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Nonlocal dispersal equations in time-periodic media: Principal spectral theory, limiting properties and long-time dynamics ☆

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Abstract

The present paper is devoted to the investigation of the following nonlocal dispersal equation

$$u_t(t,x)=\frac{D}{\sigma^m}\left[\int_{\Omega}J_\sigma(x-y)u(t,y)dy-u(t,x)\right]+f(t,x,u(t,x)),\quad t>0,\quad x\in\overline{\Omega},$$

where $\Omega\subset\mathbb{R}^N$ is a bounded and connected domain with smooth boundary, $m\in[0,2)$ is the cost parameter, $D>0$ is the dispersal rate, $\sigma>0$ characterizes the dispersal range, $J_\sigma=\frac{1}{\sigma^N}J\left(\frac{\cdot}{\sigma}\right)$ is the scaled dispersal kernel, and f is a time-periodic nonlinear function of generalized KPP type. This paper is a continuation of the works of Berestycki et al. [3,4], where f was assumed to be time-independent. We first study the principal spectral theory of the linear operator associated to the linearization of the equation at $u\equiv 0$. We establish an easily verifiable, general and sharp sufficient condition for the existence of the principal eigenvalue as well as important sup-inf characterizations of the principal eigenvalue. Next, we study the influences of the principal spectrum point on the global dynamics and confirm that the principal spectrum point being zero is critical. It is followed by the investigation of the effects of the dispersal rate D and the dispersal

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range characterized by σ on the principal spectrum point and the positive time-periodic solution. In particular, we prove various limiting properties of the principal spectrum point and the positive time-periodic solution as $D, \sigma \to 0^+$ or ∞. To achieve these, we develop new techniques to overcome fundamental difficulties caused by the lack of the usual L^2 variational formula for the principal eigenvalue, the lack of the regularizing effects of the semigroup generated by the nonlocal dispersal operator, and the presence of the time-dependence of the nonlinearity f. Finally, we establish the maximum principle for time-periodic nonlocal dispersal operators.

MSC: primary 35K57, 47G20, 92D25; secondary 37L15

Keywords: Nonlocal dispersal equation; Principal spectrum point; Generalized principal eigenvalue; Positive solution; Global dynamics; Maximum principle

Contents

1. Introduction and main results

The present paper is devoted to the investigation of the following nonlocal dispersal equation (or, integro-differential equation) in spatio-temporally heterogeneous environments

$$u_t(t,x) = \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)u(t,y)dy - u(t,x)\right] + f(t,x,u(t,x)), \quad t>0, \quad x\in\overline{\Omega}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, connected with smooth boundary, $m \in [0,2)$, $D > 0$, $\sigma > 0$ and $J_\sigma(x) = \frac{1}{\sigma^N} J\left(\frac{x}{\sigma}\right)$ for $x \in \mathbb{R}^N$. The operator

$$u \mapsto \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(\cdot - y)u(y)dy - u\right] \tag{1.2}$$

is often called the elliptic-type nonlocal dispersal operator. The dispersal kernel J and the nonlinearity $f(t,x,u)$ satisfy the following assumptions.

(H1) The dispersal kernel $J \in C(\mathbb{R}^N)$ is nonnegative and supported in $B_\gamma(0)$ for some $\gamma > 0$, and satisfies $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$, where $B_\gamma(0) \subset \mathbb{R}^N$ is the open ball centered at 0 with radius γ.

(H2) The nonlinear function $f : \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the following conditions.
(1) $f(\cdot, x, u) \in C(\mathbb{R})$, $f(t, \cdot, u) \in C^1(\overline{\Omega})$ and $f(t, x, \cdot) \in C^1(\mathbb{R})$.
(2) $f(t, x, 0) = 0$ for all $(t, x) \in \mathbb{R} \times \overline{\Omega}$ and there is $T > 0$ such that

$$f(t + T, x, u) = f(t, x, u), \quad \forall (t, x, u) \in \mathbb{R} \times \overline{\Omega} \times \mathbb{R}.$$

(3) For all $(t, x) \in \mathbb{R} \times \overline{\Omega}$, the function $u \mapsto \frac{f(t,x,u)}{u}$ is decreasing on $(0, \infty)$.
(4) There exists $M > 0$ such that

$$f(t, x, u) \le 0, \quad \forall (t, x, u) \in \mathbb{R} \times \overline{\Omega} \times [M, \infty).$$

The equation (1.1) is often used to model the evolution of a species that disperses over long distances and is subject to seasonal effects and spatial variations (see e.g. [3,4,21,22,16,18,30,35]). In this context, whether the species can survive or not in the long run, and the eventual distributions of the species, if it survives, are fundamental issues. In terms of the equation (1.1), these issues are closely related to the global dynamics of the solutions of (1.1) and the effects of the *dispersal rate* and the *dispersal range* characterized by D and σ, respectively, on the global dynamics. The number m is referred to as the *cost parameter* (see e.g. [3,4,16,18,35]). At this point, we mention that the nonlocal dispersal operator (1.2) corresponds to an elliptic operator with zero Dirichlet boundary condition. In fact, the operator (1.2) on $\overline{\Omega}$ is derived assuming that the habitat on $\mathbb{R}^{\mathbb{N}} \setminus \overline{\Omega}$ is so hostile that the individuals of the species die immediately after they land (see e.g. [18]). As a result, the operator (1.2) is not normalized as there in general holds

$$\int_{\Omega} \left[\int_{\Omega} J_\sigma(x-y)u(y)dy - u(x) \right] dx \lneqq 0$$

for non-negative u on Ω. If the habitat $\mathbb{R}^{\mathbb{N}} \setminus \overline{\Omega}$ is not so hostile, then the nonlocal dispersal operator reads

$$u \mapsto \frac{D}{\sigma^m} \left[\int_{\mathbb{R}^N} J_\sigma(\cdot - y)u(y)dy - u \right] \quad \text{on} \quad \mathbb{R}^N,$$

which is normalized.

It is known from [3,4,32,35] and references therein that the principal spectral theory of the linear operator associated to the equation (1.1) linearized at zero, namely, the parabolic-type nonlocal dispersal operator

$$v \mapsto -v_t(t, x) + \frac{D}{\sigma^m} \left[\int_{\Omega} J_\sigma(x-y)v(t, y)dy - v(t, x) \right] + f_u(t, x, 0)v(t, x),$$

plays vital roles in the investigation of (1.1). To study the principal spectral theory of the above operator for fixed m and σ, it is more convenient to replace $\frac{D}{\sigma^m}$, J_σ and $f_u(t,x,0)$ by D, J and $a(t,x)$, respectively, and therefore, we consider the following operator

$$L_\Omega[v](t,x) = -v_t(t,x) + D\left[\int_\Omega J(x-y)v(t,y)dy - v(t,x)\right] + a(t,x)v(t,x),$$
$$(t,x) \in \mathbb{R}\times\overline{\Omega}, \tag{1.3}$$

where $a \in C_T(\mathbb{R}\times\overline{\Omega})$ and

$$C_T(\mathbb{R}\times\overline{\Omega}) = \left\{v \in C(\mathbb{R}\times\overline{\Omega}) : v(t+T,x) = v(t,x),\ (t,x)\in\mathbb{R}\times\overline{\Omega}\right\}. \tag{1.4}$$

For convenience, we define the spaces $\mathcal{X}_\Omega$, $\mathcal{X}_\Omega^+$ and $\mathcal{X}_\Omega^{++}$ as follows:

$$\begin{aligned}
\mathcal{X}_\Omega &= \left\{v \in C^{1,0}(\mathbb{R}\times\overline{\Omega}) : v(t+T,x) = v(t,x),\ (t,x)\in\mathbb{R}\times\overline{\Omega}\right\},\\
\mathcal{X}_\Omega^+ &= \left\{v \in \mathcal{X}_\Omega : v(t,x) \ge 0,\ (t,x)\in\mathbb{R}\times\overline{\Omega}\right\}, \quad \text{and}\\
\mathcal{X}_\Omega^{++} &= \left\{v \in \mathcal{X}_\Omega : v(t,x) > 0,\ (t,x)\in\mathbb{R}\times\overline{\Omega}\right\},
\end{aligned} \tag{1.5}$$

where $C^{1,0}(\mathbb{R}\times\overline{\Omega})$ denotes the class of functions that are C^1 in t and continuous in x. Clearly, $\mathcal{X}_\Omega^+$ is the positive cone of $\mathcal{X}_\Omega$ and $\mathcal{X}_\Omega^{++}$ is the interior of $\mathcal{X}_\Omega^+$. The operator L_Ω is then considered as an unbounded linear operator on the space $C_T(\mathbb{R}\times\overline{\Omega})$ with domain $\mathcal{X}_\Omega$, namely,

$$L_\Omega : \mathcal{X}_\Omega(\subset C_T(\mathbb{R}\times\overline{\Omega})) \to C_T(\mathbb{R}\times\overline{\Omega}).$$

We set

$$a_T(x) := \frac{1}{T}\int_0^T a(t,x)dt, \quad x \in \overline{\Omega}.$$

The principal spectral theory for elliptic-type nonlocal dispersal operators and their properties have been extensively investigated in [10,12,3,34] and references therein. In particular, Coville et al. proved in [12,10] a sharp sufficient condition for the existence of the principal eigenvalue using the generalized principal spectral theory developed in [5], while Shen and Xie proved in [34] a necessary and sufficient spectral condition for the existence of the principal eigenvalue using a dynamical system approach. Due to the non-compactness of nonlocal operators and their resolvents, principal eigenvalues do not exist in general. The notion *principal spectrum point* (see Definition 1.1 for its definition as well as the definition of principal eigenvalue), in place of the principal eigenvalue, was used by Rawal and Shen in [32]. More precisely, they proved a necessary and sufficient spectral condition for the principal spectrum point becoming the principal eigenvalue using the compactness criterion established in [8]. We recall the definition of the principal spectrum point and the result of Rawal and Shen established in [32].

Definition 1.1 *(Principal spectrum point and principal eigenvalue).* The *principal spectrum point* of $-L_\Omega$ is defined by

$$\lambda_1(-L_\Omega) = \inf\{\Re\lambda : \lambda \in \sigma(-L_\Omega)\},$$

where $\sigma(-L_\Omega)$ is the spectrum of $-L_\Omega$. If $\lambda_1(-L_\Omega)$ is an isolated eigenvalue of $-L_\Omega$ with an eigenfunction in $\mathcal{X}_\Omega^+$, then it is called the *principal eigenvalue* of $-L_\Omega$.

Theorem 1.2 *([32, Theorem A]). Suppose* **(H1)** *and let* $a \in C_T(\mathbb{R} \times \overline{\Omega})$. *Then* $\lambda_1(-L_\Omega)$ *is the principal eigenvalue of* $-L_\Omega$ *if and only if*

$$\lambda_1(-L_\Omega) < \lambda_* := \min_{\overline{\Omega}}[D - a_T].$$

Moreover, when $\lambda_1(-L_\Omega)$ *is the principal eigenvalue of* $-L_\Omega$, *it is geometrically simple and has an eigenfunction in* $\mathcal{X}_\Omega^{++}$.

Although the condition $\lambda_1(-L_\Omega) < \lambda_*$ in the above theorem is both necessary and sufficient, it turns out to be rather hard to check when this condition is true, since it is related to both $\lambda_1(-L_\Omega)$, a number of almost no computability, and $a(t,x)$. For the sake of applications, it is expected to find an easily verifiable and general sufficient condition for $\lambda_1(-L_\Omega)$ being the principal eigenvalue of $-L_\Omega$. This leads to our first main result. Besides this, we prove sup-inf characterizations of the principal eigenvalue. These results are stated in the following theorem.

Theorem A *(Principal eigenvalue and sup-inf characterizations). Suppose* **(H1)** *and let* $a \in C_T(\mathbb{R} \times \overline{\Omega})$.

(1) *If*

$$\frac{1}{\max_{y\in\overline{\Omega}} a_T(y) - a_T} \notin L^1_{loc}(\overline{\Omega}), \tag{1.6}$$

then $\lambda_1(-L_\Omega)$ *is the principal eigenvalue of* $-L_\Omega$.

(2) *If* $\lambda_1(-L_\Omega)$ *is the principal eigenvalue of* $-L_\Omega$, *then*

$$\lambda_1(-L_\Omega) = \lambda_p(-L_\Omega) = \lambda_p'(-L_\Omega),$$

where

$$\begin{cases} \lambda_p(-L_\Omega) := \sup\left\{\lambda \in \mathbb{R} : \exists\phi \in \mathcal{X}_\Omega^{++} \ s.t.\ (L_\Omega + \lambda)[\phi] \le 0 \ in\ \mathbb{R} \times \overline{\Omega}\right\}, \\ \lambda_p'(-L_\Omega) := \inf\left\{\lambda \in \mathbb{R} : \exists\phi \in \mathcal{X}_\Omega^{++} \ s.t.\ (L_\Omega + \lambda)[\phi] \ge 0 \ in\ \mathbb{R} \times \overline{\Omega}\right\}. \end{cases} \tag{1.7}$$

Note that the condition (1.6) concerns the smoothness of a_T near its maximum points. Moreover, it is independent of the dispersal kernel J and the dispersal rate D, and hence, independent of the dispersal operator $u \mapsto D\left[\int_\Omega J(\cdot - y)u(y)dy - u\right]$. Such a dispersal-independent sufficient condition is expected for the reason that $a(t,x)$ more or less determines the existence or non-existence of the principal eigenvalue under the current assumptions on J. This can be seen from the fact that the principal eigenvalue always exists when $a \equiv 0$, which is implied by our sufficient condition and also a simple consequence of the facts that the operator $\mathcal{J} : u \mapsto D \int_\Omega J(\cdot - y)u(y)dy$ on $C(\overline{\Omega})$ is compact and $\mathcal{J}^i$ is strongly positive for some positive integer i. We further mention that the condition (1.6) becomes very useful when we study the equation (1.1) with scaled kernels later. Indeed, it allows us to prove a result on the uniform-in-parameters approximation of the principal spectrum point (see Theorem 3.3), which says that $\lambda_1(-L_\Omega)$ is almost the principal eigenvalue and is of technical importance in the study of effects of parameters on $\lambda_1(-L_\Omega)$.

Although the condition (1.6) is only a sufficient condition, it is sharp in the sense that a function $a(t, x)$ unfulfilling (1.6) can be constructed so that $-L_\Omega$ does not admit a principal eigenvalue. In fact, we construct in Appendix A a class of operators of the form (1.3) that admit no eigenvalue with an eigenfunction in $\mathcal{X}_\Omega^+ \backslash \{0\}$. Our construction is inspired by the work of Coville [10], where operators of the form $v \mapsto D\left[\int_\Omega J(\cdot - y)v(y)dy - v\right] + a(x)v$ admitting no eigenvalue with a non-negative eigenfunction are constructed. It is worthwhile to point out that the situation when (1.6) fails to be true does happen in some applications (see e.g. [15]).

The quantities $\lambda_p(-L_\Omega)$ and $\lambda_p'(-L_\Omega)$, always well-defined, are often called the *generalized principal eigenvalues* of $-L_\Omega$. These notions are originally introduced in the celebrated work of Berestycki, Nirenberg and Varadhan [5] to study the principal spectral theory of elliptic operators on general domains. Since then, they are widely used to study the principal spectral theory of various linear operators associated to reaction-diffusion equations and nonlocal dispersal equations (see [3,6,7,10,28,37,38] and references therein). Not only does the equivalence of $\lambda_1(-L_\Omega)$, $\lambda_p(-L_\Omega)$ and $\lambda_p'(-L_\Omega)$ under the existence of the principal eigenvalue give sup-inf characterizations of $\lambda_1(-L_\Omega)$, but also it provides alternative and powerful tools to study deep qualitative properties of $\lambda_1(-L_\Omega)$ in terms of the parameters. In addition, it bridges the dynamical system approach and the partial differential equation (PDE) approach, which are often separately used in literature, to study the problems where one of the two approaches alone cannot give sufficient information. We highlight that unlike elliptic-type nonlocal operators studied in [3,4,10,36,34], the principal eigenvalue for a parabolic-type nonlocal operator as in our case does not admit the usual L^2 variational formula due to the presence of the time derivative in the operator. This fact further explains the significance of the sup-inf characterizations of $\lambda_1(-L_\Omega)$.

In the presence of the principal spectral theory, namely, Theorem A, we move forward to study the global dynamics of solutions of (1.1) in the non-scaled case with $m = 0$ and $\sigma = 1$, that is,

$$u_t(t,x) = D\left[\int_\Omega J(x-y)u(t,y)dy - u(t,x)\right] + f(t,x,u(t,x)), \quad t>0, \quad x \in \overline{\Omega}. \tag{1.8}$$

To do so, we need to investigate Liouville-type results, namely, the existence/non-existence of positive entire solutions of the equation

$$u_t(t,x) = D\left[\int_\Omega J(x-y)u(t,y)dy - u(t,x)\right] + f(t,x,u(t,x)), \quad t \in \mathbb{R}, \quad x \in \overline{\Omega}. \tag{1.9}$$

From now on, we let

$$a(t,x) = f_u(t,x,0), \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

unless otherwise specified. Then, L_Ω, defined in (1.3), is the linear operator associated to the linearization of (1.9) at $u \equiv 0$. We prove the following theorem.

Theorem B *(Global dynamics). Assume* **(H1)** *and* **(H2)**. *Let $u(t,x;u_0)$ be a solution of* (1.8) *with initial data $u_0 \in C(\overline{\Omega})$, which is non-negative and not identically zero. The following statements hold.*

(i) *If $\lambda_1(-L_\Omega) < 0$, then the equation* (1.9) *admits a unique solution u^* in $\mathcal{X}_\Omega^+ \setminus \{0\}$ (which actually belongs to $\mathcal{X}_\Omega^{++}$), and there holds*

$$\|u(t,\cdot;u_0) - u^*(t,\cdot)\|_\infty \to 0 \quad as \quad t \to \infty,$$

where $\|\cdot\|_\infty$ is the sup norm on $C(\overline{\Omega})$;

(ii) *If $\lambda_1(-L_\Omega) > 0$, then the equation* (1.9) *admits no solution in $\mathcal{X}_\Omega^+ \setminus \{0\}$, and there holds*

$$\|u(t,\cdot;u_0)\|_\infty \to 0 \quad as \quad t \to \infty.$$

(iii) *If $\lambda_1(-L_\Omega) = 0$ is the principle eigenvalue, then the equation* (1.9) *admits no solution in $\mathcal{X}_\Omega^+ \setminus \{0\}$.*

In the case of Theorem B(i) (resp. (ii)), we say that u^* (resp. 0) is *globally asymptotically stable*. Theorem B(i) was proven [32, Theorem E]. Our contribution to the results are Theorem B(ii) and (iii). In particular, if $\lambda_1(-L_\Omega)$ is the principal eigenvalue, we give a full characterization of the existence and non-existence of non-negative T-periodic solutions of the equation (1.9) by the sign of $\lambda_1(-L_\Omega)$. It should be mentioned that in the case $f(t,x,u) = f(x,u)$, the global dynamics of (1.8) have attracted a lot of attention recently due to their significance in applications and underlying mathematical challenges (see e.g. [3,4,10,34,36,37]). In particular, the authors in [3,4] took a PDE approach to investigate the problem, while the authors in [34,36] dealt with the problem from a dynamical system viewpoint. In the proof of Theorem B, we take advantage of both PDE and dynamical system approaches to tackle those difficulties stemming from the lack of regularizing effects of the semigroup generated by the nonlocal dispersal operator and the presence of the time-dependence of f.

It remains an interesting *open question* to study the global dynamics of (1.8) in the critical case $\lambda_1(-L_\Omega) = 0$. We remark that in the case $f(t,x,u) = f(x,u)$ treated in [3,4], the authors used a Harnack-type inequality for nonlocal elliptic-type equations (see [11]) and bootstrap arguments to confirm the global vanishing dynamics in the critical case. But, for nonlocal parabolic-type equations as in our case, no Harnack-type inequality is known, and bootstrap arguments together with the variation of constants formula are not helping due to the lack of regularizing effects of the semigroup generated by the nonlocal dispersal operator as just mentioned. We further remark that the energy and Fourier methods used in [1,9,20,27] do not apply in this framework because of the time-dependence of the nonlinearity.

We turn to the study of the effects of the dispersal rate D and the dispersal range characterized by σ on the principal spectrum point and the positive T-periodic solution associated to the equation (1.1).

We first study the effects of the dispersal rate D. For this purpose, it is more convenient to consider the non-scaled equations (1.8) and (1.9). In the next result, we write $\lambda_1^D(-L_\Omega)$ for $\lambda_1(-L_\Omega)$ to highlight the dependence on D.

Theorem C *(Effects of the dispersal rate). Suppose* **(H1)** *and* **(H2)**. *The following hold.*

(1) *The function* $D \mapsto \lambda_1^D(-L_\Omega)$ *is continuous on* $(0,\infty)$ *and satisfies*

$$\lambda_1^D(-L_\Omega) \to \begin{cases} -\max_{\overline{\Omega}} a_T & \text{as} \quad D \to 0^+, \\ \infty & \text{as} \quad D \to \infty. \end{cases}$$

(2) *Suppose, in addition,* J *is symmetric with respect to each component. If* $a(t,x) = \alpha(t) + \beta(x)$, *then* $D \mapsto \lambda_1^D(-L_\Omega)$ *is non-decreasing. If, in addition, the operator*

$$v \mapsto D\left[\int_\Omega J(\cdot - y)v(y)dy - v\right] + \beta(x)v : C(\overline{\Omega}) \to C(\overline{\Omega})$$

admits a principal eigenvalue, then $D \mapsto \lambda_1^D(-L_\Omega)$ *is increasing.*

(3) *If* $\max_{\overline{\Omega}} a_T > 0$, *then the equation* (1.9) *admits a unique solution* u_D^* *in* $\mathcal{X}_\Omega^+ \setminus \{0\}$ *(which actually belongs to* $\mathcal{X}_\Omega^{++}$*) that is globally asymptotically stable for each* $0 < D \ll 1$. *The equation* (1.9) *admits no solution in* $\mathcal{X}_\Omega^+ \setminus \{0\}$ *for each* $D \gg 1$.

(4) *If* $\min_{\overline{\Omega}} a_T > 0$, *then there holds the limit*

$$\lim_{D\to 0^+} u_D^*(t,x) = v^*(t,x) \quad \text{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

where $v^*(t,x)$ *is the unique positive and* T*-periodic solution of the equation* $v_t = f(t,x,v)$ *for every* $x \in \overline{\Omega}$.

We emphasize that due to the unboundedness of L_Ω and the non-self-adjointness of L_Ω resulting in the lack of the usual $L^2(\Omega)$ variational formula for the principal eigenvalue if exists, we cannot invoke the techniques used in the papers [3,10,34] to derive estimates of the principal spectrum point needed in the proof of Theorem C. Fortunately, the characterizations (1.7) open a new way to derive desired limits of the principal spectrum point in Theorem C as well as in Theorem D and Theorem E below. We see from Theorem C(1)(3) that if $\max_{\overline{\Omega}} a_T > 0$, then the small dispersal rates are favored, while the large dispersal rates are always unfavored. It would be interesting to know whether $\lambda_1^D(-L_\Omega)$ is monotone with respect to D. It is referred to [16] for the construction of a non-monotone sequence of principal eigenvalues of parabolic operators with homogeneous Neumann boundary condition, and therefore, we believe that there is no monotonicity in general. Nevertheless, we prove in Theorem C(2) the monotonicity in a special case.

Now, we study the effects of the dispersal range characterized by σ. To do so, we consider the following operator

$$L_{\Omega,m,\sigma}[v](t,x) = -v_t(t,x) + \frac{D}{\sigma^m}\left[\int_{\Omega} J_\sigma(x-y)v(t,y)dy - v(t,x)\right] + a(t,x)v(t,x),$$
$$(t,x) \in \mathbb{R}\times\overline{\Omega}$$

associated to the linearization of (1.1) at $u \equiv 0$. We prove the following result.

Theorem D *(Scaling limits of the principal spectrum point). Assume* **(H1)** *and* **(H2)**.

(1) *As $\sigma \to \infty$, there holds*

$$\lambda_1(-L_{\Omega,m,\sigma}) \to \begin{cases} D - \max\limits_{\overline{\Omega}} a_T, & m = 0, \\ -\max\limits_{\overline{\Omega}} a_T, & m > 0. \end{cases}$$

(2) *Suppose, in addition, J is symmetric with respect to each component. As $\sigma \to 0^+$, there holds*

$$\lambda_1(-L_{\Omega,m,\sigma}) \to -\max_{\overline{\Omega}} a_T, \qquad \forall m \in [0,2).$$

(3) *In the case $m = 0$, if Ω contains the origin and $a(t,x)$ is radially symmetric and radially non-increasing with respect to x, namely, $a(t,x) = a(t,y)$ if $|x| = |y|$ and $a(t,x) \le a(t,y)$ if $|x| \ge |y|$ for all $t \in \mathbb{R}$, then $\sigma \mapsto \lambda_1(-L_{\Omega,0,\sigma})$ is non-decreasing.*

Remark 1.3. In the case of Theorem D(3), if $\max\limits_{\overline{\Omega}} a_T \in (0, D)$, then the monotonicity and continuity (see Proposition 6.1(5)) of $\sigma \mapsto \lambda_1(-L_{\Omega,0,\sigma})$ together with Theorem D(1)(2) imply the existence of a threshold value $\sigma^* > 0$ such that $\lambda_1(-L_{\Omega,0,\sigma}) < 0$ if and only if $\sigma < \sigma^*$, and hence, (1.9) admits a unique solution $u^*_\sigma \in \mathcal{X}^{++}_\Omega$ that is globally asymptotically stable if and only if $\sigma < \sigma^*$. This σ^* is usually referred to as the *critical range for persistence*. Below, we study the asymptotic behaviors of u^*_σ in terms of σ, and it remains an interesting *open problem* to study the limit of u^*_σ as $\sigma \to \sigma^-_*$.

Results as in Theorem D have been obtained in [3,34] in the case of elliptic-type nonlocal operators. The lack of the usual $L^2(\Omega)$ variational characterization for the principal eigenvalue in our case indeed yields substantial difficulties, especially, in the study of the limit of $\lambda_1(-L_{\Omega,m,\sigma})$ as $\sigma \to 0^+$, that cannot be solved by methods developed in [3,4,34,7] and references therein. To overcome the difficulties, we take advantage of the sup-inf characterizations of the principal eigenvalue established in Theorem A and develop new techniques involving delicate analysis of decaying rates in terms of σ of various terms (see the proof of Theorem D in Section 5 for more details). It is also generally understood (see e.g. [16]) that the time-dependence of $f(t,x,u)$ largely complicates the behavior of $\lambda_1(-L_{\Omega,m,\sigma})$ in term of various parameters. We remark that $0 < \sigma \ll 1$ and $\sigma \gg 1$ represent two completely different dispersal strategies. The former says that the dispersal is essentially localized, while the latter supports the dispersal over very long distances. It is interesting to see that the behaviors of the principal spectrum point are intrinsically different between the cases $m = 0$ and $m \in (0, 2)$. More precisely, in the case $m \in (0, 2)$, both small and large dispersal ranges are favored provided $\max_{\overline{\Omega}} a_T > 0$. The situation in the case $m = 0$ is more complicated. If $\max_{\overline{\Omega}} a_T \in (0, D)$, small dispersal ranges are favored and large dispersal ranges are unfavored, while if $\max_{\overline{\Omega}} a_T > D$, both small and large dispersal ranges are favored. From this, we see the involved global dynamics of (1.1) with respect to the dispersal range.

Note that in the case $m \in (0, 2)$, while $0 < \sigma \ll 1$ and $\sigma \gg 1$ represent two completely different dispersal strategies, the limits of $\lambda_1(-L_{\Omega,m,\sigma})$ as $\sigma \to 0^+$ and $\sigma \to \infty$ turn out to be the same. This can be intuitively understood as follows. Since the dispersal term $\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)u(t,y)dy - u(t,x)\right]$ vanishes as $\sigma \to 0^+$ or $\sigma \to \infty$, the dynamics of the equation (1.1) when $0 < \sigma \ll 1$ or $\sigma \gg 1$ is more or less governed by the dynamics of the limiting equation

$$v_t = f(t, x, v), \tag{1.10}$$

which actually is a family of periodic ODEs indexed by $x \in \overline{\Omega}$. For each fixed $x \in \overline{\Omega}$, the dynamics of (1.10) is determined by the sign of $a_T(x)$: if $a_T(x) \le 0$ then $v \equiv 0$ is globally asymptotically stable, and if $a_T(x) > 0$ then there is a unique positive periodic solution that is globally asymptotically stable (see the proof of Lemma 6.2 for more details). Note that $-\max_{\overline{\Omega}} a_T < 0$ is equivalent to the existence of some $x_0 \in \overline{\Omega}$ such that $a_T(x_0) > 0$, and $-\max_{\overline{\Omega}} a_T > 0$ is equivalent to $a_T(x) < 0$ for all $x \in \overline{\Omega}$. Thus, (i) if there is $x_0 \in \overline{\Omega}$ such that $a_T(x_0) > 0$, then the species near the location x_0 persists; this guarantees the persistence of the overall species when $0 < \sigma \ll 1$ or $\sigma \gg 1$ for the following reasons: if $0 < \sigma \ll 1$, then the species disperses over very short distances with very high rates as $\frac{D}{\sigma^m} \gg 1$; as a result, the species residing near the location x_0 would stay there for a rather long period resulting in the persistence of the species near the location x_0 as well as the overall species; if $\sigma \gg 1$, the species disperses over very long distances with very low rates as $0 < \frac{D}{\sigma^m} \ll 1$, and thus, the species grows during a rather long period when they stay near the location x_0; this again results in the persistence of the overall species; (ii) if $a_T(x) < 0$ for all $x \in \overline{\Omega}$, then the habitat is hostile everywhere leading to the extinction of the species.

We further investigate the behaviors of the positive T-periodic solution of the following equation

$$u_t(t,x) = \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)u(t,y)dy - u(t,x)\right] + f(t,x,u(t,x)), \quad t \in \mathbb{R}, \quad x \in \overline{\Omega} \tag{1.11}$$

in favored cases and prove the following theorem.

Theorem E *(Scaling limits of the positive T-periodic solution). Suppose* **(H1)** *and* **(H2)**.

(1) *Suppose, in addition, J is symmetric with respect to each component. For each $m \in [0, 2)$, the following statements hold.*

(a) *If* $\max_{\overline{\Omega}} a_T > 0$, *then there exist* $0 < \sigma_1 \ll 1$ *such that for each* $\sigma \in (0, \sigma_1)$, *the equation* (1.11) *has a unique positive* T*-periodic solution* u^*_σ *that is globally asymptotically stable.*

(b) *If* $\min_{\overline{\Omega}} a_T(x) > 0$, *there holds the limit*

$$\lim_{\sigma \to 0^+} u^*_\sigma(t,x) = v^*(t,x) \quad \textit{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

where $v^*(t,x)$ *is the unique positive and* T*-periodic solution of the equation* $v_t = f(t,x,v)$ *for every* $x \in \overline{\Omega}$.

(2) *For each* $m > 0$, *the following statements hold.*

(a) *If* $\max_{\overline{\Omega}} a_T > 0$, *then there exist* $1 \ll \sigma_2 < \infty$ *such that for each* $\sigma > \sigma_2$, *the equation* (1.11) *has a unique positive* T*-periodic solution* u^*_σ *that is globally asymptotically stable.*

(b) *If* $\min_{\overline{\Omega}} a_T > 0$, *there holds the limit*

$$\lim_{\sigma \to \infty} u^*_\sigma(t,x) = v^*(t,x) \quad \textit{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

where $v^*(t,x)$ *is the same as the one in (1)(b).*

Results as in Theorem E have been obtained by Berestycki, Coville and Vo in [4] in the case $f(t,x,u) = f(x,u)$. More precisely, it was shown in [4] that for $f(x,u) = a(x)u - u^2$ and $m \in [0,2)$, the unique positive solution of the following equation

$$\frac{1}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)u(y)dy - u(x)\right] + a(x)u(x) - u(x)^2 = 0, \quad x \in \Omega$$

converges, as $\sigma \to 0^+$, to some nonnegative solution of

$$u(x)[a(x) - u(x)] = 0, \quad x \in \Omega.$$

Very recently, Shen and Xie studied in [35] the case with $m = 2$ and $\sigma \to 0^+$ and proved that the principal eigenvalue and the positive T-periodic solution converge to that of the corresponding reaction-diffusion equation

$$\begin{cases} u_t(t,x) = d\Delta u(t,x) + f(t,x,u(t,x)), & x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega \end{cases} \tag{1.12}$$

for some $d > 0$.

In this paper, Theorem A-Theorem E are established in the case of a bounded domain Ω. These results are expected to hold for a general domain Ω with appropriate assumptions on $a(t,x) := f_u(t,x,0)$. But, we are unable to do so mainly due to the lack of a Harnack-type inequality preventing us from successfully executing the procedure of using bounded domains to approximate a general domain. We refer the reader to [3,4] for the investigation of (1.1) on a general domain Ω with $f(t,x,u) = f(x,u)$. In this case, a Harnack-type inequality is known (see [11]).

Finally, we establish a maximum principle for the operator L_Ω defined in (1.3) with a general $a \in C_T(\mathbb{R} \times \overline{\Omega})$, which is of fundamental importance and independent interest.

Definition 1.4 *(Maximum principle).* We say that L_Ω admits the *maximum principle* if for any function $u \in C^{1,0}([0,T] \times \overline{\Omega})$ satisfying

$$\begin{cases} L_\Omega[u] \le 0 & \text{in } (0,T] \times \Omega, \\ u \ge 0 & \text{on } (0,T] \times \partial\Omega, \\ u(0,\cdot) \ge u(T,\cdot) & \text{in } \Omega, \end{cases} \tag{1.13}$$

there must hold $u > 0$ in $[0,T] \times \Omega$ unless $u \equiv 0$ in $[0,T] \times \Omega$.

Theorem F *(Maximum principle). Suppose* **(H1)**. *If $\lambda_1(-L_\Omega)$ is the principal eigenvalue, then L_Ω admits the maximum principle if and only if $\lambda_1(-L_\Omega) \ge 0$.*

A similar result in the case $a(t,x) = a(x)$ has been obtained by Coville in [10]. We point out that the maximum principle for elliptic or parabolic operators holds if and only if the principal eigenvalue is strictly positive (see e.g. [5,25,2,31]). This interesting difference is caused by the nonlocality of the operator. The maximum principle has many applications in the context of reaction-diffusion equations and nonlocal dispersal equations. For instance, the maximum principle was used to study deep qualitative properties of the principal eigenvalue in [5,7], and the non-existence of positive solutions of nonlocal KPP-type equations in unbounded domains in [4].

It remains an interesting problem to prove the maximum principle after dropping the assumption that $\lambda_1(-L_\Omega)$ is the principal eigenvalue. This seems to be an extremely difficult problem because even in the elliptic case, Berestycki and Rossi cannot completely characterize the validity of the maximum principle by λ_1 without the help of λ_1' and λ_1'' in their recent work (see [7, Theorem 1.6]). However, the equivalence of different generalized eigenvalues, namely, $\lambda_1 = \lambda_1' = \lambda_1''$, is not always the case. In the proof of [7, Theorem 1.7 and 1.9], Berestycki and Rossi need the existence of generalized eigenfunctions associated to λ_1. Generalized eigenfunctions associated to λ_1 were also used in the celebrated work of Berestycki, Nirenberg and Varadhan [5] to prove the maximum principle. The existence of generalized eigenfunctions for local operators comes from the regularity theory. For elliptic-type nonlocal operators, generalized eigenfunctions can be obtained thanks to a weak version of Harnack's inequality [11]. However, generalized eigenfunctions cannot be found in general in our case due to the lack of regularity and Harnack-type inequalities. This is why we work under the assumption that $\lambda_1(-L_\Omega)$ is the principal eigenvalue. In this case, we give a complete characterization of the maximum principle by the principal eigenvalue $\lambda_1(-L_\Omega)$.

To this end, let us mention that the study of (1.1) serves as the first step to the understanding of the global dynamics of the following mathematically and biologically significant competitive system proposed in [18],

$$\begin{cases} u_t(t,x) = \frac{D_1}{\sigma_1^m}\left[\int_\Omega J_{\sigma_1}(x-y)u(t,y)dy - u(t,x)\right] + u(t,x)(a(t,x)-u(t,x)-v(t,x)), \\ \quad x\in\overline{\Omega}, \\ v_t(t,x) = \frac{D_2}{\sigma_2^m}\left[\int_\Omega J_{\sigma_2}(x-y)v(t,y)dy - v(t,x)\right] + v(t,x)(a(t,x)-u(t,x)-v(t,x)), \\ \quad x\in\overline{\Omega}. \end{cases} \tag{1.14}$$

In fact, the investigation of the global dynamics of (1.14) relies on the detailed stability analysis of semi-trivial states that mainly comes from the analysis of (1.1). We refer the reader to [13,16,17] for the treatment of similar parabolic competitive systems. A well-known result for parabolic competitive systems with time-independent coefficients is that the slow disperser wipes out the fast disperser (see e.g. [13,29]). But, this is in general not the case when the coefficients are time-dependent (see [16]). In particular, the dynamics of (1.14) is in general complicated even when $\sigma_1 = \sigma_2$.

Organization of the paper. The paper is organized as follows. In Section 2, we study the existence of the principal eigenvalue of $-L_\Omega$ as well as its characterizations. In particular, we prove Theorem A. In Section 3, we prove a result on the approximation of the principal spectrum point. In Section 4, we study the existence and non-existence of non-negative T-periodic solutions of (1.9) and the global dynamics of (1.8) in terms of the principal eigenvalue $\lambda_1(-L_\Omega)$. In particular, Theorem B is proven. In Section 5, we study the effects of the dispersal rate D on the principal spectrum point $\lambda_1(-L_\Omega)$ and the positive T-periodic solution, and prove Theorem C. In Section 6, we study the effects of the dispersal range characterized by σ on the principal spectrum point and the positive T-periodic solution associated to the (1.1) and (1.11). In particular, we prove Theorem D and Theorem E. The last section, Section 7, is devoted to the proof of the maximum principle stated in Theorem F. In Appendix A, we construct examples of $-L_\Omega$ admitting no principal eigenvalue.

2. Principal eigenvalue and sup-inf characterizations

In this section, we investigate the principal spectral theory of the operator L_Ω defined in (1.3) with a general $a \in C_T(\mathbb{R}\times\overline{\Omega})$ and prove Theorem A. Recall that the spaces $C_T(\mathbb{R}\times\overline{\Omega})$, $\mathcal{X}_\Omega$, $\mathcal{X}_\Omega^+$ and $\mathcal{X}_\Omega^{++}$ are defined in (1.4) and (1.5), respectively. The operator L_Ω is considered as an unbounded linear operator on $C_T(\mathbb{R}\times\overline{\Omega})$ with domain $\mathcal{X}_\Omega$.

It is known (see e.g. [10,36]) that, due to the nonlocality, neither the operator (1.3) nor its resolvent is compact, and therefore, the Krein-Rutmann theorem [24] and its generalized versions [33,26] cannot be applied to ensure the existence of the principal eigenvalue. As a matter of fact, $-L_\Omega$ does not admit a principal eigenvalue in general (see Appendix A). It is then of vital significance to know when $\lambda_1(-L_\Omega)$ is indeed the principal eigenvalue of $-L_\Omega$. For nonlocal parabolic-type operators, the first result was obtained by Rawal and Shen in [32] (see Theorem 1.2). Moreover, they proved that any eigenvalue $\lambda\in\mathbb{R}$ of $-L_\Omega$ having an eigenfunction in $\mathcal{X}_\Omega^+\setminus\{0\}$ coincides with $\lambda_1(-L_\Omega)$, and therefore, must be the principal eigenvalue. Theorem 1.2 gives a necessary and sufficient spectral condition to determine whether $\lambda_1(-L_\Omega)$ is the principal eigenvalue of $-L_\Omega$. However, this spectral condition is hard to verify in general as $\lambda_1(-L_\Omega)$ is not computable and can be barely estimated in general. Although some sufficient conditions based on this spectral condition have been derived in [32, Theorem B], they are more or less restricted. Therefore, it is expected to find a more verifiable condition for $\lambda_1(-L_\Omega)$ becoming the principal eigenvalue of $-L_\Omega$. Here, we provide a sufficient condition that only requires mild smoothness of $a_T(x)$ near its maximum points. We recall from Theorem 1.2 that $\lambda_* = \min_{\overline{\Omega}}[D - a_T]$.

Theorem 2.1. *Suppose* **(H1)**. *If* (1.6) *holds, then* $\lambda_1(-L_\Omega) < \lambda_*$. *In particular,* $\lambda_1(-L_\Omega)$ *is the principal eigenvalue of* $-L_\Omega$.

We remark that (1.6) is independent of the dispersal kernel J, and hence, becomes very useful later when we study the equation (1.1) with scaled kernels. Theorem 2.1 is the first part of Theorem A.

Before proving Theorem 2.1, let us write $L_\Omega = H_\Omega + K_\Omega$, where

$$H_\Omega[v](t,x) = -v_t(t,x) - Dv(t,x) + a(t,x)v(t,x), \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

$$K_\Omega[v](t,x) = D \int_\Omega J(x-y)v(t,y)dy, \quad (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

They are considered as operators on $C_T(\mathbb{R} \times \overline{\Omega})$. Hence, H_Ω is unbounded with domain $\mathcal{X}_\Omega$ and K_Ω is bounded. We recall the following results from [32].

Proposition 2.2 *([32, Proposition 3.5 and Proposition 3.7]). Suppose* **(H1)**.

(1) *For any* $\alpha > -\lambda_*$, *the inverse* $(\alpha - H_\Omega)^{-1} : C_T(\mathbb{R} \times \overline{\Omega}) \to C_T(\mathbb{R} \times \overline{\Omega})$ *exists. Moreover, there exists* $M > 0$ *such that the estimate*

$$\big((\alpha - H_\Omega)^{-1} v\big)(t,x) \ge \frac{M}{\alpha - (-D + a_T(x))} v(x), \quad (t,x) \in \mathbb{R} \times \overline{\Omega}$$

holds for any $\alpha \in (-\lambda_*, -\lambda_* + 1]$ *and any* $v \in \mathcal{X}_\Omega^+$ *with* $v(t,x) \equiv v(x)$.

(2) *The inequality* $\lambda_1(-L_\Omega) < \lambda_*$ *holds if and only if there is* $\alpha_0 > -\lambda_*$ *such that* $r(K_\Omega(\alpha_0 - H_\Omega)^{-1}) > 1$, *where* $r(K_\Omega(\alpha_0 - H_\Omega)^{-1})$ *is the spectral radius of* $K_\Omega(\alpha - H_\Omega)^{-1}$.

We point out that the proof of Proposition 2.2(1) follows from the Floquet theory and an elementary analysis of the resolvent $(\alpha - H_\Omega)^{-1}$ given by the variation of constants formula, and the proof of Proposition 2.2(2) follows from a simple application of [8, Theorem 2.2]. It is referred to [32, Proposition 3.5 and Proposition 3.7] for more details.

We now prove Theorem 2.1.

Proof of Theorem 2.1. By contradiction, we assume $\lambda_1(-L_\Omega) \ge \lambda_*$. Proposition 2.2(2) yields

$$r(K_\Omega(\alpha - H_\Omega)^{-1}) \le 1, \quad \forall \alpha > -\lambda_*. \tag{2.1}$$

It is known from the variation of constants formula that for any $\alpha > -\lambda_*$ and $v \in \mathcal{X}_\Omega$, $(\alpha - H_\Omega)^{-1} v$ is given by

$$\big((\alpha - H_\Omega)^{-1}v\big)(t,x) = \int_{-\infty}^{t} e^{\int_s^t(-D+a(\tau,x)-\alpha)d\tau} v(s,x)ds.$$

In particular, there holds the monotonicity of the operator $(\alpha - H_\Omega)^{-1}$ in the sense that

$$v_1, v_2 \in \mathcal{X}_\Omega \text{ with } v_1 \geq v_2 \text{ implies } (\alpha - H_\Omega)^{-1}v_1 \geq (\alpha - H_\Omega)^{-1}v_2.$$

Now, Proposition 2.2(1) implies that for each $\alpha \in (-\lambda_*, -\lambda_* + 1]$,

$$\big((\alpha - H_\Omega)^{-1}1\big)(t,x) \geq \frac{M}{\alpha - (-D + a_T(x))} > 0, \quad (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Applying K_Ω to both sides of the above estimate, we find

$$\begin{aligned}\big(K_\Omega(\alpha - H_\Omega)^{-1}1\big)(t,x) &= D\int_\Omega J(x-y)\big((\alpha - H_\Omega)^{-1}1\big)(t,y)dy \\ &\geq \int_\Omega J(x-y)\frac{DM}{\alpha - (-D + a_T(y))}dy, \quad (t,x) \in \mathbb{R} \times \overline{\Omega}.\end{aligned} \tag{2.2}$$

By the monotonicity of $(\alpha - H_\Omega)^{-1}$, (2.2) and Proposition 2.2(1), we find for each $(t,x) \in \mathbb{R} \times \overline{\Omega}$

$$\begin{aligned}\big((\alpha - H_\Omega)^{-1}K_\Omega(\alpha - H_\Omega)^{-1}1\big)(t,x) &\geq \left((\alpha - H_\Omega)^{-1}\int_\Omega J(\cdot - y)\frac{DM}{\alpha - (-D + a_T(y))}dy\right)(t,x) \\ &\geq \frac{M}{\alpha - (-D + a_T(x))}\int_\Omega J(x-y)\frac{DM}{\alpha - (-D + a_T(y))}dy.\end{aligned}$$

Applying K_Ω to both sides of the above estimate, we find

$$\begin{aligned}&\big((K_\Omega(\alpha - H_\Omega)^{-1})^2 1\big)(t,x) \\ &\geq \int_\Omega J(x-y)\frac{DM}{\alpha - (-D + a_T(y))}\int_\Omega J(y-z)\frac{DM}{\alpha - (-D + a_T(z))}dzdy.\end{aligned}$$

Repeating the above arguments, we find for each $(t,x_0) \in \mathbb{R} \times \overline{\Omega}$ the following estimate

$$\big((K_\Omega(\alpha - H_\Omega)^{-1})^n 1\big)(t,x_0) \geq \int_\Omega \cdots \int_\Omega \prod_{m=1}^{n}\left[J(x_{m-1} - x_m)\frac{DM}{\alpha - (-D + a_T(x_m))}\right]dx_n \cdots dx_1.$$

As a result,

$$\begin{aligned}\|(K_\Omega(\alpha - H_\Omega)^{-1})^n\| &\geq \max_{(t,x_0)\in\mathbb{R}\times\overline{\Omega}} \big((K_\Omega(\alpha - H_\Omega)^{-1})^n 1\big)(t,x_0)\\ &\geq \max_{x_0\in\overline{\Omega}} \int_\Omega \cdots \int_\Omega \prod_{m=1}^n \left[J(x_{m-1}-x_m)\frac{DM}{\alpha - (-D + a_T(x_m))}\right] dx_n \cdots dx_1,\end{aligned}$$

which implies that for any $x_0 \in \overline{\Omega}$ and $\delta > 0$,

$$\begin{aligned}&\|(K_\Omega(\alpha - H_\Omega)^{-1})^n\|\\ &\geq \int_{\Omega\cap B_\delta(x_0)} \cdots \int_{\Omega\cap B_\delta(x_0)} \prod_{m=1}^n \left[J(x_{m-1}-x_m)\frac{DM}{\alpha - (-D + a_T(x_m))}\right] dx_n \cdots dx_1\\ &\geq \left[\inf_{x\in\Omega\cap B_\delta(x_0)} \int_{\Omega\cap B_\delta(x_0)} J(x-y)\frac{DM}{\alpha - (-D + a_T(y))} dy\right]^n,\end{aligned}$$

where $B_\delta(x_0)$ is the open ball in $\mathbb{R}^N$ centered at x_0 with radius δ. We then use (2.1) and Gelfand's formula for the spectral radius of a bounded linear operator to find

$$1 \geq \inf_{x\in\Omega\cap B_\delta(x_0)} \int_{\Omega\cap B_\delta(x_0)} J(x-y)\frac{DM}{\alpha - (-D + a_T(y))} dy =: I(x_0,\delta,\alpha) \tag{2.3}$$

for all $x_0 \in \overline{\Omega}$, $\delta > 0$ and $\alpha \in (-\lambda_*, -\lambda_* + 1]$.

Since J is continuous and $J(0) > 0$, there exists $\delta_* > 0$ and $c_* > 0$ such that $J \geq c_*$ on $B_{\delta_*}(0)$, the open ball in $\mathbb{R}^N$ centered at 0 with radius δ_*. Hence,

$$\begin{aligned}I(x_0,\delta,\alpha) &\geq \inf_{x\in\Omega\cap B_\delta(x_0)} \int_{\Omega\cap B_\delta(x_0)\cap B_{\delta_*}(x)} J(x-y)\frac{DM}{\alpha - (-D + a_T(y))} dy\\ &\geq c_* \inf_{x\in\Omega\cap B_\delta(x_0)} \int_{\Omega\cap B_\delta(x_0)\cap B_{\delta_*}(x)} \frac{DM}{\alpha - (-D + a_T(y))} dy\\ &= c_* \int_{\Omega\cap B_\delta(x_0)} \frac{DM}{\alpha - (-D + a_T(y))} dy,\end{aligned}$$

provided $2\delta \leq \delta_*$ so that $B_\delta(x_0) \subset B_{\delta_*}(x)$ whenever $x \in \overline{B_\delta(x_0)}$. In particular, for any $x_0 \in \overline{\Omega}$ and $\alpha \in (-\lambda_*, -\lambda_* + 1]$,

$$I(x_0, \delta_*/2, \alpha) \geq c_* \int_{\Omega\cap B_{\delta_*/2}(x_0)} \frac{DM}{\alpha - (-D + a_T(y))} dy.$$

Since $\frac{1}{\max_{y\in\overline{\Omega}} a_T(y)-a_T} \notin L^1_{loc}(\overline{\Omega})$, or equivalently $\frac{1}{-\lambda_*-(-D+a_T)} \notin L^1_{loc}(\overline{\Omega})$, there exists $x_* \in \overline{\Omega}$ such that

$$\frac{1}{-\lambda_* - (-D + a_T)} \notin L^1(\overline{\Omega} \cap B_{\delta_*/2}(x_*)),$$

which implies the existence of some $\epsilon_* \in (0, 1)$ such that

$$c_* \int\limits_{\Omega\cap B_{\delta_*/2}(x_*)} \frac{DM}{-\lambda_* + \epsilon_* - (-D + a_T(y))} dy \geq 2$$

for all $\epsilon \in (0, \epsilon_*]$. In particular, $I(x_*, \delta_*/2, -\lambda_* + \epsilon_*) \geq 2$, which contradicts to (2.3). □

To further investigate the properties of $\lambda_1(-L_\Omega)$, we prove the equivalence of the notions introduced in (1.7), which is restated in the following theorem.

Theorem 2.3. *Suppose* **(H1)**. *If $\lambda_1(-L_\Omega)$ is the principal eigenvalue, then*

$$\lambda_p(-L_\Omega) = \lambda'_p(-L_\Omega) = \lambda_1(-L_\Omega).$$

Proof. For simplicity, we write $\lambda_p = \lambda_p(-L_\Omega)$, $\lambda'_p = \lambda'_p(-L_\Omega)$ and $\lambda_1 = \lambda_1(-L_\Omega)$.

First, we prove $\lambda_1 = \lambda_p$. By Theorem 1.2, there exists $\phi_1 \in \mathcal{X}^{++}_\Omega$ such that

$$L_\Omega[\phi_1] + \lambda_1\phi_1 = 0 \quad \text{in} \quad \mathbb{R} \times \overline{\Omega}. \tag{2.4}$$

Since $\inf\limits_{\mathbb{R}\times\overline{\Omega}} \phi_1 > 0$, one has $\lambda_1 \leq \lambda_p$. We suppose by contradiction that $\lambda_1 < \lambda_p$. From the definition of λ_p, there are $\lambda \in (\lambda_1, \lambda_p)$ and $\phi \in \mathcal{X}^{++}_\Omega$ such that

$$L_\Omega[\phi] + \lambda\phi \leq 0 \quad \text{in} \quad \mathbb{R} \times \overline{\Omega}. \tag{2.5}$$

Clearly, $w := \frac{\phi_1}{\phi} \in \mathcal{X}^{++}_\Omega$.

Rewriting (2.5) as

$$-\phi_t + a(t,x)\phi \leq -\lambda\phi - D\left[\int\limits_\Omega J(x-y)\phi(t,y)dy - \phi(t,x)\right],$$

we deduce

$$\begin{aligned} L_\Omega[\phi_1] = -w_t\phi + D&\left[\int\limits_\Omega J(x-y)\phi(t,y)w(t,y)dy - w(t,x)\phi(t,x)\right] \\ &+ [-\phi_t + a(t,x)\phi(x)]w \end{aligned}$$

$$\leq -w_t\phi + D\left[\int_\Omega J(x-y)\phi(t,y)w(t,y)dy - w(t,x)\phi(t,x)\right]$$
$$+\left[-\lambda\phi - D\left(\int_\Omega J(x-y)\phi(t,y)dy - \phi(t,x)\right)\right]w$$
$$= -w_t\phi + D\int_\Omega J(x-y)\phi(t,y)[w(t,y)-w(t,x)]dy - \lambda\phi_1.$$

Using (2.4), we find

$$-(\lambda_1-\lambda)\phi_1 \leq -w_t\phi + D\int_\Omega J(x-y)\phi(t,y)[w(t,y)-w(t,x)]dy. \tag{2.6}$$

As $w \in \mathcal{X}_\Omega^{++}$, there exists $(t_0, x_0) \in \mathbb{R}\times\overline{\Omega}$ such that $w(t_0,x_0) = \max_{\mathbb{R}\times\overline{\Omega}} w$. Then, $w_t(t_0,x_0) = 0$. Hence, setting $(t,x) = (t_0,x_0)$ in (2.6) yields $-(\lambda_1-\lambda)\phi_1(t_0,x_0) \leq 0$, which leads to $\lambda_1 \geq \lambda$. This contradiction confirms $\lambda_1 = \lambda_p$.

Next, we prove $\lambda_1 = \lambda'_p$. Obviously, $\lambda_1 \geq \lambda'_p$. Assume that $\lambda_1 > \lambda'_p$. There are $\tilde{\lambda} \in (\lambda'_p, \lambda_1)$ and $\tilde{\phi} \in \mathcal{X}_\Omega^{++}$ such that $L[\tilde{\phi}] + \tilde{\lambda}\tilde{\phi} \geq 0$. Set $\tilde{w} := \frac{\phi_1}{\tilde{\phi}}$. Arguing as above, we derive

$$0 > -(\lambda_1-\tilde{\lambda})\phi_1 \geq -\tilde{w}_t\phi + D\int_\Omega J(x-y)\tilde{\phi}(t,y)[\tilde{w}(t,y)-\tilde{w}(t,x)]dy. \tag{2.7}$$

Let $(t_1,x_1) \in \mathbb{R}\times\overline{\Omega}$ be such that $\tilde{w}(t_1,x_1) = \min_{\mathbb{R}\times\overline{\Omega}} \tilde{w}$. Substituting (t_1,x_1) into the right-hand side of (2.7), we derive a contradiction. □

We remark that the parabolic-type operator $-L_\Omega$ is not self-adjoint, and thus, we lack the usual $L^2(\Omega)$ variational formula for the principal eigenvalue $\lambda_1(-L_\Omega)$. The sup-inf characterizations of $\lambda_1(-L_\Omega)$ given in Theorem 2.3 remedy the situation and play crucial roles in the sequel.

3. Approximating the principal spectrum point

In this section, we prove a result on the approximation of the principal spectrum point $\lambda_1(-L_\Omega)$. We define the following spaces:

$$\begin{aligned} X_\Omega &= C(\overline{\Omega}), \\ X_\Omega^+ &= \left\{v \in X_\Omega : v(x) \geq 0,\ x \in \overline{\Omega}\right\}, \quad \text{and} \\ X_\Omega^{++} &= \left\{v \in X_\Omega : v(x) > 0,\ x \in \overline{\Omega}\right\}. \end{aligned} \tag{3.1}$$

Denote by $\|\cdot\|_\infty$ the max norm on X_Ω. Clearly, X_Ω^+ is the positive cone of X_Ω and X_Ω^{++} is the interior of X_Ω^+.

Consider the following equation

$$u_t(t,x) = D\left[\int_{\Omega} J(x-y)u(t,y)dy - u(t,x)\right] + a(t,x)u(t,x), \quad t>0, \quad x \in \overline{\Omega}, \tag{3.2}$$

where $a \in C_T(\mathbb{R}\times\overline{\Omega})$. Denote by $\{\Phi(t;s)\}_{t\ge s\ge 0}$ the evolution family of bounded linear operators on X_Ω generated by the solutions of (3.2), that is, if $u(t,x;s,u_0)$ is the unique solution of (3.2) with initial data $u(s,\cdot;s,u_0) = u_0 \in X_\Omega$, then

$$u(t,\cdot;s,u_0) = \Phi(t;s)u_0 \in X_\Omega, \quad t \ge s.$$

If $u_0 \in X_\Omega^+$, the comparison principle implies that $\Phi(t;s)u_0 \in X_\Omega^+$ for all $t > s$. Moreover, if $u_0 \in X_\Omega^+\backslash\{0\}$, then $\Phi(t;s)u_0 \in X_\Omega^{++}$ for all $t > s$. Also, the time-periodicity ensures that

$$\Phi(t+T, s+T) = \Phi(t,s), \quad t \ge s \ge 0.$$

The operator norm of $\Phi(t,s)$ is denoted by $\|\Phi(t,s)\|$.

The next result connects $\Phi(t,s)$ with $\lambda_1(-L_\Omega)$.

Lemma 3.1. *Suppose* **(H1)** *and let* $a \in C_T(\mathbb{R}\times\overline{\Omega})$. *There hold*

$$-\lambda_1(-L_\Omega) = \frac{\ln r(\Phi(T,0))}{T} = \limsup_{t-s\to\infty} \frac{\ln\|\Phi(t,s)\|}{t-s},$$

where $r(\Phi(T,0))$ *is the spectral radius of* $\Phi(T,0)$.

Proof. See [32, Proposition 3.3 and Proposition 3.10]. □

We recall the following result from [32].

Lemma 3.2 *([32, Lemma 4.1]). Suppose* **(H1)** *and let* $a \in C_T(\mathbb{R}\times\overline{\Omega})$. *For any* $\epsilon > 0$, *there exists* $a^\epsilon \in C_T(\mathbb{R}\times\overline{\Omega})$, *independent of* D *and* J, *such that the following hold:*

(1) *the function* $a_T^\epsilon := \frac{1}{T}\int_0^T a^\epsilon(t,\cdot)dt$ *belongs to* $C^N(\overline{\Omega})$, *attains its maximum value at some* $x_0 \in \Omega$, *and has zero partial derivatives up to order* $N-1$ *at* x_0*;*
(2) *there holds* $\max_{\mathbb{R}\times\overline{\Omega}} |a^\epsilon - a| \le \frac{\epsilon}{2}$*;*
(3) *for any integers* $p, q \ge 0$, *if* $a \in C^{p,q}(\mathbb{R}\times\overline{\Omega})$, *then* $a^\epsilon \in C^{p,q}(\mathbb{R}\times\overline{\Omega})$.

Proof. The statements (1) and (2) are included in the statement of [32, Lemma 4.1]. The independence of a^ϵ from D and J, and the statement (3) can be easily obtained from the proof of [32, Lemma 4.1]. □

As an application of Theorem A(1), we prove the following approximation result. To highlight the dependence on $a(t,x)$, we write L_Ω as $L_\Omega(a)$, and $\Phi(t,s)$ as $\Phi(t,s;a)$.

Theorem 3.3. *Suppose* **(H1)** *and let $a \in C_T(\mathbb{R} \times \overline{\Omega})$. For any $\epsilon > 0$ and any integers $p, q \geq 0$, there exists $a^\epsilon \in C_T(\mathbb{R} \times \overline{\Omega}) \cap C^{p,q}(\mathbb{R} \times \overline{\Omega})$, independent of D and J, such that the following hold:*

(1) *the function a_T^ϵ belongs to $C^N(\overline{\Omega})$, attains its maximum value at some $x_0 \in \Omega$, and has zero partial derivatives up to order $N-1$ at x_0;*
(2) *there holds $\max_{\mathbb{R}\times\overline{\Omega}} |a^\epsilon - a| \leq \epsilon$;*
(3) *$\lambda_1(-L(a^\epsilon))$ is the principal eigenvalue of $-L(a^\epsilon)$;*
(4) *there holds $|\lambda_1(-L_\Omega(a^\epsilon)) - \lambda_1(-L_\Omega(a))| \leq \epsilon$.*

Proof. Fix $\epsilon > 0$ and integers $p, q \geq 0$. There exists $\tilde{a}^\epsilon \in C_T(\mathbb{R} \times \overline{\Omega}) \cap C^{p,q}(\mathbb{R} \times \overline{\Omega})$ such that $\max_{\mathbb{R}\times\overline{\Omega}} |\tilde{a}^\epsilon - a| \leq \frac{\epsilon}{2}$. Clearly, $\tilde{a}^\epsilon$ can be taken to be independent of D and J. Applying Lemma 3.2 to $\tilde{a}^\epsilon$, we find some $a^\epsilon \in C_T(\mathbb{R} \times \overline{\Omega}) \cap C^{p,q}(\mathbb{R} \times \overline{\Omega})$ satisfying (1) and $\max_{\mathbb{R}\times\overline{\Omega}} |a^\epsilon - a| \leq \frac{\epsilon}{2}$. The statement (2) follows.

By (1) and the N-th order Taylor expansion with remainder, we readily verify that

$$\frac{1}{\max_{y\in\overline{\Omega}} a_T^\epsilon(y) - a_T^\epsilon} \notin L^1_{loc}(\overline{\Omega}).$$

It follows from Theorem A(1) that $\lambda_1(-L(a^\epsilon))$ is the principal eigenvalue of $-L(a^\epsilon)$. This proves (3).

It remains to show (4). By (2) and the comparison principle, we find for any $u_0 \in X_\Omega^+$

$$\Phi(t,s;a^\epsilon - \epsilon)u_0 \leq \Phi(t,s;a)u_0 \leq \Phi(t,s;a^\epsilon + \epsilon)u_0, \quad \forall t \geq s.$$

As $\Phi(t,s;a^\epsilon \pm \epsilon)u_0 = e^{\pm\epsilon(t-s)}\Phi(t,s;a^\epsilon)u_0$, we find

$$\frac{\ln\|\Phi(t,s;a^\epsilon)\|}{t-s} - \epsilon \leq \frac{\ln\|\Phi(t,s;a)\|}{t-s} \leq \frac{\ln\|\Phi(t,s;a^\epsilon)\|}{t-s} + \epsilon, \quad \forall t \geq s.$$

The result then follows from Lemma 3.1. □

Remark 3.4. We point out that the functions $\{a^\epsilon\}_{\epsilon>0}$ in Theorem 3.3 are independent of D and J. As a result, if we consider the operator

$$L_{\Omega,m,\sigma}[v](t,x) := -v_t(t,x) + \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy - v(t,x)\right] + a(t,x)v(t,x),$$
$$(t,x) \in \mathbb{R} \times \overline{\Omega},$$

and want to study the dependence of the principal spectrum point $\lambda_1(-L_{\Omega,m,\sigma})$ on D, σ and m, we could use an approximating argument, which allows us to assume, without loss of generality, that $\lambda_1(-L_{\Omega,m,\sigma})$ is the principal eigenvalue of $-L_{\Omega,m,\sigma}$.

4. Global dynamics

In this section, we study the long-time dynamics of solutions of (1.8), namely,

$$u_t(t,x) = D\left[\int_{\Omega} J(x-y)u(t,y)dy - u(t,x)\right] + f(t,x,u(t,x)), \quad t>0, \quad x\in\overline{\Omega},$$

and prove Theorem B. Recall the spaces X_Ω, X_Ω^+ and X_Ω^{++} from (3.1).

For $u_0 \in X_\Omega$, we denote by $u(t,\cdot;u_0) \in X_\Omega$ for all $t>0$ the unique solution of (1.8) with initial data $u(0,\cdot;u_0)=u_0$. If $u_0\in X_\Omega^+$, the comparison principle (see e.g. [19,32]) yields that $u(t,\cdot;u_0)\in X_\Omega^+$ for all $t>0$. Moreover, if $u_0 \in X_\Omega^+\backslash\{0\}$, then $u(t,\cdot;u_0)\in X_\Omega^{++}$ for all $t>0$.

Note that the linearization of (1.8) at $u\equiv 0$ is just the equation (3.2) with $a(t,x)=f_u(t,x,0)$. Let $\{\Phi(t;s)\}_{t\geq s\geq 0}$ be the evolution family as defined after (3.2).

Before proving Theorem B, we prove the following comparison principle.

Proposition 4.1. *Let $u\in\mathcal{X}_\Omega^{++}$ be a sub-solution of (1.9) and $v\in\mathcal{X}_\Omega^{++}$ be a super-solution of (1.9). Then, $u\leq v$ in $\mathbb{R}\times\overline{\Omega}$.*

Proof. Let $\alpha_* := \sup\left\{\alpha>0 : \alpha u\leq v \text{ in } \mathbb{R}\times\overline{\Omega}\right\}$. By assumptions on u and v, the number α_* is well-defined and positive. If $\alpha_*\geq 1$, then we are done. So, we assume $\alpha_*<1$.

Set $w := v-\alpha_* u$. Then, $w\geq 0$ and there exists $(t_0,x_0)\in\mathbb{R}\times\overline{\Omega}$ such that $w(t_0,x_0)=0$. Obviously, w satisfies

$$\begin{aligned} w_t(t,x) &\geq D\left[\int_{\Omega} J(x-y)w(t,y)dy - w(t,x)\right] + f(t,x,v(t,x)) - \alpha_* f(t,x,u(t,x)) \\ &> D\left[\int_{\Omega} J(x-y)w(t,y)dy - w(t,x)\right] + f(t,x,v(t,x)) - f(t,x,\alpha_* u(t,x)), \\ &\quad (t,x)\in\mathbb{R}\times\overline{\Omega}, \end{aligned}$$

where we used **(H2)**-(3) and $\alpha_*<1$ in the second inequality. Considering the above inequality at (t_0,x_0), we immediately deduce a contradiction. □

We are now in the position to prove Theorem B.

Proof of Theorem B. We point out that case (i) is taken from [32, Theorem E]. Our contribution to the results are (ii) and (iii). Let $\lambda_1=\lambda_1(-L_\Omega)$ for simplicity.

(i) If $\lambda_1<0$, results in [32, Theorem E] using a contraction argument confirms the existence, uniqueness and global asymptotic stability of a solution $u^*\in\mathcal{X}_\Omega^{++}$ of (1.9). For the sake of completeness, we outline the arguments.

On one hand, it is easy to see from **(H2)**-(4) that for any $M\gg 1$, $u(t,x)\equiv M$ is a super-solution of (1.9), and then, the time-periodicity implies that $\{u(nT,\cdot;M)\}_n$ is a non-increasing sequence. Therefore, the function

$$u^+(x) := \lim_{n\to\infty} u(nT, x; M), \quad x \in \overline{\Omega}$$

is well-defined and upper semi-continuous. On the other hand, for any $0 < \epsilon \ll 1$, it can be shown using the assumption $\lambda_1 < 0$ that $\epsilon\phi_1$ is a sub-solution of (1.9), where ϕ_1 is a fixed principal eigenfunction of $-L_\Omega$, and then, $\{u(nT, \cdot; \epsilon\phi_1(0, \cdot))\}_n$ is a non-decreasing sequence. Therefore, the function

$$u^-(x) := \lim_{n\to\infty} u(nT, x; \epsilon\phi_1(0, \cdot)), \quad x \in \overline{\Omega}$$

is well-defined and lower semi-continuous. Clearly, $u^- \le u^+$.

To show $u^+ = u^-$, we define

$$\rho_n := \inf\left\{\ln\alpha : \frac{1}{\alpha}u(nT, \cdot; M) \le u(nT, \cdot; \epsilon\phi_1(0, \cdot)) \le \alpha u(nT, \cdot; M)\right\}.$$

As $\{u(nT, \cdot; M)\}_n$ is a non-increasing sequence and $\{u(nT, \cdot; \epsilon\phi_1(0, \cdot))\}_n$ is a non-decreasing sequence, $u(nT, \cdot; M)$ and $u(nT, \cdot; \epsilon\phi_1(0, \cdot))$ are getting closer to each other as n increases. As a result, the sequence $\{\rho_n\}_n$ is non-increasing, and thus, $\rho_* := \lim_{n\to\infty} \rho_n$ is well-defined. If $\rho_* > 0$, then arguments using the comparison principle would allow us to construct some $\alpha_* > 1$ and $0 < \delta \ll 1$ such that $\frac{1}{\alpha_*}u(nT, \cdot; M) \le u(nT, \cdot; \epsilon\phi_1(0, \cdot)) \le \alpha_* u(nT, \cdot; M)$ and $\ln\alpha_* < \rho_n - \delta$ for all large n. This contradicts the definition of ρ_*, and hence, $\rho_* = 0$, which implies $u^+ = u^-$.

Hence, $v^* := u^+$ is continuous and satisfies $\inf_{x\in\overline{\Omega}} v^* > 0$. Clearly, $u(T, \cdot; v^*) = v^*$. Then, v^* can be easily extended to be a solution $u^* \in \mathcal{X}_\Omega^{++}$ of (1.9) such that $u^*(t, \cdot) = u(t, \cdot; v^*)$ for $t \in [0, T]$.

By the above contraction argument, the uniqueness of solutions of (1.9) in the space $\mathcal{X}_\Omega^{++}$ follows. The uniqueness also follows directly from Proposition 4.1. The global stability of u^* follows again from the contraction argument.

(ii) Suppose $\lambda_1 > 0$. Since $f(t, x, u(t, x; u_0)) \le a(t, x)u(t, x; u_0)$, there holds

$$u_t(t, x; u_0) \le D\left[\int_\Omega J(x - y)u(t, y; u_0)dy - u(t, x; u_0)\right] + a(t, x)u(t, x; u_0).$$

The comparison principle yields $u(t, \cdot; u_0) \le \Phi(t, 0)u_0$.

We claim $\|\Phi(t, 0)u_0\|_\infty \to 0$ as $t \to \infty$. Write $t = [t] + r_t$, where $[t]$ is the largest number of the form nT not great than t and $r_t \in [0, T)$. By the time-periodicity, we find

$$\Phi(t, 0)u_0 = \Phi(t, [t])\Phi([t], [t] - T) \cdots \Phi(T, 0)u_0 = \Phi(t_r, 0)\Phi(T, 0)^{\frac{[t]}{T}} u_0.$$

Obviously, there is $C = C(T) > 0$ such that $\|\Phi(t_r, 0)\| \le C$. It is well-known that

$$r(\Phi(T, 0)) = \lim_{n\to\infty} \|\Phi(T, 0)^n\|^{\frac{1}{n}} \quad \text{(Gelfand's formula)}.$$

Moreover, by Lemma 3.1, there holds $-\lambda_1 = \frac{\ln r(\Phi(T,0))}{T}$, which implies $e^{-\lambda_1 T} = \lim_{n\to\infty} \|\Phi(T,0)^n\|^{\frac{1}{n}}$. In particular, we find $\|\Phi(T,0)^n\| \le e^{-\frac{\lambda_1}{2}Tn}$ for $n \gg 1$. Hence,

$$\|\Phi(t,0)u_0\|_\infty \le C\|u_0\|_\infty \|\Phi(T,0)^{\frac{[t]}{T}}\| \le C\|u_0\|_\infty e^{-\frac{\lambda_1}{2}[t]} \to 0 \quad \text{as} \quad t \to \infty.$$

This proves the claim and confirms the statement.

(iii) We show that if $\lambda_1 = 0$, then the equation (1.9) admits no solution in $\mathcal{X}_\Omega^{++}$. For contradiction, suppose that $v^* \in \mathcal{X}_\Omega^{++}$ is a solution of (1.9). Let ϕ_1 be the principal eigenfunction associated to λ_1 with the normalization $\phi_1 < v^*$ in $\mathbb{R} \times \overline{\Omega}$. We see from **(H2)**-(3) that

$$\begin{aligned}
0 &= \lambda_1 \phi_1(t,x) \\
&= \partial_t \phi_1(t,x) - D\left[\int_\Omega J(x-y)\phi_1(t,y)dy - \phi_1(t,x)\right] - a(t,x)\phi_1(t,x) \\
&\le \partial_t \phi_1(t,x) - D\left[\int_\Omega J(x-y)\phi_1(t,y)dy - \phi_1(t,x)\right] - f(t,x,\phi_1(t,x)), \quad (t,x) \in \mathbb{R} \times \overline{\Omega},
\end{aligned}$$

that is, ϕ_1 is a super-solution of (1.9). By Proposition 4.1, there holds $v^* \le \phi_1$ in $\mathbb{R} \times \overline{\Omega}$, which contradicts the normalization. □

5. Effects of the dispersal rate

In this section, we study the effects of the dispersal rate D on $\lambda_1(D) := \lambda_1(-L_\Omega)$ and the positive T-periodic solution associated to the equations (1.8) and (1.9). In particular, we prove Theorem C.

We first prove Theorem C(1) concerning the effects of D on $\lambda_1(D)$.

Proof of Theorem C(1). The proof is done within two steps.

Step 1. We prove the results under the additional assumption that $\lambda_1(D)$ is the principal eigenvalue for all $D > 0$.

As $\lambda_1(D)$ is an isolated eigenvalue, the continuous differentiability of $D \mapsto \lambda_1(D)$ follows from the classical perturbation theory (see e.g. [23]).

For the limits, we first claim that for each $\epsilon > 0$, there exists $D_\epsilon > 0$ such that

$$-\max_{\overline{\Omega}} a_T - \epsilon \le \lambda_1(D) \le -\min_{\overline{\Omega}} a_T + \epsilon, \quad \forall D \in (0, D_\epsilon). \tag{5.1}$$

Indeed, it is easy to check that the function $\phi(t,x) := e^{\int_0^t [a(s,x)-a_T(x)]ds}$, $(t,x) \in \mathbb{R} \times \overline{\Omega}$ is a positive T-periodic solution of $\phi_t = a(t,x)\phi - a_T(x)\phi$. In particular, $\phi \in \mathcal{X}_\Omega^{++}$. For each $\epsilon > 0$, we set $\lambda_\epsilon^{\max} = -\max_{\overline{\Omega}} a_T - \epsilon$ and $\lambda_\epsilon^{\min} = -\min_{\overline{\Omega}} a_T + \epsilon$. It is easy to see that

$$\left(L_\Omega + \lambda_\epsilon^{\max}\right)[\phi](t,x) = D\left[\int_\Omega J(x-y)\phi(t,y)dy - \phi(t,x)\right] + \left[a_T(x) - \max_{\overline{\Omega}} a_T - \epsilon\right]\phi(t,x),$$

$$\left(L_\Omega + \lambda_\epsilon^{\min}\right)[\phi](t,x) = D\left[\int_\Omega J(x-y)\phi(t,y)dy - \phi(t,x)\right] + \left[a_T(x) - \min_{\overline{\Omega}} a_T + \epsilon\right]\phi(t,x).$$

Since $\min_{[0,T]\times\overline{\Omega}} \phi > 0$ and $\max_{[0,T]\times\overline{\Omega}} \phi < \infty$, it is straightforward to check that for each $\epsilon > 0$, there exists $D_\epsilon > 0$ such that for each $D \in (0, D_\epsilon)$, there hold

$$\left(L_\Omega + \lambda_\epsilon^{\max}\right)[\phi] \le 0 \quad \text{and} \quad \left(L_\Omega + \lambda_\epsilon^{\min}\right)[\phi] \ge 0. \tag{5.2}$$

It then follows from (5.2), the definitions of $\lambda_p(-L_\Omega)$ and $\lambda'_p(-L_\Omega)$, and Theorem 2.3 that for each $\epsilon > 0$, there holds $\lambda_\epsilon^{\max} \le \lambda_1(D) \le \lambda_\epsilon^{\min}$ for all $D \in (0, D_\epsilon)$. This is exactly (5.1).

Next, we prove

$$\lambda_1(D) \to -\max_{\overline{\Omega}} a_T \quad \text{as} \quad D \to 0^+. \tag{5.3}$$

By Theorem 2.1 and (5.1), for each $\epsilon > 0$ there exists $D_\epsilon > 0$ such that

$$-\max_{\overline{\Omega}} a_T - \epsilon \le \lambda_1(D) \le \min_{\overline{\Omega}}[D - a_T], \quad \forall D \in (0, D_\epsilon).$$

Setting $D \to 0^+$, we find

$$-\max_{\overline{\Omega}} a_T - \epsilon \le \liminf_{D\to 0^+} \lambda_1(D) \le \limsup_{D\to 0^+} \lambda_1(D) \le -\max_{\overline{\Omega}} a_T, \quad \forall \epsilon > 0,$$

which leads to (5.3).

Finally, to show

$$\lambda_1(D) \to \infty \quad \text{as} \quad D \to \infty, \tag{5.4}$$

we consider the following operator

$$L_\Omega^0[\psi] := \int_\Omega J(\cdot - y)\psi(y)dy - \psi(x), \quad \psi \in X_\Omega,$$

where X_Ω is defined in (3.1). It is known from [34, Theorem 2.1 and Proposition 3.4] that the principal eigenvalue of $-L_\Omega^0$ exists and is positive. Let $\lambda^0 > 0$ be the principal eigenvalue of $-L_\Omega^0$, and $\psi^0 \in X_\Omega^{++}$ be an associated eigenfunction.

Let $\lambda_D = D\lambda^0 - \max_{[0,T]\times\overline{\Omega}} a$. We see that

$$(L_\Omega + \lambda_D)[\psi^0] = DL_\Omega^0[\psi^0] + a\psi^0 + \lambda_D\psi^0 = \left[-D\lambda^0 + a + \lambda_D\right]\psi^0 \le 0.$$

That is, (λ_D, ψ^0) is a test pair for $\lambda_p(-L_\Omega)$. It follows that $\lambda_1(D) = \lambda_p(-L_\Omega) \ge \lambda_D$. Setting $D \to \infty$, we arrive at (5.4).

Step 2. If $\lambda_1(D)$ is not the principal eigenvalue for some $D > 0$, we can use an approximating argument as already pointed out in Remark 3.4. More precisely, applying Theorem 3.3, we find that for each $\epsilon > 0$, there exists $a^\epsilon \in C_T(\mathbb{R} \times \overline{\Omega})$ such that

(i) $\lambda_D^\epsilon := \lambda_1(-L_\Omega(a^\epsilon))$ is the principal eigenvalue for all $D > 0$, where $L_\Omega(a^\epsilon)$ is L_Ω with a replaced by a^ϵ;
(ii) $|\lambda_D^\epsilon - \lambda_D| < \epsilon$ for all $D > 0$.

We then apply **Step 1** to conclude that for each $\epsilon > 0$, the function $D \mapsto \lambda_D^\epsilon$ is continuously differentiable on $(0, \infty)$ and satisfies the limits

$$\lambda_D^\epsilon \to \begin{cases} -\max_{\overline{\Omega}} a_T^\epsilon & \text{as} \quad D \to 0^+, \\ \infty & \text{as} \quad D \to \infty, \end{cases}$$

where $a_T^\epsilon = \frac{1}{T}\int_0^T a^\epsilon(t,\cdot)dt$. The results for the function $D \mapsto \lambda_D$ now is a consequence of (ii). □

Next, we prove the monotonicity of the function $D \mapsto \lambda_1(D)$ for a special class of $a(t,x)$ as in Theorem C(2).

Proof of Theorem C(2). We first prove the result under the assumption that the operator

$$v \mapsto D\left[\int_\Omega J(\cdot - y)v(y)dy - v\right] + \beta(x)v : C(\overline{\Omega}) \to C(\overline{\Omega}) \tag{5.5}$$

admits a principal eigenvalue for all $D > 0$. We write $L_\Omega = L_\Omega^{\mathcal{S}} + L_\Omega^{\mathcal{T}}$, where the superscripts $\mathcal{S}$ and $\mathcal{T}$ stand for space and time, respectively, and

$$L_\Omega^{\mathcal{S}}[v](x) = D\left[\int_\Omega J(x-y)v(y)dy - v(x)\right] + \beta(x)v(x),$$

$$L_\Omega^{\mathcal{T}}[v](t) = -v_t(t) + \alpha(t)v(t).$$

Let $(\lambda_1^{\mathcal{S}}(D), \phi_D^{\mathcal{S}})$ be the principal eigen-pair of $-L_\Omega^{\mathcal{S}}$. It is known from [34, Theorem 2.2 (1)] that $D \mapsto \lambda_1^{\mathcal{S}}(D)$ is non-decreasing, and increasing if β is not a constant function. Let us show that the function $D \mapsto \lambda_1^{\mathcal{S}}(D)$ is increasing even if β is a constant function. From the classical perturbation theory (see e.g. [23]), we know that $D \mapsto (\lambda_1^{\mathcal{S}}(D), \phi_D^{\mathcal{S}})$ is continuously differentiable, and therefore, we can differentiate the equation $L_\Omega^{\mathcal{S}}[\phi_D^{\mathcal{S}}] = \lambda_1^{\mathcal{S}}(D)\phi_D^{\mathcal{S}}$ with respect to D to find

$$\int_\Omega J(x-y)\phi_D^{\mathcal{S}}(y)dy - \phi_D^{\mathcal{S}}(x) + L_\Omega^{\mathcal{S}}[\partial_D \phi_D^{\mathcal{S}}](x) + \partial_D \lambda_1^{\mathcal{S}}(D)\phi_D^{\mathcal{S}}(x) + \lambda_1^{\mathcal{S}}(D)\partial_D \phi_D^{\mathcal{S}}(x) = 0.$$

Multiplying the above equation by $\phi_D^{\mathcal{S}}$ and integrating the resulting equation over Ω, we find

$$\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(y)dxdy - \int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx + \partial_D \lambda_1^{\mathcal{S}}(D) \int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx = 0, \tag{5.6}$$

where we used the fact that

$$\begin{aligned}\int_\Omega L_\Omega^{\mathcal{S}}[\partial_D \phi_D^{\mathcal{S}}](x)\phi_D^{\mathcal{S}}(x)dx + \int_\Omega \lambda_1^{\mathcal{S}}(D)\partial_D \phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(x)dx \\ = \int_\Omega \partial_D \phi_D^{\mathcal{S}}(x)\left[L_\Omega^{\mathcal{S}}[\phi_D^{\mathcal{S}}](x) + \lambda_1^{\mathcal{S}}(D)\phi_D^{\mathcal{S}}(x)\right]dx = 0.\end{aligned}$$

It then follows from (5.6), the symmetry of J and $\int_\Omega J(\cdot - y)dy \lneqq 1$ on Ω that

$$\begin{aligned}
&\partial_D \lambda_1^{\mathcal{S}}(D) = -\frac{\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(y)dxdy - \int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx}{\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx} \\
&= \frac{1}{2}\frac{2\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx - 2\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(y)dxdy}{\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx} \\
&> \frac{1}{2}\frac{2\int_{\Omega\times\Omega} J(y-x)\phi_D^{\mathcal{S}}(x)^2 dxdy - 2\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(y)dxdy}{\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx} \\
&= \frac{1}{2}\frac{\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(y)^2 dxdy + \int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)^2 dxdy - 2\int_{\Omega\times\Omega} J(x-y)\phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{S}}(y)dxdy}{\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx} \\
&= \frac{1}{2}\frac{\int_{\Omega\times\Omega} J(x-y)\left[\phi_D^{\mathcal{S}}(y) - \phi_D^{\mathcal{S}}(x)\right]^2 dxdy}{\int_\Omega \phi_D^{\mathcal{S}}(x)^2 dx} \geq 0.
\end{aligned}$$

Hence, $D \mapsto \lambda_1^{\mathcal{S}}(D)$ is increasing.

Now, let us define $\phi_D^{\mathcal{T}}(t) = e^{\int_0^t [\alpha(s) - \alpha_T]ds}$ for $t \in \mathbb{R}$, where $\alpha_T = \frac{1}{T}\int_0^T \alpha(t)dt$. Clearly, $\phi_D^{\mathcal{T}}$ is continuously differentiable, positive and T-periodic, and satisfies

$$L_\Omega^{\mathcal{T}}[\phi_D^{\mathcal{T}}] + \alpha_T \phi_D^{\mathcal{T}} = 0.$$

It follows that $\lambda_1(D) = \lambda_1^{\mathcal{S}}(D) + \alpha_T$ is the principal eigenvalue of $-L_\Omega$ with the principal eigenfunction $\phi_D(t,x) = \phi_D^{\mathcal{S}}(x)\phi_D^{\mathcal{T}}(t)$. As $D \mapsto \lambda_1^{\mathcal{S}}(D)$ is increasing, so is $D \mapsto \lambda_1^{\mathcal{S}}(D)$.

If the operator defined in (5.5) does not admit a principal eigenvalue for some $D > 0$, then we can use an approximating argument as in **Step 2** in the proof of Theorem C(1) to deduce the result. This completes the proof. □

Finally, we study the effects of the dispersal rate D on the positive T-periodic solutions of (1.9) by proving Theorem C(3)(4).

Proof of Theorem C(3)(4). (3) By Theorem C(1), $\lambda_1(D) < 0$ for all $0 < D \ll 1$ and $\lambda_1(D) > 0$ for all $D \gg 1$. The result then follows from Theorem B.

(4) We refer the reader to the proof of Lemma 6.2 and Theorem 6.3, where similar problems are treated. In fact, we can follow the lines as in the proof of Theorem 6.3(2) except that we need to set $\sigma = 1$ and replace the limit

$$\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy - v^*(t,x)\right] \to 0 \quad \text{as} \quad \sigma \to 0^+$$
$$\text{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega}$$

in the proof of Theorem 6.3(2) by the following limit

$$D(1-\delta)\left[\int_\Omega J(x-y)v^*(t,y)dy - v^*(t,x)\right] \to 0 \quad \text{as} \quad D \to 0^+$$
$$\text{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega}. \quad \square$$

6. Effects of the dispersal range and scaling limits

In this section, we study the effects of the dispersal range characterized by σ on the principal spectrum point and the positive T-periodic solution associated to (1.1) and (1.11). In particular, we prove Theorem D and Theorem E.

Note that for each fixed $\sigma > 0$ and $m \geq 0$, introducing the new dispersal rate $\tilde{D} := \frac{D}{\sigma^m}$ and the new dispersal kernel $\tilde{J} := J_\sigma$, we are completely in the situation of the operator (1.3). Therefore, studies and results in previous sections apply here. In particular, we have the following proposition collecting some basic facts about $\lambda_1(-L_{\Omega,m,\sigma})$, $\lambda_p(-L_{\Omega,m,\sigma})$ and $\lambda'_p(-L_{\Omega,m,\sigma})$. For convenience, we define

$$\mathcal{M}_{\Omega,m,\sigma}[v](t,x) := -v_t(t,x) + \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy - v(t,x)\right], \quad (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Therefore,

$$L_{\Omega,m,\sigma}[v](t,x) := \mathcal{M}_{\Omega,m,\sigma}[v](t,x) + a(t,x)v(t,x), \quad (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Proposition 6.1. *Suppose* **(H1)**. *Let* $m \geq 0$ *and* $\sigma > 0$.

(1) *The principal spectrum point* $\lambda_1(-L_{\Omega,m,\sigma})$ *is the principal eigenvalue if and only if*

$$\lambda_1(-L_{\Omega,m,\sigma}) < \frac{D}{\sigma^m} - \max_{\overline{\Omega}} a_T.$$

Moreover, when $\lambda_1(-L_{\Omega,m,\sigma})$ *is the principal eigenvalue of* $-L_{\Omega,m,\sigma}$*, it is geometrically simple and has an eigenfunction in* $\mathcal{X}_\Omega^{++}$.

(2) *If* (1.6) *holds, then* $\lambda_1(-L_{\Omega,m,\sigma})$ *is the principal eigenvalue of* $-L_{\Omega,m,\sigma}$*. If* $\lambda_1(-L_{\Omega,m,\sigma})$ *is the principal eigenvalue of* $-L_{\Omega,m,\sigma}$*, then*

$$\lambda_p(-L_{\Omega,m,\sigma}) = \lambda_p'(-L_{\Omega,m,\sigma}) = \lambda_1(-L_{\Omega,m,\sigma}).$$

(3) $\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a)$ *is Lipschitz continuous with respect to* $a \in C_T(\mathbb{R}\times\overline{\Omega})$*. More precisely,*

$$|\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a) - \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - b)| \le \sup_{t\in[0,T]} \|a(t,\cdot) - b(t,\cdot)\|_\infty,$$
$$\forall a, b \in C_T(\mathbb{R}\times\overline{\Omega}).$$

(4) *If* $\Omega_1 \subset \Omega_2$*, then* $\lambda_p(-L_{\Omega_1,m,\sigma}) \ge \lambda_p(-L_{\Omega_2,m,\sigma})$*. If, in addition,* $\lambda_1(-L_{\Omega_1,m,\sigma})$ *and* $\lambda_1(-L_{\Omega_2,m,\sigma})$ *are principal eigenvalues of* $-L_{\Omega_1,m,\sigma}$ *and* $-L_{\Omega_2,m,\sigma}$*, respectively, then*

$$|\lambda_p(-L_{\Omega_2,m,\sigma}) - \lambda_p(-L_{\Omega_1,m,\sigma})| \le C_0|\Omega_2 \setminus \Omega_1|,$$

where $C_0 > 0$ *depends on* a*,* σ*,* D*,* m*,* J_σ *and* Ω_2.

(5) *The function* $\sigma \mapsto \lambda_1(-L_{\Omega,m,\sigma})$ *is continuous.*

Proof. (1) It is a direct consequence of Theorem 1.2.

(2) It follows from Theorem 1.2 and Theorem A.

(3) By Theorem 3.3, Remark 3.4 and an approximating argument as in **Step 2** in the proof of Theorem C(1), we may assume, without loss of generality, that both $\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a)$ and $\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - b)$ are principal eigenvalues. Let us fix $\lambda < \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a)$. By (2), there exists $\phi \in \mathcal{X}_\Omega^{++}$ such that

$$\mathcal{M}_{\Omega,m,\sigma}[\phi](t,x) + (a(t,x)+\lambda)\phi(t,x) \le 0, \quad \forall (t,x) \in \mathbb{R}\times\overline{\Omega}.$$

Clearly,

$$\begin{aligned} 0 &\ge \mathcal{M}_{\Omega,m,\sigma}[\phi] + (a+\lambda)\phi = \mathcal{M}_{\Omega,m,\sigma}[\phi] + (b+\lambda+a-b)\phi \\ &\ge \mathcal{M}_{\Omega,m,\sigma}[\phi] + \left(b + \lambda - \sup_{t\in[0,T]} \|a(t,\cdot) - b(t,\cdot)\|_\infty\right)\phi. \end{aligned}$$

Again, by (2),

$$\lambda - \sup_{t\in[0,T]} \|a(t,\cdot) - b(t,\cdot)\|_\infty \le \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - b).$$

As this holds for any $\lambda < \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a)$, we arrive at

$$\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a) - \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - b) \leq \sup_{t\in[0,T]} \|a(t,\cdot) - b(t,\cdot)\|_\infty.$$

Switching the role of a and b, we find

$$\lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - b) - \lambda_1(-\mathcal{M}_{\Omega,m,\sigma} - a) \leq \sup_{t\in[0,T]} \|a(t,\cdot) - b(t,\cdot)\|_\infty.$$

The result follows.

(4) Let (λ, ψ) be a test pair for $\lambda_p(-L_{\Omega_2,m,\sigma})$, that is, $(\lambda, \psi) \in \mathbb{R} \times \mathcal{X}^{++}_{\Omega_2}$ satisfies $(L_{\Omega_2,m,\sigma} + \lambda)[\psi] \leq 0$ in $\mathbb{R} \times \overline{\Omega}_2$. Define

$$\psi_{\overline{\Omega}_1}(t,x) = \psi(t,x), \quad (t,x) \in \mathbb{R} \times \overline{\Omega}_1 \subset \mathbb{R} \times \overline{\Omega}_2.$$

Then, $\psi_{\overline{\Omega}_1} \in \mathcal{X}^{++}_{\Omega_1}$. Moreover, for any $(t,x) \in \mathbb{R} \times \overline{\Omega}_1$

$$\begin{aligned}(L_{\Omega_1,m,\sigma} + \lambda)[\psi_{\overline{\Omega}_1}](t,x) &= -\partial_t \psi_{\overline{\Omega}_1}(t,x) + \frac{D}{\sigma^m}\left[\int_{\Omega_1} J_\sigma(x-y)\psi_{\overline{\Omega}_1}(t,y)dy - \psi_{\overline{\Omega}_1}(t,x)\right] \\ &\quad + a(t,x)\psi_{\overline{\Omega}_1}(t,x) + \lambda\psi_{\overline{\Omega}_1}(t,x) \\ &\leq -\psi_t(t,x) + \frac{D}{\sigma^m}\left[\int_{\Omega_2} J_\sigma(x-y)\psi(t,y)dy - \psi(t,x)\right] \\ &\quad + a(t,x)\psi(t,x) + \lambda\psi(t,x) \\ &= (L_{\Omega_2,m,\sigma} + \lambda)[\psi](t,x) \leq 0.\end{aligned}$$

That is, $(\lambda, \psi_{\overline{\Omega}_1})$ is a test pair for $\lambda_p(-L_{\Omega_1,m,\sigma})$, and hence, $\lambda \leq \lambda_p(-L_{\Omega_1,m,\sigma})$. Taking the supremum over all such λ, we arrive at

$$\lambda_p(-L_{\Omega_2,m,\sigma}) \leq \lambda_p(-L_{\Omega_1,m,\sigma}). \tag{6.1}$$

To prove the other statement, let $(\lambda_p(-L_{\Omega_2,m,\sigma}), \psi)$ be the eigen-pair of $-L_{\Omega_2,m,\sigma}$ with the normalization $\sup\limits_{[0,T]\times\overline{\Omega}_2} \psi = 1$. Direct calculations yield

$$\begin{aligned}\mathcal{M}_{\Omega_1,m,\sigma}[\psi] + (a + \lambda_p(-L_{\Omega_2,m,\sigma}))\psi &= -\frac{D}{\sigma^m}\int_{\Omega_2\setminus\Omega_1} J_\sigma(\cdot - y)\psi(t,y)dy \\ &\geq -\frac{D\|J_\sigma\|_\infty}{\sigma^m}|\Omega_2 \setminus \Omega_1| \\ &\geq -\frac{D\|J_\sigma\|_\infty}{\sigma^m \min\limits_{\overline{\Omega}_1}\psi}|\Omega_2 \setminus \Omega_1|\psi \quad \text{in} \quad \mathbb{R} \times \overline{\Omega}_1.\end{aligned}$$

That is,

$$L_{\Omega_1,m,\sigma}[\psi] + \left[\lambda_p(-L_{\Omega_2,m,\sigma}) + C_0|\Omega_2 \setminus \Omega_1|\right]\psi \geq 0 \quad \text{in} \quad \mathbb{R} \times \overline{\Omega}_1,$$

where $C_0 = \frac{D\|J_\sigma\|_\infty}{\sigma^m \min\limits_{\overline{\Omega}_1} \psi}$. By (2),

$$\lambda_p(-L_{\Omega_1,m,\sigma}) = \lambda'_p(-L_{\Omega_1,m,\sigma}) \leq \lambda_p(-L_{\Omega_2,m,\sigma}) + C_0|\Omega_2 \setminus \Omega_1|.$$

This together with (6.1) lead to the result.

(5) By Theorem 3.3, Remark 3.4 and an approximating arguments as in **Step 2** in the proof of Theorem C(1), we may assume, without loss of generality, that $\lambda_1(-L_{\Omega,m,\sigma})$ is the principal eigenvalue of $L_{\Omega,m,\sigma}$ for all $\sigma > 0$. The result then follows from the classical perturbation theory of isolated eigenvalues (see e.g. [23]). In fact, for each fixed $\sigma_0 > 0$, we can write $L_{\Omega,m,\sigma}$ as $L_{\Omega,m,\sigma} = L_{\Omega,m,\sigma_0} + U_{\sigma,\sigma_0}$, where

$$U_{\sigma,\sigma_0}[v](t,x) = \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy - v(t,x)\right] - \frac{D}{\sigma_0^m}\left[\int_\Omega J_{\sigma_0}(x-y)v(t,y)dy - v(t,x)\right].$$

The result then follows from the facts that U_{σ,σ_0} is bounded and linear, and $U_{\sigma,\sigma_0} \to 0$ in norm as $\sigma \to \sigma_0$. $\square$

Next, we prove Theorem D concerning the scaling limits of the principal spectrum point.

Proof of Theorem D. By Theorem 3.3, Remark 3.4 and an approximating argument as in **Step 2** in the proof of Theorem C(1), we may assume, without loss of generality, that $a \in C^{1,4}(\mathbb{R} \times \overline{\Omega})$ and $\lambda_1(-L_{\Omega,m,\sigma})$ is the principal eigenvalue of $-L_{\Omega,m,\sigma}$ for all $\sigma > 0$.

(1) We first prove the result in the case $m > 0$. By Proposition 6.1(1), we find $\lambda_1(-L_{\Omega,m,\sigma}) < \frac{D}{\sigma^m} - \max\limits_{\overline{\Omega}} a_T$, which implies $\limsup\limits_{\sigma\to\infty} \lambda_1(-L_{\Omega,m,\sigma}) \leq -\max\limits_{\overline{\Omega}} a_T$. It remains to show that

$$\liminf_{\sigma\to\infty} \lambda_1(-L_{\Omega,m,\sigma}) \geq -\max_{\overline{\Omega}} a_T. \tag{6.2}$$

To do so, let us fix some constant $\phi_0 > 0$. It is straightforward to check that for each $x \in \overline{\Omega}$, the function

$$\phi(t,x) = e^{\int_0^t [a(s,x)-a_T(x)]ds}\phi_0, \quad t \in \mathbb{R} \tag{6.3}$$

is a positive T-periodic solution of the ODE $v_t = a(t,x)v - a_T(x)v$. Clearly, $\phi \in \mathcal{X}_{\overline{\Omega}}^{++}$ and we may choose ϕ_0 such that $\sup\limits_{\mathbb{R}\times\overline{\Omega}} \phi = 1$. For any $\epsilon > 0$, we see that for each $(t,x) \in \mathbb{R} \times \overline{\Omega}$,

$$\left(L_{\Omega,m,\sigma}-\max_{\overline{\Omega}}a_T-\epsilon\right)[\phi](t,x)$$
$$=-\phi_t(t,x)+\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)\phi(t,y)dy-\phi(t,x)\right]+\left[a(t,x)-\max_{\overline{\Omega}}a_T-\epsilon\right]\phi(t,x)$$
$$\le\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)\phi(t,y)dy-\phi(t,x)\right]-\epsilon\phi(t,x). \tag{6.4}$$

As $\min_{\mathbb{R}\times\overline{\Omega}}\phi>0$ and $\left\|\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(\cdot-y)\phi(t,y)dy-\phi(t,\cdot)\right]\right\|_\infty\to 0$ as $\sigma\to\infty$, there is $\sigma_\epsilon>0$ such that $(L_{\Omega,m,\sigma}-\max_{\overline{\Omega}}a_T-\epsilon)[\phi]\le 0$ for all $\sigma\ge\sigma_\epsilon$, which implies that

$$\lambda_1(-L_{\Omega,m,\sigma})=\lambda_p(-L_{\Omega,m,\sigma})\ge-\max_{\overline{\Omega}}a_T-\epsilon,\quad \sigma\ge\sigma_\epsilon.$$

The arbitrariness of ϵ then yields (6.2). Hence, $\lim_{\sigma\to\infty}\lambda_1(-L_{\Omega,m,\sigma})=-\max_{\overline{\Omega}}a_T$.

Now, we prove the result in the case $m=0$. Proposition 6.1 ensures $\lambda_1(-L_{\Omega,0,\sigma})<D-\max_{\overline{\Omega}}a_T$. To finish the proof, it suffices to show that

$$\liminf_{\sigma\to\infty}\lambda_1(-L_{\Omega,0,\sigma})\ge D-\max_{\overline{\Omega}}a_T. \tag{6.5}$$

Let ϕ be defined as in (6.3). For any $\epsilon>0$, we see that for each $(t,x)\in\mathbb{R}\times\overline{\Omega}$,

$$\left(L_{\Omega,0,\sigma}-\max_{\overline{\Omega}}a_T-\epsilon\right)[\phi](t,x)$$
$$=-\phi_t(t,x)+D\left[\int_\Omega J_\sigma(x-y)\phi(t,y)dy-\phi(t,x)\right]+\left[a(t,x)-\max_{\overline{\Omega}}a_T-\epsilon\right]\phi(t,x)$$
$$\le D\left[\int_\Omega J_\sigma(x-y)\phi(t,y)dy-\phi(t,x)\right]-\epsilon\phi(t,x).$$

Hence, for each $\epsilon>0$, there holds

$$\left(L_{\Omega,0,\sigma}+D-\max_{\overline{\Omega}}a_T-\epsilon\right)[\phi](t,x)\le D\int_\Omega J_\sigma(x-y)\phi(t,y)dy-\epsilon\phi(t,x),$$
$$\forall (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

As $\left\|\int_\Omega J_\sigma(\cdot-y)\phi(t,y)dy\right\|_\infty\to 0$ as $\sigma\to\infty$, we can follow the arguments as in the case $m>0$ to conclude (6.5).

(2) Recall that $m \in [0, 2)$. Let ϕ be as in (6.3). Since $a \in C^{1,4}(\mathbb{R} \times \overline{\Omega})$, there holds $\phi \in C^{1,4}(\mathbb{R} \times \overline{\Omega})$. Let $\tilde{\phi} \in C^{1,4}(\mathbb{R} \times \mathbb{R}^N)$ be positive and T-periodic, and satisfy $\tilde{\phi}(t, x) = \phi(t, x)$ for $(t, x) \in \mathbb{R} \times \overline{\Omega}$. For any $\epsilon > 0$, similar arguments as in (6.4) lead to

$$
\begin{aligned}
&\left(L_{\Omega,m,\sigma} - \max_{\overline{\Omega}} a_T - \epsilon\right)[\phi](t, x) \\
&\quad \le \frac{D}{\sigma^m}\left[\int_{\Omega} J_\sigma(x - y)\phi(t, y)dy - \phi(t, x)\right] - \epsilon\phi(t, x) \\
&\quad \le \frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J_\sigma(x - y)\tilde{\phi}(t, y)dy - \tilde{\phi}(t, x)\right] - \epsilon\phi(t, x) \\
&\quad = \frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J(z)\tilde{\phi}(t, x + \sigma z)dz - \tilde{\phi}(t, x)\right] - \epsilon\phi(t, x) \\
&\quad = \frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J(z)\left(\tilde{\phi}(t, x + \sigma z) - \tilde{\phi}(t, x)\right)dz\right] - \epsilon\phi(t, x), \qquad (t, x) \in \mathbb{R} \times \overline{\Omega}.
\end{aligned}
$$

By the fourth-order Taylor's expansion with remainder, we find

$$
\tilde{\phi}(t, x + \sigma z) - \tilde{\phi}(t, x) = \sum_{1 \le |\alpha| \le 3} \frac{\partial^\alpha \tilde{\phi}(t, x)}{\alpha!}\sigma^{|\alpha|}z^\alpha + \sigma^4 \sum_{|\alpha|=4} R_\alpha(t, x)z^\alpha,
$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is the usual multiple index, and

$$
R_\alpha(t, x) = \frac{4}{\alpha!}\int_0^1 (1 - s)^3 \partial^\alpha \tilde{\phi}(t, x + s\sigma z)ds.
$$

Since J is symmetric with respect to each component, there hold $\int_{\mathbb{R}^N} J(z)z^\alpha dz = 0$ for $|\alpha| = 1$ or 3 and $\int_{\mathbb{R}^N} J(z)z_i z_j dz = 0$ for $i \ne j$. Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} J(z)\left(\tilde{\phi}(t, x + \sigma z) - \tilde{\phi}(t, x)\right)dz \\
&\qquad = \sigma^2 \sum_{i=1}^N \frac{\partial_{x_i}^2 \tilde{\phi}(t, x)}{2} \int_{\mathbb{R}^N} J(z)z_i^2 dz + \sigma^4 \sum_{|\alpha|=4} R_\alpha(t, x) \int_{\mathbb{R}^N} J(z)z^\alpha dz.
\end{aligned}
$$

It follows that

$$\frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J(z)\left(\tilde{\phi}(t,x+\sigma z)-\tilde{\phi}(t,x)\right)dz\right]$$
$$=\frac{D\sigma^{2-m}}{2}\sum_{i=1}^{N}\partial_{x_i}^2\tilde{\phi}(t,x)\int_{\mathbb{R}^N} J(z)z_i^2dz+D\sigma^{4-m}\sum_{|\alpha|=4}R_\alpha(t,x)\int_{\mathbb{R}^N} J(z)z^\alpha dz.$$

Since $\tilde{\phi}\in C^{1,4}(\mathbb{R}\times\mathbb{R}^N)$ and J is compactly supported, there holds the boundedness of the functions $(t,x)\mapsto\sum_{i=1}^N\partial_{x_i}^2\tilde{\phi}(t,x)\int_{\mathbb{R}^N}J(z)z_i^2dz$ and $(t,x)\mapsto\sum_{|\alpha|=4}R_\alpha(t,x)\int_{\mathbb{R}^N}J(z)z^\alpha dz$. It follows from the assumption $m\in[0,2)$ that

$$\frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J(z)\left(\tilde{\phi}(t,x+\sigma z)-\tilde{\phi}(t,x)\right)dz\right]\to 0\quad\text{as}\quad\sigma\to 0\quad\text{uniformly in }(t,x)\in\mathbb{R}\times\overline{\Omega}.$$

As $\min_{\mathbb{R}\times\overline{\Omega}}\phi>0$, for any $\epsilon>0$, there exists $\sigma_\epsilon>0$ such that

$$\left(L_{\Omega,m,\sigma}-\max_{\overline{\Omega}}a_T-\epsilon\right)[\phi]\le 0\quad\text{in}\quad\mathbb{R}\times\overline{\Omega},\quad 0<\sigma\le\sigma_\epsilon.$$

By the definition of $\lambda_p(-L_{\Omega,m,\sigma})$ and Proposition 6.1(2), we obtain

$$\liminf_{\sigma\to 0^+}\lambda_1(-L_{\Omega,m,\sigma})=\liminf_{\sigma\to 0^+}\lambda_p(-L_{\Omega,m,\sigma})\ge-\max_{\overline{\Omega}}a_T. \tag{6.6}$$

We show the reverse inequality, namely,

$$\limsup_{\sigma\to 0^+}\lambda_1(-L_{\Omega,m,\sigma})\le-\max_{\overline{\Omega}}a_T. \tag{6.7}$$

For any $\epsilon>0$, there exists an open ball $B_\epsilon\subset\Omega$ of radius ϵ such that $a_T+\epsilon>\max_{\overline{\Omega}}a_T$ in B_ϵ. Let $\tilde{\phi}_\epsilon\in C^{1,4}(\mathbb{R}\times\mathbb{R}^N)$ be non-negative and T-periodic, and satisfy

$$\tilde{\phi}_\epsilon=\phi\text{ in }\mathbb{R}\times\overline{B}_\epsilon,\quad\tilde{\phi}_\epsilon=0\text{ in }\mathbb{R}\times(\mathbb{R}^N\backslash B_{2\epsilon})\quad\text{and}\quad\sup_{\mathbb{R}\times\mathbb{R}^N}\tilde{\phi}_\epsilon\le\sup_{\mathbb{R}\times\mathbb{R}^N}\phi=1.$$

Using the fourth-order Taylor's expansion with remainder as above, we find

$$\frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J_\sigma(x-y)\tilde{\phi}_\epsilon(t,y)dy-\tilde{\phi}_\epsilon(t,x)\right]$$
$$=\frac{D\sigma^{2-m}}{2}\sum_{i=1}^{N}\partial_{x_i}^2\tilde{\phi}_\epsilon(t,x)\int_{\mathbb{R}^N} J(z)z_i^2dz+O(\sigma^{4-m}),\quad(t,x)\in\mathbb{R}\times\mathbb{R}^N,$$

where

$$\mathcal{I}^1_{\epsilon,\sigma}(t,x) = D\sigma^{4-m} \sum_{|\alpha|=4} \frac{4}{\alpha!} \int_0^1 (1-s)^3 \partial^\alpha \tilde{\phi}_\epsilon(t, x+s\sigma z) ds \int_{\mathbb{R}^N} J(z) z^\alpha dz.$$

Therefore, we find that for $(t,x) \in \mathbb{R} \times B_\epsilon$,

$$\begin{aligned}
&\left(L_{B_\epsilon,m,\sigma} - \max_{\overline{\Omega}} a_T + \epsilon + \frac{1}{|\ln \epsilon|}\right)[\phi](t,x) \\
&= -\phi_t(t,x) + \frac{D}{\sigma^m}\left[\int_{B_\epsilon} J_\sigma(x-y)\phi(t,y)dy - \phi(t,x)\right] \\
&\quad + \left[a(t,x) - \max_{\overline{\Omega}} a_T + \epsilon + \frac{1}{|\ln \epsilon|}\right]\phi(t,x) \\
&\geq \frac{D}{\sigma^m}\left[\int_{B_\epsilon} J_\sigma(x-y)\phi(t,y)dy - \phi(t,x)\right] + \frac{\phi(t,x)}{|\ln \epsilon|}. \\
&= \frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J_\sigma(x-y)\tilde{\phi}_\epsilon(t,y)dy - \tilde{\phi}_\epsilon(t,x) - \int_{B_{2\epsilon}\setminus B_\epsilon} J_\sigma(x-y)\tilde{\phi}_\epsilon(t,y)dy\right] + \frac{\phi(t,x)}{|\ln \epsilon|} \\
&= \frac{D\sigma^{2-m}}{2}\sum_{i=1}^N \partial^2_{x_i}\tilde{\phi}_\epsilon(t,x)\int_{\mathbb{R}^N} J(z) z_i^2 dz + \mathcal{I}^1_{\epsilon,\sigma}(t,x) \\
&\quad - \frac{D}{\sigma^{m+N}} \int_{B_{2\epsilon}\setminus B_\epsilon} J\left(\frac{x-y}{\sigma}\right)\tilde{\phi}_\epsilon(t,y)dy + \frac{\phi(t,x)}{|\ln \epsilon|} \\
&= \mathcal{I}^1_{\epsilon,\sigma}(t,x) + \mathcal{I}^2_{\epsilon,\sigma}(t,x) + \mathcal{I}^3_{\epsilon,\sigma}(t,x) + \mathcal{I}^4_\epsilon(t,x),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}^2_{\epsilon,\sigma}(t,x) &= \frac{D\sigma^{2-m}}{2}\sum_{i=1}^N \partial^2_{x_i}\tilde{\phi}_\epsilon(t,x)\int_{\mathbb{R}^N} J(z) z_i^2 dz, \\
\mathcal{I}^3_{\epsilon,\sigma}(t,x) &= -\frac{D}{\sigma^{m+N}} \int_{B_{2\epsilon}\setminus B_\epsilon} J\left(\frac{x-y}{\sigma}\right)\tilde{\phi}_\epsilon(t,y)dy \quad \text{and} \\
\mathcal{I}^4_\epsilon(t,x) &= \frac{\phi(t,x)}{|\ln \epsilon|}.
\end{aligned}$$

Note that $\min_{\mathbb{R}\times\overline{B}_\epsilon} \phi(t,x) \geq \min_{\mathbb{R}\times\overline{\Omega}} \phi > 0$ for all $0<\epsilon\ll 1$. Choosing $\epsilon=\sigma^k$ with $k=\frac{m+2N}{N}$, we see that the following estimates hold: for $0<\sigma\ll 1$,

$$\sup_{\mathbb{R}\times\mathbb{R}^N} |\mathcal{I}^1_{\epsilon,\sigma}| \leq C_4\sigma^{4-m}, \quad \sup_{\mathbb{R}\times\mathbb{R}^N} |\mathcal{I}^2_{\epsilon,\sigma}| \leq C_1\sigma^{2-m}, \quad \sup_{\mathbb{R}\times\mathbb{R}^N} |\mathcal{I}^3_{\epsilon,\sigma}| \leq \frac{C_2\sigma^{kN}}{\sigma^{m+N}},$$
$$\inf_{\mathbb{R}\times\mathbb{R}^N} \mathcal{I}^4_\epsilon \geq \frac{C_3}{|\ln(\sigma^k)|},$$

where $C_i>0$ for $i=1,2,3,4$. As $\lim_{\sigma\to 0^+}\sigma^\beta|\ln\sigma|=0$ for any $\beta>0$, the term $\mathcal{I}^4_\epsilon$ dominates $\mathcal{I}^1_{\epsilon,\sigma}$, $\mathcal{I}^2_{\epsilon,\sigma}$ and $\mathcal{I}^3_{\epsilon,\sigma}$ for all small σ. Thus, there holds

$$\left(L_{B_{\sigma^k},m,\sigma} - \max_{\overline{\Omega}} a_T + \sigma^k + \frac{1}{|\ln(\sigma^k)|}\right)[\phi] \geq 0 \quad \text{in} \quad \mathbb{R}\times B_{\sigma^k}, \quad 0<\sigma\ll 1.$$

It then follows from the definition of $\lambda'_p(-L_{B_{\sigma^k},m,\sigma})$ and Proposition 6.1(2) that

$$\lambda_1(-L_{B_{\sigma^k},m,\sigma}) = \lambda'_p(-L_{B_{\sigma^k},m,\sigma}) \leq -\max_{\overline{\Omega}} a_T + \sigma^k + \frac{1}{|\ln(\sigma^k)|}, \quad 0<\sigma\ll 1.$$

By Proposition 6.1(2)(4), $\lambda_1(-L_{\Omega,m,\sigma}) \leq \lambda_1(-L_{B_{\sigma^k},m,\sigma})$, which yields

$$\lambda_1(-L_{\Omega,m,\sigma}) \leq -\max_{\overline{\Omega}} a_T + \sigma^k + \frac{1}{|\ln(\sigma^k)|}, \quad 0<\sigma\ll 1.$$

Sending $\sigma\to 0$, we prove (6.7).

Combining (6.6) and (6.7), we conclude the expected limit $\lim_{\sigma\to 0^+}\lambda_1(-L_{\Omega,m,\sigma}) = -\max_{\overline{\Omega}} a_T$.

(3) Recall that a is a radially non-increasing function of x. For $\sigma_1\geq\sigma_2$, we show $\lambda_1(-L_{\Omega,0,\sigma_1}) \geq \lambda_1(-L_{\Omega,0,\sigma_2})$. It is equivalent to show $\lambda_p(-L_{\Omega,0,\sigma_1}) \geq \lambda_p(-L_{\Omega,0,\sigma_2})$.

Set $\Omega_\sigma=\frac{1}{\sigma}\Omega$ and $a_\sigma(t,\omega)=a(t,\sigma\omega)$ for $t\in\mathbb{R}$ and $\omega\in\Omega_\sigma$. Clearly, $\lambda_p(-L_{\Omega,0,\sigma}) = \lambda_p(-\mathcal{M}_{\Omega_\sigma,0,1}-a_\sigma)$. Therefore, we need to show that

$$\lambda_p(-\mathcal{M}_{\Omega_{\sigma_1},0,1}-a_{\sigma_1}) \geq \lambda_p(-\mathcal{M}_{\Omega_{\sigma_2},0,1}-a_{\sigma_2}).$$

To do so, it suffices to prove the inequality $\lambda_p(-\mathcal{M}_{\Omega_{\sigma_1},0,1}-a_{\sigma_1}) \geq \lambda$ for any $\lambda < \lambda_p(-\mathcal{M}_{\Omega_{\sigma_2},0,1}-a_{\sigma_2})$.

Fix such a λ. By Proposition 6.1, there exists a positive function $\phi\in\mathcal{X}^{++}_{\Omega_{\sigma_2}}$ such that

$$\mathcal{M}_{\Omega_{\sigma_2},0,1}[\phi] + (a_{\sigma_2}+\lambda)[\phi] \leq 0 \quad \text{in} \quad \mathbb{R}\times\overline{\Omega}_{\sigma_2}.$$

Since Ω contains the origin, there holds $\Omega_{\sigma_1}\subset\Omega_{\sigma_2}$. Moreover, $a_{\sigma_1}(t,x)\leq a_{\sigma_2}(t,x)$ for all $(t,x)\in\mathbb{R}\times\Omega_{\sigma_1}$. Direct computations yield

$$\mathcal{M}_{\Omega_{\sigma_1},0,1}[\phi] + (a_{\sigma_1}(t,x) + \lambda)[\phi] \le \mathcal{M}_{\Omega_{\sigma_2},0,1}[\phi] + (a_{\sigma_2}(t,x) + \lambda)[\phi] \le 0, \quad (t,x) \in \mathbb{R} \times \Omega_{\sigma_1}.$$

This implies $\lambda_p(-\mathcal{M}_{\Omega_{\sigma_1},0,1} - a_{\sigma_1}) \ge \lambda$.

The proof is complete. □

Finally, we study the scaling limits of the positive T-periodic solution of (1.11) and we prove Theorem E. We need the following lemma.

Lemma 6.2. *Suppose* **(H2)**. *If* $\min_{\overline{\Omega}} a_T > 0$*, then for each* $x \in \overline{\Omega}$*, the equation*

$$v_t = f(t,x,v)$$

has a unique positive T*-periodic solution, denoted by* $v^*(t,x)$*, that is continuous in* x*.*

Proof. Fix any $x \in \overline{\Omega}$. Clearly, $v \equiv 0$ is a solution of the equation, and the linearized equation at $v \equiv 0$ reads $\dot{w} = a(t,x)w$. Then, the sign of $a_T(x)$ determines the local stability of $v \equiv 0$. In particular, if $a_T(x) > 0$, $v \equiv 0$ is linearly unstable. By the assumptions on $f(t,x,s)$, the comparison principle and contraction arguments as in the proof of Theorem B, we are able to show the existence of a unique positive T-periodic solution, which is globally asymptotically stable. This is actually also necessary, as it can be shown that $a_T(x) \le 0$ would ensure the global asymptotic stability of $v \equiv 0$, and therefore, the non-existence of positive T-periodic solutions.

If $\min_{\overline{\Omega}} a_T > 0$, then for each $x \in \overline{\Omega}$, there exists a unique positive T-periodic solution $v^*(t,x)$. It is not hard to see that it is continuous in x. □

Theorem E is divided into the following two theorems.

Theorem 6.3. *Suppose* **(H1)** *and* **(H2)**. *Let* $m \in [0,2)$.

(1) *If* $\max_{\overline{\Omega}} a_T > 0$*, then there exists* $\sigma_1 > 0$ *such that for each* $\sigma \in (0,\sigma_1]$*, equation* (1.11) *admits a unique positive solution* $u^*_\sigma \in \mathcal{X}^{++}_\Omega$ *that is globally asymptotically stable.*
(2) *If* $\min_{\overline{\Omega}} a_T > 0$*, then the limit*

$$\lim_{\sigma \to 0^+} u^*_\sigma(t,x) = v^*(t,x) \quad \textit{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega}$$

holds, where v^* *is given in Lemma* 6.2.

Theorem 6.4. *Suppose* **(H1)** *and* **(H2)**. *Let* $m > 0$.

(1) *If* $\max_{\overline{\Omega}} a_T > 0$*, then there exists* $\sigma_2 > 0$ *such that for each* $\sigma \in [\sigma_2,\infty)$*, equation* (1.11) *admits a unique positive solution* $u^*_\sigma \in \mathcal{X}^{++}_\Omega$ *that is globally asymptotically stable.*
(2) *If* $\min_{\overline{\Omega}} a_T > 0$*, then the limit*

$$\lim_{\sigma \to \infty} u^*_\sigma(t,x) = v^*(t,x) \quad \textit{uniformly in} \quad (t,x) \in \mathbb{R} \times \overline{\Omega}$$

holds, where v^* *is given in Lemma* 6.2.

In the rest of this section, we prove Theorem 6.3 and Theorem 6.4.

Proof of Theorem 6.3. (1) It follows from Theorem D(2) and Theorem B.

(2) We claim that for each $0<\epsilon\ll 1$, there exists $\sigma_\epsilon>0$ such that for each $\sigma\in(0,\sigma_\epsilon)$ there holds

$$v^*(t,x)-\epsilon\le u^*_\sigma(t,x)\le v^*(t,x)+\epsilon,\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Let us prove the lower bound; the upper bound follows from similar arguments. Let $0<\epsilon\ll 1$. Since $\min\limits_{\mathbb{R}\times\overline{\Omega}} v^*>0$, there exists $\delta=\delta(\epsilon)>0$ such that

$$v(t,x):=(1-\delta)v^*(t,x)\ge v^*(t,x)-\epsilon>0,\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Note that for each $(t,x)\in\mathbb{R}\times\overline{\Omega}$,

$$\begin{aligned}
&-v_t(t,x)+\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy-v(t,x)\right]+f(t,x,v(t,x))\\
&=-(1-\delta)v^*_t(t,x)+\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy-v^*(t,x)\right]\\
&\quad+(1-\delta)f(t,x,v^*(t,x))\\
&\quad+\left[f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))\right]\\
&=\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy-v^*(t,x)\right]\\
&\quad+\left[f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))\right].
\end{aligned}$$

We see that

$$\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy-v^*(t,x)\right]\to 0\quad\text{as}\quad \sigma\to 0^+$$
$$\text{uniformly in}\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Since for each $(t,x)\in\mathbb{R}\times\overline{\Omega}$,

$$f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))=v(t,x)\left[\frac{f(t,x,v(t,x))}{v(t,x)}-\frac{f(t,x,v^*(t,x))}{v^*(t,x)}\right]>0,$$

where we used **(H2)**-(3), there holds

$$\inf_{(t,x)\in\mathbb{R}\times\overline{\Omega}}\left[f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))\right]>0.$$

Hence, there exists $\sigma_\epsilon > 0$ such that for each $\sigma \in (0, \sigma_\epsilon)$, there holds

$$v_t(t,x) < \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy - v(t,x)\right] + f(t,x,v(t,x)), \quad (t,x) \in \mathbb{R}\times\overline{\Omega}. \tag{6.8}$$

It remains to show that for each $\sigma \in (0, \sigma_\epsilon)$, there holds $v(t,x) \le u^*_\sigma(t,x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$. To do so, let us fix any $\sigma \in (0,\sigma_\epsilon)$ and define

$$\alpha_* = \inf\left\{\alpha > 0 : v(t,x) \le \alpha u^*_\sigma(t,x) \text{ for all } (t,x) \in \mathbb{R}\times\overline{\Omega}\right\}.$$

Since $\min\limits_{\mathbb{R}\times\overline{\Omega}} u^*_\sigma > 0$ and $v(t,x)$ is bounded, α_* is well-defined and positive. Due to the continuity of $v(t,x)$ and $u^*_\sigma(t,x)$, there holds $v(t,x) \le \alpha_* u^*_\sigma(t,x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$. Moreover, there exists $(t_0,x_0) \in \mathbb{R}\times\overline{\Omega}$ such that $v(t_0,x_0) = \alpha_* u^*_\sigma(t_0,x_0)$.

Clearly, if $\alpha_* \le 1$, then we are done. Therefore, let us assume $\alpha_* > 1$. By (6.8) and the equation satisfied by $u^*_\sigma(t,x)$, we see that $w(t,x) := v(t,x) - \alpha_* u^*_\sigma(t,x)$ satisfies

$$w_t(t,x) < \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)w(t,y)dy - w(t,x)\right] + f(t,x,v(t,x)) - \alpha_* f(t,x,u^*_\sigma(t,x)),$$
$$(t,x) \in \mathbb{R}\times\overline{\Omega}.$$

However, since $w_t(t_0,x_0) = 0$, $\int_\Omega J_\sigma(x_0-y)w(t_0,y)dy - w(t_0,x_0) \le 0$ and

$$f(t_0,x_0,v(t_0,x_0)) - \alpha_* f(t_0,x_0,u^*_\sigma(t_0,x_0)) < f(t_0,x_0,v(t_0,x_0)) - f(t_0,x_0,\alpha_* u^*_\sigma(t_0,x_0)) = 0,$$

where we used $\alpha_* > 1$ so that $\alpha_* u^*_\sigma(t_0,x_0) > u^*_\sigma(t_0,x_0)$, and hence,

$$\frac{f(t_0,x_0,u^*_\sigma(t_0,x_0))}{u^*_\sigma(t_0,x_0)} > \frac{f(t_0,x_0,\alpha_* u^*_\sigma(t_0,x_0))}{\alpha_* u^*_\sigma(t_0,x_0)}$$

by **(H2)**-(3), we arrive at

$$w_t(t_0,x_0) < \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x_0-y)w(t_0,y)dy - w(t_0,x_0)\right] + f(t_0,x_0,v(t_0,x_0)) - \alpha_* f(t_0,x_0,u^*_\sigma(t_0,x_0)),$$

which leads to a contradiction. Hence, $\alpha_* \le 1$ and the proof is completed. □

Proof of Theorem 6.4. (1) It follows from Theorem D(1) and Theorem B.

(2) It suffices to show that for each $0 < \epsilon \ll 1$, there exists $\sigma_\epsilon > 0$ such that for each $\sigma \in (0, \sigma_\epsilon)$ there holds

$$v^*(t,x) - \epsilon \le u^*_\sigma(t,x) \le v^*(t,x) + \epsilon, \quad (t,x) \in \mathbb{R}\times\overline{\Omega}.$$

As the proof follows from arguments as in the proof of Theorem 6.3, we here outline the proof of the lower bound.

Let $0<\epsilon\ll 1$. Since $\min_{\mathbb{R}\times\overline{\Omega}} v^*>0$, there exists $\delta=\delta(\epsilon)>0$ such that

$$v(t,x):=(1-\delta)v^*(t,x)\geq v^*(t,x)-\epsilon>0,\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Direct calculations using the equation satisfied by $v^*(t,x)$ ensure that for each $(t,x)\in\mathbb{R}\times\overline{\Omega}$,

$$\begin{aligned}&-v_t(t,x)+\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy-v(t,x)\right]+f(t,x,v(t,x))\\&=\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy-v^*(t,x)\right]\\&\quad+\left[f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))\right].\end{aligned}$$

Obviously,

$$\frac{D(1-\delta)}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v^*(t,y)dy-v^*(t,x)\right]\to 0\quad\text{as}\quad\sigma\to\infty$$
$$\text{uniformly in}\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Arguing as in the proof of Theorem 6.3, we find

$$\inf_{(t,x)\in\mathbb{R}\times\overline{\Omega}}\left[f(t,x,v(t,x))-(1-\delta)f(t,x,v^*(t,x))\right]>0.$$

Hence, there exists $\sigma_\epsilon>0$ such that for each $\sigma>\sigma_\epsilon$, there holds

$$v_t(t,x)<\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)v(t,y)dy-v(t,x)\right]+f(t,x,v(t,x)),\quad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

Now, we can argue as in the proof of Theorem 6.3 to conclude that for each $\sigma>\sigma_\epsilon$, there holds $v(t,x)\leq u^*_\sigma(t,x)$ for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$. This completes the proof. □

Remark 6.5. The asymptotic behaviors of the positive T-periodic solution of the reaction-diffusion equation (1.12) for small diffusion rates, i.e., as $d\to 0^+$, has been studied by Daners and López-Gómez in [14].

7. Maximum principle

We prove the maximum principle, namely, Theorem F, for the operator L_Ω defined in (1.3) with a general $a \in C_T(\mathbb{R} \times \overline{\Omega})$.

Proof of Theorem F. By Theorem 1.2, $\lambda_1 := \lambda_1(-L_\Omega)$ is the principal eigenvalue of $-L_\Omega$. Let $\phi \in \mathcal{X}_\Omega^{++}$ be an associated eigenfunction. Then,

$$L_\Omega[\phi] + \lambda_1 \phi = 0. \tag{7.1}$$

We first prove the sufficiency, that is, $\lambda_1 \geq 0$ implies the maximum principle. To do so, let $u \in C^{1,0}([0,T] \times \overline{\Omega})$ be nonzero and satisfy (1.13). Define $w := \frac{u}{\phi}$. Then, simple calculations using (7.1) give

$$0 \geq L_\Omega[u] = L_\Omega[w\phi] = -w_t\phi + D\int_\Omega J(x-y)\phi(t,y)[w(t,y) - w(t,x)]dy - \lambda_1 w\phi.$$

Arguing by contradiction, let us assume that there exists $(t_0, x_0) \in (0,T] \times \Omega$ such that $u(t_0,x_0) = \min\limits_{[0,T]\times\Omega} u \leq 0$. Then, there exists $(t_1,x_1) \in (0,T] \times \Omega$ such that $w(t_1,x_1) = \min\limits_{[0,T]\times\Omega} w \leq 0$. It then follows

$$\begin{aligned} -w_t(t_1,x_1)\phi(t_1,x_1) &\geq 0, \\ D\int_\Omega J(x_1-y)\phi(t_1,y)[w(t_1,y) - w(t_1,x_1)]dy &> 0 \quad \text{and} \\ -\lambda_1 w(t_1,x_1)\phi(t_1,x_1) &\geq 0. \end{aligned}$$

Hence, we conclude that $L_\Omega[u](t_1,x_1) > 0$, which leads to a contradiction. This proves the sufficiency.

Now, we prove the necessity, that is, the maximum principle implies $\lambda_1 \geq 0$. For contradiction, let us assume $\lambda_1 < 0$. Let $\Omega_0 \subset\subset \Omega$. The size of Ω_0 will be specified later.

Let $\eta : \overline{\Omega} \to [0,1]$ be continuous and satisfy

$$\eta(x) = \begin{cases} 0, & x \in \partial\Omega, \\ 1, & x \in \Omega_0. \end{cases}$$

We calculate

$$L_\Omega[\eta\phi](t,x) = D\int_\Omega J(x-y)\phi(t,y)[\eta(y) - \eta(x)]dy - \lambda_1 \eta(x)\phi(t,x).$$

There are three cases.

(1) If $x \in \Omega_0$, we have

$$L_\Omega[\eta\phi](t,x) = D \int\limits_{\Omega\setminus\Omega_0} J(x-y)\phi(t,y)[\eta(y)-1]dy - \lambda_1\phi(t,x)$$
$$\geq -D\|J\|_\infty\|\phi\|_\infty|\Omega\setminus\Omega_0| - \lambda_1 \inf_{[0,T]\times\Omega}\phi(t,x).$$

(2) If $x \in \Omega\setminus\Omega_0$ and $\eta(x) \geq \frac{1}{2}$, we find

$$L_\Omega[\eta\phi](t,x) \geq D \int\limits_{\{y:\eta(y)\leq\eta(x)\}} J(x-y)\phi(t,y)[\eta(y)-\eta(x)]dy - \frac{1}{2}\lambda_1\phi(t,x)$$
$$\geq -D \int\limits_{\{y:\eta(y)\leq\eta(x)\}} J(x-y)\phi(t,y)dy - \frac{\lambda_1}{2}\phi(t,x)$$
$$\geq -D\|J\|_\infty\|\phi\|_\infty|\Omega\setminus\Omega_0| - \frac{\lambda_1}{2}\inf_{[0,T]\times\Omega}\phi(t,x),$$

where we used the fact that $\{y : \eta(y) \leq \eta(x)\} \subset \Omega\setminus\Omega_0$.

(3) If $x \in \Omega\setminus\Omega_0$ and $\eta(x) \leq \frac{1}{2}$, then

$$L_\Omega[\eta\phi](t,x) \geq D\int\limits_{\Omega_0} J(x-y)\phi(t,y)[\eta(y)-\eta(x)]dy$$
$$+ D\int\limits_{\Omega\setminus\Omega_0} J(x-y)\phi(t,y)[\eta(y)-\eta(x)]dy$$
$$\geq \frac{D}{2}\int\limits_{\Omega_0} J(x-y)\phi(t,y)dy - \frac{D}{2}\int\limits_{\Omega\setminus\Omega_0} J(x-y)\phi(t,y)dy$$
$$\geq \frac{D}{2}\left[\inf_{(t,x)\in[0,T]\times(\Omega\setminus\Omega_0)}\int\limits_{\Omega_0} J(x-y)\phi(t,y)dy - \|J\|_\infty\|\phi\|_\infty|\Omega\setminus\Omega_0|\right]$$

As $J(0) > 0$, it is easy to choose Ω_0, say sufficiently close to Ω, such that $L_\Omega[\eta\phi] \geq 0$. Now, we have the following:

$$\begin{cases} L_\Omega[-\eta\phi] \leq 0 & \text{in } (0,T]\times\Omega, \\ -\eta\phi = 0 & \text{on } (0,T]\times\partial\Omega, \\ (-\eta\phi)(0,\cdot) = (-\eta\phi)(T,\cdot). \end{cases}$$

Then, applying the comparison principle, we conclude that $-\eta\phi > 0$ in $[0,T]\times\Omega$, which clearly is a contradiction. □

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Appendix A. Counterexamples

We construct examples of J and $a(t,x)$ so that the operator $-L_\Omega$ defined in (1.3) admits no eigenvalue with an eigenfunction in $\mathcal{X}_\Omega^+\backslash\{0\}$. In particular, $-L_\Omega$ admits no principal eigenvalue. To do so, let

$$a(t,x) = \alpha(t) + \beta(x), \quad (t,x) \in \mathbb{R} \times \overline{\Omega},$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function and $\beta : \overline{\Omega} \to \mathbb{R}$ is a continuous function. Set $\alpha_T = \frac{1}{T}\int_0^T \alpha(t)dt$. Then, $a_T(x) = \alpha_T + \beta(x)$ for $x \in \overline{\Omega}$.

Let us suppose that $-L_\Omega$ admits an eigenvalue λ_1 with an eigenfunction $\phi \in \mathcal{X}_\Omega^+\backslash\{0\}$. Then, there must hold $\phi \in \mathcal{X}_\Omega^{++}$. Let us define $\psi(t,x) = e^{\int_0^t [\alpha(s)-\alpha_T]ds}\phi(t,x)$ for $(t,x) \in \mathbb{R} \times \overline{\Omega}$. It is easy to see that $\psi \in \mathcal{X}_\Omega^{++}$. Moreover, multiplying the equation $L_\Omega[\phi] + \lambda_1\phi = 0$ by the function $t \mapsto e^{\int_0^t [\alpha(s)-\alpha_T]ds}$, we find

$$-\psi_t(t,x) + \int_\Omega J(x-y)\psi(t,y)dy - \psi(t,x) + [\alpha_T + \beta(x)]\,\psi(t,x) + \lambda_1\psi(t,x) = 0,$$
$$(t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Setting $\psi_T(x) = \frac{1}{T}\int_0^T \psi(t,x)dx$ for $x \in \overline{\Omega}$, and integrating the above equation over $[0,T]$ with respect to t results in

$$\int_\Omega J(x-y)\psi_T(y)dy - \psi_T(x) + [\alpha_T + \beta(x)]\psi_T(x) + \lambda_1\psi_T(x) = 0, \quad x \in \overline{\Omega}.$$

That is, $-\lambda_1$ is an eigenvalue of the following (bounded) nonlocal elliptic-type operator

$$L_{J,\alpha_T,\beta} : X_\Omega \to X_\Omega,$$
$$v \mapsto \int_\Omega J(\cdot - y)v(y)dy - v(x) + [\alpha_T + \beta]v$$

with an eigenfunction $\psi_T \in X_\Omega^{++}$.

We know from [10] (also see [36]) that J and β can be explicitly constructed so that the operator $L_{J,\alpha_T,\beta}$ admits no eigenvalue with an eigenfunction in $X_\Omega^+\backslash\{0\}$. Therefore, for such J and β, the operator $-L_\Omega$ does not admit an eigenvalue with an eigenfunction in $\mathcal{X}_\Omega^+\backslash\{0\}$.

We summarize the above analysis in the following result.

Theorem A.1. *Let J satisfy $J \equiv \rho$ on Ω, and set $\beta_{\max} = \max_{x\in\bar{\Omega}} \beta(x)$. If $\rho \int_\Omega \frac{1}{\beta_{\max}-\beta(x)} dx < 1$, then $-L_\Omega$ admits no eigenvalue with an eigenfunction in $\mathcal{X}_\Omega^+\backslash\{0\}$. In particular, $-L_\Omega$ admits no principal eigenvalue.*

Proof. By [10, Theorem 5.1], the operator $L_{J,\alpha_T,\beta}$ admits no eigenvalue with an eigenfunction in $X_\Omega^+\backslash\{0\}$. The theorem then follows from the above analysis. □

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