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Convergence to Equilibrium in Fokker–Planck Equations

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Abstract
The present paper is devoted to the investigation of long-time behaviors of global probability solutions of Fokker–Planck equations with rough coefficients. In particular, we prove the convergence of probability solutions under a Lyapunov condition in terms of the Markov semigroup associated to the stationary one. A generalization of earlier results on the existence and uniqueness of global probability solutions is also given.

Keywords Fokker–Planck equation · Probability solutions · Convergence

Mathematics Subject Classification Primary 37C40 · 37C75 · 34F05 · 60H10; Secondary 35Q84 · 60J60

1 Introduction

Let $\mathcal{U} \subset \mathbb{R}^n$ be an open and connected domain which can be bounded, unbounded, or the whole space $\mathbb{R}^n$. Stochastic differential equations of the form

$$dx = V(x)dt + G(x)dW, \quad x \in \mathcal{U}, \tag{1.1}$$

Dedicated to the memory of Professor George R. Sell.

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where $V : \mathcal{U} \to \mathbb{R}^n$, $G : \mathcal{U} \to \mathbb{R}^{n\times m}$ and W is a standard m-dimensional Wiener process, are often used to model and describe the uncertainties or impurities in a dynamical system in $\mathcal{U}$. If $GG^\top$ is everywhere positive definite, V and G are locally Lipschitz continuous, and solutions of (1.1) exist globally in the sense of distributions, then it is well-known that (1.1) generates a Markov semigroup on $B_b(\mathcal{U})$, the space of bounded measurable functions on $\mathcal{U}$, which is strongly Feller and irreducible. Consequently, if the Markov semigroup admits a unique invariant measure, then it follows from classical convergence results of Markov semigroups (see e.g. [13,18,22]) that the transition probability functions of (1.1) converge to the invariant measure as $t \to \infty$. We note that the existence and uniqueness of an invariant measure for the Markov semigroup requires certain dissipative or Lyapunov conditions in $\mathcal{U}$ (see e.g. [1,18,22,25]).

The present paper aims at studying a similar convergence problem when the coefficients of (1.1) are rough, which is actually the case in many applications. To do so, we consider the following Fokker–Planck equation

$$\partial_t u = Lu := \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad t > 0,\ x \in \mathcal{U}, \tag{1.2}$$

where $\partial_i = \partial_{x_i}$, $\partial^2_{ij} = \partial^2_{x_i x_j}$ and the usual summation convention is used. We note that, though the Fokker–Planck equation has interests in its own right in describing diffusive processes in general, it can be generated from the stochastic differential equation (1.1) when $(a^{ij}) := \frac{GG^\top}{2}$ and $V = (V^i)$ in the sense that the probability distribution of solutions of (1.1) formally satisfies (1.2).

We make the following standard hypothesis:

(H) The diffusion matrix $A = (a^{ij})$ is pointwise positive-definite, and $a^{ij} \in W^{1,p}_{loc}(\mathcal{U})$, $V^i \in L^p_{loc}(\mathcal{U})$, $i, j \in \{1, \ldots, n\}$, where $p > n + 2$ is fixed.

We note that as $p > n$ the diffusion matrix $A = (a^{ij})$ is in fact locally uniformly positive-definite by Sobolev imbedding.

Under the regularity conditions in **(H)**, we will consider weak solutions of (1.2) in the sense of measure. Denote

$$\mathcal{L} = a^{ij}\partial^2_{ij} + V^i\partial_i$$

as the formal L^2 adjoint of the Fokker–Planck operator L.

Definition 1.1 Let ν be a Borel probability measure on $\mathcal{U}$ and $T > 0$ be given. We say that a family of Borel measures $\mu = (\mu_t)_{t\in[0,T)}$ on $\mathcal{U}$ is a *measure solution* of (1.2) in $[0, T)$ with initial condition ν if $a^{ij}, V^i \in L^1_{loc}(\mathcal{U}, d\mu_t dt)$ for any $i, j \in \{1, \ldots, n\}$,

$$\mu_0 = \nu, \tag{1.3}$$

and the identities

$$\int_{\mathcal{U}} \phi d\mu_t = \int_{\mathcal{U}} \phi d\nu + \lim_{\epsilon\to 0^+} \int_\epsilon^t \int_{\mathcal{U}} \mathcal{L}\phi d\mu_s ds, \quad \phi \in C_0^\infty(\mathcal{U}),$$

hold a.e. $t \in [0, T)$. If, in addition, $\mu_t(\mathcal{U}) \le 1$ for a.e. $t \in [0, T)$ (respectively, $\mu_t(\mathcal{U}) = 1$ for a.e. $t \in [0, T)$), then $\mu = (\mu_t)_{t\in[0,T)}$ is called a *sub-probability solution* (respectively, *probability solution*). In the case $T = \infty$, a measure solution, a sub-probability solution and a probability solution are called a *global measure solution*, a *global sub-probability solution* and a *global probability solution*, respectively.

In literature (see e.g. [2,3,24]), the following equivalent definition has been used: A family of Borel measures $\mu = (\mu_t)_{t\in[0,T)}$ on $\mathcal{U}$ is called a measure solution of (1.2) in $[0, T)$ satisfying the initial condition (1.3) if $a^{ij}, V^i \in L^1_{loc}(\mathcal{U}, d\mu_t dt)$ for any $i, j \in \{1, \dots, n\}$, and

$$\int_0^T \int_{\mathcal{U}} (\partial_t u + \mathcal{L}u) d\mu_t dt = 0, \quad \forall u \in C_0^\infty((0, T) \times \mathcal{U}),$$

and for any $\phi \in C_0^\infty(\mathcal{U})$, there exists a set $J_\phi \subset (0, T)$ with $|(0, T)\backslash J_\phi| = 0$ such that

$$\lim_{t\in J_\phi, t\to 0} \int_{\mathcal{U}} \phi d\mu_t = \int_{\mathcal{U}} \phi d\nu.$$

The existence and uniqueness of global probability solutions of Fokker–Planck equations have been studied by many authors (see e.g. [2–4,10,24,26]) even for time-dependent coefficients. In particular, such existence and uniqueness result has been recently shown for (1.2) in [24] under the existence of a Lyapunov-like function $U \in C^2(\mathcal{U})$ which is non-negative and satisfies

$$\lim_{x\to\partial\mathcal{U}} U(x) = \infty \tag{1.4}$$

and

$$\mathcal{L}U(x) \le C_1 + C_2 U(x), \quad x \in \mathcal{U} \tag{1.5}$$

for some constants $C_1, C_2 \ge 0$.

As opposed to the issue of global existence and uniqueness, not much results on the convergence of global probability solutions of (1.2) are available or explicitly stated even in the case of $\mathcal{U} = \mathbb{R}^n$. For the case of uniformly Hölder continuous and bounded coefficients of (1.2) in $\mathcal{U} = \mathbb{R}^n$, certain strong dissipative condition is assumed in [21] to yield the convergence of global probability solutions of (1.2) to the stationary one (also see [14] and references therein).

In this paper, we would like to give some results on the convergence of global probability solutions of (1.2) in a general domain $\mathcal{U}$ in $\mathbb{R}^n$ which can be bounded, unbounded, or the whole space $\mathbb{R}^n$. We will do so under the assumption **(H)** and certain Lyapunov type of conditions imposed only near $\partial\mathcal{U}$. It turns out that we are also able to obtain the existence and uniqueness of global probability solutions of (1.2) by imposing conditions like (1.5) but only near $\partial\mathcal{U}$.

For any non-negative continuous function U on $\mathcal{U}$ satisfying (1.4), we note that all sub-level sets

$$\Omega_\rho = \{x \in \mathcal{U} : U(x) < \rho\}, \quad \rho > 0$$

are pre-compact. In fact, such a function is a special compact function on $\mathcal{U}$ defined in [19,20]. We refer the reader to [19,20] for the meaning of $x \to \partial\mathcal{U}$ when $\mathcal{U}$ is unbounded. In particular, when $\mathcal{U} = \mathbb{R}^n$, $x \to \partial\mathbb{R}^n$ simply means $|x| \to \infty$.

Making use of such compactness, we will work with Lyapunov types of functions as follows.

Definition 1.2 Let $U \in C^2(\mathcal{U})$ be a non-negative function.

(1) U is said to be an *unbounded Lyapunov-like function* associated to $\mathcal{L}$ if (1.4) holds and there are constants $C_1, C_2, \rho_m > 0$ such that

$$\mathcal{L}U(x) \le C_1 + C_2 U(x), \quad x \in \mathcal{U}\backslash\overline{\Omega}_{\rho_m}. \tag{1.6}$$

(2) U is said to be an *unbounded Lyapunov function* associated to $\mathcal{L}$ if (1.4) holds and there are constants $\gamma, \rho_m > 0$ such that

$$\mathcal{L}U(x) \le -\gamma, \quad x \in \mathcal{U}\backslash\overline{\Omega}_{\rho_m}. \tag{1.7}$$

(3) U is said to be an *unbounded strong Lyapunov function* associated to $\mathcal{L}$ if (1.4) holds and there are constants $C_1, C_2, \rho_m > 0$ such that

$$\mathcal{L}U(x) \le C_1 - C_2 U(x), \quad x \in \mathcal{U}\backslash\overline{\Omega}_{\rho_m}. \tag{1.8}$$

The condition (1.6) is weaker than (1.5) under the assumption **(H)** on the drift term $V = (V^i)$. They are of course equivalent if V is continuous on $\mathcal{U}$. Due to the condition (1.4), it is clear that a function of class (3) is stronger than that of (2) which is further stronger than that of (1).

Our result concerning the existence and uniqueness of global probability solutions of (1.2) states as follows.

Theorem A *Suppose* **(H)** *and that there exists an unbounded Lyapunov-like function associated to $\mathcal{L}$. Let ν be a Borel probability measure on $\mathcal{U}$. Then, the following statements hold.*

(1) The equation (1.2) *admits a global probability solution* $(\mu_t)_{t\in[0,\infty)}$ *with initial condition ν satisfying the property that the function $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ is continuous on* $[0, \infty)$ *for any* $\phi \in C_0^\infty(\mathcal{U})$.

(2) If $(\tilde{\mu})_{t\in[0,\infty)}$ *is a global sub-probability solution of* (1.2) *with initial condition ν satisfying the function $t \mapsto \int_{\mathcal{U}} \phi d\tilde{\mu}_t$ being continuous on* $[0, \infty)$ *for any* $\phi \in C_0^\infty(\mathcal{U})$*, then* $\tilde{\mu}_t = \mu_t$ *for all* $t > 0$.

Among probability solutions of (1.2), the stationary ones, defined as follows, are of particular importance.

Definition 1.3 Under the assumption **(H)**, a Borel probability measure μ on $\mathcal{U}$ is called a *stationary measure* of the Fokker–Planck equation (1.2) if $L\mu = 0$ in the sense that

$$V^i \in L^1_{loc}(\mathcal{U}, \mu), \ i = 1, \ldots, n, \quad \text{and} \quad \int_{\mathcal{U}} \mathcal{L}\phi d\mu = 0, \quad \forall \phi \in C_0^\infty(\mathcal{U}).$$

We note that the above definition actually only requires $p > n$ in **(H)**. The existence and uniqueness of stationary measures of (1.2) under Lyapunov type of conditions have been extensively studied in literature (see e.g. [8,9,11,12,18,20,28]). In particular, it is shown in [20] that stationary measures of (1.2) exist under the condition **(H)** with $p > n + 2$ replaced by $p > n$ and the existence of a Lyapunov function associated to $\mathcal{L}$ which needs not be unbounded. The unboundedness of the Lyapunov function however guarantees the uniqueness of stationary measures (see also [11] for the case $\mathcal{U} = \mathbb{R}^n$).

As mentioned at the beginning of this section, if (1.1) generates a Markov semigroup which admits a unique invariant measure, then the transition probability functions of (1.1), i.e., the global probability solutions of (1.2), converge to the invariant measure which is necessarily a stationary measure of (1.2).

Under the condition **(H)** and in the presence of an unbounded Lyapunov function associated to $\mathcal{L}$, a natural question is whether one can have a similar theory for the convergence of global probability solutions of (1.2) to the unique stationary measure. Thanks to the recent works [12,27] which show the existence of a unique strongly continuous Markov semigroup associated to the unique stationary measure of (1.2), we have the following convergence results for global probability solutions of (1.2) which also indicate certain exponential mixing property of the unique stationary measure.

Theorem B *Suppose* **(H)**. *Then, the following statements hold.*

(1) If there exists an unbounded Lyapunov function associated to $\mathcal{L}$, then, as $t \to \infty$, any global probability solution $(\mu_t)_{t\in[0,\infty)}$ of (1.2) *strongly converges to the unique stationary measure μ_*, i.e., for any Borel set $B \subset \mathcal{U}$, there holds*

$$\mu_t(B) \to \mu_*(B) \quad as \quad t \to \infty.$$

(2) If there exists an unbounded strong Lyapunov function U associated to $\mathcal{L}$, then, as $t \to \infty$, any global probability solution $(\mu_t)_{t\in[0,\infty)}$ of (1.2) *with initial condition ν satisfying $U \in L^1(\mathcal{U}, \nu)$ exponentially converges to the unique stationary measure μ_* which is exponentially mixing, i.e., there are constants $C, r > 0$ such that*

$$\|\mu_t - \mu_*\|_{TV} \le Ce^{-rt}, \quad t \ge 0,$$

where $\|\cdot\|_{TV}$ denotes the total variation distance, and

$$\left|\int_{\mathcal{U}} \psi T_t \phi d\mu_* - \int_{\mathcal{U}} \psi d\mu_* \int_{\mathcal{U}} \phi d\mu_*\right| \le Ce^{-rt} \left\|(\phi - \langle \mu_*, \phi\rangle)^2\right\|_U^{1/2} \left\|\psi^2\right\|_U^{1/2}, \quad t \ge 0,$$

for any pair of measurable functions ϕ and ψ with $\|\phi^2\|_U < \infty$ and $\|\psi^2\|_U < \infty$, where $\langle \mu_*, \phi\rangle = \int_{\mathcal{U}} \phi d\mu_*$ and

$$\|\phi\|_U := \left\|\frac{\phi}{1+U}\right\|_{L^\infty(\mathcal{U},\mu_*)} = \left\|\frac{\phi}{1+U}\right\|_{L^\infty(\mathcal{U},dx)}.$$

The rest of the paper is organized as follows. In Sect. 2, we study the existence and uniqueness of global probability solutions and prove Theorem A. In Sect. 3, we investigate the long-time behaviors of global probability solutions and prove the results of convergence as stated in Theorem B. In Sect. 4, some examples are given as applications of Theorems A, B. In Appendix A, we collect some facts from [12,27] about Markov semigroups associated to stationary measures.

Through the rest of the paper, for short, we refer a measure solution, a sub-probability solution or a probability solution of (1.2) with initial condition ν as that of the initial value problem (1.2)–(1.3), or Cauchy problem (1.2)–(1.3), or simply, (1.2)–(1.3).

2 Existence and Uniqueness of Global Probability Solutions

In this section, we prove the existence and uniqueness of global probability solutions of the initial value problem (1.2)–(1.3). We first construct a global sub-probability solution, which is further shown to be a global probability solution in the presence of an unbounded Lyapunov-like function, which also ensures the uniqueness. In particular, Theorem A is proven by combining Propositions 2.1, 2.2 and 2.3 below.

2.1 Existence of Global Sub-probability Solutions

We first prove the existence of global sub-probability solutions of the Cauchy problem (1.2)–(1.3).

Proposition 2.1 *Assume* **(H)**. *Then, the Cauchy problem* (1.2)–(1.3) *admits a global sub-probability solution $(\mu_t)_{t\in[0,\infty)}$ in the sense of* Definition 1.1. *Moreover, the following properties of $(\mu_t)_{t\in[0,\infty)}$ hold.*

(i) $\mu_t(\mathcal{U}) \le 1$ *for all* $t \in (0,\infty)$*;*
(ii) There exists a positive and locally Hölder continuous function $u \in L^p_{loc}((0,\infty), W^{1,p}_{loc}(\mathcal{U}))$ *with* $\partial_t u \in L^p_{loc}((0,\infty), W^{-1,p}_{loc}(\mathcal{U}))$ *such that* $d\mu_t dt = u(t,x)dxdt$*;*
(iii) For any $\phi \in C_0^\infty(\mathcal{U})$*, the map* $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ *is a continuous function on* $[0,\infty)$*, and*

$$\int_{\mathcal{U}} \phi d\mu_t = \int_{\mathcal{U}} \phi d\mu_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi d\mu_\tau d\tau, \quad 0 < s < t < \infty.$$

Proof This theorem is more or less proven in [2,3,24] by means of the regularity theory established in [5]. For completeness, we recall the essential arguments below.

By [3, Theorem 3.1], [5, Section 3] and [24, Theorem 2.3], for any $T > 1$, there exists a sub-probability solution $(\mu_t^T)_{t\in[0,T)}$ of (1.2)–(1.3) on $[0,T)$ such that (i)-(iii) with $[0,\infty)$ replaced by $[0,T)$ hold. Let u^T be the function on $(0,T)\times\mathcal{U}$ given in (ii), so $d\mu_t^T dt = u^T(t,x)dxdt$.

We then apply the regularity theory established in [5, Section 3] to conclude that there are a sequence $\{T_k\}_{k\ge1}$ with $T_k \to \infty$ as $k \to \infty$ and a nonnegative function u on $(0,\infty)\times\mathcal{U}$ such that $u^{T_k}(t,x)$ converges to u locally uniformly as $k \to \infty$. Then, the measures $(\mu_t)_{t\in[0,\infty)}$ defined by u, i.e., $d\mu_t dt = u(t,x)dxdt$, are a global measure solution of (1.2)–(1.3). Now, (i) follows readily, (ii) follows from the regularity theory (see [5, Corollary 3.9]), Harnack's inequality, Sobolev imbedding (see e.g. [23, Section 7]) and the connectivity of $\mathcal{U}$, and (iii), except the continuity of $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ at $t = 0$, is a consequence of Lebesgue dominated convergence theorem and the locally uniform convergence of u^{T_k} to u as $k \to \infty$.

It remains to show the continuity of $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ at $t = 0$, namely, $\int_{\mathcal{U}} \phi d\mu_t \to \int_{\mathcal{U}} \phi d\nu$ as $t \to 0^+$. As in the proof of [24, Theorem 2.3], we can prove the existence of some $C > 0$, $\alpha > 0$ and $\epsilon_0 > 0$ such that for any $T > 0$ there holds

$$\left| \int_{\mathcal{U}} \phi u^T(t,x)dx - \int_{\mathcal{U}} \phi d\nu \right| \le Ct^\alpha, \quad t \in (0,\epsilon_0],$$

which leads to the result. □

Denote by $C_c(\mathcal{U})$ the space of compactly supported continuous functions on $\mathcal{U}$.

Lemma 2.1 *Assume* **(H)**. *Let* $(\mu_t)_{t\in[0,\infty)}$ *be the global sub-probability solution of* (1.2)–(1.3) *given in* Proposition 2.1. *Then, for any* $\phi \in C_c(\mathcal{U})$*, the function* $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ *is continuous on* $[0,\infty)$*. If, in addition,* $\mu_t(\mathcal{U}) = 1$ *for all* $t > 0$*, then for any* $\phi \in C(\mathcal{U})$ *that is constant outside some compact set, the function* $t \mapsto \int_{\mathcal{U}} \phi d\mu_t$ *is continuous on* $[0,\infty)$*.*

Proof Fix $\phi \in C_c(\mathcal{U})$ and $t_0 \in [0,\infty)$. Let Ω_* be an open bounded set such that $\text{supp}(\phi) \subset\subset \Omega_*$. By smooth approximation and the continuity of ϕ, we can find a sequence $(\phi_n)_{n\ge1} \subset C_0^\infty(\mathcal{U})$ with $\text{supp}(\phi_n) \subset \Omega_*$ for all n such that $\phi_n \to \phi$ as $n \to \infty$ locally uniformly on $\mathcal{U}$. In particular, $\phi_n \to \phi$ as $n \to \infty$ uniformly on $\overline{\Omega}_*$.

It then follows that for any $n \ge 1$,

$$\begin{aligned}
&\left| \int_{\mathcal{U}} \phi d\mu_t - \int_{\mathcal{U}} \phi d\mu_{t_0} \right| \\
&\quad \le \left| \int_{\mathcal{U}} \phi d\mu_t - \int_{\mathcal{U}} \phi_n d\mu_t \right| + \left| \int_{\mathcal{U}} \phi_n d\mu_t - \int_{\mathcal{U}} \phi_n d\mu_{t_0} \right| + \left| \int_{\mathcal{U}} \phi_n d\mu_{t_0} - \int_{\mathcal{U}} \phi d\mu_{t_0} \right| \\
&\quad \le 2\sup_{\overline{\Omega}_*} |\phi - \phi_n| + \left| \int_{\mathcal{U}} \phi_n d\mu_t - \int_{\mathcal{U}} \phi_n d\mu_{t_0} \right|.
\end{aligned}$$

The result now follows by first setting $t \to t_0$ and then $n \to \infty$. □

2.2 Existence of Global Probability Solutions

In this subsection, we prove that the global sub-probability solution $(\mu_t)_{t\in[0,\infty)}$ of (1.2)–(1.3) constructed in Proposition 2.1 is actually a global probability solution in the presence of an unbounded Lyapunov-like function.

The following result generalizes [3, Theorem 3.1] and [24, Theorem 2.7] in the case of time-independent coefficients.

Proposition 2.2 *Suppose* **(H)** *and that there exists an unbounded Lyapunov-like function associated to* $\mathcal{L}$. *Then, the global sub-probability solution* $(\mu_t)_{t\in[0,\infty)}$ *of* (1.2)–(1.3) *given in* Proposition 2.1 *is a global probability solution of* (1.2)–(1.3).

Proof Let $U \in C^2(\mathcal{U})$ be the unbounded Lyapunov-like function associated to $\mathcal{L}$ as in the statement and $C_1, C_2, > 0$, $\rho_m > 0$ be as in (1.6) for the present U. We may assume, without loss of generality, that $U \in L^1(\mathcal{U}, \nu)$. In fact, arguing exactly the same as in the proof of [3, Corollary 2.3], an C^2 function θ (depending on ν) satisfying

$$\theta(0) = 0, \quad \theta(\infty) = \infty, \quad 0 \le \theta' \le 1, \quad \theta'' \le 0 \quad \text{and} \quad \theta \circ U \in L^1(\nu). \tag{2.1}$$

can be found. Then, for $x \in \mathcal{U}\setminus\overline{\Omega}_{\rho_m}$,

$$\begin{aligned}\mathcal{L}(\theta \circ U) &= \theta'(U)\mathcal{L}U + \theta''(U)a^{ij}\partial_i U\partial_j U \\ &\le C_1 + C_2\theta'(U)U \\ &\le C_1 + C_2\theta \circ U,\end{aligned}$$

where we used $a^{ij}\partial_i U\partial_j U \ge 0$ due to the positive-definiteness, and $\theta'(t)t \le \theta(t)$ due to the concavity. Hence, $U \in L^1(\mathcal{U}, \nu)$ is assumed in the rest of the proof.

To proceed, we fix $\rho_0 \in (\rho_m, \infty)$, and for $\rho \ge \rho_0 + 1$, we define $\eta_\rho \in C^2[0, \infty)$ such that it is non-decreasing and satisfies

$$\eta_\rho(t) = \begin{cases} 0, & t \in [0, \rho_m], \\ t, & t \in [\rho_0, \rho], \\ \rho + 1, & t \in [\rho + 2, \infty). \end{cases}$$

We can further fix the shape of $\eta_\rho(t)$ for $t \in [\rho_m, \rho_0] \cup [\rho, \rho + 2]$ so that

$$\begin{aligned} &C := \sup_{\rho\ge\rho_0+1}\sup_{t\ge0}\max\{\eta'_\rho(t), |\eta''_\rho(t)|\} < \infty, \\ &\eta''_\rho(t) \le 0 \text{ for all } t \in [\rho, \rho + 2] \text{ and } \rho \ge \rho_0 + 1. \end{aligned} \tag{2.2}$$

Set $\eta(t) = \lim_{\rho\to\infty}\eta_\rho(t)$.

By Proposition 2.1, there holds

$$\int_{\mathcal{U}}\phi d\mu_t = \int_{\mathcal{U}}\phi d\mu_s + \int_s^t\int_{\mathcal{U}}\mathcal{L}\phi d\mu_\tau d\tau, \quad 0 < s < t < \infty$$

for any $\phi \in C_0^\infty(\mathcal{U})$. By assumptions on the coefficients of $\mathcal{L}$, the above equality also holds for any $\phi \in C_0^2(\mathcal{U})$. In particular, since $\eta_\rho(U) - (\rho + 1) \in C_0^2(\mathcal{U})$, we have

$$\int_{\mathcal{U}}[\eta_\rho(U) - (\rho + 1)]d\mu_t = \int_{\mathcal{U}}[\eta_\rho(U) - (\rho + 1)]d\mu_s + \int_s^t\int_{\mathcal{U}}\mathcal{L}(\eta_\rho(U))d\mu_\tau d\tau.$$

Denote Ω_ρ as ρ-sub-level set of U for each $\rho > 0$. To treat the second term on the right-hand side of the above equality, we write

$$\begin{aligned}
&\int_s^t \int_{\mathcal{U}} \mathcal{L}(\eta_\rho(U)) d\mu_\tau d\tau \\
&= \int_s^t \int_{\mathcal{U}} \eta'_\rho(U) \mathcal{L} U d\mu_\tau d\tau + \int_s^t \int_{\mathcal{U}} \eta''_\rho(U) a^{ij} \partial_i U \partial_j U d\mu_\tau d\tau \\
&= \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_{\rho m}} \eta'_\rho(U) \mathcal{L} U d\mu_\tau d\tau + \int_s^t \int_{\Omega_{\rho_0} \setminus \overline{\Omega}_{\rho m}} \eta''_\rho(U) a^{ij} \partial_i U \partial_j U d\mu_\tau d\tau \\
&+ \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_\rho} \eta''_\rho(U) a^{ij} \partial_i U \partial_j U d\mu_\tau d\tau.
\end{aligned} \tag{2.3}$$

For the first term on the right-hand side of the second equality in (2.3), we find from (1.6) and the fact $\mu_t(\mathcal{U}) \leq 1$ that

$$\begin{aligned}
&\int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_{\rho m}} \eta'_\rho(U) \mathcal{L} U d\mu_\tau d\tau \leq C_1(t-s) + \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_{\rho m}} \eta'_\rho(U) U d\mu_\tau d\tau \\
&\leq C_1(t-s) + C\rho_0(t-s) + \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_{\rho_0}} \eta_\rho(U) d\mu_\tau d\tau \\
&= C_1(t-s) + \int_s^t \int_{\Omega_{\rho_0} \setminus \overline{\Omega}_{\rho m}} \eta'_\rho(U) U d\mu_\tau d\tau \\
&+ \int_s^t \int_{\Omega_\rho \setminus \overline{\Omega}_{\rho_0}} U d\mu_\tau d\tau + \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_\rho} \eta'_\rho(U) U d\mu_\tau d\tau \\
&\leq C_1(t-s) + C\rho_0(t-s) + \int_s^t \int_{\Omega_\rho \setminus \overline{\Omega}_{\rho_0}} U d\mu_\tau d\tau + \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_\rho} \eta_\rho(U) d\mu_\tau d\tau \\
&= C_1(t-s) + C\rho_0(t-s) + \int_s^t \int_{\Omega_{\rho+2} \setminus \overline{\Omega}_{\rho_0}} \eta_\rho(U) d\mu_\tau d\tau,
\end{aligned}$$

where $C > 0$ is as in (2.2). In the above estimates, we used the facts that $a^{ij} \partial_i U \partial_j U \geq 0$ and $\eta''_\rho(t) \leq 0$ for $t \in (\rho, \rho+2)$, $\eta'_\rho(t)t \leq \eta(t)$ for $t \in (\rho, \rho+1)$, which are simple consequences of the construction of η_ρ.

Setting

$$C_3 := C_1 + C\rho_0 + C \sup_{\Omega_{\rho_0} \setminus \overline{\Omega}_{\rho m}} a^{ij} \partial_i U \partial_j U,$$

we arrive at

$$\int_s^t \int_{\mathcal{U}} \mathcal{L}(\eta_\rho(U)) d\mu_\tau d\tau \leq C_3(t-s) + \int_s^t \int_{\mathcal{U}} \eta_\rho(U) d\mu_\tau d\tau,$$

and hence,

$$\int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\mu_t \leq \int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\mu_s + C_3(t-s) + \int_s^t \int_{\mathcal{U}} \eta_\rho(U) d\mu_\tau d\tau. \tag{2.4}$$

Since $\eta_\rho(U) - (\rho+1) \in C_0^2(\mathcal{U})$, Lemma 2.1 ensures that

$$\lim_{s \to 0^+} \int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\mu_s = \int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\nu.$$

Therefore, passing to the limit $s \to 0^+$ in (2.4) yields

$$\int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\mu_t \le \int_{\mathcal{U}} [\eta_\rho(U) - (\rho+1)] d\nu + C_3 t + \int_0^t \int_{\mathcal{U}} \eta_\rho(U) d\mu_\tau d\tau,$$

which leads to

$$\int_{\mathcal{U}} \eta_\rho(U) d\mu_t + (\rho+1)[1-\mu_t(\mathcal{U})] \le \int_{\mathcal{U}} \eta_\rho(U) d\nu + C_3 t + \int_0^t \int_{\mathcal{U}} \eta_\rho(U) d\mu_\tau d\tau. \quad (2.5)$$

Since $\mu_t(\mathcal{U}) \le 1$, Grönwall's inequality gives

$$\int_{\mathcal{U}} \eta_\rho(U) d\mu_t \le \left[\int_{\mathcal{U}} \eta_\rho(U) d\nu + C_3 t \right] e^t.$$

Setting $\rho \to \infty$, we find

$$\int_{\mathcal{U}} \eta(U) d\mu_t \le \left[\int_{\mathcal{U}} \eta(U) d\nu + C_3 t \right] e^t. \quad (2.6)$$

Hence, for fixed $t > 0$, the right-hand side of (2.5) is bounded uniformly in $\rho \ge \rho_0 + 1$, and therefore, if $\mu_t(\mathcal{U}) < 1$, we set $\rho \to \infty$ in (2.5) to deduce a contradiction. Thus, $\mu_t(\mathcal{U}) = 1$ for all $t > 0$.

Remark 2.1 Note that we obtained the estimate (2.6) in the proof of Proposition 2.2. This in particular implies that under the assumption of Proposition 2.2, if U is an unbounded Lyapunov-like function associated to $\mathcal{L}$ and the initial condition ν satisfies $U \in L^1(\mathcal{U}, \nu)$, then the global probability solution $(\mu_t)_{t\in[0,\infty)}$ of (1.2)–(1.3) given in Proposition 2.1 and Proposition 2.2 satisfies $\int_{\mathcal{U}} U d\mu_t < \infty$ for all $t > 0$.

2.3 Uniqueness of Global Probability Solutions

In this subsection, we prove the uniqueness of global probability solutions of the Cauchy problem (1.2)–(1.3) in the class of global sub-probability solutions.

Proposition 2.3 *Suppose* **(H)** *and that there is an unbounded Lyapunov-like function associated to* $\mathcal{L}$. *Let* $(\mu_t)_{t\in[0,\infty)}$ *be the global probability solution of* (1.2)–(1.3) *given in Propositions* 2.1 *and* 2.2. *If* $(\tilde{\mu})_{t\in[0,\infty)}$ *is a global sub-probability solution of* (1.2)–(1.3), *then* $\tilde{\mu}_t = \mu_t$ *for all* $t > 0$.

Proof We adapt the proof of [24, Theorem 3.5]. Let U be the unbounded Lyapunov-like function associated to $\mathcal{L}$ and C_1, C_2, ρ_m be as in (1.6) for the present U. Let $u(t,x)$ and $\tilde{u}(t,x)$ be the densities of $(\mu_t)_{t\in[0,\infty)}$ and $(\tilde{\mu}_t)_{t\in[0,\infty)}$, respectively, and set $v(t,x) = \frac{\tilde{u}(t,x)}{u(t,x)}$. By the regularity theorem established in [5], $v(t,x)$ is continuous and positive. It is proven in [26, Lemma 2.2] (see also [24, Lemma 3.4]) that for any $\phi \in C_0^2(\mathcal{U})$ with $\phi \ge 0$ and $\lambda > 0$, there holds

$$\begin{aligned} &\int_{\mathcal{U}} \left[e^{\lambda(1-v(t,x))} - e^\lambda \right] \phi(x) d\mu_t \\ &\quad \le (1-e^\lambda) \int_{\mathcal{U}} \phi d\nu + \int_0^t \int_{\mathcal{U}} \left[e^{\lambda(1-v(s,x))} - e^\lambda \right] \mathcal{L}\phi(x) d\mu_s ds, \quad t > 0. \end{aligned} \quad (2.7)$$

Fix $\rho_0 \in (\rho_m, \infty)$. Let $\eta \in C^2([0,\infty))$ be a non-decreasing function satisfying

$$\eta(t) = \begin{cases} 0, & t \in [0, \rho_m], \\ t, & t \in [\rho_0, \infty). \end{cases}$$

Let $\zeta \in C^2([0,\infty))$ be a non-increasing and convex function satisfying $\zeta(0) = 1$ and $\zeta(t) = 0$ for $t \geq 1$.

For $N \gg 1$, an application of (2.7) with $\phi = \zeta(N^{-1}\eta(U))$ yields

$$\int_{\mathcal{U}} \left[e^{\lambda(1-v(t,x))} - e^{\lambda}\right] \zeta(N^{-1}\eta(U(x)))d\mu_t \leq (1-e^{\lambda}) \int_{\mathcal{U}} \zeta(N^{-1}\eta(U))d\nu + \underbrace{\int_0^t \int_{\mathcal{U}} [e^{\lambda(1-v(s,x))} - e^{\lambda}]\mathcal{L}(\zeta(N^{-1}\eta(U)))d\mu_s ds}_{(I)}, \tag{2.8}$$

where $\mathcal{L}(\zeta(N^{-1}\eta(U))$ is given by

$$\begin{aligned}\mathcal{L}(\zeta(N^{-1}\eta(U))) = \zeta'(N^{-1}\eta(U))N^{-1}\eta'(U)\mathcal{L}U \\ + \zeta''(N^{-1}\eta(U))[N^{-1}\eta'(U)]^2 a^{ij}\partial_i U \partial_j U \\ + \zeta'(N^{-1}\eta(U))N^{-1}\eta''(U)a^{ij}\partial_i U \partial_j U.\end{aligned}$$

It follows from the dominated convergence theorem and the fact $\zeta(N^{-1}\eta(U(x))) \to 1$ pointwise as $N \to \infty$ that

$$\int_{\mathcal{U}} \left[e^{\lambda(1-v(t,x))} - e^{\lambda}\right] \zeta(N^{-1}\eta(U(x)))d\mu_t \to \int_{\mathcal{U}} \left[e^{\lambda(1-v(t,x))} - e^{\lambda}\right] d\mu_t$$

and

$$(1-e^{\lambda}) \int_{\mathcal{U}} \zeta(N^{-1}\eta(U))d\nu \to 1 - e^{\lambda},$$

as $N \to \infty$. To treat the second term on the right hand side of (2.8), we write $(I) = (I1) + (I2) + (I3)$, where

$$\begin{aligned}(I1) &= \int_0^t \int_{\mathcal{U}} \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta'(N^{-1}\eta(U))N^{-1}\eta'(U)\mathcal{L}U d\mu_s ds, \\ (I2) &= \int_0^t \int_{\mathcal{U}} \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta''(N^{-1}\eta(U))[N^{-1}\eta'(U)]^2 a^{ij}\partial_i U \partial_j U d\mu_s ds, \\ (I3) &= \int_0^t \int_{\mathcal{U}} \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta'(N^{-1}\eta(U))N^{-1}\eta''(U)a^{ij}\partial_i U \partial_j U d\mu_s ds.\end{aligned}$$

As $[e^{\lambda(1-v(s,x))} - e^{\lambda}]\zeta'(N^{-1}\eta(U)) \geq 0$, we find

$$\begin{aligned}(I1) &\leq \frac{\sup \eta'}{N} \int_0^t \int_{\mathcal{U}\setminus\overline{\Omega}_{\rho m}} \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta'(N^{-1}\eta(U))(C_1 + C_2 U(x))d\mu_s ds \\ &\leq \frac{C \sup \eta'}{N} \int_0^t \int_{\{\eta(U)\leq N\}} (C_1 + C_2 U(x))d\mu_s ds,\end{aligned}$$

where

$$C = \sup_{[0,t]\times\mathcal{U}} \left| \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta'(N^{-1}\eta(U)) \right|.$$

Since

$$\left| \frac{1}{N} \chi_{\{\eta(U)\leq N\}}(C_1 + C_2 U(x)) \right| \leq C_1 + C_2$$

and

$$\frac{1}{N}\chi_{\{\eta(U)\le N\}}(C_1 + C_2U(x)) \to 0 \quad \text{as} \quad N \to \infty \quad \text{pointwise,}$$

the dominated convergence theorem yields $\limsup_{N\to\infty}(I1) \le 0$.

It is clear that $(I2) \le 0$. For $(I3)$, the fact $\eta''(t) = 0$ for $t \in [0, \rho_m] \cup [\rho_0, \infty)$ implies that

$$(I3) = \int_0^t \int_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho_m}} \left[e^{\lambda(1-v(s,x))} - e^{\lambda}\right] \zeta'(N^{-1}\eta(U))N^{-1}\eta''(U)a^{ij}\partial_i U\partial_j U d\mu_s ds.$$

The dominated convergence theorem then yields $\lim_{N\to\infty}(I3) = 0$. Hence, we have shown $\limsup_{N\to\infty}(I) \le 0$, which then implies that

$$\int_{\mathcal{U}} \left[e^{\lambda(1-v(t,x))} - e^{\lambda}\right] d\mu_t \le 1 - e^{\lambda},$$

i.e.,

$$\int_{\mathcal{U}} e^{\lambda(1-v(t,x))} d\mu_t \le 1, \quad \forall\lambda > 0.$$

If there is a Boreal set B in $\mathcal{U}$ with positive Lebesgue measure and $\delta > 0$ such that $v(t,x) \le 1-\delta$ for $x \in B$, then

$$e^{\lambda\delta}|B| \le \int_B e^{\lambda(1-v(t,x))} d\mu_t \le 1,$$

which leads to a contradiction by setting $\lambda \to \infty$. Therefore, $v(t,x) \ge 1$ a.e.. But, since $\int_{\mathcal{U}} \tilde{u}(t,x)dx \le 1 = \int_{\mathcal{U}} u(t,x)dx$, we conclude $v(t,x) = 1$ a.e., and hence, $v(t,x) = 1$ by continuity. □

3 Convergence to Equilibrium

In this section, we study the convergence of global probability solutions of (1.2)–(1.3) and prove, in particular, Theorem B. We first recall the following result concerning the existence and uniqueness of stationary measures.

Proposition 3.1 *Suppose* **(H)** *with $p > n+2$ replaced by $p > n$ and that there exists an unbounded Lyapunov function associated to $\mathcal{L}$. Then,* (1.2) *admits a unique stationary measure μ_* with density function lying in $W^{1,p}_{loc}(\mathcal{U})$.*

The existence part in Proposition 3.1 is proven in [20] even without the unboundedness assumption of a Lyapunov function (see [19,20] for definition of a general Lyapunov function). The uniqueness part in Proposition 3.1 follows from [12, Example 5.1] and [20, Theorem A] in which it is actually shown that the unique stationary measure $\mu_* \in \mathcal{M}_{md}$, where $\mathcal{M}_{md}$ is defined in (A.1). Hence, by Proposition A.2, the semigroup $(T_t)_{t\ge0}$ given in Proposition A.1 (1) is a Markov semigroup and the only C_0-semigroup in $L^1(\mathcal{U}, \mu_*)$ whose generator $(\overline{\mathcal{L}}, \mathcal{D}(\overline{\mathcal{L}}))$ extending $(\mathcal{L}, C_0^\infty(\mathcal{U}))$, and moreover, μ_* is the unique invariant measure of $(T_t)_{t\ge0}$. If **(H)** is assumed, then $(T_t)_{t\ge0}$ is joint continuous (see Proposition A.1 (2)), and hence,

$$T_t\phi(x) = \int_{\mathcal{U}} \phi(y)p(t,x,y)dy = \int_{\mathcal{U}} \phi(y)p_t(x,y)d\mu_*(y), \quad \phi \in L^1(\mathcal{U},\mu_*), \quad t>0, \tag{3.1}$$

where $p(t,x,y)$ and $p_t(x,y)$ satisfy properties described in Remark A.1 (3).

In what follows in this section, we assume **(H)** and the existence of an unbounded Lyapunov function associated to $\mathcal{L}$, and let $(\mu_t)_{t\in[0,\infty)}$ be the unique global probability solution of (1.2)–(1.3) obtained in Theorem A (or, in Propositions 2.1, 2.2 and 2.3) for a given Borel probability measure ν on $\mathcal{U}$. We use $\mathcal{B}(\mathcal{U})$ to denote the Borel σ-algebra of $\mathcal{U}$.

3.1 Strong Convergence to Equilibrium

To study the convergence of the global probability solution $(\mu_t)_{t\in[0,\infty)}$ as $t\to\infty$, we imbed it into the dual semigroup of $(T_t)_{t\geq 0}$. To do so, we note that for each $t>0$, $\phi\mapsto\int_{\mathcal{U}} T_t\phi d\nu$ defines a bounded linear functional on $C_b(\mathcal{U})$, and therefore, by Riesz representation theorem, for any $t>0$, there exists a unique Borel probability measure, denoted by $T_t^*\nu$, such that

$$\left\langle T_t^*\nu, \phi\right\rangle := \int_{\mathcal{U}} \phi dT_t^*\nu = \int_{\mathcal{U}} T_t\phi d\nu, \quad \forall \phi\in C_b(\mathcal{U}).$$

The next two lemmas summarize some properties of $T_t^*\nu$.

Lemma 3.1 *$T_t^*\nu$ strongly converges to μ_* as $t\to\infty$, i.e., for any $B\in\mathcal{B}(\mathcal{U})$, there holds*

$$T_t^*\nu(B)\to\mu_*(B) \quad \text{as} \quad t\to\infty.$$

Proof Note that

$$T_t^*\nu(B) = \langle T_t^*\nu, \chi_B\rangle = \int_{\mathcal{U}} T_t\chi_B d\nu, \quad t\geq 0.$$

By Remark A.1 (2), for any $B\in\mathcal{B}(\mathcal{U})$, there holds

$$\lim_{t\to\infty} T_t\chi_B(x) = \mu_*(B), \quad \forall x\in\mathcal{U}.$$

It follows that

$$T_t^*\nu(B) = \int_{\mathcal{U}} T_t\chi_B d\nu \to \mu_*(B) \quad \text{as} \quad t\to\infty,$$

i.e., $T_t^*\nu$ converges to μ_* strongly as $t\to\infty$. □

Lemma 3.2 *For any $\phi\in C_0^\infty(\mathcal{U})$, the function $t\mapsto\langle T_t^*\nu,\phi\rangle$ is continuous on $[0,\infty)$, and*

$$\left\langle T_t^*\nu, \phi\right\rangle = \left\langle T_s^*\nu, \phi\right\rangle + \int_s^t \left\langle T_\tau^*\nu, \mathcal{L}\phi\right\rangle d\tau \tag{3.2}$$

for all $0<s<t<\infty$.

Proof The continuity of $t\mapsto\langle T_t^*\nu,\phi\rangle$ at $t\neq 0$ is obvious. The continuity at $t=0$ follows from Remark A.1 (4). More precisely, since $T_t\phi(x)\to\phi(x)$ as $t\to 0^+$ locally uniformly in $x\in\mathcal{U}$, the dominated convergence theorem then yields

$$\left|\left\langle T_t^*\nu, \phi\right\rangle - \langle\nu,\phi\rangle\right| \leq \int_{\mathcal{U}} |T_t\phi-\phi|d\nu \to 0 \quad \text{as} \quad t\to 0^+.$$

The identity (3.2) follows from Fubini's theorem, the fact $\overline{\mathcal{L}}\phi = \mathcal{L}\phi$ and the formula

$$T_t\phi - T_s\phi = \int_s^t T_\tau \overline{\mathcal{L}}\phi d\tau,$$

which is a general fact of strongly continuous semigroups. □

The following result implies Theorem B (1).

Theorem 3.1 *Suppose* **(H)** *and that there exists an unbounded Lyapunov function associated to $\mathcal{L}$. Then, μ_t strongly converges to μ_* as $t \to \infty$.*

Proof We note from Lemma 3.2 that $T_t^*\nu$ is a global probability solution of (1.2)–(1.3) with additional continuity properties. Hence, by Proposition 2.3, $\mu_t = T_t^*\nu$ for all $t > 0$. The theorem now follows from Lemma 3.1. □

3.2 Exponential Convergence to Equilibrium

We now study the convergence of $(\mu_t)_{t\in[0,\infty)}$ to μ_* under stronger conditions. We start with the following result giving convergence results of the Markov semigroup $(T_t)_{t\geq 0}$ under appropriate assumptions.

Lemma 3.3 *Suppose* **(H)** *and that there exists an unbounded Lyapunov function U associated to $\mathcal{L}$. Also, suppose that there exist constants $t_0, c, b > 0$ and $\kappa \in (0, 1)$ such that*

$$T_tU(x) \leq c(1 + U(x)), \quad x \in \mathcal{U} \tag{3.3}$$

for any $t \in (0, t_0)$, and

$$T_{t_0}U(x) \leq \kappa U(x) + b, \quad x \in \mathcal{U}. \tag{3.4}$$

Then, there exist constants $C, r > 0$ such that

$$\|T_t\phi - \langle\mu_*, \phi\rangle\|_U \leq Ce^{-rt}\|\phi - \langle\mu_*, \phi\rangle\|_U, \quad t \geq t_0$$

for every measurable function $\phi : \mathcal{U} \to \mathbb{R}$ with $\|\phi\|_U < \infty$, where $\langle\mu_*, \phi\rangle = \int_{\mathcal{U}} \phi d\mu_*$ and

$$\|\phi\|_U := \left\|\frac{\phi}{1+U}\right\|_{L^\infty(\mathcal{U},\mu_*)} = \left\|\frac{\phi}{1+U}\right\|_{L^\infty(\mathcal{U},dx)}.$$

Proof We apply a version of Harris's theorem stated in [17, Theorem 3.6] to $T_{t_0}^n = T_{nt_0}$. To do so, we need to verify two conditions. The first condition involved in that theorem is just (3.4). The second condition involved is that for every $R > 0$, there exists $0 < \alpha = \alpha(R) < 1$ such that

$$\sup \|T_{t_0}(x, \cdot) - T_{t_0}(y, \cdot)\|_{TV} \leq 2(1 - \alpha), \tag{3.5}$$

where the supremum is taken over all $x, y \in \mathcal{U}$ such that $U(x) + U(y) \leq R$.

To verify this condition, we fix $R > 0$ and set

$$U_R = \{(x, y) \in \mathcal{U} \times \mathcal{U} : U(x) + U(y) \leq R\} \subset \overline{\Omega}_R \times \overline{\Omega}_R,$$

where $\Omega_R = \{x \in \mathcal{U} : U(x) < R\}$ is the R-sub-level set of U. Note that

$$\begin{aligned}\|T_{t_0}(x, \cdot) - T_{t_0}(y, \cdot)\|_{TV} &= 2 \sup_{B\in\mathcal{B}(\mathcal{U})} |T_{t_0}(x, B) - T_{t_0}(y, B)| \\ &= 2 \sup_{B\in\mathcal{B}(\mathcal{U})} \left|\int_B [p(t_0, x, z) - p(t_0, y, z)]dz\right|.\end{aligned}$$

We claim that

$$\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}<2,\qquad \forall x,y\in\mathcal{U}. \tag{3.6}$$

Obviously, $\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}\le 2$ for any $x,y\in\mathcal{U}$. If there are $x,y\in\mathcal{U}$ such that $\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}=2$, then there exists a sequence $(B_n)_n\subset\mathcal{B}(\mathcal{U})$ such that either

$$T_{t_0}(x,B_n)\to 1 \text{ and } T_{t_0}(y,B_n)\to 0 \text{ as } n\to\infty,\quad \text{or}$$
$$T_{t_0}(x,B_n)\to 0 \text{ and } T_{t_0}(y,B_n)\to 1 \text{ as } n\to\infty.$$

Suppose, without loss of generality, that the former is true. Let us fix some compact set $B_0\subset\mathcal{U}$ such that $T_{t_0}(x,B_0)=\frac{1}{2}$. Since $p(t_0,y,z)$ is positive and continuous, $c_y:=\min_{z\in B_0}p(t_0,y,z)>0$. Since $\max_{z\in B_0}p(t_0,x,z)<\infty$, the Lebesgue measure of $B_0\cap B_n$, denoted by $|B_0\cap B_n|$, does not go to 0 as $n\to\infty$, for otherwise, $T_{t_0}(x,B_n)$ can not converge to 1 as $n\to\infty$. It then follows that

$$T_{t_0}(y,B_n)\ge T_{t_0}(y,B_0\cap B_n)\ge c_y|B_0\cap B_n|\ge c_y\epsilon_0,\quad n\gg 1,$$

where $\epsilon_0>0$ is such that $|B_0\cap B_n|\ge\epsilon_0$ for all $n\gg 1$. This leads to a contradiction. Thus (3.6) is true.

Denote

$$h(x_1,x_2)=2\sup_{B\in\mathcal{B}(\mathcal{U})}\int_B|p(t_0,x_1,z)-p(t_0,x_2,z)|\,dz,\quad x_1,x_2\in\mathcal{U}.$$

Then for any $x,x',y,y'\in\overline{\Omega}_R$,

$$\begin{aligned}
&\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}\\
&=2\sup_{B\in\mathcal{B}(\mathcal{U})}\bigg|\int_B[p(t_0,x,z)-p(t_0,x',z)]dz+\int_B[p(t_0,x',z)-p(t_0,y',z)]dz\\
&+\int_B[p(t_0,y',z)-p(t_0,y,z)]dz\bigg|\\
&\le 2\sup_{B\in\mathcal{B}(\mathcal{U})}\int_B|p(t_0,x,z)-p(t_0,x',z)|dz+2\sup_{B\in\mathcal{B}(\mathcal{U})}\int_B|p(t_0,y',z)-p(t_0,y,z)|dz\\
&+\|T_{t_0}(x',\cdot)-T_{t_0}(y',\cdot)\|_{TV}\\
&\le h(x,x')+h(y,y')+\|T_{t_0}(x',\cdot)-T_{t_0}(y',\cdot)\|_{TV}.
\end{aligned}$$

By symmetry, we find that

$$\Big|\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}-\|T_{t_0}(x',\cdot)-T_{t_0}(y',\cdot)\|_{TV}\Big|\le h(x,x')+h(y,y'). \tag{3.7}$$

We note from Remark A.1 (3) that there is $c_{t_0,R}>0$ such that

$$h(x,x')\le c_{t_0,R}|x-x'|^\alpha \quad\text{and}\quad h(y,y')\le c_{t_0,R}|y-y'|^\alpha.$$

This, together with (3.7) implies the continuity of the map $(x,y)\mapsto\|T_{t_0}(x,\cdot)-T_{t_0}(y,\cdot)\|_{TV}$ on $\overline{\Omega}_R\times\overline{\Omega}_R$. With this continuity, the condition (3.5) now follows from the claim (3.6) and the compactness of $\overline{\Omega}_R$.

It now follows from Harris's theorem that $(T_{nt_0})_{n\ge1}$ admits a unique invariant measure, which must be μ_*, and there are $C_1>0$ and $r_1\in(0,1)$ such that

$$\|T_{nt_0}\phi-\langle\mu_*,\phi\rangle\|_U\le C_1r_1^n\|\phi-\langle\mu_*,\phi\rangle\|_U,\quad n\ge1 \tag{3.8}$$

holds for every measurable function $\phi : \mathcal{U} \to \mathbb{R}$ with $\|\phi\|_U < \infty$. For any $t \geq t_0$, write $t = [t] + r_t$, where $[t]$ is the largest number of the form nt_0 not greater than t, and $t_r \in [0, t_0)$. Then by (3.8) and (3.3),

$$\begin{aligned}\|T_t\phi - \langle\mu_*, \phi\rangle\|_U &= \|T_t(\phi - \langle\mu_*, \phi\rangle)\|_U \\ &= \left\| \frac{T_{r_t}T_{[t]}(\phi - \langle\mu_*, \phi\rangle)}{1+U} \right\|_{L^\infty} \\ &\leq \left\| \frac{T_{r_t}(1+U)}{1+U} \right\|_{L^\infty} \|T_{[t]}(\phi - \langle\mu_*, \phi\rangle)\|_U \\ &\leq (c+1)C_1 r_1^{\frac{[t]}{t_0}} \|\phi - \langle\mu_*, \phi\rangle\|_U.\end{aligned}$$

This completes the proof. □

Next, we provide sufficient conditions for the assumptions in Lemma 3.3.

Lemma 3.4 *Suppose* **(H)** *and that there exists an unbounded strong Lyapunov function U associated to $\mathcal{L}$. Then, the following statements hold.*

(1) $U \in L^1(\mathcal{U}, \mu_*)$.
(2) If ν satisfies $U \in L^1(\mathcal{U}, \nu)$, then conditions of Lemma 3.3 *hold.*

Proof Let $C_1, C_2, \rho_m > 0$ be as in (1.8) for the unbounded strong Lyapunov function U as in the statement.

(1) Let $\tilde{\nu}$ be an Borel probability measure on $\mathcal{U}$ such that $U \in L^1(\mathcal{U}, \tilde{\nu})$. Let $(\tilde{\mu})_{t\in[0,\infty)}$ be the global probability solution of (1.2)–(1.3) with $\nu = \tilde{\nu}$.

We claim that

$$\sup_{t>0} \int_{\mathcal{U}} U d\tilde{\mu}_t < \infty. \tag{3.9}$$

Let us assume (3.9) for the moment and show that $U \in L^1(\mathcal{U}, \mu_*)$. Clearly, we can find a sequence of non-negative functions $\{U_n\}_n \subset C_b(\mathcal{U})$ satisfying $U_n \leq U_{n+1}$ and $U_n(x) \to U(x)$ locally uniformly in $x \in \mathcal{U}$ as $n \to \infty$. By the monotone convergence theorem,

$$\int_{\mathcal{U}} U d\mu_* = \lim_{n\to\infty} \int_{\mathcal{U}} U_n d\mu_*.$$

The strong convergence of $\tilde{\mu}_t$ to μ_* as $t \to \infty$ given in Theorem 3.1 implies that

$$\lim_{t\to\infty} \int_{\mathcal{U}} U_n d\tilde{\mu}_t = \int_{\mathcal{U}} U_n d\mu_*, \quad \forall n.$$

It then follows from (3.9) and the monotonicity of the sequence $\{U_n\}_n$ that $\sup_n \int_{\mathcal{U}} U_n d\mu_* < \infty$, which leads to the result.

To finish the proof, it remains to show (3.9). To do so, we fix $\rho_0 \in (\rho_m, \infty)$. For $\rho \geq \rho_0 + 1$, we define $\eta_\rho \in C^2[0, \infty)$ such that it is non-decreasing and satisfies

$$\eta_\rho(t) = \begin{cases} 0, & t \in [0, \rho_m], \\ t, & t \in [\rho_0, \rho], \\ \rho + 1, & t \in [\rho + 2, \infty). \end{cases}$$

Moreover, the shape of $\eta_\rho(t)$ for $t \in [\rho_m, \rho_0] \cup [\rho, \rho+2]$ can be fixed so that

$$\begin{aligned} &C_3 := \sup_{\rho \geq \rho_0+1} \sup_{t\geq 0} \max\{\eta_\rho'(t), |\eta_\rho''(t)|\} < \infty, \\ &\eta_\rho''(t) \leq 0 \text{ for all } t \in [\rho, \rho+2] \text{ and } \rho \geq \rho_0 + 1. \end{aligned} \tag{3.10}$$

We set $\eta(t) := \lim_{\rho\to\infty} \eta_\rho(t)$.

We estimate $\int_{\mathcal{U}} \eta_\rho(U) d\tilde{\mu}_t$. By Proposition 2.1(iii), the equalities

$$\int_{\mathcal{U}} \phi d\tilde{\mu}_t = \int_{\mathcal{U}} \phi d\tilde{\mu}_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi d\tilde{\mu}_\tau d\tau, \quad 0 < s < t < \infty$$

hold for any $\phi \in C_0^\infty(\mathcal{U})$, which, by the assumption **(H)**, actually hold for any $\phi \in C_0^2(\mathcal{U})$. Since $\tilde{\mu}_t(\mathcal{U}) = 1 = \tilde{\mu}_s(\mathcal{U})$, they are also true for any $\phi + c$, where $\phi \in C_0^2(\mathcal{U})$ and $c \in \mathbb{R}$. That is, they are true for any C^2 function that is constant outside some compact set. In particular, they can be applied to the function $\eta_\rho(U)$ for any $\rho \geq \rho_0 + 1$. Hence, we find

$$\frac{d}{dt} \int_{\mathcal{U}} \eta_\rho(U) d\tilde{\mu}_t = \int_{\mathcal{U}} \mathcal{L}(\eta_\rho(U)) d\tilde{\mu}_t = \int_{\mathcal{U}} \eta_\rho'(U)\mathcal{L}U d\tilde{\mu}_t + \int_{\mathcal{U}} \eta_\rho''(U) a^{ij} \partial_i U \partial_j U d\tilde{\mu}_t. \tag{3.11}$$

For the first term on the right-hand side of (3.11), we have

$$\begin{aligned} \int_{\mathcal{U}} \eta_\rho'(U)\mathcal{L}U d\tilde{\mu}_t &= \int_{\Omega_{\rho+1}\setminus\overline{\Omega}_{\rho m}} \eta_\rho'(U)\mathcal{L}U d\tilde{\mu}_t \\ &\leq C_1 \int_{\Omega_{\rho+2}\setminus\overline{\Omega}_{\rho m}} \eta_\rho'(U) d\tilde{\mu}_t - C_2 \int_{\Omega_{\rho+2}\setminus\overline{\Omega}_{\rho m}} \eta_\rho'(U) U d\tilde{\mu}_t \\ &\leq C_1 C_3 - C_2 \int_{\Omega_\rho\setminus\overline{\Omega}_{\rho_0}} U d\tilde{\mu}_t \\ &= C_1 C_3 - C_2 \int_{\mathcal{U}} \eta_\rho(U) d\tilde{\mu}_t + C_2 \int_{\overline{\Omega}_{\rho_0}} \eta_\rho(U) d\tilde{\mu}_t + C_2 \\ &\quad \int_{\mathcal{U}\setminus\Omega_\rho} \eta_\rho(U) d\tilde{\mu}_t \\ &\leq C_4 - C_2 \int_{\mathcal{U}} \eta_\rho(U) d\tilde{\mu}_t + C_2(\rho+1)\tilde{\mu}_t(\mathcal{U}\setminus\Omega_\rho), \end{aligned}$$

where $C_4 = C_1 C_3 + C_2 \rho_0$.

For the second term on the right-hand side of (3.11), we deduce from the positive-definiteness of (a^{ij}) and the negativity of $\eta_\rho''(t)$ on $[\rho, \rho+2]$ that

$$\begin{aligned} \int_{\mathcal{U}} \eta_\rho''(U) a^{ij} \partial_i U \partial_j U d\tilde{\mu}_t &= \int_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} \eta_\rho''(U) a^{ij} \partial_i U \partial_j U d\tilde{\mu}_t \\ &\quad + \int_{\Omega_{\rho+2}\setminus\overline{\Omega}_\rho} \eta_\rho''(U) a^{ij} \partial_i U \partial_j U d\tilde{\mu}_t \\ &\leq \int_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} \eta_\rho''(U) a^{ij} \partial_i U \partial_j U d\tilde{\mu}_t \\ &\leq C_3 \sup_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} a^{ij} \partial_i U \partial_j U. \end{aligned}$$

With $C_5 := C_4 + C_3 \sup_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} a^{ij}\partial_i U \partial_j U$, the equality (3.11) gives

$$\frac{d}{dt}\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_t \le -C_2\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_t + C_5 + C_2(\rho+1)\tilde{\mu}_t(\mathcal{U}\setminus\Omega_\rho),$$

which implies that for any $t > t_0$

$$\begin{aligned}\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_t &\le e^{-C_2(t-t_0)}\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_{t_0} + \int_{t_0}^t e^{-C_2(t-s)}C_5 ds\\ &\quad + \int_{t_0}^t e^{-C_2(t-s)}C_2(\rho+1)\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho)ds\\ &\le \int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_{t_0} + \frac{C_5}{C_2} + \int_{t_0}^t e^{-C_2(t-s)}C_2(\rho+1)\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho)ds.\end{aligned} \tag{3.12}$$

Since $\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_{t_0} \to \int_{\mathcal{U}} \eta_\rho(U)d\tilde{\nu}$ as $t_0 \to 0^+$ by Lemma 2.1, we pass to the limit $t_0 \to 0^+$ in (3.12) to obtain

$$\int_{\mathcal{U}} \eta_\rho(U)d\tilde{\mu}_t \le \int_{\mathcal{U}} \eta_\rho(U)d\tilde{\nu} + \frac{C_5}{C_2} + \int_0^t e^{-C_2(t-s)}C_2(\rho+1)\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho)ds. \tag{3.13}$$

By Remark 2.1, we have $\int_{\mathcal{U}} U d\tilde{\mu}_s < \infty$. Since

$$\int_{\mathcal{U}} \min\{U,\rho\}d\tilde{\mu}_s = \int_{\Omega_\rho} U d\tilde{\mu}_s + \rho\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho),$$

and as $\rho \to \infty$, both $\int_{\mathcal{U}} \min\{U,\rho\}d\tilde{\mu}_s$ and $\int_{\Omega_\rho} U d\tilde{\mu}_s$ converge to $\int_{\mathcal{U}} U d\tilde{\mu}_s$ due to the monotone convergence theorem, we find $\rho\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho) \to 0$ as $\rho \to \infty$, which together with the dominated convergence theorem imply that for any fixed $t > 0$,

$$\int_0^t e^{-C_2(t-s)}C_2(\rho+1)\tilde{\mu}_s(\mathcal{U}\setminus\Omega_\rho)ds \to 0 \quad \text{as} \quad \rho \to \infty.$$

Hence, for any fixed $t > 0$, we pass to the limit $\rho \to \infty$ in (3.13) to find

$$\int_{\mathcal{U}} \eta(U)\tilde{\mu}_t \le \int_{\mathcal{U}} \eta(U)d\tilde{\nu} + \frac{C_5}{C_2},$$

which leads to (3.9).

(2) Note that $\overline{\mathcal{L}} = \mathcal{L}$ on $C_0^2(\mathcal{U})$ and the constant function 1 belongs to $\mathcal{D}(\overline{\mathcal{L}})$ with $\overline{\mathcal{L}}1 = 0$. Hence, $\overline{\mathcal{L}}\phi = \mathcal{L}\phi$ for any C^2 function that is constant outside some compact set. Let η_ρ be as in the proof of (1). Then $\eta_\rho(U)$ is a C^2 function that equals $\rho+1$ on $\mathcal{U}\setminus\overline{\Omega}_{\rho+2}$, and therefore, $\overline{\mathcal{L}}(\eta_\rho(U)) = \mathcal{L}(\eta_\rho(U))$. It follows that

$$\begin{aligned}\frac{d}{dt}T_t\eta_\rho(U) &= T_t\overline{\mathcal{L}}\eta_\rho(U)\\ &= T_t\mathcal{L}\eta_\rho(U)\\ &= T_t[\eta_\rho'(U)\mathcal{L}U + \eta_\rho''(U)a^{ij}\partial_i U\partial_j U].\end{aligned}$$

Clearly, $\eta''_\rho(U)a^{ij}\partial_i U\partial_j U \le C_3 \sup_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} a^{ij}\partial_i U\partial_j U$, where $C_3 > 0$ is as in (3.10). Since

$$\begin{aligned}
-C_2\eta'_\rho(U)U + C_2\eta_\rho(U) &= -C_2\eta'_\rho(U)U[\chi_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} + \chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}] - C_2U\chi_{\Omega_\rho\setminus\overline{\Omega}_{\rho_0}}\\
&\quad + C_2\eta_\rho(U)[\chi_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} + \chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}] + C_2U\chi_{\Omega_\rho\setminus\overline{\Omega}_{\rho_0}}\\
&\le C_2\eta_\rho(U)[\chi_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} + \chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]\\
&\le C_2\rho_0 + C_2(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho},
\end{aligned}$$

we have

$$\begin{aligned}
\eta'_\rho(U)\mathcal{L}U &\le C_1\eta'_\rho(U) - C_2\eta'_\rho(U)U\\
&\le C_1C - C_2\eta'_\rho(U)U + C_2\eta_\rho(U) - C_2\eta_\rho(U)\\
&\le C_1C + C_2\rho_0 + C_2(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho} - C_2\eta_\rho(U).
\end{aligned}$$

Thus, setting

$$C_6 := C_3 \sup_{\Omega_{\rho_0}\setminus\overline{\Omega}_{\rho m}} a^{ij}\partial_i U\partial_j U + C_1C_3 + C_2\rho_0,$$

we find

$$\frac{d}{dt}T_t\eta_\rho(U) \le C_6 + C_2T_t[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}] - C_2T_t\eta_\rho(U),$$

which leads to

$$\begin{aligned}
T_t\eta_\rho(U) &\le e^{-C_2t}\eta_\rho(U) + \int_0^t e^{-C_2(t-s)}C_6ds + C_2\int_0^t e^{-C_2(t-s)}T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]ds\\
&\le e^{-C_2t}\eta_\rho(U) + \frac{C_6}{C_2} + C_2\int_0^t e^{-C_2(t-s)}T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]ds.
\end{aligned} \tag{3.14}$$

Multiplying the above inequality by a function $\phi \in C_0^\infty(\mathcal{U})$ with $\phi \ge 0$, and then integrating the resulting one over $\mathcal{U}$ with respect to μ_* yields

$$\begin{aligned}
\int_{\mathcal{U}} T_t\eta_\rho(U)\phi d\mu_* &\le e^{-C_2t}\int_{\mathcal{U}}\eta_\rho(U)\phi d\mu_* + \frac{C_6}{C_2}\int_{\mathcal{U}}\phi d\mu_*\\
&\quad + C_2\int_0^t e^{-C_2(t-s)}\left(\int_{\mathcal{U}} T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]\phi d\mu_*\right)ds.
\end{aligned} \tag{3.15}$$

Since $U \in L^1(\mathcal{U},\mu_*)$ by (1), we have $(\rho+1)\mu_*(\mathcal{U}\setminus\overline{\Omega}_\rho) < \infty$, i.e., $(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho} \in L^1(\mathcal{U},\mu_*)$, and hence $(\rho+1)\mu_*(\mathcal{U}\setminus\overline{\Omega}_\rho) \to 0$ as $\rho\to\infty$. Using the invariance of μ_* with respect to $(T_t)_{t\ge0}$, we have

$$\begin{aligned}
0 \le \int_{\mathcal{U}} T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]\phi d\mu_* &\le \left(\sup_{\mathcal{U}}\phi\right)\int_{\mathcal{U}} T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]d\mu_*\\
&= \left(\sup_{\mathcal{U}}\phi\right)\int_{\mathcal{U}}(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}d\mu_*\\
&= \left(\sup_{\mathcal{U}}\phi\right)(\rho+1)\mu_*(\mathcal{U}\setminus\overline{\Omega}_\rho).
\end{aligned}$$

It follows that

$$\int_{\mathcal{U}} T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]\phi d\mu_* \to 0 \quad \text{as} \quad \rho\to\infty \quad \text{uniform for } s>0.$$

Setting $\rho \to \infty$ in (3.15), with $\eta(t) = \lim_{\rho\to\infty} \eta_\rho(t)$, we find

$$\int_{\mathcal{U}} T_t \eta(U)\phi d\mu_* \le e^{-C_2 t} \int_{\mathcal{U}} \eta(U)\phi d\mu_* + \frac{C_6}{C_2} \int_{\mathcal{U}} \phi d\mu_*,$$

which then implies

$$T_t \eta(U) \le e^{-C_2 t} \eta(U) + \frac{C_6}{C_2}.$$

Since $\eta(U)$ is also an unbounded strong Lyapunov function, we can simply take a $t_0 > 0$ so that $e^{-C_2 t_0} < 1$, for which conditions of Lemma 3.3 are satisfied. □

Remark 3.1 If we would use (3.1) to treat the term $T_s[(\rho+1)\chi_{\mathcal{U}\setminus\overline{\Omega}_\rho}]$ on the last line of the inequality (3.14), some singularity would occur because $p(s, x, y)$ is singular at $s = 0$. This is why we put the inequality (3.14) into the space $L^1(\mathcal{U}, \mu_*)$ in the proof of the lemma.

Finally, we prove the following theorem which implies Theorem B (2).

Theorem 3.2 *Suppose* **(H)** *and that there exists an unbounded strong Lyapunov function U associated to $\mathcal{L}$. If ν satisfies $U \in L^1(\mathcal{U}, \nu)$, then there exist $t_0 > 0$ and constants $C, r > 0$ such that the followings hold for all $t \ge t_0$:*

$$\|\mu_t - \mu_*\|_{TV} = \|T_t^* \nu - \mu_*\|_{TV} \le Ce^{-rt}, \tag{3.16}$$

$$\left| \int_{\mathcal{U}} \psi T_t \phi d\mu_* - \int_{\mathcal{U}} \psi d\mu_* \int_{\mathcal{U}} \phi d\mu_* \right| \le Ce^{-rt} \|(\phi - \langle \mu_*, \phi\rangle)^2\|_U^{1/2} \|\psi^2\|_U^{1/2}, \tag{3.17}$$

for any pair of measurable functions ϕ and ψ with $\|\phi^2\|_U < \infty$ and $\|\psi^2\|_U < \infty$.

Proof By Lemma 3.4 (2), we can apply Lemma 3.3 to deduce that

$$\begin{aligned}
\|T_t^* \nu - \mu_*\|_{TV} &= \sup_{\phi:|\phi|\le 1} |\langle T_t^* \nu, \phi\rangle - \langle \mu_*, \phi\rangle| \\
&= \sup_{\phi:|\phi|\le 1} |\langle \nu, T_t \phi - \langle \mu_*, \phi\rangle\rangle| \\
&\le \sup_{\phi:|\phi|\le 1} \int_{\mathcal{U}} \frac{|T_t\phi - \langle \mu_*, \phi\rangle|}{1+U} (1+U) d\nu \\
&\le \int_{\mathcal{U}} (1+U) d\nu \sup_{\phi:|\phi|\le 1} \|T_t \phi - \langle \mu_*, \phi\rangle\|_U \\
&\le Ce^{-rt} \int_{\mathcal{U}} (1+U) d\nu \sup_{\phi:|\phi|\le 1} \|\phi - \langle \mu_*, \phi\rangle\|_U.
\end{aligned}$$

This proves (3.16).

Since $p(t, x, y)dy$ is a probability measure and $a \mapsto \sqrt{a}$ is a concave function, Jensen's inequality yields

$$T_t \sqrt{U}(x) = \int_{\mathcal{U}} \sqrt{U}(y) p(t, x, y) dy \le \sqrt{\int_{\mathcal{U}} U(y) p(t, x, y) dy} = \sqrt{T_t U(x)}.$$

By Lemma 3.4 (2), conditions of Lemma 3.3 hold. It follows that there are constants $\tilde{c} > 0$, $\tilde{\kappa} \in (0, 1)$ and $\tilde{b} > 0$ such that

$$T_t \sqrt{U}(x) \le \tilde{c}(1 + \sqrt{U}(x)), \quad x \in \mathcal{U}$$

for any $t \in (0, t_0)$, and

$$T_{t_0}\sqrt{U}(x) \le \tilde{\kappa}\sqrt{U}(x) + \tilde{b}, \quad x \in \mathcal{U}.$$

Applying Lemma 3.3 concludes that there exist constants $\tilde{C}, \tilde{r} > 0$ such that

$$\|T_t\phi - \langle\mu_*, \phi\rangle\|_{\sqrt{U}} \le \tilde{C}e^{-\tilde{r}t}\|\phi - \langle\mu_*, \phi\rangle\|_{\sqrt{U}}, \quad t \ge t_0 \tag{3.18}$$

holds for every measurable function $\phi : \mathcal{U} \to \mathbb{R}$ with $\|\phi\|_{\sqrt{U}} < \infty$, or equivalently, $\|\phi^2\|_U < \infty$. Clearly, (3.18) implies that

$$|T_t\phi(x) - \langle\mu_*, \phi\rangle| \le \tilde{C}e^{-\tilde{r}t}(1 + \sqrt{U}(x))\|\phi - \langle\mu_*, \phi\rangle\|_{\sqrt{U}}, \quad x \in \mathcal{U}, \ t \ge t_0.$$

It then follows that

$$\begin{aligned}
&\left|\int_{\mathcal{U}} \psi T_t\phi d\mu_* - \int_{\mathcal{U}} \psi d\mu_* \int_{\mathcal{U}} \phi d\mu_*\right| \\
&\quad \le \int_{\mathcal{U}} |\psi||T_t\phi - \langle\mu_*, \phi\rangle| d\mu_* \\
&\quad \le \tilde{C}e^{-\tilde{r}t}\|\phi - \langle\mu_*, \phi\rangle\|_{\sqrt{U}}\sqrt{\int_{\mathcal{U}} |\psi|^2 d\mu_*}\sqrt{\int_{\mathcal{U}} (1 + \sqrt{U})^2 d\mu_*} \\
&\quad \le \sqrt{2}\tilde{C}e^{-\tilde{r}t}\|\phi - \langle\mu_*, \phi\rangle\|_{\sqrt{U}}\|\psi^2\|_U^{1/2}\int_{\mathcal{U}} (1 + U) d\mu_*.
\end{aligned}$$

This proves (3.17). □

4 Examples

In this section, we apply Theorems A and B to some examples.

Example 4.1 Consider the following Itô stochastic differential equation

$$dx = bxdt + \sqrt{2\sigma(x^2 + 1)}dW, \quad x \in \mathbb{R},$$

where $\sigma > b > 0$. The corresponding Fokker–Planck equation reads

$$\partial_t u = Lu := \partial^2_{xx}(\sigma(x^2 + 1)u) - \partial_x(bxu), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.1}$$

and the adjoint Fokker–Planck operator is simply $\mathcal{L} = \sigma(x^2 + 1)\partial^2_{xx} + bx\partial_x$.

Consider $U(x) = \ln(x^2 + 1)$. Since

$$\mathcal{L}U(x) = -\frac{2(\sigma - b)x^2 - 2\sigma}{x^2 + 1},$$

$\mathcal{L}U(x) \le -(\sigma - b)$ for $|x| \gg 1$, i.e., U is an unbounded Lyapunov function. Hence, *Proposition* 3.1 ensures the existence of a unique stationary measure μ_* of the Fokker–Planck equation (4.1). It follows from Theorem A that for any initial Borel probability measure, (4.1) has a unique global probability solution $(\mu_t)_{t\in[0,\infty)}$. Moreover, Theorem B (1) asserts the strong convergence of μ_t to μ_* as $t \to \infty$.

Example 4.2 Consider the Itô stochastic differential equation

$$dx = b(x)dt + \sqrt{2}dW, \quad x \in \mathbb{R},$$

whose corresponding Fokker–Planck equation is given by

$$\partial_t u = Lu := \partial^2_{xx} u - \partial_x (b(x)u), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.2}$$

where $b : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $b(x) = -\frac{x}{|x|^{\frac{3}{2}}}$ for $|x| \gg 1$. The adjoint Fokker–Planck operator is simply $\mathcal{L} = \partial^2_{xx} + b(x)\partial_x$.

Consider a C^2 function $U : \mathbb{R} \to [0, \infty)$ such that

$$U(x) = e^{|x|^{\frac{1}{4}}}, \quad |x| \gg 1.$$

Simple calculation yields that

$$\mathcal{L}U(x) = e^{|x|^{\frac{1}{4}}} \left[\frac{1}{16}|x|^{-\frac{3}{2}} - \frac{3}{16}|x|^{-\frac{7}{4}} - \frac{1}{4}|x|^{-\frac{5}{4}} \right], \quad |x| \gg 1,$$

which implies that $\mathcal{L}U(x) \to -\infty$ as $|x| \to \infty$. Hence, U is an unbounded Lyapunov function.

It follows from Proposition 3.1 that (4.2) admits a unique stationary measure μ_*, and from Theorem A that for any initial Borel probability measure, (4.2) admits a unique global probability solution $(\mu_t)_{t\in[0,\infty)}$. Theorem B (1) then guarantees that μ_t converges strongly to μ_* as $t \to \infty$.

Example 4.3 For given constant $b < 0$, consider the Itô stochastic differential equation

$$dx = bxdt + \sqrt{2}(1 - x^2)dW, \quad x \in \mathcal{U} = (-1, 1),$$

whose corresponding Fokker–Planck equation is given by

$$\partial_t u = Lu := \partial^2_{xx}[(1 - x^2)^2 u] + \partial_x(bxu), \quad t > 0, \quad x \in (-1, 1). \tag{4.3}$$

The adjoint Fokker–Planck operator reads $\mathcal{L} = (1 - x^2)^2 \partial^2_{xx} + bx\partial_x$.

Let $U(x) = -\ln(1 - x^2)$, $x \in (-1, 1)$. Then $U(x) \to \infty$ as $|x| \to 1^-$. Since

$$\mathcal{L}U(x) = 2 + 2x^2 + \frac{2bx^2}{1 - x^2}, \quad x \in (-1, 1),$$

we can find constants $C_1, C_2, \rho_m > 0$ such that

$$\mathcal{L}U(x) \le C_1 - C_2 U(x), \quad x \in (-1, 1)\setminus\overline{\Omega}_{\rho_m},$$

where $\Omega_{\rho_m} = \{x \in (-1, 1) : U(x) < \rho_m\}$. Hence, U is an unbounded strong Lyapunov function of (4.3).

Applying Proposition 3.1, Theorems A and B (1), we conclude that if ν is a Borel probability measure on $(-1, 1)$, then the unique global probability solution $(\mu_t)_{t\in[0,\infty)}$ of (4.3) with initial condition ν converges strongly to the unique stationary measure of (4.3) as $t \to \infty$. If, in addition, ν satisfies $U \in L^1((-1, 1), \nu)$, then Theorem B (2) ensures that $(\mu_t)_{t\in[0,\infty)}$ converges exponentially fast in the total variation distance to the unique stationary measure of (4.3) as $t \to \infty$.

Example 4.4 Consider the Itô stochastic differential equation

$$dx = \left[\tan\left(-\frac{\pi}{2}x\right) + \mathrm{sign}(x)\right]dt + |1 - |x||^{\alpha}\, dW, \quad x \in (-1, 1)$$

studied in [16], where $\alpha > 0$. The corresponding Fokker–Planck equation reads

$$\partial_t u = Lu := \frac{1}{2}\partial_{xx}^2(|1-|x||^{2\alpha}u) - \partial_x\left(\left[\tan\left(-\frac{\pi}{2}x\right) + \text{sign}(x)\right]u\right), \quad t > 0, \quad x \in (-1, 1). \tag{4.4}$$

Hence, the adjoint Fokker–Planck operator is given by

$$\mathcal{L} = \frac{1}{2}(|1 - |x||^{2\alpha})\partial_{xx}^2 + \left[\tan\left(-\frac{\pi}{2}x\right) + \text{sign}(x)\right]\partial_x.$$

Note that the drift coefficient is discontinuous at $x = 0$ and the diffusion coefficient is not Hölder continuous with exponent $\frac{1}{2}$ if $\alpha < \frac{1}{2}$.

It was shown in [24, Example 3.10] that $U(x) = \frac{2-x^2}{1-x^2}$, $x \in (-1, 1)$ is a positive C^2 function and satisfies $U(x) \to \infty$ as $|x| \to 1^-$ and

$$\frac{\mathcal{L}U(x)}{U(x)} \to -\infty \quad \text{as} \quad |x| \to 1.$$

In particular, U is an unbounded strong Lyapunov function, and therefore, the assumptions in Proposition 3.1, Theorems A and B are satisfied. Hence, if ν is a Borel probability measure on $(-1, 1)$, then the unique global probability solution $(\mu_t)_{t\in[0,\infty)}$ of (4.4) with initial condition ν converges strongly to the unique stationary measure of (4.4) as $t \to \infty$. If, in addition, ν satisfies $\int_{(-1,1)} U d\nu < \infty$, then the convergence is exponentially fast in the total variation distance.

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Appendix A. Stationary Measures and Associated Markov Semigroups

In this section, we recall some results obtained in [12] (also see [6,7]) concerning Markov semigroups associated to stationary measures.

Let $\mathcal{P}$ be the set of Borel probability measures on $\mathcal{U}$. Let

$$\mathcal{M} := \{\mu \in \mathcal{P} : L\mu = 0 \text{ in the sense of Definition 1.3}\}$$

be the set of stationary measures of (1.2).

The following results describe the existence and some properties of sub-Markov semigroups associated to a given stationary measure of (1.2).

Proposition A.1 *Let $\mu \in \mathcal{M}$.*

(1) If **(H)** *is assumed with $p > n+2$ replaced by $p > n$, then there exists a closed extension $(\overline{\mathcal{L}}, \mathcal{D}(\overline{\mathcal{L}}))$ of $(\mathcal{L}, C_0^\infty(\mathcal{U}))$ generating a sub-Markov contractive C_0-semigroup $(T_t)_{t\geq 0}$ on $L^1(\mathcal{U}, \mu)$ such that μ is sub-invariant for $(T_t)_{t\geq 0}$, i.e.,*

$$\int_{\mathcal{U}} T_t\phi d\mu \leq \int_{\mathcal{U}} \phi d\mu, \quad t \geq 0$$

for all $\phi \in L^\infty(\mathcal{U}, \mu)$ with $\phi \geq 0$.

(2) If **(H)** *is assumed, then there exist unique sub-probability kernels $K_t(\cdot, dy)$, $t > 0$, on $\mathcal{U}$ such that*

$$K_t(x, dy) = p(t, x, y)dy,$$

where $p(t, x, y)$ is locally Hölder continuous and positive on $(0, \infty) \times \mathcal{U} \times \mathcal{U}$, and for each $\phi \in L^1(\mathcal{U}, \mu)$, the function

$$x \mapsto K_t\phi(x) := \int_{\mathcal{U}} \phi(y)p(t, x, y)dy, \quad \mathcal{U} \to \mathbb{R}$$

is a μ-version of $T_t\phi$ such that $(t, x) \to K_t\phi(x)$ is continuous on $(0, \infty) \times \mathcal{U}$. In addition, if $\tilde{\mu} \in \mathcal{P}$ is invariant for $(K_t)_{t\geq 0}$, i.e.,

$$\tilde{\mu} = K_t^*\tilde{\mu}(dy) := \int_{\mathcal{U}} K_t(x, dy)d\nu(x), \quad t \geq 0,$$

then $\tilde{\mu} = \mu$.

Proof See [12, Theorem 2.3] or [26] for (1), and [12, Theorem 4.4] or [5, Theorem 4.1, Corollary 4.3] for (2). □

Here are some remarks, implied by Proposition A.1(2), about the semigroup $(K_t)_{t\geq 0}$ given in Proposition A.1(2) (see [6, Remark 1.7.6]).

Remark A.1 Assume **(H)**.

(1) The semigroup $(K_t)_{t\geq 0}$ is strongly Feller and stochastically continuous.
(2) The probability measures

$$B \mapsto K_t\chi_B(x) := \int_B p(t, x, y)dy, \quad t > 0, \quad x \in \mathcal{U}$$

are equivalent. In particular, if μ is invariant for $(K_t)_{t\geq 0}$, i.e.,

$$\int_{\mathcal{U}} K_t\phi d\mu = \int_{\mathcal{U}} \phi d\mu, \quad t \geq 0,$$

for all $\phi \in L^1(\mathcal{U}, \mu)$, Doob's theorem (see e.g. [15, Theorem 4.2.1]) yields

$$\lim_{t\to\infty} K_t\chi_B(x) = \mu(B), \quad \forall x \in \mathcal{U}$$

for any Borel set $B \subset \mathcal{U}$.
(3) By the proof of [5, Theorem 4.1], the transition density function $p(t, x, y)$ of $(K_t)_{t\geq 0}$ is given by

$$p(t, x, y) = p_t(x, y)\varrho(y), \quad (t, x, y) \in (0, \infty) \times \mathcal{U} \times \mathcal{U},$$

where $\varrho \in W^{1,p}_{loc}(\mathcal{U})$ is the density of μ, and $(t, x, y) \mapsto p_t(x, y)$ is continuous on $(0, \infty) \times \mathcal{U} \times \mathcal{U}$ and satisfies

$$\sup_{x,y\in\mathcal{K}} \sup_{z\in\mathcal{U}} \frac{|p_t(x, z) - p_t(y, z)|}{|x - y|^\alpha} < \infty$$

for any $t > 0$ and any compact set $\mathcal{K} \subset \mathcal{U}$, where $\alpha > 0$ is some constant.
(4) By [12, Theorem 2.3(iii)], or [6, Theorem 1.5.7(iii)], for any $\phi \in C_0^\infty(\mathcal{U})$, $T_t\phi$ has a continuous modification, which must be $K_t\phi$, such that

$$K_t\phi(x) \to \phi(x) \quad \text{as} \quad t \to 0^+ \quad \text{locally uniformly in} \quad x \in \mathcal{U}.$$

In the next result, sufficient conditions for the sub-Markov semigroup $(T_t)_{t\geq 0}$ in Proposition A.1 being a Markov semigroup are provided.

Proposition A.2 *Assume* **(H)** *with $p > n + 2$ replaced by $p > n$. Let $\mu \in \mathcal{M}$. Then, the following two assertions are equivalent:*

(1) For some (and therefore all) $\lambda > 0$, there holds $L^1(\mathcal{U}, \mu) = \overline{(\mathcal{L} - \lambda I)(C_0^\infty(\mathcal{U}))}$;
(2) There exists a unique C_0-semigroup in $L^1(\mathcal{U}, \mu)$ whose generator extending $(\mathcal{L}, C_0^\infty(\mathcal{U}))$.

If one of the above two equivalent assertions holds, then the semigroup $(T_t)_{t\geq 0}$ given in Proposition *A.1(1) is a Markov semigroup and μ is invariant for $(T_t)_{t\geq 0}$.*

Proof See [12, Proposition 2.6]. □

Set

$$\mathcal{M}_{md} = \left\{\mu \in \mathcal{M} : L^1(\mathcal{U}, \mu) = \overline{(\mathcal{L} - I)(C_0^\infty(\mathcal{U}))}\right\}. \tag{A.1}$$

The following result holds.

Proposition A.3 *Assume* **(H)** *with $p > n+2$ replaced by $p > n$. If $\mathcal{M}_{md} \neq \emptyset$, then $\#\mathcal{M} = 1$.*

Proof See [12, Theorem 4.1]. □

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