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One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation

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ABSTRACT

This paper is devoted to the study of the asymptotic dynamics of the stochastic damped sine-Gordon equation with homogeneous Neumann boundary condition. It is shown that for any positive damping and diffusion coefficients, the equation possesses a random attractor, and when the damping and diffusion coefficients are sufficiently large, the random attractor is a one-dimensional random horizontal curve regardless of the strength of noise. Hence its dynamics is not chaotic. It is also shown that the equation has a rotation number provided that the damping and diffusion coefficients are sufficiently large, which implies that the solutions tend to oscillate with the same frequency eventually and the so-called frequency locking is successful.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where

$$\Omega = \big\{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C\big(\mathbb{R}, \mathbb{R}^m\big)\colon\ \omega(0) = 0\big\},$$

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the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology (see [1]), and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$. Define $(\theta_t)_{t\in\mathbb{R}}$ on Ω via

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$

Thus, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system.

Consider the following stochastic damped sine-Gordon equation with additive noise:

$$du_t + \alpha\, du + (-K\Delta u + \sin u)\, dt = f\, dt + \sum_{j=1}^{m} h_j\, dW_j \quad \text{in } U \times \mathbb{R}^+ \tag{1.1}$$

complemented with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial U \times \mathbb{R}^+, \tag{1.2}$$

where $U \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary ∂U, $u = u(x,t)$ is a real function of $x \in U$ and $t \geqslant 0$, α, $K > 0$ are damping and diffusion coefficients, respectively, $f \in H^1(U)$, $h_j \in H^2(U)$ with $\frac{\partial h_j}{\partial n} = 0$ on ∂U, $j = 1, \dots, m$, and $\{W_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on $(\Omega, \mathcal{F}, \mathbb{P})$. We identify $\omega(t)$ with $(W_1(t), W_2(t), \dots, W_m(t))$, i.e.,

$$\omega(t) = \big(W_1(t), W_2(t), \dots, W_m(t)\big), \quad t \in \mathbb{R}.$$

Sine-Gordon equations describe the dynamics of continuous Josephson junctions (see [18]) and have been widely studied (see [3–5,11,13–15,17–19,25,26,29–31], etc.). Various interesting dynamical scenarios such as subharmonic bifurcation and chaotic behavior are observed in damped and driven sine-Gordon equations (see [3,4,19], etc.). Note that interesting dynamics of a dissipative system occurs in its global attractor (if it exists). It is therefore of great importance to study the existence and the structure/dimension of a global attractor of a damped sine-Gordon equation.

As it is known, under various boundary conditions, a deterministic damped sine-Gordon equation possesses a finite-dimensional global attractor (see [15,16,27,29–31]). Moreover, some upper bounds of the dimension of the attractor were obtained in [15,29–31]. In [26,27], the authors proved that under Neumann boundary condition, when the damping is sufficiently large, the dimension of the global attractor is one, which justifies the folklore that there is no chaotic dynamics in a strongly damped sine-Gordon equation.

Recently, the existence of attractors of stochastic damped sine-Gordon equations has been studied by several authors (see [5,13,14]). For example, for Eq. (1.1) with Dirichlet boundary condition considered in [13], the author proved the existence of a finite-dimensional attractor in the random sense. However, the existing works on stochastic damped sine-Gordon equations deal with Dirichlet boundary conditions only. The case of a Neumann boundary condition is of great physical interest. It is therefore important to investigate both the existence and structure of attractors of stochastic damped sine-Gordon equations with Neumann boundary conditions. Observe that there is no bounded attracting sets in such case in the original phase space due to the uncontrolled space average of the solutions, which leads to nontrivial dynamics and also some additional difficulties. Nevertheless, it is still expected that (1.1)–(1.2) possesses an attractor in the original phase space in proper sense.

The objective of the current paper is to provide a study on the existence and structure of random attractors (see Definition 2.2 for the definition of random attractor) of stochastic damped sine-Gordon equations with Neumann boundary conditions, i.e. (1.1)–(1.2). We will do so in terms of the random dynamical system generated by (1.1)–(1.2) (see Definition 2.1 for the definition of random dynamical system).

The following are the main results of this paper:

(1) For any $\alpha > 0$ and $K > 0$, (1.1)–(1.2) possesses a random attractor (see Theorem 4.1 and Corollary 4.2).
(2) When K and α are sufficiently large, the random attractor of (1.1)–(1.2) is a one-dimensional random horizontal curve (and hence is one-dimensional) (see Theorem 5.3 and Corollary 5.4).
(3) When K and α are sufficiently large, the rotation number of (1.1) exists (see Theorem 6.4 and Corollary 6.5).

The above results make an important contribution to the understanding of the nonlinear dynamics of stochastic damped sine-Gordon equations with Neumann boundary conditions. Property (1) extends the existence result of random attractor in the Dirichlet boundary case to the Neumann boundary case and shows that system (1.1)–(1.2) is dissipative. By property (2), the asymptotic dynamics of (1.1)–(1.2) with sufficiently large α and K is one-dimensional regardless of the strength of noise and hence is not chaotic. Observe that $\rho \in \mathbb{R}$ is called the *rotation number* of (1.1)–(1.2) (see Definition 6.1 for detail) if for any solution $u(t,x)$ of (1.1)–(1.2) and any $x \in U$, the limit $\lim_{t\to\infty} \frac{u(t,x)}{t}$ exists almost surely and

$$\lim_{t\to\infty} \frac{u(t,x)}{t} = \rho \quad \text{for a.e. } \omega \in \Omega.$$

Property (3) then shows that all the solutions of (1.1)–(1.2) tend to oscillate with the same frequency eventually almost surely and hence frequency locking is successful in (1.1)–(1.2) provided that α and K are sufficiently large.

We remark that the results in the current paper also hold for stochastic damped sine-Gordon equations with periodic boundary conditions.

It should be pointed out that the dynamical behavior of variety of systems of the form (1.1) have been studied in [22–25] for ordinary differential equations, [26,27] for partial differential equations and [6,21,28] for stochastic (random) ordinary differential equations. In above literatures, two main aspects considered are the structure of the attractor and the phenomenon of frequency locking. For example, in [28], the authors studied a class of nonlinear noisy oscillators. They proved the existence of a random attractor which is a family of horizontal curves and the existence of a rotation number which implies the frequency locking.

The rest of the paper is organized as follows. In Section 2, we present some basic concepts and properties for general random dynamical systems. In Section 3, we provide some basic settings about (1.1)–(1.2) and show that it generates a random dynamical system in proper function space. We prove in Section 4 the existence of a unique random attractor of the random dynamical system ϕ generated by (1.1)–(1.2) for any $\alpha, K > 0$. We show in Section 5 that the random attractor of ϕ is a random horizontal curve provided that α and K are sufficiently large. In Section 6, we prove the existence of a rotation number of (1.1)–(1.2) provided that α and K are sufficiently large.

2. General random dynamical systems

In this section, we collect some basic knowledge about general random dynamical systems (see [1,8] for details). Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be the ergodic metric dynamical system mentioned in Section 1.

Definition 2.1. A continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X \quad (t, \omega, u) \mapsto \varphi(t, \omega, u)$$

such that the following properties hold:

(1) $\varphi(0,\omega,u)=u$ for all $\omega\in\Omega$ and $u\in X$;
(2) $\varphi(t+s,\omega,\cdot)=\varphi(t,\theta_s\omega,\varphi(s,\omega,\cdot))$ for all $s,t\geqslant 0$ and $\omega\in\Omega$;
(3) φ is continuous in t and u.

For given $u\in X$ and $E,F\subset X$, we define

$$d(u,E)=\inf_{v\in E}\|u-v\|_X$$

and

$$d_H(E,F)=\sup_{u\in E}d(u,F).$$

$d_H(E,F)$ is called the *Hausdorff semi-distance* from E to F.

Definition 2.2.

(1) A set-valued mapping $\omega\mapsto D(\omega):\Omega\to 2^X$ is said to be a *random set* if the mapping $\omega\mapsto d(u,D(\omega))$ is measurable for any $u\in X$. If $D(\omega)$ is also closed (compact) for each $\omega\in\Omega$, the mapping $\omega\mapsto D(\omega)$ is called a *random closed* (*compact*) *set*. A random set $\omega\mapsto D(\omega)$ is said to be *bounded* if there exist $u_0\in X$ and a random variable $R(\omega)>0$ such that

$$D(\omega)\subset\bigl\{u\in X\colon \|u-u_0\|_X\leqslant R(\omega)\bigr\}\quad\text{for all } \omega\in\Omega.$$

(2) A random set $\omega\mapsto D(\omega)$ is called *tempered* provided for $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$\lim_{t\to\infty}e^{-\beta t}\sup\bigl\{\|b\|_X\colon b\in D(\theta_{-t}\omega)\bigr\}=0\quad\text{for all } \beta>0.$$

(3) A random set $\omega\mapsto B(\omega)$ is said to be a *random absorbing set* if for any tempered random set $\omega\mapsto D(\omega)$, there exists $t_0(\omega)$ such that

$$\varphi\bigl(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\bigr)\subset B(\omega)\quad\text{for all } t\geqslant t_0(\omega),\ \omega\in\Omega.$$

(4) A random set $\omega\mapsto B_1(\omega)$ is said to be a *random attracting set* if for any tempered random set $\omega\mapsto D(\omega)$, we have

$$\lim_{t\to\infty}d_H\bigl(\varphi\bigl(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\bigr),B_1(\omega)\bigr)=0\quad\text{for all } \omega\in\Omega.$$

(5) A random compact set $\omega\mapsto A(\omega)$ is said to be a random attractor if it is an random attracting set and $\varphi(t,\omega,A(\omega))=A(\theta_t\omega)$ for all $\omega\in\Omega$ and $t\geqslant 0$.

Theorem 2.3. *(See [8, Theorem 1.8.1].) Let φ be a continuous random dynamical system over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. If there is a tempered random compact attracting set $\omega\mapsto B_1(\omega)$ of φ, then $\omega\mapsto A(\omega)$ is a random attractor of φ, where*

$$A(\omega)=\bigcap_{t>0}\overline{\bigcup_{\tau\geqslant t}\varphi\bigl(\tau,\theta_{-\tau}\omega,B_1(\theta_{-\tau}\omega)\bigr)},\quad \omega\in\Omega.$$

Moreover, $\omega\mapsto A(\omega)$ is the unique random attractor of ϕ.

3. Basic settings

In this section, we give some basic settings about (1.1)–(1.2) and show that it generates a random dynamical system. Define an unbounded operator

$$A : D(A) \equiv \left\{ u \in H^2(U)\colon \left.\frac{\partial u}{\partial n}\right|_{\partial U} = 0 \right\} \to L^2(U), \quad u \mapsto -K\Delta u. \tag{3.1}$$

Clearly, A is nonnegative definite and self-adjoint. Its spectral set consists of only nonnegative eigenvalues, denoted by λ_i, satisfying

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \quad (\lambda_i \to +\infty \quad \text{as } i \to \infty). \tag{3.2}$$

It is well known that $-A$ generates an analytic semigroup of bounded linear operators $\{e^{-At}\}_{t\geqslant 0}$ on $L^2(U)$ (and $H^1(U)$). Let $E = H^1(U) \times L^2(U)$, endowed with the usual norm

$$\|Y\|_{H^1\times L^2} = \left(\|\nabla u\|^2 + \|u\|^2 + \|v\|^2\right)^{\frac{1}{2}} \quad \text{for } Y = (u, v)^\top, \tag{3.3}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(U)$ and $\top$ stands for the transposition.

The existence of solutions to problem (1.1)–(1.2) follows from [10]. We next transform the problem (1.1)–(1.2) to a deterministic system with a random parameter, and then show that it generates a random dynamical system.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be the ergodic metric dynamical system in Section 1. For $j \in \{1, 2, \ldots, m\}$, consider the one-dimensional Ornstein–Uhlenbeck equation

$$dz_j + z_j\,dt = dW_j(t).$$

Its unique stationary solution is given by

$$z_j(\theta_t\omega_j) = \int_{-\infty}^{0} e^s(\theta_t\omega_j)(s)\,ds = -\int_{-\infty}^{0} e^s\omega_j(s+t)\,ds + \omega_j(t), \quad t \in \mathbb{R}.$$

Note that the random variable $|z_j(\omega_j)|$ is tempered and the mapping $t \mapsto z_j\ (\theta_t\omega_j)$ is $\mathbb{P}$-a.s. continuous (see [2,12]). More precisely, there is a θ_t-invariant $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \mapsto z_j\ (\theta_t\omega_j)$ is continuous for $\omega \in \Omega_0$ and $j = 1, 2, \ldots, m$. Putting $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$, which solves $dz + z\,dt = \sum_{j=1}^m h_j\,dW_j$.

Now, let $v = u_t - z(\theta_t\omega)$ and take the functional space E into consideration, we obtain the equivalent system of (1.1)–(1.2),

$$\begin{cases} \dot u = v + z(\theta_t\omega), \\ \dot v = -Au - \alpha v - \sin u + f + (1-\alpha)z(\theta_t\omega). \end{cases} \tag{3.4}$$

Let $Y = (u, v)^\top$, $C = \begin{pmatrix} 0 & I \\ -A & -\alpha I \end{pmatrix}$, $F(\theta_t\omega, Y) = (z(\theta_t\omega), -\sin u + f + (1-\alpha)z(\theta_t\omega))^\top$, problem (3.4) has the following simple matrix form

$$\dot Y = CY + F(\theta_t\omega, Y). \tag{3.5}$$

We will consider (3.4) or (3.5) for $\omega \in \Omega_0$ and write Ω_0 as Ω from now on.

Clearly, C is an unbounded closed operator on E with domain $D(C) = D(A) \times H^1(U)$. It is not difficult to check that the spectral set of C consists of only following points [27]

$$\mu_i^{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\lambda_i}}{2}, \quad i = 0, 1, 2, \ldots$$

and C generates a C_0-semigroup of bounded linear operators $\{e^{Ct}\}_{t \geqslant 0}$ on E. Furthermore, let $F^{\omega}(t, Y) := F(\theta_t \omega, Y)$, it is easy to see that $F^{\omega}(\cdot,\cdot) : \mathbb{R}^+ \times E \to E$ is continuous in t and globally Lipschitz continuous in Y for each $\omega \in \Omega$. By the classical theory concerning the existence and uniqueness of the solutions, we obtain (see [20,29]).

Theorem 3.1. *Consider* (3.5). *For each $\omega \in \Omega$ and each $Y_0 \in E$, there exists a unique function $Y(\cdot, \omega, Y_0) \in C([0, +\infty); E)$ such that $Y(0, \omega, Y_0) = Y_0$ and $Y(t, \omega, Y_0)$ satisfies the integral equation*

$$Y(t, \omega, Y_0) = e^{Ct} Y_0 + \int_0^t e^{C(t-s)} F\big(\theta_s \omega, Y(s, \omega, Y_0)\big)\, ds. \tag{3.6}$$

Furthermore, if $Y_0 \in D(C)$, there exists $Y(\cdot, \omega, Y_0) \in C([0, +\infty); D(C)) \cap C^1((\mathbb{R}^+, +\infty); E)$ which satisfies (3.6) *and $Y(t, \omega, Y_0)$ is jointly continuous in t, Y_0, and is measurable in ω. Then, $Y : \mathbb{R}^+ \times \Omega \times E \to E$ (or $\mathbb{R}^+ \times \Omega \times D(C) \to D(C)$) is a continuous random dynamical system.*

We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times E \to E$ (or $\mathbb{R}^+ \times \Omega \times D(C) \to D(C)$) by

$$\phi(t, \omega, \phi_0) = Y\big(t, \omega, Y_0(\omega)\big) + \big(0, z(\theta_t \omega)\big)^{\top}, \tag{3.7}$$

where $\phi_0 = (u_0, u_1)^{\top}$ and $Y_0(\omega) = (u_0, u_1 - z(\omega))^{\top}$. It is easy to see that ϕ is a continuous random dynamical system associated with the problem (1.1)–(1.2) on E (or $D(C)$). We next show a useful property of just defined random dynamical systems.

Lemma 3.2. *Suppose that $p_0 = (2\pi, 0)^{\top}$. The random dynamical system Y defined in* (3.6) *is p_0-translation invariant in the sense that*

$$Y(t, \omega, Y_0 + p_0) = Y(t, \omega, Y_0) + p_0, \quad t \geqslant 0,\ \omega \in \Omega,\ Y_0 \in E.$$

Proof. Since $Cp_0 = 0$ and $F(t, \omega, Y)$ is p_0-periodic in Y, $Y(t, \omega, Y_0) + p_0$ is a solution of (3.5) with initial data $Y_0 + p_0$. Thus, $Y(t, \omega, Y_0) + p_0 = Y(t, \omega, Y_0 + p_0)$. $\square$

Note that $\mu_1^+ \to 0$ as $\alpha \to +\infty$, which will cause some difficulty. In order to overcome it, we introduce a new norm which is equivalent to the usual norm $\|\cdot\|_{H^1 \times L^2}$ on E in (3.3). Here, we only collect some results about the new norm (see [27] for details). Since C has at least two real eigenvalues 0 and $-\alpha$ with corresponding eigenvectors $\eta_0 = (1, 0)^{\top}$ and $\eta_{-1} = (1, -\alpha)^{\top}$, let $E_1 = \operatorname{span}\{\eta_0\}$, $E_{-1} = \operatorname{span}\{\eta_{-1}\}$ and $E_{11} = E_1 + E_{-1}$. For any $u \in L^2(U)$, define $\bar{u} = \frac{1}{|U|} \int_U u(x)\, dx$, i.e., the spatial average of u, let $\dot{L}^2(U) = \{u \in L^2(U){:}\ \bar{u} = 0\}$, $\dot{H}^1(U) = H^1(U) \cap \dot{L}^2(U)$ and $E_{22} = \dot{H}^1(U) \times \dot{L}^2(U)$. It's easy to see that $E = E_{11} \oplus E_{22}$ and E_1 is invariant under C. We now define two bilinear forms on E_{11} and E_{22} respectively. For $Y_i = (u_i, v_i)^{\top} \in E_{11}$, $i = 1, 2$, let

$$\langle Y_1, Y_2 \rangle_{E_{11}} = \frac{\alpha^2}{4} \langle u_1, u_2 \rangle + \left\langle \frac{\alpha}{2} u_1 + v_1, \frac{\alpha}{2} u_2 + v_2 \right\rangle, \tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(U)$, and for $Y_i = (u_i, v_i)^\top \in E_{22}$, $i = 1, 2$, let

$$\langle Y_1, Y_2\rangle_{E_{22}} = \left\langle A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_2\right\rangle + \left(\frac{\alpha^2}{4} - \delta\lambda_1\right)\langle u_1, u_2\rangle + \left\langle \frac{\alpha}{2}u_1 + v_1, \frac{\alpha}{2}u_2 + v_2\right\rangle, \tag{3.9}$$

where $A^{\frac{1}{2}} = \sqrt{K}\nabla$ (see (3.1) for the definition of A) and $\delta \in (0, 1]$. By the Poincaré inequality

$$\left\| A^{\frac{1}{2}} u\right\|^2 \geqslant \lambda_1 \|u\|^2, \quad \forall u \in \dot{H}^1(U),$$

(3.9) is then positive definite. Note that for any $Y \in E$, $\bar{Y} = \int_U Y(x)\,dx \in E_{11}$ and $Y - \bar{Y} \in E_{22}$, thus we define

$$\langle Y_1, Y_2\rangle_E = \langle \bar{Y}_1, \bar{Y}_2\rangle_{E_{11}} + \langle Y_1 - \bar{Y}_1, Y_2 - \bar{Y}_2\rangle_{E_{22}} \quad \text{for } Y_1, Y_2 \in E. \tag{3.10}$$

Lemma 3.3. *(See [27].)*

(1) (3.8) *and* (3.9) *define inner products on E_{11} and E_{22}, respectively.*
(2) (3.10) *defines an inner product on E, and the corresponding norm $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^1\times L^2}$ in* (3.3)*, where*

$$\begin{aligned} \|Y\|_E &= \left(\frac{\alpha^2}{4}\|u\|^2 + \left\|\frac{\alpha}{2}u + v\right\|^2 + \left\|A^{\frac{1}{2}}(u - \bar{u})\right\|^2 - \delta\lambda_1\|u - \bar{u}\|^2\right)^{\frac{1}{2}} \\ &= \left(\frac{\alpha^2}{4}\|u\|^2 + \left\|\frac{\alpha}{2}u + v\right\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2 - \delta\lambda_1\|u - \bar{u}\|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.11}$$

for $Y = (u, v)^\top \in E$.
(3) *In terms of the inner product $\langle\cdot,\cdot\rangle_E$, E_1 and E_{11} are orthogonal to E_{-1} and E_{22}, respectively.*
(4) *In terms of the norm $\|\cdot\|_E$, the Lipschitz constant L_F of F in* (3.5) *satisfies*

$$L_F \leqslant \frac{2}{\alpha}. \tag{3.12}$$

Now let $E_2 = E_{-1} \oplus E_{22}$, then E_2 is orthogonal to E_1 and $E = E_1 \oplus E_2$. Thus, E_2 is also invariant under C. Denote by P and Q $(= I - P)$ the projections from E into E_1 and E_2, respectively.

Lemma 3.4.

(1) *For any $Y \in D(C) \cap E_2$, $\langle CY, Y\rangle_E \leqslant -a\|Y\|_E^2$, where*

$$a = \frac{\alpha}{2} - \left|\frac{\alpha}{2} - \frac{\delta\lambda_1}{\alpha}\right|. \tag{3.13}$$

(2) *$\|e^{Ct}Q\| \leqslant e^{-at}$ for $t \geqslant 0$.*
(3) *$e^{Ct}PY = PY$ for $Y \in E$, $t \geqslant 0$.*

Proof. See Lemma 3.3 and Corollary 3.3.1 in [27] for (1) and (2). We now show (3). For $Y \in D(C) \cap E_1$, since $\frac{d}{dt}e^{Ct}Y = e^{Ct}CY = 0$, we have $e^{Ct}Y = e^{C0}Y = Y$. Then, by approximation, $e^{Ct}Y = Y$ for $u \in E_1$, $t \geqslant 0$, since $D(A) \cap E_1$ is dense in E_1. Thus, $e^{Ct}PY = PY$ for $Y \in E$, $t \geqslant 0$. □

We will need the following lemma and its corollaries.

Lemma 3.5. *For any $\epsilon > 0$, there is a tempered random variable $r : \Omega \mapsto \mathbb{R}^+$ such that*

$$\left\| z(\theta_t\omega) \right\| \leqslant e^{\epsilon|t|} r(\omega) \quad \text{for all } t \in \mathbb{R},\ \omega \in \Omega, \tag{3.14}$$

where $r(\omega)$, $\omega \in \Omega$ satisfies

$$e^{-\epsilon|t|} r(\omega) \leqslant r(\theta_t\omega) \leqslant e^{\epsilon|t|} r(\omega), \quad t \in \mathbb{R},\ \omega \in \Omega. \tag{3.15}$$

Proof. For $j \in \{1, 2, \dots, m\}$, since $|z_j(\omega_j)|$ is a tempered random variable and the mapping $t \mapsto \ln|z_j(\theta_t\omega_j)|$ is $\mathbb{P}$-a.s. continuous, it follows from Proposition 4.3.3 in [1] that for any $\epsilon_j > 0$ there is a tempered random variable $r_j(\omega_j) > 0$ such that

$$\frac{1}{r_j(\omega_j)} \leqslant \left| z_j(\omega_j) \right| \leqslant r_j(\omega_j),$$

where $r_j(\omega_j)$ satisfies, for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$e^{-\epsilon_j|t|} r_j(\omega_j) \leqslant r_j(\theta_t\omega_j) \leqslant e^{\epsilon_j|t|} r_j(\omega_j), \quad t \in \mathbb{R}. \tag{3.16}$$

Taking $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_m = \epsilon$, then we have

$$\left\| z(\theta_t\omega) \right\| \leqslant \sum_{j=1}^{m} \left| z_j(\theta_t\omega_j) \right| \cdot \|h_j\| \leqslant \sum_{j=1}^{m} r_j(\theta_t\omega_j)\|h_j\| \leqslant e^{\epsilon|t|} \sum_{j=1}^{m} r_j(\omega_j)\|h_j\|.$$

Let $r(\omega) = \sum_{j=1}^{m} r_j(\omega_j)\|h_j\|$, (3.14) is satisfied and (3.15) is trivial from (3.16). $\square$

Corollary 3.6. *For any $\epsilon > 0$, there is a tempered random variable $r' : \Omega \mapsto \mathbb{R}^+$ such that*

$$\left\| A^{\frac{1}{2}} z(\theta_t\omega) \right\| \leqslant e^{\epsilon|t|} r'(\omega) \quad \text{for all } t \in \mathbb{R},\ \omega \in \Omega,$$

where $r'(\omega) = \sum_{j=1}^{m} r_j(\omega_j)\|A^{\frac{1}{2}} h_j\|$ satisfies

$$e^{-\epsilon|t|} r'(\omega) \leqslant r'(\theta_t\omega) \leqslant e^{\epsilon|t|} r'(\omega), \quad t \in \mathbb{R},\ \omega \in \Omega.$$

Corollary 3.7. *For any $\epsilon > 0$, there is a tempered random variable $r'' : \Omega \mapsto \mathbb{R}^+$ such that*

$$\left\| A z(\theta_t\omega) \right\| \leqslant e^{\epsilon|t|} r''(\omega) \quad \text{for all } t \in \mathbb{R},\ \omega \in \Omega,$$

where $r''(\omega) = \sum_{j=1}^{m} r_j(\omega_j)\|A h_j\|$ satisfies

$$e^{-\epsilon|t|} r''(\omega) \leqslant r''(\theta_t\omega) \leqslant e^{\epsilon|t|} r''(\omega), \quad t \in \mathbb{R},\ \omega \in \Omega.$$

4. Existence of random attractor

In this section, we study the existence of a random attractor. Throughout this section we assume that $p_0 = 2\pi\eta_0 = (2\pi, 0)^\top \in E_1$ and $\delta \in (0, 1]$ is such that $a > 0$, where a is as in (3.13). We remark in the end of this section that such δ always exists.

The space $D(C)$ can be endowed with the graph norm,

$$\|Y\|_{\tilde{E}} = \|Y\|_E + \|CY\|_E \quad \text{for } Y \in D(C).$$

Since C is a closed operator, $D(C)$ is a Banach space under the graph norm. We denote $(D(C), \|\cdot\|_{\tilde{E}})$ by $\tilde{E}$ and let $\tilde{E}_1 = \tilde{E} \cap E_1$, $\tilde{E}_2 = \tilde{E} \cap E_2$.

By Lemma 3.2 and the fact that operator C has a zero eigenvalue, we will define a random dynamical system $\mathbf{Y}$ defined on torus induced from Y. Then by properties of Y restricted on E_2, we can prove the existence of a random attractor of $\mathbf{Y}$. Thus, we can say that Y has a unbounded random attractor. Now, we define $\mathbf{Y}$.

Let $\mathbb{T}^1 = E_1/p_0\mathbb{Z}$ and $\mathbf{E} = \mathbb{T}^1 \times E_2$. For $Y_0 \in E$, let $\mathbf{Y}_0 := Y_0 \pmod{p_0} = Y_0 + p_0\mathbb{Z} \subset E$ denotes the equivalence class of Y_0, which is an element of $\mathbf{E}$. And the norm on $\mathbf{E}$ is denoted by

$$\|\mathbf{Y}_0\|_{\mathbf{E}} = \inf_{y \in p_0\mathbb{Z}} \|Y_0 + y\|_E.$$

Note that, by Lemma 3.2, $Y(t, \omega, Y_0 + kp_0) = Y(t, \omega, Y_0) + kp_0$, $\forall k \in \mathbb{Z}$ for $t \geqslant 0$, $\omega \in \Omega$ and $Y_0 \in E$. With this, we define $\mathbf{Y} : \mathbb{R}^+ \times \Omega \times \mathbf{E} \to \mathbf{E}$ by setting

$$\mathbf{Y}(t, \omega, \mathbf{Y}_0) = Y(t, \omega, Y_0) \pmod{p_0}, \tag{4.1}$$

where $\mathbf{Y}_0 = Y_0 \pmod{p_0}$. It is easy to see that $\mathbf{Y} : \mathbb{R}^+ \times \Omega \times \mathbf{E} \to \mathbf{E}$ is a random dynamical system.

Similarly, the random dynamical system ϕ defined in (3.7) also induces a random dynamical system $\mathbf{\Phi}$ on $\mathbf{E}$. By (3.7) and (4.1), $\mathbf{\Phi}$ is defined by

$$\mathbf{\Phi}(t, \omega, \mathbf{\Phi}_0) = \mathbf{Y}(t, \omega, \mathbf{Y}_0) + \tilde{z}(\theta_t\omega) \pmod{p_0}, \tag{4.2}$$

where $\mathbf{\Phi}_0 = \phi_0 \pmod{p_0}$, $\tilde{z}(\theta_t\omega) = (0, z(\theta_t\omega))^\top$ and $\mathbf{Y}_0 = \mathbf{\Phi}_0 - \tilde{z}(\omega) \pmod{p_0}$.

The main result of this section can now be stated as follows.

Theorem 4.1. *The random dynamical system* $\mathbf{Y}$ *defined in* (4.1) *has a unique random attractor* $\omega \mapsto \mathbf{A}_0(\omega)$, *where*

$$\mathbf{A}_0(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau \geqslant t} \mathbf{Y}\big(\tau, \theta_{-\tau}\omega, \mathbf{B}_1(\theta_{-\tau}\omega)\big)}, \quad \omega \in \Omega,$$

in which $\omega \mapsto \mathbf{B}_1(\omega)$ *is a tempered random compact attracting set for* $\mathbf{Y}$.

Corollary 4.2. *The induced random dynamical system* $\mathbf{\Phi}$ *defined in* (4.2) *has a random attractor* $\omega \mapsto \mathbf{A}(\omega)$, *where* $\mathbf{A}(\omega) = \mathbf{A}_0(\omega) + \tilde{z}(\omega) \pmod{p_0}$ *for all* $\omega \in \Omega$.

Proof. It follows from (4.2) and Theorem 4.1. □

To prove Theorem 4.1, we first introduce the concept of random pseudo-balls and prove a lemma on the existence of a pseudo-tempered random absorbing pseudo-ball.

Definition 4.3. Let $R:\Omega\to\mathbb{R}^+$ be a random variable. A random pseudo-ball $\omega\in\Omega\mapsto B(\omega)\subset E$ with random radius $\omega\mapsto R(\omega)$ is a set of the form

$$\omega\mapsto B(\omega)=\big\{b(\omega)\in E\colon\ \big\|Q\,b(\omega)\big\|_E\leqslant R(\omega)\big\}.$$

Furthermore, a random set $\omega\mapsto B(\omega)\subset E$ is called pseudo-tempered provided $\omega\mapsto Q\,B(\omega)$ is a tempered random set in E, i.e., for $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$\lim_{t\to\infty}e^{-\beta t}\sup\big\{\|Q\,b\|_E\colon\ b\in B(\theta_{-t}\omega)\big\}=0\quad\text{for all }\beta>0.$$

Notice that any random pseudo-ball $\omega\mapsto B(\omega)$ in E has the form $\omega\mapsto E_1\times Q\,B(\omega)$, where $\omega\mapsto Q\,B(\omega)$ is a random ball in E_2, which implies the measurability of $\omega\mapsto B(\omega)$.

By Definition 4.3, if $\omega\mapsto B(\omega)$ is a random pseudo-ball in E, then $\omega\mapsto B(\omega)$ (mod p_0) is random bounded set in $\mathbf{E}$. And if $\omega\mapsto B(\omega)$ is a pseudo-tempered random set in E, then $\omega\mapsto B(\omega)$ (mod p_0) is tempered random set in $\mathbf{E}$.

Lemma 4.4. *Let $a>0$. Then there exists a tempered random set $\omega\mapsto\mathbf{B}_0(\omega):=B_0(\omega)$ (mod p_0) in $\mathbf{E}$ such that, for any tempered random set $\omega\mapsto\mathbf{B}(\omega):=B(\omega)$ (mod p_0) in $\mathbf{E}$, there is a $T_{\mathbf{B}}(\omega)>0$ such that*

$$\mathbf{Y}\big(t,\theta_{-t}\omega,\mathbf{B}(\theta_{-t}\omega)\big)\subset\mathbf{B}_0(\omega)\quad\textit{for all }t\geqslant T_{\mathbf{B}}(\omega),\ \omega\in\Omega,$$

where $\omega\mapsto B_0(\omega)$ is a random pseudo-ball in E with random radius $\omega\mapsto R_0(\omega)$ and $\omega\mapsto B(\omega)$ is any pseudo-tempered random set in E.

Proof. For $\omega\in\Omega$, we obtain from (3.6) that

$$Y\big(t,\omega,Y_0(\omega)\big)=e^{Ct}Y_0(\omega)+\int_0^t e^{C(t-s)}F\big(\theta_s\omega,Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds.\tag{4.3}$$

The projection of (4.3) to E_2 is

$$Q\,Y\big(t,\omega,Y_0(\omega)\big)=e^{Ct}Q\,Y_0(\omega)+\int_0^t e^{C(t-s)}Q\,F\big(\theta_s\omega,Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds.\tag{4.4}$$

By replacing ω by $\theta_{-t}\omega$, it follows from (4.4) that

$$Q\,Y\big(t,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)=e^{Ct}Q\,Y_0(\theta_{-t}\omega)+\int_0^t e^{C(t-s)}Q\,F\big(\theta_{s-t}\omega,Y\big(s,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)\big)\,ds,$$

and it then follows from Lemma 3.4 and $Q^2=Q$ that

$$\begin{aligned}&\big\|Q\,Y\big(t,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)\big\|_E\\&\quad\leqslant e^{-at}\big\|Q\,Y_0(\theta_{-t}\omega)\big\|_E+\int_0^t e^{-a(t-s)}\big\|F\big(\theta_{s-t}\omega,Y\big(s,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)\big)\big\|_E\,ds.\end{aligned}\tag{4.5}$$

By (3.11), Lemma 3.5 and Corollary 3.6 with $\epsilon = \frac{a}{2}$,

$$\begin{aligned}
&\left\| F\big(\theta_{s-t}\omega, Y\big(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\big)\right\|_E \\
&\quad = \left(\frac{\alpha^2}{4}\left\|z(\theta_{s-t}\omega)\right\|^2 + \left\|\left(1-\frac{\alpha}{2}\right)z(\theta_{s-t}\omega) - \sin(Y_u) + f\right\|^2 + \left\|A^{\frac{1}{2}}z(\theta_{s-t}\omega)\right\|^2 \right. \\
&\qquad \left. - \delta\lambda_1\left\|z(\theta_{s-t}\omega) - \overline{z(\theta_{s-t}\omega)}\right\|^2\right)^{\frac{1}{2}} \\
&\quad \leqslant \left(\big(\alpha^2 - 3\alpha + 3\big)\left\|z(\theta_{s-t}\omega)\right\|^2 + 3\left\|\sin(Y_u)\right\|^2 + 3\|f\|^2 + \left\|A^{\frac{1}{2}}z(\theta_{s-t}\omega)\right\|^2\right)^{\frac{1}{2}} \\
&\quad \leqslant \left(\big(\alpha^2 - 3\alpha + 3\big)e^{a(t-s)}\big(r(\omega)\big)^2 + e^{a(t-s)}\big(r'(\omega)\big)^2 + 3|U| + 3\|f\|^2\right)^{\frac{1}{2}} \\
&\quad \leqslant a_1 e^{\frac{a}{2}(t-s)} r(\omega) + e^{\frac{a}{2}(t-s)} r'(\omega) + a_2,
\end{aligned}$$

where Y_u satisfies $Y(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)) = (Y_u, Y_v)^\top$, $a_1 = \sqrt{\alpha^2 - 3\alpha + 3}$, $a_2 = \sqrt{3|U| + 3\|f\|^2}$ and $|U|$ is the Lebesgue measure of U. We find from (4.5) that

$$\left\|QY\big(t,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\right\|_E \leqslant e^{-at}\left\|QY_0(\theta_{-t}\omega)\right\|_E + \frac{2}{a}\big(1 - e^{-\frac{a}{2}t}\big)\big(a_1 r(\omega) + r'(\omega)\big) + \frac{a_2}{a}\big(1 - e^{-at}\big).$$

Now for $\omega \in \Omega$, define

$$R_0(\omega) = \frac{4}{a}\big(a_1 r(\omega) + r'(\omega)\big) + \frac{2a_2}{a}.$$

Then, for any pseudo-tempered random set $\omega \mapsto B(\omega)$ in E and any $Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there is a $T_B(\omega) > 0$ such that for $t \geqslant T_B(\omega)$,

$$\left\|QY\big(t,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\right\|_E \leqslant R_0(\omega), \quad \omega \in \Omega,$$

which implies

$$Y\big(t,\theta_{-t}\omega, B(\theta_{-t}\omega)\big) \subset B_0(\omega) \quad \text{for all } t \geqslant T_B(\omega),\ \omega \in \Omega,$$

where $\omega \mapsto B_0(\omega)$ is the random pseudo-ball centered at origin with random radius $\omega \mapsto R_0(\omega)$. In fact, $\omega \mapsto R_0(\omega)$ is a tempered random variable since $\omega \mapsto r(\omega)$ and $\omega \mapsto r'(\omega)$ are tempered random variables. Then the measurability of random pseudo-tempered ball $\omega \mapsto B_0(\omega)$ is obtained from Definition 4.3 and $\omega \mapsto B_0(\omega)$ is a random pseudo-ball. Hence, $\omega \mapsto \mathbf{B}_0(\omega) := B_0(\omega) \pmod{p_0}$ is a tempered random ball in $\mathbf{E}$. It then follows from the definition of $\mathbf{Y}$ that

$$\mathbf{Y}\big(t,\theta_{-t}\omega, \mathbf{B}(\theta_{-t}\omega)\big) \subset \mathbf{B}_0(\omega) \quad \text{for all } t \geqslant T_{\mathbf{B}}(\omega),\ \omega \in \Omega,$$

where $T_{\mathbf{B}}(\omega) = T_B(\omega)$ for $\omega \in \Omega$. This complete the proof. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 2.3, it suffices to prove the existence of a random attracting set which restricted on E_2 is tempered and compact, i.e., there exists a random set $\omega \mapsto B_1(\omega)$ such that $\omega \mapsto QB_1(\omega)$ is tempered and compact in E_2 and for any pseudo-tempered random set $\omega \mapsto B(\omega)$ in E,

$$d_H\big(Y\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big),B_1(\omega)\big)\to 0\quad \text{as } t\to\infty,\ \omega\in\Omega,$$

where d_H is the Hausdorff semi-distance. Since pseudo-tempered random sets in E are absorbed by the random absorbing set $\omega\mapsto B_0(\omega)$, it suffices to prove that

$$d_H\big(Y\big(t,\theta_{-t}\omega,B_0(\theta_{-t}\omega)\big),B_1(\omega)\big)\to 0\quad \text{as } t\to\infty,\ \omega\in\Omega. \tag{4.6}$$

Clearly, if such a $\omega\mapsto B_1(\omega)$ exists, then $\omega\mapsto \mathbf{B}_1(\omega):=B_1(\omega)\ (\mathrm{mod}\ p_0)$ is a tempered random compact attracting set for $\mathbf{Y}$. We next show that (4.6) holds.

By the superposition principle, (3.5) with initial data $Y_0(\omega)$ can be decomposed into

$$\dot Y_1=CY_1+F\big(\theta_t\omega,Y\big(t,\omega,Y_0(\omega)\big)\big),\qquad Y_{10}(\omega)=0 \tag{4.7}$$

and

$$\dot Y_2=CY_2,\qquad Y_{20}(\omega)=Y_0(\omega), \tag{4.8}$$

where $Y(t,\omega,Y_0(\omega))$ is the solution of (3.5) with initial data $Y_0(\omega)\in B_0(\omega)$. Let $Y_1(t,\omega,Y_{10}(\omega))$ and $Y_2(t,Y_{20}(\omega))$ be solutions of (4.7) and (4.8), respectively. We now give some estimations of $Y_1(t,\omega,Y_{10}(\omega))$ and $Y_2(t,Y_{20}(\omega))$, which ensure the existence of a random attracting set which restricted on E_2 is tempered and compact.

We first estimate $Y_2(t,Y_{20}(\omega))$. Clearly, (4.8) is a linear problem. It is easy to see that

$$Y_2\big(t,Y_{20}(\omega)\big)=e^{Ct}Y_{20}(\omega),$$

which implies (with ω being replaced by $\theta_{-t}\omega$) that

$$\big\|QY_2\big(t,Y_{20}(\theta_{-t}\omega)\big)\big\|_E\leqslant \big\|e^{Ct}Q\big\|\cdot\big\|QY_{20}(\theta_{-t}\omega)\big\|_E\leqslant e^{-at}R_0(\theta_{-t}\omega)\to 0\quad \text{as } t\to\infty. \tag{4.9}$$

For $Y_1(t,\omega,Y_{10}(\omega))$, we show that it is bounded by a tempered random bounded closed set in $\tilde E$, which then is compact in E since $\tilde E$ is compactly imbedded in E. Note that

$$Y_1\big(t,\omega,Y_{10}(\omega)\big)=\int_0^t e^{C(t-s)}F\big(\theta_s\omega,Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds, \tag{4.10}$$

it then follows that

$$\big\|QY_1\big(t,\theta_{-t}\omega,Y_{10}(\theta_{-t}\omega)\big)\big\|_E\leqslant \frac{2}{a}\big(1-e^{-\frac{a}{2}t}\big)\big(a_1r(\omega)+r'(\omega)\big)+\frac{a_2}{a}\big(1-e^{-at}\big), \tag{4.11}$$

where $a_1=\sqrt{\alpha^2-3\alpha+3}$ and $a_2=\sqrt{3|U|+3\|f\|^2}$ are the same as in the proof of Lemma 4.4, $|U|$ denotes the Lebbesgue measure of U.

We next estimate $CQY_1(t,\theta_{-t}\omega,Y_{10}(\theta_{-t}\omega))$. We find from (4.10) that

$$\begin{aligned}CQY_1\big(t,\theta_{-t}\omega,Y_{10}(\theta_{-t}\omega)\big)&=\int_0^t e^{C(t-s)}CQF\big(\theta_{s-t}\omega,Y\big(s,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)\big)\,ds\\&=\int_0^t e^{C(t-s)}CF\big(\theta_{s-t}\omega,Y\big(s,\theta_{-t}\omega,Y_0(\theta_{-t}\omega)\big)\big)\,ds.\end{aligned}$$

Then,

$$\big\| CQY_1\big(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\big)\big\|_E \leqslant \int_0^t e^{-a(t-s)}\big\| CF\big(\theta_{s-t}\omega, Y\big(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\big)\big\|_E\,ds. \quad (4.12)$$

Obviously,

$$CF\big(\theta_{s-t}\omega, Y\big(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\big) = \begin{pmatrix} -\sin(Y_u) + f + (1-\alpha)z(\theta_{s-t}\omega) \\ \alpha\sin(Y_u) - \alpha f - \alpha(1-\alpha) - Az(\theta_{s-t}\omega) \end{pmatrix},$$

where Y_u satisfies $Y(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)) = (Y_u, Y_v)^\top$. By (3.11), Lemma 3.5, Corollaries 3.6 and 3.7 with $\epsilon = \frac{a}{2}$,

$$\begin{aligned}
&\big\| CF\big(\theta_{s-t}\omega, Y\big(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\big)\big\|_E^2 \\
&\quad\leqslant \frac{7}{4}\alpha^2\big\|\sin(Y_u)\big\|^2 + \frac{7}{4}\alpha^2\|f\|^2 + \frac{7}{4}\alpha^2(1-\alpha)^2\big\|z(\theta_{s-t}\omega)\big\|^2 + 4\big\|Az(\theta_{s-t}\omega)\big\|^2 \\
&\qquad + 3\big\|A^{\frac{1}{2}}\sin(Y_u)\big\|^2 + 3\big\|A^{\frac{1}{2}}f\big\|^2 + 3(1-\alpha)^2\big\|A^{\frac{1}{2}}z(\theta_{s-t}\omega)\big\|^2 \\
&\quad\leqslant a_3^2 + \frac{7}{4}\alpha^2(1-\alpha)^2 e^{a(t-s)}\big(r(\omega)\big)^2 + 3(1-\alpha)^2 e^{a(t-s)}\big(r'(\omega)\big)^2 \\
&\qquad + 4e^{a(t-s)}\big(r''(\omega)\big)^2 + 3\big\|A^{\frac{1}{2}}\sin(Y_u)\big\|^2 \\
&\quad\leqslant \bigg(a_3 + \frac{\sqrt{7}}{2}\alpha|1-\alpha|e^{\frac{a}{2}(t-s)}r(\omega) + \sqrt{3}|1-\alpha|e^{\frac{a}{2}(t-s)}r'(\omega) \\
&\qquad + 2e^{\frac{a}{2}(t-s)}r''(\omega) + \sqrt{3}\big\|A^{\frac{1}{2}}\sin(Y_u)\big\|\bigg)^2,
\end{aligned}$$

where $a_3 = \sqrt{\frac{7}{4}\alpha^2|U| + \frac{7}{4}\alpha^2\|f\|^2 + 3\|A^{\frac{1}{2}}f\|^2}$. Then, (4.12) implies

$$\begin{aligned}
&\big\| CQY_1\big(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\big)\big\|_E \\
&\quad\leqslant \frac{a_3}{a}\big(1-e^{-at}\big) + \sqrt{3}\int_0^t e^{-a(t-s)}\big\|A^{\frac{1}{2}}\sin(Y_u)\big\|\,ds \\
&\qquad + \frac{2}{a}\bigg(\frac{\sqrt{7}}{2}\alpha|1-\alpha|r(\omega) + \sqrt{3}|1-\alpha|r'(\omega) + 2r''(\omega)\bigg)\big(1-e^{-\frac{a}{2}t}\big). \qquad (4.13)
\end{aligned}$$

For the integral on the right-hand side of (4.13), we note that

$$\big\|A^{\frac{1}{2}}\sin(Y_u)\big\| \leqslant \big\|A^{\frac{1}{2}}Y_u\big\| \leqslant a_4\big\|QY\big(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big)\big\|_E,$$

where $a_4 = \sqrt{2/(2-\delta)}$. Since

$$\begin{aligned}
&\left\| QY\left(s,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\right)\right\|_E \\
&\quad \leqslant e^{-as}\left\|QY_0(\theta_{-t}\omega)\right\|_E + \int_0^s e^{-a(s-\tau)}\left\|F\left(\theta_{\tau-t}\omega, Y\left(\tau,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\right)\right)\right\|_E d\tau,
\end{aligned}$$

we find that

$$\begin{aligned}
&\int_0^t e^{-a(t-s)}\left\|A^{\frac{1}{2}}\sin(Y_u)\right\| ds \\
&\quad \leqslant a_4 t e^{-at}\left\|QY_0(\theta_{-t}\omega)\right\|_E + a_4\int_0^t\int_0^s e^{-a(t-\tau)}\left\|F\left(\theta_{\tau-t}\omega, Y\left(\tau,\theta_{-t}\omega, Y_0(\theta_{-t}\omega)\right)\right)\right\|_E d\tau\, ds \\
&\quad \leqslant a_4 t e^{-at}\left\|QY_0(\theta_{-t}\omega)\right\|_E \\
&\qquad + a_4\int_0^t \left(\frac{2}{a}\left(a_1 r(\omega)+r'(\omega)\right)\left(e^{-\frac{a}{2}(t-s)}-e^{-\frac{a}{2}t}\right)+\frac{a_2}{a}\left(e^{-a(t-s)}-e^{-at}\right)\right) ds \\
&\quad = a_4 t e^{-at}\left\|QY_0(\theta_{-t}\omega)\right\|_E + \frac{2a_4}{a}\left(a_1 r(\omega)+r'(\omega)\right)\left(\frac{2}{a}\left(1-e^{-\frac{a}{2}t}\right)-te^{-\frac{a}{2}t}\right) \\
&\qquad + \frac{a_2 a_4}{a}\left(\frac{1}{a}\left(1-e^{-at}\right)-te^{-at}\right).
\end{aligned} \tag{4.14}$$

Combining (4.11), (4.13) and (4.14), there is a $T(\omega) > 0$ such that for all $t \geqslant T(\omega)$,

$$\begin{aligned}
\left\|QY_1\left(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\right)\right\|_{\tilde{E}} &= \left\|QY_1\left(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\right)\right\|_E + \left\|CQY_1\left(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\right)\right\|_E \\
&\leqslant R_1(\omega),
\end{aligned} \tag{4.15}$$

where $R_1(\omega) = a_5 r(\omega) + a_6 r'(\omega) + \frac{8}{a} r''(\omega) + a_7$ is a tempered random variable, in which $a_5 = \frac{4a_1+2\sqrt{7}\alpha|1-\alpha|}{a} + \frac{8\sqrt{3}a_1 a_4}{a^2}$, $a_6 = \frac{4+4\sqrt{3}|1-\alpha|}{a} + \frac{8\sqrt{3}a_4}{a^2}$ and $a_7 = \frac{2a_2+2a_3}{a} + \frac{2\sqrt{3}a_2 a_4}{a^2}$.

Now, let $\omega \mapsto B_1(\omega)$ be the random pseudo-ball in $\tilde{E}$ centered at origin with random radius $\omega \mapsto R_1(\omega)$, then $\omega \mapsto B_1(\omega)$ is tempered and measurable. By (4.9), (4.15) and

$$QY\left(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega)\right) = QY_1\left(t,\theta_{-t}\omega, Y_{10}(\theta_{-t}\omega)\right) + QY_2\left(t, Y_{20}(\theta_{-t}\omega)\right),$$

we have for $\omega \in \Omega$,

$$d_H\left(Y\left(t,\theta_{-t}\omega, B_0(\theta_{-t}\omega)\right), B_1(\omega)\right) \to 0 \quad \text{as } t \to \infty.$$

Then by the compact embedding of $\tilde{E}$ into E, $\omega \mapsto QB_1(\omega)$ is compact in E_2, which implies that $\omega \mapsto \mathbf{B}_1(\omega) := B_1(\omega) \pmod{p_0}$ is a tempered random compact attracting set for $\mathbf{Y}$. Thus by Theorem 2.3, $\mathbf{Y}$ has a unique random attractor $\omega \mapsto \mathbf{A}_0(\omega)$, where

$$\mathbf{A}_0(\omega) = \bigcap_{t>0}\overline{\bigcup_{\tau\geqslant t}\mathbf{Y}\left(\tau,\theta_{-\tau}\omega, \mathbf{B}_1(\theta_{-\tau}\omega)\right)}, \quad \omega \in \Omega.$$

This completes the proof. □

Remark 4.5.

(1) For any $\alpha > 0$ and $\lambda_1 = K\tilde{\lambda}_1 > 0$ (see (3.2)), there is a $\delta \in (0, 1]$ such that $a > 0$ holds, where a is as in (3.13) and $\tilde{\lambda}_1$ is the smallest positive eigenvalue of $-\Delta$ and a constant.

(2) We can say that the random dynamical Y (or ϕ) has a unique random attractor in the sense that the induced random dynamical system $\mathbf{Y}$ (or $\mathbf{\Phi}$) has a unique random attractor, and we will say that Y (or ϕ) has a unique random attractor directly in the sequel. We denote the random attractor of Y and ϕ by $\omega \mapsto A_0(\omega)$ and $\omega \mapsto A(\omega)$ respectively. Indeed, $\omega \mapsto A_0(\omega)$ and $\omega \mapsto A(\omega)$ satisfy

$$\mathbf{A}_0(\omega) = A_0(\omega) \pmod{p_0}, \qquad \mathbf{A}(\omega) = A(\omega) \pmod{p_0}, \quad \omega \in \Omega.$$

(3) For the deterministic damped sine-Gordon equation with homogeneous Neumann boundary condition, the authors proved in [27] that the random attractor is a horizontal curve provided that α and K are sufficiently large. Similarly, we expect that the random attractor $\omega \mapsto A(\omega)$ of ϕ has the similar property, i.e., $A(\omega)$ is a horizontal curve for each $\omega \in \Omega$ provided that α and K are sufficiently large. We prove that this is true in next section.

(4) By (2), system (1.1)–(1.2) is dissipative (i.e. it possesses a random attractor). In Section 6, we will show that (1.1)–(1.2) with sufficiently large α and K also has a rotation number and hence all the solutions tend to oscillate with the same frequency eventually.

5. One-dimensional random attractor

In this section, we apply the theory established in [7] to show that the random attractor of Y (or ϕ) is one-dimensional provided that α and K are sufficiently large. This method has been used by Chow, Shen and Zhou [6] to systems of coupled noisy oscillators. Throughout this section we assume that $p_0 = 2\pi\eta_0 = (2\pi, 0)^\top \in E_1$ and $a > 4L_F$ (see (3.13) for the definition of a and see (3.12) for the upper bound of L_F). We remark in the end of this section that this condition can be satisfied provided that α and K are sufficiently large.

Definition 5.1. Suppose $\{\Phi^\omega\}_{\omega\in\Omega}$ is a family of maps from E_1 to E_2 and $n \in \mathbb{N}$. A family of graphs $\omega \mapsto \ell(\omega) \equiv \{(p, \Phi^\omega(p)):\ p \in E_1\}$ is said to be a random np_0-periodic horizontal curve if $\omega \mapsto \ell(\omega)$ is a random set and $\{\Phi^\omega\}_{\omega\in\Omega}$ satisfy the Lipschitz condition

$$\left\|\Phi^\omega(p_1) - \Phi^\omega(p_2)\right\|_E \leqslant \|p_1 - p_2\|_E \quad \text{for all } p_1, p_2 \in E_1,\ \omega \in \Omega$$

and the periodic condition

$$\Phi^\omega(p + np_0) = \Phi^\omega(p) \quad \text{for all } p \in E_1,\ \omega \in \Omega.$$

Clearly, for any $\omega \in \Omega$, $\ell(\omega)$ is a deterministic np_0-periodic horizontal curve. When $n = 1$, we simply call it a horizontal curve.

Lemma 5.2. *Let $a > 4L_F$. Suppose that $\omega \mapsto \ell(\omega)$ is a random np_0-periodic horizontal curve in E. Then, $\omega \mapsto Y(t, \omega, \ell(\omega))$ is also a random np_0-periodic horizontal curve in E for all $t > 0$. Moreover, $\omega \mapsto Y(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega))$ is a random np_0-periodic horizontal curve for all $t > 0$.*

Proof. First, since Y is a random dynamical system and $\omega \mapsto \ell(\omega)$ is a random set in E, $\omega \mapsto Y(t, \omega, \ell(\omega))$ and $\omega \mapsto Y(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega))$ are random sets in E for all $t > 0$. We next show the Lipschitz condition and periodic condition.

It is sufficient to prove the Lipschitz condition and periodic condition valid for $\omega \mapsto \ell(\omega)$ in $D(C)$ since $D(C)$ is dense in E. Clearly, for $\omega \in \Omega$ and $t > 0$,

$$Y\big(t,\omega,\ell(\omega)\big)=\big\{\big(PY\big(t,\omega,p+\Phi^{\omega}(p)\big),\,QY\big(t,\omega,p+\Phi^{\omega}(p)\big)\big):\ p\in E_1\cap D(C)\big\}.$$

For $p_1,p_2\in E_1\cap D(C)$, $p_1\neq p_2$, let $Y_i(t,\omega)=Y(t,\omega,p_i+\Phi^{\omega}(p_i))$, $i=1,2$, $p(t,\omega)=P(Y_1(t,\omega)-Y_2(t,\omega))$ and $q(t,\omega)=Q(Y_1(t,\omega)-Y_2(t,\omega))$, where P, Q are defined as in Section 1. We have by Lemma 3.4

$$\begin{aligned}PY_i(t,\omega)&=e^{Ct}P\big(p_i+\Phi^{\omega}(p_i)\big)+\int_0^t e^{C(t-s)}PF\big(\theta_s\omega,Y_i(s,\omega)\big)\,ds\\&=P\big(p_i+\Phi^{\omega}(p_i)\big)+\int_0^t PF\big(\theta_s\omega,Y_i(s,\omega)\big)\,ds,\quad i=1,2,\end{aligned}$$

and then, $\frac{d}{dt}PY_i(t,\omega)=PF(\theta_t\omega,Y_i(t,\omega))$, $i=1,2$, it then follows that

$$\begin{aligned}\frac{d}{dt}p(t,\omega)&=\frac{d}{dt}P\big(Y_1(t,\omega)-Y_2(t,\omega)\big)\\&=P\big(F\big(\theta_t\omega,Y_1(t,\omega)\big)-F\big(\theta_t\omega,Y_2(t,\omega)\big)\big).\end{aligned}\tag{5.1}$$

Since $p(t,\omega)+q(t,\omega)=Y_1(t,\omega)-Y_2(t,\omega)$,

$$\begin{aligned}\frac{d}{dt}\big(p(t,\omega)+q(t,\omega)\big)&=\frac{d}{dt}\big(Y_1(t,\omega)-Y_2(t,\omega)\big)\\&=C\big(Y_1(t,\omega)-Y_2(t,\omega)\big)+F\big(\theta_t\omega,Y_1(t,\omega)\big)-F\big(\theta_t\omega,Y_2(t,\omega)\big),\end{aligned}$$

then, by the orthogonal decomposition,

$$\begin{aligned}\frac{d}{dt}q(t,\omega)&=C\big(Y_1(t,\omega)-Y_2(t,\omega)\big)+Q\left(F\big(\theta_t\omega,Y_1(t,\omega)\big)-F\big(\theta_t\omega,Y_2(t,\omega)\big)\right)\\&=Cq(t,\omega)+Q\left(F\big(\theta_t\omega,Y_1(t,\omega)\big)-F\big(\theta_t\omega,Y_2(t,\omega)\big)\right).\end{aligned}\tag{5.2}$$

We find from (5.1) that

$$\begin{aligned}\frac{d}{dt}\|p(t,\omega)\|_E^2&=2\left\langle p(t,\omega),\frac{d}{dt}p(t,\omega)\right\rangle_E\\&\geqslant-2\|p(t,\omega)\|_E\cdot\big\|P\big(F\big(\theta_t\omega,Y_1(t,\omega)\big)-F\big(\theta_t\omega,Y_2(t,\omega)\big)\big)\big\|_E\\&\geqslant-2L_F\big(\|p(t,\omega)\|_E^2+\|p(t,\omega)\|_E\|q(t,\omega)\|_E\big).\end{aligned}$$

Similarly, by (5.2) and Lemma 3.4,

$$\frac{d}{dt}\|q(t,\omega)\|_E^2\leqslant-2a\|q(t,\omega)\|_E^2+2L_F\big(\|p(t,\omega)\|_E\|q(t,\omega)\|_E+\|q(t,\omega)\|_E^2\big).$$

Because $a>4L_F$, if there is a $t_0\geqslant 0$ such that $\|q(t_0,\omega)\|_E=\|p(t_0,\omega)\|_E$ and since $\|p(t,\omega)\|_E\neq 0$ for $t\geqslant 0$, then

$$\frac{d}{dt}\bigg|_{t=t_0}\big(\|q(t,\omega)\|_E^2-\|p(t,\omega)\|_E^2\big)\leqslant(8L_F-2a)\|q(t_0,\omega)\|_E^2<0,$$

which means that there is a $\bar{t}_0 > t_0$ such that for $t \in (t_0, \bar{t}_0)$,

$$\begin{aligned}\|q(t,\omega)\|_E^2 - \|p(t,\omega)\|_E^2 &< \|q(0,\omega)\|_E^2 - \|p(0,\omega)\|_E^2 \\ &= \|\Phi^\omega(p_1) - \Phi^\omega(p_2)\|_E^2 - \|p_1 - p_2\|_E^2 \\ &\leqslant 0,\end{aligned}$$

namely, $\|q(t,\omega)\|_E < \|p(t,\omega)\|_E$ for $t \in (t_0, \bar{t}_0)$.

If there is another $t_1 \geqslant \bar{t}_0$ such that $\|q(t_1,\omega)\|_E = \|p(t_1,\omega)\|_E$, then

$$\frac{d}{dt}\bigg|_{t=t_1} \left(\|q(t,\omega)\|_E^2 - \|p(t,\omega)\|_E^2\right) \leqslant (8L_F - 2a)\|q(t_1,\omega)\|_E^2 < 0,$$

which means that there is a $\bar{t}_1 > t_1$ such that for $t \in (t_1, \bar{t}_1)$, $\|q(t,\omega)\|_E < \|p(t,\omega)\|_E$. Continue this process, we have for all $t \geqslant 0$, $\|q(t,\omega)\|_E \leqslant \|p(t,\omega)\|_E$, i.e.,

$$\left\|Q\big(Y_1(t,\omega) - Y_2(t,\omega)\big)\right\|_E \leqslant \left\|P\big(Y_1(t,\omega) - Y_2(t,\omega)\big)\right\|_E,$$

which shows that $\omega \mapsto Y(t,\omega,\ell(\omega))$ satisfies the Lipschitz condition in Definition 5.1.

We next show the periodic condition. We find from Lemma 3.2 that

$$Y\big(t,\omega,p + \Phi^\omega(p)\big) + np_0 = Y\big(t,\omega,p + np_0 + \Phi^\omega(p)\big).$$

Since $\Phi^\omega(p) = \Phi^\omega(p + np_0)$, $Y(t,\omega,p + \Phi^\omega(p)) + np_0 = Y(t,\omega,p + np_0 + \Phi^\omega(p + np_0))$. It follows that

$$QY\big(t,\omega,p + \Phi^\omega(p)\big) = QY\big(t,\omega,p + np_0 + \Phi^\omega(p + np_0)\big).$$

Consequently, $\omega \mapsto Y(t,\omega,\ell(\omega))$ is a random np_0-periodic horizontal curve for all $t > 0$.

Moreover, for any fixed $\omega \in \Omega$ and $t > 0$, $\bar{\omega} = \theta_{-t}\omega \in \Omega$ is fixed. Then, $Y(t,\bar{\omega},\ell(\bar{\omega}))$ is a deterministic np_0-periodic horizontal curve, which yields the assertion. $\square$

Choose $\gamma \in (0, \frac{a}{2})$ such that

$$\frac{2}{\alpha}\left(\frac{1}{\gamma} + \frac{1}{a - 2\gamma}\right) < 1, \tag{5.3}$$

where $\frac{2}{\alpha}$ is the upper bound of the Lipschitz constant of F (see (3.12)). We remark in the end of this section that such a γ exists provided that α and K are sufficiently large. We next show the main result in this section.

Theorem 5.3. *Assume that $a > 4L_F$ and that there is a $\gamma \in (0, \frac{a}{2})$ such that* (5.3) *holds. Then the random attractor $\omega \mapsto A_0(\omega)$ of the random dynamical system Y is a random horizontal curve.*

Proof. By the equivalent relation between ϕ and Y, we mainly focus on Eq. (3.5), which can be viewed as a deterministic system with a random parameter $\omega \in \Omega$. We write it here as $(3.5)_\omega$ for some fixed $\omega \in \Omega$.

Observe that the linear part of $(3.5)_\omega$, i.e.

$$\dot{Y} = CY \tag{5.4}$$

has a one-dimensional center space $E^c = \operatorname{span}\{(1,0)\} = E_1$ and a one co-dimensional stable space $E^s = E_2$. We first show that $(3.5)_\omega$ has a one-dimensional invariant manifold, denoted by $W(\omega)$, and will show later that $W(\omega)$ exponentially attracts all the solutions of $(3.5)_\omega$.

Let $F^\omega(t,Y) = F(\theta_t\omega, Y)$, $\omega \in \Omega$. For fixed $\omega \in \Omega$, consider the following integral equation

$$\tilde{Y}(t) = e^{Ct}\xi + \int_0^t e^{C(t-s)} P F^\omega\big(s, \tilde{Y}(s)\big) ds + \int_{-\infty}^t e^{C(t-s)} Q\, F^\omega\big(s, \tilde{Y}(s)\big)\, ds, \quad t \leqslant 0, \tag{5.5}$$

where $\xi = P\tilde{Y}(0) \in E_1$. For $g : (-\infty, 0] \to E$ such that $\sup_{t\leqslant 0} \|e^{\gamma t} g(t)\|_E < \infty$, define

$$(Lg)(t) = \int_0^t e^{C(t-s)} P g(s)\, ds + \int_{-\infty}^t e^{C(t-s)} Q\, g(s)\, ds, \quad t \leqslant 0.$$

It is easy to see that

$$\sup_{t\leqslant 0}\big\|e^{\gamma t}(Lg)(t)\big\|_E \leqslant \left(\frac{1}{\gamma} + \frac{1}{a-\gamma}\right)\sup_{t\leqslant 0}\big\|e^{\gamma t} g(t)\big\|_E \leqslant \left(\frac{1}{\gamma} + \frac{1}{a-2\gamma}\right)\sup_{t\leqslant 0}\big\|e^{\gamma t} g(t)\big\|_E,$$

which means that $\|L\| \leqslant \frac{1}{\gamma} + \frac{1}{a-2\gamma}$. Then, Theorem 3.3 in [7] shows that for any $\xi \in E_1$, Eq. (5.5) has a unique solution $\tilde{Y}^\omega(t,\xi)$ satisfying $\sup_{t\leqslant 0} \|e^{\gamma t}\tilde{Y}^\omega(t,\xi)\|_E < \infty$. Let

$$h(\omega,\xi) = Q\,\tilde{Y}^\omega(0,\xi) = \int_{-\infty}^0 e^{-Cs} Q\, F^\omega\big(s, \tilde{Y}^\omega(s,\xi)\big)\, ds, \quad \omega \in \Omega.$$

Let

$$W(\omega) = \big\{\xi + h(\omega,\xi) \colon \xi \in E_1\big\}, \quad \omega \in \Omega.$$

For any $\epsilon \in (0,\gamma)$ in Lemma 3.5 and Corollary 3.6, we have

$$\big\|h(\theta_{-t}\omega,\xi)\big\|_E \leqslant \frac{1}{a-\epsilon}\big(a_1 r(\omega) + r'(\omega)\big)e^{\epsilon t} + \frac{a_2}{a}, \quad t \geqslant 0. \tag{5.6}$$

Observe that

$$\begin{aligned}\tilde{Y}^\omega(t,\xi) &= e^{Ct}\xi + \int_0^t e^{C(t-s)} P F^\omega\big(s, \tilde{Y}^\omega(s,\xi)\big)\, ds + \int_{-\infty}^t e^{C(t-s)} Q\, F^\omega\big(s, \tilde{Y}(s,\omega,\xi)\big)\, ds \\ &= e^{Ct}\big(\xi + h(\omega,\xi)\big) + \int_0^t e^{C(t-s)} F^\omega\big(s, \tilde{Y}^\omega(s,\xi)\big)\, ds,\end{aligned}$$

i.e., $\tilde{Y}^{\omega}(t,\xi)$ is the solution of (3.5) with initial data $\xi + h(\omega,\xi)$ for $t \leqslant 0$. Thus, for $Y_0(\omega) = \xi + h(\omega,\xi) \in W(\omega)$, there is a negative continuation of $Y(t,\omega,Y_0(\omega))$, i.e.,

$$Y\big(t,\omega,Y_0(\omega)\big) = \tilde{Y}^{\omega}(t,\xi), \quad t \leqslant 0. \tag{5.7}$$

Moreover, for $t \geqslant 0$, we obtain from (3.6) and (5.7) that

$$\begin{aligned}
&Y\big(t,\omega,Y_0(\omega)\big)\\
&= e^{Ct}\big(\xi + h(\omega,\xi)\big) + \int_0^t e^{C(t-s)} F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds\\
&= e^{Ct}\xi + \int_0^t e^{C(t-s)} F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds + \int_{-\infty}^0 e^{C(t-s)} Q\,F^{\omega}\big(s, \tilde{Y}^{\omega}(s,\xi)\big)\,ds\\
&= e^{Ct}\xi + \int_0^t e^{C(t-s)} F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds + \int_{-\infty}^0 e^{C(t-s)} Q\,F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds\\
&= e^{Ct}\xi + \int_0^t e^{C(t-s)} P\,F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds + \int_{-\infty}^t e^{C(t-s)} Q\,F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds\\
&= e^{Ct}\left(\xi + \int_0^t e^{-Cs} P\,F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds\right) + \int_{-\infty}^0 e^{-Cs} Q\,F\big(\theta_{t+s}\omega, Y\big(t+s,\omega,Y_0(\omega)\big)\big)\,ds.
\end{aligned}$$

Then by the uniqueness of solution of (5.5) for fixed $\omega \in \Omega$, we have

$$\begin{aligned}
&h\left(\theta_t\omega, e^{Ct}\left(\xi + \int_0^t e^{-Cs} P\,F\big(\theta_s\omega, Y\big(s,\omega,Y_0(\omega)\big)\big)\,ds\right)\right)\\
&\quad = \int_{-\infty}^0 e^{-Cs} Q\,F\big(\theta_{t+s}\omega, Y\big(t+s,\omega,Y_0(\omega)\big)\big)\,ds,
\end{aligned}$$

and then for $t \geqslant 0$,

$$Y\big(t,\omega,W(\omega)\big) = W(\theta_t\omega). \tag{5.8}$$

By (5.7) and (5.8), $W(\omega)$ is an invariant manifold of $(3.5)_{\omega}$.

Next, we show that $W(\omega)$ attracts the solutions of $(3.5)_{\omega}$, more precisely, for the given $\omega \in \Omega$, we prove the existence of a stable foliation $\{W_s(\omega,Y_0)\colon\ Y_0 \in W(\omega)\}$ of the invariant manifold $W(\omega)$ of $(3.5)_{\omega}$. Consider the following integral equation

$$\hat{Y}(t) = e^{Ct}\eta + \int_0^t e^{C(t-s)} Q\,\big(F^{\omega}\big(s, \hat{Y}(s) + Y^{\omega}\big(s, \xi + h(\omega,\xi)\big)\big)$$

$$\begin{aligned}&- F^\omega\big(s, Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\big)\,ds\\&+ \int_\infty^t e^{C(t-s)} P\big(F^\omega\big(s, \hat{Y}(s) + Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\\&- F^\omega\big(s, Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\big)\,ds, \quad t \geqslant 0, \end{aligned} \tag{5.9}$$

where $\xi + h(\omega,\xi) \in W(\omega)$, $\eta = Q\hat{Y}(0) \in E_2$ and $Y^\omega(t, \xi + h(\omega,\xi)) := Y(t, \omega, \xi + h(\omega,\xi))$, $t \geqslant 0$ is the solution of (3.5) with initial data $\xi + h(\omega,\xi)$ for fixed $\omega \in \Omega$. Theorem 3.4 in [7] shows that for any $\xi \in E_1$ and $\eta \in E_2$, Eq. (5.9) has a unique solution $\hat{Y}^\omega(t,\xi,\eta)$ satisfying $\sup_{t\geqslant 0} \|e^{\gamma t}\hat{Y}^\omega(t,\xi,\eta)\|_E < \infty$ and for any $\xi \in E_1$, η_1, $\eta_2 \in E_2$,

$$\sup_{t\geqslant 0} e^{\gamma t}\big\|\hat{Y}^\omega(t,\xi,\eta_1) - \hat{Y}^\omega(t,\xi,\eta_2)\big\|_E \leqslant M\|\eta_1 - \eta_2\|_E, \tag{5.10}$$

where $M = \frac{1}{1-\frac{2}{\alpha}(\frac{1}{\gamma}+\frac{1}{a-2\gamma})}$. Let

$$\begin{aligned}\hat{h}(\omega,\xi,\eta) &= \xi + P\hat{Y}^\omega(0,\xi,\eta)\\&= \xi + \int_\infty^0 e^{-Cs} P\big(F^\omega\big(s, \hat{Y}^\omega(s,\xi,\eta) + Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\\&\quad - F^\omega\big(s, Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\big)\,ds.\end{aligned}$$

Then, $W_s(\omega, \xi + h(\omega,\xi)) = \{\eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)\colon\ \eta \in E_2\}$ is the stable foliation of $W(\omega)$ at $\xi + h(\omega,\xi)$.

Observe that

$$\begin{aligned}&\hat{Y}^\omega(t,\xi,\eta) + Y^\omega\big(t, \xi + h(\omega,\xi)\big) - Y^\omega\big(t, \xi + h(\omega,\xi)\big)\\&\quad = \hat{Y}^\omega(t,\xi,\eta)\\&\quad = e^{Ct}\big(\eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta) - \xi - h(\omega,\xi)\big)\\&\qquad + \int_0^t e^{C(t-s)}\big(F^\omega\big(s, \hat{Y}^\omega(s,\xi,\eta) + Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\\&\qquad - F^\omega\big(s, Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\big)\,ds\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}&Y^\omega\big(t, \eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)\big) - Y^\omega\big(t, \xi + h(\omega,\xi)\big)\\&\quad = e^{Ct}\big(\eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta) - \xi - h(\omega,\xi)\big)\\&\qquad + \int_0^t e^{C(t-s)}\big(F^\omega\big(s, Y^\omega\big(s, \eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)\big)\big)\\&\qquad - F^\omega\big(s, Y^\omega\big(s, \xi + h(\omega,\xi)\big)\big)\big)\,ds.\end{aligned} \tag{5.12}$$

Comparing (5.11) with (5.12), we find that

$$\hat{Y}^{\omega}(t,\xi,\eta) = Y^{\omega}\big(t, \eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)\big) - Y^{\omega}\big(t, \xi + h(\omega,\xi)\big), \quad t \geqslant 0. \tag{5.13}$$

In addition, if $\eta = 0$, then by the uniqueness of solution of (5.9), $\hat{Y}^{\omega}(t,\xi,0) \equiv 0$ for $t \geqslant 0$, which associates with (5.10) and (5.13) show that

$$\sup_{t \geqslant 0} e^{\gamma t} \big\| Y^{\omega}\big(t, \eta + h(\omega,\xi) + \hat{h}(\omega,\xi,\eta)\big) - Y^{\omega}\big(t, \xi + h(\omega,\xi)\big)\big\|_E \leqslant M\|\eta\|_E \tag{5.14}$$

for any $\xi \in E_1$ and $\eta \in E_2$.

We now claim that $\omega \mapsto W(\omega)$ is the random attractor of Y. Let $\omega \mapsto B(\omega)$ be any pseudo-tempered random set in E. For any $\omega \mapsto Y_0(\omega) \in \omega \mapsto B(\omega)$, there is $\omega \mapsto \xi(\omega) \in E_1$ such that

$$Y_0(\theta_{-t}\omega) \in W_s\big(\theta_{-t}\omega, \xi(\theta_{-t}\omega) + h\big(\theta_{-t}\omega, \xi(\theta_{-t}\omega)\big)\big).$$

Let $\eta(\theta_{-t}\omega) = Q\,Y_0(\theta_{-t}\omega) - h(\theta_{-t}\omega, \xi(\theta_{-t}\omega))$. By (5.6), it is easy to see that

$$\sup_{Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)} \big\|\eta(\theta_{-t}\omega)\big\| \leqslant \sup_{Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)} \big\|Q\,Y_0(\theta_{-t}\omega)\big\| + \frac{1}{a-\epsilon}\big(a_1 r(\omega) + r'(\omega)\big)e^{\epsilon t} + \frac{a_2}{a}.$$

It then follows from (5.14) and the fact that $\omega \mapsto Q\,B(\omega)$ is tempered that

$$\begin{aligned}
&\sup_{Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)} \big\|Y\big(t, \theta_{-t}\omega, Y_0(\theta_{-t}\omega)\big) - Y\big(t, \theta_{-t}\omega, \xi(\theta_{-t}\omega) + h\big(\theta_{-t}\omega, \xi(\theta_{-t}\omega)\big)\big)\big\|_E \\
&\quad \leqslant Me^{-\gamma t} \sup_{Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)} \big\|\eta(\theta_{-t}\omega)\big\|_E \\
&\quad \leqslant Me^{-\gamma t} \sup_{Y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)} \big\|Q\,Y_0(\theta_{-t}\omega)\big\| + \frac{M}{a-\epsilon}\big(a_1 r(\omega) + r'(\omega)\big)e^{(\epsilon-\gamma)t} + \frac{a_2 M}{a}e^{-\gamma t} \\
&\quad \to 0 \quad \text{as } t \to \infty,
\end{aligned}$$

which associates with (5.8) lead to

$$d_H\big(Y\big(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\big), W(\omega)\big) \to 0 \quad \text{as } t \to \infty.$$

Therefore, $A_0(\omega) = W(\omega)$ for $\omega \in \Omega$. Next, we show that $\omega \mapsto A_0(\omega)$ is a random horizontal curve. In fact, for some random horizontal curve $\omega \mapsto \ell(\omega)$ in E, for example, $\ell(\omega) \equiv \{(p, \Phi^{\omega}(p))\colon\ \Phi^{\omega}(p) = c,\ p \in E_1\}$, $\omega \in \Omega$, where $c \in E_2$ is constant, it must be contained in some pseudo-tempered random set, for example $\omega \mapsto B_{2\|c\|_E}(\omega)$, where $B_{2\|c\|_E}(\omega)$ is a pseudo-ball with radius $2\|c\|_E$. Then, for $\omega \in \Omega$,

$$d_H\big(Y\big(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega)\big), A_0(\omega)\big) \to 0 \quad \text{as } t \to \infty,$$

which means that $\lim_{t\to\infty} Y(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega)) \subset A_0(\omega)$. Since $A_0(\omega)$ is one-dimensional, we have for $\omega \in \Omega$,

$$A_0(\omega) = \lim_{t \to \infty} Y\big(t, \theta_{-t}\omega, \ell(\theta_{-t}\omega)\big).$$

It then follows from Lemma 5.2 that $\omega \mapsto A_0(\omega)$ is a random horizontal curve. $\square$

Corollary 5.4. *Assume that $a > 4L_F$ and that there is a $\gamma \in (0, \frac{a}{2})$ such that* (5.3) *holds. Then the random attractor $\omega \mapsto A(\omega)$ of the random dynamical system ϕ is a random horizontal curve.*

Proof. It follows from Corollary 4.2, Remark 4.5 and Theorem 5.3. $\square$

Remark 5.5. At the beginning of this section, we assume that $a > 4L_F$. Since $a = \frac{\alpha}{2} - |\frac{\alpha}{2} - \frac{\delta\lambda_1}{\alpha}|$ and $L_F \leqslant \frac{2}{\alpha}$, we can take α, λ_1 satisfying $\frac{\alpha}{2} - |\frac{\alpha}{2} - \frac{\delta\lambda_1}{\alpha}| > \frac{8}{\alpha}$, where λ_1 is the smallest positive eigenvalue of A and its value is determined by the diffusion coefficient K. On the other hand, we need some $\gamma \in (0, \frac{a}{2})$ such that (5.3) holds. Note that

$$\min_{\gamma \in (0, \frac{a}{2})} \left(\frac{1}{\gamma} + \frac{1}{a - 2\gamma} \right) = \left(\frac{1}{\gamma} + \frac{1}{a - 2\gamma} \right) \bigg|_{\gamma = \frac{(2-\sqrt{2})a}{2}} = \frac{\sqrt{2}}{(3\sqrt{2} - 4)a},$$

which implies that there exist α, λ_1 satisfying

$$\frac{\alpha}{2} - \left| \frac{\alpha}{2} - \frac{\delta\lambda_1}{\alpha} \right| > \frac{2\sqrt{2}}{(3\sqrt{2} - 4)\alpha} > \frac{8}{\alpha}. \tag{5.15}$$

Indeed, let $c = \frac{2\sqrt{2}}{3\sqrt{2}-4}$, then for any $\alpha > \sqrt{2c}$ and $\lambda_1 > c$, there is a $\delta > 0$ satisfying

$$\frac{c}{\lambda_1} < \delta < \min\left\{ \frac{\alpha^2 - c}{\lambda_1}, 1 \right\}$$

such that (5.15) holds.

6. Rotation number

In this section, we study the phenomenon of frequency locking, i.e., the existence of a rotation number of the stochastic damped sine-Gordon equation (1.1)–(1.2), which characterizes the speed that the solution of (1.1)–(1.2) moves around the one-dimensional random attractor.

Definition 6.1. The stochastic damped sine-Gordon equation (1.1) with boundary condition (1.2) is said to have a rotation number $\rho \in \mathbb{R}$ if, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and each $\phi_0 = (u_0, u_1)^\top \in E$, the limit $\lim_{t\to\infty} \frac{P\phi(t,\omega,\phi_0)}{t}$ exists and

$$\lim_{t\to\infty} \frac{P\phi(t, \omega, \phi_0)}{t} = \rho\eta_0,$$

where $\eta_0 = (1, 0)^\top$ is the basis of E_1.

We remark that the rotation number of (1.1)–(1.2) (if exists) is unique. In fact, assume that ρ_1 and ρ_2 are rotation numbers of (1.1)–(1.2). Then there is $\omega \in \Omega$ such that for any $\phi_0 \in E$,

$$\rho_1\eta_0 = \lim_{t\to\infty} \frac{P\phi(t, \omega, \phi_0)}{t} = \rho_2\eta_0.$$

Therefore, $\rho_1 = \rho_2$ and then the rotation number of (1.1)–(1.2) (if exists) is unique.

From (3.7), we have

$$\frac{P\phi(t,\omega,\phi_0)}{t} = \frac{PY(t,\omega,Y_0(\omega))}{t} + \frac{P(0,z(\theta_t\omega))^\top}{t}, \tag{6.1}$$

where $\phi_0 = (u_0, u_1)^\top$ and $Y_0(\omega) = (u_0, u_1 - z(\omega))^\top$. By Lemma 2.1 in [12], it is easy to prove that $\lim_{t\to\infty} \frac{P(0,z(\theta_t\omega))^\top}{t} = (0,0)^\top$. Thus, it sufficient to prove the existence of the rotation number of the random system (3.5).

By the random dynamical system $\mathbf{Y}$ defined in (4.1), we define the corresponding skew-product semiflow $\mathbf{\Theta}_t : \Omega \times \mathbf{E} \to \Omega \times \mathbf{E}$ for $t \geqslant 0$ by setting

$$\mathbf{\Theta}_t(\omega, \mathbf{Y}_0) = \big(\theta_t\omega, \mathbf{Y}(t,\omega,\mathbf{Y}_0)\big).$$

Obviously, $(\Omega \times \mathbf{E}, \mathcal{F} \times \mathcal{B}, (\mathbf{\Theta}_t)_{t\geqslant 0})$ is a measurable dynamical system, where $\mathcal{B} = \mathcal{B}(\mathbf{E})$ is the Borel σ-algebra of $\mathbf{E}$.

Lemma 6.2. *There is a measure μ on $\Omega \times \mathbf{E}$ such that $(\Omega \times \mathbf{E}, \mathcal{F} \times \mathcal{B}, \mu, (\mathbf{\Theta}_t)_{t\geqslant 0})$ becomes an ergodic metric dynamical system.*

Proof. Let $\Pr_\Omega(\mathbf{E})$ be the set of all random probability measures on $\mathbf{E}$ and $\Pr_{\mathbb{P}}(\Omega \times \mathbf{E})$ be the set of all probability measures on $\Omega \times \mathbf{E}$ with marginal $\mathbb{P}$. We know from Propositions 3.3 and 3.6 in [9] that $\Pr_\Omega(\mathbf{E})$ and $\Pr_{\mathbb{P}}(\Omega \times \mathbf{E})$ are isomorphism. Moreover, both $\Pr_\Omega(\mathbf{E})$ and $\Pr_{\mathbb{P}}(\Omega \times \mathbf{E})$ are convex, and the convex structure is preserved by this isomorphism.

Let $\Gamma = \{\omega \mapsto \mu_\omega \in \Pr_\Omega(\mathbf{E})\colon \mathbb{P}\text{-a.s. } \mu_\omega(\mathbf{A}_0(\omega)) = 1,\ \omega \mapsto \mu_\omega \text{ is invariant for } \mathbf{Y}\}$. Clearly, Γ is convex. Since $\omega \mapsto \mathbf{A}_0(\omega)$ is the random attractor of $\mathbf{Y}$, we obtain from Corollary 6.13 in [9] that $\Gamma \neq \emptyset$. Let $\omega \mapsto \mu_\omega$ be an extremal point of Γ. Then, by the isomorphism between $\Pr_\Omega(\mathbf{E})$ and $\Pr_{\mathbb{P}}(\Omega \times \mathbf{E})$ and Lemma 6.19 in [9], the corresponding measure μ on $\Omega \times \mathbf{E}$ of $\omega \mapsto \mu_\omega$ is $(\mathbf{\Theta}_t)_{t\geqslant 0}$-invariant and ergodic. Thus, $(\Omega \times \mathbf{E}, \mathcal{F} \times \mathcal{B}, \mu, (\mathbf{\Theta}_t)_{t\geqslant 0})$ is an ergodic metric dynamical system. $\square$

We next show a simple lemma which will be used. For any $p_i = (s_i, 0)^\top \in E_1$, $i = 1, 2$, we define

$$p_1 \leqslant p_2 \quad \text{if } s_1 \leqslant s_2.$$

Then we have

Lemma 6.3. *Suppose that $a > 4L_F$. Let ℓ be any deterministic np_0-periodic horizontal curve (ℓ satisfies the Lipschitz and periodic condition in Definition 5.1). For any $Y_1, Y_2 \in \ell$ with $PY_1 \leqslant PY_2$, there holds*

$$PY(t,\omega,Y_1) \leqslant PY(t,\omega,Y_2) \quad \textit{for } t > 0,\ \omega \in \Omega. \tag{6.2}$$

Proof. Clearly, if $PY_1 = PY_2$, then (6.2) holds. We now prove that (6.2) holds for $PY_1 < PY_2$. If not, then by the continuity of Y with respect to t, there is a $t_0 > 0$ such that $PY(t_0,\omega,Y_1) = PY(t_0,\omega,Y_2)$, which implies that $Y(t_0,\omega,Y_1) = Y(t_0,\omega,Y_2)$ since $Y(t_0,\omega,Y_1)$ and $Y(t_0,\omega,Y_2)$ belong to the same deterministic np_0-periodic horizontal curve $Y(t_0,\omega,\ell)$, which leads to a contradiction. The lemma is thus proved. $\square$

We now show the main result in this section.

Theorem 6.4. *Assume that $a > 4L_F$. Then the rotation number of (3.5) exists.*

Proof. Note that

$$\frac{PY(t,\omega,Y_0)}{t}=\frac{PY_0}{t}+\frac{1}{t}\int_0^t PF\big(\theta_s\omega, Y(s,\omega,Y_0)\big)\,ds.$$

Since $F(\theta_s\omega, Y(s,\omega,Y_0)+kp_0)=F(\theta_s\omega, Y(s,\omega,Y_0))$, $\forall k\in\mathbb{Z}$, we can identify $F(\theta_s\omega,\mathbf{Y}(s,\omega,\mathbf{Y}_0))$ with $F(\theta_s\omega, Y(s,\omega,Y_0))$. Precisely, define $h: E\to\mathcal{E}$, $Y\mapsto\{Y\}$, where $\mathcal{E}$ is the collection of all singleton sets of E, i.e. $\mathcal{E}=\{\{Y\}\colon\ Y\in E\}$ (see Remark 6.6 for more details of the space $\mathcal{E}$). Clearly, h is a homeomorphism from E to $\mathcal{E}$. Then,

$$F\big(\theta_s\omega, Y(s,\omega,Y_0)\big)=h^{-1}\big(F\big(\theta_s\omega,\mathbf{Y}(s,\omega,\mathbf{Y}_0)\big)\big).$$

Thus,

$$\begin{aligned}\frac{PY(t,\omega,Y_0)}{t}&=\frac{PY_0}{t}+\frac{1}{t}\int_0^t Ph^{-1}\big(F\big(\theta_s\omega,\mathbf{Y}(s,\omega,\mathbf{Y}_0)\big)\big)\,ds\\&=\frac{PY_0}{t}+\frac{1}{t}\int_0^t \mathbf{F}\big(\boldsymbol{\Theta}_s(\omega,\mathbf{Y}_0)\big)\,ds,\end{aligned}\tag{6.3}$$

where $\mathbf{F}=P\circ h^{-1}\circ F\in L^1(\Omega\times\mathbf{E},\mathcal{F}\times\mathcal{B},\mu)$. Let $t\to\infty$ in (6.3), $\lim_{t\to\infty}\frac{PY_0}{t}=(0,0)^\top$ and by Lemma 5.2 and Ergodic Theorems in [1], there exist a constant $\rho\in\mathbb{R}$ such that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{F}\big(\boldsymbol{\Theta}_s(\omega,\mathbf{Y}_0)\big)\,ds=\rho\eta_0,$$

which means

$$\lim_{t\to\infty}\frac{PY(t,\omega,Y_0)}{t}=\rho\eta_0$$

for μ-a.e. $(\omega,Y_0)\in\Omega\times E$. Thus, there is $\Omega^*\subset\Omega$ with $\mathbb{P}(\Omega^*)=1$ such that for any $\omega\in\Omega^*$, there is $Y_0^*(\omega)\in E$ such that

$$\lim_{t\to\infty}\frac{PY(t,\omega,Y_0^*(\omega))}{t}=\rho\eta_0.$$

By Lemma 3.2, we have that for any $n\in\mathbb{N}$ and $\omega\in\Omega^*$,

$$\lim_{t\to\infty}\frac{PY(t,\omega,Y_0^*(\omega)\pm np_0)}{t}=\lim_{t\to\infty}\frac{PY(t,\omega,Y_0^*(\omega))\pm np_0}{t}=\rho\eta_0.\tag{6.4}$$

Now for any $\omega\in\Omega^*$ and any $Y_0\in E$, there is $n_0(\omega)\in\mathbb{N}$ such that

$$PY_0^*(\omega)-n_0(\omega)p_0\leqslant PY_0\leqslant PY_0^*(\omega)+n_0(\omega)p_0$$

and there is a $n_0(\omega)p_0$-periodic horizontal curve $l_0(\omega)$ such that $Y_0^*(\omega) - n_0(\omega)p_0$, Y_0, $Y_0^*(\omega) + n_0(\omega)p_0 \in l_0(\omega)$. Then by Lemma 6.3, we have

$$PY\big(t,\omega,Y_0^*(\omega) - n_0(\omega)p_0\big) \leqslant PY(t,\omega,Y_0) \leqslant PY\big(t,\omega,Y_0^*(\omega) + n_0(\omega)p_0\big),$$

which together with (6.4) implies that for any $\omega \in \Omega^*$ and any $Y_0 \in E$,

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y_0)}{t} = \rho\eta_0.$$

Consequently, for any a.e. $\omega \in \Omega$ and any $Y_0 \in E$,

$$\lim_{t\to\infty} \frac{PY(t,\omega,Y_0)}{t} = \rho\eta_0.$$

The theorem is thus proved. □

Corollary 6.5. *Assume that $a > 4L_F$. Then the rotation number of the stochastic damped sine-Gordon equation* (1.1) *with the boundary condition* (1.2) *exists.*

Proof. It follows from (6.1) and Theorem 6.4. □

Remark 6.6. We first note that the space $\mathcal{E} = \{\{Y\}: Y \in E\}$ in the proof of Theorem 6.4 is a linear space according to the linear structure defined by

$$\alpha\{X\} + \beta\{Y\} = \{\alpha X + \beta Y\}, \quad \text{for } \alpha, \beta \in \mathbb{R},\ \{X\}, \{Y\} \in \mathcal{E}.$$

Also, for $\{X\}, \{Y\} \in \mathcal{E}$, we define

$$\big\langle\{X\},\{Y\}\big\rangle_{\mathcal{E}} = \langle X, Y\rangle_E. \tag{6.5}$$

It is easy to verify that the functional $\langle\cdot,\cdot\rangle_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ defined by (6.5) is bilinear, symmetric and positive, thus defining the scalar product in $\mathcal{E}$ over $\mathbb{R}$. Moreover, the completeness of $\mathcal{E}$ is from the completeness of E. Hence, $\mathcal{E}$ is a Hilbert space.

Remark 6.7. In the proof of Theorem 6.4, we used an ergodic invariant measure μ of $(\Omega \times \mathbf{E}, \mathcal{F} \times \mathcal{B}, \mu, (\mathbf{\Theta}_t)_{t\geqslant 0})$. It should be pointed out that the measure μ on $\Omega \times \mathbf{E}$ that makes $(\Omega \times \mathbf{E}, \mathcal{F} \times \mathcal{B}, \mu, (\mathbf{\Theta}_t)_{t\geqslant 0})$ becomes an ergodic metric dynamical system may not be unique, because the convex set Γ in the proof of Lemma 6.2 may have more than one extremal points. However, as mentioned above, the rotation number in Theorem 6.4 and Corollary 6.5 are independent of μ and are unique.

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