsilon > 0$ the modified free resolvent and the modified perturbed resolvent, that is,

$$F(\lambda + i\epsilon) = |V|^{1/2}(H_0 - \lambda - i\epsilon)^{-1} V^{1/2},$$

$$P(\lambda + i\epsilon) = |V|^{1/2}(H - \lambda - i\epsilon)^{-1} V^{1/2}.$$

Our first result is summarized in

Theorem 1.1. *Let $d \geq 2$ and consider*

$$H = H_0 + V = H_0 + \sum_{n=1}^{\infty} v_n(\cdot - x_n) \quad \text{on } L^2(\mathbb{R}^d),$$

where $\{v_n\}_{n \in \mathbb{N}}$ are real-valued, uniformly bounded, i.e., $\sup_{n \in \mathbb{N}} \|v_n\|_{\infty} < \infty$, and uniformly compactly supported, i.e., there's some bounded set $\Omega \subset \mathbb{R}^d$ such that

$$\text{supp}(v_n) \subset \Omega \quad \text{for all } n \in \mathbb{N},$$

and $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ satisfies the sparse condition

$$\text{dist}(x_n, \{x_m\}_{m \neq n}) \geq Cn^{\gamma}, \quad n \in \mathbb{N}$$

for some $C > 0$ and $\gamma > \frac{2}{d-1}$. Then, the following statements hold

(i) For any $0 < a < b < \infty$, we have

$$\sup_{\lambda \in [a, b], \epsilon \in (0, 1]} \|F(\lambda + i\epsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < \infty \quad \text{and} \quad \sup_{\lambda \in [a, b], \epsilon \in (0, 1]} \|P(\lambda + i\epsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < \infty,$$

where $\mathcal{L}(L^2(\mathbb{R}^d))$ is the space of bounded linear operators on $L^2(\mathbb{R}^d)$.

(ii) For any $I = [a, b] \subset (0, \infty)$, the local wave operators

$$\Omega_{\pm}(H, H_0; I) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0 t} \chi_I(H_0)$$

exist and are complete. In particular, the wave operators

$$\Omega_{\pm}(H, H_0) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0 t}$$

exist.

Theorem 1.1(i) is the analog of the classical limiting absorption principle (see, e.g., Refs. 1 and 27), which, for short-range potentials, implies the supremums as in Theorem 1.1(i). Therefore, we here actually obtain a weak form. It is easy to see that this weak form closely relates to Kato's smooth method, and therefore, Theorem 1.1(ii) is almost a direct consequence of Theorem 1.1(i). Note that while the results in Refs. 8 and 17 hold for $d \geq 3$, our results are also true in the dimension $d = 2$, which, for continuum models, seems unknown, even for the existence of wave operators.

Next, we put more conditions on $\{v_n\}_{n \in \mathbb{N}}$, which includes the case that $v_n, n \in \mathbb{N}$ are the same, to ensure the completeness of wave operators.

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Suppose further that $\{v_n\}_{n \in \mathbb{N}}$ satisfies one of the following two conditions:*

(i) $\|v_n\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$, that is, V is decaying;

(ii) the set \mathcal{E} has Lebesgue measure zero, where

$$\mathcal{E} = \{E < 0 \mid \exists n_k \text{ and } E_{n_k} \in \sigma(H_0 + v_{n_k}) \text{ s.t. } E_{n_k} \rightarrow E \text{ as } k \rightarrow \infty\}.$$

Then, the wave operators $\Omega_{\pm}(H, H_0)$ exist and are complete. In particular, $\sigma_{ac}(H) = \mathbb{R}_+$.

The assumption(ii) in the statement of Theorem 1.2 is made according to Klaus's theorem (see, e.g., Refs. 2, 8, and 15). We remark that the set \mathcal{E} can have positive Lebesgue measure (see Remark 5.1(i)).

As an immediate result of Theorem 1.2, we obtain

Theorem 1.3. Let $d \geq 2$. Consider the Schrödinger operator

$$H = H_0 + \sum_{n=1}^{\infty} v(\cdot - x_n) \quad \text{on } L^2(\mathbb{R}^d),$$

where $v : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded and compactly supported, and $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ satisfies the sparse condition

$$\text{dist}(x_n, \{x_m\}_{m \neq n}) \geq Cn^{\gamma}, \quad n \in \mathbb{N}$$

for some $C > 0$ and $\gamma > \frac{2}{d-1}$. Then, the wave operators $\Omega_{\pm}(H, H_0)$ exist and are complete.

Here, we want to point out the work of Poulin (see Ref. 24) and Jakšić and Poulin (see Ref. 11) for discrete Schrödinger operators with random sparse potentials. With the presence of randomness, one is only interested in almost sure properties, and therefore, the completeness of wave operators is always expected. This is indeed a big difference between deterministic and random operators. Actually, both the above two papers obtain the existence and completeness of wave operators. In terms of the methods, while both papers are based on the Jakšić-Last criterion of completeness (see Ref. 10), which holds only for discrete models, the former uses probabilistic arguments and the later uses analytic Fredholm theorem and is deterministic in nature.

There is a random version of the operator (1.1), that is,

$$H_{\omega} = H_0 + \sum_{n=1}^{\infty} \omega_n v_n(\cdot - x_n),$$

where $\{\omega_n\}_{n \in \mathbb{N}}$ are random variables. The existence of wave operators has been established in Ref. 17. Due to the present of randomness, we also expect the completeness. Moreover, considering the localization phenomenon, the scenario for H_{ω} should be: almost surely, wave operators exist and are complete, and outside \mathbb{R}_+ , the spectrum is pure point. This has been established for discrete models (see Ref. 24), and therefore, it is believed to be true for continuum models. We hope to prove it in our further work.

The rest of the paper is organized as follows. In Sec. II, we give some basic settings about the model. Sections III and IV are the main parts of the paper. In Sec. III, we establish the limiting absorption principle of $F(z)$. In Sec. IV, we first show the invertibility of $1 + F(z)$, then prove the uniform boundedness of $(1 + F(z))^{-1}$, and finally, establish the limiting absorption principle of $P(z)$. The final section, Sec. V, is devoted to the proof of Theorems 1.1, 1.2, and 1.3. We also make some remarks about the results of the present paper there.

Throughout the paper, we use the following notations: for two nonnegative numbers a and b , by writing $a \lesssim b$ we mean there is some $C > 0$ such that $a \leq Cb$; similarly, we also use $a \gtrsim b$; $B_R(x)$ denotes the open ball in \mathbb{R}^d centered at x with radius R ; its closure is denoted by $\bar{B}_R(x)$; we write B_R and \bar{B}_R for $B_R(0)$ and $\bar{B}_R(0)$, respectively; χ_B is the characteristic function of $B \subset \mathbb{R}^d$; the norm and the inner product on $L^2 = L^2(\mathbb{R}^d)$ are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$; for two Banach spaces X and Y , we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y and the norm on $\mathcal{L}(X, Y)$ is denoted by $\|\cdot\| = \|\cdot\|_{\mathcal{L}(X, Y)}$; if $X = Y$, we write $\mathcal{L}(X) = \mathcal{L}(X, Y)$; the set \mathbb{N} stands for positive integers; $L^{\infty} = L^{\infty}(\mathbb{R}^d)$ with norm $\|\cdot\|_{\infty}$; $H^2 = H^2(\mathbb{R}^d)$; $\ell^p = \ell^p(\mathbb{N})$.

II. BASIC SETTINGS

This section serves as a preparation for the proof of main results. As in the assumptions of Theorem 1.1, $\{v_n\}_{n \in \mathbb{N}}$ are real-valued and satisfy $\sup_{n \in \mathbb{N}} \|v_n\|_\infty < \infty$ and $\cup_{n \in \mathbb{N}} \text{supp}(v_n) \subset \Omega$ for some bounded set $\Omega \subset \mathbb{R}^d$. The sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ satisfies the sparse condition

$$\text{dist}(x_n, \{x_m\}_{m \neq n}) \gtrsim n^\gamma, \quad n \in \mathbb{N} \quad (2.1)$$

for some $\gamma > \frac{2}{d-1}$. Then, the sparse potential

$$V = \sum_{n=1}^{\infty} v_n(\cdot - x_n)$$

is real-valued and bounded, and hence, Kato-Rellich theorem (see, e.g., Theorem X.12 of Ref. 25) ensures that $H = H_0 + V = -\Delta + V$ defines a lower bounded self-adjoint operator on L^2 with domain H^2 .

Let $R > 0$ be such that $\text{supp}(v_n) \subset \bar{B}_R$ for all $n \in \mathbb{N}$. Also, for $n \in \mathbb{N}$, let

$$\Sigma_n = \text{supp}(v_n(\cdot - x_n)).$$

Then, $\Sigma_n \subset \bar{B}_R(x_n)$. We will use the following simple result.

Lemma 2.1. For any $m, n \in \mathbb{N}$, there holds

$$\Sigma_m \times \Sigma_n \subset \bar{B}_R(x_m) \times \bar{B}_R(x_n) \subset \bar{B}_{\sqrt{2}R}(x_m, x_n) \subset \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |x - y| - |x_m - x_n| \leq 2R \right\}.$$

Proof. The first two inclusions are trivial. If $(x, y) \in \bar{B}_{\sqrt{2}R}(x_m, x_n)$, then

$$\begin{aligned} |x - y| &\leq |x - x_m| + |x_m - x_n| + |x_n - y| \\ &\leq \sqrt{2}|(x, y) - (x_m, x_n)| + |x_m - x_n| \\ &\leq 2R + |x_m - x_n|. \end{aligned}$$

Similarly, $|x_m - x_n| \leq 2R + |x - y|$. This proves the third one. \square

Remark 2.2. Since, by Kuroda-Birman theorem (see, e.g., Ref. 26, Theorem XI.9), only the behavior of V at infinity matters when we study the scattering theory of H , we may drop $v_n(\cdot - x_n)$, $n = 1, \dots, N - 1$ for some N large. For later use, this large N is chosen to satisfy

- (i) Σ_n , $n \geq N$ are pairwise disjoint;
- (ii) $(\Sigma_m \times \Sigma_n) \cap \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |x - y| \leq 2R\} = \emptyset$ for any $m, n \geq N$ with $m \neq n$.

Condition (i) is easily satisfied by the sparseness condition (2.1), which together with Lemma 2.1, ensures (ii). Also, (ii) simply means that if $m \neq n$, then $\Sigma_m \times \Sigma_n$ is away from the diagonal $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid x = y\}$.

From now on, we fix some large N as in the above remark and write

$$V = \sum_{n=N}^{\infty} v_n(\cdot - x_n) \quad \text{and} \quad \Sigma = \bigcup_{n=N}^{\infty} \Sigma_n.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, we set $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$, the free resolvent and the perturbed resolvent. It is well-known (see, e.g., Ref. 6, p. 288) that $R_0(z)$ is an integral operator with an explicit integral kernel given by

$$k_{0,z}(x, y) = k_{0,z}(|x - y|) = c_d \left(\frac{\sqrt{z}}{|x - y|} \right)^{(d-2)/2} K_{(d-2)/2}(-i\sqrt{z}|x - y|), \quad (2.2)$$

where c_d is a constant depends only on d , \sqrt{z} satisfies $\Im\sqrt{z} > 0$ and K_\bullet is the Bessel potential. In particular, if $d = 3$, then $k_{0,z}(x, y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$. Using Kato's smooth method to prove the existence and completeness of wave operators, the continuous extension of $R_0(z)$ and $R(z)$ to \mathbb{R}_+ plays a crucial role. For $R_0(z)$, we have the well-known limiting absorption principle, that is, for $z \in \mathbb{C} \setminus \mathbb{R}$ with $\Re z > 0$ and $\sigma > \frac{1}{2}$

$$\|R_0(z)\|_{\mathcal{L}(L^2_\sigma, L^2_{-\sigma})} \leq C_\sigma (\Im z)^{-1/2},$$

where $C_\sigma > 0$ depends only on σ and $L^2_\sigma = \{\phi \mid \|(1 + |\cdot|^2)^{\sigma/2} \phi\| < \infty\}$ is the weighted space. As for $R(z)$ with short-range V , i.e., $(1 + |\cdot|)^{1+\epsilon} V \in L^\infty$ for some $\epsilon > 0$, Agmon's fundamental work (see Ref. 1) established the following bound

$$\sup_{\Re z > \lambda, \Im z > 0} \|R(z)\|_{\mathcal{L}(L^2_\sigma, L^2_{-\sigma})} < \infty \quad (2.3)$$

provided $\lambda > 0$ and $\sigma > \frac{1}{2}$. Later, similar results were obtained for other classes of decaying potentials (see, e.g., Refs. 7, 9, 27, and 30).

However, the potential V discussed in the present paper may not decay fast enough at infinity or even not decay, and therefore, (2.3) is not expected here. To overcome this difficulty, we modify both the free resolvent $R_0(z)$ and the perturbed resolvent $R(z)$ by defining the following operators: for $z \in \mathbb{C} \setminus \mathbb{R}$, we set

$$\begin{aligned} F(z) &= |V|^{1/2} R_0(z) V^{1/2}, \\ P(z) &= |V|^{1/2} R(z) V^{1/2}, \end{aligned}$$

where $V^{1/2} = \text{sgn}(V)|V|^{1/2} = |V|^{1/2}\text{sgn}(V)$ so that $V = V^{1/2}|V|^{1/2}$. Since V is bounded, both $F(z)$ and $P(z)$ are bounded on L^2 . By means of the resolvent identity $R(z) - R_0(z) = -R(z)V R_0(z)$, we get

$$|V|^{1/2} R(z) V^{1/2} - |V|^{1/2} R_0(z) V^{1/2} = -|V|^{1/2} R(z) V^{1/2} |V|^{1/2} R_0(z) V^{1/2},$$

that is,

$$P(z)(1 + F(z)) = F(z).$$

Therefore, the limiting absorption principle, i.e., Theorem 1.1(i), may be established if enough information about $F(z)$ can be acquired.

In the sequel, we fix $0 < a < b < \infty$ and set

$$\begin{aligned} \mathcal{S} &= \left\{ z = \lambda + i\epsilon \in \mathbb{C}_+ \mid \lambda \in [a, b] \text{ and } \epsilon \in (0, 1] \right\}, \\ \overline{\mathcal{S}} &= \left\{ z = \lambda + i\epsilon \in \mathbb{C}_+ \mid \lambda \in [a, b] \text{ and } \epsilon \in [0, 1] \right\}. \end{aligned}$$

In Secs. III and IV, we prove for $z \in \mathcal{S}$ the uniform boundedness of $F(z)$ and of $(1 + F(z))^{-1}$, which then allows us to establish the uniform boundedness of $P(z)$.

III. ANALYSIS OF $F(z)$

Recall that for $z \in \mathcal{S}$, $F(z)$ is defined by

$$F(z) = |V|^{1/2} R_0(z) V^{1/2}.$$

Since $|V|^{1/2}$ and $V^{1/2}$ are integral operators with integral kernels $|V|^{1/2}(x)\delta(x - y)$ and $V^{1/2}(x)\delta(x - y)$, respectively, $F(z)$ is also an integral operator with integral kernel

$$\begin{aligned} k_{F(z)}(x, y) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V|^{1/2}(x)\delta(x - u)k_{0,z}(u, v)V^{1/2}(v)\delta(v - y)dudv \\ &= |V|^{1/2}(x)k_{0,z}(x, y)V^{1/2}(y), \end{aligned} \quad (3.1)$$

where $k_{0,z}(x, y)$ is the integral kernel of $R_0(z)$ given in (2.2). We recall the following property of $k_{0,z}(x, y)$ (see, e.g., Ref. 9, p. 418), which comes from the standard estimates of Bessel potentials.

Lemma 3.1. There's a constant $C = C(a, b)$ such that for any $z \in \bar{\mathcal{S}}$

$$|k_{0,z}(x, y)| = |k_{0,z}(|x - y|)| \leq \begin{cases} C \ln \left(\frac{2}{|x - y|} \right), & \text{if } |x - y| \leq 2R \text{ and } d = 2, \\ \frac{C}{|x - y|^{d-2}}, & \text{if } |x - y| \leq 2R \text{ and } d \geq 3, \\ \frac{C}{|x - y|^{(d-1)/2}}, & \text{if } |x - y| > 2R \text{ and } d \geq 2. \end{cases}$$

We now prove the uniform boundedness of $F(z)$ for $z \in \mathcal{S}$.

Theorem 3.2. $F(z)$ is uniformly bounded on \mathcal{S} , that is,

$$\sup_{z \in \mathcal{S}} \|F(z)\| < \infty.$$

Proof. Let $\phi, \psi \in L^2$, we estimate $\langle \phi, F(z)\psi \rangle$. Using (3.1), we have

$$\begin{aligned} \langle \phi, F(z)\psi \rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{\phi(x)} |V|^{1/2}(x) k_{0,z}(x, y) V^{1/2}(y) \psi(y) dx dy \\ &= \sum_{n=N}^{\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{\phi(x)} |v_n(x - x_n)|^{1/2} k_{0,z}(x, y) v_n(y - x_n)^{1/2} \psi(y) dx dy \\ &\quad + \sum_{m, n \geq N, m \neq n} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{\phi(x)} |v_m(x - x_m)|^{1/2} k_{0,z}(x, y) v_n(y - x_n)^{1/2} \psi(y) dx dy \\ &= I + II, \end{aligned} \tag{3.2}$$

where $v_n(\cdot - x_n)^{1/2} = \text{sgn}(v_n(\cdot - x_n)) |v_n(\cdot - x_n)|^{1/2}$. Note by Lemma 2.1 and Remark 2.2, if $x, y \in \Sigma_n$ then $|x - y| \leq 2R$, and if $x \in \Sigma_m, y \in \Sigma_n$ with $m \neq n$ then $|x - y| > 2R$.

For the term I in the last line of (3.2), we claim that

$$|I| \lesssim \|\chi_{\Sigma} \phi\| \|\chi_{\Sigma} \psi\|. \tag{3.3}$$

Considering Lemma 3.1, we distinguish between $d = 2$ and $d \geq 3$. We first deal with the case $d \geq 3$. If $d \geq 3$, we obtain from Lemma 3.1 and the boundedness of V that

$$\begin{aligned} |I| &\lesssim \sum_{n=N}^{\infty} \iint_{\Sigma_n \times \Sigma_n} \frac{|\phi(x)| |\psi(y)|}{|x - y|^{d-2}} dx dy \\ &= \sum_{n=N}^{\infty} \iint_{\Sigma_n \times \Sigma_n} \frac{|\phi(x)|}{|x - y|^{(d-2)/2}} \frac{|\psi(y)|}{|x - y|^{(d-2)/2}} dx dy \\ &\leq \sum_{n=N}^{\infty} \left(\iint_{\Sigma_n \times \Sigma_n} \frac{|\phi(x)|^2}{|x - y|^{d-2}} dx dy \right)^{1/2} \left(\iint_{\Sigma_n \times \Sigma_n} \frac{|\psi(x)|^2}{|x - y|^{d-2}} dx dy \right)^{1/2} \end{aligned}$$

Using the inequality $\chi_{\Sigma_n}(x)\chi_{\Sigma_n}(y) \leq \chi_{\Sigma_n - \Sigma_n}(x - y)$, we find

$$\begin{aligned} \iint_{\Sigma_n \times \Sigma_n} \frac{|\phi(x)|^2}{|x - y|} dx dy &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\chi_{\Sigma_n}(x)\phi(x)|^2 \frac{\chi_{\Sigma_n}(x)\chi_{\Sigma_n}(y)}{|x - y|^{d-2}} dx dy \\ &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\chi_{\Sigma_n}(x)\phi(x)|^2 \frac{\chi_{\Sigma_n - \Sigma_n}(x - y)}{|x - y|^{d-2}} dx dy \\ &= \|\chi_{\Sigma_n}\phi\|^2 \int_{\Sigma_n - \Sigma_n} \frac{1}{|x|^{d-2}} dx. \end{aligned}$$

Similarly,

$$\iint_{\Sigma_n \times \Sigma_n} \frac{|\psi(x)|^2}{|x - y|^{d-2}} dx dy \leq \|\chi_{\Sigma_n}\psi\|^2 \int_{\Sigma_n - \Sigma_n} \frac{1}{|x|^{d-2}} dx.$$

It then follows that

$$\begin{aligned} |I| &\lesssim \sum_{n=N}^{\infty} \|\chi_{\Sigma_n}\phi\| \|\chi_{\Sigma_n}\psi\| \int_{\Sigma_n - \Sigma_n} \frac{1}{|x|^{d-2}} dx \\ &\leq \left(\int_{\bar{B}_{2R}} \frac{1}{|x|^{d-2}} dx \right) \sum_{n=N}^{\infty} \|\chi_{\Sigma_n}\phi\| \|\chi_{\Sigma_n}\psi\| \\ &\leq \left(\int_{\bar{B}_{2R}} \frac{1}{|x|^{d-2}} dx \right) \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\|, \end{aligned}$$

where we used the fact that for any $n \geq N$, $\Sigma_n - \Sigma_n \subset \bar{B}_{2R}$, and so $\int_{\Sigma_n - \Sigma_n} \frac{1}{|x|^{d-2}} dx \leq \int_{\bar{B}_{2R}} \frac{1}{|x|^{d-2}} dx$, and Hölder's inequality for sequence in the last step. Since $\int_{\bar{B}_{2R}} \frac{1}{|x|^{d-2}} dx < \infty$, we obtain (3.3) in the case $d \geq 3$.

The estimate (3.3) in the case $d = 2$ can be treated similarly. In fact, Lemma 3.1, the boundedness of V and arguments as in the case $d \geq 3$ give

$$\begin{aligned} |I| &\lesssim \left(\int_{\bar{B}_{2R}} \left| \ln \frac{2}{|x|} \right| dx \right) \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\| \\ &= \left(2\pi \int_0^{2R} \left| \ln \frac{2}{r} \right| r dr \right) \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\| \\ &\lesssim \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\|, \end{aligned}$$

since the integral $\int_0^{2R} \left| \ln \frac{2}{r} \right| r dr < \infty$. Thus, (3.3) also holds in the case $d = 2$.

For the term II in the last line of (3.2), we claim that

$$|II| \lesssim \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\|. \tag{3.4}$$

In fact, Hölder's inequality applied to integrals with respect to x and y successively gives

$$\begin{aligned} |II| &\lesssim \sum_{m,n \geq N, m \neq n} \left(\int_{\Sigma_m} |\phi(x)|^2 dx \right)^{1/2} \left(\iint_{\Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy \right)^{1/2} \left(\int_{\Sigma_n} |\psi(y)|^2 dy \right)^{1/2} \\ &\leq \left(\sum_{m,n \geq N, m \neq n} \iint_{\Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy \right)^{1/2} \left(\sum_{m,n \geq N, m \neq n} \int_{\Sigma_m} |\phi(x)|^2 dx \int_{\Sigma_n} |\psi(y)|^2 dy \right)^{1/2} \\ &\leq \left(\iint_{\cup_{m,n \geq N, m \neq n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy \right)^{1/2} \|\chi_{\Sigma}\phi\| \|\chi_{\Sigma}\psi\|. \end{aligned}$$

For the integral in the last line of the above estimate, symmetry gives

$$\begin{aligned} \iint_{\cup_{m,n \geq N, m \neq n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy &\lesssim \iint_{\cup_{m,n \geq N, m < n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy \\ &= \sum_{n \geq N} \iint_{\cup_{m: N \leq m < n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy. \end{aligned}$$

Since $|x_m - x_n| \gtrsim n^\gamma$ for $m < n$ by the sparseness condition (2.1), we deduce from Remark 2.2 that $|x - y| \gtrsim n^\gamma$ for $(x, y) \in \Sigma_m \times \Sigma_n$ if $m < n$. Thus, for any $n \geq N$

$$\iint_{\cup_{m: N \leq m < n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^{d-1}} dx dy \lesssim n^{-(d-1)\gamma} \left| \bigcup_{m: N \leq m < n} \Sigma_m \times \Sigma_n \right| \lesssim n^{-(d-1)\gamma+1},$$

which implies that

$$\iint_{\cup_{m,n \geq N, m \neq n} \Sigma_m \times \Sigma_n} \frac{1}{|x - y|^2} dx dy \lesssim \sum_{n \geq N} n^{-(d-1)\gamma+1}.$$

Since $\gamma > \frac{2}{d-1}$, i.e., $(d - 1)\gamma - 1 > 1$, the above series converges. Hence, (3.4) follows.

Combining (3.2), (3.3), and (3.4), we find for any $\phi, \psi \in L^2(\mathbb{R}^3)$ the estimate

$$|\langle \phi, F(z)\psi \rangle| \lesssim \|\chi_\Sigma \phi\| \|\chi_\Sigma \psi\|,$$

which leads to the result since the above estimate is clearly independent of $z \in \mathcal{S}$. □

Theorem 3.2 suggests us to define for $\lambda \in [a, b]$ the boundary operator

$$F(\lambda + i0) = \lim_{\epsilon \downarrow 0} F(\lambda + i\epsilon) \quad \text{in } \mathcal{L}(L^2). \tag{3.5}$$

Note for $\lambda \in [a, b]$, the integral kernel $k_{F(\lambda + i\epsilon)}$ given in (3.1) converges pointwise to

$$k_{F(\lambda+i0)}(x, y) := |V|^{1/2}(x)k_{0,\lambda}(x, y)V^{1/2}(y)$$

as $\epsilon \downarrow 0$. Moreover, since $k_{F(\lambda + i0)}$ clearly defines a bounded linear operator on L^2 by the proof of Theorem 3.2, $F(\lambda + i0)$ must be the integral operator with integral kernel $k_{F(\lambda + i0)}$, that is,

$$F(\lambda + i0)\phi(x) = \int_{\mathbb{R}^3} |V|^{1/2}(x)k_{0,\lambda}(x, y)V^{1/2}(y)\phi(y)dy, \quad \phi \in L^2. \tag{3.6}$$

Then, considering the continuity of the integral kernels $k_{F(z)}$ in $z \in \overline{\mathcal{S}}$, we obtain the continuity of $F(z)$ on $\overline{\mathcal{S}}$. The above scenarios are summarized in

Lemma 3.3. The following statements hold:

- (i) For each $\lambda \in [a, b]$, $F(\lambda + i\epsilon)$ is norm Cauchy as $\epsilon \downarrow 0$. In particular, $F(\lambda + i0)$, given in (3.5), is well-defined.
- (ii) For each $\lambda \in [a, b]$, $F(\lambda + i0)$ is an integral operator with integral kernel $k_{F(\lambda + i0)}$, that is, (3.6) is true.
- (iii) $F(z)$ is analytic in $\text{int}(\mathcal{S})$ and continuous on $\overline{\mathcal{S}}$. In particular,

$$\sup_{z \in \overline{\mathcal{S}}} \|F(z)\| < \infty.$$

Proof. Let $z_1, z_2 \in \mathcal{S}$, the proof of Theorem 3.2 gives

$$\begin{aligned} \|F(z_1) - F(z_2)\| &\lesssim \int_{\overline{B}_{2R}} |k_{0,z_1}(|x|) - k_{0,z_2}(|x|)| dx \\ &\quad + \left(\iint_{\cup_{m,n \geq N, m \neq n} \Sigma_m \times \Sigma_n} |k_{0,z_1}(x, y) - k_{0,z_2}(x, y)|^2 dx dy \right)^{1/2}. \end{aligned}$$

Since, by the proof of Theorem 3.2, the integrals $\int_{\bar{B}_{2R}} \frac{1}{|x-y|^{d-2}} dx dy$ (if $d \geq 3$), $\int_{\bar{B}_{2R}} \left| \ln \frac{2}{|x|} \right| dx$ (if $d = 2$) and $\iint_{\cup_{m,n \geq N, m \neq n} \Sigma_m \times \Sigma_n} \frac{1}{|x-y|^{d-1}} dx dy$ are finite, (i) and (ii) follow from Lebesgue's dominated convergence theorem. This also implies the continuity of $F(z)$ on $\bar{\mathcal{S}}$. The analyticity of $F(z)$ in $\text{int}(\mathcal{S})$, the interior of \mathcal{S} , is a simple result of the analyticity of $R_0(z)$ in $\text{int}(\mathcal{S})$. Thus, (iii) is true. \square

IV. ANALYSIS OF $P(z)$

Recall that for $z \in \mathcal{S}$,

$$P(z) = |V|^{1/2} R(z) V^{1/2}$$

and it satisfies

$$P(z)(1 + F(z)) = F(z), \quad (4.1)$$

where $F(z) = |V|^{1/2} R_0(z) V^{1/2}$. The aim of this section is to establish the uniform boundedness of $P(z)$ on \mathcal{S} . The main result is following

Theorem 4.1. *$P(z)$ is uniformly bounded on \mathcal{S} , that is,*

$$\sup_{z \in \mathcal{S}} \|P(z)\| < \infty.$$

Considering Eq. (4.1) and Theorem 3.2, to prove Theorem 4.1, it suffices to prove the invertibility of $1 + F(z)$ for each $z \in \mathcal{S}$ and the uniform boundedness of their inverses on \mathcal{S} , which are accomplished through several lemmas.

We first establishes the invertibility of $1 + F(z)$ for $z \in \mathcal{S}$.

Lemma 4.2. *For each $z \in \mathcal{S}$, the operator $1 + F(z)$ is boundedly invertible. Moreover, $(1 + F(z))^{-1}$ is continuous on \mathcal{S} .*

Proof. Fix any $z \in \mathcal{S}$. We first show that -1 is neither an eigenvalue nor in the residue spectrum of $F(z)$. Clearly, it suffices to show that $1 + F(z)$ is one-to-one and has dense range. To do so, let $\phi \in L^2$ be such that $(1 + F(z))\phi = 0$, then $\text{sgn}(V)(1 + F(z))\phi = 0$, that is,

$$\text{sgn}(V)\phi + V^{1/2} R_0(z) V^{1/2} \phi = 0.$$

Writing $V^{1/2} R_0(z) V^{1/2} = \Re(V^{1/2} R_0(z) V^{1/2}) + i \Im(V^{1/2} R_0(z) V^{1/2})$, we have

$$\langle \phi, \text{sgn}(V)\phi \rangle + \langle \phi, \Re(V^{1/2} R_0(z) V^{1/2})\phi \rangle + i \langle \phi, \Im(V^{1/2} R_0(z) V^{1/2})\phi \rangle = 0,$$

which implies that $\langle \phi, \Im(V^{1/2} R_0(z) V^{1/2})\phi \rangle = 0$. Since

$$\Im(V^{1/2} R_0(z) V^{1/2}) = V^{1/2} \Im R_0(z) V^{1/2} = \Im V^{1/2} R_0(\bar{z}) R_0(z) V^{1/2},$$

we conclude that $\Im V^{1/2} R_0(z) V^{1/2} \phi = 0$, which implies $R_0(z) V^{1/2} \phi = 0$, and so, $V^{1/2} \phi = 0$. Therefore, $F(z)\phi = 0$ and

$$\phi = (1 + F(z))\phi = 0.$$

This shows that $1 + F(z)$ is one-to-one. A similar argument shows that $(1 + F(z))^* = 1 + F(z)^*$ is also one-to-one. From the fact that $\text{ran}(1 + F(z)) \oplus \ker(1 + F(z)^*) = L^2$, we conclude that $1 + F(z)$ has dense range. Hence, $1 + F(z)$ is one-to-one and has dense range.

As a result of the fact that -1 is neither an eigenvalue nor in the residue spectrum of $F(z)$, we obtain that if $1 + F(z)$ is not boundedly invertible, then $(1 + F(z))^{-1}$ is densely defined and unbounded.

Now, we show that $1 + F(z)$ is boundedly invertible. For contradiction, we assume that $1 + F(z)$ is not boundedly invertible, that is, $-1 \in \sigma(F(z))$. Then, the above analysis says that there exists $\{\phi_n\}_{n \in \mathbb{N}} \subset L^2$ such that $\|\phi_n\| = 1$ for all n and $\|(1 + F(z))\phi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Define

$$\psi_n = R_0(z) V^{1/2} \phi_n.$$

Since V is bounded, $\psi_n \in H^2$. Moreover, there is some $C > 0$ such that $\inf_{n \in \mathbb{N}} \|\psi_n\| \geq C$. In fact, if this is not true, then we can find some subsequence $\{\phi_{n_k}\}_{k \in \mathbb{N}}$ such that $\|\psi_{n_k}\| = \|R_0(z)V^{1/2}\phi_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. It then follows that

$$\begin{aligned} 1 = \|\phi_{n_k}\| &\leq \|(1 + F(z))\phi_{n_k}\| + \|F(z)\phi_{n_k}\| \\ &\leq \|(1 + F(z))\phi_{n_k}\| + \| |V|^{1/2} \|_{\infty} \|R_0(z)V^{1/2}\phi_{n_k}\| \\ &\rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which leads to the contradiction.

Also, setting $\varphi_n = (1 + F(z))\phi_n$, i.e., $\phi_n = \varphi_n - F(z)\phi_n$, we have

$$(H_0 - z)\psi_n = V^{1/2}\phi_n = V^{1/2}\varphi_n - V^{1/2}F(z)\phi_n = V^{1/2}\varphi_n - V\psi_n,$$

that is, $(H_0 + V - z)\psi_n = V^{1/2}\varphi_n$, or

$$(H_0 + V - z) \frac{\psi_n}{\|\psi_n\|} = \frac{V^{1/2}\varphi_n}{\|\psi_n\|}.$$

Since $\inf_{n \in \mathbb{N}} \|\psi_n\| \geq C$, we have $\|\frac{V^{1/2}\varphi_n}{\|\psi_n\|}\| \leq \frac{\|V^{1/2}\|_{\infty}\|\varphi_n\|}{C} \rightarrow 0$ as $n \rightarrow \infty$. The bounded invertibility of $H_0 + V - z$ then ensures that

$$\frac{\psi_n}{\|\psi_n\|} = (H_0 + V - z)^{-1} \frac{V^{1/2}\varphi_n}{\|\psi_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which leads to a contradiction. Consequently, $-1 \in \rho(F(z))$, that is, $1 + F(z)$ is boundedly invertible.

For the continuity of $(1 + F(z))^{-1}$ on \mathcal{S} , we fix any $z_0 \in \mathcal{S}$, and then for any $z \in \mathcal{S}$ we have the formal expansion

$$\begin{aligned} (1 + F(z))^{-1} &= [1 + F(z_0) + F(z) - F(z_0)]^{-1} \\ &= [(1 + F(z_0))(1 + (1 + F(z_0))^{-1}(F(z) - F(z_0)))]^{-1} \\ &= [1 - (1 + F(z_0))^{-1}(F(z_0) - F(z))]^{-1}(1 + F(z_0))^{-1} \\ &= \left(1 + \sum_{n=1}^{\infty} [(1 + F(z_0))^{-1}(F(z_0) - F(z))]^n\right) (1 + F(z_0))^{-1}. \end{aligned}$$

The above series converges if $\|(1 + F(z_0))^{-1}(F(z_0) - F(z))\| < 1$, which, by Lemma 3.3, is true for all z close to z_0 . Thus, for any $z \in \mathcal{S}$ close to z_0 , we deduce

$$\begin{aligned} &\|(1 + F(z))^{-1} - (1 + F(z_0))^{-1}\| \\ &\leq \left\| \left(\sum_{n=1}^{\infty} [(1 + F(z_0))^{-1}(F(z_0) - F(z))]^n \right) (1 + F(z_0))^{-1} \right\| \\ &\leq \frac{\|(1 + F(z_0))^{-1}(F(z_0) - F(z))\|}{1 - \|(1 + F(z_0))^{-1}(F(z_0) - F(z))\|} \|(1 + F(z_0))^{-1}\| \\ &\rightarrow 0 \quad \text{as } z \rightarrow z_0. \end{aligned}$$

This establishes the continuity of $(1 + F(z))^{-1}$ at z_0 , and thus, $(1 + F(z))^{-1}$ is continuous on \mathcal{S} . \square

Next, we prove the invertibility of $1 + F(\lambda + i0)$ for $\lambda \in [a, b]$. To do so, we need the following result about approximate inverse (see, e.g., Ref. 5).

Proposition 4.3. Let X, Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. Suppose that there exist $B_1, B_2 \in \mathcal{L}(Y, X)$, $R_1 \in \mathcal{L}(Y)$ with $\|R_1\|_{\mathcal{L}(Y)} < 1$ and $R_2 \in \mathcal{L}(X)$ with $\|R_2\|_{\mathcal{L}(X)} < 1$ such that

$$AB_1 = I_Y + R_1 \text{ on } Y$$

$$B_2A = I_X + R_2 \text{ on } X,$$

where I_X and I_Y are identity operators on X and Y , respectively. Then, A is boundedly invertible with $A^{-1}: Y \rightarrow X$ given by

$$A^{-1} = B_1(I_Y + R_1)^{-1} = (I_X + R_2)^{-1}B_2.$$

Now, we prove

Lemma 4.4. For each $\lambda \in [a, b]$, the operator $1 + F(\lambda + i0)$ is boundedly invertible.

Proof. Fix any $\lambda \in [a, b]$. We first claim that there exists $C > 0$ such that

$$\inf_{\epsilon \in (0,1]} \|1 + F(\lambda + i\epsilon)\| \geq C. \quad (4.2)$$

In fact, if this is not true, then, by Lemma 4.2, there exists $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$\|1 + F(\lambda + i\epsilon_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This means that $\{F(\lambda + i\epsilon_n)\}_{n \in \mathbb{N}}$ converges in norm to the operator $-I$, where I is the identity operator on L^2 . Since $\{F(\lambda + i\epsilon_n)\}_{n \in \mathbb{N}}$ also converges in norm to the operator $F(\lambda + i0)$ by Lemma 3.3(i), we have $F(\lambda + i0) = -I$. But, clearly, this is not the case since $F(\lambda + i0)$ is the integral operator with the integral kernel $|V|^{1/2}(x)k_{0,\lambda}(x, y)V^{1/2}(y)$, and thus, if $\phi = 0$ on Σ , then $F(\lambda + i0)\phi = 0$. Hence, (4.2) is true. By (4.2) and Lemma 4.2, we find

$$\sup_{\epsilon \in (0,1]} \|(1 + F(\lambda + i\epsilon))^{-1}\| \leq \frac{1}{C}. \quad (4.3)$$

Next, we apply Proposition 4.3 to prove the bounded invertibility of $1 + F(\lambda + i0)$. Note

$$\begin{aligned} & (1 + F(\lambda + i0))(1 + F(\lambda + i\epsilon))^{-1} \\ &= [1 + F(\lambda + i\epsilon) + F(\lambda + i0) - F(\lambda + i\epsilon)](1 + F(\lambda + i\epsilon))^{-1} \\ &= 1 + [F(\lambda + i0) - F(\lambda + i\epsilon)](1 + F(\lambda + i\epsilon))^{-1}. \end{aligned}$$

Similarly,

$$(1 + F(\lambda + i\epsilon))^{-1}(1 + F(\lambda + i0)) = 1 + (1 + F(\lambda + i\epsilon))^{-1}[F(\lambda + i0) - F(\lambda + i\epsilon)]$$

Thus, if we fix some $\epsilon \in (0, 1]$ such that $\|F(\lambda + i0) - F(\lambda + i\epsilon)\| < C$ (this can be done by Lemma 3.3), (4.3) implies

$$\|(F(\lambda + i0) - F(\lambda + i\epsilon))(1 + F(\lambda + i\epsilon))^{-1}\| < 1$$

and

$$\|(1 + F(\lambda + i\epsilon))^{-1}(F(\lambda + i0) - F(\lambda + i\epsilon))\| < 1.$$

By Proposition 4.3, we conclude that $1 + F(\lambda + i0)$ is boundedly invertible. \square

In Lemmas 4.2 and 4.4, we proved that $1 + F(z)$ is boundedly invertible for each $z \in \overline{\mathcal{S}}$. In the next result, we prove the uniform boundedness of their inverses.

Lemma 4.5. $(1 + F(z))^{-1}$ is continuous on $\overline{\mathcal{S}}$. In particular, there holds

$$\sup_{z \in \overline{\mathcal{S}}} \|(1 + F(z))^{-1}\| < \infty.$$

Proof. Considering the compactness of $\overline{\mathcal{S}}$, it suffices to show the continuity of $(1 + F(z))^{-1}$ on $\overline{\mathcal{S}}$. By Lemma 4.2, we only need to show the continuity of $(1 + F(z))^{-1}$ at $z = \lambda + i0$ for each $\lambda \in [a, b]$. The latter actually follows from the continuity of $F(z)$ on $\overline{\mathcal{S}}$ by Lemma 3.3 as in the proof

of Lemma 4.2. Indeed, fixing any $\lambda \in [a, b]$, for any $z \in \overline{\mathcal{S}}$, we have the formal expansion

$$(1 + F(z))^{-1} = \left(1 + \sum_{n=1}^{\infty} [(1 + F(\lambda + i0))^{-1}(F(\lambda + i0) - F(z))]^n \right) (1 + F(\lambda + i0))^{-1}.$$

The series converges if $\|(1 + F(\lambda + i0))^{-1}(F(\lambda + i0) - F(z))\| < 1$, which, by Lemma 3.3, is true for all z close to λ . This completes the proof. \square

Finally, we prove Theorem 4.1.

Proof of Theorem 4.1. The result follows from (4.1) and Lemmas 3.3, 4.2, 4.4, and 4.5. In fact, $P(z)$ can be continuously extended to $\overline{\mathcal{S}}$. In particular, there holds

$$\sup_{z \in \overline{\mathcal{S}}} \|P(z)\| < \infty. \quad \square$$

V. PROOF OF MAIN RESULTS AND REMARKS

In this section, we first prove Theorems 1.1, 1.2, and 1.3. Then, we make some remarks.

Proof of Theorem 1.1. The first part of Theorem 1.1 is a direct consequence of Theorems 3.2 and 4.1.

For the existence and completeness of local wave operators, we invoke Proposition A.2. To do so, writing $H - H_0 = |V|^{1/2}V^{1/2}$, since clearly $|V|^{1/2}$ is H -bounded and $V^{1/2}$ is H_0 -bounded, we only need to show that for any compact interval $I \subset (0, \infty)$, $|V|^{1/2}$ is H -smooth on I and $V^{1/2}$ is H_0 -smooth on I .

Let $I \subset (0, \infty)$ be a compact interval. For the H -smoothness of $|V|^{1/2}$ on I , Proposition A.1 says that it suffices to show that

$$\sup_{\lambda \in I, \epsilon \in (0, 1]} \| |V|^{1/2} R(\lambda + i\epsilon) |V|^{1/2} \| < \infty. \quad (5.1)$$

Now, by $|V|^{1/2} = V^{1/2} \operatorname{sgn}(V)$, we have

$$\| |V|^{1/2} R(\lambda + i\epsilon) |V|^{1/2} \| \leq \| |V|^{1/2} R(\lambda + i\epsilon) V^{1/2} \| \| \operatorname{sgn}(V) \| = \| |V|^{1/2} R(\lambda + i\epsilon) V^{1/2} \|.$$

Theorem 4.1 then implies (5.1). Similarly, the H_0 -smoothness of $V^{1/2}$ on I follows from Theorem 3.2. This completes the proof. \square

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. We invoke Proposition A.3. To do so, let $\{I_n\}_{n \in \mathbb{N}}$ be a family of compact intervals in $(0, \infty)$ such that $(0, \infty) = \cup_{n=1}^{\infty} I_n$. By the proof of Theorem 1.1, we know that $|V|^{1/2}$ is H -smooth on each I_n and $V^{1/2}$ is H_0 -smooth on each I_n . To apply Proposition A.3, we only need to show that $\sigma(H) \setminus (0, \infty)$ has Lebesgue measure zero.

Clearly, by Weyl's theorem (see, e.g., Theorem XIII.14 of Ref. 27), if $\{v_n\}_{n \in \mathbb{N}}$ satisfies condition (i) in the statement of Theorem 1.2, then $\sigma(H) \setminus (0, \infty)$ has Lebesgue measure zero. If $\{v_n\}_{n \in \mathbb{N}}$ satisfies condition (ii) in the statement of Theorem 1.2, we need to use Klaus's theorem (see, e.g., Refs. 2, 8, and 15) about the structure of the essential spectrum of H , that is,

$$\sigma_{\text{ess}}(H) = \mathcal{E} \cup \mathbb{R}_+.$$

Since \mathcal{E} has Lebesgue measure zero by assumption, we find

$$|\sigma(H) \setminus (0, \infty)| = |\sigma_{\text{disc}}(H)| + |\sigma_{\text{ess}}(H) \cap (-\infty, 0]| = 0.$$

Thus, the wave operators exist and are complete.

The conclusion $\sigma_{ac}(H) = \mathbb{R}_+$ follows from general results of the existence and completeness of wave operators (see, e.g., Ref. 4, Corollary 3.6.6). This completes the proof. \square

We now prove Theorem 1.3.

Proof of Theorem 1.3. Under the assumption of Corollary 1.3, the set \mathcal{E} is given by

$$\mathcal{E} = \sigma_{\text{disc}}(H_0 + v),$$

and thus, it has zero Lebesgue measure. □

Finally, we make remarks about the results of the present paper.

Remark 5.1.

- (i) *We construct an example showing that the set \mathcal{E} in the statement of Theorem 1.2 can have positive Lebesgue measure. Let v be a nonpositive, bounded and compactly supported function and suppose $E(1) < 0$ is an eigenvalue of $H_0 + v$. Then, by analytic perturbation theory (see, e.g., Refs. 13 and 27), there is a small $\beta_0 > 0$ such that for any $\beta \in (1 - \beta_0, 1 + \beta_0)$ there is a unique point $E(\beta) \in \sigma(H_0 + \beta v)$ near $E(1)$. Moreover, $E(\beta)$ is analytic. Also, since $v \leq 0$, $E(\beta)$ is strictly decreasing by Feynman-Hellmann theorem (see, e.g., Ref. 28, Theorem 4.1.29). We define the set $\{\beta_n\}_{n \in \mathbb{N}}$ by*

$$\{\beta_n\}_{n \in \mathbb{N}} := E^{-1}((E(1 + \beta_0), E(1 - \beta_0)) \cap \mathbb{Q}).$$

It is countable and for any $E \in (E(1 + \beta_0), E(1 - \beta_0))$, we can find a subsequence $\{\beta_{n_k}\}_{k \in \mathbb{N}}$ such that $E(\beta_{n_k}) \rightarrow E$ as $k \rightarrow \infty$. Now, consider the operator

$$H = H_0 + \sum_{n=1}^{\infty} v_n(\cdot - x_n)$$

with $v_n = \beta_n v$ for $n \in \mathbb{N}$. Since $\{\beta_n\}_{n \in \mathbb{N}} \subset (1 - \beta_0, 1 + \beta_0)$, $\{v_n\}_{n \in \mathbb{N}}$ are uniformly bounded and uniformly compactly supported. Moreover, above analysis says that \mathcal{E} contains the set $(E(1 + \beta_0), E(1 - \beta_0))$, and therefore has positive Lebesgue measure.

- (ii) *Note that I and II in (3.2) correspond to the diagonal and off-diagonal part of $F(z)$. The proof of Theorem 3.2 clearly implies that the off-diagonal part of $F(z)$ is Hilbert-Schmidt. Although this compactness does not play a role in our proof, it does play a key role in the work of Jakšić and Poulin (see Ref. 11) for the discrete model. Moreover, if we write $F(z) = F_{\text{diag}}(z) + F_{\text{offdiag}}(z)$. Then, while $\Im F_{\text{diag}}(z)$ is strictly positive for the discrete model (this directly leads to the bounded invertibility of $1 + F_{\text{diag}}(z)$), it is not true for the continuum model discussed here. In fact, assuming $v_n, n \in \mathbb{N}$ are nonnegative, then*

$$F_{\text{diag}}(z) = \sum_{n=N}^{\infty} v_n(\cdot - x_n)^{1/2} R_0(z) v_n(\cdot - x_n)^{1/2},$$

which leads to

$$\Im F_{\text{diag}}(z) = \Im z \sum_{n=N}^{\infty} v_n(\cdot - x_n)^{1/2} R_0(\bar{z}) R_0(z) v_n(\cdot - x_n)^{1/2}$$

by resolvent identity. Thus, for any $z \in \mathcal{S}$, $\Im F_{\text{diag}}(z) \geq 0$. But, note that each $v_n(\cdot - x_n)^{1/2} R_0(\bar{z}) R_0(z) v_n(\cdot - x_n)^{1/2}$ is compact, and so, its spectrum contains 0. Hence, $0 \in \sigma(\Im F_{\text{diag}}(z))$, and $\Im F_{\text{diag}}(z)$ is not strictly positive.

- (iii) *We remark that the proof of Theorem 4.1, that is,*

$$\sup_{z \in \mathcal{S}} \|P(z)\| < \infty,$$

can be greatly simplified if V is decaying. In fact, if V is decaying, that is, $\|v_n\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$, then Theorem 3.2 actually reads

$$\sup_{z \in \mathcal{S}} \|F(z)\| \lesssim \sup_{n \geq N} \|v_n\|_{\infty}$$

Therefore, we can fix some large N so that $\sup_{z \in \mathcal{S}} \|F(z)\| \leq \frac{1}{2}$. By stability of bounded invertibility (see, e.g., Ref. 19), for each $z \in \mathcal{S}$, $1 + F(z)$ is boundedly invertible with

$$\|(1 + F(z))^{-1}\| \leq \frac{1}{1 - \|F(z)\|} \leq 2.$$

It then follows that for each $z \in \mathcal{S}$, $\|P(z)\| = \|F(z)(1 + F(z))^{-1}\| \leq 1$.

APPENDIX: SMOOTH PERTURBATION

We collect some results of Kato's smooth method used in Sec. V. The following material is taken from Section XIII.7 of Ref. 27. We also refer to Ref. 29, Chap. 4.

Let H_0 and H be self-adjoint operators on some separable Hilbert space \mathcal{H} . Let A be a closed operator on \mathcal{H} . We recall that A is called H -smooth if for each $\phi \in \mathcal{H}$ and each $\epsilon \neq 0$, $(H - \lambda - i\epsilon)^{-1}\phi \in \mathcal{D}(A)$ for almost all $\lambda \in \mathbb{R}$ and

$$\sup_{\|\phi\|=1, \epsilon > 0} \int_{\mathbb{R}} (\|A(H - \lambda - i\epsilon)^{-1}\phi\|^2 + \|A(H - \lambda + i\epsilon)^{-1}\phi\|^2) d\lambda < \infty.$$

We also recall that A is called H -smooth on Ω , a Borel set, if $A\chi_{\Omega}(H)$ is H -smooth. The following result gives a sufficient condition for A being H -smooth on Ω .

Proposition A.1. Let $\Omega \subset \mathbb{R}$ and suppose $\mathcal{D}(H) \subset \mathcal{D}(A)$. Then, A is H -smooth on $\overline{\Omega}$ provided

$$\sup_{\lambda \in \Omega, \epsilon \in (0, 1)} \|A(H - \lambda - i\epsilon)A^*\| < \infty.$$

Regarding the existence and completeness of local wave operators and wave operators, we have the following two results.

Proposition A.2. Suppose that there are closed operators A, B on \mathcal{H} such that

$$H - H_0 = A^*B$$

in the sense of quadratic forms. Suppose further that A is H -bounded and H -smooth on some bounded open interval $I \subset \mathbb{R}$ and that B is H_0 -bounded and H_0 -smooth on I . Then, the local wave operators $\Omega_{\pm}(H, H_0; I) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0t} P_{ac}(H_0)\chi_I(H_0)$ exist and are complete.

Proposition A.3. Suppose that there are closed operators A, B on \mathcal{H} such that A is H -bounded and B is H_0 -bounded, and that

$$H - H_0 = A^*B$$

in the sense of quadratic forms. Let $\{I_i\}_{i=1}^{\infty}$ be a family of bounded open intervals and let $S = \cup_{i=1}^{\infty} I_i$. Suppose that

- (i) A is H -smooth on each I_i and B is H_0 -smooth on each I_i .
- (ii) Both $\sigma(H) \setminus S$ and $\sigma(H_0) \setminus S$ have Lebesgue measure zero.

Then, the wave operators $\Omega_{\pm}(H, H_0) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0t} P_{ac}(H_0)$ exist and are complete.

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