



# Regularity and stability of transition fronts in nonlocal equations with time heterogeneous ignition nonlinearity

Wenxian Shen, Zhongwei Shen <sup>\*,1</sup>

*Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA*

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## Abstract

The present paper is devoted to the investigation of various properties of transition fronts in one-dimensional nonlocal equations in heterogeneous media of ignition type, whose existence has been established by the authors of the present paper in a previous work. It is first shown that transition fronts are continuously differentiable in space with uniformly bounded and uniformly Lipschitz continuous space partial derivative. This is the first time that space regularity of transition fronts in nonlocal equations is ever studied. It is then shown that transition fronts are uniformly steep. Finally, asymptotic stability, in the sense of exponentially attracting front-like initial data, of transition fronts is studied.

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<sup>\*</sup> Corresponding author.

*E-mail addresses:* [wenxish@auburn.edu](mailto:wenxish@auburn.edu) (W. Shen), [zws0004@auburn.edu](mailto:zws0004@auburn.edu), [zhongwei@ualberta.ca](mailto:zhongwei@ualberta.ca) (Z. Shen).

<sup>1</sup> Current address: Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada.

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**1. Introduction**

Consider

$$u_t = J * u - u + f(t, x, u), \tag{1.1}$$

where  $J$  is the dispersal kernel and  $[J * u](x) = \int_{\mathbb{R}} J(x - y)u(y)dy = \int_{\mathbb{R}} J(y)u(x - y)dy$ , and the reaction term  $f$  is of monostable type, bistable type or ignition type. Such an equation, introduced as a substitute for the classical reaction–diffusion equation

$$u_t = u_{xx} + f(t, x, u), \tag{1.2}$$

has been used to model various diffusive processes with jumps (see e.g. [12] for some background). While a large amount of literature has been carried out to the understanding of (1.2), its nonlocal version (1.1) has attracted a lot of attention recently and some results have been established. For (1.1) in the homogeneous media, traveling waves, i.e., solutions of the form  $u(t, x) = \phi(x - ct)$  with  $(c, \phi)$  satisfying

$$J * \phi - \phi + c\phi_x + f(\phi) = 0, \quad \phi(-\infty) = 1, \quad \phi(\infty) = 0,$$

have been obtained (see [1,6–10,21] and references therein). The study of (1.1) in the heterogeneous media is rather recent and results concerning front propagation are very limited. In [11, 27–29], the authors investigated (1.1) in space periodic monostable media and proved the existence of spreading speeds and periodic traveling waves. In [20], Rawal, Shen and Zhang studied the existence of spreading speeds and traveling waves of (1.1) in space–time periodic monostable media. For (1.1) in space heterogeneous monostable media, Berestycki, Coville and Vo studied in [2] the principal eigenvalue, positive solution and long-time behavior of solutions, while Lim and Zlatoš proved in [13] the existence of transition fronts in the sense of Berestycki–Hamel (see [3,4]). In [5], Berestycki and Rodríguez studied (1.1) with a barrier nonlinearity of monostable type or bistable type, and proved that while propagation always occurs in the monostable case, it may be obstructed in the bistable case. For (1.1) in time heterogeneous media of ignition type, the authors of the present paper proved in [26] the existence of transition fronts.

In the present paper, we continue to study (1.1) in time heterogeneous media based on the work done in [26]. Recall that, an entire solution  $u(t, x)$  of (1.1) is called a *transition front* in the sense of Berestycki–Hamel (see [3,4]) if  $u(t, -\infty) = 1$  and  $u(t, \infty) = 0$  for any  $t \in \mathbb{R}$ , and for any  $\epsilon \in (0, 1)$  there holds

$$\sup_{t \in \mathbb{R}} \text{diam}\{x \in \mathbb{R} | \epsilon \leq u(t, x) \leq 1 - \epsilon\} < \infty.$$

Equivalently, an entire solution  $u(t, x)$  of (1.1) is called a transition front if there exists a function  $X : \mathbb{R} \rightarrow \mathbb{R}$ , called the *interface location function*, such that

$$\lim_{x \rightarrow -\infty} u(t, x + X(t)) = 1 \text{ and } \lim_{x \rightarrow \infty} u(t, x + X(t)) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

We remark that neither the definition of transition fronts nor the equation (1.1) itself guarantees any space regularity of transition fronts beyond continuity. This is because the semigroup generated by the nonlocal dispersal operator  $u \mapsto J * u - u$  has no regularizing effect, and therefore, a solution of (1.1) does not become smoother as time elapses. Moreover, transition fronts constructed in [13] and [26] are only uniformly Lipschitz continuous in space; it is not known whether they are continuously differentiable in space. One of the main goals of the present paper is to investigate the space regularity of transition fronts constructed in [26]. It should be pointed out that space regularity is of fundamental importance in further studying various important properties, such as uniform steepness and stability, of transition fronts.

Now, let us focus on (1.1) in time heterogeneous media of ignition type, i.e.,

$$u_t = J * u - u + f(t, u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.3}$$

where the dispersal kernel  $J$  satisfies

**(H1)**  $J \in C^1(\mathbb{R})$ ,  $J(x) = J(-x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $\int_{\mathbb{R}} J(x) dx = 1$ ,  $\int_{\mathbb{R}} |J'(x)| dx < \infty$  and

$$\int_{\mathbb{R}} J(x) e^{\lambda x} dx < \infty, \quad \forall \lambda > 0; \tag{1.4}$$

and the time heterogeneous nonlinearity  $f(t, u)$  satisfies

**(H2)**  $f : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  is continuously differentiable and satisfies the following conditions:

- there are  $\theta \in (0, 1)$  (the ignition temperature),  $f_{\min} \in C^{1,\alpha}([0, 1])$  and a Lipschitz continuous function  $f_{\max} : [0, 1] \rightarrow \mathbb{R}$  satisfying

$$\begin{aligned} f_{\min}(u) = 0 = f_{\max}(u), \quad u \in [0, \theta] \cup \{1\}, \\ 0 < f_{\min}(u) \leq f_{\max}(u), \quad u \in (\theta, 1), \\ f'_{\min}(1) < 0 \end{aligned}$$

such that

$$f_{\min}(u) \leq f(t, u) \leq f_{\max}(u), \quad (t, u) \in \mathbb{R} \times [0, 1];$$

- $f(t, u) < 0$  for  $(t, u) \in \mathbb{R} \times (1, \infty)$ ;
- first-order partial derivatives are uniformly bounded, i.e.,

$$\sup_{(t,u) \in \mathbb{R} \times [0, \infty)} |f_t(t, u)| < \infty \text{ and } \sup_{(t,u) \in \mathbb{R} \times [0, \infty)} |f_u(t, u)| < \infty;$$

- there exists  $\tilde{\theta} \in (\theta, 1)$  such that  $f_u(t, u) \leq 0$  for all  $t \in \mathbb{R}$  and  $u \in [\tilde{\theta}, 1]$ .

We remark that, in the present paper, only solutions  $u(t, x)$  with  $0 \leq u(t, x) \leq 1$  will be considered. The assumptions on  $f(t, u)$  for  $u > 1$  are only for certain technical convenience.

For convenience and later use, let us first summarize the main results obtained in [26]. To this end, let us consider the following homogeneous equation

$$u_t = J * u - u + f_{\min}(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.5}$$

where  $f_{\min}$ , given in (H2), is of ignition type. Assume (H1) and (H2). It is proven in [8] that there are a unique  $c_{\min}^* > 0$  and a unique  $C^1$  function  $\phi = \phi_{\min} : \mathbb{R} \rightarrow (0, 1)$  satisfying

$$\begin{cases} J * \phi - \phi + c_{\min}^* \phi' + f_{\min}(\phi) = 0, \\ \phi' < 0, \phi(0) = \theta, \phi(-\infty) = 1 \text{ and } \phi(\infty) = 0. \end{cases} \tag{1.6}$$

That is,  $\phi_{\min}$  is the normalized wave profile and  $\phi_{\min}(x - c_{\min}^* t)$  is the traveling wave of (1.5). Moreover, using the equation in (1.6), it is not hard to see that  $\phi'_{\min}$  is uniformly Lipschitz continuous, that is,

$$\sup_{x \neq y} \left| \frac{\phi'_{\min}(x) - \phi'_{\min}(y)}{x - y} \right| < \infty. \tag{1.7}$$

The following proposition is proven in [26].

**Proposition 1.1** ([26]). *Suppose (H1)–(H2).*

(1) *For  $s < 0$ , there exists a unique  $y_s \in \mathbb{R}$  with  $y_s \rightarrow -\infty$  as  $s \rightarrow -\infty$  such that the classical solution  $u(t, x; s)$  of (1.3) with initial data  $u(s, x; s) = \phi_{\min}(x - y_s)$  satisfies the normalization  $u(0, 0; s) = \theta$  and the following properties:*

- (i)  $u(t, -\infty; s) = 1, u(t, \infty; s) = 0$  and  $u(t, x; s)$  is strictly decreasing in  $x$ ;
- (ii) let  $X_\lambda(t; s)$  be such that  $u(t, X_\lambda(t; s); s) = \lambda$  for any  $\lambda \in (0, 1)$ ; there exist  $c_{\min} > 0, c_{\max} > 0$ , and a twice continuously differentiable function  $X(\cdot; s) : [s, \infty) \rightarrow \mathbb{R}$  satisfying

$$0 < c_{\min} \leq \dot{X}(t; s) \leq c_{\max} < \infty, \quad s < 0, \quad t \geq s \quad \text{and} \quad \sup_{s < 0, t \geq s} |\ddot{X}(t; s)| < \infty$$

such that

$$\forall \lambda \in (0, 1), \quad \sup_{s < 0, t \geq s} |X(t; s) - X_\lambda(t; s)| < \infty$$

and there exist exponents  $c_\pm > 0$  and shifts  $h_\pm > 0$  such that

$$\begin{aligned} u(t, x; s) &\geq 1 - e^{c_-(x - X(t; s) + h_-)} && \text{if } x \leq X(t; s) - h_-, \\ u(t, x; s) &\leq e^{-c_+(x - X(t; s) - h_+)} && \text{if } x \geq X(t; s) + h_+ \end{aligned}$$

for all  $s < 0, t \geq s$ ;

(iii)  $u(t, x; s)$  is uniformly Lipschitz continuous in space, that is,

$$\sup_{\substack{x \neq y \\ s < 0, t \geq s}} \left| \frac{u(t, y; s) - u(t, x; s)}{y - x} \right| < \infty. \tag{1.8}$$

(2) There is a transition front  $u(t, x)$  that is strictly decreasing in space and uniformly Lipschitz continuous in space, that is,

$$\sup_{\substack{x \neq y \\ t \in \mathbb{R}}} \left| \frac{u(t, y) - u(t, x)}{y - x} \right| < \infty,$$

and a continuously differentiable function  $X : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following properties:

(i) there hold

$$X(t; s) \rightarrow X(t), \quad u(t, x; s) \rightarrow u(t, x) \quad \text{and} \quad u_t(t, x; s) \rightarrow u_t(t, x)$$

locally uniformly in  $(t, x) \in \mathbb{R} \times \mathbb{R}$  as  $s \rightarrow -\infty$  along some subsequence;

(ii)  $\dot{X}(t) \in [c_{\min}, c_{\max}]$  for all  $t \in \mathbb{R}$ , where  $c_{\min}$  and  $c_{\max}$  are as in (1)(ii);

(iii) there hold

$$\begin{aligned} u(t, x) &\geq 1 - e^{c_-(x-X(t)+h_-)} \quad \text{if } x \leq X(t) - h_-, \\ u(t, x) &\leq e^{-c_+(x-X(t)-h_+)} \quad \text{if } x \geq X(t) + h_+ \end{aligned}$$

for all  $s < 0, t \geq s$ , where  $c_{\pm}$  and  $h_{\pm}$  are as in (1)(ii).

In the present paper, we intend to improve the uniform Lipschitz continuity in space of  $u(t, x)$  in Proposition 1.1(2), and then, study other important properties of  $u(t, x)$  such as uniform steepness and stability. To do so, we further assume

(H3)  $f(t, u)$  is twice continuously differentiable in  $u$  and satisfies

$$\sup_{(t,u) \in \mathbb{R} \times [0,1]} |f_{uu}(t, u)| < \infty.$$

Our first main result concerning space regularity of  $u(t, x)$  is stated in the following theorem.

**Theorem 1.2.** Suppose (H1)–(H3). Let  $u(t, x)$  be the transition front in Proposition 1.1(2). Then, for any  $t \in \mathbb{R}$ ,  $u(t, x)$  is continuously differentiable in  $x$ . Moreover,  $u_x(t, x)$  is uniformly bounded and uniformly Lipschitz continuous in  $x$ , that is,

$$\sup_{(t,x) \in \mathbb{R} \times \mathbb{R}} |u_x(t, x)| < \infty \quad \text{and} \quad \sup_{\substack{x \neq y \\ t \in \mathbb{R}}} \left| \frac{u_x(t, x) - u_x(t, y)}{x - y} \right| < \infty, \tag{1.9}$$

respectively.

We remark that since  $u(t, x)$  is strictly decreasing in  $x$ , the uniform bound of  $u_x(t, x)$  in (1.9) is equivalent to  $\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}} u_x(t, x) > -\infty$ . With the continuous differentiability of  $u(t, x)$  in  $x$ , the profile function  $\phi(t, x) = u(t, x + X(t))$  satisfies the following evolution equation

$$\phi_t = J * \phi - \phi + \dot{X}(t)\phi_x + f(t, \phi),$$

which could be used to construct transition fronts if  $X(t)$  can be first constructed (see [17] for the work on (1.2) in time heterogeneous monostable media).

Next, we study the uniform steepness of the transition front. We prove

**Theorem 1.3.** *Suppose (H1)–(H3). Let  $u(t, x)$  and  $X(t)$  be as in Proposition 1.1(2). Then, for any  $M > 0$ , there holds*

$$\sup_{t \in \mathbb{R}} \sup_{x \in [X(t)-M, X(t)+M]} u_x(t, x) < 0.$$

A simple consequence of Theorem 1.2 and Theorem 1.3 is that the interface location at any constant value between 0 and 1 is continuously differentiable and moves with finite speed. More precisely, for  $\lambda \in (0, 1)$ , let  $X_\lambda(t)$  be such that  $u(t, X_\lambda(t)) = \lambda$  for all  $t \in \mathbb{R}$ . It is well-defined by the monotonicity of  $u(t, x)$  in  $x$ . Then,

**Corollary 1.4.** *Suppose (H1)–(H3). For any  $\lambda \in (0, 1)$ ,  $X_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable and satisfies  $\sup_{t \in \mathbb{R}} |\dot{X}_\lambda(t)| < \infty$ .*

**Proof.** Let  $\lambda \in (0, 1)$ . By Theorem 1.3 and the fact that  $\sup_{t \in \mathbb{R}} |X_\lambda(t) - X(t)| < \infty$  due to Proposition 1.1(2)(ii), there exists some  $\alpha_\lambda > 0$  such that

$$\sup_{t \in \mathbb{R}} u_x(t, X_\lambda(t)) \leq -\alpha_\lambda. \tag{1.10}$$

Then, since  $u(t, X_\lambda(t)) = \lambda$ , implicit function theorem says that  $X_\lambda(t)$  is continuously differentiable. Differentiating the equation  $u(t, X_\lambda(t)) = \lambda$  with respect to  $t$ , we find

$$\dot{X}_\lambda(t) = -\frac{u_t(t, X_\lambda(t))}{u_x(t, X_\lambda(t))}.$$

The result then follows from (1.10) and the fact  $\sup_{(t,x) \in \mathbb{R} \times \mathbb{R}} |u_t(t, x)| < \infty$ .  $\square$

Finally, we study the stability of transition fronts. Note that for any  $\zeta \in \mathbb{R}$ ,  $u(t, x - \zeta)$  is also a transition front of (1.3). Therefore, we study the stability of the family  $\{u(t, x - \xi)\}_{\xi \in \mathbb{R}}$  instead of a single transition front  $u(t, x - \zeta_0)$  for some  $\zeta_0 \in \mathbb{R}$ . Let  $C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$  be the space of bounded and uniformly continuous functions on  $\mathbb{R}$ . For  $u_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$ , denote by  $u(t, x; t_0, u_0)$  the unique solution of (1.3) with initial data  $u(t_0, \cdot; t_0, u_0) = u_0$ . To state the result, we put more restrictions on the last assumption in (H2) and assume

**(H4)** *There exist  $\tilde{\theta} \in (\theta, 1)$  and  $\tilde{\beta} > 0$  such that  $f_u(t, u) \leq -\tilde{\beta}$  for all  $(t, u) \in \mathbb{R} \times [\tilde{\theta}, 2]$ . Also,  $f(t, u) = 0$  for  $(t, u) \in \mathbb{R} \times (-\infty, 0)$ .*

Let  $M_1 > 0$  be such that for any  $t \in \mathbb{R}$

$$u(t, x) \geq \frac{1 + \tilde{\theta}}{2} \text{ if } x - X(t) \leq -M_1 \quad \text{and} \quad u(t, x) \leq \frac{\theta}{2} \text{ if } x - X(t) \geq M_1, \tag{1.11}$$

where  $\tilde{\theta}$  is as in (H4). Such an  $M_1$  exists by Proposition 1.1(2)(iii). For given  $\alpha > 0$ , let  $\Gamma_\alpha : \mathbb{R} \rightarrow [0, 1]$  be a smooth nonincreasing function satisfying

$$\Gamma_\alpha(x) = \begin{cases} 1, & x \leq -M_1 - 1, \\ e^{-\alpha(x-M_1)}, & x \geq M_1 + 1. \end{cases} \tag{1.12}$$

We prove

**Theorem 1.5.** *Suppose (H1)–(H4).*

- (1) *There is  $\alpha_0 > 0$  such that for any  $0 < \alpha \leq \alpha_0$ , there are  $\epsilon_0 = \epsilon_0(\alpha)$ ,  $\omega = \omega(\alpha)$ , and  $A = A(\alpha)$  satisfying that for any  $u_0 : \mathbb{R} \rightarrow [0, 1]$ ,  $u_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$ , if there exist  $t_0 \in \mathbb{R}$ ,  $\epsilon \in (0, \epsilon_0]$ ,  $\zeta_0^\pm$  such that*

$$\begin{aligned} u(t_0, x - \zeta_0^-) - \epsilon \Gamma_\alpha(x - \zeta_0^- - X(t_0)) \\ \leq u_0(x) \leq u(t_0, x - \zeta_0^+) + \epsilon \Gamma_\alpha(x - \zeta_0^+ - X(t_0)) \end{aligned} \tag{1.13}$$

for all  $x \in \mathbb{R}$ , then, there holds

$$\begin{aligned} u(t, x - \zeta^-(t)) - q(t) \Gamma_\alpha(x - \zeta^-(t) - X(t)) \\ \leq u(t, x; t_0, u_0) \leq u(t, x - \zeta^+(t)) + q(t) \Gamma_\alpha(x - \zeta^+(t) - X(t)) \end{aligned}$$

for all  $x \in \mathbb{R}$  and  $t \geq t_0$ , where

$$\zeta^\pm(t) = \zeta_0^\pm \pm \frac{A\epsilon}{\omega} (1 - e^{-\omega(t-t_0)}) \quad \text{and} \quad q(t) = \epsilon e^{-\omega(t-t_0)}.$$

- (2) *Let  $u(t, x)$  and  $X(t)$  be as in Proposition 1.1(2). Let  $\beta_0 > 0$ . Suppose  $t_0 \in \mathbb{R}$  and  $u_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$  satisfy*

$$\begin{cases} u_0 : \mathbb{R} \rightarrow [0, 1], & u_0(-\infty) = 1; \\ \exists C > 0 \text{ s.t. } |u_0 - u(t_0, x)| \leq C e^{-\beta_0(x-X(t_0))} \text{ for } x \in \mathbb{R}. \end{cases}$$

Then, there exist  $\omega > 0$  and  $\tilde{\epsilon}_0 > 0$  such that for any  $\epsilon \in (0, \tilde{\epsilon}_0]$  there are  $\zeta^\pm = \zeta^\pm(\epsilon, u_0) \in \mathbb{R}$  such that

$$u(t, x - \zeta^-) - \epsilon e^{-\omega(t-t_0)} \leq u(t, x; t_0, u_0) \leq u(t, x - \zeta^+) + \epsilon e^{-\omega(t-t_0)}$$

for all  $x \in \mathbb{R}$  and  $t \geq t_0$ .

Based on [Theorem 1.5](#) and the “squeezing technique” (see e.g. [\[7,15,22,25\]](#)), we obtain the asymptotic stability.

**Theorem 1.6.** *Suppose (H1)–(H4). Let  $u(t, x)$  and  $X(t)$  be as in [Proposition 1.1\(2\)](#). Let  $\beta_0 > 0$ . Suppose  $t_0 \in \mathbb{R}$  and  $u_0 \in C^b_{\text{unif}}(\mathbb{R}, \mathbb{R})$  satisfy*

$$\begin{cases} u_0 : \mathbb{R} \rightarrow [0, 1], & u_0(-\infty) = 1; \\ \exists C > 0 \text{ s.t. } |u_0 - u(t_0, x)| \leq C e^{-\beta_0(x - X(t_0))} \text{ for } x \in \mathbb{R}. \end{cases}$$

Then, there exist  $C = C(u_0) > 0$ ,  $\zeta_* = \zeta_*(u_0) \in \mathbb{R}$  and  $r = r(\beta_0) > 0$  such that

$$\sup_{x \in \mathbb{R}} |u(t, x; t_0, u_0) - u(t, x - \zeta_*)| \leq C e^{-r(t - t_0)}$$

for all  $t \geq t_0$ .

Allowing the solution to evolve for a period of time, it will develop into the shape satisfying the condition in [Theorem 1.5](#) at a later time due to the stability of the stationary solution  $u \equiv 1$ . Therefore, [Theorem 1.5\(2\)](#) and [Theorem 1.6](#) are true for more general initial data (see [Corollary 4.2](#) and [Corollary 5.3](#)).

To this end, we point out that transition fronts for the reaction–diffusion equation [\(1.2\)](#) have been studied in the monostable case (see [\[16,17,19,30\]](#)), the bistable case (see [\[23,32\]](#)) and the ignition case (see [\[14,15,18,24,25,31,32\]](#)). In terms of the ignition case, the existence, stability and uniqueness of transition fronts for [\(1.2\)](#) have been obtained in [\[14,15,18,31,32\]](#) when  $f(t, x, u) = f(x, u)$  and in [\[24,25\]](#) when  $f(t, x, u) = f(t, u)$ .

The rest of the paper is organized as follows. In [Section 2](#), we study the space regularity of  $u(t, x)$  and prove [Theorem 1.2](#). In [Section 3](#), we study the uniform steepness of  $u(t, x)$  and prove [Theorem 1.3](#). In [Section 4](#), we study the stability of  $u(t, x)$  and prove [Theorem 1.5](#). In [Section 5](#), we study the asymptotic stability of  $u(t, x)$  and prove [Theorem 1.6](#). We also include an appendix, [Appendix A](#), on comparison principles for convenience.

## 2. Regularity of transition fronts

In this section, we study the regularity of  $u(t, x)$  in  $x$  and prove [Theorem 1.2](#). Throughout this section, we assume (H1)–(H3). To prove [Theorem 1.2](#), we first investigate the space regularity of  $u(t, x; s)$ . We have

**Theorem 2.1.** *For any  $s < 0$  and  $t \geq s$ ,  $u(t, x; s)$  is continuously differentiable in  $x$ . Moreover,*

(i)  $u_x(t, x; s)$  is uniformly bounded, that is,

$$\sup_{\substack{x \in \mathbb{R} \\ s < 0, t \geq s}} |u_x(t, x; s)| < \infty;$$

(ii)  $u_x(t, x; s)$  is uniformly Lipschitz continuous in space, that is,

$$\sup_{\substack{x \neq y \\ s < 0, t \geq s}} \left| \frac{u_x(t, x; s) - u_x(t, y; s)}{x - y} \right| < \infty.$$



Assuming [Theorem 2.1](#), let us prove [Theorem 1.2](#).

**Proof of Theorem 1.2.** It follows from [Proposition 1.1\(2\)\(i\)](#), [Theorem 2.1](#), Arzelà–Ascoli theorem and the diagonal argument. More precisely, besides  $u(t, x; s) \rightarrow u(t, x)$  and  $u_t(t, x; s) \rightarrow u_t(t, x)$  locally uniformly as in [Proposition 1.1\(2\)\(i\)](#), we also have

$$u_x(t, x; s) \rightarrow u_x(t, x) \quad \text{locally uniformly in } (t, x) \in \mathbb{R} \times \mathbb{R} \tag{2.1}$$

as  $s \rightarrow -\infty$  along some subsequence. Then,  $u(t, x)$  inherits the properties of  $u(t, x; s)$ .  $\square$

In the rest of this section, we prove [Theorem 2.1](#).

**Proof of Theorem 2.1.** (i) Set

$$v^\eta(t, x; s) := \frac{u(t, x + \eta; s) - u(t, x; s)}{\eta}.$$

By [\(1.8\)](#),  $\sup_{\substack{x \in \mathbb{R}, \eta \neq 0 \\ s < 0, t \geq s}} |v^\eta(t, x; s)| < \infty$ . Clearly,  $v^\eta(t, x; s)$  satisfies

$$v_t^\eta(t, x; s) = \int_{\mathbb{R}} J(x - y)v^\eta(t, y; s)dy - v^\eta(t, x; s) + a^\eta(t, x; s)v^\eta(t, x; s), \tag{2.2}$$

where

$$a^\eta(t, x; s) = \frac{f(t, u(t, x + \eta; s)) - f(t, u(t, x; s))}{u(t, x + \eta; s) - u(t, x; s)}$$

is uniformly bounded by (H2). Setting

$$b^\eta(t, x; s) := \int_{\mathbb{R}} J(x - y)v^\eta(t, y; s)dy = \int_{\mathbb{R}} \frac{J(x - y + \eta) - J(x - y)}{\eta} u(t, y; s)dy,$$

we see that  $\sup_{\substack{x \in \mathbb{R}, \eta \neq 0 \\ s < 0, t \geq s}} |b^\eta(t, x; s)| < \infty$ , since  $J' \in L^1(\mathbb{R})$  and  $u(t, x; s) \in (0, 1)$ .

The solution of [\(2.2\)](#) is given by

$$v^\eta(t, x; s) = v^\eta(s, x; s)e^{-\int_s^t (1 - a^\eta(\tau, x; s))d\tau} + \int_s^t b^\eta(r, x; s)e^{-\int_r^t (1 - a^\eta(\tau, x; s))d\tau} dr. \tag{2.3}$$

Notice that as  $\eta \rightarrow 0$ , the following pointwise limits hold:

$$\begin{aligned} v^\eta(s, x; s) &= \frac{\phi_{\min}(x + \eta - y_s) - \phi_{\min}(x - y_s)}{\eta} \rightarrow \phi'_{\min}(x - y_s), \\ a^\eta(t, x; s) &\rightarrow f_u(t, u(t, x; s)) \quad \text{and} \\ b^\eta(t, x; s) &\rightarrow \int_{\mathbb{R}} J'(x - y)u(t, y; s)dy, \end{aligned}$$

where  $\phi_{\min}$  is as in (1.6). Then, letting  $\eta \rightarrow 0$  in (2.3), we conclude from the dominated convergence theorem that for any  $s < 0, t \geq s$  and  $x \in \mathbb{R}$ , the limit  $u_x(t, x; s) = \lim_{\eta \rightarrow 0} v^\eta(t, x; s)$  exists and

$$u_x(t, x; s) = \phi'_{\min}(x - y_s)e^{-\int_s^t (1 - f_u(\tau, u(\tau, x; s)))d\tau} + \int_s^t b(r, x; s)e^{-\int_r^t (1 - f_u(\tau, u(\tau, x; s)))d\tau} dr, \tag{2.4}$$

where  $b(t, x; s) = \int_{\mathbb{R}} J'(x - y)u(t, y; s)dy = \int_{\mathbb{R}} J'(y)u(t, x - y; s)dy$ . In particular, for any  $s < 0$  and  $t \geq s, u(t, x; s)$  is continuously differentiable in  $x$ . The uniform boundedness of  $u_x(t, x; s)$ , i.e.,  $\sup_{x \in \mathbb{R}, s < 0, t \geq s} |u_x(t, x; s)| < \infty$ , then follows from (1.8).

(ii) Since  $u_x(t, x; s)$  is uniformly bounded by (i), we trivially have

$$\forall \delta > 0, \sup_{\substack{|x-y| \geq \delta \\ s < 0, t \geq s}} \left| \frac{u_x(t, x; s) - u_x(t, y; s)}{x - y} \right| < \infty.$$

Thus, to show the uniform Lipschitz continuity of  $u_x(t, x; s)$  in  $x$ , it suffices to show the local uniform Lipschitz continuity, i.e.,

$$\forall \delta > 0, \sup_{\substack{|x-y| \leq \delta \\ s < 0, t \geq s}} \left| \frac{u_x(t, x; s) - u_x(t, y; s)}{x - y} \right| < \infty. \tag{2.5}$$

To this end, we fix  $\delta > 0$ . Let  $X(t; s)$  and  $X_\lambda(t; s)$  for  $\lambda \in (0, 1)$  be as in Proposition 1.1(1)(ii) and define

$$L_1 = \delta + \sup_{s < 0, t \geq s} |X_\theta(t; s) - X(t; s)| \quad \text{and} \quad L_2 = \delta + \sup_{s < 0, t \geq s} |X_{\tilde{\theta}}(t; s) - X(t; s)|,$$

where  $\tilde{\theta} \in (\theta, 1)$  is given in (H2). Notice that  $L_1 < \infty$  and  $L_2 < \infty$  by the estimates in Proposition 1.1(1)(ii). Then, for any  $x \in \mathbb{R}$  and  $|\eta| \leq \delta$  we have

- if  $x \geq X(t; s) + L_1$ , then  $x + \eta \geq x - \delta \geq X_\theta(t; s)$ , which implies that  $u(t, x + \eta; s) \leq \theta$  by monotonicity, and hence

$$f_u(t, u(t, x + \eta; s)) = 0; \tag{2.6}$$

- if  $x \leq X(t; s) - L_2$ , then  $x + \eta \leq x + \delta \leq X_{\tilde{\theta}}(t; s)$ , which implies that  $u(t, x + \eta; s) \geq \tilde{\theta}$  by monotonicity, and hence by (H2),

$$f_u(t, u(t, x + \eta; s)) \leq 0. \tag{2.7}$$

According to (2.6) and (2.7), we consider time-dependent and disjoint decompositions of  $\mathbb{R}$  into

$$\mathbb{R} = R_l(t; s) \cup R_m(t; s) \cup R_r(t; s),$$

where

$$\begin{aligned}
 R_l(t; s) &= (-\infty, X(t; s) - L_2), \\
 R_m(t; s) &= [X(t; s) - L_2, X(t; s) + L_1] \quad \text{and} \\
 R_r(t; s) &= (X(t; s) + L_1, \infty).
 \end{aligned}
 \tag{2.8}$$

For  $s < 0$  and  $x_0 \in \mathbb{R}$ , let  $t_{\text{first}}(x_0; s)$  be the first time that  $x_0$  is in  $R_m(t; s)$ , that is,

$$t_{\text{first}}(x_0; s) = \min \{t \geq s \mid x_0 \in R_m(t; s)\},$$

and  $t_{\text{last}}(x_0; s)$  be the last time that  $x_0$  is in  $R_m(t; s)$ , that is,

$$t_{\text{last}}(x_0; s) = \max \{t_0 \in \mathbb{R} \mid x_0 \in R_m(t_0; s) \text{ and } x_0 \notin R_m(t, s) \text{ for } t > t_0\}.$$

Since  $\dot{X}(t; s) \geq c_{\min} > 0$  by Proposition 1.1(1)(ii), if  $x_0 \in R_l(s; s)$ , then  $x_0 \in R_l(t; s)$  for all  $t > s$ . In this case,  $t_{\text{first}}(x_0; s)$  and  $t_{\text{last}}(x_0; s)$  are not well-defined, but it will not cause any trouble. We see that  $t_{\text{first}}(x_0; s)$  and  $t_{\text{last}}(x_0; s)$  are well-defined only if  $x_0 \notin R_l(s; s)$ . As a simple consequence of  $\dot{X}(t; s) \in [c_{\min}, c_{\max}]$  in Proposition 1.1(1)(ii) and the fact that the length of  $R_m(t; s)$  is  $L_1 + L_2$ , we have

$$T = T(\delta) := \sup_{s < 0, x_0 \notin R_l(s; s)} [t_{\text{last}}(x_0; s) - t_{\text{first}}(x_0; s)] < \infty. \tag{2.9}$$

Moreover, we see that for any  $|\eta| \leq \delta$ ,

$$\begin{aligned}
 f_u(t, u(t, x_0 + \eta; s)) &= 0 \quad \text{if } t \in [s, t_{\text{first}}(x_0; s)], \\
 f_u(t, u(t, x_0 + \eta; s)) &\leq 0 \quad \text{if } t \geq t_{\text{last}}(x_0; s).
 \end{aligned}
 \tag{2.10}$$

We now show that

$$\sup_{\substack{x_0 \in \mathbb{R}, 0 < |\eta| \leq \delta \\ s < 0, t \geq s}} \left| \frac{u_x(t, x_0 + \eta; s) - u_x(t, x_0; s)}{\eta} \right| < \infty. \tag{2.11}$$

Using (2.4), we have

$$\begin{aligned}
 &\frac{u_x(t, x_0 + \eta; s) - u_x(t, x_0; s)}{\eta} \\
 &= \underbrace{\frac{\phi'_{\min}(x_0 + \eta - y_s) - \phi'_{\min}(x_0 - y_s)}{\eta} e^{-\int_s^t (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau}}_{\text{(I)}} \\
 &\quad + \underbrace{\phi'_{\min}(x_0 - y_s) \frac{e^{-\int_s^t (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau} - e^{-\int_s^t (1 - f_u(\tau, u(\tau, x_0; s))) d\tau}}{\eta}}_{\text{(II)}}
 \end{aligned}$$

$$\begin{aligned}
 &+ \underbrace{\int_s^t \frac{b(r, x_0 + \eta; s) - b(r, x_0; s)}{\eta} e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0; s))) d\tau} dr}_{(III)} \\
 &+ \underbrace{\int_s^t b(r, x_0; s) \frac{e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau} - e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0; s))) d\tau}}{\eta} dr}_{(IV)}.
 \end{aligned}$$

Hence, it suffice to bound terms (I)–(IV). To do so, we need to consider three cases:  $x_0 \in R_l(s; s)$ ,  $x_0 \in R_m(s; s)$  and  $x_0 \in R_r(s; s)$ . We here focus on the last case, i.e.,  $x_0 \in R_r(s; s)$ , which is the most involved one. The other two cases are simpler and can be treated similarly. Also, for fixed  $s < 0$  and  $x_0 \in R_r(s; s)$ , we will focus on the case  $t \geq t_{\text{last}}(x_0; s)$ ; the case with  $t \in [t_{\text{first}}(x_0; s), t_{\text{last}}(x_0; s)]$  or  $t \leq t_{\text{first}}(x_0; s)$  will be clear. Thus, we assume  $x_0 \in R_r(s; s)$  and  $t \geq t_{\text{last}}(x_0; s)$ .

We will frequently use the following estimates: for any  $|\tilde{\eta}| \leq \delta$  there hold

$$\begin{aligned}
 e^{-\int_r^{t_{\text{first}}(x_0; s)} (1 - f_u(\tau, u(\tau, x_0 + \tilde{\eta}; s))) d\tau} &= e^{-(t_{\text{first}}(x_0; s) - r)}, \quad r \in [s, t_{\text{first}}(x_0; s)] \\
 e^{-\int_r^{t_{\text{last}}(x_0; s)} (1 - f_u(\tau, u(\tau, x_0 + \tilde{\eta}; s))) d\tau} &\leq e^{T \sup_{(t, u) \in \mathbb{R} \times [0, 1]} |1 - f_u(t, u)|}, \quad r \in [t_{\text{first}}(x_0; s), t_{\text{last}}(x_0; s)] \\
 e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \tilde{\eta}; s))) d\tau} &\leq e^{-(t - r)}, \quad r \in [t_{\text{last}}(x_0; s), t].
 \end{aligned} \tag{2.12}$$

They are simple consequences of (2.9) and (2.10). Set

$$\begin{aligned}
 C_0 &:= \sup_{(t, u) \in \mathbb{R} \times [0, 1]} |1 - f_u(t, u)|, \quad C_1 := \sup_{x \neq y} \left| \frac{\phi'_{\min}(x) - \phi'_{\min}(y)}{x - y} \right|, \quad C_2 := \sup_{x \in \mathbb{R}} |\phi'_{\min}(x)| \\
 C_3 &:= \sup_{(t, u) \in \mathbb{R} \times [0, 1]} |f_{uu}(t, u)| \times \sup_{\substack{x \in \mathbb{R} \\ s < 0, t \geq s}} |u_x(t, x; s)|, \quad C_4 = \sup_{\substack{x \neq y \\ s < 0, t \geq s}} \left| \frac{u(t, x; s) - u(t, y; s)}{x - y} \right|.
 \end{aligned}$$

Note that all these constants are finite. In fact,  $C_0 < \infty$  by (H2),  $C_1 < \infty$  by (1.7),  $C_3 < \infty$  by (H3) and Theorem 2.1 (i), and  $C_4 < \infty$  by Proposition 1.1 (1) (iii).

We are ready to bound (I)–(IV). For the term (I), using (1.7) and (2.12), we see that

$$\begin{aligned}
 |(I)| &\leq C_1 e^{-\int_s^t (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau} \\
 &= C_1 e^{-[\int_s^{t_{\text{first}}(x_0; s)} + \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} + \int_{t_{\text{last}}(x_0; s)}^t] (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau} \\
 &\leq C_1 e^{-(t_{\text{first}}(x_0; s) - s)} e^{C_0 T} e^{-(t - t_{\text{last}}(x_0; s))} \leq C_1 e^{C_0 T}.
 \end{aligned} \tag{2.13}$$

For the term (II), we have from Taylor expansion of the function  $\eta \mapsto e^{-\int_s^t (1 - f_u(\tau, u(\tau, x_0 + \eta; s))) d\tau}$  at  $\eta = 0$  that

$$\begin{aligned}
 |(\text{II})| &\leq C_2 \left| \frac{e^{-\int_s^t (1-f_u(\tau, u(\tau, x_0+\eta; s)))d\tau} - e^{-\int_s^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau}}{\eta} \right| \\
 &\leq C_2 e^{-\int_s^t (1-f_u(\tau, u(\tau, x_0+\eta_*; s)))d\tau} \int_s^t \left| f_{uu}(\tau, u(\tau, x_0 + \eta_*; s)) u_x(\tau, x_0 + \eta_*; s) \right| d\tau,
 \end{aligned}$$

where  $\eta_*$  is between 0 and  $\eta$ , and hence,  $|\eta_*| \leq \delta$ . We see

$$\int_s^t \left| f_{uu}(\tau, u(\tau, x_0 + \eta_*; s)) u_x(\tau, x_0 + \eta_*; s) \right| d\tau \leq C_3(t - s).$$

It then follows from (2.12) that

$$\begin{aligned}
 |(\text{II})| &\leq C_2 C_3 e^{-(t_{\text{first}}(x_0; s) - s)} e^{-(t - t_{\text{last}}(x_0; s))} (t - s) \\
 &= C_2 C_3 e^{-(t_{\text{first}}(x_0; s) - s)} e^{-(t - t_{\text{last}}(x_0; s))} \\
 &\quad \times \left[ (t - t_{\text{last}}(x_0; s)) + (t_{\text{last}}(x_0; s) - t_{\text{first}}(x_0; s)) + (t_{\text{first}}(x_0; s) - s) \right] \\
 &\leq C_2 C_3 \left[ e^{-(t - t_{\text{last}}(x_0; s))} (t - t_{\text{last}}(x_0; s)) + T + e^{-(t_{\text{first}}(x_0; s) - s)} (t_{\text{first}}(x_0; s) - s) \right] \\
 &\leq C_2 C_3 \left( \frac{2}{e} + T \right).
 \end{aligned} \tag{2.14}$$

For the term (III), we first see that

$$\begin{aligned}
 \left| \frac{b(r, x_0 + \eta; s) - b(r, x_0; s)}{\eta} \right| &= \left| \int_{\mathbb{R}} J'(y) \frac{u(r, x_0 + \eta - y; s) - u(r, x_0 - y; s)}{\eta} dy \right| \\
 &\leq C_4 \|J'\|_{L^1(\mathbb{R})}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |(\text{III})| &\leq C_4 \|J'\|_{L^1(\mathbb{R})} \int_s^t e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr \\
 &= C_4 \|J'\|_{L^1(\mathbb{R})} \left[ \underbrace{\int_s^{t_{\text{first}}(x_0; s)} e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\text{III-1})} + \underbrace{\int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\text{III-2})} \right. \\
 &\quad \left. + \underbrace{\int_{t_{\text{last}}(x_0; s)}^t e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\text{III-3})} \right].
 \end{aligned}$$

We estimate (III-1), (III-2) and (III-3). For (III-1), we obtain from (2.12) that

$$\begin{aligned}
 \text{(III-1)} &= \int_s^{t_{\text{first}}(x_0; s)} e^{-[ \int_r^{t_{\text{first}}(x_0; s)} + \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} + \int_{t_{\text{last}}(x_0; s)}^t ](1 - f_u(\tau, u(\tau, x_0; s)))} d\tau dr \\
 &\leq e^{C_0 T} \int_s^{t_{\text{first}}(x_0; s)} e^{-(t_{\text{first}}(x_0; s) - r)} e^{-(t - t_{\text{last}}(x_0; s))} dr \\
 &= e^{C_0 T} e^{-(t - t_{\text{last}}(x_0; s))} (1 - e^{-(t_{\text{first}}(x_0; s) - s)}) \leq e^{C_0 T}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \text{(III-2)} &= \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} e^{-[ \int_r^{t_{\text{last}}(x_0; s)} + \int_{t_{\text{last}}(x_0; s)}^t ](1 - f_u(\tau, u(\tau, x_0; s)))} d\tau dr \\
 &\leq e^{C_0 T} \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} e^{-(t - t_{\text{last}}(x_0; s))} dr \leq e^{C_0 T} T e^{-(t - t_{\text{last}}(x_0; s))} \leq T e^{C_0 T}
 \end{aligned}$$

and  $\text{(III-3)} \leq \int_{t_{\text{last}}(x_0; s)}^t e^{-(t-r)} dr = 1 - e^{-(t - t_{\text{last}}(x_0; s))} \leq 1$ . Hence,

$$\text{(III)} \leq C_4 \|J'\|_{L^1(\mathbb{R})} (e^{C_0 T} + T e^{C_0 T} + 1). \tag{2.15}$$

For the term (IV), using  $|b(r, x_0; s)| \leq \|J'\|_{L^1(\mathbb{R})}$  and Taylor expansion as in the treatment of the term (II), we have

$$\begin{aligned}
 |\text{(IV)}| &\leq \|J'\|_{L^1(\mathbb{R})} \int_s^t e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \eta_*; s)))} d\tau \\
 &\quad \times \left( \int_r^t |f_{uu}(\tau, u(\tau, x_0 + \eta_*; s)) u_x(\tau, x_0 + \eta_*; s)| d\tau \right) dr \\
 &\leq C_3 \|J'\|_{L^1(\mathbb{R})} \int_s^t (t - r) e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \eta_*; s)))} d\tau dr \\
 &= C_3 \|J'\|_{L^1(\mathbb{R})} \underbrace{\left[ \int_s^{t_{\text{first}}(x_0; s)} (t - r) e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \eta_*; s)))} d\tau dr \right]}_{\text{(IV-1)}} \\
 &\quad + \underbrace{\int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} (t - r) e^{-\int_r^t (1 - f_u(\tau, u(\tau, x_0 + \eta_*; s)))} d\tau dr}_{\text{(IV-2)}}
 \end{aligned}$$

$$+ \underbrace{\int_{t_{\text{last}}(x_0; s)}^t (t-r)e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0+\eta_*; s)))d\tau} dr}_{\text{(IV-3)}}$$

where  $|\eta_*| \leq |\eta| \leq \delta$ . Similar to (III-1), (III-2) and (III-3), we have

$$\begin{aligned} \text{(IV-1)} &= \int_s^{t_{\text{first}}(x_0; s)} (t-r)e^{-[\int_r^{t_{\text{first}}(x_0; s)} + \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} + \int_{t_{\text{last}}(x_0; s)}^t] (1-f_u(\tau, u(\tau, x_0+\eta_*; s)))d\tau} dr \\ &\leq e^{C_0 T} \int_s^{t_{\text{first}}(x_0; s)} [(t-t_{\text{last}}(x_0; s)) + T + (t_{\text{first}}(x_0; s) - r)] e^{-(t_{\text{first}}(x_0; s)-r)} e^{-(t-t_{\text{last}}(x_0; s))} dr \\ &\leq e^{C_0 T} \left[ (t-t_{\text{last}}(x_0; s)) e^{-(t-t_{\text{last}}(x_0; s))} \int_s^{t_{\text{first}}(x_0; s)} e^{-(t_{\text{first}}(x_0; s)-r)} dr \right. \\ &\quad \left. + T \int_s^{t_{\text{first}}(x_0; s)} e^{-(t_{\text{first}}(x_0; s)-r)} dr + \int_s^{t_{\text{first}}(x_0; s)} (t_{\text{first}}(x_0; s) - r) e^{-(t_{\text{first}}(x_0; s)-r)} dr \right] \\ &\leq e^{C_0 T} \left[ \frac{1 - e^{-(t_{\text{first}}(x_0; s)-s)}}{e} + T(1 - e^{-(t_{\text{first}}(x_0; s)-s)}) \right. \\ &\quad \left. + \left( 1 - (1 + t_{\text{first}}(x_0; s) - s) e^{-(t_{\text{first}}(x_0; s)-s)} \right) \right] \\ &\leq e^{C_0 T} \left( \frac{1}{e} + T + 1 \right), \end{aligned}$$

$$\begin{aligned} \text{(IV-2)} &= \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} (t-r)e^{-[\int_r^{t_{\text{last}}(x_0; s)} + \int_{t_{\text{last}}(x_0; s)}^t] (1-f_u(\tau, u(\tau, x_0+\eta_*; s)))d\tau} dr \\ &\leq e^{C_0 T} \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} [(t-t_{\text{last}}(x_0; s)) + (t_{\text{last}}(x_0; s) - r)] e^{-(t-t_{\text{last}}(x_0; s))} dr \\ &\leq e^{C_0 T} \left[ T(t-t_{\text{last}}(x_0; s)) e^{-(t-t_{\text{last}}(x_0; s))} + \int_{t_{\text{first}}(x_0; s)}^{t_{\text{last}}(x_0; s)} (t_{\text{last}}(x_0; s) - r) dr \right] \\ &\leq e^{C_0 T} \left( \frac{T}{e} + \frac{T^2}{2} \right) \end{aligned}$$

and

$$(IV-3) \leq \int_{t_{\text{last}}(x_0; s)}^t (t-r)e^{-(t-r)} dr = 1 - (1+t-t_{\text{last}}(x_0; s))e^{-(t-t_{\text{last}}(x_0; s))} \leq 1.$$

Hence,

$$|(IV)| \leq C_3 \|J'\|_{L^1(\mathbb{R})} \left[ e^{C_0 T} \left( \frac{1}{e} + T + 1 \right) + e^{C_0 T} \left( \frac{T}{e} + \frac{T^2}{2} \right) + 1 \right]. \tag{2.16}$$

Consequently, (2.11) follows from (2.13), (2.14), (2.15) and (2.16).  $\square$

### 3. Uniform steepness

In this section, we study the steepness of transition fronts and prove [Theorem 1.3](#). Throughout this section, we assume (H1)–(H3). [Theorem 1.3](#) will be a simple result of the following theorem.

**Theorem 3.1.** *For any  $M > 0$ , there exists  $\alpha_M > 0$  such that*

$$\sup_{x \in [X(t; s) - M, X(t; s) + M]} u_x(t, x; s) \leq -\alpha_M$$

for all  $s < 0, t \geq s$ .

Assuming [Theorem 3.1](#), we prove [Theorem 1.3](#).

**Proof of [Theorem 1.3](#).** It follows from [Proposition 1.1](#) (2)(i), (2.1) and [Theorem 3.1](#).  $\square$

To finish the proof of [Theorem 1.3](#), we prove [Theorem 3.1](#), which is based on the following Lemma, whose proof is inspired by the proof of [[7, Theorem 5.1](#)] and [[22, Lemma 3.2](#)].

**Lemma 3.2.** *For any  $t > t_0 \geq s, h > 0$  and  $z \in \mathbb{R}$ , there holds*

$$u_x(t, x; s) \leq C \int_{z-h}^{z+h} u_x(t_0, y; s) dy, \quad \forall x \in \mathbb{R},$$

where  $C = C(t - t_0, |x - z|, h) > 0$  satisfies

- (i)  $C \rightarrow 0$  polynomially as  $t - t_0 \rightarrow 0$  and  $C \rightarrow 0$  exponentially as  $t - t_0 \rightarrow \infty$ ;
- (ii)  $C : (0, \infty) \times [0, \infty) \times (0, \infty) \rightarrow (0, \infty)$  is locally uniformly positive in the sense that for any  $0 < t_1 < t_2 < \infty, M_1 > 0$  and  $h_1 > 0$ , there holds

$$\inf_{t \in [t_1, t_2], M \in [0, M_1], h \in (0, h_1]} C(t, M, h) > 0.$$



**Proof.** Let  $\epsilon > 0$ . Let  $v_1(t, x; s) = u(t, x + \epsilon; s)$  and  $v_2(t, x; s) = u(t, x; s)$ . We see that  $v(t, x; s) := v_1(t, x; s) - v_2(t, x; s) < 0$  by monotonicity and satisfies

$$v_t = J * v - v + f(t, v_1) - f(t, v_2).$$

By (H2), we can find  $K > 0$  such that  $f(t, v_1) - f(t, v_2) \leq -K(v_1 - v_2)$ , which implies that

$$v_t \leq J * v - v - K v.$$

Setting  $\tilde{v}(t, x; s) = e^{(1+K)(t-t_0)}v(t, x; s)$ , we see

$$\tilde{v}_t \leq J * \tilde{v}. \tag{3.1}$$

Since  $v < 0$ , we have  $\tilde{v} < 0$ , which implies  $J * \tilde{v} < 0$  by the nonnegativity of  $J$  by (H1), and therefore,  $\tilde{v}_t < 0$  by (3.1). In particular,  $\tilde{v}(t, x; s) < \tilde{v}(t_0, x; s)$ . It then follows from the nonnegativity of  $J$  and (3.1) that

$$\tilde{v}_t(t, x; s) \leq [J * \tilde{v}(t, \cdot; s)](x) \leq [J * \tilde{v}(t_0, \cdot; s)](x). \tag{3.2}$$

For each  $x \in \mathbb{R}$ , (3.2) is an ordinary differential inequality. Integrating (3.1) over  $[t_0, t]$  with respect to the time variable, we find from  $\tilde{v}(t_0, x; s) < 0$  that

$$\tilde{v}(t, x; s) \leq (t - t_0)[J * \tilde{v}(t_0, \cdot; s)](x) + \tilde{v}(t_0, x; s) < (t - t_0)[J * \tilde{v}(t_0, \cdot; s)](x).$$

In particular, for any  $T > 0$ , we have

$$\tilde{v}(t_0 + T, x; s) < T[J * \tilde{v}(t_0, \cdot; s)](x). \tag{3.3}$$

Then, considering (3.1) with initial time at  $t_0 + T$  and repeating the above arguments, we find

$$\tilde{v}(t_0 + T + T, x; s) < T[J * \tilde{v}(t_0 + T, \cdot; s)](x) < T^2[J * J * \tilde{v}(t_0, \cdot; s)](x),$$

where we used (3.3) in the second inequality. Repeating this, we conclude that for any  $T > 0$  and any  $N = 1, 2, 3, \dots$ , there holds

$$\tilde{v}(t_0 + NT, x; s) < T^N [J^N * \tilde{v}(t_0, \cdot; s)](x), \tag{3.4}$$

where  $J^N = \underbrace{J * J * \dots * J}_{N \text{ times}}$ . Since  $J$  is nonnegative and positive on some open interval,  $J^N$  can be positive on any fixed bounded interval if  $N$  is large. Moreover, since  $J$  is symmetric, so is  $J^N$ .

Now, let  $x \in \mathbb{R}$ ,  $z \in \mathbb{R}$  and  $h > 0$ , and let  $N := N(|x - z|, h)$  be large enough so that

$$\tilde{C} = \tilde{C}(|x - z|, h) := \inf_{y \in [x-z-h, x-z+h]} J^N(y) > 0.$$

Note that the dependence of  $N$  on  $x - z$  through  $|x - z|$  is due to the symmetry of  $J^N$ . Moreover, the positivity of  $\tilde{C} : [0, \infty) \times (0, \infty) \rightarrow (0, \infty)$  is uniform on compact sets, which is because  $N$  can be chosen to be nondecreasing in  $|x - z|$  and in  $h$ .

Then, for  $t > t_0$ , we see from (3.4) with  $T = \frac{t-t_0}{N}$  that

$$\begin{aligned} \tilde{v}(t, x; s) &< \left(\frac{t-t_0}{N}\right)^N \int_{\mathbb{R}} J^N(x-y)\tilde{v}(t_0, y; s)dy \\ &\leq \left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h} J^N(x-y)\tilde{v}(t_0, y; s)dy \\ &\leq \tilde{C} \left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h} \tilde{v}(t_0, y; s)dy, \end{aligned}$$

since  $x - y \in [x - z - h, x - z + h]$  when  $y \in [z - h, z + h]$ . Going back to  $u(t, x; s)$ , we find

$$u(t, x + \epsilon; s) - u(t, x; s) \leq \tilde{C}e^{-(1+K)(t-t_0)} \left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h} [u(t_0, y + \epsilon; s) - u(t_0, y; s)]dy.$$

Dividing the above estimate by  $\epsilon$ , we conclude the result from dominated convergence theorem with  $C = \tilde{C}e^{-(1+K)(t-t_0)} \left(\frac{t-t_0}{N}\right)^N$ . From which, we obtain the properties of  $C$  and finish the proof.  $\square$

Now, we prove [Theorem 3.1](#).

**Proof of Theorem 3.1.** Set

$$h_\theta := \max \left\{ \sup_{s < 0, t \geq s} |X(t; s) - X_{\frac{\theta}{2}}(t; s)|, \sup_{s < 0, t \geq s} |X(t; s) - X_{\frac{1+\theta}{2}}(t; s)| \right\}.$$

By [Proposition 1.1](#)(1)(ii),  $h_\theta < \infty$ . Then,

$$X(t_0; s) + h_\theta \geq X_{\frac{\theta}{2}}(t_0; s), \quad X(t_0; s) - h_\theta \leq X_{\frac{1+\theta}{2}}(t_0; s) \tag{3.5}$$

for all  $t_0 \geq s$ . Now, for any  $\tau \geq 0$  and  $t_0 \geq s$ , we apply [Lemma 3.2](#) with  $z = X(t_0; s)$  and  $h = h_\theta$  to see that if  $|x - X(t_0; s)| \leq M$ , then

$$\begin{aligned} u_x(\tau + t_0, x; s) &\leq C(\tau, M, h_\theta) \int_{X(t_0; s)-h_\theta}^{X(t_0; s)+h_\theta} u_x(t_0, y; s)dy \\ &= C(\tau, M, h_\theta)[u(t_0, X(t_0; s) + h_\theta; s) - u(t_0, X(t_0; s) - h_\theta; s)] \tag{3.6} \\ &\leq C(\tau, M, h_\theta)[u(t_0, X_{\frac{\theta}{2}}(t_0; s); s) - u(t_0, X_{\frac{1+\theta}{2}}(t_0; s); s)] \\ &= -\frac{C(\tau, M, h_\theta)}{2}, \end{aligned}$$

where we used (3.5) and the monotonicity in the second inequality. Notice  $C(\tau, M, h_\theta) \rightarrow 0$  as  $\tau \rightarrow 0$ .

To apply (3.6), we see that if  $|x - X(t_0 + 1; s)| \leq M$ , then

$$|x - X(t_0; s)| \leq |x - X(t_0 + 1; s)| + |X(t_0 + 1; s) - X(t_0; s)| \leq M + c_{\max},$$

where we used Proposition 1.1(1)(ii). We then apply (3.6) with  $M$  replaced by  $M + c_{\max}$  and  $\tau$  replaced by 1 to conclude that

$$u_x(t_0 + 1, x; s) \leq -\frac{C(1, M + c_{\max}, h_\theta)}{2}.$$

Since  $t_0 \geq s$  is arbitrary, we have shown

$$\sup_{s < 0, t-s \geq 1} \sup_{x \in [X(t; s) - M, X(t; s) + M]} u_x(t, x; s) < 0.$$

To finish the proof, we only need to show

$$\sup_{s < 0, 0 \leq t-s \leq 1} \sup_{x \in [X(t; s) - M, X(t; s) + M]} u_x(t, x; s) < 0. \tag{3.7}$$

To this end, we recall

$$u_x(t, x; s) = \phi'_{\min}(x - y_s) e^{-\int_s^t (1 - f_u(\tau, u(\tau, x; s))) d\tau} + \int_s^t b(r, x; s) e^{-\int_r^t (1 - f_u(\tau, u(\tau, x; s))) d\tau} dr, \tag{3.8}$$

where  $b(t, x; s) = \int_{\mathbb{R}} J'(x - y) u(t, y; s) dy = \int_{\mathbb{R}} J(x - y) u_x(t, y; s) dy$ . It is just the solution of the initial-value problem

$$(u_x)_t = J * u_x - u_x + f_u(t, u) u_x, \quad u_x(s, x; s) = \phi'_{\min}(x - y_s).$$

Set  $a := \inf_{(t, x) \in \mathbb{R} \times [0, 1]} f_u(t, u) < 0$ . Since  $\phi'_{\min} < 0$  and  $b(t, x; s) < 0$ , (3.8) implies

$$u_x(t, x; s) \leq \phi'_{\min}(x - y_s) e^{-(1-a)(t-s)} + \int_s^t b(r, x; s) e^{-(1-a)(t-r)} dr.$$

In particular,

$$u_x(t, x; s) \leq \phi'_{\min}(x - y_s) e^{-(1-a)} \quad \text{if } 0 \leq t - s \leq 1 \tag{3.9}$$

For  $0 \leq t - s \leq 1$ , we have from  $\dot{X}(t; s) \in [c_{\min}, c_{\max}]$  by Proposition 1.1(1)(ii) that

$$X(t; s) - X(s; s) \in [c_{\min}(t - s), c_{\max}(t - s)] \subset [0, c_{\max}]. \tag{3.10}$$

Recall that for any  $\lambda \in (0, 1)$ ,  $X_\lambda(t; s)$  is such that  $u(t, X_\lambda(t; s); s) = \lambda$ . In particular,  $X_\lambda(s; s)$  is such that  $\phi_{\min}(X_\lambda(s; s) - y_s) = \lambda$ . Thus,  $X_\lambda(s; s) - y_s$  is independent of  $s$ . From the construction

of  $X(t; s)$  in Theorem [26, Theorem 4.1], we know  $X(s; s) = X_{\lambda_*}(s; s) + C_1$  for some  $\lambda_* \in (\theta, 1)$  and  $C_1 > 0$ . Hence, there exists  $C_2 \in \mathbb{R}$  such that  $X(s; s) = y_s + C_2$  for all  $s < 0$ , which, together with (3.10), implies

$$X(t; s) - y_s = X(t; s) - X(s; s) + X(s; s) - y_s \in [C_2, C_2 + c_{\max}].$$

Now, if  $x \in [X(t; s) - M, X(t; s) + M]$ , then

$$x - y_s = x - X(t; s) + X(t; s) - y_s \in [-M, M] + [C_2, C_2 + c_{\max}] \subset [C_2 - M, C_2 + c_{\max} + M].$$

In particular, there exists  $c_M > 0$  such that

$$\sup_{x \in [X(t; s) - M, X(t; s) + M]} \phi'_{\min}(x - y_s) \leq \sup_{x \in [C_2 - M, C_2 + c_{\max} + M]} \phi'_{\min}(x) \leq -c_M,$$

since  $\phi'_{\min}$  is continuous and negative everywhere. It then follows from (3.9) that

$$\sup_{s < 0, 0 \leq t - s \leq 1} \sup_{x \in [X(t; s) - M, X(t; s) + M]} u_x(t, x; s) \leq -c_M e^{-(1-a)}.$$

In particular, (3.7) follows. This completes the proof.  $\square$

#### 4. Stability of transition fronts

In this section, we study the stability of transition fronts and prove Theorem 1.5. Throughout this section, we assume (H1)–(H4).

We first prove a Lemma. Let  $c_{\min}, c_{\max}$  be as in (1.6),  $M_1$  be as in (1.11), and  $\Gamma = \Gamma_\alpha$  be as in (1.12).

##### Lemma 4.1.

(i) Let  $I(r) = \int_{\mathbb{R}} J(x)e^{-rx} dx$  for  $r \in \mathbb{R}$ . Then,

$$I(r) = 1 + \frac{I''(\tilde{r})}{2} r^2$$

for some  $\tilde{r} = \tilde{r}(r)$  satisfying  $|\tilde{r}| \leq |r|$ .

(ii) There exists  $\alpha_0 > 0$  satisfying that for any  $0 < \alpha \leq \alpha_0$ , there exists  $M_2 = M_2(\alpha) > M_1 + 1$  such that

$$|e^{\alpha(x-M_1)} [J * \Gamma](x) - 1| \leq \frac{\alpha c_{\min}}{4}, \quad \forall x \geq M_2.$$

In particular,

$$|[J * \Gamma](x) - e^{-\alpha(x-M_1)}| \leq \frac{\alpha c_{\min}}{4} e^{-\alpha(x-M_1)}, \quad \forall x \geq M_2. \tag{4.1}$$

**Proof.** (i) By (H2),  $I(r)$  is well-defined for any  $r \in \mathbb{R}$  and it is smooth in  $r$ . We see  $I(0) = 1$ . Since

$$I'(r) = - \int_{\mathbb{R}} J(y)ye^{-ry} dy \rightarrow 0 \text{ as } r \rightarrow 0$$

due to the symmetry of  $J$ , we have  $I'(0) = 0$ . The result then follows from second-order Taylor expansion at  $r = 0$ .

(ii) Since  $\Gamma(x) = e^{-\alpha(x-M_1)}$  for  $x \geq M_1 + 1$ , we deduce

$$\begin{aligned} & e^{\alpha(x-M_1)}[J * \Gamma](x) - 1 \\ &= e^{\alpha(x-M_1)} \int_{-\infty}^{M_1+1} J(x-y)\Gamma(y)dy + \int_{M_1+1}^{\infty} J(x-y)e^{\alpha(x-y)} dy - 1 \end{aligned} \tag{4.2}$$

For the first term on the right hand side of (4.2), we see that since  $J(x)$  decays faster than exponential functions by (H2) at  $-\infty$ , it is not hard to check that  $e^{\alpha(x-M_1)} \int_{-\infty}^{M_1+1} J(x-y)\Gamma(y)dy \rightarrow 0$  as  $x \rightarrow \infty$ . Notice this limit is locally uniform in  $\alpha \in [0, \infty)$ . Thus, there exists  $\tilde{M}_2 = \tilde{M}_2(\alpha) > 0$  such that

$$\left| e^{\alpha(x-M_1)} \int_{-\infty}^{M_1+1} J(x-y)\Gamma(y)dy \right| \leq \frac{\alpha c_{\min}}{8}, \quad \forall x \geq \tilde{M}_2.$$

For the last two terms on the right hand side of (4.2), we have

$$\begin{aligned} \left| \int_{M_1+1}^{\infty} J(x-y)e^{\alpha(x-y)} dy - 1 \right| &= \left| \int_{M_1+1-x}^{\infty} J(y)e^{-\alpha y} dy - 1 \right| \\ &\leq \left| \int_{\mathbb{R}} J(y)e^{-\alpha y} dy - 1 \right| + \left| \int_{-\infty}^{M_1+1-x} J(y)e^{-\alpha y} dy \right| \end{aligned}$$

By (i), we conclude that there is  $\alpha_0 > 0$  such that for any  $0 < \alpha \leq \alpha_0$ ,  $\left| \int_{\mathbb{R}} J(y)e^{-\alpha y} dy - 1 \right| \leq \frac{\alpha c_{\min}}{16}$ . Since  $\left| \int_{-\infty}^{M_1+1-x} J(y)e^{-\alpha y} dy \right| \rightarrow 0$  locally uniformly in  $\alpha \in [0, \infty)$  as  $x \rightarrow \infty$ , we can find some  $\tilde{M}_2 = \tilde{M}_2(\alpha) > 0$  such that

$$\left| \int_{-\infty}^{M_1+1-x} J(y)e^{-\alpha y} dy \right| \leq \frac{\alpha c_{\min}}{16}, \quad \forall x \geq \tilde{M}_2.$$

Hence,

$$\left| \int_{M_1+1}^{\infty} J(x-y)e^{\alpha(x-y)}dy - 1 \right| \leq \frac{\alpha c_{\min}}{8}, \quad \forall x \geq \bar{M}_2.$$

The result follows with  $M_2 = \max\{\bar{M}_2, \bar{M}_2\}$ .  $\square$

Now, we prove **Theorem 1.5**.

**Proof of Theorem 1.5.** (1) Let  $\alpha_0$  be as in **Lemma 4.1**. For given  $0 < \alpha \leq \alpha_0$ , set

$$C_{\text{steep}} = C_{\text{steep}}(\alpha) := - \sup_{t \in \mathbb{R}} \sup_{x \in [X(t)-M_2, X(t)+M_2]} u_x(t, x) > 0$$

by **Theorem 1.3**. Set

$$C_{f_u} := \sup_{(t,u) \in \mathbb{R} \times [0, \infty)} |f_u(t, u)| \quad \text{and} \quad A = A(\alpha) := \frac{2C_{f_u} + 1}{C_{\text{steep}}}. \tag{4.3}$$

Finally, set

$$\epsilon_0 = \epsilon_0(\alpha) := \min \left\{ \frac{1 - \tilde{\theta}}{2}, \frac{\theta}{2}, \frac{1}{4A}, \frac{c_{\min}}{4A} \right\} \quad \text{and} \quad \omega = \omega(\alpha) := \min \left\{ \tilde{\beta}, \frac{\alpha c_{\min}}{4} \right\}, \tag{4.4}$$

where  $\tilde{\beta} > 0$  is as in (H4). Clearly,  $\tilde{\beta} \leq C_{f_u}$ .

We are going to prove (1) by constructing appropriate sub-solution and super-solution. We first construct a sub-solution. Let

$$\underline{u}(t, x; t_0) = u(t, x - \zeta^-(t)) - q(t)\Gamma(x - \zeta^-(t) - X(t)), \quad t \geq t_0, x \in \mathbb{R},$$

where

$$\zeta^-(t) = \zeta_0^- - \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-t_0)}) \quad \text{and} \quad q(t) = \epsilon e^{-\omega(t-t_0)}.$$

Clearly,  $\dot{\zeta}^-(t) = -Aq(t)$  and  $\dot{q}(t) = -\omega q(t)$ . We claim that  $\underline{u} = \underline{u}(t, x; t_0)$  is a sub-solution, that is,  $\underline{u}_t \leq J * \underline{u} - \underline{u} + f(t, \underline{u})$ . To show this, we consider three cases.

**Case 1.**  $x - \zeta^-(t) - X(t) \leq -M_2$ . For such  $x$ ,  $\underline{u} = u(t, x - \zeta^-(t)) - q(t)$ . We see

$$\begin{aligned} & \underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \\ &= u_t(t, x - \zeta^-(t)) - \dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) - \dot{q}(t) \\ & \quad - [[J * u(t, \cdot - \zeta^-(t))](x) - u(t, x - \zeta^-(t))] \\ & \quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - 1]q(t) - f(t, \underline{u}) \\ &= f(t, u(t, x - \zeta^-(t))) - f(t, \underline{u}) - \dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) + \omega q(t) \\ & \quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - 1]q(t). \end{aligned}$$

Notice  $\dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) \geq 0$ . We see that  $u(t, x - \zeta^-(t)) \geq \frac{1+\tilde{\theta}}{2}$  by the choice of  $M_1$  in (1.11) and  $M_2$ . Since  $\epsilon \leq \epsilon_0 \leq \frac{1-\tilde{\theta}}{2}$ , there holds  $\underline{u} \geq \tilde{\theta}$  because of the restriction on  $x$  in this case. Thus, by (H4), we find

$$f(t, u(t, x - \zeta^-(t))) - f(t, \underline{u}) \leq -\tilde{\beta}q(t).$$

Trivially,  $[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) \leq 1$ , since  $\Gamma \in [0, 1]$ . Hence,

$$\underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \leq -\tilde{\beta}q(t) + \omega q(t) \leq 0.$$

**Case 2.**  $x - \zeta^-(t) - X(t) \geq M_2$ . For such  $x$ ,  $\underline{u} = u(t, x - \zeta^-(t)) - q(t)e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)}$ . We see

$$\begin{aligned} & \underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \\ &= u_t(t, x - \zeta^-(t)) - \dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) \\ & \quad - [\dot{q}(t) + \alpha q(t)(\dot{\zeta}^-(t) + \dot{X}(t))]e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)} \\ & \quad - [[J * u(t, \cdot - \zeta^-(t))](x) - u(t, x - \zeta^-(t))] \\ & \quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)}]q(t) - f(t, \underline{u}) \\ &= f(t, u(t, x - \zeta^-(t))) - f(t, \underline{u}) - \dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) \\ & \quad - [\dot{q}(t) + \alpha q(t)(\dot{\zeta}^-(t) + \dot{X}(t))]e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)} \\ & \quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)}]q(t) \end{aligned}$$

Again,  $\dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) \geq 0$ . We see that  $u(t, x - \zeta^-(t)) \leq \frac{\theta}{2}$  by the choice of  $M_1$  and  $M_2$ , and therefore,  $\underline{u} \leq u(t, x - \zeta^-(t)) \leq \frac{\theta}{2}$ . Thus,  $f(t, u(t, x - \zeta^-(t))) = 0 = f(t, u^-)$ . Moreover,

$$\dot{q}(t) + \alpha q(t)(\dot{\zeta}^-(t) + \dot{X}(t)) = (-\omega - A\alpha q(t) + \alpha \dot{X}(t))q(t) \geq (-\omega - A\alpha\epsilon_0 + \alpha c_{\min})q(t).$$

Also, by (4.1), we have

$$\begin{aligned} & \left| [J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)} \right| \\ &= \left| \int_{\mathbb{R}} J(x - \zeta^-(t) - X(t) - y)\Gamma(y)dy - e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)} \right| \\ &\leq \frac{\alpha c_{\min}}{4} e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)}. \end{aligned}$$

It then follows that

$$\underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \leq \left( \omega + A\alpha\epsilon_0 - \alpha c_{\min} + \frac{\alpha c_{\min}}{4} \right) q(t) e^{-\alpha(x-\zeta^-(t)-X(t)-M_1)} \leq 0,$$

where we used (4.4).

**Case 3.**  $|x - \zeta^-(t) - X(t)| \leq [-M_2, M_2]$ . We compute

$$\begin{aligned} & \underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \\ &= u_t(t, x - \zeta^-(t)) - \dot{\zeta}^-(t)u_x(t, x - \zeta^-(t)) \\ &\quad - \dot{q}(t)\Gamma(x - \zeta^-(t) - X(t)) + q(t)\Gamma'(x - \zeta^-(t) - X(t))[\dot{\zeta}^-(t) + \dot{X}(t)] \\ &\quad - [[J * u(t, \cdot - \zeta^-(t))](x) - u(t, x - \zeta^-(t))] \\ &\quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - \Gamma(x - \zeta^-(t) - X(t))]q(t) - f(t, \underline{u}) \\ &= f(t, u(t, x - \zeta^-(t))) - f(t, \underline{u}) + Aq(t)u_x(t, x - \zeta^-(t)) \\ &\quad + \omega q(t)\Gamma(x - \zeta^-(t) - X(t)) + q(t)\Gamma'(x - \zeta^-(t) - X(t))[\dot{X}(t) - Aq(t)] \\ &\quad + [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - \Gamma(x - \zeta^-(t) - X(t))]q(t). \end{aligned}$$

We see that

$$\begin{aligned} & |f(t, u(t, x - \zeta^-(t))) - f(t, \underline{u})| \leq C_{f_u}q(t), \\ & Aq(t)u_x(t, x - \zeta^-(t)) \leq -AC_{\text{steep}}q(t), \\ & \omega q(t)\Gamma(x - \zeta^-(t) - X(t)) \leq \omega q(t), \\ & q(t)\Gamma'(x - \zeta^-(t) - X(t))[\dot{X}(t) - Aq(t)] \leq q(t)\Gamma'(x - \zeta^-(t) - X(t))[c_{\min} - A\epsilon_0] \leq 0, \\ & [[J * \Gamma(\cdot - \zeta^-(t) - X(t))](x) - \Gamma(x - \zeta^-(t) - X(t))]q(t) \leq q(t), \end{aligned}$$

where we used (4.4) in the fourth estimate. It then follows that

$$\underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \leq (C_{f_u} - AC_{\text{steep}} + \omega + 1)q(t) \leq 0,$$

where we used (4.4).

Hence, we have shown  $\underline{u}_t - [J * \underline{u} - \underline{u}] - f(t, \underline{u}) \leq 0$ , that is,  $\underline{u}$  is a sub-solution. By the first inequality in (1.13) and comparison principle, we conclude that

$$u(t, x - \zeta^-(t)) - q(t)\Gamma(x - \zeta^-(t) - X(t)) = \underline{u}(t, x; t_0) \leq u(t, x; t_0, u_0). \tag{4.5}$$

For the super-solution, we set

$$\bar{u}(t, x; t_0) = u(t, x - \zeta^+(t)) + q(t)\Gamma(x - \zeta^+(t) - X(t)), \quad t \geq t_0, x \in \mathbb{R},$$

where

$$\zeta^+(t) = \zeta_0^+ + \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-t_0)}) \quad \text{and} \quad q(t) = \epsilon e^{-\omega(t-t_0)}.$$

The proof of  $\bar{u} = \bar{u}(t, x; t_0)$  being a super-solution, that is,  $\bar{u}_t \geq J * \bar{u} - \bar{u} + f(t, \bar{u})$ , follows from arguments for the sub-solution. We outline the proof for completeness.



**Case 1.**  $x - \zeta^+(t) - X(t) \leq -M_2$ . We compute

$$\begin{aligned} &\bar{u}_t - [J * \bar{u} - \bar{u}] - f(t, \bar{u}) \\ &= f(t, u(t, x - \zeta^+(t))) - f(t, \bar{u}) - \dot{\zeta}^+(t)u_x(t, x - \zeta^+(t)) - \omega q(t) \\ &\quad - [[J * \Gamma(\cdot - \zeta^+(t) - X(t))](x) - 1]q(t) \\ &\geq \tilde{\beta}q(t) - \omega q(t) \geq 0. \end{aligned}$$

**Case 2.**  $x - \zeta^+(t) - X(t) \geq M_2$ . We compute

$$\begin{aligned} &\bar{u}_t - [J * \bar{u} - \bar{u}] - f(t, \bar{u}) \\ &= f(t, u(t, x - \zeta^+(t))) - f(t, \bar{u}) - \dot{\zeta}^+(t)u_x(t, x - \zeta^+(t)) \\ &\quad + [\alpha(\dot{\zeta}^+(t) + \dot{X}(t)) - \omega]q(t)e^{-\alpha(x - \zeta^+(t) - X(t) - M_1)} \\ &\quad - [[J * \Gamma(\cdot - \zeta^+(t) - X(t))](x) - e^{-\alpha(x - \zeta^+(t) - X(t) - M_1)}]q(t). \end{aligned}$$

We see  $u(t, x - \zeta^+(t)) \leq \frac{\theta}{2}$ , and therefore,  $\bar{u} \leq \theta$  since  $\epsilon_0 \leq \frac{\theta}{2}$ . In particular,  $f(t, u(t, x - \zeta^+(t))) - f(t, \bar{u}) = 0$ . Since  $-\dot{\zeta}^+(t)u_x(t, x - \zeta^+(t)) \geq 0$ ,

$$\alpha(\dot{\zeta}^+(t) + \dot{X}(t)) - \omega \geq \alpha Aq(t) + \alpha c_{\min} - \omega \geq 0$$

and

$$[J * \Gamma(\cdot - \zeta^+(t) - X(t))](x) - e^{-\alpha(x - \zeta^+(t) - X(t) - M_1)} \leq \frac{\alpha c_{\min}}{4} e^{-\alpha(x - \zeta^-(t) - X(t) - M_1)},$$

we have

$$\bar{u}_t - [J * \bar{u} - \bar{u}] - f(t, \bar{u}) \geq \left( \alpha Aq(t) + \alpha c_{\min} - \omega - \frac{\alpha c_{\min}}{4} \right) q(t) e^{-\alpha(x - \zeta^+(t) - X(t) - M_1)} \geq 0.$$

**Case 3.**  $|x - \zeta^+(t) - X(t)| \leq M_2$ . We compute

$$\begin{aligned} &\bar{u}_t - [J * \bar{u} - \bar{u}] - f(t, \bar{u}) \\ &= f(t, u(t, x - \zeta^+(t))) - f(t, \bar{u}) - \dot{\zeta}^+(t)u_x(t, x - \zeta^+(t)) \\ &\quad - \omega q(t)\Gamma(x - \zeta^+(t) - X(t)) - q(t)\Gamma'(x - \zeta^+(t) - X(t))(Aq(t) + \dot{X}(t)) \\ &\quad - [[J * \Gamma(\cdot - \zeta^+(t) - X(t))](x) - \Gamma(x - \zeta^+(t) - X(t))]q(t). \end{aligned}$$

We see that

$$\begin{aligned}
 f(t, u(t, x - \zeta^+(t))) - f(t, \bar{u}) &\geq -C_{f_u}q(t), \\
 -\dot{\zeta}^+(t)u_x(t, x - \zeta^+(t)) &\geq 0, \\
 -\omega q(t)\Gamma(x - \zeta^+(t) - X(t)) &\geq -\omega q(t), \\
 -q(t)\Gamma'(x - \zeta^+(t) - X(t))(Aq(t) + \dot{X}(t)) &\geq 0, \\
 -[[J * \Gamma(\cdot - \zeta^+(t) - X(t))](x) - \Gamma(x - \zeta^+(t) - X(t))]q(t) &\geq -q(t).
 \end{aligned}$$

It then follows that

$$\bar{u}_t - [J * \bar{u} - \bar{u}] - f(t, \bar{u}) \geq (AC_{\text{steep}} - C_{f_u} - \omega - 1)q(t) \geq 0.$$

Hence,  $\bar{u}$  is a super-solution. By the second inequality in (1.13) and comparison principle, we conclude that

$$u(t, x; t_0, u_0) \leq \bar{u}(t, x; t_0) = u(t, x - \zeta^+(t)) + q(t)\Gamma(x - \zeta^+(t) - X(t)). \tag{4.6}$$

The result then follows from (4.5) and (4.6).

(2) Note first that there is  $0 < \alpha = \alpha(\beta_0) \leq \alpha_0$  satisfying that for any  $\epsilon \in (0, \epsilon_0(\alpha))$ , there exists  $\zeta_0^\pm = \zeta_0^\pm(\epsilon, u_0) \in \mathbb{R}$  with  $\zeta_0^- < \zeta_0^+$  such that

$$u(t_0, x - \zeta_0^-) - \epsilon\Gamma_\alpha(x - \zeta_0^- - X(t_0)) \leq u_0(x) \leq u(t_0, x - \zeta_0^+) + \epsilon\Gamma_\alpha(x - \zeta_0^+ - X(t_0)). \tag{4.7}$$

We then conclude (2) by applying (1) and noticing that  $\lim_{t \rightarrow \infty} \zeta^\pm(t)$  exist and  $\Gamma \in [0, 1]$ .  $\square$

Note that the proof of Theorem 1.5(2) does not depend explicitly on the condition on  $u_0$  as in the statement of Theorem 1.5(2); instead, it only needs (4.7). This observation allows us to prove the following corollary, which generalizes Theorem 1.5(2) to more general initial data.

**Corollary 4.2.** *Let  $u(t, x)$  and  $X(t)$  be as in Proposition 1.1(2). Let  $\beta_0 > 0$ . Suppose  $t_0 \in \mathbb{R}$  and  $\tilde{u}_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$  satisfy*

$$\begin{cases} \tilde{u}_0 : \mathbb{R} \rightarrow [0, 1], & \liminf_{x \rightarrow -\infty} \tilde{u}_0(x) > \theta; \\ \exists C > 0 \text{ s.t. } |\tilde{u}_0 - u(t_0, x)| \leq Ce^{-\beta_0(x - X(t_0))} & \text{for } x \in \mathbb{R}. \end{cases}$$

*Then, there exist  $\omega > 0$  and  $\tilde{\epsilon}_0 > 0$  such that for any  $\epsilon \in (0, \tilde{\epsilon}_0]$ , there are  $\zeta_\pm = \zeta_\pm(\epsilon, u_0) \in \mathbb{R}$  and  $t_1 = t_1(\epsilon, u_0)$  such that*

$$u(t, x - \zeta_-) - \epsilon e^{-\omega(t-t_1)} \leq u(t, x; t_0, \tilde{u}_0) \leq u(t, x - \zeta_+) + \epsilon e^{-\omega(t-t_1)}$$

*for all  $x \in \mathbb{R}$  and  $t \geq t_1$ .*

**Proof.** The idea is that we allow the solution  $u(t, x; t_0, \tilde{u}_0)$  to evolve for some time. Due to the asymptotical stability of 1, it will develop into some shape satisfying (4.7). Then, we apply Theorem 1.5(2) at that time to conclude the result.

Modifying  $\tilde{u}_0$  near  $-\infty$ , we can find  $u_0 \in C^b_{\text{unif}}(\mathbb{R}, \mathbb{R})$  satisfying  $u_0 \geq \tilde{u}_0$  and

$$\begin{cases} u_0 : \mathbb{R} \rightarrow [0, 1], & u_0(-\infty) = 1; \\ \exists C > 0 \text{ s.t. } |u_0 - u(t_0, x)| \leq C e^{-\beta_0(x - X(t_0))} & \text{for } x \in \mathbb{R}. \end{cases}$$

In particular, we can apply [Theorem 1.5\(2\)](#) to  $u_0$  to conclude that

$$u(t, x; t_0, u_0) \leq u(t, x - \zeta^+) + q(t)\Gamma_\alpha(x - \zeta^+ - X(t)), \tag{4.8}$$

where  $q(t) = e^{-\omega(t-t_0)}$  and  $\Gamma_\alpha$  is the same as in the proof of [Theorem 1.5\(2\)](#). Notice [\(4.8\)](#) holds for some  $0 < \alpha \leq \alpha_0$ . Since  $u_0 \geq \tilde{u}_0$ , we have from comparison and [\(4.8\)](#) that

$$u(t, x; t_0, \tilde{u}_0) \leq u(t, x - \zeta^+) + q(t)\Gamma_\alpha(x - \zeta^+ - X(t)). \tag{4.9}$$

Thus, for a fixed small  $\epsilon > 0$ , we can find some  $t_1 = t_1(\epsilon) \gg t_0$  such that  $q(t) \leq \epsilon$ , and then,

$$u(t_1, x; t_0, \tilde{u}_0) \leq u(t_1, x - \zeta^+) + \epsilon\Gamma_\alpha(x - \zeta^+ - X(t_1)). \tag{4.10}$$

Next, we construct an appropriate lower bound for  $u(t_1, x; t_0, \tilde{u}_0)$ . This actually follows from the asymptotic stability of the equilibrium 1. More precisely, since  $\liminf_{x \rightarrow -\infty} \tilde{u}_0(x) > \theta$ , there exist  $\lambda_0 \in (\theta, \liminf_{x \rightarrow -\infty} \tilde{u}_0(x))$  and a function  $\bar{u}_0 \in C^b_{\text{unif}}(\mathbb{R}, \mathbb{R})$  satisfying

$$\exists x_1 < x_2 \quad \text{s.t.} \quad \bar{u}_0(x) = \begin{cases} \lambda_0 & \text{if } x \leq x_1, \\ 0 & \text{if } x \geq x_2 \end{cases}$$

such that  $\bar{u}_0 \leq \tilde{u}_0$ . Now, we consider the solution  $u_B(t, x; \bar{u}_0)$  with initial data  $u_B(0, \cdot; \bar{u}_0) = \bar{u}_0$  of the following homogeneous equation

$$u_t = J * u - u + f_B(u) \tag{4.11}$$

where  $f_B : [0, 1] \rightarrow \mathbb{R}$  is a bistable nonlinearity satisfying the following conditions

$$\begin{cases} f_B \in C^2([0, 1]), \quad f_B(0) = 0, \quad f_B(\theta) = 0, \quad f_B(1) = 0, \\ f'_B(0) < 0, \quad f'_B(1) < 0, \\ f_B(u) < 0 \text{ for } u \in (0, \theta), \quad 0 < f_B(u) \leq f_{\min}(u) \text{ for } u \in (\theta, 1), \\ \int_0^1 f_B(u)du > 0 \text{ and } 1 + f'_B(u) > 0 \text{ for } u \in [0, 1]. \end{cases}$$

Let  $c_B > 0$  be the unique speed of the traveling waves of [\(4.11\)](#), and we fix some profile  $\phi_B$ . Since  $f_B \leq f_{\min} \leq f(t, u)$  on  $[0, 1]$ , we conclude from the comparison principle that

$$u_B(t - t_0, x; \bar{u}_0) \leq u(t, x; t_0, u_0).$$

It is known (see [\[1, Theorem 4.2\]](#)) that there exist  $\zeta_B^\pm \in \mathbb{R}$ ,  $\epsilon_B > 0$  and  $\omega_B > 0$  such that

$$\phi_B(x - c_B t - \zeta_B^-) - \epsilon_B e^{-\omega_B(t-t_0)} \leq u_B(t - t_0, x; \bar{u}_0) \leq \phi_B(x - c_B t - \zeta_B^+) + \epsilon_B e^{-\omega_B(t-t_0)}.$$

In particular,  $u_B(t - t_0, -\infty; \bar{u}_0)$ , and hence,  $u(t, x; t_0, u_0)$ , approaches to 1 exponentially fast. Thus, making  $\alpha > 0$  so small that  $\omega$  is small and choosing  $t_1$  larger if necessary, we can guarantee that

$$u(t_1, -\infty; t_0, u_0) \geq 1 - \frac{\epsilon}{2}.$$

Thus, choosing  $\alpha > 0$  further small if necessary, we can find  $\zeta^- \in \mathbb{R}$  such that

$$u(t_1, x - \zeta^-) - \epsilon \Gamma_\alpha(x - \zeta^- - X(t_1)) \leq u(t_1, x; t_0, \bar{u}_0). \tag{4.12}$$

Finally, in the presence of (4.9) and (4.12), we can apply Theorem 1.5(2) to  $u(t_1, x; t_0, \bar{u}_0)$  to conclude the result.  $\square$

### 5. Asymptotic stability of transition fronts

In this section, we study the asymptotic stability of transition fronts and prove Theorem 1.6. We assume (H1)–(H4) throughout this section.

We first prove two lemmas. The first one concerns the exponential decay of  $u_x(t, x + X(t))$  at  $\pm\infty$ .

**Lemma 5.1.** *There exist  $\tilde{c}_\pm > 0$ ,  $\tilde{C}_\pm > 0$  and  $\tilde{h}_\pm > 0$  such that*

$$\begin{aligned} 0 > u_x(t, x) &\geq -\tilde{C}_+ e^{-\tilde{c}_+(x-X(t)-\tilde{h}_+)}, & \forall x \geq X(t) + \tilde{h}_+, \\ 0 > u_x(t, x) &\geq -\tilde{C}_- e^{\tilde{c}_-(x-X(t)+\tilde{h}_-)}, & \forall x \leq X(t) - \tilde{h}_- \end{aligned}$$

for all  $t \in \mathbb{R}$ .

**Proof.** We prove the first estimate for  $u_x(t, x)$ . By monotonicity,  $u_x(t, x) < 0$ . Since  $X(t; s)$  and  $u_x(t, x; s)$  converge locally uniformly to  $X(t)$  and  $u_x(t, x)$ , respectively, it suffices to show

$$u_x(t, x; s) \geq -\tilde{C} e^{-\tilde{c}(x-X(t;s)-\tilde{h})}, \quad \forall x \geq X(t; s) + \tilde{h} \tag{5.1}$$

for all  $s < 0, t \geq s$ . To this end, we set

$$\tilde{C} = \sup_{\substack{s < 0, t \geq s \\ x \in \mathbb{R}}} |u_x(t, x; s)| \quad \text{and} \quad \tilde{h} \geq \sup_{s < 0, t \geq s} |X(t; s) - X_\theta(t; s)|.$$

By the choice of  $\tilde{h}$ , we have  $f(t, u(t, x; s)) = 0$  for  $x \geq X(t; s) + \tilde{h}$ . Since  $u_x(t, x; s)$  satisfies  $(u_x)_t = J * u_x - u_x + f_u(t, u(t, x; s))u_x$ , we see that  $u_x(t, x; s)$  satisfies

$$(u_x)_t = J * u_x - u_x, \quad x \geq X(t; s) + \tilde{h}. \tag{5.2}$$

Define

$$N[v] = v_t - [J * v - v].$$

We compute

$$\begin{aligned} N[-\tilde{C}e^{-c(x-X(t)-\tilde{h})}] &= -\tilde{C}e^{-c(x-X(t)-\tilde{h})} \left[ c\dot{X}(t; s) - \int_{\mathbb{R}} J(y)e^{cy} dy + 1 \right] \\ &\leq -\tilde{C}e^{-c(x-X(t)-\tilde{h})} \left[ cc_{\min} - \int_{\mathbb{R}} J(y)e^{cy} dy + 1 \right]. \end{aligned}$$

Setting  $g(c) = cc_{\min} - \int_{\mathbb{R}} J(y)e^{cy} dy + 1$ , we see  $g(0) = 0$  and  $g'(c) = c_{\min} - \int_{\mathbb{R}} yJ(y)e^{cy} dy$ . Since  $c_{\min} > 0$  and  $\int_{\mathbb{R}} yJ(y)e^{cy} dy \rightarrow 0$  as  $c \rightarrow 0$  by the symmetry of  $J$ , we are able to find some  $\tilde{c} > 0$  such that  $g(\tilde{c}) > 0$ . It then follows that  $N[-\tilde{C}e^{-\tilde{c}(x-X(t)-\tilde{h})}] \leq 0$ . In particular, by (5.2), we have

$$N[u_x] = 0 \geq N[-\tilde{C}e^{-\tilde{c}(x-X(t)-\tilde{h})}], \quad x \geq X(t; s) + \tilde{h}, \quad t \geq s. \tag{5.3}$$

Moreover, we trivially have

$$u_x(t, x; s) \geq -\tilde{C} \geq -\tilde{C}e^{-\tilde{c}(x-X(t)-\tilde{h})}, \quad x \leq X(t; s) + \tilde{h}, \quad t \geq s. \tag{5.4}$$

Also, at the initial moment  $s$ , choosing  $\tilde{c}$  smaller and  $\tilde{h}$  larger (if necessary), we have

$$u_x(s, x; s) = \phi'_{\min}(x - y_s) \geq -\tilde{C}e^{-\tilde{c}(x-X(s)-\tilde{h})}. \tag{5.5}$$

We then conclude from (5.3), (5.4), (5.5) and the comparison principle (see Proposition A.1) that (5.1) holds. We point out that the above arguments work due to the fact that  $\dot{X}(t; s) \geq c_{\min} > 0$ .

For the second estimate for  $u_x(t, x)$ , we notice that if we choose  $\hat{h}$  be such that

$$\hat{h} \geq \sup_{s < 0, t \geq s} |X(t; s) - X_{\tilde{\theta}}(t; s)|,$$

where  $\tilde{\theta}$  is as in (H4). Then,  $f_u(t, u(t, x; s)) \leq -\tilde{\beta}$  for  $x \leq X(t; s) - \hat{h}$ . It then follows that  $u_x(t, x; s)$  satisfies

$$(u_x)_t \geq J * u_x - u_x - \tilde{\beta}u_x, \quad x \leq X(t; s) - \hat{h}.$$

The rest of the proof then follows from similar arguments as above if we consider

$$N[v] = v_t - [J * v - v] + \tilde{\beta}v.$$

This completes the proof.  $\square$

The second lemma, improving Theorem 1.5(1), is the key to Theorem 1.6. Shall not cause any confusion with  $u(t, x; s)$ , we will use  $u(t, x; t_0)$  to denote a solution of (1.3) with initial condition at time  $t_0$ . Recall  $\alpha > 0$  is small, and  $\Gamma = \Gamma_\alpha$ ,  $A = A(\alpha)$ ,  $\epsilon_0 = \epsilon_0(\alpha)$  and  $\omega = \omega(\alpha)$  are as in (1.12), (4.3) and (4.4).

**Lemma 5.2.** *Suppose there exist  $\zeta \in \mathbb{R}$ ,  $\delta > 0$  and  $\epsilon \in (0, \epsilon_0]$  such that*

$$u(\tau, x - \zeta) - \epsilon\Gamma(x - \zeta - X(\tau)) \leq u(\tau, x; t_0) \leq u(\tau, x - \zeta - \delta) + \epsilon\Gamma(x - \zeta - \delta - X(\tau)) \tag{5.6}$$

*for some  $\tau \geq t_0$ . Then, there exist large  $\sigma = \sigma(\alpha) > 0$  and small  $\tilde{\epsilon} = \tilde{\epsilon}(\alpha, \epsilon_0) > 0$  such that*

$$\begin{aligned} &u(t, x - \zeta(t)) - q(t)\Gamma(x - \zeta(t) - X(t)) \\ &\leq u(t, x; t_0) \leq u(t, x - \zeta(t) - \delta(t)) - q(t)\Gamma(x - \zeta(t) - \delta(t) - X(t)) \end{aligned}$$

*for all  $t \geq \tau + \sigma$ , where*

$$\begin{aligned} \zeta(t) &\in [\zeta - \frac{2A\epsilon}{\omega}, \zeta + \tilde{\epsilon} \min\{1, \delta\}], \\ 0 \leq \delta(t) &\leq \delta - \tilde{\epsilon} \min\{1, \delta\} + \frac{4A\epsilon}{\omega}, \\ 0 \leq q(t) &\leq (\frac{\epsilon}{2} + \tilde{C}\tilde{\epsilon} \min\{1, \delta\})e^{-\omega(t-\tau-\sigma)}, \end{aligned}$$

*where  $\tilde{C} > 0$  is some constant and  $\tilde{C}\tilde{\epsilon} \leq \frac{\epsilon_0}{2}$ .*

**Proof.** Applying [Theorem 1.5\(1\)](#) to (5.6), we find

$$\begin{aligned} &u(t, x - \zeta_\tau^-(t)) - q_\tau(t)\Gamma(x - \zeta_\tau^-(t) - X(t)) \\ &\leq u(t, x; t_0) \leq u(t, x - \zeta_\tau^+(t) - \delta) + q_\tau(t)\Gamma(x - \zeta_\tau^+(t) - \delta - X(t)) \end{aligned} \tag{5.7}$$

for all  $t \geq \tau$ , where  $\zeta_\tau^\pm(t) = \zeta \pm \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-\tau)})$  and  $q_\tau(t) = \epsilon e^{-\omega(t-\tau)}$ .

We modify (5.7) at the moment  $t = \tau + \sigma$  for some  $\sigma > 0$  to be chosen to obtain a new estimate for  $u(\tau + \sigma, x; t_0)$ , and then apply [Theorem 1.5\(1\)](#) to this new estimate to conclude the result of the lemma. To this end, we set

$$\tilde{\delta} = \min\{\delta, 1\} \quad \text{and} \quad \tilde{C}_{\text{steep}} = \frac{1}{2} \sup \{u_x(t, x) \mid |x - X(t)| \leq 2, t \geq t_0\} < 0.$$

Then, for  $t \geq t_0$ , we deduce from Taylor expansion that

$$\int_{X(t)-\frac{1}{2}}^{X(t)+\frac{1}{2}} [u(t, y - \tilde{\delta}) - u(t, y)] dy \geq -2\tilde{C}_{\text{steep}}\tilde{\delta}.$$

In particular, at the moment  $t = \tau$ , either

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y - \tilde{\delta}) - u(\tau, y + \zeta; t_0)] dy \geq -\tilde{C}_{\text{steep}}\tilde{\delta} \tag{5.8}$$

or

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y + \zeta; t_0) - u(\tau, y)] dy \geq -\tilde{C}_{\text{steep}} \tilde{\delta} \tag{5.9}$$

must be the case.

We first consider the problem when (5.9) holds. We are about to establish an appropriate lower bound for

$$u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}),$$

where  $\tilde{\epsilon} > 0$  and  $\sigma > 0$  are to be chosen. To do so, let  $M > 0$  be a large number to be chosen, and consider three cases: (i)  $x - \zeta - X(\tau) \in [-M, M]$ ; (ii)  $x - \zeta - X(\tau) \leq -M$ ; (iii)  $x - \zeta - X(\tau) \geq -M$ .

**Case (i).**  $x - \zeta - X(\tau) \in [-M, M]$ . We write

$$\begin{aligned} & u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\ &= [u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma))] \\ & \quad + [u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta})]. \end{aligned} \tag{5.10}$$

For the first difference on the right hand side of (5.10), we argue

$$\begin{aligned} & u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_{\tau}^-(\tau + \sigma)) + q_{\tau}(\tau + \sigma)\Gamma(x - \zeta_{\tau}^-(\tau + \sigma) - X(\tau + \sigma)) \\ &= u(\tau + \sigma, x; t_0) \\ & \quad - [u(\tau + \sigma, x - \zeta + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma})) - q_{\tau}(\tau + \sigma)\Gamma(x - \zeta + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma}) - X(\tau + \sigma))] \\ &= u(\tau + \sigma, y + \zeta; t_0) \\ & \quad - [u(\tau + \sigma, y + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma})) - q_{\tau}(\tau + \sigma)\Gamma(y + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma}) - X(\tau + \sigma))] \\ & \hspace{15em} (\text{by } y = x - \zeta \in X(\tau) + [-M, M]) \end{aligned}$$

$$= u(\tau + \sigma, y + \zeta; t_0) - \tilde{u}(\tau + \sigma, y)$$

$$(\text{where } \tilde{u}(t, y) = u(t, y + \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-\tau)})) - q_{\tau}(t)\Gamma(y + \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-\tau)}) - X(t)))$$

$$\geq C(\sigma, M) \int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y + \zeta; t_0) - \tilde{u}(\tau, y)] dy$$

$$\geq C(\sigma, M) \int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y + \zeta; t_0) - u(\tau, y)] dy \quad (\text{by } \tilde{u}(\tau, y) \leq u(\tau, y))$$

$$\geq -C(\sigma, M)\tilde{C}_{\text{steep}}\tilde{\delta} \quad (\text{by (5.9)}),$$

where the first inequality follows from the arguments as in the proof of Lemma 3.2. In fact, we know  $u(t, y + \zeta; t_0)$  is a solution of  $v_t = J * v - v + f(t, v)$ , while  $\tilde{u}(t, y)$  is a subsolution by the proof of Theorem 1.5. Moreover,  $u(t, y + \zeta; t_0) \geq \tilde{u}(t, y)$  by (5.7). Based on these information, we can repeat the arguments in the proof of Lemma 3.2 to conclude the inequality. Here,  $C(t - \tau, M) > 0$  satisfies  $C(t - \tau, M) \rightarrow 0$  polynomially as  $t - \tau \rightarrow 0$  and exponentially as  $t - \tau \rightarrow \infty$ . Thus, we have shown

$$\begin{aligned} &u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) \\ &\geq -C(\sigma, M)\tilde{C}_{\text{steep}}\tilde{\delta} - q_\tau(\tau + \sigma)\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)). \end{aligned} \tag{5.11}$$

For the second difference on the right hand side of (5.10), Taylor expansion gives

$$\begin{aligned} &u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\ &= u_x(\tau + \sigma, x - \zeta + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma}) - x_*)\tilde{\epsilon}\tilde{\delta}, \end{aligned}$$

where  $x_* \in [0, \tilde{\epsilon}\tilde{\delta}] \subset [0, 1]$ . Setting

$$\tilde{\epsilon} = \tilde{\epsilon}(\sigma, M) := \min \left\{ 1, \frac{-\tilde{C}_{\text{steep}}C(\sigma, M)}{\sup_{(t,x) \in \mathbb{R} \times \mathbb{R}} |u_x(t, x)|} \right\} > 0, \tag{5.12}$$

we deduce

$$u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \geq C(\sigma, M)\tilde{C}_{\text{steep}}\tilde{\delta}. \tag{5.13}$$

It then follows from (5.10), (5.11) and (5.13) that

$$u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \geq -q_\tau(\tau + \sigma)\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)). \tag{5.14}$$

**Case (ii).**  $x - \zeta - X(\tau) \leq -M$ . We write

$$\begin{aligned} &u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\ &= [u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma))] \\ &\quad + [u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta})] \\ &\geq -q_\tau(\tau + \sigma)\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)) \\ &\quad + [u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta})], \end{aligned} \tag{5.15}$$

where we used the first inequality in (5.7). For the term in the bracket, we first choose  $M = M(\alpha)$  such that  $-M + \frac{A\epsilon}{\omega} \leq -\tilde{h}_-$ , where  $\tilde{h}_-$  is as in Lemma 5.1. Then, we have



$$\begin{aligned}
 x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma) &\leq x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau) \\
 &= x - \zeta - X(\tau) + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma}) \\
 &\leq -M + \frac{A\epsilon}{\omega} \leq -\tilde{h}_-.
 \end{aligned}$$

It then follows from Lemma 5.1 that

$$\begin{aligned}
 &u(\tau + \sigma, x - \zeta_{\tau}^{-}(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_{\tau}^{-}(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\
 &= u_x(\tau + \sigma, x - \zeta_{\tau}^{-}(\tau + \sigma) - x_*)\tilde{\epsilon}\tilde{\delta} \quad (\text{where } x_* \in [0, \tilde{\epsilon}\tilde{\delta}] \subset [0, 1]) \\
 &\geq -\tilde{C}_-e^{\tilde{c}_-(x-\zeta_{\tau}^{-}(\tau+\sigma)-x_*-X(\tau+\sigma)+\tilde{h}_-)}\tilde{\epsilon}\tilde{\delta} \\
 &= -\tilde{C}_-e^{\tilde{c}_-(x-\zeta_{\tau}^{-}(\tau+\sigma)-x_*-X(\tau)+\tilde{h}_-)}e^{-\tilde{c}_-(X(\tau+\sigma)-X(\tau))}\tilde{\epsilon}\tilde{\delta} \\
 &\geq -\tilde{C}_-e^{-\tilde{c}_-(X(\tau+\sigma)-X(\tau))}\tilde{\epsilon}\tilde{\delta} \\
 &\geq -\tilde{C}_-e^{-\tilde{c}_-c_{\min}\sigma}\tilde{\epsilon}\tilde{\delta}.
 \end{aligned}$$

Going back to (5.15), we find

$$\begin{aligned}
 &u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_{\tau}^{-}(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\
 &\geq -q_{\tau}(\tau + \sigma)\Gamma(x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma)) - \tilde{C}_-e^{-\tilde{c}_-c_{\min}\sigma}\tilde{\epsilon}\tilde{\delta} \\
 &= -[q_{\tau}(\tau + \sigma) + \tilde{C}_-e^{-\tilde{c}_-c_{\min}\sigma}\tilde{\epsilon}\tilde{\delta}]\Gamma(x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma)) \\
 &\geq -[q_{\tau}(\tau + \sigma) + \tilde{C}_-\tilde{\epsilon}\tilde{\delta}]\Gamma(x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma))
 \end{aligned} \tag{5.16}$$

if we choose  $M$  large so that  $-M + \frac{A\epsilon}{\omega} \leq -M_1 - 1$ , and hence,  $\Gamma(x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma)) = 1$ .

**Case (iii).**  $x - \zeta - X(\tau) \geq M$ . Choosing  $M = M(\alpha, \sigma)$  larger, say  $M - c_{\max}\sigma \geq \max\{M_1 + 1, \tilde{h}_+ + 1\}$ , we have

$$\begin{aligned}
 x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma) &= x - \zeta - X(\tau) + \frac{A\epsilon}{\omega}(1 - e^{-\omega\sigma}) - (X(\tau + \sigma) - X(\tau)) \\
 &\geq M - c_{\max}\sigma \geq \max\{M_1 + 1, \tilde{h}_+ + 1\}.
 \end{aligned}$$

As a result,

$$\Gamma(x - \zeta_{\tau}^{-}(\tau + \sigma) - X(\tau + \sigma)) = e^{-\alpha(x-\zeta_{\tau}^{-}(\tau+\sigma)-X(\tau+\sigma)-M_1)}$$

and by Lemma 5.1

$$u_x(\tau + \sigma, x - \zeta_{\tau}^{-}(\tau + \sigma) - x_*) \geq \tilde{C}_+e^{-\tilde{c}_+(x-\zeta_{\tau}^{-}(\tau+\sigma)-x_*-X(\tau+\sigma)-\tilde{h}_+)}, \quad \forall x_* \in [0, 1].$$

Together with the first inequality in (5.7), we deduce

$$\begin{aligned}
 & u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\
 & \geq u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma)) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\
 & \quad - q_\tau(\tau + \sigma)\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)) \\
 & = u_x(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - x_*)\tilde{\epsilon}\tilde{\delta} - q_\tau(\tau + \sigma)e^{-\alpha(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma) - M_1)} \\
 & \geq -\tilde{C}_+e^{-\tilde{c}_+(x - \zeta_\tau^-(\tau + \sigma) - x_* - X(\tau + \sigma) - \tilde{h}_+)}\tilde{\epsilon}\tilde{\delta} - q_\tau(\tau + \sigma)e^{-\alpha(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma) - M_1)} \\
 & \geq -[\tilde{C}_+\tilde{\epsilon}\tilde{\delta} + q_\tau(\tau + \sigma)]e^{-\alpha(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma) - M_1)},
 \end{aligned}$$

if we choosing  $\alpha$  smaller so that  $\alpha \leq \tilde{c}_+$ . Hence,

$$\begin{aligned}
 & u(\tau + \sigma, x; t_0) - u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) \\
 & \geq -[\tilde{C}_+\tilde{\epsilon}\tilde{\delta} + q_\tau(\tau + \sigma)]\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)).
 \end{aligned} \tag{5.17}$$

Thus, combining (5.14), (5.16), (5.17) and the second inequality in (5.7), we find

$$\begin{aligned}
 & u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) - \bar{q}_\tau(\tau + \sigma, \tilde{\epsilon})\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)) \\
 & \leq u(\tau + \sigma, x; t_0) \\
 & \leq u(\tau + \sigma, x - \zeta_\tau^+(\tau + \sigma) - \delta) + q_\tau(\tau + \sigma)\Gamma(x - \zeta_\tau^+(\tau + \sigma) - \delta - X(\tau + \sigma)),
 \end{aligned} \tag{5.18}$$

where

$$\bar{q}_\tau(\tau + \sigma, \tilde{\epsilon}) = \begin{cases} q_\tau(\tau + \sigma), & x - \zeta - X(\tau) \in [-M, M], \\ q_\tau(\tau + \sigma) + \tilde{C}_-\tilde{\epsilon}\tilde{\delta}, & x - \zeta - X(\tau) \leq -M, \\ \tilde{C}_+\tilde{\epsilon}\tilde{\delta} + q_\tau(\tau + \sigma), & x - \zeta - X(\tau) \geq M. \end{cases}$$

Observe that the first  $\Gamma$  in (5.18) is not in its right form, but from the monotonicity, we see

$$\Gamma(x - \zeta_\tau^-(\tau + \sigma) - X(\tau + \sigma)) \leq \Gamma(x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta} - X(\tau + \sigma)).$$

Since clearly  $q_\tau(\tau + \sigma) \leq \bar{q}_\tau(\tau + \sigma, \tilde{\epsilon})$ , we conclude from (5.18) that

$$\begin{aligned}
 & u(\tau + \sigma, x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta}) - \bar{q}(\sigma)\Gamma(x - \zeta_\tau^-(\tau + \sigma) - \tilde{\epsilon}\tilde{\delta} - X(\tau + \sigma)) \\
 & \leq u(\tau + \sigma, x; t_0) \\
 & \leq u(\tau + \sigma, x - \zeta_\tau^+(\tau + \sigma) - \delta) + \bar{q}(\sigma)\Gamma(x - \zeta_\tau^+(\tau + \sigma) - \delta - X(\tau + \sigma)),
 \end{aligned} \tag{5.19}$$

where  $\bar{q}(\sigma, \tilde{\epsilon}) = \bar{q}_\tau(\tau + \sigma, \tilde{\epsilon})$  is independent of  $\tau$ . To apply Theorem 1.5(1), we choose  $\sigma = \sigma(\alpha)$  sufficiently large and  $\tilde{\epsilon} = \tilde{\epsilon}(\sigma, M, \epsilon_0) = \tilde{\epsilon}(\alpha, \epsilon_0)$  sufficiently small so that

$$e^{-\omega\sigma} \leq \frac{1}{2} \quad \text{and} \quad (\tilde{C}_- + \tilde{C}_+)\tilde{\epsilon} \leq \frac{\epsilon_0}{2}.$$

Of course, for  $\tilde{\epsilon}$ , we should also take (5.12) into consideration. As a result  $\bar{q}(\sigma, \tilde{\epsilon}) \leq \epsilon_0$ . We then apply Theorem 1.5(1) to (5.19) to conclude that

$$\begin{aligned}
 &u(t, x - \zeta^-(t)) - \bar{q}(\sigma, \tilde{\epsilon})e^{-\omega(t-\tau-\sigma)}\Gamma(x - \zeta^-(t) - X(t)) \\
 &\leq u(t, x; t_0) \leq u(t, x - \zeta^+(t)) + \bar{q}(\sigma, \tilde{\epsilon})e^{-\omega(t-\tau-\sigma)}\Gamma(x - \zeta^+(t) - X(t))
 \end{aligned}
 \tag{5.20}$$

for  $t \geq \tau + \sigma$ , where

$$\begin{aligned}
 \zeta^-(t) &= \zeta_\tau^-(\tau + \sigma) + \tilde{\epsilon}\tilde{\delta} - \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-\tau-\sigma)}) = \zeta - \frac{2A\epsilon}{\omega} + \tilde{\epsilon}\tilde{\delta} + \frac{A\epsilon}{\omega}[e^{-\omega\sigma} + e^{-\omega(t-\tau-\sigma)}], \\
 \zeta^+(t) &= \zeta_\tau^+(\tau + \sigma) + \delta + \frac{A\epsilon}{\omega}(1 - e^{-\omega(t-\tau-\sigma)}) = \zeta + \frac{2A\epsilon}{\omega} + \delta - \frac{A\epsilon}{\omega}[e^{-\omega\sigma} + e^{-\omega(t-\tau-\sigma)}].
 \end{aligned}$$

Let  $\tilde{C} = \tilde{C}_- + \tilde{C}_+$ . Setting

$$\begin{aligned}
 q(t) &= \left(\frac{\epsilon}{2} + \tilde{C}\tilde{\epsilon}\tilde{\delta}\right)e^{-\omega(t-\tau-\sigma)}, \quad \zeta(t) = \zeta^-(t) \quad \text{and} \\
 \delta(t) &= \delta - \tilde{\epsilon}\tilde{\delta} + \frac{4A\epsilon}{\omega} - \frac{2A\epsilon}{\omega}[e^{-\omega\sigma} + e^{-\omega(t-\tau-\sigma)}],
 \end{aligned}$$

we can rewrite (5.20) as

$$\begin{aligned}
 &u(t, x - \zeta(t)) - q(t)\Gamma(x - \zeta(t) - X(t)) \\
 &\leq u(t, x; t_0) \leq u(t, x - \zeta(t) - \delta(t)) - q(t)\Gamma(x - \zeta(t) - \delta(t) - X(t))
 \end{aligned}
 \tag{5.21}$$

for  $t \geq \tau + \sigma$ .

The estimate (5.21) is established under the assumption (5.9). If (5.8) holds, then similar arguments lead also to (5.21) with  $q(t)$  and  $\delta(t)$  of the same form and

$$\zeta(t) = \zeta - \frac{2A\epsilon}{\omega} + \frac{A\epsilon}{\omega}[e^{-\omega\sigma} + e^{-\omega(t-\tau-\sigma)}].$$

We just remark that the choice of  $\sigma$  in this case is still independent of  $\delta$ , which follows from the observation that replacing  $\delta$  by  $\tilde{\delta}$  at appropriate steps when estimating the lower bound for the term

$$u(\tau + \sigma, x - \zeta_\tau^+(\tau + \sigma) - \delta + \tilde{\epsilon}\tilde{\delta}) - u(\tau + \sigma, x; t_0).$$

The lemma then follows.  $\square$

Now, we prove Theorem 1.6.

**Proof of Theorem 1.6.** By Theorem 1.5(2), we have

$$\begin{aligned}
 &u(t, x - \zeta^-) - \epsilon e^{-\omega(t-t_0)}\Gamma(x - \zeta^- - X(t)) \\
 &\leq u(t, x; t_0, u_0) \leq u(t, x - \zeta^+) + \epsilon e^{-\omega(t-t_0)}\Gamma(x - \zeta^+ - X(t))
 \end{aligned}$$

for all  $t \geq t_0$ . In particular,

$$\begin{aligned}
 &u(t_0 + T_0, x - \zeta_0) - q_0\Gamma(x - \zeta_0 - X(t_0 + T_0)) \\
 &\leq u(t_0 + T_0, x; t_0, u_0) \leq u(t, x - \zeta_0 - \delta_0) + q_0\Gamma(x - \zeta_0 - \delta_0 - X(t_0 + T_0)),
 \end{aligned}
 \tag{5.22}$$

where  $\zeta_0 = \zeta^-$ ,  $\delta_0 = \zeta^+ - \zeta^-$ ,  $q_0 = \epsilon e^{-\omega T_0}$  and  $T_0 > 0$  is chosen so that

$$\frac{4A}{\omega} \epsilon_0 e^{-\omega T_0} \leq \frac{\tilde{\epsilon}}{2}.
 \tag{5.23}$$

Here, we may assume, without loss of generality, that  $\delta_0 > 1$ . We now use iteration arguments to reduce  $\delta_0$ .

Let  $T \geq T_0$ . Applying Lemma 5.2 to (5.22), we find at the moment  $t_0 + T_0 + \sigma + T$ ,

$$\begin{aligned}
 &u(t_0 + T_0 + \sigma + T, x - \zeta_1) - q_1\Gamma(x - \zeta_1 - X(t_0 + T_0 + \sigma + T)) \\
 &\leq u(t_0 + T_0 + \sigma + T, x; t_0, u_0) \\
 &\leq u(t_0 + T_0 + \sigma + T, x - \zeta_1 - \delta_1) - q_1\Gamma(x - \zeta_1 - \delta_1 - X(t_0 + T_0 + \sigma + T)),
 \end{aligned}
 \tag{5.24}$$

where

$$\begin{aligned}
 &\zeta_1 \in [\zeta_0 - \frac{2Aq_0}{\omega}, \zeta_0 + \tilde{\epsilon} \min\{1, \delta_0\}] \subset [\zeta_0 - \frac{2Aq_0}{\omega}, \zeta_0 + \tilde{\epsilon}], \\
 &0 \leq \delta_1 \leq \delta_0 - \tilde{\epsilon} \min\{1, \delta_0\} + \frac{4Aq_0}{\omega} = \delta_0 - \tilde{\epsilon} + \frac{4Aq_0}{\omega} \leq \delta_0 - \frac{\tilde{\epsilon}}{2}, \\
 &0 \leq q_1 \leq (\frac{q_0}{2} + \tilde{C}\tilde{\epsilon} \min\{1, \delta_0\})e^{-\omega T} \leq \epsilon_0 e^{-\omega T}.
 \end{aligned}$$

If  $\delta_1 \leq 1$ , we stop. Otherwise, applying Lemma 5.2 to (5.24), we find at the moment  $t_0 + T_0 + 2(\sigma + T)$ ,

$$\begin{aligned}
 &u(t_0 + T_0 + 2(\sigma + T), x - \zeta_2) - q_2\Gamma(x - \zeta_2 - X(t_0 + T_0 + 2(\sigma + T))) \\
 &\leq u(t_0 + T_0 + 2(\sigma + T), x; t_0, u_0) \\
 &\leq u(t_0 + T_0 + 2(\sigma + T), x - \zeta_2 - \delta_2) - q_2\Gamma(x - \zeta_2 - \delta_2 - X(t_0 + T_0 + 2(\sigma + T))),
 \end{aligned}
 \tag{5.25}$$

where

$$\begin{aligned}
 &\zeta_2 \in [\zeta_1 - \frac{2Aq_1}{\omega}, \zeta_1 + \tilde{\epsilon} \min\{1, \delta_1\}] \subset [\zeta_1 - \frac{2Aq_1}{\omega}, \zeta_1 + \tilde{\epsilon}], \\
 &0 \leq \delta_2 \leq \delta_1 - \tilde{\epsilon} \min\{1, \delta_1\} + \frac{4Aq_1}{\omega} \leq \delta_0 - 2\tilde{\epsilon} + 2\frac{4Aq_1}{\omega} \leq \delta_0 - 2\frac{\tilde{\epsilon}}{2}, \\
 &0 \leq q_2 \leq (\frac{q_1}{2} + \tilde{C}\tilde{\epsilon} \min\{1, \delta_1\})e^{-\omega T} \leq \epsilon_0 e^{-\omega T}.
 \end{aligned}$$

If  $\delta_2 \leq 1$ , we stop. Otherwise, applying Lemma 5.2 to (5.25). Repeating this, if  $\delta_1, \delta_2, \dots, \delta_{N-1}$  are all greater than one 1, then we will have

$$\begin{aligned}
 &u(t_0 + T_0 + N(\sigma + T), x - \zeta_N) - q_N \Gamma(x - \zeta_N - X(t_0 + T_0 + N(\sigma + T))) \\
 &\leq u(t_0 + T_0 + N(\sigma + T), x; t_0, u_0) \\
 &\leq u(t_0 + T_0 + N(\sigma + T), x - \zeta_N - \delta_N) - q_N \Gamma(x - \zeta_N - \delta_N - X(t_0 + T_0 + N(\sigma + T))),
 \end{aligned} \tag{5.26}$$

where

$$\begin{aligned}
 &\zeta_N \in [\zeta_{N-1} - \frac{2Aq_{N-1}}{\omega}, \zeta_{N-1} + \tilde{\epsilon} \min\{1, \delta_{N-1}\}] \subset [\zeta_{N-1} - \frac{2Aq_{N-1}}{\omega}, \zeta_{N-1} + \tilde{\epsilon}], \\
 &0 \leq \delta_N \leq \delta_{N-1} - \tilde{\epsilon} \min\{1, \delta_{N-1}\} + \frac{4Aq_{N-1}}{\omega} \leq \delta_0 - N\frac{\tilde{\epsilon}}{2}, \\
 &0 \leq q_N \leq (\frac{q_{N-1}}{2} + \tilde{C}\tilde{\epsilon} \min\{1, \delta_{N-1}\})e^{-\omega T} \leq \epsilon_0 e^{-\omega T}.
 \end{aligned}$$

Observe that there must be an  $N$  such that  $\delta_N \leq \delta_0 - N\frac{\tilde{\epsilon}}{2} \leq 1$ . We then stop here. Setting  $\tilde{T}_0 = T_0 + N(\sigma + T)$ ,  $\tilde{\zeta}_0 = \zeta_N$ ,  $\tilde{\delta}_0 = \delta_N$  and  $\tilde{q}_0 = q_N$  in (5.26), we have

$$\begin{aligned}
 &u(t_0 + \tilde{T}_0, x - \tilde{\zeta}_0) - \tilde{q}_0 \Gamma(x - \tilde{\zeta}_0 - X(t_0 + \tilde{T}_0)) \\
 &\leq u(t_0 + \tilde{T}_0, x; t_0, u_0) \\
 &\leq u(t_0 + \tilde{T}_0, x - \tilde{\zeta}_0 - \tilde{\delta}_0) - \tilde{q}_0 \Gamma(x - \tilde{\zeta}_0 - \tilde{\delta}_0 - X(t_0 + \tilde{T}_0)),
 \end{aligned} \tag{5.27}$$

where  $\tilde{\delta}_0 \in [0, 1]$ .

We now apply the above iteration arguments to (5.27), as a new initial step, to conclude the result. Recall (5.23),  $\tilde{q}_0 \leq \epsilon_0 e^{-\omega T_0}$  and  $\tilde{C}\tilde{\epsilon} \leq \frac{\epsilon_0}{2}$ . We now choose  $T$  so larger that

$$(\frac{\epsilon_0}{2} e^{-\omega T_0} + \tilde{C}\tilde{\epsilon})e^{-\omega T} \leq (1 - \frac{\tilde{\epsilon}}{2})\epsilon_0 e^{-\omega T_0}.$$

Applying Lemma 5.2 to (5.27), we find

$$\begin{aligned}
 &u(t_0 + \tilde{T}_0 + \sigma + T, x - \tilde{\zeta}_1) - \tilde{q}_1 \Gamma(x - \tilde{\zeta}_1 - X(t_0 + \tilde{T}_0 + \sigma + T)) \\
 &\leq u(t_0 + \tilde{T}_0 + \sigma + T, x; t_0, u_0) \\
 &\leq u(t_0 + \tilde{T}_0 + \sigma + T, x - \tilde{\zeta}_1 - \tilde{\delta}_1) - \tilde{q}_1 \Gamma(x - \tilde{\zeta}_1 - \tilde{\delta}_1 - X(t_0 + \tilde{T}_0 + \sigma + T)),
 \end{aligned} \tag{5.28}$$

where

$$\begin{aligned}
 &\tilde{\zeta}_1 \in [\tilde{\zeta}_0 - \frac{2A\tilde{q}_0}{\omega}, \tilde{\zeta}_0 + \tilde{\epsilon}\tilde{\delta}_0], \\
 &0 \leq \tilde{\delta}_1 \leq \tilde{\delta}_0 - \tilde{\epsilon}\tilde{\delta}_0 + \frac{4A\tilde{q}_0}{\omega} \leq 1 - \tilde{\epsilon} + \frac{\tilde{\epsilon}}{2} = 1 - \frac{\tilde{\epsilon}}{2}, \\
 &0 \leq \tilde{q}_1 \leq (\frac{\tilde{q}_0}{2} + \tilde{C}\tilde{\epsilon}\tilde{\delta}_0)e^{-\omega T} \leq (\frac{\epsilon_0}{2} e^{-\omega T} + \tilde{C}\tilde{\epsilon})e^{-\omega T} \leq (1 - \frac{\tilde{\epsilon}}{2})\epsilon_0 e^{-\omega T_0}.
 \end{aligned}$$

Applying Lemma 5.2 to (5.28), we find

$$\begin{aligned} &u(t_0 + \tilde{T}_0 + 2(\sigma + T), x - \tilde{\zeta}_2) - \tilde{q}_2 \Gamma(x - \tilde{\zeta}_2 - X(t_0 + \tilde{T}_0 + 2(\sigma + T))) \\ &\leq u(t_0 + \tilde{T}_0 + 2(\sigma + T), x; t_0, u_0) \\ &\leq u(t_0 + \tilde{T}_0 + 2(\sigma + T), x - \tilde{\zeta}_2 - \tilde{\delta}_2) - \tilde{q}_2 \Gamma(x - \tilde{\zeta}_2 - \tilde{\delta}_2 - X(t_0 + \tilde{T}_0 + 2(\sigma + T))), \end{aligned}$$

where

$$\begin{aligned} \tilde{\zeta}_2 &\in [\tilde{\zeta}_1 - \frac{2A\tilde{q}_1}{\omega}, \tilde{\zeta}_1 + \tilde{\epsilon}\tilde{\delta}_1], \\ 0 \leq \tilde{\delta}_2 &\leq \tilde{\delta}_1 - \tilde{\epsilon}\tilde{\delta}_1 + \frac{4A\tilde{q}_1}{\omega} \leq (1 - \frac{\tilde{\epsilon}}{2})(1 - \tilde{\epsilon} + \frac{4A}{\omega}\epsilon_0 e^{-\omega T_0}) \leq (1 - \frac{\tilde{\epsilon}}{2})^2, \\ 0 \leq \tilde{q}_2 &\leq (\frac{\tilde{q}_1}{2} + \tilde{C}\tilde{\epsilon}\tilde{\delta}_1)e^{-\omega T} \leq (1 - \frac{\tilde{\epsilon}}{2})(\frac{\epsilon_0}{2}e^{-\omega T} + \tilde{C}\tilde{\epsilon})e^{-\omega T} \leq (1 - \frac{\tilde{\epsilon}}{2})^2\epsilon_0 e^{-\omega T_0}. \end{aligned}$$

Then, applying Lemma 5.2 repeatedly, we find for  $n \geq 3$

$$\begin{aligned} &u(t_0 + \tilde{T}_0 + n(\sigma + T), x - \tilde{\zeta}_n) - \tilde{q}_n \Gamma(x - \tilde{\zeta}_n - X(t_0 + \tilde{T}_0 + n(\sigma + T))) \\ &\leq u(t_0 + \tilde{T}_0 + n(\sigma + T), x; t_0, u_0) \\ &\leq u(t_0 + \tilde{T}_0 + n(\sigma + T), x - \tilde{\zeta}_n - \tilde{\delta}_n) - \tilde{q}_n \Gamma(x - \tilde{\zeta}_n - \tilde{\delta}_n - X(t_0 + \tilde{T}_0 + n(\sigma + T))), \end{aligned}$$

where

$$\begin{aligned} \tilde{\zeta}_n &\in [\tilde{\zeta}_{n-1} - \frac{2A\tilde{q}_{n-1}}{\omega}, \tilde{\zeta}_{n-1} + \tilde{\epsilon}\tilde{\delta}_{n-1}], \\ 0 \leq \tilde{\delta}_n &\leq \tilde{\delta}_{n-1} - \tilde{\epsilon}\tilde{\delta}_{n-1} + \frac{4A\tilde{q}_{n-1}}{\omega} \leq (1 - \frac{\tilde{\epsilon}}{2})^n, \\ 0 \leq \tilde{q}_n &\leq (\frac{\tilde{q}_{n-1}}{2} + \tilde{C}\tilde{\epsilon}\tilde{\delta}_{n-1})e^{-\omega T} \leq (1 - \frac{\tilde{\epsilon}}{2})^n \epsilon_0 e^{-\omega T_0}. \end{aligned}$$

This clearly implies that  $\tilde{\zeta}_n \rightarrow \tilde{\zeta}_\infty$ ,  $\tilde{\delta}_n \rightarrow 0$  and  $\tilde{q}_n \rightarrow 0$  exponentially as  $n \rightarrow \infty$ , where  $\tilde{\zeta}_\infty \in \mathbb{R}$ . The theorem then follows readily.  $\square$

Finally, as a simple consequence of Theorem 1.6 and Corollary 4.2, we have

**Corollary 5.3.** *Let  $u(t, x)$  and  $X(t)$  be as in Proposition 1.1(2). Let  $\beta_0 > 0$ . Suppose  $t_0 \in \mathbb{R}$  and  $\tilde{u}_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R})$  satisfy*

$$\begin{cases} \tilde{u}_0 : \mathbb{R} \rightarrow [0, 1], & \liminf_{x \rightarrow -\infty} \tilde{u}_0(x) > \theta; \\ \exists C > 0 \text{ s.t. } |\tilde{u}_0 - u(t_0, x)| \leq C e^{-\beta_0(x - X(t_0))} \text{ for } x \in \mathbb{R}. \end{cases}$$

Then, there exist  $C = C(\tilde{u}_0) > 0$ ,  $\zeta_* = \zeta_*(\tilde{u}_0) \in \mathbb{R}$  and  $r = r(\beta_0) > 0$  such that

$$\sup_{x \in \mathbb{R}} |u(t, x; t_0, \tilde{u}_0) - u(t, x - \zeta_*)| \leq C e^{-r(t-t_0)}$$

for all  $t \geq t_0$ .

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### Appendix A. Comparison principles

We state comparison principles used in the previous sections. See [26, Proposition A.1] for the proof.

**Proposition A.1.** *Let  $K : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  be continuous and satisfy  $\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} K(x, y) dy < \infty$ . Let  $a : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous and uniformly bounded.*

- (i) *Suppose that  $X : [0, \infty) \rightarrow \mathbb{R}$  is continuous and that  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following:  $u, u_t : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous, the limit  $\lim_{x \rightarrow \infty} u(t, x) = 0$  is locally uniformly in  $t$ , and*

$$\begin{cases} u_t(t, x) \geq \int_{\mathbb{R}} K(x, y) u(t, y) dy + a(t, x) u(t, x), & x > X(t), t > 0, \\ u(t, x) \geq 0, & x \leq X(t), t > 0, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Then  $u(t, x) \geq 0$  for  $(t, x) \in (0, \infty) \times \mathbb{R}$ .

- (ii) *Suppose that  $X : [0, \infty) \rightarrow \mathbb{R}$  is continuous and that  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following:  $u, u_t : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous, the limit  $\lim_{x \rightarrow -\infty} u(t, x) = 0$  is locally uniformly in  $t$ , and*

$$\begin{cases} u_t(t, x) \geq \int_{\mathbb{R}} K(x, y) u(t, y) dy + a(t, x) u(t, x), & x < X(t), t > 0, \\ u(t, x) \geq 0, & x \geq X(t), t > 0, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Then  $u(t, x) \geq 0$  for  $(t, x) \in (0, \infty) \times \mathbb{R}$ .

- (iii) *Suppose that  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following:  $u, u_t : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous,  $\inf_{t \geq 0, x \in \mathbb{R}} u(t, x) > -\infty$ , and*

$$\begin{cases} u_t(t, x) \geq \int_{\mathbb{R}} K(x, y) u(t, y) dy + a(t, x) u(t, x), & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Then  $u(t, x) \geq 0$  for  $(t, x) \in (0, \infty) \times \mathbb{R}$ . Moreover, if  $u_0(x) \not\equiv 0$ , then  $u(t, x) > 0$  for  $(t, x) \in (0, \infty) \times \mathbb{R}$ .

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