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Global dynamics of an infinite dimensional epidemic model with nonlocal state structures

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Abstract

In this paper, we derive and analyze a state-structured epidemic model for infectious diseases in which the state structure is nonlocal. The state is a measure of infectivity of infected individuals or the intensity of viral replications in infected cells. The model gives rise to a system of nonlinear integro-differential equations with a nonlocal term. We establish the well-posedness and dissipativity of the associated nonlinear semigroup. We establish an equivalent principal spectral condition between the linearized operator and the next-generation operator and show that the basic reproduction number $\mathcal{R}_0$ is a sharp threshold: if $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable, and if $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and a unique endemic equilibrium is globally asymptotically stable. The proof of global stability of the endemic equilibrium utilizes a global Lyapunov function whose construction was motivated by the graph-theoretic method for coupled systems on networks developed in [24].

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1. Introduction

In mathematical modeling of transmission dynamics of infectious diseases, the state of the infected individuals is a main source of population heterogeneity and model complexity. The state can be measured in the degree of individual infectivity. In the HIV infection, which has a very long infectious period, individual infectivity typically varies with the level of viral load. For Tuberculosis, the infectivity varies with the bacteria load in the lung that can be released through coughing. For Malaria, infectivity of human host varies with the parasitoid load in the blood that can be taken in a mosquito bite. In many common viral infections such as influenza and hepatitis, infected individuals can be in an asymptomatic state or a state of active disease. In in-host models of viral dynamics, the state of infected target cell can be measured by the level of viral replication in the cell. There are natural variations of the infectivity in an infected host or viral replications in an infected cell as the infection progresses or due to changes in a cell's life cycle. Such variations cause a change in the state of the individual host or cell. A change in the state can also be the result of medical interventions that change load of pathogens or levels of viral replication. In a more general sense, spatial location of an individual can also be regarded as a state, and the spatial movements of hosts create state changes.

Discrete state structures in epidemic models have been extensively modeled in the literature using large-scale systems of ordinary or delay differential equations. These include but not limited to multi-stage models for disease progression and amelioration [12,15,16,18,29,31], multi-patch models for discrete dispersal among populations on patches [3,23,24,41,44], and many classical multi-compartment models for epidemics and for viral dynamics. The transfer or movement of individuals among compartments in these models are not necessarily limited to the nearest neighbors and are often nonlocal in nature. Threshold conditions in terms of the basic reproduction number, existence and uniqueness, as well as local and global stability of endemic equilibrium are well studied in the setting of discrete state structure. The development of graph-theoretic approach to the construction of Lyapunov functions [14,15,24] have proven to be successful for establishing the global stability of the endemic equilibrium for this type of complex models. It is natural to generalize discrete state structure to continuous structures and consider continuous models using integro-differential equations and investigate the impact of nonlocal changes in the states on the global dynamics.

To derive a continuous state model for the disease infection in a host population, we partition the population into compartments of susceptible, infected, and recovered individuals. The number of susceptible and recovered individuals at time t is denoted by $S(t)$ and $R(t)$, respectively, and the number of infected individuals at state $x \in \Omega$ is denoted by $I(t, x)$, where $\Omega \subset \mathbb{R}^N$. In the absence of infection, we assume that the population dynamics are described by a nonlinear differential equation $S'(t) = \Lambda(S(t))$ for a typical demography function $\Lambda(S)$. We assume that the disease transmission is horizontal through direct contact among individuals, and describe the rate of new infections at state x by a nonlinear function $f(x, S(t), I(t, x))$, so that the rate of total number of new infections at time t from all infected individuals is given by

$\int_\Omega f(y, S(t), I(t, y))dy$. The transfer of individuals, per unit time, to the state x from all other states is described by $\int_\Omega \theta(x, y)I(t, y)dy$ with a kernel function $\theta(x, y)$. The total transfer of individuals from state x to all other states is given by $-\gamma(x)I(t, x)$. We can derive the following state-structured SIR model:

$$\begin{aligned}\dot{S}(t) &= \Lambda(S(t)) - \int_\Omega f(y, S(t), I(t, y))dy,\\ I_t(t, x) &= -\kappa(x)I(t, x) + \alpha(x)\int_\Omega f(y, S(t), I(t, y))dy\\ &\quad + \int_\Omega \theta(y, x)I(t, y)dy - \gamma(x)I(t, x) - \delta(x)I(t, x),\\ \dot{R}(t) &= \int_\Omega \delta(x)I(t, x)dx - dR(t).\end{aligned} \tag{1.1}$$

The term $-\kappa(x)I(t, x)$ is the removal from the infected individuals at state x with a state-dependent removal rate $\kappa(x)$ that includes death by natural causes and due to disease. The function $\alpha(x)$ is the fraction of newly infected individuals that enter state x, and it satisfies $\int_\Omega \alpha(x)dx = 1$. The term $\delta(x)I(t, x)$ represents the recovery of people at state x, and the total number of recovered people is given by $\int_\Omega \delta(x)I(t, x)dx$. Recovered individuals are assumed to be permanently immune from the disease. The death among the recovered individuals is given by dR with a constant death rate d. All the constant and state-dependent parameters are assumed to be nonnegative.

Since the first two equations in (1.1) do not contain the variable R, we can consider the following closed subsystem:

$$\begin{aligned}\dot{S}(t) &= \Lambda(S(t)) - \int_\Omega f(y, S(t), I(t, y))dy,\\ I_t(t, x) &= -\kappa(x)I(t, x) + \alpha(x)\int_\Omega f(y, S(t), I(t, y))dy\\ &\quad + \int_\Omega \theta(y, x)I(t, y)dy - \gamma(x)I(t, x).\end{aligned} \tag{1.2}$$

In this system, we have absorbed $\delta(x)$ into $\kappa(x)$ for simplicity of notation. Once behaviors of $S(t)$ and $I(t, x)$ are known, those of $R(t)$ can be derived from the third equation in (1.1).

We denote $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. Let $C(\Omega)$ be the space of real-valued continuous functions on Ω and denote

$$C_+(\Omega) = \{I \in C(\Omega) \ :\ I \geq 0\}, \quad \text{and} \quad C_{++}(\Omega) = \{I \in C(\Omega) \ :\ \inf_\Omega I > 0\},$$

where $C_+(\Omega)$ is the positive cone of $C(\Omega)$ and $C_{++}(\Omega)$ is the interior of $C_+(\Omega)$.

We make the following basic assumptions throughout the paper:

- $\Omega \subset \mathbb{R}^N$ is compact and connected with a smooth boundary and satisfies $\Omega = \overline{\mathrm{int}(\Omega)}$;
- $\Lambda \in C^1(\mathbb{R}, \mathbb{R})$ is non-increasing on $\mathbb{R}_+$, and there exists a unique $S^0 > 0$ such that $\Lambda(S^0) = 0$ and $\Lambda(S) > 0$ for $0 < S < S^0$ and $\Lambda(S) < 0$ for $S > S^0$, and that $\Lambda'(S^0) < 0$;
- $f \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ and satisfies the following properties:
  - $f(x, 0, I) = f(x, S, 0) = 0$ for all $(x, S, I) \in \Omega \times \mathbb{R}^2_+$;
  - for $x \in \Omega$ and $S > 0$, $f(x, S, I)$ is non-decreasing with respect to $I \in \mathbb{R}_+$;
  - for $x \in \Omega$ and $I > 0$, $f(x, S, I)$ is increasing with respect to $S \in \mathbb{R}_+$;
  - for $x \in \Omega$ and $S \geq 0$, $\frac{f(x,S,I)}{I}$ is non-increasing with respect to $I \in \mathbb{R}_{++}$;
- $\kappa \in C_{++}(\Omega)$, and $\alpha \in C_+(\Omega)$ satisfies $\int_\Omega \alpha(x)dx = 1$;
- $\theta \in C(\Omega \times \Omega, \mathbb{R}_+)$ and satisfies $\theta(x, x) > 0$ for all $x \in \Omega$;
- $\gamma \in C_+(\Omega)$ satisfies the balance condition

$$\int_\Omega \theta(x, y)dy = \gamma(x), \quad \forall x \in \Omega. \tag{1.3}$$

These assumptions are biologically motivated. The assumptions on $\Lambda(S)$ are satisfied by simple immigration-death function $\Lambda(S) = \lambda - dS$ and more general class of growth functions

$$\Lambda(S) = \lambda + rS\Big(1 - \frac{S}{K}\Big) - dS$$

with $d \geq r$, in which case S^0 is the positive root of the quadratic equation $\frac{rS^2}{K} - (r - d)S - \lambda = 0$. Assumptions on $f(x, S, I)$ are satisfied by common incidence functions such as nonlinear incidence $\beta(x)S^pI^q$, $p, q \geq 0$, with a state-dependent transmission coefficient $\beta(x)$, and saturation incidences $\frac{\beta(x)S^pI}{1+aS^q}$ or $\frac{\beta(x)SI^p}{1+aI^q}$, $p \geq q > 0$. We also note that assumptions on θ and the balance condition (1.3) implies $\gamma \in C_{++}(\Omega)$. Moreover, the balance condition (1.3) is not essential and it can be replaced by the following weaker condition

$$\int_\Omega \theta(x, y)dy \leq \gamma(x), \quad \forall x \in \Omega. \tag{1.4}$$

In fact, if (1.4) instead of (1.3) is true, then we can replace $\gamma(x)$ by $\tilde{\gamma}(x) = \int_\Omega \theta(x, y)dy$ and absorb $\gamma(x) - \tilde{\gamma}(x) \geq 0$ into $\kappa(x)$.

System (1.1) can be considered as the continuous limit of several classes of discrete epidemic models studied in the literature. If the state of the infected individuals are partitioned into n discrete infected stages $[a_{i-1}, a_i]$, with $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = \infty$, we can denote $I_i(t) = \int_{a_{i-1}}^{a_i} I(t, x)dx$. If the state-dependent parameters are assumed to be constant in each stage $[a_{i-1}, a_i]$, namely, $f(x, S, I) = f_j(S, I)$, $\kappa(x) = \kappa_j$, and $\gamma(x) = \gamma_j$ for $x \in [a_{j-1}, a_j]$, and denote $\int_{a_{i-1}}^{a_i} \alpha(x)dx = \alpha_i$, $\int_{a_{i-1}}^{a_i} \theta(x, y)dy = \phi_{ij}$ for $x \in [a_{j-1}, a_j]$, and $\zeta_j(I_j) = (\kappa_j + \gamma_j)I_j$, we obtain the following system of ODEs for $S(t)$, $I_i(t)$, $R(t)$:

$$
\begin{aligned}
\dot{S} &= \Lambda(S) - \sum_{j=1}^{n} f_j(S, I_j), \\
\dot{I}_1 &= \alpha_1 \sum_{j=1}^{n} f_j(S, I_j) - \sum_{j=1}^{n} \phi_{j1} I_1 + \sum_{j=1}^{n} \phi_{1j} I_j - \zeta_1(I_1) \\
\dot{I}_i &= \alpha_i \sum_{j=1}^{n} f_j(S, I_j) + \sum_{j=1}^{n} \phi_{ij} I_j - \sum_{j=1}^{n} \phi_{ji} I_j - \zeta_i(I_i), \quad i = 2, 3, \ldots, n, \\
\dot{R} &= \zeta_n(I_n) - dR.
\end{aligned}
\tag{1.5}
$$

This is a general multi-stage model in which infected individuals are allowed to transfer among different stages, and new infections are distributed into all stages with probabilities $\alpha_j \geq 0$, $\sum_{j=1}^{n} \alpha_j = 1$. An example of diseases that can be described by this model is the HIV infection, which progresses through an infectious period that can be as long as a decade without antiretroviral treatment (ART). The rate of HIV progression can vary greatly among individuals. It is known that some HIV infected individuals are rapid progressors who can progress to AIDS and death within 1–4 years after the primary infection if not treated [32], while others are long-term non-progressors who show no signs of disease progression for over 12 years and remain asymptomatic (and infectious) [8]. The discrete model (1.5) allows the possibility of both slow and rapid progression since the primary infection. Biologically plausible values of the parameters in model (1.5) may be found in [18] and [31]. Distributing newly infected individuals among different disease stages and classes was also used in models for Tuberculosis in [34] and differential-infectivity models for HIV [18]. Transfers from late stages to earlier stages can be due to the ART treatment which suppresses the viral load and allows the CD4 count to rebound.

In the special case when $\alpha_1 = 1$, and $\alpha_j = 0$ for $j = 2, \cdots, n$, model (1.5) reduces to a multi-stage model considered in [16], which includes as special cases many earlier models of stage progression and stage progression with amelioration, see e.g. [15,18,29,31]. Special cases of stage progression include the classical SEIR models and its variations with multiple E and I compartments, see e.g. [4] and references therein.

The state x can also be considered as a spatial variable. Suppose that the kernel function $\theta(x, y)$ takes the special form $\theta(x, y) = J(x - y)$, where J is a dispersal kernel such as the density of a Gaussian distribution, and denote the convolution operator $\mathcal{J} \star I(x) = \int_\Omega J(x - y)I(y)dy$, then system (1.2) becomes a spatial epidemic model with nonlocal dispersal:

$$
\begin{aligned}
\dot{S} &= \Lambda(S) - \int_\Omega f(y, S, I(t, y))dy, \\
I_t &= -\kappa(x)I + \alpha(x) \int_\Omega f(y, S, I(t, y))dy + \mathcal{J} * I - \gamma(x)I.
\end{aligned}
\tag{1.6}
$$

For treatment of nonlocal dispersal equations, we refer the reader to [13,19,26] and references therein. If the spatial domain consists of finite number of patches, and population on each patch are assumed to be spatially uniform, then model (1.6) can be further reduced to a multi-patch SIR model with dispersal among patches that is described by a system of ODEs, see e.g. [23,24] and references therein.

Other types of continuous structured epidemic models have been proposed and studied in the literature, including age-structured models using PDEs (see e.g. [5,43]) and group-structured models using integral equations (see e.g. [39]). None of these models have incorporated biologically relevant nonlocal effects. Our study was inspired in part by the work of H.R. Thieme [39]. Due to the integral term in the I equation of (1.2), little regularity can be expected for the associated linear semigroup, which results in the lack of regularity of the I component of solutions of (1.2). The dissipativity of I in $L^1(\Omega)$ needs to be passed to the dissipativity in $C(\Omega)$ at each state. Because of the nonlocal term $\int_\Omega \theta(y,x)I(t,y)dy$, several challenges need to be overcome in the mathematical analysis of (1.2). The principal spectral theory of the linearized operator at the disease-free equilibrium needs special treatment due to the lack of regularity. In addition, an equivalent principal spectral condition between the linearized operator and the generalized next-generation operator is obtained. This establishes a direct connection between the stability of the disease-free equilibrium and the threshold condition in term of the basic reproduction number $\mathcal{R}_0$. Due to the nonlocality, the proof of global stability also becomes highly nontrivial. The choice of Lyapunov functions for proving global stability of an endemic equilibrium is informed by those of the corresponding discrete models in [22,15,16,24]. A key assumption in those work on discrete models is the irreducibility of the nonlocal contact matrix, often expressed equivalently as the strong connectedness of the nonlocal contact network. Our conditions on the kernel $\theta(x,y)$ imply the irreducibility for a nonnegative integral operator as discussed in [2], under which the integral operator satisfies the generalized Perron–Frobenius Theorem [2,30].

We provide a summary of our result concerning the global dynamics of (1.2) and give some discussions.

Theorem A. *Let $\mathcal{R}_0$ be the spectral radius of the operator $\mathcal{L}: C(\Omega) \to C(\Omega)$ defined by*

$$\mathcal{L}[I](x) = \frac{\alpha(x)}{\kappa(x)+\gamma(x)} \int_\Omega f_I(y, S^0, 0) I(y)dy + \frac{\int_\Omega \theta(y,x)I(y)dy}{\kappa(x)+\gamma(x)}, \quad x \in \Omega.$$

Then, the following statements hold.

(1) *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium $(S^0, 0)^T$ of (1.2) is globally asymptotically stable in $X_+ := \mathbb{R}_+ \times C_+(\Omega)$.*
(2) *If $\mathcal{R}_0 > 1$, then (1.2) is uniformly persistent and admits an endemic equilibrium $(S^*, I^*)^T$ in X_+ with $I^* \in C_{++}(\Omega)$. If, in addition,*

$$\left[\frac{S^*}{S} - \frac{f(x,S^*,I^*(x))}{f(x,S,I)}\right]\left[\frac{S^*}{S} - \frac{\frac{f(x,S^*,I^*(x))}{I^*(x)}}{\frac{f(x,S,I)}{I}}\right] \le 0, \quad \forall x \in \Omega, \ S, I > 0, \tag{1.7}$$

then $(S^*, I^*)^T$ is globally asymptotically stable in $X_+ \backslash \{(S,0)^T : S \ge 0\}$.

This theorem is a sharp threshold result that characterizes the global dynamics of (1.2) in term of the basic reproduction number $\mathcal{R}_0$. Since our incidence function $f(x,S,I)$ is very general, a condition such as (1.7) is typically required for the uniqueness and global stability of the endemic equilibrium when $\mathcal{R}_0 > 1$. Condition (1.7) is automatically satisfied when $f(x,S,I)$ takes common incidence forms including nonlinear incidence $f(x,S,I) = \beta(x)I^pS$ and saturated nonlinear incidence $f(x,S,I) = \beta(x)\frac{I^pS}{1+\kappa I^p}$, $0 < p \le 1$. It is also known that when such a condition is not satisfied, multiple endemic states are possible and oscillations can occur in simple SEIR epidemic models, see e.g. [25].

The main mathematical tool for proving the global stability of the endemic equilibrium $(S^*, I^*)^T$ is the method of Lyapunov functionals. Our Lyapunov functional

$$V(S, I(\cdot)) := \int_\Omega \eta(x)\left[\alpha(x)S^*\mathscr{G}\left(\frac{S}{S^*}\right) + I^*(x)\mathscr{G}\left(\frac{I(x)}{I^*(x)}\right)\right]dx, \tag{1.8}$$

where $\mathscr{G}(a) = a - 1 - \ln a$ and η is an appropriate function, is inspired by those for the discrete models (e.g. [22,15,16,24]) and for infinite dimensional models ([39]). We note that, similar to the discrete case, the Lyapunov functional in (1.8) is not defined on the whole phase space X_+, but only on the interior region $X_{++} := \mathbb{R}_{++} \times C_{++}(\Omega)$. For this functional to be well-defined, establishing the uniform persistence of the semi-flow $\{\Sigma_t\}_{t\geq 0}$ on X_+ generated by the solutions of (1.2) is a crucial step. The uniform persistence, together with the dissipativity and the asymptotic smoothness of $\{\Sigma_t\}_{t\geq 0}$, ensure that $\{\Sigma_t\}_{t\geq 0}$, when considered as a semi-flow on X_{++}, has a global attractor $\mathcal{A}_0$, which attracts solutions of (1.2) with initial data in $X_+\backslash\{(S^0, 0)^T\}$, and hence, reduces the problem to the dynamics in a bounded neighborhood of $\mathcal{A}_0$, on which the Lyapunov functional is well-defined.

The paper is organized as follows: the well-posedness of the model, dissipativity and positivity of the nonlinear semigroup are shown in the next section, Section 2. In Section 3, we discuss the threshold operator and the basic reproduction number $\mathcal{R}_0$. Section 4 is devoted to the existence and local stability of stationary solutions, including the disease-free equilibrium and endemic equilibria. In Section 5, we establish the global stability of the disease-free equilibrium when $\mathcal{R}_0 < 1$, uniform persistence when $\mathcal{R}_0 > 1$, as well as the uniqueness and global stability of the endemic equilibrium. In Section 6, we present some examples to illustrate our general results.

2. Global well-posedness, dissipativity and positivity

Set $u(t) = (S(t), I(\cdot, t))^T$. Then, we can rewrite the system (1.2) in the following abstract form:

$$\dot{u}(t) = Au(t) + F(u(t)), \tag{2.1}$$

where operators A and F are defined as

$$A\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \Lambda'(S^0)S \\ -\kappa I - \gamma I \end{pmatrix}$$

and

$$F\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \Lambda(S) - \Lambda'(S^0)S - \int_\Omega f(y, S, I(y))dy \\ \alpha \int_\Omega f(y, S, I(y))dy + \int_\Omega \theta(y, \cdot)I(y)dy \end{pmatrix}.$$

We consider equation (2.1) in the Banach space $X = \mathbb{R} \times C(\Omega)$ equipped with the norm

$$\|(S, I)^T\|_X = |S| + \|I\|_{C(\Omega)} = |S| + \sup_{x\in\Omega} |I(x)|, \quad (S, I)^T \in X.$$

Let $X_+ := \mathbb{R}_+ \times C_+(\Omega)$ denote the closed positive cone of X and $X_{++} := \mathbb{R}_{++} \times C_{++}(\Omega)$ the interior of X_+.

The following result establishes the global well-posedness, positivity and dissipativity of the solutions of (2.1) in the space X_+.

Theorem 2.1. *For any $u_0 \in X_+$, there exists a unique global classical solution $u(\cdot\,; u_0) : [0, \infty) \to X_+$ of (2.1) with $u(0\,; u_0) = u_0$. Moreover, the semi-flow defined by*

$$\Sigma_t u_0 = u(t\,; u_0), \quad t \geq 0, \quad u_0 \in X_+$$

is bounded dissipative and asymptotically smooth, and hence, it admits a global attractor in X_+.

In our notation, surpassing the dependence on initial data, the unique global classical solution $u : [0, \infty) \to X_+$ of (2.1) in Theorem 2.1 is written as

$$u(t) = (S(t), I(t, \cdot))^T, \quad t \geq 0,$$

with initial data $u_0 = (S_0, I_0(\cdot))^T$. In the rest of this section, we prove Theorem 2.1.

Lemma 2.2. *The operator A generates a uniformly continuous and strictly contractive semigroup $\{e^{At}\}_{t\geq 0}$ on X preserving X_+, namely, $e^{At} X_+ \subset X_+$ for all $t \geq 0$.*

Proof. As $\Lambda'(S^0) < 0$ by assumption and the semigroup $\{e^{At}\}_{t\geq 0}$ generated by A is given by

$$e^{At}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} e^{\Lambda'(S^0)t} S \\ e^{-(\kappa+\gamma)t} I \end{pmatrix}, \quad t \geq 0,$$

the lemma follows. □

The next lemma establishes the local well-posedness and positivity of (2.1).

Lemma 2.3. *For any $u_0 \in X$, there is a $T_{\max} = T_{\max}(u_0) > 0$ such that (2.1) has a unique classical solution $u(\cdot; u_0) \in C([0, T_{\max}), X) \cap C^1((0, T_{\max}), X)$. Moreover, the following statements hold.*

(1) *If $T_{max} < \infty$, then $\lim_{t\to T_{\max}^-} \|u(t; u_0)\|_X = \infty$.*
(2) *Let $T \in (0, T_{\max})$. If $\{u_n\}_n \subset X$ with $u_n \to u_0$ in X as $n \to \infty$, then $u(t; u_n)$ exists for all $t \in [0, T]$ for all large n, and $\sup_{t\in[0,T]} \|u(t; u_n) - u(t; u_0)\|_X \to 0$ as $n \to \infty$.*
(3) *If $u_0 \in X_+$, then $u(t; u_0) \in X_+$ for all $t \in (0, T_{\max})$.*

Proof. By the assumptions on Λ and f, it is easy to see that $F : X \to X$ is locally Lipschitz. Then, all statements except (3) follow from Lemma 2.2 and classical results (see e.g. [33,7,27]).

The statement (3) follows from well-known results (see e.g. [27]) if we can verify

(i) the resolvent $(\lambda - A)^{-1}$ is positive for all $\lambda \gg 1$;
(ii) for any $R > 0$, there is $\lambda_R > 0$ such that

$$F(u(t)) + \lambda_R u(t) \geq 0, \quad t \geq 0,$$

for all $u \in C(\mathbb{R}_+, X_+ \cap B(0, R))$, where $B(0, R)$ is the open ball in X centered at 0 with radius R.

Since $\{e^{At}\}_{t\geq 0}$ is positive by Lemma 2.2, it is known (see e.g. [11]) that $(\lambda - A)^{-1}$ is positive if and only if $\lambda > s(A)$, where $s(A)$ is the spectral bound of A. Since A is bounded, $s(A) < \infty$, and hence, (i) follows.

For $R > 0$, by the assumptions on Λ and f, we can find a sufficiently large $\lambda_R \gg 1$ such that

$$\begin{aligned} F\begin{pmatrix} S(t) \\ I(t,\cdot) \end{pmatrix} + \lambda_R \begin{pmatrix} S(t) \\ I(t,\cdot) \end{pmatrix} &= \begin{pmatrix} \Lambda(S(t)) - \Lambda'(S^0)S(t) - \int_\Omega f(y, S(t), I(t,y))dy + \lambda_R S(t) \\ \alpha \int_\Omega f(y, S(t), I(t,y))dy + \int_\Omega \theta(y,\cdot)I(t,y)dy + \lambda_R I(t,\cdot) \end{pmatrix} \\ &\geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

for all $(S, I)^T \in C(\mathbb{R}_+, X_+ \cap B(0, R))$. This proves (ii), and completes the proof of the lemma. □

To prove the global well-posedness in X_+, we need the following lemma.

Lemma 2.4. *For $u_0 \in X_+$, let $u(t; u_0)$, $t \in [0, T_{\max}(u_0))$, be the positive classical solution of* (2.1) *given in Lemma 2.3. Then,*

$$\sup_{t\in[0,T_{\max}(u_0))} \|u(t; u_0)\|_X < \infty.$$

Proof. We first establish the uniform estimate in $\mathbb{R} \times L^1(\Omega)$ and then pass it to the uniform estimate in X.

Let $u_0 = (S_0, I_0(\cdot))^T$ and $u(t; u_0) = (S(t), I(t,\cdot))^T$. Then, $(S(t), I(t,\cdot))^T$ satisfies (1.2). Integrating the I-equation in (1.2) over Ω, we obtain from the balance condition (1.3) that

$$\frac{d}{dt}\|I(t,\cdot)\|_{L^1} \leq -(\inf_\Omega \kappa)\|I(t,\cdot)\|_{L^1} + \int_\Omega f(y, S(t), I(t,y))dy.$$

Then, using the S equation in (1.2), we find

$$\frac{d}{dt}(S(t) + \|I(t,\cdot)\|_{L^1}) \leq \Lambda(S(t)) - (\inf_\Omega \kappa)\|I(t,\cdot)\|_{L^1}.$$

It follows that there exist constants $M > 0$ and $C > 0$, independent of u_0, such that, for any $t \in (0, T_{\max}(u_0))$,

$$S(t) + \|I(t,\cdot)\|_{L^1} \geq M \quad \text{implies} \quad \frac{d}{dt}(S(t) + \|I(t,\cdot)\|_{L^1}) \leq -C.$$

In fact, we may choose $M = 2S^0 + \frac{2\Lambda(0)}{\inf_\Omega \kappa}$. Then, when $S(t) + \|I(t,\cdot)\|_{L^1} \geq M$, one of the following holds:

- if $S(t) \geq \frac{M}{2}$, then

$$\frac{d}{dt}(S(t) + \|I(t,\cdot)\|_{L^1}) \leq \Lambda(S(t)) \leq \Lambda(\frac{M}{2}) < 0;$$

- if $\|I(t,\cdot)\|_{L^1} \geq \frac{M}{2}$, then

$$\frac{d}{dt}(S(t) + \|I(t,\cdot)\|_{L^1}) \leq \Lambda(0) - (\inf_\Omega \kappa)(S^0 + \frac{\Lambda(0)}{\inf_\Omega \kappa}) = -S^0 \inf_\Omega \kappa < 0.$$

Setting $C = \min\{-\Lambda(\frac{M}{2}), S^0 \inf_\Omega \kappa\}$, we have verified the claim. As a direct consequence, we have

$$C_1(u_0) := \sup_{t \in [0, T_{\max}(u_0))} (S(t) + \|I(t,\cdot)\|_{L^1}) < \infty. \tag{2.2}$$

In particular, $\sup_{t \in [0, T_{\max}(u_0))} S(t) < \infty$.

It remains to show that

$$\sup_{t \in [0, T_{\max}(u_0))} \|I(t,\cdot)\|_{C(\Omega)} < \infty. \tag{2.3}$$

Note that $\int_\Omega \theta(y,x) I(t,y) dy \leq (\sup_{\Omega\times\Omega} \theta) \|I(t,\cdot)\|_{L^1} \leq (\sup_{\Omega\times\Omega} \theta) C_1(u_0) =: C_2(u_0)$ for all $x \in \Omega$, and

$$\begin{aligned} \alpha(x) \int_\Omega f(y, S(t), I(t,y)) dy &\leq \sup_\Omega \alpha \int_\Omega f_I(y, S(t), 0) I(t,y) dy \\ &\leq \sup_\Omega \alpha \times \sup_{(y,s) \in \Omega \times [0, C_1(u_0)]} f_I(y,s,0) \times C_1(u_0) =: C_3(u_0). \end{aligned}$$

It then follows from the I equation in (1.2) that

$$I_t(t,x) \leq -(\kappa(x) + \gamma(x)) I(t,x) + C_2(u_0) + C_3(u_0),$$

which yields (2.3) as $\inf_\Omega(\kappa + \gamma) > 0$. This completes the proof. $\square$

The following result gives the global well-posedness and dissipativity.

Lemma 2.5. *For any $u_0 \in X_+$, there exists a unique global classical solution $u(\cdot\,; u_0) : [0,\infty) \to X_+$ of (2.1) with $u(0; u_0) = u_0$. Moreover, the semi-flow defined by*

$$\Sigma_t u_0 = u(t\,; u_0), \quad t \geq 0, \quad u_0 \in X_+$$

is bounded dissipative.

Proof. By Lemmas 2.3 and 2.4, we have global well-posedness in X_+. It remains to show the bounded dissipativity, that is, there exists a bounded subset $\mathcal{B} \subset X_+$ such that, for each bounded subset $B \subset X_+$, there exists $t_0 = t_0(B) > 0$ such that $\Sigma_t B \subset \mathcal{B}$ for all $t \geq t_0$.

Let $(S(t), I(t,x))^T$ be the unique solution of (1.2) with initial data $(S_0, I_0(\cdot))^T \in X_+$, namely, $\Sigma_t (S_0, I_0(\cdot))^T = (S(t), I(t,x))^T$. Using the same argument as in the proof of Lemma 2.4, we can show that there exist $M > 0$ and $C > 0$, both independent of u_0, such that, for $t \in (0,\infty)$,

$$S(t) + \|I(t,\cdot)\|_{L^1} \geq M \quad \text{implies} \quad \frac{d}{dt}(S(t) + \|I(t,\cdot)\|_{L^1}) \leq -C.$$

As a result, if $B \subset X_+$ is a bounded set, then there exists $\tilde{t}_0 = \tilde{t}_0(B) > 0$ such that

$$\sup_{(S_0, I_0(\cdot))^T \in B} (S(t) + \|I(t,\cdot)\|_{L^1}) \leq M, \quad t \geq \tilde{t}_0,$$

which implies that, for all $x \in \Omega$ and $t \geq \tilde{t}_0$

$$\begin{aligned} I_t(t,x) &= -\kappa(x) I(t,x) + \alpha(x) \int_\Omega f(y, S(t), I(t,y))dy + \int_\Omega \theta(y,x) I(t,y)dy - \gamma(x) I(t,x) \\ &\leq -\inf_\Omega (\kappa + \gamma) I(t,x) + \sup_\Omega \alpha \int_\Omega f_I(y, S(t), 0) I(t,y)dy + \sup_{\Omega\times\Omega} \theta \int_\Omega I(t,y)dy \\ &\leq -\zeta_0 I(t,x) + M_1, \end{aligned}$$

where $\zeta_0 = \inf_\Omega (\kappa + \gamma)$ and $M_1 = \left[\sup_\Omega \alpha \sup_{(y,S)\in\Omega\times[0,M]} f_I(y, S, 0) + \sup_{\Omega\times\Omega} \theta\right] M$. It then follows that

$$I(t,x) \leq e^{-\zeta_0 (t - \tilde{t}_0)} I(\tilde{t}_0, x) + \frac{M_1}{\zeta_0}, \quad x \in \Omega, \quad t \geq \tilde{t}_0.$$

Since $\sup_{(S_0, I_0(\cdot))^T \in B} \|I(\tilde{t}_0, \cdot)\|_{C(\Omega)} < \infty$ (note that the supremum depends only on B), then we can easily find some $t_0 = t_0(B) \gg \tilde{t}_0$ such that $\|I(t,\cdot)\|_{C(\Omega)} \leq \frac{M_1}{\zeta_0} + 1$ for all $t \geq t_0$. Setting $\mathcal{B} = \{(S, I) \in X_+ : S + \|I\|_{C(\Omega)} \leq M + \frac{M_1}{\zeta_0} + 1\}$, we find $\Sigma_t B \subset \mathcal{B}$ for all $t \geq t_0$. This completes the proof. □

To finish the proof of Theorem 2.1, it remains to show that $\{\Sigma_t\}_{t\geq 0}$ is *asymptotically smooth*, that is, for any closed, bounded and positively invariant set $B \subset X_+$, there exists a compact set $K \subset X_+$ such that $d_H(\Sigma_t B, K) \to 0$ as $t \to \infty$, where d_H is the Hausdorff semi-distance, see e.g. [17].

Lemma 2.6. *The semi-flow $\{\Sigma_t\}_{t\geq 0}$ is asymptotically smooth.*

Proof. We see that

$$\Sigma_t u_0 = e^{At}u_0 + \int_0^t e^{A(t-s)}F(\Sigma_s u_0)ds, \quad t \geq 0, \quad u_0 \in X_+.$$

Since $\{e^{At}\}_{t\geq 0}$ is strictly contractive and F is compact, the lemma follows from results established in [42]. □

3. The threshold operator and the basic reproduction number

In this section, we consider two operators: the first, denoted by L, is derived from the linearization of (1.2) at the disease-free equilibrium $(S^0, 0)^T$; the second is the threshold operator $\mathcal{L}$, which will be used to define the basic reproduction number. We first prove a sufficient and necessary condition for the spectral bound $s(L)$ of L to be the principal eigenvalue of L (Theorem 3.2), which characterizes the local stability of $(S^0, 0)^T$, and then establish a relation between $s(L)$ and the spectral radius $r(\mathcal{L})$ of $\mathcal{L}$ (Theorem 3.6).

3.1. Principal spectral theory

The linearization of (1.2) at the disease-free equilibrium $(S^0, 0)^T$ reads

$$\dot{S}(t) = \Lambda'(S^0)S(t) - \int_\Omega f_I(y, S^0, 0)I(t, y)dy,$$

$$I_t(t, x) = -\kappa(x)I(t, x) + \alpha(x)\int_\Omega f_I(y, S^0, 0)I(t, y)dy + \int_\Omega \theta(y, x)I(t, y)dy - \gamma(x)I(t, x).$$

Note that the second equation in the above system only involves I. It is more efficient to consider the following linear operator associated to the second equation:

$$L[I](x) = -\kappa(x)I(x) + \alpha(x)\int_\Omega f_I(y, S^0, 0)I(y)dy + \int_\Omega \theta(y, x)I(y)dy - \gamma(x)I(x), \quad x \in \Omega. \tag{3.1}$$

We denote by $s(L) := \sup\{\Re\lambda : \lambda \in \sigma(L)\}$ the spectral bound of L and write $L = T + U$, where

$$T[I](x) = -[\kappa(x) + \gamma(x)]I(x), \quad x \in \Omega \tag{3.2}$$

and

$$U[I](x) = \alpha(x)\int_\Omega f_I(y, S^0, 0)I(y)dy + \int_\Omega \theta(y, x)I(y)dy, \quad x \in \Omega. \tag{3.3}$$

Both T and U are bounded linear operators on $C(\Omega)$. Since T is a multiplication operator, $\sigma(T)$ coincides with the range of $-(\kappa+\gamma)$, and $s(T)<0$.

Some basic properties of T and U are presented in the following lemma.

Lemma 3.1. *The following statements hold.*

(1) *T generates a uniformly continuous, positive and uniformly exponentially stable semigroup $\{e^{Tt}\}_{t\geq 0}$ on $C(\Omega)$.*
(2) *U is positive and compact.*

Proof. (1) Since T is bounded, it generates a uniformly continuous semigroup $\{e^{Tt}\}_{t\geq 0}$ given by

$$e^{Tt}[I](x)=e^{-[\kappa(x)+\gamma(x)]t}I(x),\quad x\in\Omega,\quad I\in C(\Omega),\quad t\geq 0.$$

(2) Positivity follows from the definition of U in (3.3). To show compactness, let $B\subset C(\Omega)$ be a bounded set. Then, for $x,z\in\Omega$ and $I\in B$

$$\begin{aligned}|U[I](x)-U[I](z)|&\leq|\alpha(x)-\alpha(z)|\int_\Omega f_I(y,S^0,0)I(y)dy+\int_\Omega|\theta(y,x)-\theta(y,z)||I(y)|dy\\&\leq C_1(|\alpha(x)-\alpha(z)|+\int_\Omega|\theta(y,x)-\theta(y,z)|dy)\end{aligned}$$

for some $C_1>0$ (uniformly in $I\in B$). This shows the equi-continuity of the family $U[B]$, and the compactness of U follows from the Arzelà–Ascoli theorem. $\square$

Theorem 3.2. *The following statements hold.*

(1) *If $s(L)>s(T)$, then $s(L)$ is an isolated and simple eigenvalue of L, whose eigen-space is spanned by some $\phi\in C_{++}(\Omega)$, and if $\lambda\in\sigma(L)$ and $\lambda\neq s(L)$, then $\Re\lambda<s(L)$.*
(2) *Conversely, if there exist $\lambda_p\in\mathbb{R}$ and $\phi_p\in C_{++}(\Omega)$ such that $L[\phi_p]=\lambda_p\phi_p$, then $s(L)=\lambda_p>s(T)$.*

Proof. (1) By Lemma 3.1 and the assumptions on θ, which imply the irreducibility of U, the results follow immediately from [6, Theorem 2.2].

(2) Let us assume that there are $\lambda_p\in\mathbb{R}$ and $\phi_p\in C_{++}(\Omega)$ such that $L\phi_p=\lambda_p\phi_p$. Let $\{e^{Lt}\}_{t\geq 0}$ be the uniformly continuous semigroup on $C(\Omega)$ generated by L. Then, $e^{Lt}\phi_p=e^{\lambda_p t}\phi_p$ for all $t\geq 0$. By Lemma 3.1 and general perturbation results (see e.g. [11, Corollary VI.1.11]), $\{e^{Lt}\}_{t\geq 0}$ is a positive semigroup. As a result, for any $I_0\in C_+(\Omega)$ with $\|I_0\|_{C(\Omega)}\leq 1$, there holds

$$e^{Lt}I_0\leq\frac{1}{\inf_\Omega\phi_p}e^{Lt}\phi_p=\frac{1}{\inf_\Omega\phi_p}e^{\lambda_p t}\phi_p\leq\frac{\sup_\Omega\phi_p}{\inf_\Omega\phi_p}e^{\lambda_p t},\quad t\geq 0,$$

which implies that $\|e^{Lt}\|\leq\frac{\sup_\Omega\phi_p}{\inf_\Omega\phi_p}e^{\lambda_p t}$ for all $t\geq 0$. Since the growth bound of $\{e^{Lt}\}_{t\geq 0}$ coincides with $s(L)$ (see e.g. [1, Theorem C-IV 1.1]), we arrive at $s(L)\leq\lambda_p$, and hence, $s(L)=\lambda_p$.

It remains to show that $\lambda_p > s(T)$. Since

$$\lambda_p\phi_p(x) = -\kappa(x)\phi_p(x) + \alpha(x)\int_\Omega f_I(y, S^0, 0)\phi_p(y)dy + \int_\Omega \theta(y, x)\phi_p(y)dy - \gamma(x)\phi_p(x)$$
$$> -\kappa(x)\phi_p(x) - \gamma(x)\phi_p(x),$$

we find $\lambda_p > \sup_\Omega(-\kappa - \gamma) = s(T)$. $\square$

We remark that by Theorem 3.2, $s(L) > s(T)$ is a sufficient and necessary condition for $s(L)$ to be the principal eigenvalue of L. Similar results have been established for nonlocal operators of the form $u \mapsto \int_\Omega J(\cdot - y)u(y)dy - u + au$, where J is the dispersal kernel and $a = a(x)$ is the essential growth rate, see e.g. [9,37].

Since the local stability of the disease-free equilibrium $(S^0, 0)^T$ is determined by the sign of the principal eigenvalue $s(L)$ of the operator L, we have the following corollary.

Corollary 3.3. *The disease-free equilibrium $(S^0, 0)^T$ is asymptotically stable if $s(L) < 0$ and unstable if $s(L) > 0$.*

3.2. The threshold operator

Motivated by the work [39], we consider the threshold operator, or the generalized next generation operator, $\mathcal{L} : C(\Omega) \to C(\Omega)$ defined by

$$\mathcal{L}[I](x) = \frac{\alpha(x)}{\kappa(x) + \gamma(x)}\int_\Omega f_I(y, S^0, 0)I(y)dy + \frac{\int_\Omega \theta(y, x)I(y)dy}{\kappa(x) + \gamma(x)}, \quad x \in \Omega. \tag{3.4}$$

It can be verified that $\mathcal{L}$ is well-defined, continuous and positive. Note that $\mathcal{L}$ needs not to be strongly positive. The operator $\mathcal{L}$ is said to be *irreducible* if, for any $I \in C_+(\Omega)\backslash\{0\}$ and $\mu \in (C(\Omega))^*_+\backslash\{0\}$, there exists $n_0 = n_0(I, \mu)$ such that $\int_\Omega \mathcal{L}^n[I](x)d\mu(x) > 0$ for all $n \geq n_0$.

Lemma 3.4. *The operator $\mathcal{L}$ is compact and irreducible.*

Proof. We first prove the compactness. Let $B \subset C_+(\Omega)$ be bounded. It follows from our assumptions on f, θ, and α that $\mathcal{L}[B]$ is bounded. We will show that $\mathcal{L}[B]$ is equi-continuous. For $x, z \in \Omega$ and $I \in B$, we have

$$|\mathcal{L}[I](x) - \mathcal{L}[I](z)| \leq \left|\frac{\alpha(x)}{\kappa(x) + \gamma(x)} - \frac{\alpha(z)}{\kappa(z) + \gamma(z)}\right| \int_\Omega f_I(y, S^0, 0)I(y)dy$$
$$+ \frac{\int_\Omega |\theta(y, x) - \theta(y, z)|I(y)dy}{\kappa(x) + \gamma(x)}$$
$$+ \left|\frac{1}{\kappa(x) + \gamma(x)} - \frac{1}{\kappa(z) + \gamma(z)}\right| \int_\Omega \theta(y, z)I(y)dy.$$

It then follows that, for $x, z \in \Omega$

$$\sup_{I\in B}|\mathcal{L}[I](x)-\mathcal{L}[I](z)| \le C_1\left[\left|\frac{\alpha(x)}{\kappa(x)+\gamma(x)}-\frac{\alpha(z)}{\kappa(z)+\gamma(z)}\right|+\int_\Omega|\theta(y,x)-\theta(y,z)|dy\right.$$
$$\left.+\left|\frac{1}{\kappa(x)+\gamma(x)}-\frac{1}{\kappa(z)+\gamma(z)}\right|\right],$$

where constant $C_1>0$ is uniform in $I\in B$. This shows that $\mathcal{L}[B]$ is equi-continuous, and thus pre-compact, by Arzelà–Ascoli Theorem. Hence, $\mathcal{L}$ is compact.

We show that $\mathcal{L}$ is irreducible. By (3.4), we find for $I\in C_+(\Omega)$

$$\mathcal{L}[I](x)\ge\frac{\int_\Omega\theta(y,x)I(y)dy}{\max_\Omega(\kappa+\gamma)},\quad \forall x\in\Omega,$$

which implies that

$$\mathcal{L}^n[I](x)\ge\frac{\int_\Omega\cdots\int_\Omega\int_\Omega\theta(y_{n-1},x)\theta(y_{n-2},y_{n-1})\cdots\theta(y_1,y_2)\theta(y,y_1)I(y)dydy_1\cdots dy_{n-1}}{[\max_\Omega(\kappa+\gamma)]^n},$$
$$\forall x\in\Omega$$

for all $n\ge1$.

By assumptions on θ and the connectedness of Ω, it is easy to see that for each $I\in C_+(\Omega)\backslash\{0\}$, there exists $n_0=n_0(I)$ such that $\mathcal{L}^n[I](x)>0$ for $x\in\Omega$ for all $n\ge n_0$. In fact, let $E_n:=\{x\in\Omega:\mathcal{L}^n[I](x)>0\}$. Then, $E_n\subset E_{n+1}$ for all n and there is n_0 such that $E_n=\Omega$. Hence, if $\mu\in(C(\Omega))_+^*\backslash\{0\}$, then $\int_\Omega\mathcal{L}^n[I](x)d\mu(x)>0$ for all $n\ge n_0$. This, in particular, says that $\mathcal{L}$ is irreducible. □

Due to Lemma 3.4, we can apply results in [35,30] (see [20, Proposition 4.4] for a summary) and arrive at the following result. We point out that in [35,30,20], the irreducibility of $\mathcal{L}$ is said to be non-supporting. Let $r(\mathcal{L})$ denote the spectral radius of operator $\mathcal{L}$.

Lemma 3.5. *The spectral radius $r(\mathcal{L})$ is a positive and algebraically simple eigenvalue of $\mathcal{L}$ with an eigenfunction in $C_{++}(\Omega)$. Moreover, if λ is an eigenvalue of $\mathcal{L}$ with an eigenfunction in $C_+(\Omega)\backslash\{0\}$, then $\lambda=r(\mathcal{L})$.*

We now establish a relation between $s(L)$ and $r(\mathcal{L})$. Note that $\mathcal{L}=id+\frac{1}{\kappa+\gamma}L$ or $L=(\kappa+\gamma)(\mathcal{L}-id)$, where id is the identity operator on $C(\Omega)$.

Theorem 3.6. *$s(L)>0$, $s(L)=0$ and $s(L)<0$ if and only if $r(\mathcal{L})>1$, $r(\mathcal{L})=1$ and $r(\mathcal{L})<1$, respectively.*

Proof. We first show that $s(L)=0$ if and only if $r(\mathcal{L})=1$. Suppose $s(L)=0$. Write $L=T+U$, where T and U are as in (3.2) and (3.3), respectively. Since $s(T)<0$, we can apply Theorem 3.2 (1) to L. In particular, there is $\phi\in C_{++}(\Omega)$ such that $L\phi=0$. It then follows that $\mathcal{L}\phi=\phi+\frac{1}{\kappa+\gamma}L\phi=\phi$, that is, $(1,\phi)$ is an eigen-pair of $\mathcal{L}$, and hence, $r(\mathcal{L})=1$ by Lemma 3.5.

Conversely, suppose $r(\mathcal{L})=1$. Let $\psi\in C_{++}(\Omega)$ be an eigenfunction. Then, $(\mathcal{L}-id)\psi=0$, which implies that $L\psi=(\kappa+\gamma)(\mathcal{L}-id)\psi=0$. We then apply Theorem 3.2(2) to conclude that $s(L)=0$.

It suffices to show that $s(L) > 0$ if and only if $r(\mathcal{L}) > 1$. Suppose $s(L) > 0$. By Theorem 3.2 (1), we have, in particular, some $\phi \in C_{++}(\Omega)$ such that $L\phi = s(L)\phi$. It then follows that

$$\mathcal{L}\phi = \phi + \frac{1}{\kappa+\gamma} L\phi = \phi + s(L)\frac{\phi}{\kappa+\gamma} \geq (1+c_0)\phi,$$

where $c_0 = \frac{s(L)}{\sup_\Omega(\kappa+\gamma)} > 0$. Iterating the above inequality, we find $\mathcal{L}^n\phi \geq (1+c_0)^n\phi$ for all $n \geq 1$, which implies that $\|\mathcal{L}^n\| \geq (1+c_0)^n$, and hence, $\|\mathcal{L}^n\|^{1/n} \geq 1+c_0$ for all $n \geq 1$. As a result, $r(\mathcal{L}) \geq 1+c_0 > 1$.

Conversely, let $r(\mathcal{L}) > 1$ and $\psi \in C_{++}(\Omega)$ be an eigenfunction. Then, $L\psi = (\kappa+\gamma)(\mathcal{L} - id)\psi = (\kappa+\gamma)(r(\mathcal{L})-1)\psi \geq c_1\psi$, where $c_1 = (r(\mathcal{L})-1)\inf_\Omega(\kappa+\gamma) > 0$.

Suppose $s(L) < 0$ for contradiction. Then, $0 \in \rho(L)$ and $(-L)^{-1}$ is positive as L generates a positive semigroup. Therefore,

$$\psi = (-L)^{-1}(-L\psi) \leq (-L)^{-1}(-c_1\psi) = -c_1(-L)^{-1}\psi.$$

As $(-L)^{-1}\psi \geq 0$, we find $\psi \leq 0$, which leads to a contradiction. Hence, $s(L) \geq 0$, and there must hold $s(L) > 0$. □

3.3. *The basic reproduction number*

The basic reproduction number $\mathcal{R}_0$ for an infectious disease is the average number of secondary infections caused by an infectious host during the infectious period within an entirely susceptible host population. For our model (1.2), the basic reproduction number is defined as the spectral radius of the threshold operator $\mathcal{R}_0 := r(\mathcal{L})$. This agrees with the definition of $\mathcal{R}_0$ using the next generation operator in [10,39]. Our definition of $\mathcal{R}_0$ generalizes the definition for the finite-stage model (1.5) in [16], which was derived using the method of next generation matrix in [40].

The relationship between $\mathcal{R}_0$ and $s(L)$ in Theorem 3.6 allows us to derive the following threshold result from Corollary 3.3.

Theorem 3.7. *The disease-free equilibrium $(S^0, 0)^T$ is asymptotically stable if the basic reproduction number $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

4. Stationary solutions

An element $(S, I)^T \in X_+$ is called a *stationary solution* of (1.2) if it satisfies

$$\begin{cases} \Lambda(S) - \displaystyle\int_\Omega f(y, S, I(y))dy = 0, \\ -\kappa(x)I(x) + \alpha(x)\displaystyle\int_\Omega f(y, S, I(y))dy + \displaystyle\int_\Omega \theta(y,x)I(y)dy - \gamma(x)I(x) = 0, \quad x \in \Omega. \end{cases}$$

We see that the disease-free equilibrium $(S^0, 0)^T$ is always a stationary solution of (1.2). Here, we are interested in *endemic equilibria*, namely, stationary solutions $(S^*, I^*)^T \in X_+$ of (1.2) with $I^* \not\equiv 0$. It turns out that the existence or non-existence of endemic equilibria is related to the stability of the disease-free equilibrium $(S^0, 0)^T$. Therefore, let us look at the linearization of (1.2) at $(S^0, 0)^T$,

$$\dot{S}(t) = \Lambda'(S^0)S(t) - \int_\Omega f_I(y, S^0, 0)I(t, y)dy,$$

$$I_t(t, x) = -\kappa(x)I(t, x) + \alpha(x)\int_\Omega f_I(y, S^0, 0)I(t, y)dy + \int_\Omega \theta(y, x)I(t, y)dy - \gamma(x)I(t, x).$$

Notice the second equation in the above system only involves I, and therefore, instead of considering the spectral problem associated to the system, it is more convenient to consider the one related to the operator L defined in (3.1). Concerning L, the sign of $s(L)$ is of the most interest. In Theorem 3.6, we have shown that the sign of $s(L)$ is the same as $r(\mathcal{L}) - 1$, where $r(\mathcal{L})$ is the spectral radius of the threshold operator $\mathcal{L}$ given in (3.4). Hence, $s(L)$ has the same sign as that of $\mathcal{R}_0 - 1$. Therefore, $\mathcal{R}_0 = 1$ is the critical case. More precisely, we will show that the disease-free equilibrium $(S^0, 0)^T$ is the only stationary solution of (1.2) in X_+ when $\mathcal{R}_0 < 1$ (see Theorem 4.1), while endemic equilibrium exists when $\mathcal{R}_0 > 1$ (see Theorem 4.2).

4.1. Disease-free equilibrium

Theorem 4.1. *Suppose $\mathcal{R}_0 < 1$. Then, the disease-free equilibrium $(S^0, 0)^T$ is the unique stationary solution of* (1.2) *in X_+.*

Proof. Let $g : X_+ \to \mathbb{R}$ be defined by $g(S, I) = \Lambda(S) - \int_\Omega f(y, S, I(y))dy$. Clearly, $g_S(S, I) = \Lambda'(S) - \int_\Omega f_S(y, S, I(y))dy < 0$, $g(0, I) > 0$ and $g(S^0 + 1, I) < 0$. Hence, for each $\tilde{I} \in C_+(\Omega)$, there exists a unique $\tilde{S} > 0$ such that $g(\tilde{S}, \tilde{I}) = 0$. Let us define the map $G : C_+(\Omega) \to (0, \infty)$ by setting $G(\tilde{I}) = \tilde{S}$. Clearly, $G(0) = S^0$. Moreover, G is continuous and non-increasing in the sense that if $I_1, I_2 \in C_+(\Omega)$ with $I_1 \geq I_2$, then $G(I_1) \geq G(I_2)$. In particular, $G(I) \leq S^0$ for all $I \in C_+(\Omega)$.

By Lemma 3.6, $s(L) < 0$. Consider the nonlinear evolutionary equation

$$I_t(t, x) = -\kappa(x)I(x) + \alpha(x)\int_\Omega f(y, G(I), I(y))dy + \int_\Omega \theta(y, x)I(y)dy - \gamma(x)I(x),$$
$$x \in \Omega,\ t \geq 0. \tag{4.1}$$

By the assumptions on f, it is not hard to see that for each $I_0 \in C_+(\Omega)$, (4.1) admits a unique global classical solution $I(t, x; I_0)$ with $I(0, \cdot) = I_0$ satisfying $I(t, \cdot; I_0) \in C_+(\Omega)$ for all $t \geq 0$.

As $f(y, G(I), I(y)) \leq f(y, S^0, I(y)) \leq f_I(y, S^0, 0)I(y)$, we conclude from the comparison principle that

$$0 \leq I(t, x; I_0) \leq e^{Lt}I_0(x) \quad \text{for } x \in \Omega,\ t \geq 0.$$

Since $\{e^{Lt}\}_{t\geq 0}$ is a positive semigroup on $C(\Omega)$, $s(L)$ is the same as the growth bound of $\{e^{Lt}\}_{t\geq 0}$ (see e.g. [1, Theorem C-IV 1.1]). We remark that this also follows from the fact that L is a bounded linear operator (see e.g. [11, Corollary IV.2.4]). Then, we can find some $\epsilon_0 > 0$ with $s(L) + \epsilon_0 < 0$ such that

$$\|e^{Lt}\| \le M_0 e^{(s(L)+\epsilon_0)t}, \quad t \ge 0$$

for some $M_0 > 0$. Therefore,

$$\|I(t, \cdot; I_0)\|_{C(\Omega)} \le \|e^{Lt} u_0\|_{C(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

In particular, any nonzero element in $C_+(\Omega)$ can not be a stationary solution of (4.1). Equivalently, $(S^0, 0)^T$ is the only stationary solution of (1.2) in X_+. $\square$

4.2. Endemic equilibrium

Theorem 4.2. *Suppose $\mathcal{R}_0 > 1$. Then, (1.2) admits a stationary solution $(S^*, I^*)^T$ in X_+ with $I^* \in C_{++}(\Omega)$.*

Before proving the above lemma, let us do some preparation. For $n \ge 1$, let us define

$$f_n(x, S, I) := f(x, S, \min\{I, n\}), \quad (x, S, I) \in \Omega \times \mathbb{R}_+^2,$$

and consider the stationary system

$$\begin{cases} \Lambda(S) - \int_\Omega f_n(y, S, I(y))dy = 0, \\ -\kappa(x)I(x) + \alpha(x)\int_\Omega f_n(y, S, I(y))dy + \int_\Omega \theta(y, x)\min\{I(y), n\}dy \\ \quad - \gamma(x)I(x) + \epsilon_n(\kappa(x) + \gamma(x)) = 0, \quad x \in \Omega, \end{cases} \tag{4.2}$$

where $\{\epsilon_n\}_n$ is a bounded and decreasing sequence in $(0, \infty)$ satisfying $\epsilon_n \to 0$ as $n \to \infty$.

Let $g_n : X_+ \to \mathbb{R}$ be defined by

$$g_n(S, I) = \Lambda(S) - \int_\Omega f_n(y, S, I(y))dy, \quad (S, I) \in X_+.$$

Clearly, $\partial_S g_n(S, I) = \Lambda'(S) - \int_\Omega \partial_S f_n(y, S, I(y))dy < 0$, $g_n(0, I) > 0$ and $g_n(S^0 + 1, I) < 0$. Hence, for each $\tilde{I} \in C_+(\Omega)$, there exists a unique $\tilde{S} > 0$ such that $g_n(\tilde{S}, \tilde{I}) = 0$. Let us define the map $G_n : C_+(\Omega) \to (0, \infty)$ by setting $G_n(\tilde{I}) = \tilde{S}$. Clearly, $G_n(0) = S^0$. Moreover, we have for each $n \ge 1$, G_n is continuous and non-increasing in the sense that if $I_1, I_2 \in C_+(\Omega)$ with $I_1 \ge I_2$, then $G_n(I_1) \ge G_n(I_2)$.

For each $n \ge 1$, we consider the map $\mathscr{F}_n : C_+(\Omega) \to C_+(\Omega)$ by setting

$$\mathscr{F}_n[I](x) = \frac{\alpha(x)}{\kappa(x) + \gamma(x)} \int_\Omega f_n(y, G_n(I), I(y))dy + \frac{\int_\Omega \theta(y, x)\min\{I(y), n\}dy}{\kappa(x) + \gamma(x)} + \epsilon_n, \quad x \in \Omega.$$

Clearly, it is well-defined, continuous and compact (see the arguments as in the proof of Lemma 3.4).

We now prove Theorem 4.2.

Proof of Theorem 4.2. Clearly,

$$\mathscr{F}_n[I](x) \leq \frac{\alpha(x)}{\kappa(x)+\gamma(x)} \int_\Omega f_n(y, S^0, n)dy + \frac{n\int_\Omega \theta(y,x)dy}{\kappa(x)+\gamma(x)} + \epsilon_n, \quad x \in \Omega$$

for all $I \in C_+(\Omega)$. Therefore, for $r_n \gg 1$, $\mathscr{F}_n(C_n) \subset C_n$, where $C_n = \{I \in C_+(\Omega) : \sup_\Omega I \leq r_n\}$. Since C_n is closed and convex, we can apply Schauder fixed point theorem to conclude the existence of some $I_n \in C_n$ such that $\mathscr{F}_n[I_n] = I_n$. Setting $S_n = G_n(I_n)$, we find that $(S_n, I_n)^T$ solves (4.2), that is,

$$\begin{cases} \Lambda(S_n) - \displaystyle\int_\Omega f_n(y, S_n, I_n(y))dy = 0, \\ -\kappa(x)I_n(x) + \alpha(x)\displaystyle\int_\Omega f_n(y, S_n, I_n(y))dy + \int_\Omega \theta(y,x)\min\{I_n(y), n\}dy \\ \quad -\gamma(x)I_n(x) + \epsilon_n(\kappa(x)+\gamma(x)) = 0. \end{cases} \tag{4.3}$$

Integrating the second equation in (4.3) over Ω with respect to x and adding the resulting equation to the first equation in (4.3), we find

$$\begin{aligned} &\Lambda(S_n) - \int_\Omega \kappa(x)I_n(x)dx + \int_\Omega \gamma(x)\min\{I_n(x), n\}dx - \int_\Omega \gamma(x)I_n(x)dx \\ &+ \epsilon_n \int_\Omega (\kappa(x)+\gamma(x))dx = 0. \end{aligned}$$

Since $\int_\Omega \gamma(x)\min\{I_n(x), n\}dx \leq \int_\Omega \gamma(x)I_n(x)dx$, we find

$$\int_\Omega \kappa(x)I_n(x)dx \leq \Lambda(S_n) + \epsilon_n \int_\Omega (\kappa(x)+\gamma(x))dx \leq \Lambda(0) + \epsilon_1 \int_\Omega (\kappa(x)+\gamma(x))dx.$$

As $\inf_\Omega \kappa > 0$, we find that the sequence $\{\int_\Omega I_n(x)dx\}_n$ is bounded. It then follows from the second equation in (4.3) or the fact $I_n = \mathscr{F}[I_n]$ that the sequence $\{I_n\}_n$ is bounded in $C_+(\Omega)$. Moreover, using the fact $I_n = \mathscr{F}[I_n]$ and the same arguments as in the proof of Lemma 3.4, we readily see that $\{I_n\}_n$ is equi-continuous, and hence, pre-compact in $C_+(\Omega)$. Therefore, up to choose a subsequence, we may assume that there exists $(S^*, I^*)^T \in X_+$ such that $(S_n, I_n)^T \to (S^*, I^*)^T$ as $n \to \infty$. Passing to the limit $n \to \infty$ in (4.3), we conclude that $(S^*, I^*)^T$ is a stationary solution of (1.2).

It remains to show that $I^* \in C_+(\Omega)\backslash\{0\}$ (then there must hold $I^* \in C_{++}(\Omega)$). This actually follows from the arguments in [39, Page 3784], we include the arguments for the convenience of the reader. For contradiction, let us suppose $I^* \equiv 0$. Then, $S^* = S^0$. In what follows, we only consider all sufficiently large n, and thus, $I_n \le n$. Setting

$$\begin{aligned}\mathscr{G}[S_n, I_n](x) &:= \frac{\alpha(x)}{\kappa(x)+\gamma(x)}\int_\Omega f_n(y, S_n, I_n(y))dy + \frac{\int_\Omega \theta(y,x)\min\{I_n(y), n\}dy}{\kappa(x)+\gamma(x)},\\ &= \frac{\alpha(x)}{\kappa(x)+\gamma(x)}\int_\Omega f(y, S_n, I_n(y))dy + \frac{\int_\Omega \theta(y,x) I_n(y)dy}{\kappa(x)+\gamma(x)}, \quad x\in\Omega,\end{aligned}$$

we see $I_n = \mathscr{F}[I_n] = \mathscr{G}[S_n, I_n] + \epsilon_n$. Then,

$$\frac{I_n}{\|I_n\|_{C(\Omega)}} = \frac{\mathscr{G}[S_n, I_n] - \mathcal{L}[I_n]}{\|I_n\|_{C(\Omega)}} + \mathcal{L}[\frac{I_n}{\|I_n\|_{C(\Omega)}}] + \frac{\epsilon_n}{\|I_n\|_{C(\Omega)}}. \tag{4.4}$$

Direct arguments ensure that $\frac{\mathscr{G}[S_n, I_n] - \mathcal{L}[I_n]}{\|I_n\|_{C(\Omega)}} \to 0$ in $C(\Omega)$ as $n\to\infty$. Therefore, the sequence $\{\frac{\epsilon_n}{\|I_n\|_{C(\Omega)}}\}_n$ remains bounded. Therefore, up to a subsequence, we may assume that as $n\to\infty$

$$\frac{\epsilon_n}{\|I_n\|_{C(\Omega)}} \to \delta \quad \text{and} \quad \mathcal{L}[\frac{I_n}{\|I_n\|_{C(\Omega)}}] \to J$$

for some $\delta \ge 0$ and $J \in C_+(\Omega)$, where we used the compactness of $\mathcal{L}$. Hence, $\frac{I_n}{\|I_n\|_{C(\Omega)}} \to J + \delta =: I$ in $C(\Omega)$ as $n\to\infty$. Note that $\|I\|_{C(\Omega)} = 1$. Passing $n\to\infty$ in (4.4), we then arrive at

$$I = \mathcal{L}[I] + \delta.$$

From which, we conclude that $I \in C_{++}(\Omega)$. By Krein–Rutman theorem, there exists some $\mu \in (C(\Omega))_+^* \setminus \{0\}$ such that $\mathcal{L}^*\mu = r(\mathcal{L})\mu$. Clearly, $\int_\Omega I(x)d\mu(x) > 0$. It then follows that

$$\int_\Omega I(x)d\mu(x) = \int_\Omega \mathcal{L}[I](x)d\mu(x) + \delta\mu(\Omega) \ge \int_\Omega I(x)d\mathcal{L}^*\mu(x) = r(\mathcal{L})\int_\Omega I(x)d\mu(x).$$

It is a contradiction, as $r(\mathcal{L}) > 1$. □

We remark that Theorem 4.2 only gives the existence of an endemic equilibrium. It is not clear whether the endemic equilibrium is unique under the conditions in Theorem 4.2. We will show later in Theorem 5.5 that it is the only endemic equilibrium under some additional conditions when we establish the global stability.

5. Global dynamics

In this section, we study the global dynamics of (1.2).

5.1. Global dynamics when $\mathcal{R}_0 < 1$

We prove the global asymptotic stability of the disease-free equilibrium $(S^0, 0)^T$ when $\mathcal{R}_0 < 1$, which is part (1) in Theorem A.

Theorem 5.1. *Suppose $\mathcal{R}_0 < 1$. Then, $(S^0, 0)^T$ is globally asymptotically stable in X_+.*

Proof. We first show that $(S^0, 0)^T$ is linearly stable. To do so, let us consider the linearization of (1.2) at $(S^0, 0)^T$, namely,

$$\dot{S}(t) = \Lambda'(S^0)S(t) - \int_\Omega f_I(y, S^0, 0)I(t, y)dy,$$

$$I_t(t, x) = -\kappa(x)I(t, x) + \alpha(x)\int_\Omega f_I(y, S^0, 0)I(t, y)dy + \int_\Omega \theta(y, x)I(t, y)dy - \gamma(x)I(t, x).$$

Define the operator $\tilde{L} : X \to X$ by setting

$$\tilde{L}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \Lambda'(S^0)S - \int_\Omega f_I(y, S^0, 0)I(y)dy \\ -\kappa I + \alpha \int_\Omega f_I(y, S^0, 0)I(y)dy + \int_\Omega \theta(y, \cdot)I(y)dy - \gamma I \end{pmatrix}$$
$$= \begin{pmatrix} \Lambda'(S^0)S - \int_\Omega f_I(y, S^0, 0)I(y)dy \\ L[I] \end{pmatrix}.$$

We claim that $s(\tilde{L}) \le s(L)$. Indeed, let $\lambda \in \mathbb{C}$ with $\Re\lambda > s(L)$, and $(\tilde{S}, \tilde{I})^T \in X$, then the unique solution of the equation

$$(\lambda - \tilde{L})\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \lambda - \Lambda'(S^0)S + \int_\Omega f_I(y, S^0, 0)I(y)dy \\ (\lambda - L)[I] \end{pmatrix} = \begin{pmatrix} \tilde{S} \\ \tilde{I} \end{pmatrix}$$

is given by

$$I = (\lambda - L)^{-1}[\tilde{I}] \quad \text{and} \quad S = -\frac{1}{\Lambda'(S^0)}\left[\tilde{S} - \lambda - \int_\Omega f_I(y, S^0, 0)(\lambda - L)^{-1}[\tilde{I}](y)dy\right].$$

Moreover, it is easy to see that $\|(S, I)^T\|_X \le M\|(\tilde{S}, \tilde{I})^T\|_X$ for some $M > 0$ (independent of $(\tilde{S}, \tilde{I})^T$). That is, $(\lambda - \tilde{L})^{-1}$ is well-defined and is a bounded linear operator, and hence, $\lambda \in \rho(\tilde{L})$, which implies that $s(\tilde{L}) \le s(L) < 0$.

Since $\tilde{L}$ is bounded, it generates a uniformly continuous semigroup $\{e^{\tilde{L}t}\}_{t\geq 0}$ on X. It then follows from [11, Corollary IV.2.4] that the growth bound of $\{e^{\tilde{L}t}\}_{t\geq 0}$ coincides with $s(\tilde{L})$, and hence, $(S^0, 0)^T$ is linearly stable.

For the global asymptotic stability, we need to show for any solution $(S(t), I(t,\cdot))^T$ with initial $(S_0, I_0(\cdot))^T \in X_+\backslash\{0\}$, the following holds,

$$(S(t), I(t,\cdot))^T \to (S^0, 0)^T \text{ in } X \text{ as } t \to \infty. \tag{5.1}$$

By Theorem 2.1 and the structure of the S equation in (1.2), it suffices to prove (5.1) for any $(S_0, I_0(\cdot))^T \in X_+\backslash\{0\}$ with $\|(S(t), I(t,\cdot))^T\|_X \leq M$ and $S(t) \leq S^0 + \epsilon_0$ for all $t \geq 0$, where $M \gg 1$ and $\epsilon_0 > 0$ is a small number such that $s(L_0) < 0$, where L_0 is defined by

$$L_0[I] := -\kappa I + \alpha \int_\Omega f_I(y, S^0 + \epsilon_0, 0) I(y) dx + \int_\Omega \theta(y,\cdot) I(y) dy - \gamma I.$$

Let $\{e^{L_0 t}\}_{t\geq 0}$ be the positive semigroup generated by L_0. By comparison principle, we find $0 \leq I(t,\cdot) \leq e^{L_0 t} I_0$ for all $t \geq 0$. Since the growth bound of $\{e^{L_0 t}\}_{t\geq 0}$ is the same as $s(L_0)$, we conclude that

$$\|I(t,\cdot)\|_{C(\Omega)} \leq M_1 e^{-\omega t} \|I_0\|_{C(\Omega)}, \quad t \geq 0 \tag{5.2}$$

for some $\omega > 0$ and $M_1 > 1$.

It remains to show that $S(t) \to S^0$ as $t \to \infty$, which however is a simple consequence of (5.2) (so that $\int_\Omega f(y, S(t), I(t,y)) dy \to 0$ as $t \to \infty$) and the assumptions on Λ. □

5.2. Uniform persistence when $\mathcal{R}_0 > 1$

System (1.2) is said to be *uniformly persistent* if there exists a constant $\epsilon_1 > 0$ such that, for any $(S_0, I_0(\cdot))^T \in X_+$ with $I_0 \not\equiv 0$, the unique solution $(S(t), I(t,x))^T$ of (1.2) with initial data $(S_0, I_0(\cdot))^T$ satisfies

$$\liminf_{t\to+\infty} S(t) > \epsilon_1 \quad \text{and} \quad \liminf_{t\to+\infty} \inf_{x\in\Omega} I(t,x) > \epsilon_1. \tag{5.3}$$

Theorem 5.2. *Suppose $\mathcal{R}_0 > 1$. Then $\{\Sigma_t\}_{t\geq 0}$ is uniformly persistent.*

In order to prove Theorem 5.2, we first prove two lemmas.

Lemma 5.3. *Let $(S(t), I(t,x))^T$ be the unique solution of (1.2) with initial data $(S_0, I_0(\cdot))^T \in X_+$ with $I_0 \not\equiv 0$. Then, $I(t,x) > 0$ for all $t > 0$ and $x \in \Omega$.*

Proof. Let $J(t,x)$ be the unique solution of

$$\begin{cases} J_t(t,x) = -\kappa(x) J(t,x) + \displaystyle\int_\Omega \theta(y,x) J(t,y) dy - \gamma(x) J(t,x), \\ J(0,\cdot) = I_0. \end{cases}$$

Then, by the comparison principle, $I(t,x) \geq J(t,x) \geq 0$ for $t \geq 0$ and $x \in \Omega$. Therefore, the lemma follows if we can show

$$J(t,x) > 0, \quad t > 0, \quad x \in \Omega, \tag{5.4}$$

which follows from arguments as in the proof of [21, Theorem 2.1] or [37, Proposition 2.2]. For completeness, we provide an outline of the proof of (5.4).

Let $\Theta : C(\Omega) \to C(\Omega)$ be defined by

$$\Theta[I](x) = \int_\Omega \theta(y,x)I(y)dy, \quad x \in \Omega.$$

Then Θ is continuous. Let $\{e^{\Theta t}\}_{t\geq 0}$ be the uniformly continuous and positive semigroup on $C(\Omega)$ generated by Θ. Then, for each $t > 0$, $e^{\Theta t} = \sum_{n=0}^{\infty} \frac{t^n \Theta^n}{n!}$, where the series converges in the operator norm. In particular,

$$e^{\Theta t}[I_0] = \sum_{n=0}^{\infty} \frac{t^n \Theta^n[I_0]}{n!}, \quad t > 0. \tag{5.5}$$

Since $I_0 \not\equiv 0$, $\theta(x,x) > 0$ for all $x \in \Omega$ and

$$\Theta^{n+1}[I_0](x) = \int_\Omega \theta(y,x)\Theta^n[I_0](y)dy, \quad x \in \Omega$$

for $n \geq 1$, an iteration argument ensures the existence of n_0 such that $\Theta^{n+1}[I_0](x) > 0$ for $x \in \Omega$ and for all $n \geq n_0$. By (5.5), we then conclude that $e^{\Theta t} I_0(x) > 0$ for all $x \in \Omega$.

Let $\zeta_1 = \sup_\Omega(\kappa + \gamma)$. Then, we see

$$J(t,\cdot) = e^{-\zeta_1 t}e^{\Theta t}[I_0] + \int_0^t e^{-\zeta_1(t-s)}e^{\Theta(t-s)}[(\zeta_1 - \kappa - \gamma)J(s,\cdot)]ds \geq e^{-\zeta_1 t}e^{\Theta t}[I_0],$$

which leads to (5.4). This completes the proof. □

The following lemma establishes weak uniform persistence.

Lemma 5.4. *Suppose $\mathcal{R}_0 > 1$. Then, there exists $\epsilon_2 > 0$ such that, for $(S_0, I_0(\cdot))^T \in X_+$ with $I_0 \not\equiv 0$, the unique solution $(S(t), I(t,x))^T$ of (1.2) with initial data $(S_0, I_0(\cdot))^T$ satisfies*

$$\limsup_{t\to+\infty} \sup_{x\in\Omega} I(t,x) > \epsilon_2.$$

Proof. Suppose that the lemma fails. Then, for $\epsilon > 0$, there exists $(S_{0\epsilon}, I_{0\epsilon}(\cdot))^T \in X_+$ with $I_{0\epsilon} \not\equiv 0$ such that the unique solution $(S_\epsilon(t), I_\epsilon(t,x))^T$ of (1.2) with initial data $(S_{0\epsilon}, I_{0\epsilon}(\cdot))^T$ satisfies

$$\limsup_{t\to+\infty}\sup_{x\in\Omega} I_\epsilon(t,x) \le 2\epsilon.$$

Replacing $(S_{0\epsilon}, I_{0\epsilon}(\cdot))^T$ by $(S_\epsilon(t_\epsilon), I_\epsilon(t_\epsilon,\cdot))^T$ for some $t_\epsilon \gg 1$ and applying Lemma 5.3, without loss of generality, we may assume that $0 < I_\epsilon(t,x) < \epsilon$ for all $t \ge 0$ and $x \in \Omega$. Since $\dot{S}_\epsilon(t) \le \Lambda(S_\epsilon(t))$, the assumptions on Λ implies that $S_\epsilon(t) \le S^0 + \epsilon_0$ for all sufficiently large t, where ϵ_0 is a sufficiently small number (independent of ϵ). Therefore, choosing t_ϵ larger, we may assume, without loss of generality, that $S_\epsilon(t) \le S^0 + \epsilon_0$ for all $t \ge 0$. Since $f(x,S,I)$ is a non-decreasing function in S and I, it follows that

$$\dot{S}_\epsilon(t) \ge \Lambda(S_\epsilon(t)) - \int_\Omega f(y, S^0+\epsilon_0, \epsilon)dy, \quad t \ge 0.$$

Setting $\varphi(\epsilon) := \int_\Omega f(y, S^0+\epsilon_0,\epsilon)dy$, we find

$$\dot{S}_\epsilon(t) \ge \Lambda(S_\epsilon(t)) - \varphi(\epsilon), \quad t \ge 0.$$

Clearly, $\varphi(\epsilon)$ is a continuous and non-decreasing function in ϵ and $\varphi(0) = 0$. The assumptions on Λ imply that the equation $\Lambda(S) - \varphi(\epsilon) = 0$ has a unique solution $\Lambda^{-1}(\varphi(\epsilon))$ for all sufficiently small ϵ. Setting $\psi(\epsilon) = 2(S^0 - \Lambda^{-1}(\varphi(\epsilon)))$, we have $\Lambda^{-1}(\varphi(\epsilon)) = S^0 - \frac{1}{2}\psi(\epsilon)$. Let us consider the equation

$$\dot{\tilde{S}}(t) = \Lambda(\tilde{S}(t)) - \varphi(\epsilon), \tag{5.6}$$

which has a unique equilibrium $S^0 - \frac{1}{2}\psi(\epsilon)$ attracting all positive solutions. It then follows from the comparison principle for ODEs that

$$S_\epsilon(t) \ge S^0 - \psi(\epsilon)$$

for sufficiently large t. Hence, choosing t_ϵ larger, we can assume that $S_\epsilon(t) \ge S^0 - \psi(\epsilon)$ for all $t \ge 0$.

Since $f(x,S,I)$ is non-decreasing in S and I, it then follows that

$$\begin{aligned}
\partial_t I_\epsilon(t,x) \ge& -\kappa(x)I_\epsilon(t,x) + \alpha(x)\int_\Omega f(y, S^0-\psi(\epsilon), I_\epsilon(t,y))dy \\
&+ \int_\Omega \theta(y,x)I_\epsilon(t,y)dy - \gamma(x)I_\epsilon(t,x) \\
\ge& -\kappa(x)I_\epsilon(t,x) + \alpha(x)\int_\Omega \frac{f(y,S^0-\psi(\epsilon),\epsilon)}{\epsilon} I_\epsilon(t,y)dy \\
&+ \int_\Omega \theta(y,x)I_\epsilon(t,y)dy - \gamma(x)I_\epsilon(t,x),
\end{aligned} \tag{5.7}$$

where we used $I_\epsilon(t,x) \le \epsilon$ and the assumption that $\frac{f(x,S,I)}{I}$ is a non-increasing function of I on $(0,\infty)$.

Let us define linear operators L_ϵ and $\mathcal{L}_\epsilon$ on $C(\Omega)$ by setting

$$L_\epsilon[I](x) := -\kappa(x)I(x) + \alpha(x)\int_\Omega \frac{f(y, S^0-\psi(\epsilon),\epsilon)}{\epsilon} I(y)dy + \int_\Omega \theta(y,x)I(y)dy - \gamma(x)I(x),$$

and

$$\mathcal{L}_\epsilon[I](x) := \frac{\alpha(x)}{\kappa(x)+\gamma(x)}\int_\Omega \frac{f(y, S^0-\psi(\epsilon),\epsilon)}{\epsilon} I(y)dy + \frac{1}{\kappa(x)+\gamma(x)}\int_\Omega \theta(y,x)I(y)dy,$$

respectively. We see that $\mathcal{L}_\epsilon$ is strongly positive and compact. By arguments as in the proof of Theorem 3.6, we conclude that $s(L_\epsilon) > 0$ if and only if $r(\mathcal{L}_\epsilon) > 1$. Due to the facts that the function $\frac{f(x,S,I)}{I}$ is a non-increasing function of I on $(0,\infty)$ and that the function $f(x,S,I)$ is continuously differentiable on $\Omega \times \mathbb{R}^2_{++}$, Dini's lemma implies that

$$\sup_{y\in\Omega}\left|\frac{f(y,S^0-\psi(\epsilon),\epsilon)}{\epsilon} - f_I(y,S^0,0)\right| \to 0 \text{ uniformly as } \epsilon \to 0.$$

This implies that $\mathcal{L}_\epsilon \to \mathcal{L}$ in the operator norm as $\epsilon \to 0$. Since $\mathcal{R}_0 = r(\mathcal{L}) > 1$, we can choose ϵ_* such small that $r(\mathcal{L}_{\epsilon_*}) > 1$, as the spectral radius is a continuous function of compact linear operators. Then, $s(L_{\epsilon_*}) > 0$. By Theorem 3.2 (actually, a version of Theorem 3.2 for L^*_ϵ), $s(L_{\epsilon_*})$ is an isolated and simple eigenvalue. Let $\phi \in C_{++}(\Omega)$ be its eigenfunction. We may suppose $\|\phi\|_{C(\Omega)} = 1$. Clearly, we can choose some $\xi > 0$ such small that $\xi\phi(x) \le I_{0\epsilon_*}(x)$ for all $x \in \Omega$. Let $\{e^{L_{\epsilon_*}t}\}_{t\ge 0}$ be the uniformly continuous semigroup on $C(\Omega)$ generated by L_{ϵ_*}. By Lemma 3.1 and general perturbation results (see e.g. [11, Corollary VI.1.11]), $\{e^{L_{\epsilon_*}t}\}_{t\ge 0}$ is a positive semigroup. Then, we have

$$e^{L_{\epsilon_*}t} I_{0\epsilon_*} \ge e^{L_{\epsilon_*}t}\xi\phi = e^{s(L_{\epsilon_*})t}\xi\phi, \quad t \ge 0,$$

which implies that $\|e^{L_{\epsilon_*}t} I_{0\epsilon_*}\|_{C(\Omega)} \to \infty$ as $t \to \infty$ since $s(L_{\epsilon_*}) > 0$. By comparison principle, we have from (5.7) that $\|I_{\epsilon_*}(t,\cdot)\|_{C(\Omega)} \ge \|e^{L_{\epsilon_*}t} I_{0\epsilon_*}\|_{C(\Omega)} \to \infty$ as $t \to \infty$. This is a contraction. $\square$

We now prove Theorem 5.2.

Proof of Theorem 5.2. Since the function $f(x,S,I)$ is continuously differentiable and the system (1.2) is dissipative, there exists constant $k > 0$ such that

$$\dot{S}(t) \ge \Lambda(S(t)) - kS(t)$$

for all large t. Furthermore, since function $\Lambda(S)$ is continuously differentiable, there exists constant $l > 0$ such that $\Lambda(S(t)) \ge \Lambda(0) - lS(t)$ for all large t, which leads to

$$\dot{S}(t) \ge \Lambda(0) - (k+l)S(t) \tag{5.8}$$

for all large t. It then follows from (5.8) that, for small $\varsigma > 0$, $S(t) \geq \frac{\Lambda(0)}{k+l} - \varsigma$ holds for all large t (depending on ς). In particular,

$$\liminf_{t\to+\infty} S(t) \geq \frac{\Lambda(0)}{k+l}$$

for any solutions of (1.2).

It remains to show that there exists a constant $\epsilon_1 > 0$ such that

$$\liminf_{t\to+\infty} \inf_{x\in\Omega} I(t,x) > \epsilon_1 \tag{5.9}$$

for all nonnegative solutions of (1.2) with $I_0 \not\equiv 0$. For $(S, I)^T \in X_+$, we define

$$\rho((S,I)^T) = \sup_{x\in\Omega} I(x) \quad \text{and} \quad \tilde{\rho}((S,I)^T) = \inf_{x\in\Omega} I(x).$$

Clearly, ρ and $\tilde{\rho}$ are continuous functions on X_+. Let $(S(t), I(t,\cdot))^T$ be the solution of the system (1.2) with initial data $(S_0, I_0(\cdot))^T$. Then, $\Sigma_t(S_0, I_0(\cdot))^T = (S(t), I(t,\cdot))^T$ for $t \geq 0$. By Theorem 2.1, $(\Sigma_t)_{t\geq 0}$ has a global attractor.

We are going to apply Theorem [38, Theorem A.34]. To do so, we need to verify the following three conditions.

- $\rho((S_0, I_0(\cdot))^T) > 0$ implies $\rho(\Sigma_t(S_0, I_0(\cdot))^T) = \sup_{x\in\Omega} I(t,x) > 0$ for $t \geq 0$.
- The system (1.2) is uniformly weakly ρ-persistent, that is, there exists $\epsilon > 0$ such that $\rho((S_0, I_0(\cdot))^T) > 0$ implies $\limsup_{t\to\infty} \rho(\Sigma_t(S_0, I_0(\cdot))^T) > \epsilon$.
- Let $(\breve{S}(t), \breve{I}(t,\cdot))^T$ be any bounded total orbit of Σ_t such that $\rho((\breve{S}(t), \breve{I}(t,\cdot))^T) > 0$ for all $t \in \mathbb{R}$. Then, $\tilde{\rho}((\breve{S}(0), \breve{I}(0,\cdot)^T) > 0$.

The first two conditions follow directly from Lemma 5.3 and Lemma 5.4, respectively. For the third one, we see that there exists some $t_0 < 0$ such that

$$\rho((\breve{S}(t_0), \breve{I}(t_0,\cdot)^T) = \sup_{x\in\Omega} \breve{I}(t_0,\cdot) > 0.$$

It then follows from Lemma 5.3 that

$$\tilde{\rho}((\breve{S}(0), \breve{I}(0,\cdot)^T) = \inf_{x\in\Omega} \breve{I}(0,x) > 0.$$

Now, we apply [38, Theorem A.34] to conclude that there exists $\epsilon_1 > 0$ such that $\rho((S_0, I_0)^T) > 0$ implies $\liminf_{t\to+\infty} \rho(\Sigma_t(S_0, I_0)^T) > \epsilon_1$, namely, $\liminf_{t\to+\infty} \inf_{x\in\Omega} I(t,x) > \epsilon_1$. This proves (5.9) and completes the proof of Theorem 5.2. $\square$

5.3. *Global dynamics when $\mathcal{R}_0 > 1$*

Assume that $\mathcal{R}_0 = r(\mathcal{L}) > 1$ and let $(S^*, I^*)^T$ denote an endemic equilibrium in Theorem 4.2. We prove the global stability of $(S^*, I^*)^T$ as stated in Theorem A-(2).

Theorem 5.5. *Suppose $\mathcal{R}_0 > 1$. If*

$$\left[\frac{S^*}{S} - \frac{f(x,S^*,I^*(x))}{f(x,S,I)}\right]\left[\frac{S^*}{S} - \frac{\frac{f(x,S^*,I^*(x))}{I^*(x)}}{\frac{f(x,S,I)}{I}}\right] \le 0, \quad \forall x \in \Omega,\ S, I > 0, \tag{5.10}$$

then $(S^*, I^*)^T$ is globally asymptotically stable in $X_+ \backslash \{(S,0)^T : S \ge 0\}$. In particular, $(S^*, I^*)^T$ is the unique endemic equilibrium.

Condition (5.10) is automatically satisfied for all $x \in \Omega$, S, S^*, I, $I^* > 0$ when $f(x,S,I)$ takes common incidence forms including nonlinear incidence $f(x,S,I) = \beta(x)I^pS$ and saturated nonlinear incidence $f(x,S,I) = \beta(x)\frac{I^pS}{1+\kappa I^p}$, $0 < p \le 1$.

Proof of Theorem 5.5. Any endemic equilibrium $(S^*, I^*)^T$ of (1.2) satisfies the system

$$\begin{cases} \Lambda(S^*) = \int\limits_\Omega f(y,S^*,I^*(y))dy, \\ (\kappa(x)+\gamma(x))I^*(x) = \alpha(x)\int\limits_\Omega f(y,S^*,I^*(y))dy + \int\limits_\Omega \theta(y,x)I^*(y)dy, \quad x \in \Omega. \end{cases} \tag{5.11}$$

By Theorem 2.1, Theorem 5.2 and [28, Theorem 3.7 and Remark 3.10], $(\Sigma_t)_{t\ge0}$ as a semi-flow on $X_{++} = (0,\infty) \times C_{++}(\Omega)$ has a global attractor $\mathcal{A}_0$ with the property (see e.g. [36]) that there exist $\zeta_1, \zeta_2 > 0$ such that for any $(S_0, I_0(\cdot))^\top \in \mathcal{A}_0$, there hold

$$\zeta_1 \le S_0 \le \zeta_2 \quad \text{and} \quad \zeta_1 \le \inf_{x\in\Omega} I_0(x) \le \sup_{x\in\Omega} I_0(x) \le \zeta_2.$$

Moreover, by Lemma 5.3, any solution of (1.2) with initial data $(S_0, I_0(\cdot))^T \in X_+$ satisfying $I_0 \not\equiv 0$ is attracted by $\mathcal{A}_0$.

To prove the theorem, we only need to show that $(S^*, I^*)^T$ is the global attractor of $(\Sigma_t)_{t\ge0}$ in X_{++}, namely, $\mathcal{A}_0 = \{(S^*, I^*)^T\}$.

Let $(S(t), I(t,\cdot))^T$ be the unique solution of the system (1.2) with initial data $(S_0, I_0(\cdot))^T \in \mathcal{A}_0$. Then, $(S(t), I(t,\cdot))^T \in \mathcal{A}_0$ for all $t \ge 0$. From the properties of $\mathcal{A}_0$, we know that there exist δ_1, $\delta_2 > 0$ such that

$$\delta_1 \le \frac{S(t)}{S^*} \le \delta_2 \quad \text{and} \quad \delta_1 \le \inf_{x\in\Omega} \frac{I(t,x)}{I^*(x)} \le \sup_{x\in\Omega} \frac{I(t,x)}{I^*(x)} \le \delta_2$$

for all $t \ge 0$. Define

$$\mathcal{K}(x,y) := \theta(y,x)I^*(y) + \alpha(x)f(y,S^*,I^*(y)), \quad (x,y) \in \Omega \times \Omega.$$

Then $\mathcal{K} \in C(\Omega \times \Omega)$ and $\mathcal{K}(x,x) > 0$ for $x \in \Omega$. Then,

$$\Upsilon(x) := \int\limits_\Omega \mathcal{K}(x,y)dy, \quad x \in \Omega$$

is a continuous and strictly positive function on Ω, namely, $\Upsilon \in C_{++}(\Omega)$. By [39, Proposition 11.1], there exists an almost everywhere positive Borel function η on Ω such that $\Upsilon\eta$ is integrable and

$$\int_\Omega \eta(x)\Big(\int_\Omega \mathcal{K}(x,y)(v(x)-v(y))dy\Big)dx = 0, \quad \forall v \in L^\infty(\Omega). \tag{5.12}$$

Set $\mathscr{G}(a) = a - 1 - \ln a$. Then, $\mathscr{G}(a)$ is continuously differentiable and has its global minimum value 0 obtained at $a = 1$. Define the Lyapunov functional

$$V(t) = V(S(t), I(t,\cdot)) := \int_\Omega \eta(x)\Big[\alpha(x)S^*\mathscr{G}\Big(\frac{S(t)}{S^*}\Big) + I^*(x)\mathscr{G}\Big(\frac{I(t,x)}{I^*(x)}\Big)\Big]dx. \tag{5.13}$$

From the above discussions we know that $V(t)$ is well-defined.

Differentiating $V(t)$, we use (1.2) to obtain (for notational simplicity, we will omit the time variable in $S(t)$ and $I(t,x)$ and write S and $I(x)$ for $S(t)$ and $I(t,x)$, respectively)

$$\begin{aligned}\frac{dV}{dt} &= \int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})\Big[\Lambda(S) - \int_\Omega f(y,S,I(y))dy\Big]dx \\ &+ \int_\Omega \eta(x)\Big[1-\frac{I^*(x)}{I(x)}\Big]I^*(x)\Big[-(\kappa(x)+\gamma(x))I(x) + \alpha(x)\int_\Omega f(y,S,I(y))dy \\ &+ \int_\Omega \theta(y,x)I(y)dy\Big]dx.\end{aligned} \tag{5.14}$$

Using (5.11), we obtain, after collecting and rearranging terms, that

$$\begin{aligned}\frac{dV}{dt} &= \int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\ &+ \int_\Omega \eta(x)\int_\Omega f(y,S^*,I^*(y)\alpha(x)\Big[1-\frac{S^*}{S} - \frac{f(y,S,I(y))}{f(y^*,S^*,I^*(y))} \\ &+ \frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\Big]dydx \\ &+ \int_\Omega \eta(x)\Big[1-\frac{I^*(x)}{I(x)}\Big]\Big[\alpha(x)\int_\Omega f(y^*,S^*,I^*(y))\Big(\frac{f(y,S,I(y))}{f(y^*,S^*,I^*(y))} - \frac{I(x)}{I^*(x)}\Big)dy \\ &+ \int_\Omega \theta(y,x)I^*(y)\Big(\frac{I(y)}{I^*(y)} - \frac{I(x)}{I^*(x)}\Big)dy\Big]dx.\end{aligned} \tag{5.15}$$

We rewrite the expression (5.15) as

$$
\begin{aligned}
\frac{dV}{dt} = & \int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\
& + \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y)\alpha(x)\Big[2-\frac{S^*}{S}+\frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} \\
& - \frac{I^*(x)}{I(x)}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} - \frac{I(x)}{I^*(x)}\Big]dydx \\
& + \int_\Omega \eta(x) \int_\Omega \theta(y,x)I^*(y)\Big[\frac{I(y)}{I^*(y)} - \frac{I^*(x)}{I(x)}\frac{I(y)}{I^*(y)} - \frac{I(x)}{I^*(x)} + 1\Big]dydx.
\end{aligned}
\tag{5.16}
$$

In order to use the function $\mathscr{G}$ to group terms, we rewrite (5.16) as

$$
\begin{aligned}
\frac{dV}{dt} = & \int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\
& + \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y))\alpha(x)\Big[\Big(1-\frac{S^*}{S}+\ln\frac{S^*}{S}\Big) - \ln\frac{S^*}{S} + \frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} \\
& + \Big(1-\frac{I^*(x)}{I(x)}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} + \ln\frac{I^*(x)}{I(x)}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\Big) \\
& - \ln\frac{I^*(x)}{I(x)}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} - \frac{I(x)}{I^*(x)}\Big]dydx \\
& + \int_\Omega \eta(x) \int_\Omega \theta(y,x)I^*(y)\Big[\frac{I(y)}{I^*(y)} + \Big(1-\frac{I^*(x)}{I(x)}\frac{I(y)}{I^*(y)} + \ln\frac{I^*(x)}{I(x)}\frac{I(y)}{I^*(y)}\Big) \\
& - \ln\frac{I^*(x)}{I(x)}\frac{I(y)}{I^*(y)} - \frac{I(x)}{I^*(x)}\Big]dydx.
\end{aligned}
\tag{5.17}
$$

Then, using the function $\mathscr{G}$, (5.17) can be written as

$$
\begin{aligned}
\frac{dV}{dt} = & \int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\
& - \int_\Omega \eta(x) \int_\Omega \theta(y,x)I^*(y)\mathscr{G}\Big(\frac{I^*(x)}{I(x)}\frac{I(y)}{I^*(y)}\Big)dydx \\
& - \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y))\alpha(x)\Big[\mathscr{G}\Big(\frac{S^*}{S}\Big) + \mathscr{G}\Big(\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\frac{I^*(x)}{I(x)}\Big)\Big]dydx \\
& + \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y))\alpha(x)\Big[-\ln\frac{S^*}{S} + \frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}
\end{aligned}
$$

$$
\begin{aligned}
&- \ln \frac{I^*(x)}{I(x)} \frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} - \frac{I(x)}{I^*(x)}\Big] dydx \\
&+ \int_\Omega \eta(x) \int_\Omega \theta(y,x) I^*(y) \Big[\frac{I(y)}{I^*(y)} - \ln \frac{I^*(x)}{I(x)} \frac{I(y)}{I^*(y)} - \frac{I(x)}{I^*(x)} \Big] dydx.
\end{aligned}
\tag{5.18}
$$

Rewriting the fourth term and the fifth term on the right hand side of (5.18), we find

$$
\begin{aligned}
\frac{dV}{dt} = &\int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\
&- \int_\Omega \eta(x) \int_\Omega \theta(y,x) I^*(y) \mathscr{G}\Big(\frac{I^*(x)}{I(x)} \frac{I(y)}{I^*(y)}\Big) dydx \\
&- \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y))\alpha(x) \Big[\mathscr{G}\Big(\frac{S^*}{S}\Big) + \mathscr{G}\Big(\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} \frac{I^*(x)}{I(x)}\Big) \Big] dydx \\
&+ \int_\Omega \eta(x) \int_\Omega f(y,S^*,I^*(y))\alpha(x) \Big[\frac{S^*}{S} \frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} \\
&+ \ln \frac{S}{S^*} \frac{I(x)}{I^*(x)} \frac{f(y,S^*,I^*(y))}{f(y,S,I(y))} - \frac{I(x)}{I^*(x)} \Big] dydx \\
&+ \int_\Omega \eta(x) \int_\Omega \theta(y,x) I^*(y) \Big[\ln \frac{I(x)}{I^*(x)} - \frac{I(x)}{I^*(x)} - \ln \frac{I(y)}{I^*(y)} + \frac{I(y)}{I^*(y)} \Big] dydx.
\end{aligned}
\tag{5.19}
$$

We see that the term, in the brackets in the fourth term on the right hand side of (5.19), can be written as

$$
\begin{aligned}
&\frac{S^*}{S} \frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} + \ln \frac{I(x)}{I^*(x)} \frac{S}{S^*} \frac{f(y,S^*,I^*(y))}{f(y,S,I(y))} - \frac{I(x)}{I^*(x)} \\
&= -\mathscr{G}\Big(\frac{S^*}{S} \frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} \frac{I(y)}{I^*(y)}\Big) + \Big[\ln \frac{I(x)}{I^*(x)} - \frac{I(x)}{I^*(x)} - \ln \frac{I(y)}{I^*(y)} + \frac{I(y)}{I^*(y)} \Big] \\
&+ \Big[\frac{S^*}{S} \frac{f(y,S,I(y))}{f(y,S^*,I^*(y))} - \frac{I(y)}{I^*(y)} - 1 + \frac{S}{S^*} \frac{I(y)}{I^*(y)} \frac{f(y,S^*,I^*(y))}{f(y,S,I(y))} \Big].
\end{aligned}
\tag{5.20}
$$

Substituting (5.20) into (5.19), we find

$$
\begin{aligned}
\frac{dV}{dt} = &\int_\Omega \eta(x)\alpha(x)(1-\frac{S^*}{S})(\Lambda(S)-\Lambda(S^*))dx \\
&- \int_\Omega \eta(x) \int_\Omega \theta(y,x) I^*(y) \mathscr{G}\Big(\frac{I^*(x)}{I(x)} \frac{I(y)}{I^*(y)}\Big) dydx
\end{aligned}
$$

$$
\begin{aligned}
&-\int_\Omega \eta(x)\int_\Omega f(y,S^*,I^*(y))\alpha(x)\mathscr{G}\left(\frac{S^*}{S}\right)dydx\\
&-\int_\Omega \eta(x)\int_\Omega f(y,S^*,I^*(y))\alpha(x)\mathscr{G}\left(\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\frac{I^*(x)}{I(x)}\right)dydx\\
&-\int_\Omega \eta(x)\int_\Omega f(y,S^*,I^*(y))\alpha(x)\mathscr{G}\left(\frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\frac{I(y)}{I^*(y)}\right)dydx \qquad (5.21)\\
&+\int_\Omega \eta(x)\int_\Omega \left[f(y,S^*,I^*(y))\alpha(x)+\theta(y,x)I^*(y)\right]\left[\ln\frac{I(x)}{I^*(x)}-\frac{I(x)}{I^*(x)}\right.\\
&\left.-\ln\frac{I(y)}{I^*(y)}+\frac{I(y)}{I^*(y)}\right]dydx\\
&+\int_\Omega \eta(x)\int_\Omega f(y,S^*,I^*(y))\alpha(x)\left[\frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}-\frac{I(y)}{I^*(y)}-1\right.\\
&\left.+\frac{S}{S^*}\frac{I(y)}{I^*(y)}\frac{f(y,S^*,I^*(y))}{f(y,S,I(y))}\right]dydx.
\end{aligned}
$$

Since Λ satisfies $(S-S^*)(\Lambda(S)-\Lambda(S^*))\le 0$ and $\mathscr{G}$ is nonnegative for all $a>0$, we conclude that the sum of the first five terms on the right hand side of (5.21) is non-positive for all $t\ge 0$. By (5.12), the sixth term is zero for all $t\ge 0$. By assumption (5.10), we have

$$
\begin{aligned}
&\frac{S^*}{S}\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}-\frac{I(y)}{I^*(y)}-1+\frac{S}{S^*}\frac{I(y)}{I^*(y)}\frac{f(y,S^*,I^*(y))}{f(y,S,I(y))}\\
&=\frac{f(y,S,I(y))}{f(y,S^*,I^*(y))}\frac{S}{S^*}\left[\frac{S^*}{S}-\frac{f(y,S^*,I^*(y))}{f(y,S,I(y))}\right]\left[\frac{S^*}{S}-\frac{\frac{f(y,S^*,I^*(y))}{I^*(y)}}{\frac{f(y,S,I(y))}{I(y)}}\right]\le 0,
\end{aligned}
$$

which implies that the last term on the right hand side of (5.21) is non-positive. Thus, we have shown that $\frac{dV}{dt}\le 0$ for all $t\ge 0$. Furthermore,

$$
\frac{dV}{dt}=0 \quad \text{if and only if} \quad S(t)=S^* \text{ and } I(t,\cdot)=I^*(\cdot). \qquad (5.22)
$$

We have shown that the function $V(t)$ is decreasing ($V'(t)\le 0$) along any solution $(S(t),I(t,\cdot))^T$ with initial data in X_{++}. This implies that all limit points of $(S(t),I(t,\cdot))^T$ belong to the set where $V'(t)=0$. By (5.22), the omega limit set of $(S(t),I(t,\cdot))^T$ is the singleton $\{(S^*,I^*(\cdot))^T\}$. This implies that $(S^*,I^*(\cdot))^T$ attracts all points in X_{++}, and thus $\mathcal{A}_0=\{(S^*,I^*(\cdot))^T\}$. Therefore $(S^*,I^*(\cdot))^T$ is globally asymptotically stable and unique. □

6. Examples

In this section, we apply our results established in previous sections to two examples.

6.1. Bilinear incidence

We consider

$$\begin{aligned} \dot{S}(t) &= \Lambda - \mu S(t) - S(t) \int_\Omega \beta(y) I(t,y) dy, \\ I_t(t,x) &= -\kappa(x) I(t,x) + \alpha(x) S(t) \int_\Omega \beta(y) I(t,y) dy + \int_\Omega \theta(y,x) I(t,y) dy \\ &\quad - \gamma(x) I(t,x), \end{aligned} \tag{6.1}$$

where $\Lambda > 0$, $\mu > 0$, $\beta \in C(\Omega)$ with $\inf_\Omega \beta > 0$, and κ, α, θ and γ are as in Section 1.

It can be verified that $f(x, S, I) = \beta(x) SI$ satisfies all assumptions and the disease-free equilibrium is given by $(\frac{\Lambda}{\mu}, 0)^T$. In this case, the threshold operator $\mathcal{L}$ takes the form

$$\mathcal{L}[I](x) = \frac{\alpha(x)}{\kappa(x) + \gamma(x)} \frac{\Lambda}{\mu} \int_\Omega \beta(y) I(y) dy + \frac{\int_\Omega \theta(y,x) I(y) dy}{\kappa(x) + \gamma(x)}, \quad x \in \Omega.$$

Let $\mathcal{R}_0 = r(\mathcal{L})$. By Theorems 4.1 and 5.1, if $\mathcal{R}_0 < 1$, then the disease-free equilibrium $(\frac{\Lambda}{\mu}, 0)^T$ is the unique stationary solution of (6.1) in X_+ and it is globally asymptotically stable with respect to X_+. If $\mathcal{R}_0 > 1$, then by Theorem 4.2, system (6.1) admits an endemic equilibrium in X_{++}. We remark that the existence of an endemic equilibrium in X_+ can alternatively be proved using the theory of monotone dynamical systems. In fact, define a map $\mathscr{F} : C_+(\Omega) \to C_+(\Omega)$ by

$$\mathscr{F}[I](x) = \frac{\alpha(x)}{\kappa(x) + \gamma(x)} \frac{\Lambda \int_\Omega \beta(y) I(y) dy}{\mu + \int_\Omega \beta(y) I(y) dy} + \frac{\int_\Omega \theta(y,x) I(y) dy}{\kappa(x) + \gamma(x)}, \quad x \in \Omega. \tag{6.2}$$

The existence of an endemic equilibrium of (6.1) in X_{++} can be shown by proving the existence of a nonzero fixed point of $\mathscr{F}$ in $C_+(\Omega)$, for which we can apply results from abstract monotone dynamical systems (see e.g. [45]). However, due to the lack of sub-linearity of $\mathscr{F}$, the uniqueness can not be obtained in this way generally.

When $\mathcal{R}_0 > 1$, an endemic equilibrium $(S^*, I^*)^T$ of (6.1) satisfies $\inf_\Omega I^* > 0$.

Theorem 6.1. *Suppose $\mathcal{R}_0 > 1$. Then, there exists a unique endemic equilibrium $(S^*, I^*)^T$, and it is globally asymptotically stable in X_{++}.*

Proof. To apply Theorem 5.5, we need to verify the assumption (5.10). We note that

$$\left[\frac{S^*}{S} - \frac{\beta(x) S^* I^*(x)}{\beta(x) SI}\right]\left[\frac{S^*}{S} - \frac{\frac{\beta(x) S^* I^*(x)}{I^*(x)}}{\frac{\beta(x) SI}{I}}\right] = \left[\frac{S^*}{S} - \frac{S^* I^*(x)}{SI}\right]\left[\frac{S^*}{S} - \frac{S^*}{S}\right] \equiv 0,$$

and Theorem 5.5 applies. □

6.2. A spatial epidemic model

As mentioned in Section 1, the state variable x can be considered as a spatial variable. In this case, by choosing a dispersal kernel J satisfying the following conditions: $J:\mathbb{R}^d \to \mathbb{R}$ is continuous and nonnegative, and satisfies $J(0)>0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$ (note that if the infected population has a tendency to move toward some direction, then J is not necessarily symmetric), we obtain the spatial epidemic model with nonlocal dispersal (1.6), namely,

$$\begin{aligned} \dot{S} &= \Lambda(S) - \int_\Omega f(y,S,I(t,y))dy, \\ I_t(t,x) &= -\kappa(x)I + \alpha(x)\int_\Omega f(y,S,I(t,y))dy + \mathcal{J} * I(x) - I(x), \end{aligned} \tag{6.3}$$

where Λ, f, κ, α and γ are as in Section 1, and $\mathcal{J} \star I(x) = \int_\Omega J(x-y)I(y)dy$. Setting $\theta(y,x) = J(x-y)$, we notice that

$$\int_\Omega \theta(x,y)dy = \int_\Omega J(y-x)dy \le \int_{\mathbb{R}^N} J(x)dx = 1,$$

and therefore, the balance condition (1.3) fails in general, but the weaker condition (1.4) is satisfied. As $J(0)>0$, there holds $\theta(x,x)>0$ for $x\in\Omega$. We then apply the results established in previous sections to conclude the following result.

Theorem 6.2. *Let $\mathcal{R}_0$ be the spectral radius of the operator*

$$\mathcal{L}[I](x) := \frac{\alpha(x)}{\kappa(x)+1}\int_\Omega f_I(y,S^0,0)I(y)dy + \frac{\mathcal{J}\star I(x)}{\kappa(x)+1}, \quad x\in\Omega.$$

Then,

(1) *if $\mathcal{R}_0 < 1$, $(S^0,0)^T$ is the unique stationary solution of (6.3) in X_+ and it is globally asymptotically stable in X_+;*
(2) *if $\mathcal{R}_0 > 1$, (6.3) admits a stationary solution in X_{++}, which is globally asymptotically stable in $X_+\backslash\{0\}$.*

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