Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-026-10494-2

Quasi-Stationary Distributions of Absorbed Singular Diffusion Processes in Higher Dimensions

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Received: 6 December 2025 / Revised: 28 January 2026 / Accepted: 19 February 2026

Abstract
The present paper is devoted to the investigation of the long term behavior of a class of higher-dimensional singular diffusion processes that get absorbed by the extinction set in finite time with probability one. Our primary focus is on the analysis of quasi-stationary distributions (QSDs), which describe the long term behavior of the system conditioned on not being absorbed. Under natural Lyapunov conditions, we construct a QSD and prove the sharp exponential convergence to this QSD for compactly supported initial distributions. Under stronger Lyapunov conditions ensuring that the diffusion process comes down from infinity, we show the uniqueness of a QSD and the exponential convergence to the QSD for all initial distributions. Our results can be seen as the higher-dimensional generalization of Cattiaux et al (Ann. Prob. 2009) as well as the complement to Hening and Nguyen (Ann. Appl. Prob. 2018) which looks at the long term behavior of higher-dimensional diffusions that can only become extinct asymptotically. As applications, we show how our results can be applied to many ecological models, among which cooperative, competitive, and predator-prey Lotka-Volterra systems. The cornerstone of our approach revolves around a uniformly elliptic operator that

A.H. was supported by NSF through the grants DMS 1853463 and DMS 2147903. W.Q. was partially supported by a start-up grant from Academy of Mathematics and Systems Science. Z.S. was partially supported by NSERC RGPIN-2018-04371 and NSERC RGPIN-2024-04938. Y.Y. was partially supported by NSERC RGPIN-2020-04451, PIMS CRG grant, a faculty development grant from the University of Alberta, and a Scholarship from Jilin University.

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Published online: 05 March 2026

we relate through a two-step transform to the Fokker-Planck operator associated with the diffusion process. This operator only has singular coefficients in its zeroth-order terms and can be handled more easily than the Fokker-Planck operator, which is defined on an unbounded domain and exhibits degeneracy in the extinction set. For this operator, we establish the discreteness of its spectrum, its principal spectral theory, the stochastic representation of the semigroup generated by it, and the global regularity for the associated parabolic equation. These results extend beyond the study of QSDs and are of independent interest, especially in the context of spectral theory for degenerate elliptic operators on unbounded domains. As direct consequences, we show that the extinction rate associated with the QSD and the sharp exponential convergence rate are respectively given by the absolute value of the principal eigenvalue and the spectral gap, between the principal eigenvalue and the rest of the spectrum, of this operator. Such characterizations of the QSD and exponential convergence rate were previously unknown in the context of irreversible singular diffusion processes.

Mathematics Subject Classification Primary 60J60 · 60J70 · 34F05; secondary 92D25 · 60H10

Keywords Diffusion process · Quasi-stationary distribution · Existence · Uniqueness · Exponential convergence · Spectral theory · Semigroup · Stochastic representation · Stochastic Lotka-Volterra system

Contents

1 Introduction

Absorbed diffusion processes find frequent application in the field of population biology, serving as models for the evolution of interacting species. In such ecological systems, it is a well-established fact that the eventual extinction of all species is an inevitable outcome, driven by various factors such as finite resources, limited population sizes, mortality rates, and more. However, what is crucial to recognize is that, in practical terms and when measured against human timescales, species can persist for a significant duration [9]. This prolonged persistence of species in ecological systems motivates the need to gain insights into the behavior of the ecosystem before the eventual extinction occurs. As a result, there is a strong impetus to study the dynamics of higher-dimensional diffusion processes under the condition that they do not go extinct.

To be more specific, consider the stochastic Lotka–Volterra competition system:

$$\mathrm{d}Z_t^i = Z_t^i\left(r_i - \sum_{j=1}^{d} c_{ij} Z_t^j\right)\mathrm{d}t + \sqrt{\gamma_i Z_t^i}\mathrm{d}W_t^i, \quad i \in \{1, \ldots, d\}, \tag{1.1}$$

where $Z_t = (Z_t^i) \in \overline{\mathcal{U}} := [0, \infty)^d$ are the abundances of the species at time t, $\{r_i\}_i$ are per-capita growth rates, $\{c_{ii}\}_i$ are the intra-specific competition rates, $\{c_{ij}\}_{i\neq j}$ are inter-specific competition rates, $\{\gamma_i\}_i$ are demographic parameters describing ecological timescales (see e.g. [8, 9]), and $\{W^i\}_i$ are independent standard one-dimensional Wiener processes on some probability space. The system (1.1) can be obtained as a diffusion approximation of a suitably renormalized d-type birth-and-death process in the regime of large population and small ecological timescale. A detailed derivation in dimension $d = 1$ is presented in [8, Section 8.1] (see also [1, Theorem 3.2]). The idea in [8] can be easily generalized to the general multi-type case; see [9] for a brief discussion in the two-type case $d = 2$. For reader's convenience, we present a formal derivation of the system (1.1) in Appendix B.

It is well-known (see e.g. [9, 16]) that Z_t reaches the boundary, also called the extinction set, $\Gamma := \left\{z = (z_i) \in \overline{\mathcal{U}} : z_i = 0 \text{ for some } i \in \{1, \ldots, d\}\right\}$, of $\overline{\mathcal{U}}$ in finite time almost surely. This event corresponds to the extinction of at least one species of the considered community. It is a striking feature of (1.1) that inherits from that of the underlying birth-and-death process (see Appendix B), attributes in particular to the demographic stochasticity $\sqrt{\gamma_i Z_t^i}\mathrm{d}W_t^i$, $i \in \{1, \ldots, d\}$, and can be verified by a straightforward application of the comparison principle together with Feller's classification of boundaries for one-dimensional diffusion processes (see e.g. [40]). Nonetheless, typical trajectories or sample paths of Z_t will stay in $\mathcal{U} := (0, \infty)^d$ for a long period before hitting Γ. This can be interpreted as the temporary coexistence of species, before their ultimate extinction. To understand this type of behavior, notions such as quasi-steady states and metastable states have been put forward. These concepts are often formalized in terms of the *quasi-stationary distributions* (QSDs), which are initial distributions of Z_t on $\mathcal{U}$ such that the distribution of Z_t conditioned on not reaching Γ up to time t is independent of $t \geq 0$. In this context, it is of fundamental mathematical importance to analyze the existence, uniqueness, and domains of (exponential) attraction of QSDs.

The purpose of the present paper is to investigate the existence, uniqueness and exponential convergence to QSDs for a class of irreversible diffusion processes given by models of the form

$$\mathrm{d}Z_t^i = b_i(Z_t)\mathrm{d}t + \sqrt{a_i(Z_t^i)}\mathrm{d}W_t^i, \quad i \in \{1, \ldots, d\}, \tag{1.2}$$

where $Z_t := (Z_t^i) \in \overline{\mathcal{U}}$, $b_i : \overline{\mathcal{U}} \to \mathbb{R}$ and $a_i : [0,\infty) \to [0,\infty)$. We make the following assumptions.

(H1) $a_i \in C^2([0,\infty))$, $a_i(0) = 0$, $a_i'(0) > 0$, $a_i > 0$ on $(0,\infty)$, $\limsup_{s\to\infty}\left[\frac{|a_i'(s)|^2}{a_i(s)} + a_i''(s)\right] < \infty$ and $\int_1^\infty \frac{\mathrm{d}s}{\sqrt{a_i(s)}} = \infty$ for all $i \in \{1,\dots,d\}$.

(H2) $b_i \in C^1(\overline{\mathcal{U}})$ and $b_i|_{z_i=0} = 0$ for all $i \in \{1,\dots,d\}$, where $z_i = 0$ means the set

$$\left\{z = (z_i) \in \overline{\mathcal{U}} : z_i = 0\right\}.$$

(H3) Lyapunov condition: there exists a positive function $V \in C^2(\overline{\mathcal{U}})$ satisfying the following.

(1) $\lim_{|z|\to\infty} V(z) = \infty$ and $\lim_{|z|\to\infty}(b\cdot\nabla_z V)(z) = -\infty$.

(2) There exists a non-negative and continuous function $\tilde{V} : [0,\infty) \to [0,\infty)$ satisfying

$$\int_1^\infty \frac{e^{-\beta\tilde{V}}}{a_i}\mathrm{d}s < \infty, \quad \forall \beta > 0 \text{ and } i \in \{1,\dots,d\}$$

such that $V(z) \geq \sum_{i=1}^d \tilde{V}(z_i)$ for all $z = (z_i) \in \overline{\mathcal{U}}$.

(3) $\lim_{|z|\to\infty} \frac{1}{b\cdot\nabla_z V}\sum_{i=1}^d \left(|\partial_{z_i} b_i| + \frac{|a_i' b_i|}{a_i} + |a_i'\partial_{z_i} V| + a_i|\partial^2_{z_i z_i} V|\right) = 0$.

(4) There exist constants $C > 0$ and $R > 0$ such that

$$\sum_{i=1}^d \left(a_i|\partial_{z_i} V|^2 + \frac{b_i^2}{a_i}\right) \leq -Cb\cdot\nabla_z V \quad \text{in} \quad \mathcal{U}\setminus B_R^+,$$

where $B_R^+ := \{z = (z_i) \in \mathcal{U} : z_i \in (0,R),\ \forall i \in \{1,\dots,d\}\}$ for $R > 0$.

Assumption **(H1)** says that each $a_i(s)$ behaves like $a_i'(0)s$ near $s \approx 0$, and allows each $a_i(s)$ to behave like s^γ for some $\gamma \in (-\infty, 2]$ near $s \approx \infty$. Assumption **(H2)** is satisfied if $b_i(z) = z_i f_i(z)$ for $f_i \in C^1(\overline{\mathcal{U}})$. **(H1)** and **(H2)** ensure that (1.2) generates a diffusion process Z_t on $\overline{\mathcal{U}}$ having Γ as an absorbing set. **(H3)**(1) and the condition $\lim_{|z|\to\infty} \frac{\sum_{i=1}^d |a_i\partial^2_{z_i z_i} V|}{b\cdot\nabla_z V} = 0$ contained in **(H3)**(3) imply the dissipativity of Z_t, and hence, that it does not explode in finite time almost surely. Other assumptions in **(H3)** are technical ones, but they are made according to examples discussed in Sect. 6. We note that for a reversible system, the potential function is a natural choice for V. For irreversible systems, polynomials are usually good choices for V, especially when the coefficients are polynomials or rational functions – this is often the case in applications.

We show in Proposition 2.1 that Z_t reaches Γ in finite time almost surely under **(H1)-(H3)**, and hence, that Z_t does not admit a stationary distribution that has positive concentration in $\mathcal{U}$. It is then natural to look at Z_t before reaching Γ in order to understand the dynamics of Z_t. This drives us to examine quasi-stationary distributions of Z_t or (1.2) conditioned on coexistence, i.e., $[t < T_\Gamma]$, where $T_\Gamma := \inf\{t > 0 : Z_t \in \Gamma\}$ is the first time when Z_t hits Γ. Denote by $\mathbb{P}^\mu$ the law of Z_t with initial distribution μ, and by $\mathbb{E}^\mu$ the expectation with respect to $\mathbb{P}^\mu$.

Definition 1.1 [Quasi-stationary distribution] A Borel probability measure μ on $\mathcal{U}$ is called a *quasi-stationary distribution* (QSD) of Z_t or (1.2) if for each $f \in C_b(\mathcal{U})$, one has

$$\mathbb{E}^\mu\left[f(Z_t)\middle|t < T_\Gamma\right] = \int_{\mathcal{U}} f\,\mathrm{d}\mu, \quad \forall t \geq 0.$$

The QSDs of Z_t are simply stationary distributions of Z_t conditioned on $[t < T_\Gamma]$. This is why QSDs can be seen as governing the dynamics of Z_t before extinction. It is known from the general theory of QSDs (see e.g. [19, 56]) that if μ is a QSD of Z_t, then there exists a unique $\lambda > 0$ such that if $Z_0 \sim \mu$ the time T_Γ is exponentially distributed with rate λ, i.e., $\mathbb{P}^\mu[T_\Gamma > t] = e^{-\lambda t}$ for all $t \geq 0$. In view of this, the number λ is often called the *extinction rate* associated with μ.

Our first result concerning the existence of QSDs and the conditioned dynamics of Z_t is stated in the following theorem. Denote by $\mathcal{P}(\mathcal{U})$ the set of Borel probability measures on $\mathcal{U}$. For convenience, we use the notation $0 < \epsilon \ll 1$ meaning that ϵ is as small as we want.

Theorem A *Assume* **(H1)-(H3)**. *The process Z_t admits a QSD μ_1 satisfying $\int_{\mathcal{U}} e^{\beta V} \mathrm{d}\mu_1 < \infty$ for some $\beta > 0$, and there exists $r_1 > 0$ such that the following statements hold:*

- *For any $0 < \epsilon \ll 1$ and $\mu \in \mathcal{P}(\mathcal{U})$ with compact support in $\mathcal{U}$,*
$$\lim_{t\to\infty} e^{(r_1-\epsilon)t} \left\| \mathbb{P}^\mu[Z_t \in \bullet | t < T_\Gamma] - \mu_1 \right\|_{TV} = 0,$$
where $\|\cdot\|_{TV}$ denotes the total variation distance.
- *There is $f \in C_b(\mathcal{U})$ such that for a.e. $x \in \mathcal{U}$, there is a family of sets $\{\mathcal{K}_{x,\epsilon}\}_{0<\epsilon\ll 1}$ in $(0,\infty)$ satisfying $\mathcal{K}_{x,\epsilon_2} \subset \mathcal{K}_{x,\epsilon_1}$ for $0 < \epsilon_1 < \epsilon_2 \ll 1$ and $\lim_{\epsilon\to 0} \inf_{T>0} |\mathcal{K}_{x,\epsilon} \cap (T, T+1)| = 1$ such that*
$$\lim_{\substack{t\in\mathcal{K}_{x,\epsilon}\\ t\to\infty}} e^{(r_1+\epsilon)t} \left| \mathbb{E}^x[f(Z_t) \big| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\mu_1 \right| = \infty, \quad \forall 0 < \epsilon \ll 1.$$

Remark 1.1 We make some comments about Theorem A.

- The two convergence results stated in Theorem A assert the sharp exponential convergence with rate r_1 of the conditional distribution $\mathbb{P}^\mu[Z_t \in \bullet | t < T_\Gamma]$ to the QSD μ_1 as $t \to \infty$. While it is fairly easy to show that
$$\lim_{t\to\infty} e^{(r_1+\epsilon)t} \sup_{\substack{f\in C_b(\mathcal{U})\\ \|f\|_\infty = 1}} \sup_{x\in\mathcal{U}} \left| \mathbb{E}^x[f(Z_t) \big| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\mu_1 \right| = \infty, \quad \forall 0 < \epsilon \ll 1,$$
the second conclusion presented in Theorem A is much stronger.
- What is behind Theorem A is the spectral theory (more precisely, the discreteness of the spectrum and the principal spectral theory) of a uniformly elliptic operator with singular coefficients in its zeroth-order term that we relate to the Fokker-Planck operator associated with Z_t via equivalent transforms. This allows us to address the challenges posed by the facts that the Fokker-Planck operator associated with Z_t is defined on the unbounded domain $\mathcal{U}$ and exhibits degeneracy on its boundary Γ. The QSD μ_1 is essentially given by the positive eigenfunction associated with the principal eigenvalue $-\lambda_1 < 0$, and the associated extinction rate is just the absolute value of the principal eigenvalue $-\lambda_1$. The sharp exponential convergence rate r_1 is given by the spectral gap, between the principal eigenvalue and the rest of the spectrum. Such characterizations of the QSD and the exponential convergence rate have been obtained in [8, 9] in the reversible case. Our result is the first of this type for the general setting when Z_t is irreversible.
- In the second conclusion, the function f is essentially an arbitrary non-zero element in the range of the spectral projection of the elliptic operator associated with eigenvalues having real part $-\lambda_1 - r_1$. The set $\mathcal{K}_{x,\epsilon}$ more or less corresponds to the ϵ-superlevel set of the function $t \mapsto \left|\mathbb{E}^x[f(X_t) \big| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\nu_1\right|$, where X_t is related to Z_t via some

equivalent transform (see Subsection 2.2 for details) and ν_1 is the QSD of X_t obtained in Theorem 5.1. For irreversible systems, eigenvalues having real part $-\lambda_1 - r_1$ are generally complex, giving rise to oscillations (see (5.8)). As a result, the zeros of this function, if they exist, form a sparse set, and thus, $\bigcup_{0<\epsilon\ll 1}\mathcal{K}_{x,\epsilon}$ is densely distributed in $(0,\infty)$ as described in the statement.

- The assumptions **(H1)-(H3)** do not guarantee the uniqueness of QSDs of Z_t. In the absence of coming down from infinity [8], Z_t could admit infinitely many QSDs that can be described as follows: there exists $\lambda_* > 0$ such that
  - for any $\lambda \in (0, \lambda_*]$, there is a unique QSD μ_λ having λ as the extinction rate;
  - the QSDs $\{\mu_\lambda : \lambda \in (0, \lambda_*]\}$ are partially ordered in the sense that $0 < \lambda_1 < \lambda_2 \le \lambda_*$ implies $\mu_{\lambda_1}((x,\infty)) \ge \mu_{\lambda_2}((x,\infty))$ for all $x \in (0,\infty)$. For this reason, μ_{λ_*} is often called the *minimal* QSD.

  Such a scenario of infinitely many QSDs is known in many situations (see e.g. [19, 50, 54, 70] for one-dimensional diffusion processes, and [10, 27, 29] for jump processes). See Remark 6.3 for the higher-dimensional case.
- Theorem A applies to a large class of population models including stochastic Lotka-Volterra models, models with Holling type functional responses, and Beddington-DeAngelis models. We refer the reader to Sect. 6 for more details.

Although the QSD μ_1 obtained in Theorem A attracts all compactly supported initial distributions, there is no assertion that it is the unique QSD of the process Z_t. To study the uniqueness, we make the following additional assumption.

(H4) There exist positive constants C, γ and R_* such that

$$\lim_{|z|\to\infty} V^{-\gamma-2}\sum_{i=1}^d a_i|\partial_{z_i}V|^2 = 0 \quad\text{and}\quad \frac{1}{2}\sum_{i=1}^d a_i\partial^2_{z_iz_i}V + b\cdot\nabla_z V \le -CV^{\gamma+1} \quad\text{in}\quad \mathcal{U}\setminus B^+_{R_*}.$$

Theorem B *Assume* **(H1)**-**(H4)**. *Let μ_1 and r_1 be as in Theorem A. Then, μ_1 is the unique QSD of Z_t, and for any $0<\epsilon\ll 1$ and $\mu\in\mathcal{P}(\mathcal{U})$, there holds*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t}\left\|\mathbb{P}^\mu[Z_t\in\bullet|t<T_\Gamma]-\mu_1\right\|_{TV}=0.$$

Assumption **(H4)** concerns the strong dissipativity of Z_t near infinity, and implies in particular that Z_t *comes down from infinity* (see Remark 5.3), that is, for each $\lambda>0$, there exists $R=R(\lambda)>0$ such that $\sup_{z\in\mathcal{U}\setminus B^+_R}\mathbb{E}^z\left[e^{\lambda T_R}\right]<\infty$, where $T_R := \inf\left\{t\ge 0 : Z_t\notin\mathcal{U}\setminus B^+_R\right\}$. This is more or less inspired by [8], showing in dimension one that coming down from infinity is equivalent to the uniqueness of QSDs. This property plays a crucial role in the proof of Theorem B. It says that with high probability the process Z_t quickly enters a bounded region. This happens even if the initial distribution of Z_t has a heavy tail near ∞. As a result, it makes no difference to the QSD μ_1 whether the initial distribution of Z_t is compactly supported or not. Theorem B applies to a large class of biological models including in particular the stochastic competition system (1.1) and the stochastic weak cooperation system (i.e., the system (1.1) with $\{-c_{ij}\}_{i\ne j}$ being positive and small in comparison to $\{c_{ii}\}_i$). See Section 6 for more details.

Comparison to existing literature Due to their popularity in describing non-stationary states that are often observed in applications, QSDs have been attracting significant attention. We refer the reader to [19, 56, 60] and references therein for an overview of the theory, developments and applications of QSD. We next present the current state of the art for diffusion processes. The investigation of QSDs for one-dimensional diffusion processes has been analyzed thoroughly. We refer the reader to [12–14, 20, 41, 52, 55, 65, 72] and references therein for the analysis of the regular case. For singular diffusion processes including in particular (1.1) and (1.2) in the one-dimensional setting, the work [8] lays the foundation and is generalized in [14, 37, 49, 57].

Recently, there has been a lot of progress in the study of QSDs for higher-dimensional diffusion processes. Regular diffusion processes restricted to a bounded domain and killed on the boundary have been studied in [11, 16, 33, 44, 59] and are well-understood. The stochastic competition system (1.1) has been studied in [9] in the reversible case, and in [15] in the irreversible case. In both cases the authors established the exponential convergence to the unique QSD. In [9], the authors also deal with the model in the weak cooperation and reversible case. The model treated in [15] has a more general deterministic vector field. These models are typical singular diffusion processes arising from ecology or population biology. In [16, 28, 34], the authors study elliptic diffusion processes and show the existence of a QSD and the exponential convergence to this QSD, which is the unique QSD satisfying a mild integral condition. Similar results for hypoelliptic Hamiltonian systems are established in [5, 34, 35, 45, 61]. In [5], the authors actually work on general degenerate diffusion processes under accessibility conditions and Hörmander's condition.

The works [9, 15, 16, 28, 34], which investigate higher-dimensional singular diffusion processes, are the most relevant to our work. We comment on the approaches employed in these works. In [9], the study of QSD relies on the spectral analysis of the generator, which is assumed to be *reversible* or *self-adjoint* in the weighted space $L^2(\mathcal{U}, d\mu)$ with μ being the non-integrable Gibbs measure. Due to the degeneracy of the diffusion coefficients, the authors adopt a two-step equivalent transform: firstly, a homeomorphism over $\overline{\mathcal{U}}$ is introduced to transfer the degeneracy of the diffusion coefficients to the blow-up singularity of the drift; second, the standard *Liouville transform* is applied to convert the generator of the new SDE obtained in the first step into a Schrödinger operator, for which the spectral theory is well established. This methodology was previously developed in [8] to address the one-dimensional case, whose generator is naturally self-adjoint. Clearly, the self-adjointness plays a pivotal role in the generalizing these techniques to the higher-dimensional case. However, it is important to note that most higher-dimensional diffusion processes are *irreversible* with non-self-adjoint generators.

In [15, 16, 28, 34], the authors aim to establish a general probability framework, similar to those used in the study of stationary distributions, for investigating QSDs in diffusion processes. These frameworks typically consist of three essential ingredients: the Lyapunov condition, Doeblin-type/minorization condition and certain regularity conditions. Checking these conditions is a routine job for elliptic diffusion processes, and requires hypoelliptic conditions and controllability for degenerate ones. In [15], the authors focus on studying general absorbed continuous-time Markov processes, with a particular application to (1.1). The framework introduced in [16] is applicable to both discrete- and continuous-time absorbed Markov processes. The primary objective of [34] is to investigate QSDs of stochastic damping Hamiltonian systems when the position variable is constrained within a bounded region. The work [28], originally not intended for studying QSDs, examines sub-Markov semigroups and their results can be applied to absorbed processes. It is noteworthy that studying the essential spectral radius of the semigroup under Lyapunov conditions plays a crucial role in both [34] and [28].

Our approach is rooted in the spectral theory of the Fokker-Planck operator associated with Z_t, which is non-self-adjoint, defined on the unbounded domain $\mathcal{U}$, and exhibits degeneracy on its boundary Γ, causing significant challenges. Inspired by the methodology introduced in [8, 9] treating reversible diffusion processes, we develop a two-step equivalent transform to render the Fokker-Planck operator more manageable. In the first step, we follow a procedure akin to that outlined in [9] to eliminate the degeneracy of the Fokker-Planck operator and obtain a new Fokker-Planck operator whose first-order terms have coefficients blow up at Γ. It is the second step that our approach showcases its novelty. Carefully examining the blow-up singularities in the first-order terms of the new Fokker-Planck operator, we design a *parameter-dependent Liouville-type transform*. This transform effectively eliminate these singularities, yielding a parameter-dependent uniformly elliptic operator exhibiting blow-up coefficients only in its zeroth-order terms. The presence of the parameter expands the degree of freedom for analysis and holds significant technical importance.

Through a careful analysis of the blow-up properties of these coefficients, we introduce a weighted Sobolev space and successfully establish a priori estimates for this uniformly elliptic operator (with a specified parameter) in the weighted Sobolev space. These a priori estimates ascertain the discreteness of its spectrum, unravel the principal spectral theory, and uncover the C_0-semigroup generated by it. Leveraging the stochastic representation of this semigroup as a bridge, we are able to obtain fine dynamical properties of Z_t conditioned on non-extinction. As direct consequences of our approach, we demonstrate the following: (i) The principal eigenpair of this operator gives rise to the QSD μ_1 in Theorem A, along with its associated extinction rate. (ii) The sharp exponential convergence rate r_1 stated in Theorem A is given by the spectral gap, between the principal eigenvalue and the rest of the spectrum, of this operator. Such explicit characterizations of the QSD and the sharp exponential convergence rate to the QSD were hitherto unknown for irreversible singular diffusion processes. Moreover, given the significance of Liouville-type transforms, spectral theory, and the stochastic representation of semigroups, our results extend beyond the study of QSDs and are of independent interest, especially in the context of spectral theory for degenerate elliptic operators on unbounded domains.

To this end, we would like to emphasize that applied to specific systems that do not assume reversibility, such as stochastic Lotka-Volterra models, both the results from [15, 16, 28, 34] (even though these applications are not explicitly demonstrated in [28, 34]) and our own findings can establish the (unique) existence of the QSD and its exponential attractivity. Additionally, our results provide the precise exponential rate, which is determined by the spectral gap of the Fokker-Planck operator in a specific function space.

Demographic and environmental stochasticity Consider an isolated ecosystem of interacting species. Due to finite population effects and demographic stochasticity, extinction of all species is certain to occur in finite time for all populations. However, the time to extinction can be large and the species densities can fluctuate before extinction occurs.

One way of capturing this behaviour is ignoring the effects of demographic stochasticity (i.e. finite population effects) and focusing on models with environmental stochasticity where extinction can only be asymptotic as $t \to \infty$. This approach led to the development of the field of modern coexistence theory (MCT), started by Lotka [51] and Volterra [67], and later developed by Chesson [17, 18] and other authors [6, 31, 64, 66]. Recently, there have been powerful results that have led to a general theory of coexistence and extinction [4, 38, 39].

A second way of analyzing the long term dynamics of the species is by including demographic stochasticity and studying the QSDs of the system - this is the approach we took in this paper. Our work can be seen as complementary to the work done for systems with environmental stochasticity.

Overview of proofs The proofs of Theorem A and Theorem B use techniques from PDE, spectral theory, semigroup theory and probability theory. For the reader's convenience, we outline the strategy of the proofs with the help of Figure 1.

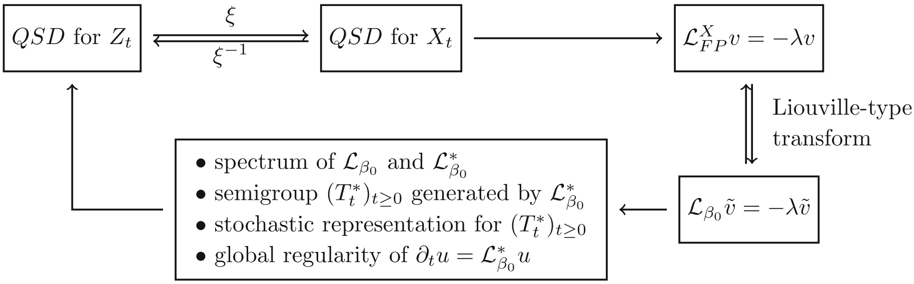

Fig. 1 Overview of proofs

- (Equivalent formalism) Theoretically, the study of QSDs of Z_t can be accomplished by investigating the (principal) spectral theory of $\mathcal{L}_{\mathbf{FP}}^Z$, the Fokker-Planck operator associated with Z_t. However, the degeneracy of $\mathcal{L}_{\mathbf{FP}}^Z$ on Γ would cause significant drawbacks. To circumvent this, we first follow [9] to introduce a homeomorphism $\xi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ and define a new process $X_t = \xi(Z_t)$ whose Fokker-Planck operator $\mathcal{L}_{\mathbf{FP}}^X$ has $\frac{1}{2}\Delta$ as its second-order term.
  Although $\mathcal{L}_{\mathbf{FP}}^X$ has the best possible second-order term, the coefficients of its first-order terms unfortunately have blow-up singularities on Γ. Introducing a parameter-dependent Liouville-type transform, we convert $\mathcal{L}_{\mathbf{FP}}^X$ into a parameter-dependent uniformly elliptic operator $\mathcal{L}_\beta := e^{\frac{Q}{2}+\beta U}\mathcal{L}_{\mathbf{FP}}^X e^{-\frac{Q}{2}-\beta U}$, whose blow-up singularities on Γ only appear in the coefficients of the zeroth-order terms. Here, $\beta > 0$ is the parameter, $U = V \circ \xi^{-1}$, and Q, given in (2.6), has singularities near Γ (see Remark A.1). The details are presented in Subsection 2.2. The parameter β is fixed to be β_0 in Lemma 3.2 (3) so that a priori estimates can be established for $\mathcal{L}_{\beta_0}$.
- (Spectral analysis) Our spectral analysis focuses on the operator $\mathcal{L}_{\beta_0}$ in $L^2(\mathcal{U};\mathbb{C})$ as well as its adjoint $\mathcal{L}_{\beta_0}^*$. According to the behavior of the coefficients of $\mathcal{L}_{\beta_0}$ near Γ and infinity, we design a weight function and define a weighted first-order Sobolev space $\mathcal{H}^1(\mathcal{U};\mathbb{C})$ that is compactly embedded into $L^2(\mathcal{U};\mathbb{C})$. Establishing a priori estimates for $\mathcal{L}_{\beta_0}$, we are able to solve the elliptic problem for $\mathcal{L}_{\beta_0} - M$ for some $M \gg 1$ in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$. The discreteness of the spectrum and principal spectral theory of $\mathcal{L}_{\beta_0}$ and $\mathcal{L}_{\beta_0}^*$ then follow. The details are given in Subsection 3.3 and Subsection 3.4.
- (Semigroup and stochastic representation) The operator $\mathcal{L}_{\beta_0}^*$ generates an analytic and eventually compact semigroup $(T_t^*)_{t\geq 0}$ on $L^2(\mathcal{U};\mathbb{C})$ that can be "block-diagonalized" according to spectral projections. We establish the stochastic representation of $(T_t^*)_{t\geq 0}$ in terms of X_t before reaching Γ, and therefore, connect the dynamics of $(T_t^*)_{t\geq 0}$ with that of X_t conditioned on $[t < S_\Gamma]$, where S_Γ is the first time that X_t hits Γ. More precisely, we show that for each $f \in C_b(\mathcal{U};\mathbb{C})$ satisfying $\tilde{f} := fe^{-\frac{Q}{2}-\beta_0 U} \in L^2(\mathcal{U};\mathbb{C})$, there holds $T_t^*\tilde{f} = e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]$ for all $t \geq 0$. The semigroups are given in Subsection 3.3 and Subsection 3.4. The stochastic representation of $(T_t^*)_{t\geq 0}$ is established in Subsection 4.3.
- (Global regularity and conclusions) The spectral theory and stochastic representation allow us to prove the results stated in Theorem A and Theorem B for the process X_t. While proving the existence of QSDs is pretty straightforward, we run into significant technical difficulties in establishing the convergence even for compactly supported initial distributions. This is due to: (i) the limitations of the stochastic representation because

of the unboundedness of the Liouville-type transform and its inverse (i.e., $e^{\frac{Q}{2}+\beta_0 U}$ and $e^{-\frac{Q}{2}-\beta_0 U}$ blow up near ∞ and at Γ, respectively); (ii) the requirement of L^∞ properties of $(T_t^*)_{t\geq 0}$. These issues are overcome by establishing the global regularity of solutions of $\partial_t u = \mathcal{L}_{\beta_0}^* u$ leading in particular to the global regularity of $(T_t^*)_{t\geq 0}$. The details are given in Sect. 5.

The rest of the paper is organized as follows. In Sect. 2, we provide some preliminaries including the proof of Z_t being absorbed by Γ in finite time almost surely, the derivation of the operator $\mathcal{L}_{\beta_0}$, and results related to the approximation of S_Γ. In Sect. 3, we study the spectral theory of $\mathcal{L}_{\beta_0}$ and its adjoint operator $\mathcal{L}_{\beta_0}^*$, and establish the associated semigroups $(T_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$. Sect. 4 is devoted to the stochastic representation of $(T_t^*)_{t\geq 0}$. In Sect. 5, we investigate the existence and uniqueness of QSDs and the exponential convergence to QSDs of X_t conditioned on the coexistence. Theorem A and Theorem B are proven in this section. In the last section, Sect. 6, we discuss applications of Theorem A and Theorem B to a wider variety of ecological models including stochastic Lotka-Volterra systems, and models with Holling type or Beddington-DeAngelis functional responses. Appendix A is included to provide the proof of some technical lemmas.

2 Preliminaries

In Subsection 2.1, we show that Z_t hits Γ in finite time almost surely. In Subsection 2.2, we present equivalent formulations for studying the existence of QSDs, and derive the operator we shall focus on in later sections. In Subsection 2.3, we fix a family of first exit times and present an approximation result.

2.1 Hitting the absorbing boundary

We prove that Z_t reaches Γ in finite time almost surely. Denote by $\mathcal{L}^Z$ the diffusion operator associated with Z_t, namely,

$$\mathcal{L}^Z = \frac{1}{2}\sum_{i=1}^{d} a_i \partial^2_{z_i z_i} + b\cdot\nabla_z. \tag{2.1}$$

Proposition 2.1 *Assume* **(H1)**-**(H3)**. *Then,* $\mathbb{P}^z[T_\Gamma < \infty] = 1$ *for each* $z \in \mathcal{U}$.

Proof The idea of the proof is more or less classical; our arguments are closer to that of [16, Proposition 4.10]. By **(H1)**-**(H2)**, $b_i \in C^1(\overline{\mathcal{U}})$ and $\sqrt{a_i}$ is locally Lipchitz in $\mathcal{U}$ and locally $\frac{1}{2}$-Hölder continuous near Γ. The classical theorem of Yamada–Watanabe [68, 69] ensures the pathwise uniqueness as well as the strong Markov property of solutions of (1.2).

Recall that for $R>0$, $B_R^+ = \{z=(z_i)\in\mathcal{U} : z_i \in (0,R),\ \forall i \in \{1,\dots,d\}\}$. The result is proven in four steps.

Step 1 We claim that for each $z\in\mathcal{U}$, Z_t does not explode in finite time $\mathbb{P}^z$-a.e., and moreover, there exists $R>0$ such that $\mathbb{P}^z[T_R<\infty]=1$ for all $z\in\mathcal{U}$, where $T_R := \hat{T}_R \wedge T_\Gamma$ and $\hat{T}_R := \inf\left\{t\geq 0 : Z_t \in \overline{B_R^+}\right\}$.

By the assumptions **(H3)**(1)(3), there is $R>0$ such that $\mathcal{L}^Z V \leq -1$ in $\overline{\mathcal{U}}\setminus\overline{B_R^+}$. This together with the Itô-Dynkin's formula implies that $\mathbb{E}^z\left[V(Z_{t\wedge\hat{T}_R})\right] \leq V(z) - \mathbb{E}^z\left[t\wedge\hat{T}_R\right]$

for all $t \geq 0$. Passing to the limit $t \to \infty$ yields $\mathbb{E}^z\left[\hat{T}_R\right] \leq V(z) < \infty$ and thus, $\mathbb{P}^z[\hat{T}_R < \infty] = 1$. The claim follows immediately.
Step 2 We prove $\mathbb{P}^z[\tau_{2R} < \infty] = 1$ for each $z \in B^+_{2R}$, where $\tau_{2R} := \inf\left\{t \geq 0 : Z_t \notin B^+_{2R}\right\}$.

For each $i \in \{1, \ldots, d\}$, we set $\overline{b}_i := \sup_{z \in B^+_{3R}} \frac{b_i(z)}{z_i}$. The assumption **(H2)** ensures $\overline{b}_i < \infty$. Denote by Y^{i,y_i}_t the solution of the SDE $\mathrm{d}Y^i_t = \overline{b}_i Y^i_t \mathrm{d}t + \sqrt{a_i(Y^i_t)}\mathrm{d}W^i_t$ in $[0, 3R]$ with an initial condition $Y^{i,y_i}_0 = y_i \in (0, 2R)$, and let $\tau^{y_i}_i$ be the first time that Y^{i,y_i}_t hits 0 or $3R$, namely, $\tau^{y_i}_i = \inf\left\{t \geq 0 : Y^{i,y_i}_t = 0 \text{ or } 3R\right\}$. The assumptions on a_i and [40, Theorem VI-3.1] guarantee that

$$\mathbb{P}\left[\tau^{y_i}_i < \infty\right] = 1 \quad \text{and} \quad \mathbb{P}\left[Y^{i,2R}_t \in [0, 3R),\ \forall t \in [0, \tau^{2R}_i]\right] > 0$$

for all $y_i \in (0, 2R)$ and $i \in \{1, \ldots, d\}$. Since

$$\mathbb{P}\left[Y^{i,y_i}_t \leq Y^{i,2R}_t,\ \forall t \in [0, \tau^{y_i}_i]\right] = 1, \quad \forall y_i \in (0, 2R)$$

due to the comparison theorem for one-dimensional SDEs (see e.g. [40, Theorem VI-1.1]), we deduce

$$p_i := \inf_{y_i \in (0,2R)} \mathbb{P}\left[Y^{i,y_i}_t \in [0, 3R)\,,\ \forall t \in [0, \tau^{y_i}_i]\right] \geq \mathbb{P}\left[Y^{i,2R}_t \in [0, 3R)\,,\ \forall t \in [0, \tau^{2R}_i]\right] > 0.$$

That is, starting at $y_i \in (0, 2R)$, the probability that Y^{i,y_i}_t exits $(0, 3R)$ through 0 has a uniform lower bound. By the comparison theorem, we find $Z^i_t \leq Y^{i,z_i}_t$ for $t \in [0, \tau^{z_i}_i]$ and thus,

$$\mathbb{P}^z\left[Z^i_t \text{ reaches 0 or } 2R \text{ before } \tau^{z_i}_i\right] \geq p_i > 0, \quad \forall i \in \{1, \ldots, d\} \text{ and } z = (z_i) \in B^+_{2R}.$$

As a result, $\inf_{z \in B^+_{2R}} \mathbb{P}^z[\tau_{2R} \leq \max_i \tau^{z_i}_i] \geq \max_i p_i > 0$. Since Z_t is time-homogeneous and strongly Markovian, in the case where $\tau_{2R} > \max_i \tau^{z_i}_i =: \tau'$, we treat $Z_{\tau'}$ as the new initial position and repeat the procedure outlined above. By iterating this process countably many times, we obtain $\mathbb{P}^z[\tau_{2R} < \infty] = 1$, for all $z \in B^+_{2R}$.
Step 3 We show that $\inf_{z \in \overline{B^+_R}} \mathbb{P}^z[Z_{\tau_{2R}} \in \Gamma] > 0$.

Fix $i \in \{1, \ldots, d\}$. We restrict Y^{i,y_i}_t on $[0, \frac{3}{2}R]$ and let $\hat{\tau}^{y_i}_i$ be the first time that Y^{i,y_i}_t hits 0 or $\frac{3}{2}R$. As in **Step 2**, we derive from [40, Theorem VI-3.1] and the comparison theorem for one-dimensional SDEs that

$$\inf_{y_i \in (0,R)} \mathbb{P}\left[Y^{i,y_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \hat{\tau}^{y_i}_i]\right] \geq \mathbb{P}\left[Y^{i,R}_t \in [0, 3R/2)\,,\ \forall t \in [0, \hat{\tau}^{R}_i]\right] > 0.$$

Let $z = (z_i) \in \overline{B^+_R}$. Since the comparison theorem ensures $Z^i_t \leq Y^{i,z_i}_t$ for $t \in [0, \hat{\tau}^{z_i}_i]$, we find

$$\begin{aligned}
\mathbb{P}^z[Z_{\tau_{2R}} \in \Gamma] &\geq \mathbb{P}\left[Y^{i,z_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \hat{\tau}^{z_i}_i],\ i \in \{1, \ldots, d\}\right] \\
&\geq \prod_{i=1}^{d} \inf_{y_i \in [0,R]} \mathbb{P}\left[Y^{i,y_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \hat{\tau}^{y_i}_i]\right] > 0,
\end{aligned}$$

where we used the independence of Y^{i,z_i}_t, $i \in \{1, \ldots, d\}$ in the second inequality. The claim follows.

Step 4 We finish the proof of the proposition. By **Step 3**, $p := \inf_{z\in\partial B_R^+\setminus\Gamma} \mathbb{P}^z[Z_{\tau_{2R}} \in \Gamma] > 0$. Set

$$T_R^{(1)} := \inf\left\{t \in [0, T_\Gamma] : Z_t \in B_R^+\right\} \quad \text{and} \quad S_{2R}^{(1)} := \inf\left\{t \geq T_R^{(1)} : Z_t \notin B_{2R}^+\right\},$$

and recursively define for each $n \geq 1$,

$$T_R^{(n+1)} := \inf\left\{t \in [S_{2R}^{(n)}, T_\Gamma] : Z_t \in B_R^+\right\} \quad \text{and} \quad S_{2R}^{(n+1)} := \inf\left\{t \geq T_R^{(n+1)} : Z_t \notin B_{2R}^+\right\}.$$

Fix $z \in \mathcal{U}$. Since **Step 1**, **Step 2** and the strong Markov property ensure $\mathbb{P}^z\left[T_R^{(n)} < \infty\right] = 1$ and $\mathbb{P}^z\left[S_{2R}^{(n)} < \infty\right] = 1$ for all $n \in \mathbb{N}$, we find $\mathbb{P}^z\left[Z_{S_{2R}^{(n)}} \in \partial B_{2R}^+ \setminus \Gamma\right] \leq (1-p)^n$ for all $n \in \mathbb{N}$. As a result

$$\mathbb{P}^z\left[T_\Gamma = \infty\right] = \mathbb{P}^z\left[S_{2R}^{(n)} < \infty, \forall n \in \mathbb{N}\right] \leq \lim_{n\to\infty} (1-p)^n = 0.$$

This completes the proof. □

Remark 2.1 The assumptions **(H3)**(2)(4) are not needed in the proof of Proposition 2.1.

2.2 Equivalent formulation

Denote by $\mathcal{L}_{\mathbf{FP}}^Z$ the Fokker-Planck operator associated with Z_t or (1.2), namely,

$$\mathcal{L}_{\mathbf{FP}}^Z u := \frac{1}{2}\sum_{i=1}^d \partial_{z_i z_i}^2 (a_i u) - \nabla_z \cdot (bu) \quad \text{in} \quad \mathcal{U}, \quad \forall u \in C^2(\mathcal{U}). \tag{2.2}$$

Proposition 2.2 *Assume* **(H1)**-**(H2)**. *Let μ be a QSD of Z_t. Then, μ admits a positive density $u \in W_{loc}^{2,p}(\mathcal{U})$ for any $p > d$ that satisfies $-\mathcal{L}_{\mathbf{FP}}^Z u = \lambda_1 u$ a.e. in $\mathcal{U}$, where λ_1 is the extinction rate associated with μ.*

Proof Following the arguments leading to [56, Proposition 4], we see that

$$\int_{\mathcal{U}} \mathbb{E}^x[f(Z_t)]d\mu = e^{-\lambda_1 t}\int_{\mathcal{U}} f d\mu, \quad \forall f \in C_0^\infty(\mathcal{U}).$$

Since $\frac{d}{dt}\mathbb{E}^\bullet[f(Z_t)] = \mathcal{L}^Z(\mathbb{E}^\bullet[f(Z_t)])$, we differentiate the above equation and evaluate at $t = 0$ to find $\int_{\mathcal{U}}(-\mathcal{L}^Z + \lambda_1) f d\mu = 0$ for all $f \in C_0^\infty(\mathcal{U})$. It follows from **(H1)-(H2)** and the classical regularity result in [7, Corollaries 2.10 and 2.11] that μ has a positive density $u \in W_{loc}^{1,p}(\mathcal{U})$ for any $p > d$. Then, we apply [32, Theorem 8.8] and [32, Theorem 9.19] to show that $u \in W_{loc}^{2,p}(\mathcal{U})$ for any $p > d$. □

Proposition 2.2 suggests studying the principal spectral theory of the operator $-\mathcal{L}^Z$ in order to find a QSD for Z_t. Direct analysis of the operator $-\mathcal{L}^Z$ is however difficult due to the degeneracy of the diffusion matrix $\mathrm{diag}\{a_1, \ldots, a_d\}$ on the boundary Γ of $\mathcal{U}$. To resolve this issue, we follow [8] to define a new process that is equivalent to Z_t and whose Fokker-Planck operator or diffusion operator is uniformly non-degenerate in $\mathcal{U}$. We proceed as follow.

For each $i \in \{1, \ldots, d\}$, we define $\xi_i : [0,\infty) \to [0,\infty)$ by setting

$$\xi_i(z_i) := \int_0^{z_i} \frac{1}{\sqrt{a_i(s)}} \mathrm{d}s, \quad z_i \in [0,\infty).$$

By **(H1)**, each ξ_i is increasing and onto, and thus, ξ_i^{-1} is well-defined. Set

$$\xi := (\xi_i) : \overline{\mathcal{U}} \to \overline{\mathcal{U}} \quad \text{and} \quad \xi^{-1} := (\xi_i^{-1}) : \overline{\mathcal{U}} \to \overline{\mathcal{U}}.$$

Clearly, $\xi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a homeomorphism with inverse ξ^{-1}, and satisfies $\xi(\Gamma) = \Gamma$ and $\xi(\mathcal{U}) = \mathcal{U}$.

Define a new process $X_t = (X_t^i)$ by setting

$$X_t^i := \xi_i(Z_t^i), \quad i \in \{1, \dots, d\}, \quad \text{or simply,} \quad X_t = \xi(Z_t), \quad t \geq 0.$$

It is clear that Γ is also an absorbing set for the process X_t, and X_t reaches Γ in finite time almost surely. Moreover, QSDs of Z_t and X_t are in an one-to-one correspondence as shown in the next result whose proof is straightforward.

Proposition 2.3 *Let μ be a Borel probability measure on $\mathcal{U}$. Then, μ is a QSD of Z_t if and only if $\xi_*\mu$ is a QSD of X_t, where ξ_* is the pushforward operator induced by ξ. Moreover, μ and $\xi_*\mu$ have the same extinction rates.*

Since $\xi \in C^2(\mathcal{U})$, we apply Itô's formula to find

$$\mathrm{d}X_t^i = \Big[p_i(X_t) - q_i(X_t^i)\Big]\mathrm{d}t + \mathrm{d}W_t^i, \quad i \in \{1, \dots, d\} \quad \text{in} \quad \mathcal{U}, \tag{2.3}$$

where $p_i : \mathcal{U} \to \mathbb{R}$ and $q_i : (0, \infty) \to \mathbb{R}$ are given by

$$p_i(x) := \frac{b_i(\xi^{-1}(x))}{\sqrt{a_i(\xi_i^{-1}(x_i))}} \quad \text{and} \quad q_i(x_i) := \frac{a_i'(\xi_i^{-1}(x_i))}{4\sqrt{a_i(\xi_i^{-1}(x_i))}}, \quad x = (x_i) \in \mathcal{U}.$$

Denote by $\mathcal{L}_{\mathbf{FP}}^X$ the Fokker-Planck operator associated with (2.3), namely,

$$\mathcal{L}_{\mathbf{FP}}^X v = \frac{1}{2}\Delta v - \nabla \cdot ((p - q)v) \quad \text{in} \quad \mathcal{U}, \quad \forall v \in C^2(\mathcal{U}),$$

where $p = (p_i)$ and $q = (q_i)$. Then, Proposition 2.2 has a counterpart for QSDs of X_t.

Proposition 2.4 *Assume* **(H1)-(H2)**. *Let ν be a QSD of X_t with extinction rate λ_1. Then, ν admits a positive density $v \in W_{loc}^{2,p}(\mathcal{U})$ for any $p > d$ that satisfies $-\mathcal{L}_{\mathbf{FP}}^X v = \lambda_1 v$ a.e. in $\mathcal{U}$.*

Remark 2.2 Note that the process X_t and the process generated by solutions of (2.3) are not really the same, as (2.3) is only defined in $\mathcal{U}$. However, the two processes agree as long as X_t stays in $\mathcal{U}$. More precisely, if we denote by S_Γ the first time that X_t reaches Γ, that is, $S_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$, then X_t satisfies (2.3) on the event $[t < S_\Gamma]$.

As indicated by Proposition 2.4, QSDs of X_t are closely related to positive eigenfunctions of $-\mathcal{L}_{\mathbf{FP}}^X$, and therefore, it is natural to investigate the associated eigenvalue problem, namely,

$$-\mathcal{L}_{\mathbf{FP}}^X v = \lambda v \quad \text{in} \quad \mathcal{U}. \tag{2.4}$$

Note that the operator $\mathcal{L}_{\mathbf{FP}}^X$ is uniformly elliptic in $\mathcal{U}$, but the functions q_i, $i \in \{1, \dots, d\}$ appearing in its first-order terms satisfy $q_i(x_i) \to \infty$ as $x_i \to 0^+$ for each $i \in \{1, \dots, d\}$. Such blow-up singularities make the investigation of the above eigenvalue problem very hard. In the following, we generalize the idea in [8] to transform (2.4) into the eigenvalue problem of another elliptic operator that has blow-up singularities only in the zeroth-order term and thus is easier to deal with.

Set

$$U := V \circ \xi^{-1} \quad \text{in} \quad \mathcal{U}, \tag{2.5}$$

where V is given in **(H3)**, and

$$Q(x) := \sum_{i=1}^{d} \int_{1}^{x_i} 2q_i(s)ds = \frac{1}{2} \sum_{i=1}^{d} \left[\ln a_i(\xi_i^{-1}(x_i)) - \ln a_i(\xi_i^{-1}(1)) \right], \quad x \in \mathcal{U}. \tag{2.6}$$

For each $\beta > 0$, we use the Liouville-type transform to define $\mathcal{L}_\beta := e^{\frac{Q}{2}+\beta U} \mathcal{L}^X_{\mathbf{FP}} e^{-\frac{Q}{2}-\beta U}$. It is straightforward to check that

$$\mathcal{L}_\beta = \frac{1}{2}\Delta - (p + \beta \nabla U) \cdot \nabla - e_\beta \quad \text{in} \quad \mathcal{U}, \tag{2.7}$$

where

$$e_\beta = \frac{1}{2}\left(\beta \Delta U - \beta^2 |\nabla U|^2\right) - \beta p \cdot \nabla U + \frac{1}{2}\sum_{i=1}^{d}(q_i^2 - q_i') - p \cdot q + \nabla \cdot p. \tag{2.8}$$

Note that the coefficient of the first-order term $-(p+\beta\nabla U)$ is continuous up to the boundary Γ, and the term $\frac{1}{2}\sum_{i=1}^{d}(q_i^2 - q_i') - p \cdot q$ blows up at the boundary Γ, but it appears in the zeroth-order term.

The following proposition establishes the "equivalence" between the eigenvalue problem (2.4) and the eigenvalue problem associated with the operator $\mathcal{L}_\beta$.

Proposition 2.5 *Suppose $v \in W^{2,1}_{loc}(\mathcal{U})$ and $\lambda \in \mathbb{R}$. Set $\tilde{v} := v e^{\frac{Q}{2}+\beta U}$. Then, (v, λ) solves (2.4) if and only if $-\mathcal{L}_\beta \tilde{v} = \lambda \tilde{v}$ in $\mathcal{U}$.*

According to Proposition 2.5, the investigation of QSDs of X_t is reduced to the exploration of the principal spectral theory of $-\mathcal{L}_\beta$ (with a fixed β), something which we will do by choosing an appropriate function space.

2.3 Approximation by first exit times

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a sequence of arbitrarily fixed bounded, connected and open sets in $\mathcal{U}$ with C^2 boundaries that satisfy $\mathcal{U}_n \subset\subset \mathcal{U}_{n+1} \subset\subset \mathcal{U}$ for all $n \in \mathbb{N}$ and $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. For each $n \in \mathbb{N}$, denote by τ_n the first time that X_t exits $\mathcal{U}_n$, namely,

$$\tau_n = \inf\{t \geq 0 : X_t \notin \mathcal{U}_n\}.$$

Recall that S_Γ is the first time that X_t hits Γ. The following result turns out to be useful.

Lemma 2.1 *Assume* **(H1)-(H3)**. *For each $x \in \mathcal{U}$, one has $\mathbb{P}^x\left[\lim_{n\to\infty} \tau_n = S_\Gamma\right] = 1$ and*

$$\lim_{n\to\infty} \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right] = \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad \forall f \in C_b(\mathcal{U}).$$

Proof Fix $x \in \mathcal{U}$. Obviously, $\tau_n < \tau_{n+1}$ for each $n \in \mathbb{N}$. Set $\tau := \lim_{n\to\infty} \tau_n$. The first conclusion follows if we show $\mathbb{P}^x[\tau = S_\Gamma] = 1$.

Clearly, $\tau_n < S_\Gamma$ for each $n \in \mathbb{N}$, leading to $\tau \leq S_\Gamma$. Since $X_t = \xi(Z_t)$ for $t \geq 0$, we find from Proposition 2.1 that $\mathbb{P}^x[S_\Gamma < \infty] = \mathbb{P}^{\xi^{-1}(x)}[T_\Gamma < \infty] = 1$. Therefore, $\mathbb{P}^x[\tau < \infty] = 1$.

Noting that arguments in the proof of Proposition 2.1 ensure that Z_t and X_t do not explode in finite time, we derive $|X_\tau| = \lim_{n\to\infty} |X_{\tau_n}| < \infty$. Moreover, since $X_{\tau_n} \in \partial\mathcal{U}_n$

and $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$, it follows that $X_\tau \in \Gamma$. As S_Γ is the first hitting time of the boundary Γ and $\tau \le S_\Gamma$, one has $\tau = S_\Gamma$.

Since τ_n increases to S_Γ $\mathbb{P}$-a.s., we find $\lim_{n\to\infty} \mathbb{1}_{\{t<\tau_n\}} = \mathbb{1}_{\{t<S_\Gamma\}}$ for each $t \ge 0$. The second conclusion then follows from the dominated convergence theorem. This completes the proof. □

3 Spectral Theory and Semigroup

This section is devoted to the spectral theory of $-\mathcal{L}_\beta$ in an appropriate function space for some appropriately fixed β, as well as the semigroup generated by $\mathcal{L}_\beta$. In Subsection 3.1 we define a weighted Hilbert space. In Subsection 3.2 we derive some important estimates and meanwhile fix a special β, denoted by β_0. In Subsection 3.3 we study the (principal) spectral theory of $-\mathcal{L}_{\beta_0}$ and the semigroup generated by $\mathcal{L}_{\beta_0}$. In Subsection 3.4 the spectral theory of $-\mathcal{L}^*_{\beta_0}$, where $\mathcal{L}^*_{\beta_0}$ is the adjoint operator of $\mathcal{L}_{\beta_0}$, and the semigroup generated by $\mathcal{L}^*_{\beta_0}$ are investigated.

3.1 A weighted Hilbert space

For $\delta \in (0, 1)$, let

$$\Gamma_\delta := \{x = (x_i) \in \mathcal{U} : x_i \le \delta \text{ for some } i \in \{1, \dots, d\}\}.$$

It is easy to see from **(H3)**(1) that there exists $R_0 > 0$ such that $\sup_{\mathcal{U}\setminus B^+_{R_0}} (b\cdot\nabla_z V)\circ\xi^{-1} < 0$, where we recall $B^+_R = \{x = (x_i) \in \mathcal{U} : x_i \in (0, R),\ \forall i \in \{1, \dots, d\}\}$ for $R > 0$. Fix some $\delta_0 \in (0, 1)$. Let $\alpha : \mathcal{U} \to \mathbb{R}$ be defined by

$$\alpha(x) := \begin{cases} \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\}, & x \in \Gamma_{\delta_0} \cap B^+_{R_0}, \\ \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - (b\cdot\nabla_z V)(\xi^{-1}(x)), & x \in \Gamma_{\delta_0} \cap (\mathcal{U} \setminus B^+_{R_0}), \\ -(b\cdot\nabla_z V)(\xi^{-1}(x)), & x \in (\mathcal{U} \setminus \Gamma_{\delta_0}) \cap (\mathcal{U} \setminus B^+_{R_0}), \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

See Figure 2 for an illustration of the subdomains used in (3.1). Obviously, $\inf_{\mathcal{U}} \alpha > 0$, $\lim_{x\to\Gamma} \alpha(x) = \infty$ and $\lim_{|x|\to\infty} \alpha(x) = \infty$. This α is defined according to the behavior of the coefficients of $-\mathcal{L}_\beta$ near Γ and ∞. Its significance is partially reflected in Lemma 3.2 below. See Remark 3.1 after Lemma 3.2 for more comments.

Denote by $\mathcal{H}^1(\mathcal{U}; \mathbb{C})$ the space of all weakly differentiable complex-valued functions $\phi : \mathcal{U} \to \mathbb{C}$ satisfying $\|\phi\|_{\mathcal{H}^1} := \left(\int_{\mathcal{U}} \alpha|\phi|^2 dx + \int_{\mathcal{U}} |\nabla\phi|^2 dx\right)^{\frac{1}{2}} < \infty$. It is not hard to verify that $\mathcal{H}^1(\mathcal{U}; \mathbb{C})$ is a Hilbert space with the inner product:

$$\langle\phi, \psi\rangle_{\mathcal{H}^1} := \int_{\mathcal{U}} \alpha\phi\overline{\psi} dx + \int_{\mathcal{U}} \nabla\phi\cdot\nabla\overline{\psi} dx, \quad \forall\phi, \psi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C}),$$

where $\overline{\psi}$ denotes the complex conjugate of ψ.

Lemma 3.1 *Assume* **(H3)**. *Then, $\mathcal{H}^1(\mathcal{U}; \mathbb{C})$ is compactly embedded into $L^2(\mathcal{U}; \mathbb{C})$.*

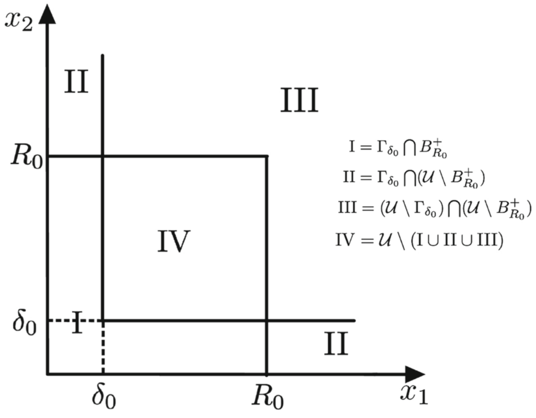

Fig. 2 Decomposition of $\mathcal{U}$ in dimension two

Proof Let $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}^1(\mathcal{U};\mathbb{C})$ satisfy $\sup_{n\in\mathbb{N}} \|\phi_n\|_{\mathcal{H}^1} \le 1$. Fix $R > 0$. Since the Rellich-Kondrachov compactness theorem ensures the compact embedding of $H^1(B_R^+;\mathbb{C})$ into $L^2(B_R^+;\mathbb{C})$, there is a subsequence, still denoted by $\{\phi_n\}_{n\in\mathbb{N}}$, and a measurable function $\phi_R \in L^2(B_R^+;\mathbb{C})$, such that $\phi_n(x) \to \phi_R(x)$ for a.e. $x \in B_R^+$ and $\lim_{n\to\infty} \int_{B_R^+} |\phi_n - \phi_R|^2 \mathrm{d}x = 0$.

Let $\{R_m\}_m \subset (0,\infty)$ satisfy $R_m \to \infty$ as $m \to \infty$. Then, the above results hold for each R_m in place of R. We apply the standard diagonal argument to find a subsequence, still denoted by $\{\phi_n\}_{n\in\mathbb{N}}$, and a measurable function $\phi : \mathcal{U} \to \mathbb{C}$ such that $\phi_n \to \phi$ a.e. in $\mathcal{U}$ as $n \to \infty$, and

$$\lim_{n\to\infty} \int_{B_R^+} |\phi_n - \phi|^2 \mathrm{d}x = 0, \quad \forall R > 0. \tag{3.2}$$

Applying Fatou's lemma, we find $\int_{\mathcal{U}} \alpha\phi^2 \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathcal{U}} \alpha\phi_n^2 \mathrm{d}x \le 1$. It follows from (3.2) that

$$\limsup_{n\to\infty} \int_{\mathcal{U}} |\phi_n - \phi|^2 \mathrm{d}x \le \limsup_{n\to\infty} \int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x, \quad \forall R > 0.$$

Note that

$$\int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x \le \frac{2}{\inf_{\mathcal{U}\setminus B_R^+} \alpha} \int_{\mathcal{U}\setminus B_R^+} \alpha \left(\phi_n^2 + \phi^2\right) \mathrm{d}x \le \frac{2}{\inf_{\mathcal{U}\setminus B_R^+} \alpha},$$

which together with the fact $\alpha(x) \to \infty$ as $|x| \to \infty$ yields $\limsup_{n\to\infty} \int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x = 0$, and hence, $\lim_{n\to\infty} \int_{\mathcal{U}} |\phi_n - \phi|^2 \mathrm{d}x = 0$. This completes the proof. □

3.2 Some estimates

We recall from (2.8) the definition of e_β and define for $N \ge 1$,

$$\begin{aligned} e_{\beta,N} &:= e_\beta - \frac{N-1}{N}(\nabla \cdot p + \beta\Delta U) \\ &= \left(\frac{1}{N} - \frac{1}{2}\right)\beta\Delta U - \frac{\beta^2}{2}|\nabla U|^2 - \beta p \cdot \nabla U + \frac{1}{2}\sum_{i=1}^{d}(q_i^2 - q_i') - p \cdot q + \frac{\nabla \cdot p}{N}. \end{aligned} \tag{3.3}$$

Obviously, $e_{\beta,1} = e_\beta$ for all $\beta > 0$. The main reason for introducing $e_{\beta,N}$ is that they arise naturally in deriving a priori estimates for both sesquilinear forms and partial differential equations related to $\mathcal{L}_\beta$ or its adjoint (see Lemma 3.3 and Lemma 4.1).

Lemma 3.2 *Assume* **(H1)-(H3)**. *Then, the following hold.*

(1) *There exists $C > 0$ such that $|\nabla U|^2 + |p|^2 \le C\alpha$ in $\mathcal{U}$, where α is defined in* (3.1).
(2) *For each $\beta > 0$, there is $C(\beta) > 0$ such that $|e_{\beta,N}| \le C(\beta)\alpha$ in $\mathcal{U}$ for all $N \ge 1$.*
(3) *There are positive constants β_0, M and C_* such that $e_{\beta_0,N} + M \ge C_*\alpha$ in $\mathcal{U}$ for all $N \ge 1$.*

Since the proof of this lemma is long and relatively independent, we postpone it to Appendix A.1 for the sake of readability.

Remark 3.1 Note that $e_{\beta,1} = e_\beta$ is the zeroth-order term of the operator $\mathcal{L}_\beta$ (see (2.7)) that has blow-up singularities at Γ as mentioned earlier. Lemma 3.2 (3) says in particular that e_β is well-controlled by the weight function α, laying the foundation for our analysis.

In what follows, the positive constants β_0, M and C_* are fixed such that the conclusion in Lemma 3.2 (3) holds.

3.3 Spectrum and semigroup

We investigate the spectral theory of $-\mathcal{L}_{\beta_0}$ and the semigroup generated by $\mathcal{L}_{\beta_0}$. Corresponding results are stated in Theorem 3.1 and Theorem 3.2.

Denote by $\mathcal{E}_{\beta_0} : \mathcal{H}^1(\mathcal{U};\mathbb{C}) \times \mathcal{H}^1(\mathcal{U};\mathbb{C}) \to \mathbb{C}$ the sesquilinear form associated with $-\mathcal{L}_{\beta_0}$, namely,

$$\mathcal{E}_{\beta_0}(\phi,\psi) = \frac{1}{2}\int_{\mathcal{U}} \nabla\phi\cdot\nabla\overline{\psi}\mathrm{d}x + \int_{\mathcal{U}} (p+\beta_0\nabla U)\cdot\nabla\phi\overline{\psi}\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\phi\overline{\psi}\mathrm{d}x, \quad \forall\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}).$$

The following lemma addresses the boundedness and “coercivity" of $\mathcal{E}_{\beta_0}$, playing crucial roles in analyzing the spectrum of $-\mathcal{L}_{\beta_0}$.

Lemma 3.3 *Assume* **(H1)**-**(H3)**.

(1) *There exists $C > 0$ such that $\left|\mathcal{E}_{\beta_0}(\phi,\psi)\right| \le C\|\phi\|_{\mathcal{H}^1}\|\psi\|_{\mathcal{H}^1}$ for all $\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$.*
(2) *For each $\phi = \phi_1 + i\phi_2 \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$, we have*

$$\mathcal{E}_{\beta_0}(\phi,\phi) = \frac{1}{2}\int_{\mathcal{U}} |\nabla\phi|^2\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0,2}|\phi|^2\mathrm{d}x + i\int_{\mathcal{U}} (p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1)\mathrm{d}x,$$

where $e_{\beta_0,2}$ is defined in (3.3). *In particular,*

$$\Re\mathcal{E}_{\beta_0}(\phi,\phi) + M\|\phi\|^2_{L^2} \ge \min\left\{\frac{1}{2}, C_*\right\}\|\phi\|^2_{\mathcal{H}^1}, \quad \forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}).$$

Proof (1) Let $\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$. Applying Hölder's inequality, we derive

$$\begin{aligned}|\mathcal{E}_{\beta_0}(\phi,\psi)| \le{}& \frac{1}{2}\left(\int_{\mathcal{U}} |\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}} |\nabla\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &+ \left(\int_{\mathcal{U}} |\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}} |p+\beta_0\nabla U|^2|\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &+ \left(\int_{\mathcal{U}} |e_{\beta_0}||\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}} |e_{\beta_0}||\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

By Lemma 3.2 (1), there is $C > 0$ such that $\int_{\mathcal{U}} |p + \beta_0 \nabla U|^2 |\psi|^2 \mathrm{d}x \leq C(1 + \beta_0^2) \int_{\mathcal{U}} \alpha |\psi|^2 \mathrm{d}x$. The conclusion then follows readily from Lemma 3.2 (2) and the definition of the norm $\|\cdot\|_{\mathcal{H}^1}$.

(2) Let $\{\eta_n\}_{n\geq 1}$ be a sequence of smooth functions on $\mathcal{U}$ taking values in $[0, 1]$ and satisfying

$$\eta_n(x) = \begin{cases} 1, & x \in \left(\mathcal{U} \setminus \Gamma_{\frac{2}{n}}\right) \cap B^+_{\frac{n}{2}}, \\ 0, & x \in \Gamma_{\frac{1}{n}} \bigcup \left(\mathcal{U} \setminus B^+_n\right), \end{cases} \quad \text{and} \quad |\nabla \eta_n(x)| \leq \begin{cases} 2n, & x \in \Gamma_{\frac{2}{n}} \setminus \Gamma_{\frac{1}{n}}, \\ 4, & x \in \left(\mathcal{U} \setminus \Gamma_{\frac{2}{n}}\right) \cap \left(B^+_n \setminus B^+_{\frac{n}{2}}\right). \end{cases}$$

Obviously, η_n has compact support and $\lim_{n\to\infty} \eta_n = 1$ locally uniform in $\mathcal{U}$.

Fix $\phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{R})$. We find from standard derivation that

$$\begin{aligned} \mathcal{E}_{\beta_0}(\phi, \eta_n^2 \phi) &= \frac{1}{2} \int_{\mathcal{U}} \eta_n^2 |\nabla \phi|^2 \mathrm{d}x + \int_{\mathcal{U}} \eta_n \overline{\phi} \nabla \phi \cdot \nabla \eta_n \mathrm{d}x + \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \nabla \phi (\eta_n^2 \overline{\phi}) \mathrm{d}x \\ &\quad + \int_{\mathcal{U}} e_{\beta_0} \eta_n^2 |\phi|^2 \mathrm{d}x =: I_1(n) + I_2(n) + I_3(n) + I_4(n). \end{aligned} \tag{3.4}$$

We find $\lim_{n\to\infty} I_1(n) = \frac{1}{2} \int_{\mathcal{U}} |\nabla \phi|^2 \mathrm{d}x$ from $\int_{\mathcal{U}} |\nabla \phi|^2 \mathrm{d}x < \infty$ and the dominated convergence theorem. Clearly, $|I_2(n)| \leq \left(\int_{\mathcal{U}} \eta_n^2 |\nabla \phi|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\nabla \eta_n|^2 |\phi|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. From the construction of η_n, we see

$$|\nabla \eta_n|^2 |\phi|^2 \leq \begin{cases} 4n^2 |\phi|^2 & \text{in} \quad \Gamma_{\frac{2}{n}} \setminus \Gamma_{\frac{1}{n}}, \\ 16 |\phi|^2 & \text{in} \quad \left(\mathcal{U} \setminus \Gamma_{\frac{2}{n}}\right) \cap \left(B^+_n \setminus B^+_{\frac{n}{2}}\right), \\ 0 & \text{otherwise}. \end{cases}$$

Since $n^2 \leq \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\}$ in $\Gamma_{\frac{2}{n}} \setminus \Gamma_{\frac{1}{n}}$ for $n \geq 1$, the definition of α yields the existence of $C_1 > 0$ such that $|\nabla \eta_n|^2 |\phi|^2 \leq C_1 \alpha |\phi|^2$ in $\mathcal{U}$ for all $n \gg 1$. Since $\lim_{n\to\infty} |\nabla \eta_n| = 0$ locally uniform in $\mathcal{U}$, we apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int_{\mathcal{U}} |\nabla \eta_n|^2 |\phi|^2 \mathrm{d}x = 0, \tag{3.5}$$

which leads to $\lim_{n\to\infty} I_2(n) = 0$.

Denote $\phi = \phi_1 + i\phi_2$. Clearly, $(\partial_j \phi)\overline{\phi} = \frac{1}{2} \partial_j |\phi|^2 + i(\phi_1 \partial_j \phi_2 - \phi_2 \partial_j \phi_1)$ for each $j \in \{1, \ldots, d\}$. This together with integration by parts yield

$$\begin{aligned} I_3(n) &= \frac{1}{2} \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \eta_n^2 \nabla |\phi|^2 \mathrm{d}x + i \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \eta_n^2 (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \mathrm{d}x \\ &= -\frac{1}{2} \int_{\mathcal{U}} (\nabla \cdot p + \beta_0 \Delta U) \eta_n^2 |\phi|^2 \mathrm{d}x - \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \nabla \eta_n (\eta_n |\phi|^2) \mathrm{d}x \\ &\quad + i \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \eta_n^2 (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \mathrm{d}x. \end{aligned}$$

It follows that

$$\begin{aligned} I_3(n) + I_4(n) &= -\int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \nabla \eta_n (\eta_n |\phi|^2) \mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0,2} \eta_n^2 |\phi|^2 \mathrm{d}x \\ &\quad + i \int_{\mathcal{U}} (p + \beta_0 \nabla U) \cdot \eta_n^2 (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \mathrm{d}x =: J_1(n) + J_2(n) + J_3(n). \end{aligned} \tag{3.6}$$

We apply Hölder's inequality and the fact $\eta_n \in [0,1]$ to find

$$\left|\int_{\mathcal{U}} (p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x\right| \le \left(\int_{\mathcal{U}} |p+\beta_0\nabla U|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\nabla\eta_n|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Note that Lemma 3.2 (1) gives $\int_{\mathcal{U}} |p+\beta_0\nabla U|^2|\phi|^2\mathrm{d}x \le C_2\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x$ for some $C_2>0$, which together with (3.5) yields $\lim_{n\to\infty} J_1(n)=0$. It follows from Lemma 3.2 (2) that $|e_{\beta_0,2}|\eta_n^2|\phi|^2 \le C_3\alpha|\phi|^2$ for some $C_3>0$. Together with the fact $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{R})$ and the dominated convergence theorem this yields $\lim_{n\to\infty} J_2(n)=\int_{\mathcal{U}} e_{\beta_0,2}|\phi|^2\mathrm{d}x$. Since Young's inequality and the fact $\eta_n\in[0,1]$ give

$$\left|(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\right| \le \frac{1}{2}|\nabla\phi|^2+\frac{1}{2}|p+\beta_0\nabla U|^2|\phi|^2,$$

the dominated convergence theorem leads to $\lim_{n\to\infty} J_3(n) = i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x$.

Letting $n\to\infty$ in (3.6), we conclude that

$$\lim_{n\to\infty}[I_3(n)+I_4(n)] = \int_{\mathcal{U}} e_{\beta_0,2}\phi^2\mathrm{d}x + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x.$$

Passing to the limit $n\to\infty$ in (3.4), we derive the expected identity from the limits of $I_1(n)$, $I_2(n)$, $I_3(n)$ and $I_4(n)$ as $n\to\infty$. The inequality in (2) is an immediate consequence of Lemma 3.2 (3). □

Remark 3.2 It can be seen from the proof of Lemma 3.3 that $\lim_{n\to\infty}\eta_n\phi=\phi$ in $\mathcal{H}^1(\mathcal{U};\mathbb{R})$ for any $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{R})$. Therefore, $C_0^\infty(\mathcal{U})$ is dense in $\mathcal{H}^1(\mathcal{U};\mathbb{R})$.

For $f\in L^2(\mathcal{U};\mathbb{C})$, we consider the following problem:

$$(-\mathcal{L}_{\beta_0}+M)u=f \quad \text{in} \quad \mathcal{U}, \tag{3.7}$$

and look for solutions in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$.

Definition 3.1 *A function $u\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$ is called a* weak solution *of (3.7) if*

$$\mathcal{E}_{\beta_0}(u,\phi)+M\langle u,\phi\rangle_{L^2} = \langle f,\phi\rangle_{L^2}, \quad \forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}),$$

where $\langle\cdot,\cdot\rangle_{L^2}$ is the usual inner product on $L^2(\mathcal{U};\mathbb{C})$.

Lemma 3.4 *Assume* **(H1)-(H3)**. *Then, for any $f\in L^2(\mathcal{U};\mathbb{C})$, (3.7) admits a unique weak solution u_f in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$. Moreover, the following hold.*

1. *There is a constant $C>0$ such that $\|u_f\|_{\mathcal{H}^1}\le C\|f\|_{L^2}$ for all $f\in L^2(\mathcal{U};\mathbb{C})$.*
2. *$u_f \in H^2_{loc}(\mathcal{U};\mathbb{C})$ satisfies $(-\mathcal{L}_{\beta_0}+M)u_f = f$ a.e. in $\mathcal{U}$, and $\mathcal{E}_{\beta_0}(u_f,\phi) = \langle-\mathcal{L}_{\beta_0}u_f,\phi\rangle_{L^2}$ for all $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$.*
3. *If $f\in L^2(\mathcal{U};\mathbb{C})$ satisfies $f\ge0$ a.e. in $\mathcal{U}$, then $u_f\ge0$ a.e. in $\mathcal{U}$. If in addition $f>0$ on a set of positive Lebesgue measure, then $u_f>0$ a.e. in $\mathcal{U}$.*

Proof Fix $f\in L^2(\mathcal{U};\mathbb{C})$. Hölder's inequality gives

$$\left|\langle f,\phi\rangle_{L^2}\right| \le \left(\int_{\mathcal{U}}\frac{1}{\alpha}|f|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}} \le \frac{1}{(\inf_{\mathcal{U}}\alpha)^{\frac{1}{2}}}\|f\|_{L^2}\|\phi\|_{\mathcal{H}^1}, \quad \forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}). \tag{3.8}$$

Hence, $\phi\mapsto\langle f,\phi\rangle_{L^2}:\mathcal{H}^1(\mathcal{U};\mathbb{C})\to\mathbb{C}$ is a continuous linear functional.

By Lemma 3.3 and the fact $\|\phi\|_{L^2} \leq (\inf_{\mathcal{U}} \alpha)^{-\frac{1}{2}} \|\phi\|_{\mathcal{H}^1}$ for $\phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C})$ one has

$$|\mathcal{E}_{\beta_0}(\phi, \psi)| + M|\langle \phi, \psi \rangle_{L^2}| \leq C_1 \|\phi\|_{\mathcal{H}^1} \|\psi\|_{\mathcal{H}^1}, \quad \forall \phi, \psi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C})$$

for some $C_1 > 0$, and

$$\Re \mathcal{E}_{\beta_0}(\phi, \phi) + M\|\phi\|_{L^2}^2 \geq \min\left\{\frac{1}{2}, C_*\right\} \|\phi\|_{\mathcal{H}^1}^2, \quad \forall \phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C}). \tag{3.9}$$

We apply the Lax-Milgram theorem (see e.g. [32]) to find a unique $u_f \in \mathcal{H}^1(\mathcal{U}; \mathbb{C})$ such that

$$\mathcal{E}_{\beta_0}(u_f, \phi) + M\langle u_f, \phi \rangle_{L^2} = \langle f, \phi \rangle_{L^2}, \quad \forall \phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C}). \tag{3.10}$$

This shows that u_f is the unique weak solution of (3.7).

(1) Setting $\phi = u_f$ in (3.10), we derive from (3.8) and (3.9) that

$$\min\left\{\frac{1}{2}, C_*\right\} \|u_f\|_{\mathcal{H}^1}^2 \leq \Re \mathcal{E}_{\beta_0}(u_f, u_f) + M\|u_f\|_{L^2}^2 \leq \frac{1}{(\inf_{\mathcal{U}} \alpha)^{\frac{1}{2}}} \|f\|_{L^2} \|u_f\|_{\mathcal{H}^1}.$$

(2) The classical regularity theory of elliptic equations ensures $u_f \in H^2_{loc}(\mathcal{U}; \mathbb{C})$. Hence, u_f is a strong solution and obeys $(-\mathcal{L}_{\beta_0} + M)u_f = f$ a.e. in $\mathcal{U}$. Multiplying this equation by $\phi \in C_0^\infty(\mathcal{U}; \mathbb{C})$ and integrating by parts result in

$$\mathcal{E}_{\beta_0}(u_f, \phi) = \langle -\mathcal{L}_{\beta_0} u_f, \phi \rangle_{L^2}. \tag{3.11}$$

Note that $C_0^\infty(\mathcal{U})$ is dense in $\mathcal{H}^1(\mathcal{U}; \mathbb{R})$ (see Remark 3.2) and both sides of (3.11) are still well-defined even if ϕ merely belongs to $\mathcal{H}^1(\mathcal{U}; \mathbb{C})$. As a result, standard approximation arguments yield that (3.11) holds for any $\phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C})$.

(3) Suppose $f \geq 0$ a.e. in $\mathcal{U}$. In this case, u_f must be real-valued. It is easy to verify that the negative part $u_f^- := -\min\{u_f, 0\} \in \mathcal{H}^1(\mathcal{U}; \mathbb{C})$. Thanks to (2), we obtain

$$\mathcal{E}_{\beta_0}(u_f, u_f^-) + M\langle u_f, u_f^- \rangle_{L^2} = \langle f, u_f^- \rangle_{L^2} \geq 0.$$

It follows from Lemma 3.3 (2), $\mathcal{E}_{\beta_0}(u_f, u_f^-) = -\mathcal{E}_{\beta_0}(u_f^-, u_f^-)$ and $\langle u_f, u_f^- \rangle_{L^2} = -\langle u_f^-, u_f^- \rangle_{L^2}$ that

$$\min\left\{\frac{1}{2}, C_*\right\} \|u_f^-\|_{\mathcal{H}^1}^2 \leq \mathcal{E}_{\beta_0}(u_f^-, u_f^-) + M\|u_f^-\|_{L^2}^2 \leq 0.$$

This implies $u_f^- = 0$, and hence that $u_f \geq 0$. If in addition $f > 0$ on a set of positive Lebesgue measure, then $u_f > 0$ at least on some set of positive Lebesgue measure. Noting that $(-\mathcal{L}_{\beta_0} + M)u_f \geq 0$, we apply the weak Harnack's inequality of weak solutions of elliptic equations (see e.g. [32, Theorem 8.18]) to find $\|u_f\|_{L^p(K)} \leq C(K, K') \inf_{K'} u_f$ for some $C(K, K') > 0$ where $p \geq 1$, and K and K' are any compact sets satisfying $K' \subset\subset K^\circ \subset\subset \mathcal{U}$. Since $u_f > 0$ on a set of positive Lebesgue measure, and K', K are arbitrary, we conclude that $u_f > 0$ a.e. in $\mathcal{U}$.

This completes the proof. □

By Lemma 3.4 and Lemma 3.1, the operator

$$(-\mathcal{L}_{\beta_0} + M)^{-1} : L^2(\mathcal{U}; \mathbb{C}) \to L^2(\mathcal{U}; \mathbb{C}), \quad f \mapsto u_f$$

is linear, positive and compact. In light of Lemma 3.4, we define the domain of $\mathcal{L}_{\beta_0}$ as follows:

$$\mathcal{D} := (-\mathcal{L}_{\beta_0} + M)^{-1} L^2(\mathcal{U}; \mathbb{C}) = \left\{\phi \in \mathcal{H}^1(\mathcal{U}; \mathbb{C}) : \mathcal{L}_{\beta_0}\phi \in L^2(\mathcal{U}; \mathbb{C})\right\}.$$

The next result collects basic spectral properties of $-\mathcal{L}_{\beta_0}$.

Theorem 3.1 *Assume* **(H1)-(H3)**. *Then, the following hold.*

(1) *The operator* $-\mathcal{L}_{\beta_0}$ *has a discrete spectrum and is contained in* $\{\lambda \in \mathbb{C} : \Re\lambda > -M\}$.
(2) *The number* $\lambda_1 := \inf\left\{\Re\lambda : \lambda \in \sigma(-\mathcal{L}_{\beta_0})\right\}$ *is a simple eigenvalue of* $-\mathcal{L}_{\beta_0}$, *and is dominating, in the sense that* $\inf\left\{\Re\lambda : \lambda \in \sigma(-\mathcal{L}_{\beta_0}) \setminus \{\lambda_1\}\right\} > \lambda_1$.
(3) *The eigenspace of* λ_1 *is spanned over* $\mathbb{C}$ *by* $\tilde{v}_1$ *for some* $\tilde{v}_1 \in \mathcal{D}$ *a.e. positive in* $\mathcal{U}$.

Proof Since $(-\mathcal{L}_{\beta_0} + M)^{-1}$ is a compact operator on $L^2(\mathcal{U})$, we apply the Fredholm alternative (see e.g. [71]) to find that

(i) the spectrum of $(-\mathcal{L}_{\beta_0} + M)^{-1}$ except 0 consists of at most countable eigenvalues with each having finite multiplicity and being a finite pole of the resolvent operator of $(-\mathcal{L}_{\beta_0} + M)^{-1}$;
(ii) 0 is the only possible accumulation point.

Denote $L^2_+(\mathcal{U}) := \{u \in L^2(\mathcal{U}) : u \geq 0 \text{ a.e.}\}$. Then, $L^2(\mathcal{U})$ becomes a Hilbert lattice with the order induced by the positive cone $L^2_+(\mathcal{U})$. Thanks to [2, Proposition 10.15] and Lemma 3.4 (3), the only possible closed lattice ideals, that are invariant under the positive operator $(-\mathcal{L}_{\beta_0} + M)^{-1}$, are $L^2(\mathcal{U})$ and $\{0\}$. Hence, $(-\mathcal{L}_{\beta_0} + M)^{-1}$ is irreducible (see e.g. [2, Definition 10.26]). Then, we can apply [23, Theorem 3] to find the spectral radius $r_1 > 0$. Since [53, Theorem 2.1] ensures $r_1 \in \sigma((-\mathcal{L}_{\beta_0} + M)^{-1})$, it follows from (i)-(ii) that r_1 is an eigenvalue of $(-\mathcal{L}_{\beta_0} + M)^{-1}$ and also a finite pole of the corresponding resolvent operator.

Thanks to Lemma 3.4 (3), we see that $\langle(-\mathcal{L}_{\beta_0} + M)^{-1} f, g\rangle_{L^2} \neq 0$ for all $f, g \in L^2_+(\mathcal{U}) \setminus \{0\}$. That is, $(-\mathcal{L}_{\beta_0} + M)^{-1}$ is nonsupporting (in the language of I. Sawashima [53, 62]). As a result, we are able to apply the results in [62] (also see [53, Theeorem 2.3]) to conclude

- r_1 is a simple eigenvalue of $(-\mathcal{L}_{\beta_0} + M)^{-1}$;
- the eigenspace of r_1 is spanned over $\mathbb{C}$ by $\tilde{v}_1$ which is quasi-interior in $L^2_+(\mathcal{U})$;
- r_1 is dominating in the sense that $\sup\left\{|\lambda| : \lambda \in \sigma((-\mathcal{L}_{\beta_0} + M)^{-1}) \setminus \{r_1\}\right\} < r_1$.

Note that a function $f \in L^2_+(\mathcal{U})$ is called quasi-interior if and only if $\langle f, g\rangle_{L^2} \neq 0$ for any $g \in L^2_+(\mathcal{U}) \setminus \{0\}$. Then, it is easy to see that $\tilde{v}_1$ is a.e. positive in $\mathcal{U}$.

By the spectral mapping theorem (see e.g. [25, Theorem IV.1.13]), there holds

$$\sigma(-\mathcal{L}_{\beta_0}) = \left\{\frac{1}{\lambda} - M : \lambda \in \sigma((-\mathcal{L}_{\beta_0} + M)^{-1}) \setminus \{0\}\right\}. \tag{3.12}$$

Hence, the spectrum of $-\mathcal{L}_{\beta_0}$ is discrete. To see $\sigma(-\mathcal{L}_{\beta_0}) \subset \{\lambda \in \mathbb{C} : \Re\lambda > -M\}$, we first claim the existence of $\theta \in (0, \frac{\pi}{2})$ such that

$$S := \left\{\langle(-\mathcal{L}_{\beta_0} + M)u, u\rangle_{L^2} : u \in \mathcal{D},\ \|u\|_{L^2} = 1\right\} \subset \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \theta\}. \tag{3.13}$$

Suppose for the moment that (3.13) is true. Since $\sigma(-\mathcal{L}_{\beta_0} + M)$ consists of eigenvalues, applying [58, Theorem 1.3.9] yields $\sigma(-\mathcal{L}_{\beta_0} + M) \subset \overline{S}$. Noting that $\overline{\{\lambda \in \mathbb{C} : |\arg\lambda| \leq \theta\}} \cap \{\lambda : \Re\lambda = 0\} = \{0\}$ and $0 \in \rho(-\mathcal{L}_{\beta_0} + M)$, we derive from (3.13) that $\sigma(-\mathcal{L}_{\beta_0} + M) \subset \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$. As a result, $\sigma(-\mathcal{L}_{\beta_0}) \subset \{\lambda \in \mathbb{C} : \Re\lambda > -M\}$, and (1) holds.

Now, we prove (3.13) for some $\theta \in (0, \frac{\pi}{2})$. Fix $u \in \mathcal{D}$. Clearly, Lemma 3.4 (2) gives

$$\langle-(\mathcal{L}_{\beta_0} - M)u, u\rangle_{L^2} = \mathcal{E}_{\beta_0}(u, u) + M\|u\|^2_{L^2}.$$

It follows from Lemma 3.3 (2) that

$$\Re\langle-(\mathcal{L}_{\beta_0} - M)u, u\rangle_{L^2} = \Re\mathcal{E}_{\beta_0}(u, u) + M\|u\|^2_{L^2} \geq \min\left\{\frac{1}{2}, C_*\right\}\|u\|^2_{\mathcal{H}^1}.$$

Applying Young's inequality, we derive from Lemma 3.3 (2) and Lemma 3.2 that

$$\begin{aligned}\left|\Im\langle-(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}\right| = |\Im\mathcal{E}_{\beta_0}(u,u)| &\le \frac{1}{2}\int_{\mathcal{U}}|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|u|^2\mathrm{d}x\\ &\le \frac{1}{2}\int_{\mathcal{U}}|\nabla u|^2\mathrm{d}x + \frac{C_2}{2}\int_{\mathcal{U}}\alpha|u|^2\mathrm{d}x,\end{aligned}$$

where $C_2>0$ is independent of $u\in\mathcal{D}$. Therefore,

$$0\le\frac{\left|\Im\langle-(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}\right|}{\Re\langle-(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}}\le\frac{\frac{1}{2}+\frac{C_2}{2}}{\min\left\{\frac{1}{2},C_*\right\}},$$

yiedling (3.13).

Finally, we prove (2)-(3). Let $\hat{\lambda}_1:=\frac{1}{r_1}-M$. Then, (3.12) implies that $\hat{\lambda}_1$ is a simple eigenvalue of $-\mathcal{L}_{\beta_0}$ with eigenspace spanned over $\mathbb{C}$ by $\tilde{v}_1$. Suppose $-\mathcal{L}_{\beta_0}$ has an eigenvalue λ with an eigenfunction $\psi\in\mathcal{D}\setminus\{0\}$. We claim that $\Re\lambda>\hat{\lambda}_1$.

If λ is real, we see from (3.12) that $\lambda=\frac{1}{r}e^{i\theta}-M$ with some $r<r_1$ and $\theta=0$ or π. Since $\sigma(-\mathcal{L}_{\beta_0})\subset\{\lambda\in\mathbb{C}:\Re\lambda>-M\}$, we find $\theta=0$ and $\lambda=\frac{1}{r}-M>\hat{\lambda}_1$.

Now, suppose λ is complex, namely, $\Im\lambda\neq0$. Following arguments in [24], we consider the product domain $\hat{\mathcal{U}}:=\mathcal{U}\times\mathcal{U}=\{(x,y):x,\ y\in\mathcal{U}\}$. Denote by $-\mathcal{L}_x$ the action of $-\mathcal{L}_{\beta_0}$ in the x variable and by $-\mathcal{L}_y$ the action of $-\mathcal{L}_{\beta_0}$ in the y variable. Set $\hat{\mathcal{L}}:=\mathcal{L}_x+\mathcal{L}_y$ and $\hat{v}_1(x,y):=\tilde{v}_1(x)\tilde{v}_1(y)$ for $(x,y)\in\hat{\mathcal{U}}$. Straightforward calculations yield $-\hat{\mathcal{L}}\hat{v}_1=2\hat{\lambda}_1\hat{v}_1$. It is not hard to see that similar results as in Lemma 3.3 hold for the sesquilinear form associated with $-\hat{\mathcal{L}}$ and hence, conclusions in Lemma 3.4 apply to $-\hat{\mathcal{L}}+2M$ in place of $-\mathcal{L}_{\beta_0}+M$. As a result,

- $2\hat{\lambda}_1$ is a simple eigenvalue of $-\hat{\mathcal{L}}$ with eigenspace spanned over $\mathbb{C}$ by $\hat{v}_1$,
- $\sigma(-\hat{\mathcal{L}})\subset\{\lambda\in\mathbb{C}:\Re\lambda>-2M\}$,
- $\frac{r_1}{2}:=(2\hat{\lambda}_1+2M)^{-1}$ is the dominating eigenvalue of $(-\hat{\mathcal{L}}+2M)^{-1}$.

Setting $\hat{\psi}:=\psi(x)\overline{\psi}(y)+\overline{\psi}(x)\psi(y)$, we find $\hat{\psi}$ is real-valued and satisfies $-\hat{\mathcal{L}}\hat{\psi}=2\Re\lambda\hat{\psi}$. Following similar arguments as in the case where λ is real, we find $\Re\lambda>\hat{\lambda}_1$.

This shows the claim. Hence, $\hat{\lambda}_1=\lambda_1$. Since $\sigma(-\mathcal{L}_{\beta_0})$ is discrete and contained in a cone (by (3.13)), it is easy to see that $\inf\left\{\Re\lambda:\lambda\in\sigma(-\mathcal{L}_{\beta_0})\setminus\{\lambda_1\}\right\}>\lambda_1$.

This proves (2)-(3) and finishes the proof. □

Remark 3.3 We point out that the positive cone $L^2_+(\mathcal{U})$ has empty interior so that the celebrated Kreĭn-Rutman theorem [42] for compact and strongly positive operators, often used to treat elliptic operators on bounded domains, does not apply here. Restricting $-\mathcal{L}_{\beta_0}$ to a smaller space does not help as $\mathcal{U}$ is unbounded.

The number λ_1 is often called *the principal eigenvalue* of $-\mathcal{L}_{\beta_0}$. So far, it is not clear whether λ_1 is positive. The positivity of λ_1 is shown later by means of the absorbing properties of the process X_t.

The following result concerns the semigroup generated by $\mathcal{L}_{\beta_0}$.

Theorem 3.2 *Assume* **(H1)**-**(H3)**. *Then,* $(\mathcal{L}_{\beta_0},\mathcal{D})$ *generates a* C_0*-semigroup* $(T_t)_{t\ge0}$ *on* $L^2(\mathcal{U};\mathbb{C})$. *Moreover,* $(T_t)_{t\ge0}$ *is positive (i.e.,* $T_tL^2_+(\mathcal{U})\subset L^2_+(\mathcal{U})$ *for all* $t\ge0$*), extends to an analytic semigroup and is immediately compact.*

Proof Note that it is equivalent to studying the operator $\mathcal{L}_{\beta_0}-M$ with domain $\mathcal{D}$. First, we show $\mathcal{L}_{\beta_0}-M$ is densely defined and closed. In fact, the density of $\mathcal{D}$ in $L^2(\mathcal{U};\mathbb{C})$ follows readily from the fact $C_0^\infty(\mathcal{U};\mathbb{C})\subset\mathcal{D}$. Since the resolvent set of $\mathcal{L}_{\beta_0}-M$ is non-empty thanks to Theorem 3.1, the closedness of $(\mathcal{L}_{\beta_0}-M,\mathcal{D})$ follows.

Next, we see from Theorem 3.1 that $(0,\infty)\subset\rho(\mathcal{L}_{\beta_0}-M)$. For fixed $\lambda>0$, we prove

$$\|(\lambda+M-\mathcal{L}_{\beta_0})^{-1}\|_{L^2\to L^2}\leq\frac{1}{\lambda},\quad\forall\lambda>0.$$

Let $f\in L^2(\mathcal{U};\mathbb{C})$ and $u\in\mathcal{D}$ be such that $(\lambda+M-\mathcal{L}_{\beta_0})u=f$. It follows from Lemma 3.4 (2) that $\mathcal{E}_{\beta_0}(u,\phi)+(\lambda+M)\langle u,\phi\rangle_{L^2}=\langle f,\phi\rangle_{L^2}$ for all $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$. As a result, Lemma 3.3 (2) ensures

$$\min\left\{\frac{1}{2},C_*\right\}\|u\|^2_{\mathcal{H}^1}+\lambda\|u\|^2_{L^2}\leq\Re\mathcal{E}_{\beta_0}(u,u)+(\lambda+M)\|u\|^2_{L^2}\leq\|f\|_{L^2}\|u\|_{L^2},$$

yielding the expected upper bound.

As a result, we apply the Hille-Yosida theorem (see e.g. [25, 58]) to find that $(\mathcal{L}_{\beta_0}-M,\mathcal{D})$ generates a C_0-semigroup of contractions $\{T_t\}_{t\geq0}$ in $L^2(\mathcal{U};\mathbb{C})$. By Lemma 3.4 (3), this semigroup must be positive. Thanks to the compactness of $(\mathcal{L}_{\beta_0}-M)^{-1}$ by Lemma 3.1, it follows from [25, Theorem II.4.29] that $(T_t)_{t\geq0}$ is immediately compact.

It remains to show that $(T_t)_{t\geq0}$ extends to an analytic semigroup. Let S be defined as in (3.13) in the proof of Theorem 3.1 and $\theta\in(0,\frac{\pi}{2})$ be such that $S\subset\{\lambda\in\mathbb{C}:|\arg\lambda|\leq\theta\}$. Then, $\sigma(-\mathcal{L}_{\beta_0}+M)\subset\{\lambda\in\mathbb{C}:|\arg\lambda|\leq\theta\}\setminus\{0\}$. Fixing $\theta_*\in(\theta,\frac{\pi}{2})$ and setting $\Sigma_{\theta_*}:=\{\lambda\in\mathbb{C}:|\arg\lambda|>\theta_*\}\subset\mathbb{C}\setminus\overline{S}$, we find $\Sigma_{\theta_*}\cup\{0\}\subset\rho(-\mathcal{L}_{\beta_0}+M)$ and there is $C_1>0$ such that $d(\lambda,\overline{S})\geq C_1|\lambda|$ for all $\lambda\in\Sigma_{\theta_*}\setminus\{0\}$. An application of [58, Theorem 1.3.9] yields

$$\|(\lambda-(-\mathcal{L}_{\beta_0}+M))^{-1}\|_{L^2\to L^2}\leq\frac{1}{d(\lambda,\overline{S})}\leq\frac{1}{C_1|\lambda|},\quad\forall\lambda\in\Sigma_{\theta_*}\setminus\{0\}.$$

As a result, an application of [58, Theorem 2.5.2 (c)] (with $A=\mathcal{L}_{\beta_0}-M$ and $\Sigma=\{\lambda\in\mathbb{C}:|\arg\lambda|<\pi-\theta_*\}$) enables us to extend $(T_t)_{t\geq0}$ to an analytic semigroup. This completes the proof. □

3.4 Adjoint operator and semigroup

Let $(\mathcal{L}^*_{\beta_0},\mathcal{D}^*)$ be the adjoint operator of $(\mathcal{L}_{\beta_0},\mathcal{D})$ in $L^2(\mathcal{U};\mathbb{C})$. Then, $\mathcal{D}^*$ is given by

$$\mathcal{D}^*:=\left\{w\in L^2(\mathcal{U};\mathbb{C}):\exists f\in L^2(\mathcal{U};\mathbb{C})\text{ s.t. }\langle w,\mathcal{L}_{\beta_0}\phi\rangle_{L^2}=\langle f,\phi\rangle_{L^2},\ \forall\phi\in\mathcal{D}\right\}.$$

For each $w\in\mathcal{D}^*$, $\mathcal{L}^*_{\beta_0}w$ is the unique element in $L^2(\mathcal{U};\mathbb{C})$ such that $\langle w,\mathcal{L}_{\beta_0}\phi\rangle_{L^2}=\langle\mathcal{L}^*_{\beta_0}w,\phi\rangle_{L^2}$ for all $\phi\in\mathcal{D}$. Integration by parts yields

$$\mathcal{L}^*_{\beta_0}w=\frac{1}{2}\Delta w+\nabla\cdot((p+\beta_0\nabla U)w)-e_{\beta_0}w,\quad w\in C_0^\infty(\mathcal{U};\mathbb{C}).\tag{3.14}$$

The following lemma summarizes some properties of the operator $-\mathcal{L}^*_{\beta_0}$.

Lemma 3.5 *Assume* **(H1)-(H3)**. *Then, the following hold.*

(1) $\sigma(-\mathcal{L}^*_{\beta_0})=\sigma(-\mathcal{L}_{\beta_0})\subset\{\lambda\in\mathbb{C}:\Re\lambda>-M\}$.
(2) $\mathcal{D}^*=\left\{w\in\mathcal{H}^1(\mathcal{U};\mathbb{C}):\mathcal{L}^*_{\beta_0}w\in L^2(\mathcal{U};\mathbb{C})\right\}$.
(3) *For each* $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$ *and* $w\in\mathcal{D}^*$ *one has* $\langle\phi,-\mathcal{L}^*_{\beta_0}w\rangle_{L^2}=\mathcal{E}_{\beta_0}(\phi,w)$.

(4) λ_1 *is a simple and dominating eigenvalue of* $-\mathcal{L}^*_{\beta_0}$ *with the associated eigenspace spanned over* $\mathbb{C}$ *by* $\tilde{v}^*_1$ *for some* $\tilde{v}^*_1 \in \mathcal{D}^*$ *a.e. positive in* $\mathcal{U}$.

Proof (1) Note that $\sigma(-\mathcal{L}^*_{\beta_0}) = \overline{\sigma(-\mathcal{L}_{\beta_0})}$. Since the spectrum of $-\mathcal{L}_{\beta_0}$ consists of eigenvalues due to Lemma 3.1 (1) and the coefficients of $-\mathcal{L}_{\beta_0}$ are real-valued, $\sigma(-\mathcal{L}_{\beta_0})$ is symmetric with respect to the real axis. Namely, $\Lambda \in \sigma(-\mathcal{L}_{\beta_0})$ if and only if $\overline{\Lambda} \in \sigma(-\mathcal{L}_{\beta_0})$. Hence, $\overline{\sigma(-\mathcal{L}_{\beta_0})} = \sigma(-\mathcal{L}_{\beta_0})$, which leads to $\sigma(-\mathcal{L}^*_{\beta_0}) = \sigma(-\mathcal{L}_{\beta_0})$.

(2) Since $-1 - M \in \rho(-\mathcal{L}^*_{\beta_0})$ by (1), we see that $\mathcal{D}^* = (-\mathcal{L}^*_{\beta_0} + 1 + M)^{-1} L^2(\mathcal{U}; \mathbb{C})$. Following similar arguments as in the proof of Lemma 3.4, we deduce

$$(-\mathcal{L}^*_{\beta_0} + M + 1)^{-1} L^2(\mathcal{U}; \mathbb{C}) = \left\{ w \in \mathcal{H}^1(\mathcal{U}; \mathbb{C}) : \mathcal{L}^*_{\beta_0} w \in L^2(\mathcal{U}; \mathbb{C}) \right\},$$

leading to the desired result.

(3) Note that $\langle \phi, -\mathcal{L}^*_{\beta_0} w \rangle_{L^2} = \langle -\mathcal{L}_{\beta_0}\phi, w \rangle_{L^2}$ for all $\phi \in \mathcal{D}$ and $w \in \mathcal{D}^*$. It follows from Lemma 3.4 (2) that $\langle \phi, -\mathcal{L}^*_{\beta_0} w \rangle_{L^2} = \mathcal{E}_{\beta_0}(\phi, w)$ for all $\phi \in \mathcal{D}$ and $w \in \mathcal{D}^*$. Since $C^\infty_0(\mathcal{U}; \mathbb{C}) \subset \mathcal{D}$ and is dense in $\mathcal{H}^1(\mathcal{U}; \mathbb{C})$ (see Remark 3.2), the conclusion follows from standard approximation arguments.

(4) This follows from (1) and arguments as in the proof of Theorem 3.1.

□

Denote by $(T^*_t)_{t \geq 0}$ the dual semigroup of $(T_t)_{t \geq 0}$. It is well-known (see e.g. [58, Corollary 1.10.6]) that $(T^*_t)_{t \geq 0}$ is a C_0-semigroup with infinitesimal generator $(\mathcal{L}^*_{\beta_0}, \mathcal{D}^*)$.

Theorem 3.3 *Assume* **(H1)**-**(H3)**. *Then,* $(T^*_t)_{t \geq 0}$ *is an analytic semigroup. Moreover, it is positive, i.e.,* $T^*_t L^2_+(\mathcal{U}) \subset L^2_+(\mathcal{U})$ *for all* $t \geq 0$, *and immediately compact.*

Proof Note that $\rho(\mathcal{L}^*_{\beta_0} - M) = \rho(\mathcal{L}_{\beta_0} - M)$. Thanks to [58, Theorem 2.5.2], the conclusion is a straightforward consequence of the analyticity of $(T_t)_{t \geq 0}$ and the fact $\|(\lambda + M - \mathcal{L}^*_{\beta_0})^{-1}\|_{L^2 \to L^2} = \|(\overline{\lambda} + M - \mathcal{L}_{\beta_0})^{-1}\|_{L^2 \to L^2}$ for each $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$. The positivity and immediate compactness follow from arguments as in the proof of Theorem 3.2. □

4 Stochastic Representation of Semigroups

In this section, we study the stochastic representation of the semigroup $(T^*_t)_{t \geq 0}$. Subsection 4.1 and Subsection 4.2 are respectively devoted to the stochastic representation and estimates of semigroups generated by $\mathcal{L}^*_{\beta_0}$ restricted to bounded domains with zero Dirichlet boundary condition. In Subsection 4.3, we establish the stochastic representation for $(T^*_t)_{t \geq 0}$.

4.1 Stochastic Representation in Bounded Domains

Let $\Omega \subset\subset \mathcal{U}$ be a connected subdomain with C^2 boundary. Denote by $\mathcal{L}^X$ the diffusion operator associated with X_t or (2.3), namely,

$$\mathcal{L}^X = \frac{1}{2}\Delta + (p - q) \cdot \nabla.$$

For each $N > 1$, let $\mathcal{L}^X_N|_\Omega$ be $\mathcal{L}^X$ considered as an operator in $L^N(\Omega; \mathbb{C})$ with domain $W^{2,N}(\Omega; \mathbb{C}) \cap W^{1,N}_0(\Omega; \mathbb{C})$. It is well-known (see e.g. [25, 32, 58]) that the spectrum of

$-\mathcal{L}_N^X|_\Omega$ is discrete and contained in $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ and $\mathcal{L}_N^X|_\Omega$ generates an analytic semigroup $(S_t^{(\Omega,N)})_{t\geq 0}$ of contractions on $L^N(\Omega;\mathbb{C})$ that satisfies $S_t^{(\Omega,N)} L_+^N(\Omega) \subset L_+^N(\Omega)$ for all $t \geq 0$. Moreover, the following stochastic representation holds: for each $f \in C(\overline{\Omega};\mathbb{C})$,

$$S_t^{(\Omega,N)} f(x) = \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<\tau_\Omega\}}\right], \quad \forall (x,t) \in \Omega \times [0,\infty), \tag{4.1}$$

where $\tau_\Omega := \inf\{t \geq 0 : X_t \notin \Omega\}$ is the first time that X_t exits Ω.

For $N > 1$, let $\mathcal{L}_{\beta_0}^{*,N}|_\Omega$ be $\mathcal{L}_{\beta_0}^*$ considered as an operator in $L^N(\Omega;\mathbb{C})$ with domain $W^{2,N}(\Omega;\mathbb{C}) \cap W_0^{1,N}(\Omega;\mathbb{C})$.

Proposition 4.1 *The following statements hold.*

(1) *The spectrum of $-\mathcal{L}_{\beta_0}^{*,N}|_\Omega$ is discrete and is contained in $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.*

(2) *$\mathcal{L}_{\beta_0}^{*,N}|_\Omega$ generates an analytic semigroup of contractions $(T_t^{(*,\Omega,N)})_{t\geq 0}$ on $L^N(\Omega;\mathbb{C})$ that is positive, namely, $T_t^{(*,\Omega,N)} L_+^N(\Omega) \subset L_+^N(\Omega)$ for all $t \geq 0$.*

(3) *For each $f \in L^N(\Omega;\mathbb{C})$ and $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$,*

$$T_t^{(*,\Omega,N)} \tilde{f} = e^{-\frac{Q}{2}-\beta_0 U} S_t^{(\Omega,N)} f, \quad \forall t \geq 0.$$

(4) *For each $f \in C(\overline{\Omega};\mathbb{C})$ and $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$,*

$$T_t^{(*,\Omega,N)} \tilde{f}(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<\tau_\Omega\}}\right], \quad \forall (x,t) \in \Omega \times [0,\infty).$$

(5) *For any $N_1, N_2 > 1$, $T_t^{(*,\Omega,N_1)}$ and $T_t^{(*,\Omega,N_2)}$ coincide on $L^{N_1}(\Omega;\mathbb{C}) \cap L^{N_2}(\Omega;\mathbb{C})$ for all $t \geq 0$.*

Proof For $f \in W^{2,N}(\Omega;\mathbb{C}) \cap W_0^{1,N}(\Omega;\mathbb{C})$, direct calculations give $\mathcal{L}_{\beta_0}^{*,N}|_\Omega \tilde{f} = e^{-\frac{Q}{2}-\beta_0 U} \mathcal{L}_N^X|_\Omega f$, where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$, the conclusions (1)-(4) follow immediately from the corresponding properties of $\mathcal{L}_N^X|_\Omega$ and $(S_t^{(\Omega,N)})_{t\geq 0}$.

In particular, for any $N_1, N_2 > 1$ we have $T_t^{(*,\Omega,N_1)} \tilde{f} = T_t^{(*,\Omega,N_2)} \tilde{f}$ for all $\tilde{f} \in C(\overline{\Omega};\mathbb{C})$. Statement (5) then follows from the density of $C(\overline{\Omega};\mathbb{C})$ in $L^N(\Omega;\mathbb{C})$ for any $N > 1$. $\square$

4.2 Estimates of Semigroups in Bounded Domains

We prove two useful lemmas concerning some estimates of the semigroup $(T_t^{(*,\Omega,N)})_{t\geq 0}$.

Lemma 4.1 *Let $N \geq 2$ and $\tilde{f} \in L^N(\Omega)$. Then, $\tilde{w} := T_\bullet^{(*,\Omega,N)} \tilde{f}$ satisfies the following inequalities:*

$$\begin{aligned}
&\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_1}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_1}^t\int_\Omega \alpha|\tilde{w}|^N\mathrm{d}x\mathrm{d}s\\
&\quad\leq \frac{1+e^{NM(t-t_1)}}{N}\int_\Omega |\tilde{w}(\cdot,t_1)|^N\mathrm{d}x, \quad \forall t > t_1 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_2}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_2}^t\int_\Omega \alpha|\tilde{w}|^N\mathrm{d}x\mathrm{d}s\\
&\quad\leq \frac{2}{N(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{w}|^N\mathrm{d}x\mathrm{d}s, \quad \forall t > t_2 > t_1 \geq 0.
\end{aligned}$$

Proof Fix $N \geq 2$ and $\tilde{f} \in L^N(\Omega)$. Then, $\tilde{w} := T_\bullet^{(*,\Omega,N)} \tilde{f}$ satisfies

$$\partial_t \tilde{w} = \mathcal{L}_{\beta_0}^{*,N}|_\Omega \tilde{w} \quad \text{in} \quad \Omega \times (0,\infty).$$

Recall $\mathcal{L}_{\beta_0}^*$ from (3.14) and $\mathcal{L}_{\beta_0}^{*,N}$ from Subsection 4.1. Multiplying the above equation by $|\tilde{w}|^{N-2}\tilde{w}$ and integrating by parts, we find, after straightforward calculations, for $t > 0$

$$\int_\Omega |\tilde{w}|^{N-2}\tilde{w}\partial_t \tilde{w} \mathrm{d}x = -\frac{N-1}{2}\int_\Omega |\tilde{w}|^{N-2}|\nabla \tilde{w}|^2 \mathrm{d}x - \int_\Omega e_{\beta_0,N}|\tilde{w}|^N \mathrm{d}x, \tag{4.2}$$

where we recall the definition of $e_{\beta_0,N}$ from (3.3).

Since $|\tilde{w}|^{N-2}\tilde{w}\partial_t\tilde{w} = \frac{1}{N}\partial_t|\tilde{w}|^N$, we integrate the above equality on $[t_1,t] \subset [0,\infty)$ to derive

$$\begin{aligned}&\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_1}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s\\ &\quad+\int_{t_1}^t\int_\Omega e_{\beta_0,N}|\tilde{w}|^N\mathrm{d}x\mathrm{d}s = \frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t_1)\mathrm{d}x.\end{aligned}$$

As Lemma 3.2 (3) gives $e_{\beta_0,N} + M \geq C_*\alpha$ for all $N \geq 2$, we find

$$\begin{aligned}&\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_1}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_1}^t\int_\Omega \alpha|\tilde{w}|^N\mathrm{d}x\mathrm{d}s\\ &\quad\leq M\int_{t_1}^t\int_\Omega |\tilde{w}|^N\mathrm{d}x\mathrm{d}s + \frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t_1)\mathrm{d}x, \quad \forall t > t_1 \geq 0.\end{aligned} \tag{4.3}$$

Setting $g(t) := \int_{t_1}^t\int_\Omega |\tilde{w}|^N\mathrm{d}x\mathrm{d}s$ for $t \geq t_1$, we arrive at $\frac{1}{N}g' \leq Mg + \frac{1}{N}\|\tilde{w}(\cdot,t_1)\|_{L^N}^N$ for all $t > t_1$. Gronwall's inequality gives $g(t) \leq \frac{e^{NM(t-t_1)}}{NM}\|\tilde{w}(\cdot,t_1)\|_{L^N}^N$ for all $t > t_1$. Inserting this into (4.3) yields the first inequality.

Now, we prove the second inequality. Fix $t_1, t_2 \in [0,\infty)$ with $t_1 < t_2$. Let $\eta \in C^\infty((0,\infty))$ be non-negative and non-decreasing such that $\eta = 0$ on $[0,t_1]$, $\eta = 1$ on $[t_2,\infty]$ and $\max_{[t_1,t_2]}\eta' \leq \frac{2}{t_2-t_1}$. Multiplying (4.2) by η and integrating by parts, we find for $t > t_2$,

$$\begin{aligned}&\frac{1}{N}\int_\Omega \eta(t)|\tilde{w}|^N(\cdot,t)\mathrm{d}x - \frac{1}{N}\int_0^t\int_\Omega \eta'|\tilde{w}|^N\mathrm{d}x\mathrm{d}s\\ &\quad= -\frac{N-1}{2}\int_0^t\int_\Omega \eta|\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega \eta e_{\beta_0,N}|\tilde{w}|^N\mathrm{d}x\mathrm{d}s.\end{aligned}$$

The definition of η then gives

$$\begin{aligned}&\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_2}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s + \int_{t_2}^t\int_\Omega e_{\beta_0,N}|\tilde{w}|^N\mathrm{d}x\mathrm{d}s\\ &\quad\leq \frac{1}{N}\int_{t_1}^{t_2}\int_\Omega \eta'|\tilde{w}|^N\mathrm{d}x\mathrm{d}s \leq \frac{2}{N(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{w}|^N\mathrm{d}x\mathrm{d}s, \quad \forall t > t_2.\end{aligned}$$

This completes the proof. □

Lemma 4.2 *For each $t > 0$, there exists $C = C(t)$, independent of the domain Ω, such that*

$$\|T_t^{(*,\Omega,2_*)}\tilde{f}\|_{L^2(\Omega)} \leq C\|\tilde{f}\|_{L^{2_*}(\Omega)}, \quad \forall \tilde{f} \in L^{2_*}(\Omega),$$

where $2_* := \frac{2(d+2)}{d+4} \in (1,2)$ is the dual exponent of $2 + \frac{4}{d}$.

Proof Take $N \in (1, 2]$. Then, $N' := \frac{N}{N-1} \geq 2$. Denote by $(T_t^{(\Omega,N')})_{t\geq 0}$ the semigroup on $L^{N'}(\Omega)$ that is dual to $(T_t^{(*,\Omega,N)})_{t\geq 0}$. Let $\mathcal{L}_{\beta_0}^{N'}|_\Omega$ be $\mathcal{L}_{\beta_0}$ considered as an operator in $L^{N'}(\Omega)$ with domain $W^{2,N'}(\Omega) \cap W_0^{1,N'}(\Omega)$. It is not hard to check that $\mathcal{L}_{\beta_0}^{N'}|_\Omega$, being $\mathcal{L}_{\beta_0}$ considered as an operator in $L^{N'}(\Omega)$ with domain $W^{2,N'}(\Omega;\mathbb{C}) \cap W_0^{1,N'}(\Omega;\mathbb{C})$, is the generator of $(T_t^{(\Omega,N')})_{t\geq 0}$.

Take $\tilde{g} \in L^{N'}(\Omega)$ and denote $\tilde{v} := T_\bullet^{(\Omega,N')}\tilde{g}$. Then, $\tilde{v}$ is the solution of

$$\partial_t \tilde{v} = \mathcal{L}_{\beta_0}^{N'}|_\Omega \tilde{v} \quad \text{in} \quad \Omega \times (0,\infty).$$

Multiplying this equation by $|\tilde{v}|^{N'-2}\tilde{v}$ and integrating by parts, we find, after straightforward calculations, for $t > 0$,

$$\int_\Omega |\tilde{v}|^{N'-2}\tilde{v}\partial_t\tilde{v}\mathrm{d}x = -\frac{N'-1}{2}\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x - \int_\Omega e^*_{\beta_0,N'}|\tilde{v}|^{N'}\mathrm{d}x,$$

where $e^*_{\beta_0,N'} := e_{\beta_0} - \frac{1}{N'}(\nabla\cdot p + \beta_0\Delta U)$. We can follow the proof of Lemma 3.2 (3) to show $e^*_{\beta_0,N} + M \geq C_*\alpha$ in $\mathcal{U}$ for all $N \geq 1$. Then, arguing as in the proof of Lemma 4.1 yields

$$\begin{aligned}\frac{1}{N'}\int_\Omega |\tilde{v}|^{N'}(\cdot,t)\mathrm{d}x + \frac{N'-1}{2}\int_0^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_0^t\int_\Omega \alpha|\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s \\ \leq \frac{1+e^{N'Mt}}{N'}\int_\Omega |\tilde{g}|^{N'}\mathrm{d}x, \quad \forall t > 0,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{1}{N'}\int_\Omega |\tilde{v}|^{N'}(\cdot,t)\mathrm{d}x + \frac{N'-1}{2}\int_{t_2}^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_2}^t\int_\Omega \alpha|\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s \\ \leq \frac{2}{N'(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s, \quad \forall t > t_2 > t_1 \geq 0.\end{aligned} \tag{4.5}$$

The Sobolev embedding theorem gives

$$\|\tilde{v}^{\frac{N'}{2}}\|_{L^{2\kappa}(\Omega\times[0,t])} \leq C_1\left(\sup_{s\in[0,t]}\|\tilde{v}^{\frac{N'}{2}}(\cdot,s)\|_{L^2(\Omega)} + \|\nabla\tilde{v}^{\frac{N'}{2}}\|_{L^2(\Omega\times[0,t])}\right),$$

where $\kappa = \frac{d+2}{d}$ and $C_1 > 0$ depends only on d. This together with (4.4) gives rise to

$$\begin{aligned}\left(\int_0^t\int_\Omega |\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{\kappa}} &\leq 2C_1^2\left(\sup_{s\in[0,t]}\int_\Omega |\tilde{v}(x,s)|^{N'}\mathrm{d}x + \frac{|N'|^2}{4}\int_0^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s\right) \\ &\leq C_2(1+e^{N'Mt})\int_\Omega |\tilde{g}|^{N'}\mathrm{d}x, \quad \forall t > 0,\end{aligned}$$

where $C_2 := 2C_1^2\left(1 + \frac{N'}{2(N'-1)}\right)$. We then deduce from (4.5) (with $\kappa N'$ instead of N') that

$$\begin{aligned}\frac{1}{\kappa N'}\int_\Omega |\tilde{v}|^{\kappa N'}(\cdot,t)\mathrm{d}x + \frac{\kappa N'-1}{2}\int_{t_2}^t\int_\Omega |\tilde{v}|^{\kappa N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_2}^t\int_\Omega \alpha|\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s \\ \leq \frac{2}{\kappa N'(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s \leq \frac{2}{\kappa N'(t_2-t_1)}C_2^\kappa(1+e^{N'Mt_2})^\kappa\|\tilde{g}\|^{\kappa N'}_{L^{N'}(\Omega)}\end{aligned}$$

Fig. 3 Illustration of the stochastic representation

$$\begin{tikzcd} f \arrow[r] \arrow[d, "\times e^{-\frac{Q}{2}-\beta_0 U}"] & \mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] \arrow[d, "\times e^{-\frac{Q}{2}-\beta_0 U}"] \\ \tilde{f} \arrow[r] & T_t^*\tilde{f} \end{tikzcd}$$

for all $t > t_2 > t_1 \geq 0$, where we used (4.4) in the second inequality.

As a consequence, for each $t > 0$, there exists $C_3 = C_3(d, N', t) > 0$ such that

$$\|T_t^{(\Omega,N')}\tilde{g}\|_{L^{\kappa N'}(\Omega)} = \|\tilde{v}(\cdot,t)\|_{L^{\kappa N'}(\Omega)} \leq C_3\|\tilde{g}\|_{L^{N'}(\Omega)}.$$

Since $T^{(\Omega,N')}$ and $T_t^{(*,\Omega,N)}$ are adjoint to each other, it follows that

$$\|T_t^{(*,\Omega,N)}\tilde{f}\|_{L^N(\Omega)} \leq C_3\|\tilde{f}\|_{L^{N_*}(\Omega)}, \quad \forall \tilde{f} \in L^{N_*}(\Omega)\cap L^N(\Omega).$$

where $N_* := \frac{\kappa N'}{\kappa N'-1}$. Thanks to Proposition 4.1 (5), the above inequality holds for all $\tilde{f} \in L^{N_*}(\Omega)$. Setting $N = 2$ yields $2_* = \frac{2(d+2)}{d+4} \in (1, 2)$. This completes the proof. □

4.3 Stochastic Representation

We prove the following theorem concerning the stochastic representation of $(T_t^*)_{t\geq 0}$.

Theorem 4.1 *Assume* **(H1)-(H3)**. *For each $f \in C_b(\mathcal{U};\mathbb{C})$ satisfying $\tilde{f} := fe^{-\frac{Q}{2}-\beta_0 U} \in L^2(\mathcal{U};\mathbb{C})$,*

$$T_t^*\tilde{f}(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)}\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad \forall (x,t) \in \mathcal{U}\times[0,\infty).$$

For the reader's convenience, we include Figure 3 to clarify the stochastic representation. Consider the following initial value problem associated with the operator $\mathcal{L}_{\beta_0}^*$:

$$\begin{cases} \partial_t\tilde{w} = \frac{1}{2}\Delta\tilde{w} + \nabla\cdot((p+\beta_0\nabla U)\tilde{w}) - e_{\beta_0}\tilde{w} & \text{in } \mathcal{U}\times[0,\infty), \\ \tilde{w}(\cdot,0) = \tilde{f} & \text{in } \mathcal{U}. \end{cases} \tag{4.6}$$

Definition 4.1 A function $\tilde{w} \in C(\mathcal{U}\times[0,\infty)) \cap L^2_{loc}([0,\infty),\mathcal{H}^1(\mathcal{U}))$ is called a *weak solution* of (4.6) if for each $\phi \in C_0^{1,1}(\mathcal{U}\times[0,\infty))$ and $t \in [0,\infty)$ one has

$$\begin{aligned} &\int_{\mathcal{U}}\tilde{w}(\cdot,t)\phi(\cdot,t)\mathrm{d}x - \int_{\mathcal{U}}\tilde{f}\phi(\cdot,0)\mathrm{d}x - \int_0^t\int_{\mathcal{U}}\tilde{w}\partial_t\phi\mathrm{d}x\mathrm{d}s \\ &\quad = -\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}\cdot\nabla\phi\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}\nabla\phi\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}e_{\beta_0}\tilde{w}\phi\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Lemma 4.3 *Assume* **(H1)-(H3)**. *For each $\tilde{f} \in C(\mathcal{U})\cap L^2(\mathcal{U})$, (4.6) admits at most one weak solution.*

The proof of the above lemma follows from energy methods and approximation arguments. Since it is somewhat standard we present its proof in Appendix A.2.

Now, we prove Theorem 4.1.

Proof of Theorem 4.1 Treating the real and imaginary parts separately, we only need to prove the theorem for $f \in C_b(\mathcal{U})$ such that $\tilde{f} := f e^{-\frac{Q}{2}-\beta_0 U} \in L^2(\mathcal{U})$. Fix such an f.

We show $T^*_\bullet \tilde{f}$ is a weak solution of (4.6). Due to the analyticity of $(T^*_t)_{t\geq 0}$ (see Theorem 3.3) and Lemma 3.5, we find

(1) $T^*_\bullet \tilde{f} \in C([0,\infty), L^2(\mathcal{U})) \cap C^1((0,\infty), L^2(\mathcal{U}))$;
(2) $T^*_t \tilde{f} \in \mathcal{D}^* \subset \mathcal{H}^1(\mathcal{U}) \cap H^2_{loc}(\mathcal{U})$ for all $t > 0$;
(3) $\frac{\mathrm{d}}{\mathrm{d}t} T^*_t \tilde{f} = \mathcal{L}^*_{\beta_0} T^*_t \tilde{f}$ for all $t > 0$.

Since $\tilde{f} \in C(\mathcal{U})$, the classical regularity theory of parabolic equations yields that $T^*_\bullet \tilde{f} \in C(\mathcal{U} \times [0,\infty))$. Applying Lemma 3.3 (2) and Lemma 3.5, we find for each $t > 0$,

$$\begin{aligned}\min\left\{\frac{1}{2}, C_*\right\} \|T^*_t \tilde{f}\|^2_{\mathcal{H}^1} &\leq \mathcal{E}_{\beta_0}(T^*_t \tilde{f}, T^*_t \tilde{f}) + M\|T^*_t \tilde{f}\|^2_{L^2} \\ &= -\langle T^*_t \tilde{f}, \mathcal{L}^*_{\beta_0} T^*_t \tilde{f}\rangle_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2} \\ &= -\langle T^*_t \tilde{f}, \frac{\mathrm{d}}{\mathrm{d}t} T^*_t \tilde{f}\rangle_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2} \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|T^*_t \tilde{f}\|^2_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2}.\end{aligned}$$

It follows that

$$\min\left\{\frac{1}{2}, C_*\right\} \int_0^t \|T^*_s \tilde{f}\|^2_{\mathcal{H}^1} \mathrm{d}s \leq \frac{1}{2}\|\tilde{f}\|^2_{L^2} + M \int_0^t \|T^*_s \tilde{f}\|^2_{L^2} \mathrm{d}s, \quad \forall t > 0.$$

This yields $T^*_\bullet \tilde{f} \in L^2_{loc}([0,\infty), \mathcal{H}^1(\mathcal{U}))$. By (3), it is easy to check that the integral identity in Definition 4.1 holds with $\tilde{w}$ replaced by $T^*_\bullet \tilde{f}$. As a consequence, $T^*_\bullet \tilde{f}$ is a weak solution of (4.6).

Define

$$\tilde{w}(x,t) := e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad (x,t) \in \mathcal{U} \times [0,\infty).$$

We claim that $\tilde{w}$ is also a weak solution of (4.6). Then, Lemma 4.3 yields $T^*_\bullet \tilde{f} = \tilde{w}$, leading to the conclusion of the theorem.

The continuity of $\tilde{w}$ in $\mathcal{U} \times [0,\infty)$ follows from the definition and continuity properties of X_t. We show $\tilde{w} \in L^2_{loc}([0,\infty), \mathcal{H}^1(\mathcal{U}))$. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be as in Subsection 2.3. It follows from Lemma 2.1 and Proposition 4.1 (4) that $\tilde{w} = \lim_{n\to\infty} T^{(*,\mathcal{U}_n,2)}_\bullet \tilde{f}|_{\mathcal{U}_n}$ in $\mathcal{U} \times [0,\infty)$, where we recall from Subsection 4.1 that $(T^{(*,\mathcal{U}_n,2)}_t)_{t\geq 0}$ is the positive analytic semigroup of contractions on $L^2(\mathcal{U}_n;\mathbb{C})$ generated by $\mathcal{L}^{*,2}_{\beta_0}|_{\mathcal{U}_n}$ with domain $W^{2,2}(\mathcal{U}_n;\mathbb{C}) \cap W^{1,2}_0(\mathcal{U}_n;\mathbb{C})$.

For convenience, we define $\tilde{w}_n := T^{(*,\mathcal{U}_n,2)}_\bullet \tilde{f}|_{\mathcal{U}_n}$ for $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \tilde{w}_n = \tilde{w}$. Lemma 4.1 (with $t_1 = 0$) gives for each $t \in [0,\infty)$ and $n \in \mathbb{N}$,

$$\frac{1}{2}\int_{\mathcal{U}_n} \tilde{w}^2_n(\cdot,t)\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathcal{U}_n} |\nabla \tilde{w}_n|^2 \mathrm{d}x\mathrm{d}s + C_*\int_0^t\int_{\mathcal{U}_n} \alpha \tilde{w}^2_n \mathrm{d}x\mathrm{d}s \leq \frac{1+e^{2Mt}}{2}\int_{\mathcal{U}_n} \tilde{f}^2 \mathrm{d}x.$$

Letting $n \to \infty$ yields $\tilde{w} \in L^2_{loc}([0,\infty), \mathcal{H}^1(\mathcal{U}))$. Since $\partial_t \tilde{w}_n = \mathcal{L}^{*,2}_{\beta_0}|_{\mathcal{U}_n} \tilde{w}_n$ in $L^2(\mathcal{U}_n)$ for all $t > 0$ and $n \in \mathbb{N}$, standard approximation arguments ensure that $\tilde{w}$ is a weak solution of (4.6). This finishes the proof. □

5 QSD: Existence, Uniqueness and Convergence

In this section, we study the existence and uniqueness of QSDs of X_t, as well as the exponential convergence of the process X_t conditioned on the event $[t < S_\Gamma]$ to QSDs. In Subsection 5.1, we show the existence of QSDs of X_t. In Subsection 5.2, we study the sharp exponential convergence of X_t with compactly supported initial distributions. In Subsection 5.3, we investigate the uniqueness of QSDs of X_t and the exponential convergence of X_t with arbitrary initial distribution. The proofs of Theorems A and B are outlined in Subsection 5.4.

5.1 Existence

We construct QSDs for X_t. Recall that λ_1 and $\tilde{v}_1$ are given in Theorem 3.1.

Theorem 5.1 *Assume* **(H1)-(H3)**. *Then, the following statements hold.*

(1) $\lambda_1 > 0$ *and* $\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta U}\mathrm{d}x < \infty$ *for any* $\beta > 0$. *Hence,* $\mathrm{d}\nu_1 := \frac{\tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}}{\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}\mathrm{d}x \in \mathcal{P}(\mathcal{U})$ *and satisfies* $\int_{\mathcal{U}} e^{\beta U}\mathrm{d}\nu_1 < \infty$ *for any* $\beta \in [0, \beta_0)$.

(2) *For each* $f \in C_b(\mathcal{U})$,

$$\mathbb{E}^{\nu_1}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] = e^{-\lambda_1 t}\int_{\mathcal{U}} f\mathrm{d}\nu_1, \quad \forall t \geq 0.$$

(3) ν_1 *is a QSD of* X_t *with extinction rate* λ_1.

We need the following lemma. Recall that the weight function α is defined in (3.1).

Lemma 5.1 *Assume* **(H1)-(H3)**. *Then,* $\tilde{v} \in L^2(\mathcal{U}, \alpha\mathrm{d}x; \mathbb{C})$ *implies* $\int_{\mathcal{U}} |\tilde{v}|e^{-\frac{Q}{2}-\beta U}\mathrm{d}x < \infty$ *for any* $\beta > 0$.

Proof Let $\beta > 0$. As $\int_{\mathcal{U}} |\tilde{v}|e^{-\frac{Q}{2}-\beta U}\mathrm{d}x \leq \left(\int_{\mathcal{U}} \alpha|\tilde{v}|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}} \frac{1}{\alpha}e^{-Q-2\beta U}\mathrm{d}x\right)^{\frac{1}{2}}$, it suffices to verify

$$\int_{\mathcal{U}} \frac{1}{\alpha}e^{-Q-2\beta U}\mathrm{d}x < \infty. \tag{5.1}$$

Let $\tilde{\alpha}(t) := \max\{\frac{1}{t^2}, 1\}$ for $t > 0$. According to the definition of α given in (3.1) and the fact that $\inf_{\mathcal{U}}\alpha > 0$, there exists $C_1 > 0$ such that $\alpha(x) \geq C_1\sum_{i=1}^d \tilde{\alpha}(x_i)$ for $x \in \mathcal{U}$. Since $U(x) = V(\xi^{-1}(x)) \geq \sum_{i=1}^d \tilde{V}(\xi_i^{-1}(x_i))$ for $x \in \mathcal{U}$ due to **(H3)**(2) and $e^{-Q} = \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{2}}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}$, we derive

$$\int_{\mathcal{U}} \frac{1}{\alpha}e^{-Q-2\beta U}\mathrm{d}x \leq \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{2}}}{C_1}\int_{\mathcal{U}} \frac{\prod_{i=1}^d \exp\left\{-2\beta\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^d \tilde{\alpha}(x_i)\right)\times\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x.$$

For each $k \in \{1, \ldots, d\}$, let Σ_k be the collection of all subsets of $\{1, \ldots, d\}$ with k elements, and set

$$A_k := \sup_{\sigma\in\Sigma_k}\int_{\{x_\sigma=(x_i)_{i\in\sigma}:x_i>0,\ \forall i\in\sigma\}} \frac{\prod_{i\in\sigma}\exp\left\{-2\beta\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i\in\sigma}\tilde{\alpha}(x_i)\right)\times\left[\prod_{i\in\sigma} a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_\sigma.$$

Clearly, (5.1) holds if $A_d < \infty$. We show this by induction.

First, we show $A_1 < \infty$. Following the arguments leading to (A.2), we can find $C_2 > 0$ such that $a_i(\xi_i^{-1}(x_i)) \geq C_2^2 x_i^2$ for $x_i \in [0,1]$ and $i \in \{1,\dots,d\}$. It follows that for each $i \in \{1,\dots,d\}$,

$$\begin{aligned}
\int_0^\infty \frac{e^{-2\beta\tilde V(\xi_i^{-1}(x_i))}}{\tilde\alpha(x_i)\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_i &\leq \frac{1}{C_2}\int_0^1 \frac{e^{-2\beta\tilde V(\xi_i^{-1}(x_i))}}{\tilde\alpha(x_i)x_i}\mathrm{d}x_i + \int_1^\infty \frac{e^{-2\beta\tilde V(\xi_i^{-1}(x_i))}}{\tilde\alpha(x_i)\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_i\\
&\leq \frac{1}{C_2}\int_0^1 x_i e^{-2\beta\tilde V(\xi_i^{-1}(x_i))}\mathrm{d}x_i + \int_1^\infty \frac{e^{-2\beta\tilde V(\xi_i^{-1}(x_i))}}{\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_i\\
&\leq \frac{1}{2C_2} + \int_{\xi_i^{-1}(1)}^\infty \frac{e^{-2\beta\tilde V(z_i)}}{a_i(z_i)}\mathrm{d}z_i,
\end{aligned}$$

where we used the definition of $\tilde\alpha$ in the second inequality, and the non-negativity of $\tilde V$ a simple change of variables in the third inequality. Since $\int_{\xi_i^{-1}(1)}^\infty \frac{e^{-2\beta\tilde V(z_i)}}{a_i(z_i)}\mathrm{d}z_i < \infty$ by **(H3)**(2), we find $A_1 < \infty$.

Suppose $A_k < \infty$ for some $k \in \{1,\dots,d-1\}$, we show $A_{k+1} < \infty$. We only prove

$$A_{k+1}^1 := \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k+1}\exp\left\{-2\beta\tilde V(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde\alpha(x_i)\right)\times\left[\prod_{i=1}^{k+1}a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1} < \infty;$$

integrals corresponding to other $\sigma \in \Sigma_{k+1}$ can be treated in exactly the same way. Note that

$$\begin{aligned}
B_0 &:= \int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{k+1}\exp\left\{-2\beta\tilde V(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde\alpha(x_i)\right)\times\left[\prod_{i=1}^{k+1}a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1}\\
&\leq \frac{1}{C_2^{k+1}}\int_0^1\cdots\int_0^1 \frac{1}{\left(\sum_{i=1}^{k+1}\frac{1}{x_i^2}\right)\times\prod_{i=1}^{k+1}x_i}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1}\\
&\leq \frac{1}{(k+1)C_2^{k+1}}\int_0^1\cdots\int_0^1 \frac{1}{\left(\prod_{i=1}^{k+1}x_i\right)^{1-\frac{2}{k+1}}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1} < \infty.
\end{aligned}$$

For $j \in \{1,\dots,k+1\}$, we see that

$$\begin{aligned}
B_j &:= \int_0^\infty \cdots \int_1^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k+1}\exp\left\{-2\beta\tilde V(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde\alpha(x_i)\right)\times\left[\prod_{i=1}^{k+1}a_i(\xi_i^{-1}(x_i))\right]^{\frac12}}\mathrm{d}x_1\cdots\mathrm{d}x_j\cdots\mathrm{d}x_{k+1}\\
&\leq A_k\int_1^\infty \frac{e^{-2\beta\tilde V(\xi_j^{-1}(x_j))}}{\left[a_j(\xi_j^{-1}(x_j))\right]^{\frac12}}\mathrm{d}x_j = A_k\int_{\xi_j^{-1}(1)}^\infty \frac{e^{-2\beta\tilde V(z_j)}}{a_j(z_j)}\mathrm{d}z_j < \infty,
\end{aligned}$$

where we used **(H3)**(2) in the last inequality. It follows that $A_{k+1}^1 = \sum_{j=0}^{k+1}B_j < \infty$. This completes the proof. □

Proof of Theorem 5.1 (1) Since $\tilde{v}_1 \in \mathcal{H}^1(\mathcal{U}) \subset L^2(\mathcal{U}, \alpha \mathrm{d}x)$, Lemma 5.1 yields $\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta U} \mathrm{d}x < \infty$ for any $\beta > 0$.

To see $\lambda_1 > 0$, we fix $f \in C_0^\infty(\mathcal{U})$ and set $\tilde{f} := f e^{-\frac{Q}{2}-\beta_0 U}$. Clearly, $\tilde{f} \in L^2(\mathcal{U})$. Theorem 4.1 gives

$$e^{\frac{-Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] = T_t^* \tilde{f}(x), \quad \forall (x,t) \in \mathcal{U} \times [0,\infty).$$

Set $v_1 := C \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}$, where $C := \left(\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U} \mathrm{d}x \right)^{-1}$. Obviously,

$$\int_{\mathcal{U}} v_1 e^{\beta U} \mathrm{d}x = C \int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-(\beta_0-\beta)U} \mathrm{d}x < \infty, \quad \forall \beta \in [0, \beta_0).$$

Moreover, we calculate

$$\int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = C \langle \tilde{v}_1, T_t^* \tilde{f} \rangle_{L^2} = C \langle T_t \tilde{v}_1, \tilde{f} \rangle_{L^2}, \quad \forall t \geq 0,$$

which together with $T_t \tilde{v}_1 = e^{-\lambda_1 t} \tilde{v}_1$ yields

$$\int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = C e^{-\lambda_1 t} \int_{\mathcal{U}} \tilde{v}_1 \tilde{f} \mathrm{d}x = e^{-\lambda_1 t} \int_{\mathcal{U}} v_1 f \mathrm{d}x, \quad \forall t \geq 0. \tag{5.2}$$

For each $x \in \mathcal{U}$, the fact $\mathbb{P}^x [S_\Gamma < \infty] = 1$ implies $\lim_{t\to\infty} \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] = 0$. This together with the fact $\sup_{x\in\mathcal{U}} \left| \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \right| \leq \|f\|_\infty$ for all $t \geq 0$ and the dominated convergence theorem implies $\lim_{t\to\infty} \int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = 0$. From which, we conclude $\lambda_1 > 0$, otherwise a contradiction can be easily derived from (5.2).

(2) Fix $f \in C_b(\mathcal{U})$ and take a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathcal{U})$ that locally uniformly converges to f as $n \to \infty$ and satisfies $\|f_n\|_\infty \leq \|f\|_\infty$ for all $n \in \mathbb{N}$. It follows from (5.2) that for each $t \geq 0$ and $n \in \mathbb{N}$, $\int_{\mathcal{U}} \mathbb{E}^\bullet \left[f_n(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}\nu_1 = e^{-\lambda_1 t} \int_{\mathcal{U}} f_n \mathrm{d}\nu_1$, where $\mathrm{d}\nu_1 := v_1 \mathrm{d}x$. Letting $n \to \infty$, we conclude the result from the dominated convergence theorem.

(3) Applying (2) with $f = \mathbb{1}_{\mathcal{U}}$, we find $\mathbb{P}^{\nu_1} [t < S_\Gamma] = \mathbb{E}^{\nu_1} \left[\mathbb{1}_{\{t<S_\Gamma\}} \right] = e^{-\lambda_1 t}$ for all $t \geq 0$. Applying (2) again, we conclude $\frac{\mathbb{E}^{\nu_1}\left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right]}{\mathbb{P}^{\nu_1}[t<S_\Gamma]} = \int_{\mathcal{U}} f \mathrm{d}\nu_1$ for all $f \in C_b(\mathcal{U})$. That is, ν_1 is a QSD of X_t and λ_1 is the associated extinction rate. □

5.2 Sharp Exponential Convergence

We study the long-time dynamics of X_t before reaching the boundary Γ. Ahead of stating the result, we recall and introduce some terminologies and notations.

Recall that the spectra of $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$ coincide, are discrete and contained in $\{\lambda \in \mathbb{C} : \Re\lambda > 0 \text{ and } \arg\lambda \leq \theta\}$ for some $\theta \in (0, \frac{\pi}{2})$. The number λ_1 is the principal eigenvalue of both $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$. Let $\tilde{v}_1^*$ be as in Lemma 3.5 (4) and suppose it satisfies the normalization

$$\langle \tilde{v}_1, \tilde{v}_1^* \rangle_{L^2} = \int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U} \mathrm{d}x, \tag{5.3}$$

where $\tilde{v}_1$ is given in Theorem 3.1. The last integral converges thanks to Lemma 5.1.

Set $\lambda_2 := \min \left\{ \Re\lambda : \lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) \text{ and } \Re\lambda > \lambda_1 \right\}$. Then, $\lambda_2 > \lambda_1$ and $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_2\}$ consists of finitely many elements. For $k = 1, 2$, let $\mathcal{P}_k^*$ and $\mathcal{P}_k$ be respectively the spectral projections of $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$ corresponding to $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_k\}$. Clearly, $\mathcal{P}_1^*$ and $\mathcal{P}_1$ are adjoint to each other and $\mathrm{ran}\mathcal{P}_1$ and $\mathrm{ran}\mathcal{P}_1^*$ of $-\mathcal{L}_{\beta_0}$ and $-\mathcal{L}_{\beta_0}^*$ corresponding to λ_1 are respectively spanned over $\mathbb{R}$ by $\tilde{v}_1$ and $\tilde{v}_1^*$. Since the coefficients of $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$ are real-valued resulting in the symmetry of the set $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_2\}$ with respect to the real axis, $\mathcal{P}_2^*$ and $\mathcal{P}_2$ are also adjoint to each other.

Suppose the set $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_2\}$ consists of N_* elements and is enumerated as

$$\lambda_{2,i}, \quad i \in \{0, \dots, N_* - 1\}.$$

Denote by $\mathcal{P}_{2,i}^*$ and $\mathcal{P}_{2,i}$ the spectral projections of $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$ corresponding to $\lambda_{2,i}$ and $\overline{\lambda_{2,i}}$, respectively. Note that $\mathcal{P}_{2,i}^*$ and $\mathcal{P}_{2,i}$ are adjoint to each other. Obviously, $\mathcal{P}_2^* = \sum_{i=0}^{N_*-1} \mathcal{P}_{2,i}^*$ and $\mathcal{P}_2 = \sum_{i=0}^{N_*-1} \mathcal{P}_{2,i}$.

For $i \in \{0, \dots, N_* - 1\}$, we let

- N_i be the order of the pole $\lambda_{2,i}$ of the resolvent of $-\mathcal{L}_{\beta_0}^*$,
- $d_i = \dim(\mathrm{ran}\mathcal{P}_{2,i}^*)$,
- $\{\tilde{v}_{i,j}^{(*,2)} : j \in \{1, \dots, d_i\}\}$ and $\{\tilde{v}_{i,j}^{(2)} : j \in \{1, \dots, d_i\}\}$ be generalized eigenfunctions of $-\mathcal{L}_{\beta_0}^*$ and $-\mathcal{L}_{\beta_0}$ that form bases of $\mathrm{ran}\mathcal{P}_{2,i}^*$ and $\mathrm{ran}\mathcal{P}_{2,i}$, respectively, and satisfy the normalization
$$\langle \tilde{v}_{i,j}^{(2)}, \tilde{v}_{i,k}^{(*,2)} \rangle_{L^2} = \delta_{jk}, \quad \forall j, k \in \{1, \dots, d_i\}. \tag{5.4}$$

Recall that ν_1 is the QSD of X_t obtained in Theorem 5.1, and $\{T_t\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ are positive and analytic semigroups of contractions on $L^2(\mathcal{U}; \mathbb{C})$ generated by $\mathcal{L}_{\beta_0}$ and $\mathcal{L}_{\beta_0}^*$, respectively.

The main result in this subsection is stated in the next theorem.

Theorem 5.2 *Assume* **(H1)-(H3)**. *For each $\nu \in \mathcal{P}(\mathcal{U})$ with compact support in $\mathcal{U}$, there holds for each $f \in C_b(\mathcal{U})$,*

$$\begin{aligned}
&\mathbb{E}^\nu[f(X_t)\big|t < S_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\nu_1 \\
&= \frac{e^{\lambda_1 t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} \tilde{v}_1^* \mathrm{d}\nu} \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} T_t^* \mathcal{P}_2^* \left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}} \int_{\mathcal{U}} f \mathrm{d}\nu_1 \right) \mathrm{d}\nu + o(e^{-(\lambda_2-\lambda_1)t}) \\
&= \frac{e^{-(\lambda_2-\lambda_1)t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} \tilde{v}_1^* \mathrm{d}\nu} \\
&\quad \times \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} \sum_{j=0}^{N_*-1} e^{-i\Im\lambda_{2,j} t} \sum_{k=0}^{N_j-1} \frac{t^k}{k!} (\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^k \mathcal{P}_{2,j}^* \left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}} \int_{\mathcal{U}} f \mathrm{d}\nu_1 \right) \mathrm{d}\nu \\
&\quad + o(e^{-(\lambda_2-\lambda_1)t}) \quad \text{as} \quad t \to \infty,
\end{aligned}$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$ and $\tilde{\mathbb{1}}_{\mathcal{U}} := e^{-\frac{Q}{2}-\beta_0 U} \mathbb{1}_{\mathcal{U}}$. In particular, the following hold:

- *For each $0 < \epsilon \ll 1$,*
$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1-\epsilon)t} \|\mathbb{P}^\nu[X_t \in \bullet | t < S_\Gamma] - \nu_1\|_{TV} = 0.$$
- *If $f \in C_b(\mathcal{U})$ is such that $\mathcal{P}_2^* \left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}} \int_{\mathcal{U}} f \mathrm{d}\nu_1 \right) \neq 0$, then for a.e. $x \in \mathcal{U}$, there is a family of sets $\{\mathcal{K}_{x,\epsilon}\}_{0<\epsilon\ll 1}$ in $(0, \infty)$ satisfying $\mathcal{K}_{x,\epsilon_2} \subset \mathcal{K}_{x,\epsilon_1}$ for $0 < \epsilon_1 < \epsilon_2 \ll 1$ and $\lim_{\epsilon\to 0} \inf_{T>0} |\mathcal{K}_{x,\epsilon} \cap (T, T+1)| = 1$ such that*
$$\lim_{\substack{t\in\mathcal{K}_{x,\epsilon} \\ t\to\infty}} e^{(\lambda_2-\lambda_1+\epsilon)t} \left| \mathbb{E}^x[f(X_t)\big|t < S_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\nu_1 \right| = \infty, \quad \forall 0 < \epsilon \ll 1.$$

Remark 5.1 We make some remarks about Theorem 5.2.

(1) Theorem 5.2 appears to be a direct consequence of the decomposition of $(T_t^*)_{t\geq 0}$ according to spectral projections ensured by Theorem 3.3 and the stochastic representation given in Theorem 4.1. This is however deceptive due to the following two reasons: (i) the stochastic representation given in Theorem 4.1 is only true for $f\in C_b(\mathcal{U})$ such that $e^{-\frac{Q}{2}-\beta_0 U}f\in L^2(\mathcal{U})$; this is indeed a restriction as $e^{-\frac{Q}{2}-\beta_0 U}$ and $e^{\frac{Q}{2}+\beta_0 U}$ are respectively unbounded near Γ and ∞; (ii) the semigroup $(T_t^*)_{t\geq 0}$ is naturally defined on $L^2(\mathcal{U})$, but we need its L^∞ properties.

(2) For $f\in C_b(\mathcal{U})$, the function $\tilde f:=e^{-\frac{Q}{2}-\beta_0 U}f$ does not necessarily belong to $L^2(\mathcal{U})$. Neither does $\tilde f-\mathbb{1}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1$. Its projections under $\mathcal{P}_2^*$ and $\mathcal{P}_{2,j}^*$ are justified in Lemma 5.2 (2).

(3) Theorem 5.2 actually holds for all initial distributions $\nu\in\mathcal{P}(\mathcal{U})$ satisfying the condition $\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu<\infty$. See Remark 5.2 for more details.

We need two lemmas before proving Theorem 5.2. The first one concerns some important properties of T_t^*, $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$.

Lemma 5.2 *Assume* **(H1)**-**(H3)**. *The following hold.*

(1) $\mathcal{P}_1^*\tilde f=\tilde v_1^*\int_{\mathcal{U}}\tilde f e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu_1$ *for all* $\tilde f\in L^2(\mathcal{U})$.

(2) *Both* $\mathcal{P}_1^*$ *and* $\mathcal{P}_2^*$ *are well-defined on* $\{fe^{-\frac{Q}{2}-\beta_0 U}: f\in C_b(\mathcal{U})\}$ *with values in* $L^2(\mathcal{U})$.

(3) $T_t^*\mathcal{P}_2^*=\sum_{j=0}^{N_*-1}T_t^*\mathcal{P}_{2,j}^*=e^{-\lambda_2 t}\sum_{j=0}^{N_*-1}e^{-i\Im\lambda_{2,j}t}\sum_{k=0}^{N_j-1}\frac{t^k}{k!}(\mathcal{L}_{\beta_0}^*+\lambda_{2,j})^k\mathcal{P}_{2,j}^*$ *for all* $t\geq 0$.

(4) *For each* $0<\epsilon\ll 1$, *there exists* $C=C(\epsilon)>0$ *such that*

$$\left\|T_t^*-e^{-\lambda_1 t}\mathcal{P}_1^*-T_t^*\mathcal{P}_2^*\right\|_{L^2\to L^2}\leq Ce^{-(\lambda_2+\epsilon)t},\quad \forall t\geq 0.$$

(5) *Let* $f\in \mathrm{ran}\mathcal{P}_2^*\setminus\{0\}$. *Then, for a.e.* $x\in\mathcal{U}$, *there is a family of sets* $\{\mathcal{K}_{x,\epsilon}\}_{0<\epsilon\ll 1}$ *in* $(0,\infty)$ *satisfying* $\mathcal{K}_{x,\epsilon_2}\subset\mathcal{K}_{x,\epsilon_1}$ *for* $0<\epsilon_1<\epsilon_2\ll 1$ *and* $\lim_{\epsilon\to 0}\inf_{T>0}|\mathcal{K}_{x,\epsilon}\cap(T,T+1)|=1$ *such that*

$$\lim_{\substack{t\in\mathcal{K}_{x,\epsilon}\\ t\to\infty}} e^{(\lambda_2+\epsilon)t}|T_t^*f|(x)=\infty,\quad \forall 0<\epsilon\ll 1.$$

Proof (1) Note that $\mathrm{ran}(\mathcal{P}_1^*|_{L^2(\mathcal{U})})$ is spanned over $\mathbb{R}$ by $\tilde v_1^*$. By the Riesz representation theorem, there exists $h\in L^2(\mathcal{U})$ such that

$$\mathcal{P}_1^*\tilde f=\langle\tilde f,h\rangle_{L^2}\tilde v_1^*,\quad\forall\tilde f\in L^2(\mathcal{U}).\tag{5.5}$$

As $\mathcal{P}_1$ and $\mathcal{P}_1^*$ are adjoint to each other it must be true that $\mathcal{P}_1\tilde v=\langle\tilde v,\tilde v_1^*\rangle_{L^2}h$ for all $\tilde v\in L^2(\mathcal{U})$. Since $\mathrm{ran}(\mathcal{P}_1|_{L^2(\mathcal{U})})$ is spanned over $\mathbb{R}$ by $\tilde v_1$, there exists $C_1\in\mathbb{R}$ such that $h=C_1\tilde v_1$. Thus, the normalization (5.3) gives

$$\tilde v_1=\mathcal{P}_1\tilde v_1=C_1\langle\tilde v_1,\tilde v_1^*\rangle_{L^2}\tilde v_1=C_1\tilde v_1\int_{\mathcal{U}}\tilde v_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x,$$

leading to $C_1=\frac{1}{\int_{\mathcal{U}}\tilde v_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}$, and hence, $h=\frac{\tilde v_1}{\int_{\mathcal{U}}\tilde v_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}$. Inserting this into (5.5) and noting the definition of ν_1 give rise to the formula for $\mathcal{P}_1^*\tilde f$.

(2) Thanks to (1), it is obvious that the statement holds for $\mathcal{P}_1^*$. Note that $\mathrm{ran}\mathcal{P}_2^*$ and $\mathrm{ran}\mathcal{P}_2$ are finite dimensional and Lemma 5.1 ensures $\int_{\mathcal{U}} e^{-\frac{Q}{2}-\beta_0 U}|v|\mathrm{d}x<\infty$ for each $v\in\mathrm{ran}\mathcal{P}_2$. Following the same proof as in (1), we arrive at the conclusion for $\mathcal{P}_2^*$ as well.

(3) and (4) are special cases of [25, Corollary V. 3.2] due to Theorem 3.3, the fact $\Re\lambda_{2,i} = \lambda_2$ for all $i \in \{1, \dots, N_*\}$, and the simplicity of the principle eigenvalue λ_1 of $-\mathcal{L}^*_{\beta_0}$.

It remains to show (5). Fix $f \in \mathrm{ran}\mathcal{P}^*_2 \setminus \{0\}$. We consider three cases.

Case 1 $N_* = 1$. In this case, $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\} = \{\lambda_2\}$. Then, f is a generalized eigenfunction of $-\mathcal{L}^*_{\beta_0}$ associated with λ_2, and thus, there exists $\tilde{N} \in \mathbb{N}$ such that $(\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}+1} f = 0$ and $(\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}} f \neq 0$ in $\mathcal{U}$. It follows from the strong unique continuation principle for elliptic equations (see e.g. [46]) that $(\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}} f \neq 0$ a.e. in $\mathcal{U}$. Since

$$T^*_t f = e^{-\lambda_2 t} \sum_{k=0}^{\tilde{N}} \frac{t^k}{k!} (\mathcal{L}^*_{\beta_0} + \lambda_2)^k f = e^{-\lambda_2 t} \left(\frac{t^{\tilde{N}}}{\tilde{N}!} (\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}} f + o(t^{\tilde{N}}) \right) \quad \text{as} \quad t \to \infty,$$

we derive $\lim_{t\to\infty} e^{(\lambda_2+\epsilon)t} |T^*_t f|(x) = \infty$ for a.e. $x \in \mathcal{U}$ and each $0 < \epsilon \ll 1$. The conclusion follows.

Case 2 $N_* = 2K + 1$ for some $K \in \mathbb{N}$. Considering the symmetry of the set $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\}$ with respect to the real axis, we can re-enumerate it as $\{\lambda_{2,j}\}_{j=-K}^{K}$ such that $\lambda_{2,0} = \lambda_2$ and $\lambda_{2,j} = \overline{\lambda}_{2,-j}$ for $j \in \{1, \dots, K\}$.

Note that $f = \sum_{j=-K}^{K} f_j$, where f_j is the projection of f onto the generalized eigenspace of $\lambda_{2,j}$. Since f is real-valued we must have $f_j = \overline{f}_{-j}$ for all $j \in \{1, \dots, K\}$. We may assume, without loss of generality, that $f_j \neq 0$ for all $j \in \{-K, \dots, K\}$.

Since $\lambda_{2,j}$ is a pole of the resolvent of $-\mathcal{L}^*_{\beta_0}$ with finite order, there exists $\tilde{N}_j \in \mathbb{N}$ such that $(\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j+1} f_j = 0$ and $(\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j} f_j \neq 0$. Applying the strong unique continuation principle for elliptic equations (see e.g. [46]), we find

$$(\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j} f_j \neq 0 \quad \text{a.e. in} \quad \mathcal{U}. \tag{5.6}$$

Clearly, $\tilde{N}_j = \tilde{N}_{-j}$ for all $j \in \{1, \dots, K\}$. Straightforward calculations then give for $t \gg 1$,

$$\begin{aligned} e^{\lambda_2 t} T^*_t f &= \sum_{j=-K}^{K} e^{-\Im\lambda_{2,j} t} \sum_{k=0}^{\tilde{N}_j} \frac{t^k}{k!} (\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^k f_j \\ &= \sum_{j=-K}^{K} e^{-i\Im\lambda_{2,j} t} \left[\frac{t^{\tilde{N}_j}}{\tilde{N}_j!} (\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j} f_j + o(t^{\tilde{N}_j}) \right] \\ &= \left[\frac{t^{\tilde{N}_0}}{\tilde{N}_0!} (\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}_0} f_0 + o(t^{\tilde{N}_0}) \right] \\ &\quad + \sum_{j=1}^{K} \left[\frac{2t^{\tilde{N}_j}}{\tilde{N}_j!} \Re\left(e^{-i\Im\lambda_{2,j} t} (\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j} f_j \right) + o(t^{\tilde{N}_j}) \right]. \end{aligned} \tag{5.7}$$

Since the asymptotics of $e^{\lambda_2 t} T^*_t f$ as $t \to \infty$ is determined by the terms with the highest degree, we may assume, without loss of generality, that $\tilde{N}_0 = \tilde{N}_1 = \dots = \tilde{N}_K$.

Set $F_0 := \frac{1}{\tilde{N}_0!}(\mathcal{L}^*_{\beta_0} + \lambda_2)^{\tilde{N}_0} f_0$ and $F_j := \frac{2}{\tilde{N}_j!}(\mathcal{L}^*_{\beta_0} + \lambda_{2,j})^{\tilde{N}_j} f_j$ for $j \in \{1, \dots, K\}$. We rewrite (5.7) as

$$\begin{aligned} \frac{e^{\lambda_2 t} T_t^* f}{t^{\tilde{N}_0}} &= F_0 + \sum_{j=1}^{K} \Re(e^{-i\Im\lambda_{2,j}t} F_j) + o(1) \\ &= F_0 + \sum_{j=1}^{K} |F_j| \sin(\Im\lambda_{2,j}t + \varphi_j) + o(1), \quad \forall t \gg 1, \end{aligned} \tag{5.8}$$

where $\varphi_j \in [0, 2\pi)$ satisfies $\tan\varphi_j = -\frac{\Re F_j}{\Im F_j}$ for $j \in \{1, \dots, K\}$.

Note that (5.6) ensures the set $\mathcal{N} := \{x \in \mathcal{U} : \exists j \in \{0, 1, \dots, K\} \text{ s.t. } |F_j|(x) = 0\}$ has zero Lebesgue measure. Fix $x \in \mathcal{U} \setminus \mathcal{N}$ and set

$$F_x(t) := F_0(x) + \sum_{j=1}^{K} |F_j|(x) \sin(\Im\lambda_{2,j}t + \varphi_j), \quad \forall t \in \mathbb{R}.$$

If $\inf_{t\gg 1} |F_x(t)| > 0$, (5.8) implies $\lim_{t\to\infty} e^{(\lambda_2+\epsilon)t}|T_t^* f|(x) = \infty$ for each $0 < \epsilon \ll 1$. Otherwise, for each $0 < \epsilon \ll 1$, we set $\mathcal{K}_{x,\epsilon} := \{t \in (0, \infty) : |F_x(t)| \geq \epsilon\}$. Then, (5.8) ensures $\lim_{\substack{t\in\mathcal{K}_{x,\epsilon} \\ t\to\infty}} e^{(\lambda_2+\epsilon)t}|T_t^* f|(x) = \infty$. It remains to show

$$\lim_{\epsilon\to 0} \inf_{T>0} |\mathcal{K}_{x,\epsilon} \cap (T, T+1)| = 1. \tag{5.9}$$

If $\{\Im\lambda_{2,j}\}_{j=1}^K$ are rationally dependent, then F_x is periodic and (5.9) follows immediately. Otherwise, F_x is quasi-periodic, or more generally, almost-periodic. Following the definition of almost-periodic functions (see e.g. [47]), it is not hard to prove (5.9).

Case 3 $N_* = 2K$ for some $K \in \mathbb{N}$. The proof is exactly the same as that in **Case 2** except that f_0 does not appear due to the fact $\lambda_2 \notin \{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\}$.

This completes the proof. □

Lemma 5.3 *Assume* **(H1)**-**(H3)**. *For each* $0 < \epsilon \ll 1$, *there exists* $C = C(\epsilon) > 0$ *such that for each* $f \in C_b(\mathcal{U})$ *and* $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$ *and* $t \geq 2$,

$$\|T_t^* \mathcal{P}_2^* \tilde{f}\|_\infty \leq Ce^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \tag{5.10}$$

$$\left|\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] - e^{\frac{Q}{2}+\beta_0 U} e^{-\lambda_1 t}\tilde{v}_1^* \int_{\mathcal{U}} f \mathrm{d}\nu_1\right| \leq Ce^{\frac{Q}{2}+\beta_0 U} e^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \tag{5.11}$$

and

$$\left|\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] - e^{\frac{Q}{2}+\beta_0 U}\left(e^{-\lambda_1 t}\tilde{v}_1^* \int_{\mathcal{U}} f \mathrm{d}\nu_1 + T_t^* \mathcal{P}_2^* \tilde{f}\right)\right| \leq Ce^{\frac{Q}{2}+\beta_0 U} e^{-(\lambda_2+\epsilon)t}\|f\|_\infty. \tag{5.12}$$

The idea of the proof is sketched in Figure 4.

Proof Fix $0 < \epsilon \ll 1$ and $f \in C_b(\mathcal{U})$. By the Markov property and homogeneity of X_t,

$$\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] = \mathbb{E}^x\left[g(X_{t-1})\mathbb{1}_{\{t-1<S_\Gamma\}}\right], \quad \forall (x, t) \in \mathcal{U} \times [1, \infty), \tag{5.13}$$

$$\begin{tikzcd}
f \in C_b \arrow[r] \arrow[d, "\times e^{-\frac{Q}{2}-\beta_0 U}"] & g := \mathbb{E}^{\bullet}\left[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}\right] \arrow[rr] \arrow[d, "\times e^{-\frac{Q}{2}-\beta_0 U}"] & & \mathbb{E}^{\bullet}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] \arrow[d, "\times e^{-\frac{Q}{2}-\beta_0 U}"] \\
\tilde{f} \in L^{2_*} \arrow[r] & \tilde{g} \in L^2 \arrow[r] & \tilde{g} = T^*_{t-2}\tilde{g} \in L^2 \arrow[r] & T^*_{t-1}\tilde{g} \in L^\infty
\end{tikzcd}$$

Fig. 4 Idea of the proof of Lemma 5.3

where $g := \mathbb{E}^{\bullet}[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}] \in C_b(\mathcal{U})$. For convenience, we set $\tilde{g} := e^{-\frac{Q}{2}-\beta_0 U} g$ and $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$.

The proof is broken into three steps.

Step 1 We claim that $\tilde{g} \in L^2(\mathcal{U})$ and there exists $D_1 > 0$ (independent of f) such that

$$\|\tilde{g}\|_{L^2} \leq D_1 \|f\|_\infty. \tag{5.14}$$

Recall that $2_* := \frac{2(d+2)}{d+4} \in (1,2)$ (see Lemma 4.2). Since $e^{-\frac{Q(x)}{2}} = \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{4}}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{4}}}$, we find

$$\int_{\mathcal{U}} |\tilde{f}|^{2_*} \mathrm{d}x = \int_{\mathcal{U}} |f|^{2_*} e^{-\frac{2_* Q}{2} - 2_* \beta_0 U} \mathrm{d}x \leq \|f\|_\infty^{2_*} \left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{2_*}{4}} \int_{\mathcal{U}} \frac{e^{-2_* \beta_0 U}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{2_*}{4}}} \mathrm{d}x.$$

Arguments as in the proof of Lemma 5.1 yield $\int_{\mathcal{U}} \frac{e^{-2_*\beta_0 U(x)}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{2_*}{4}}} \mathrm{d}x < \infty$. This implies the existence of $C_1 > 0$ (independent of f) such that

$$\|\tilde{f}\|_{L^{2_*}(\mathcal{U})} \leq C_1 \|f\|_\infty. \tag{5.15}$$

Recall $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$ from Subsection 2.3. For each $n \in \mathbb{N}$, we recall from Subsection 4.1 that $(T_t^{(*,\mathcal{U}_n,2_*)})_{t\geq 0}$ is the positive and analytic semigroup of contractions on $L^{2_*}(\mathcal{U}_n;\mathbb{C})$ generated by $(\mathcal{L}_{\beta_0}^{*,2_*}|_{\mathcal{U}_n}, W^{2,2_*}(\mathcal{U}_n;\mathbb{C}) \cap W_0^{1,2_*}(\mathcal{U}_n;\mathbb{C}))$. Since $\tilde{f} \in C(\overline{\mathcal{U}}_n)$, Proposition 4.1 ensures

$$T_t^{(*,\mathcal{U}_n,2_*)} \tilde{f}|_{\mathcal{U}_n} = e^{-\frac{Q}{2}-\beta_0 U} \mathbb{E}^{\bullet}\left[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right], \quad \forall t \in [0,\infty). \tag{5.16}$$

It follows from Lemma 4.2 the existence of $C_2 > 0$ such that $\|T_1^{(*,\mathcal{U}_n,2_*)}\tilde{f}\|_{L^2(\mathcal{U}_n)} \leq C_2 \|\tilde{f}\|_{L^{2_*}(\mathcal{U}_n)}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we derive from Lemma 2.1, (5.16) and Fatou's lemma that

$$\|\tilde{g}\|_{L^2} = \|e^{-\frac{Q}{2}-\beta_0 U} \mathbb{E}^{\bullet}\left[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}\right]\|_{L^2(\mathcal{U})} \leq C_2 \|\tilde{f}\|_{L^{2_*}(\mathcal{U})} \leq C_1 C_2 \|f\|_\infty,$$

where we used (5.15) in the last inequality.

Step 2 We claim the existence of $D_2 > 0$ such that

$$\|T_t^* \tilde{h}\|_\infty \leq D_2 \|T_{t-1}^* \tilde{h}\|_{L^2}, \quad \forall t \geq 1 \text{ and } \tilde{h} \in L^2(\mathcal{U}). \tag{5.17}$$

Setting $\tilde{h} := \tilde{g} - \mathcal{P}_1^* \tilde{g} - \mathcal{P}_2^* \tilde{g}$, we find from the above inequality, Lemma 5.2 (4) and the result in **Step 1** that for some $D_3 > 0$, there holds

$$\left\|T_{t-1}^* \tilde{g} - T_{t-1}^* \mathcal{P}_1^* \tilde{g} - T_{t-1}^* \mathcal{P}_2^* \tilde{g}\right\|_\infty \leq D_3 e^{-(\lambda_2+\epsilon)t} \|f\|_\infty, \quad \forall t \geq 2. \tag{5.18}$$

We first prove (5.17) when $\tilde{h} = e^{-\frac{Q}{2}-\beta_0 U} h \in L^2(\mathcal{U})$ for some $h \in C_b(\mathcal{U})$. The general case follows from standard approximation procedures. Note that Theorem 4.1 gives

$$T_t^* \tilde{h}(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x \left[h(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right], \quad \forall (x,t) \in \mathcal{U} \times [0,\infty). \tag{5.19}$$

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$ be as in Subsection 2.3. If we show the existence of $C_* > 0$ such that

$$\sup_{\mathcal{U}_n} e^{-\frac{Q}{2}-\beta_0 U} \left| \mathbb{E}^\bullet [h(X_t)\mathbb{1}_{\{t<\tau_n\}} \right| \le C_* \left\| e^{-\frac{Q}{2}-\beta_0 U} \mathbb{E}^\bullet \left[h(X_{t-1}) \mathbb{1}_{\{t-1<\tau_n\}} \right] \right\|_{L^2(\mathcal{U}_n)} \tag{5.20}$$

for all $t \ge 1$ and $n \in \mathbb{N}$, then (5.17) follows immediately from (5.19) and Lemma 2.1.

We show (5.20) by Moser iteration. Recall that for each $n \in \mathbb{N}$ and $N > 1$, $(T_t^{(*,\mathcal{U}_n,N)})_{t\ge 0}$ is the positive and analytic semigroup on $L^N(\mathcal{U}_n;\mathbb{C})$ generated by $(\mathcal{L}_{\beta_0}^{*,N}|_{\mathcal{U}_n}, W^{2,N}(\mathcal{U}_n;\mathbb{C}) \cap W_0^{1,N}(\mathcal{U}_n;\mathbb{C}))$. Since here for each n we only consider the action of $(T_t^{(*,\mathcal{U}_n,N)})_{t\ge 0}$ on functions in $C(\overline{\mathcal{U}}_n;\mathbb{C})$, we simply write $(T_t^{(n)})_{t\ge 0}$ for all $\left\{ (T_t^{(*,\mathcal{U}_n,N)})_{t\ge 0}, N > 1 \right\}$ in consideration of Proposition 4.1 (5). Obviously, $\tilde{h}_n := \tilde{h}|_{\mathcal{U}_n} \in C(\overline{\mathcal{U}}_n)$ for all $n \in \mathbb{N}$. It follows from Proposition 4.1 (4) that

$$T_t^{(n)} \tilde{h}_n(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x \left[h(X_t) \mathbb{1}_{\{t<\tau_n\}} \right], \quad \forall (x,t) \in \mathcal{U}_n \times [0,\infty) \text{ and } n \in \mathbb{N}. \tag{5.21}$$

Set $\tilde{w}_n := T_\bullet^{(n)} \tilde{h}_n$. We see from Lemma 4.1 that for all $n \in \mathbb{N}$ and $N \ge 2$,

$$\begin{aligned} &\frac{1}{N} \int_{\mathcal{U}_n} |\tilde{w}_n|^N(\cdot,t_2) \mathrm{d}x + \frac{N-1}{2} \int_{t_1}^{t_2} \int_{\mathcal{U}_n} |\tilde{w}_n|^{N-2} |\nabla \tilde{w}_n|^2 \mathrm{d}x\mathrm{d}s \\ &\quad \le \frac{1}{N}(1+e^{NM(t_2-t_1)}) \int_{\mathcal{U}_n} |\tilde{w}(\cdot,t_1)|^N \mathrm{d}x, \quad \forall t_2 > t_1 \ge 0, \end{aligned} \tag{5.22}$$

where we recall that $M > 0$ is fixed and independent of $n \in \mathbb{N}$ and $N \ge 2$, such that the conclusion in Lemma 3.2 (3) holds. The Sobolev embedding theorem gives

$$\|\tilde{w}_n^{\frac{N}{2}}\|_{L^{2\kappa}(\mathcal{U}_n\times[t_1,t_2])} \le C_3 \left(\sup_{s\in[t_1,t_2]} \|\tilde{w}_n^{\frac{N}{2}}(\cdot,s)\|_{L^2(\mathcal{U}_n)} + \|\nabla \tilde{w}_n^{\frac{N}{2}}\|_{L^2(\mathcal{U}_n\times[t_1,t_2])} \right),$$

where $\kappa := \frac{d+2}{d}$ and $C_3 > 0$ only depends on d. Therefore, (5.22) gives rise to

$$\left(\int_{t_1}^{t_2} \int_{\mathcal{U}_n} |\tilde{w}_n|^{\kappa N} \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{\kappa}} \le 4C_3^2 (1+e^{NM(t_2-t_1)}) \int_{\mathcal{U}_n} |\tilde{w}(\cdot,t_1)|^N \mathrm{d}x, \quad \forall t_2 > t_1 \ge 0, \tag{5.23}$$

for all $n \in \mathbb{N}$ and $N \ge 2$. We then deduce from Lemma 4.1 (with κN instead of N) and (5.23) that

$$\begin{aligned} \frac{1}{\kappa N} \int_{\mathcal{U}_n} |\tilde{w}_n|^{\kappa N}(\cdot,t_3) \mathrm{d}x &\le \frac{2}{\kappa N(t_2-t_1)} \int_{t_1}^{t_2} \int_{\mathcal{U}_n} |\tilde{w}_n|^{\kappa N} \mathrm{d}x\mathrm{d}s \\ &\le \frac{2(4C_3^2)^\kappa}{\kappa N(t_2-t_1)} \left(1+e^{NM(t_2-t_1)}\right)^\kappa \|\tilde{w}(\cdot,t_1)\|_{L^N(\mathcal{U}_n)}^{\kappa N} \end{aligned} \tag{5.24}$$

for all $t_3 > t_2 > t_1 \ge 0$, $n \in \mathbb{N}$ and $N \ge 2$.

Fix $t \geq 1$. For each $\ell \in \mathbb{N} \cup \{0\}$, we set $N = N_\ell := 2\kappa^\ell$, $t_1 := t - 2^{-\ell}$, $t_2 := t - \frac{3}{2}2^{-(\ell+1)}$ and $t_3 := t - 2^{-(\ell+1)}$ in (5.24) to find

$$\|\tilde{w}_n(\cdot, t - 2^{-(\ell+1)})\|_{L^{N_{\ell+1}}(\mathcal{U}_n)} \leq C_4^{\frac{1}{N_{\ell+1}}} 2^{\frac{\ell+2}{N_{\ell+1}}} e^{M2^{-(\ell+1)}} \|\tilde{w}_n(\cdot, t - 2^{-\ell})\|_{L^{N_\ell}(\mathcal{U}_n)} \tag{5.25}$$

for all $\ell \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, where $C_4 > 0$ is independent of ℓ and n. Set

$$A_\ell := C_4^{\frac{1}{N_{\ell+1}}} 2^{\frac{\ell+2}{N_{\ell+1}}} e^{M2^{-(\ell+1)}}, \quad \ell \in \mathbb{N} \cup \{0\}.$$

It follows from (5.25) that for each $n \in \mathbb{N}$,

$$\sup_{x \in \mathcal{U}_n} |\tilde{w}_n(x, t)| = \lim_{k \to \infty} \|\tilde{w}_n(\cdot, 1 - 2^{-(k+1)})\|_{L^{N_{k+1}}} \leq C_5 \|\tilde{w}_n(\cdot, t - 1)\|_{L^2(\mathcal{U}_n)},$$

where $C_5 := \prod_{\ell=0}^\infty A_\ell < \infty$. This, together with (5.21), gives (5.20) and hence, leads to (5.17).

Step 3 We rewrite the terms $T_{t-1}^* \tilde{g}$, $T_{t-1}^* \mathcal{P}_1^* \tilde{g}$ and $T_{t-1}^* \mathcal{P}_2^* \tilde{g}$ in (5.18) and then, finish the proof.

It follows from Theorem 4.1 and (5.13) that

$$T_{t-1}^* \tilde{g} = e^{-\frac{Q}{2} - \beta_0 U} \mathbb{E}^\bullet [g(X_{t-1}) \mathbb{1}_{\{t-1 < S_\Gamma\}}] = e^{-\frac{Q}{2} - \beta_0 U} \mathbb{E}^\bullet [f(X_t) \mathbb{1}_{\{t < S_\Gamma\}}], \quad \forall t \geq 1. \tag{5.26}$$

Noting that Lemma 5.2 (1) and Theorem 5.1 (2) give

$$\mathcal{P}_1^* \tilde{g} = \tilde{v}_1^* \int_{\mathcal{U}} \mathbb{E}^\bullet [f(X_1) \mathbb{1}_{\{1 < S_\Gamma\}}] d\nu_1 = \tilde{v}_1^* e^{-\lambda_1} \int_{\mathcal{U}} f d\nu_1,$$

we deduce

$$T_{t-1}^* \mathcal{P}_1^* \tilde{g} = T_{t-1}^* \tilde{v}_1^* e^{-\lambda_1} \int_{\mathcal{U}} f d\nu_1 = e^{-\lambda_1 t} \tilde{v}_1^* \int_{\mathcal{U}} f d\nu_1, \quad \forall t \geq 1. \tag{5.27}$$

Finally, we show $T_{t-1}^* \mathcal{P}_2^* \tilde{g} = T_t^* \mathcal{P}_2^* \tilde{f}$ for all $t \geq 1$. Obviously, it suffices to prove

$$\mathcal{P}_2^* \tilde{g} = T_1^* \mathcal{P}_2^* \tilde{f}, \quad \forall t \geq 1, \tag{5.28}$$

where $\tilde{f} := e^{-\frac{Q}{2} - \beta_0 U} f$. Note that the stochastic representation in Theorem 4.1 ensures $\tilde{g} = T_1^* \tilde{f}$ and hence, (5.28) if $f \in C_0^\infty(\mathcal{U})$. Thanks to the result in **Step 1** and Lemma 5.2 (2), both sides of (5.28) are well defined even when $f \in C_b(\mathcal{U})$. Then, (5.28) follows from standard approximation procedures.

Now, we finish the proof. Inserting (5.26), (5.27) and (5.28) into (5.18) yields

$$\left\| e^{-\frac{Q}{2} - \beta_0 U} \mathbb{E}^\bullet [f(X_t) \mathbb{1}_{\{t < S_\Gamma\}}] - e^{-\lambda_1 t} \tilde{v}_1^* \int_{\mathcal{U}} f d\nu_1 - T_t^* \mathcal{P}_2^* \tilde{f} \right\|_\infty \leq D_3 e^{-(\lambda_2 + \epsilon)t} \|f\|_\infty, \quad \forall t \geq 2.$$

Multiplying the above estimate by $e^{\frac{Q}{2} + \beta_0 U}$ gives rise to (5.12).

Thanks to Lemma 5.2 (3), we see that $\|T_t^* \mathcal{P}_2^* \tilde{h}\|_2 \leq e^{-(\lambda_2 - \epsilon)t} \|\tilde{h}\|_2$ for any $\tilde{h} \in L^2(\mathcal{U})$ and $t \geq 0$. Thus, it follows from $T_t^* \mathcal{P}_2^* \tilde{f} = T_{t-1}^* \mathcal{P}_2^* \tilde{g}$, (5.17) and (5.14) that for $t \geq 2$,

$$\begin{aligned} \|T_t^* \mathcal{P}_2^* \tilde{f}\|_\infty &= \|T_{t-1}^* \mathcal{P}_2^* \tilde{g}\|_\infty \leq D_2 \|T_{t-2}^* \mathcal{P}_2^* \tilde{g}\|_2 \\ &\leq D_2 e^{-(\lambda_2 - \epsilon)(t-2)} \|\tilde{g}\|_2 \leq D_1 D_2 e^{-(\lambda_2 - \epsilon)(t-2)} \|f\|_\infty, \end{aligned}$$

giving rise to (5.10). Finally, (5.11) is an immediate result of (5.12) and (5.10). □

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2 Let ν and f be as in the statement. For fixed $0 < \epsilon \ll 1$, we apply (5.12) in Lemma 5.3 to find $C_1 > 0$ (independent of f) such that

$$\left|\mathbb{E}^{\bullet}[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] - e^{\frac{Q}{2}+\beta_0 U}\left(e^{-\lambda_1 t}\tilde{v}_1^*\int_{\mathcal{U}} f\mathrm{d}\nu_1 + T_t^*\mathcal{P}_2^*\tilde{f}\right)\right| \le C_1 e^{\frac{Q}{2}+\beta_0 U}e^{-(\lambda_2+\epsilon)t}\|f\|_\infty$$

for all $t \ge 2$, where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$.

Since ν is compactly supported in $\mathcal{U}$, integrating the above inequality on $\mathcal{U}$ with respect to ν yields

$$\begin{aligned}&\left|\int_{\mathcal{U}} \mathbb{E}^{\bullet}[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}]\mathrm{d}\nu - e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu\int_{\mathcal{U}} f\mathrm{d}\nu_1 - \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu\right|\\ &\quad\le C_1 e^{-(\lambda_2+\epsilon)t}\|f\|_\infty\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu, \quad \forall t \ge 2.\end{aligned}$$

In particular, setting $f = \mathbb{1}_{\mathcal{U}}$ and $\tilde{\mathbb{1}}_{\mathcal{U}} := e^{-\frac{Q}{2}-\beta_0 U}\mathbb{1}_{\mathcal{U}}$ yields

$$\begin{aligned}&\left|\int_{\mathcal{U}} \mathbb{P}^x[t < S_\Gamma]\mathrm{d}\nu - e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu - \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu\right|\\ &\quad\le C_1 e^{-(\lambda_2+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu, \quad \forall t \ge 2.\end{aligned}$$

Since ν is compactly supported in $\mathcal{U}$, we find from (5.10) in Lemma 5.3 that

$$\lim_{t\to\infty} e^{(\lambda_1+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{(\lambda_1+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu = 0.$$

It follows that as $t \to \infty$,

$$\begin{aligned}&\frac{\int_{\mathcal{U}}\mathbb{E}^{\bullet}[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}]\mathrm{d}\nu}{\int_{\mathcal{U}}\mathbb{P}^{\bullet}[t < S_\Gamma]\mathrm{d}\nu}\\ &\quad= \frac{e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu\int_{\mathcal{U}} f\mathrm{d}\nu_1 + \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu + o(e^{-\lambda_2 t})}{e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu + \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu + o(e^{-\lambda_2 t})}\\ &\quad= \int_{\mathcal{U}} f\mathrm{d}\nu_1 + \frac{e^{\lambda_1 t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\mathrm{d}\nu + o(e^{-(\lambda_2-\lambda_1)t}),\end{aligned} \tag{5.29}$$

which together with Lemma 5.2 (3) leads to the result.

Thanks to (5.10) in Lemma 5.3, we derive

$$\left\|T_t^*\mathcal{P}_2^*\left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\right\|_\infty \le C_2 e^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \quad \forall t \ge 2,$$

which together with (5.29) and the fact ν is compactly supported yields

$$\left|\mathbb{E}^{\nu}[f(X_t)|t < S_\Gamma]\mathrm{d}\nu - \int_{\mathcal{U}} f\mathrm{d}\nu_1\right| \le C_3 e^{-(\lambda_2-\lambda_1-\epsilon)t}\|f\|_\infty.$$

The first statement in the "In particular" part follows readily from the arbitrariness of $f \in C_b(\mathcal{U})$. Due to Lemma 5.2 (5), it is not hard to deduce the second one. □

Remark 5.2 Recall from Lemma 3.5 (4) that $\tilde{v}_1^*$ is positive a.e. in $\mathcal{U}$ and the eigenfunction of $-\mathcal{L}_{\beta_0}^*$ associated with λ_1. Then, the result in **Step 2** in the proof of Lemma 5.3 implies $\tilde{v}_1^* \in L^\infty(\mathcal{U})$. Similarly, any other eigenfunctions of $-\mathcal{L}_{\beta_0}^*$ belong to $L^\infty(\mathcal{U}; \mathbb{C})$ and hence,

$T_t^{*}\mathcal{P}_2^{*}\tilde{f} \in L^{\infty}(\mathcal{U};\mathbb{C})$ where $\tilde{f}$ is as in the proof of Theorem 5.2. Consequently, it is not hard to check the proof of Theorem 5.2 to see that the conclusions apply to all initial distributions $\nu \in \mathcal{P}(\mathcal{U})$ satisfying $\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu < \infty$.

5.3 Uniqueness and Exponential Convergence

In this subsection, we study the uniqueness of QSDs of X_t as well as the conditioned dynamics of X_t for any initial distribution. The result is stated as follows. Recall that ν_1 is the QSD of X_t obtained in Theorem 5.1.

Theorem 5.3 *Assume* **(H1)-(H4)**. *Then, X_t admits a unique QSD, and for each $\nu \in \mathcal{P}(\mathcal{U})$ and $0 < \epsilon \ll 1$, there holds*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1-\epsilon)t}\left\|\mathbb{P}^{\nu}[X_t \in \bullet | t < S_\Gamma] - \nu_1\right\|_{TV} = 0.$$

We need the following result asserting that X_t comes down from infinity under **(H1)-(H4)**.

Lemma 5.4 *Assume* **(H1)**-**(H4)**. *For each $\lambda > 0$, there are $R = R(\lambda) > 0$ and $C_1 = C_1(\lambda) > 0$ such that $\mathbb{E}^x\left[e^{\lambda S_R}\right] \le C_1$ for all $x \in \mathcal{U}\setminus B_R^+$, where $S_R := \inf\left\{t \ge 0 : X_t \notin \mathcal{U}\setminus B_R^+\right\}$.*

Proof Recall from (2.5) that $U = V\circ\xi^{-1}$. Set $w := \exp\left\{-\frac{\epsilon}{U^\gamma}\right\}$, where $\gamma > 0$ is assumed to exist in **(H4)** and $\epsilon > 0$ is a parameter to be chosen. According to the assumptions on V, we can modify V on a bounded domain to make sure $\inf_{\mathcal{U}} V > 0$, while preserving the other properties. We thus assume without loss of generality that $\inf_{\mathcal{U}} V > 0$. This together with $\lim_{|z|\to\infty} V(z) = \infty$ implies

$$0 < \inf_{\mathcal{U}} w \le \sup_{\mathcal{U}} w \le 1. \tag{5.30}$$

Let C, R_* and γ be as in **(H4)**. Recall $\mathcal{L}^X = \frac{1}{2}\Delta + (p_i - q_i)\partial_i$. Straightforward calculations give $\mathcal{L}^X U = \left(\mathcal{L}^Z V\right)\circ\xi^{-1} \le -CU^{1+\gamma}$ in $\mathcal{U}\setminus\xi(B_{R_*}^+)$. It follows that

$$\begin{aligned}
\mathcal{L}^X w + \lambda w &= \frac{\epsilon\gamma w\mathcal{L}^X U}{U^{\gamma+1}} + \frac{1}{2}\left(a_i|\partial_{z_i}V|^2\right)\circ\xi^{-1}\left[-\frac{\epsilon\gamma(\gamma+1)}{U^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{U^{2\gamma+2}}\right]w + \lambda w\\
&\le (-C\epsilon\gamma + \lambda)w + \frac{1}{2}\left(a_i|\partial_{z_i}V|^2\right)\circ\xi^{-1}\left[-\frac{\epsilon\gamma(\gamma+1)}{U^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{U^{2\gamma+2}}\right]w \text{ in } \mathcal{U}\setminus\xi(B_{R_*}^+),
\end{aligned}$$

where we used **(H4)** in the inequality.

Set $\epsilon := \frac{3\lambda}{2C\gamma}$. As $\lim_{|z|\to\infty} a_i|\partial_{z_i}V|^2\left[-\frac{\epsilon\gamma(\gamma+1)}{V^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{V^{2\gamma+2}}\right] = 0$ (by **(H4)**), there is $R > 0$ such that

$$\mathcal{L}^X w + \lambda w \le -\frac{\lambda}{3}w \quad \text{in} \quad \mathcal{U}\setminus B_R^+. \tag{5.31}$$

We recall from Remark 2.2 that X_t satisfies the SDE (2.3) before hitting Γ. An application of Itô's formula gives

$$\mathrm{d}e^{\lambda t}w(X_t) = (\mathcal{L}^X w + \lambda w)(X_t)e^{\lambda t}\mathrm{d}t + \partial_i w(X_t)e^{\lambda t}\mathrm{d}W_t^i \quad \text{in} \quad \mathcal{U}.$$

It follows from (5.31) that for each $(x,t) \in (\mathcal{U}\setminus B_R^+)\times[0,\infty)$,

$$\mathbb{E}^x\left[e^{\lambda(t\wedge S_R)}w(X_{t\wedge S_R})\right] = w(x) + \mathbb{E}^x\left[\int_0^{t\wedge S_R}(\mathcal{L}^X w + \lambda w)(X_s)e^{\lambda s}\mathrm{d}s\right] \le w(x),$$

where S_R is as in the statement of the lemma. Thanks to (5.30), we pass to the limit $t \to \infty$ in the above inequality to conclude $\mathbb{E}^x\left[e^{\lambda S_R}\right] \le \frac{1}{\inf w}$ for all $x \in \mathcal{U} \setminus B_R^+$. This completes the proof. □

Remark 5.3 Since $Z_t = \xi^{-1}(X_t)$ and $\xi^{-1} : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a homeomorphism, we find from the above lemma that for each $\lambda > 0$, there exists $R = R(\lambda) > 0$ such that $\sup_{z\in\mathcal{U}\setminus B_R^+} \mathbb{E}^z[e^{\lambda T_R}] < \infty$, where $T_R := \inf\{t \ge 0 : Z_t \notin \mathcal{U} \setminus B_R^+\}$.

We next prove Theorem 5.3.

Proof of Theorem 5.3 Fix $\nu \in \mathcal{P}(\mathcal{U})$, $f \in C_b(\mathcal{U})$ and $0 < \epsilon \ll 1$. Set $\lambda := \lambda_1 + \lambda_2$. By Lemma 5.4, there exist $R_0 > 0$ and $C_1 > 0$ such that

$$\sup_{(x,t)\in(\mathcal{U}\setminus B_{R_0}^+)\times[0,\infty)} e^{\lambda t}\mathbb{P}^x\left[t < S_{R_0}\right] \le \sup_{x\in\mathcal{U}\setminus B_{R_0}^+} \mathbb{E}^x\left[e^{\lambda S_{R_0}}\right] \le C_1. \tag{5.32}$$

Clearly, the above inequality holds with $R > R_0$ replacing R_0. Choosing R_0 large enough, we may assume without loss of generality that $\nu(B_{R_0}^+) > 0$. We split

$$\mathbb{E}^\nu\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] = \int_{B_{R_0}^+} \mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu + \int_{\mathcal{U}\setminus B_{R_0}^+} \mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu, \quad \forall t \ge 0.$$

Applying (5.11) in Lemma 5.3, we find the existence of $C_2 > 0$ such that

$$\left|\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] - e^{\frac{Q}{2}+\beta_0 U} e^{-\lambda_1 t}\tilde{v}_1^* \int_{\mathcal{U}} f\mathrm{d}\nu_1\right| \le C_2 e^{\frac{Q}{2}+\beta_0 U} e^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \quad \forall t \ge 2, \tag{5.33}$$

and thus,

$$\begin{aligned} &\left|\int_{B_{R_0}^+} \mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu - A_1 e^{-\lambda_1 t}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right| \\ &\quad \le C_2 e^{-(\lambda_2-\epsilon)t}\|f\|_\infty \int_{B_{R_0}^+} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu, \quad \forall t \ge 2, \end{aligned} \tag{5.34}$$

where $A_1 := \int_{B_{R_0}^+} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu$. Note that $\tilde{v}_1^*$ is positive a.e. in $\mathcal{U}$ and $\tilde{v}_1^* \in L^\infty(\mathcal{U})$ (see Remark 5.2). Then, we see from $\nu(B_{R_0}^+) > 0$ that $0 < A_1 < \infty$.

We claim the existence of a bounded function $A_2 : [0,\infty) \to [0,\infty)$ and a $C_3 > 0$ such that

$$\left|\int_{\mathcal{U}\setminus B_{R_0}^+} \mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu - A_2(t)e^{-\lambda_1 t}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right| \le C_3 e^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \quad \forall t \gg 1. \tag{5.35}$$

This together with (5.34) leads to the existence of $C_4 > 0$ such that

$$\left|\mathbb{E}^\nu\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] - (A_1 + A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right| \le C_4 e^{-(\lambda_2-\epsilon)t}\|f\|_\infty, \quad \forall t \gg 1.$$

In particular, setting $f = \mathbb{1}_{\mathcal{U}}$ yields $\left|\mathbb{P}^\nu[t < S_\Gamma] - (A_1 + A_2(t))e^{-\lambda_1 t}\right| \le C_4 e^{-(\lambda_2-\epsilon)t}$ for all $t \gg 1$ and thus, $\mathbb{P}^\nu[t < S_\Gamma] \ge A_1 e^{-\lambda_1 t}$ for $t \gg 1$. Consequently, we deduce

$$\begin{aligned}
&\left|\mathbb{E}^\nu\left[f(X_t)\middle|t < S_\Gamma\right] - \int_{\mathcal{U}} f \mathrm{d}\nu_1\right| \\
&\quad\le \frac{1}{\mathbb{P}^\nu[t < S_\Gamma]}\left|\mathbb{E}^\nu\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] - (A_1 + A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}} f \mathrm{d}\nu_1\right| \\
&\qquad+ \frac{\int_{\mathcal{U}}|f|\mathrm{d}\nu_1}{\mathbb{P}^\nu[t < S_\Gamma]}\left|(A_1 + A_2(t))e^{-\lambda_1 t} - \mathbb{P}^\nu[t < S_\Gamma]\right| \\
&\quad\le \frac{2C_4 e^{-(\lambda_2-\epsilon)t}\|f\|_\infty}{A_1 e^{-\lambda_1 t}}, \quad \forall t \gg 1.
\end{aligned}$$

Since f is arbitrary in $C_b(\mathcal{U})$, it follows that

$$\left\|\mathbb{P}^\nu[X_t \in \bullet|t < S_\Gamma] - \nu_1\right\|_{TV} \le \frac{2C_4}{A_1}e^{-(\lambda_2-\lambda_1-\epsilon)t}, \quad \forall t \gg 1,$$

leading to the desired result.

It remains to prove (5.35). To do so, we write for $(x, t) \in (\mathcal{U} \setminus B^+_{R_0}) \times [0, \infty)$,

$$\mathbb{E}^x[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] = \mathbb{E}^x[f(X_t)\mathbb{1}_{\{t<S_{R_0}\}}] + \mathbb{E}^x[f(X_t)\mathbb{1}_{\{S_{R_0}\le t<S_\Gamma\}}] =: E_1(x, t) + E_2(x, t).$$

It follows from (5.32) that

$$\begin{aligned}
\int_{\mathcal{U}\setminus B^+_{R_0}} |E_1(\cdot, t)|\mathrm{d}\nu &\le \|f\|_\infty \sup_{x\in\mathcal{U}\setminus B^+_{R_0}} \mathbb{P}^x\left[t < S_{R_0}\right]\mathrm{d}\nu \\
&\le \|f\|_\infty e^{-\lambda t} \sup_{x\in\mathcal{U}\setminus B^+_{R_0}} \mathbb{E}^x[e^{\lambda S_{R_0}}] \le C_1\|f\|_\infty e^{-\lambda t}, \quad \forall t \ge 0.
\end{aligned} \tag{5.36}$$

To treat E_2, we set $h(x, t) := \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]$ for $(x, t) \in \overline{\mathcal{U}} \times [0, \infty)$. Obviously, $\|h\|_\infty \le \|f\|_\infty$ and $h(x, t) = 0$ for $(x, t) \in \Gamma \times [0, \infty)$. The strong Markov property and homogeneity of X_t yield that for each $(x, t) \in (\mathcal{U} \setminus B^+_{R_0}) \times [0, \infty)$,

$$\begin{aligned}
E_2(x, t) &= \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{S_{R_0}\le t<S_\Gamma\}}\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{S_{R_0}\le t<S_\Gamma\}}\middle|\mathcal{F}_{S_{R_0}}\right]\mathbb{1}_{\{S_{R_0}\le t\}}\right] \\
&= \mathbb{E}^x\left[h(X_{S_{R_0}}, t - S_{R_0})\mathbb{1}_{\{S_{R_0}\le t\}}\right] \\
&= \mathbb{E}^x\left[h(X_{S_{R_0}}, t - S_{R_0})\mathbb{1}_{\{S_{R_0}\le t\le S_{R_0}+2\}}\right] + \mathbb{E}^x\left[h(X_{S_{R_0}}, t - S_{R_0})\mathbb{1}_{\{t>S_{R_0}+2\}}\right] \\
&=: E_{21}(x, t) + E_{22}(x, t).
\end{aligned}$$

Note that (5.32) ensures

$$\begin{aligned}
\int_{\mathcal{U}\setminus B^+_{R_0}} |E_{21}(\cdot, t)|\mathrm{d}\nu &\le \|h\|_\infty \int_{\mathcal{U}\setminus B^+_{R_0}} \mathbb{P}^x[t < S_{R_0} + 2]\mathrm{d}\nu \\
&\le \|f\|_\infty e^{-\lambda t}\int_{\mathcal{U}\setminus B^+_{R_0}} \mathbb{E}^x[e^{\lambda(S_{R_0}+2)}]\mathrm{d}\nu \le C_1\|f\|_\infty e^{2\lambda-\lambda t}, \quad \forall t \ge 0.
\end{aligned} \tag{5.37}$$

Fix $0<\epsilon\ll 1$. Setting $\Phi:=\exp\left\{\frac{Q(X_{S_{R_0}})}{2}+\beta_0 U(X_{S_{R_0}})\right\}$, we see from (5.33) that on the event $\{t\geq S_{R_0}+2\}$ there holds

$$\left|h(X_{S_{R_0}},t-S_{R_0})-\Phi e^{-\lambda_1(t-S_{R_0})}\tilde{v}_1^*(X_{S_{R_0}})\int_{\mathcal{U}} f\mathrm{d}\nu_1\right|\leq C_2\Phi e^{-(\lambda_2-\epsilon)(t-S_{R_0})}\|f\|_\infty. \tag{5.38}$$

Since $S_{R_0}\leq S_\Gamma$ and $h(X_{S_{R_0}},t-S_{R_0})=0$ if $S_{R_0}=S_\Gamma$, we deduce

$$E_{22}(x,t)=\mathbb{E}^x\left[h(X_{S_{R_0}},t-S_{R_0})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right],\quad \forall (x,t)\in(\mathcal{U}\setminus B_{R_0}^+)\times[0,\infty),$$

which together with (5.38) yields

$$\begin{aligned}
&\left|\int_{\mathcal{U}\setminus B_{R_0}^+}E_{22}(\cdot,t)\mathrm{d}\nu-e^{-\lambda_1 t}\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[\Phi e^{\lambda_1 S_{R_0}}\tilde{v}_1^*(X_{S_{R_0}})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\\
&\leq C_2e^{-(\lambda_2-\epsilon)t}\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[\Phi e^{(\lambda_2-\epsilon)S_{R_0}}\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu\|f\|_\infty\\
&\leq C_2e^{-(\lambda_2-\epsilon)t}\|f\|_\infty\left(\max_{\mathcal{U}\cap\partial B_{R_0}^+}e^{\frac{Q}{2}+\beta_0U}\right)\left(\sup_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[e^{(\lambda_2-\epsilon)S_{R_0}}\right]\right)\leq C_5e^{-(\lambda_2-\epsilon)t}\|f\|_\infty
\end{aligned} \tag{5.39}$$

for all $t\geq 0$, where we used (5.32) and the fact $\max_{\mathcal{U}\cap\partial B_{R_0}^+}e^{\frac{Q}{2}+\beta_0U}<\infty$ to conclude the existence of $C_5>0$ in the last inequality.

Set

$$A_2(t):=\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[\Phi e^{\lambda_1 S_{R_0}}\tilde{v}_1^*(X_{S_{R_0}})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu,\quad \forall t\geq 0.$$

Thanks to (5.32), the boundedness of $\tilde{v}_1^*$ and the fact $|X_{S_{R_0}}|=R_0$ when $S_{R_0}<S_\Gamma$, it is clear that A_2 is non-negative and bounded. Since

$$\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{t<S_\Gamma\}}\right]\mathrm{d}\nu=\int_{\mathcal{U}\setminus B_{R_0}^+}\left[E_1(\cdot,t)+E_{21}(\cdot,t)+E_{22}(\cdot,t)\right]\mathrm{d}\nu,\quad\forall t\geq 0,$$

we deduce from (5.36), (5.37) and (5.39) that

$$\begin{aligned}
&\left|\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{t<S_\Gamma\}}\right]\mathrm{d}\nu-A_2(t)e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\\
&\leq\left[C_5e^{-(\lambda_2-\epsilon)t}+C_1(1+e^{2\lambda})e^{-\lambda t}\right]\|f\|_\infty
\end{aligned}$$

for all $t\geq 0$. Since $\lambda=\lambda_1+\lambda_2$ and $0<\epsilon\ll 1$, (5.35) follows. This completes the proof. □

5.4 Proof of Theorem A and Theorem B

Because of the fact $X_t=\xi(Z_t)$ and Proposition 2.3, conclusions in Theorem A and Theorem B follow directly from Theorem 5.1, Theorem 5.2 and Theorem 5.3.

6 Applications

In this section, we discuss a series of important applications of Theorem A and Theorem B. We first provide a general result that holds for most ecological models and then show how to apply this result to specific situations, including: stochastic Lotka-Volterra systems of competitive, predator-prey or cooperative type, systems modelled by Holling type functional responses and predator-prey systems modelled by Beddington-DeAngelis functional responses.

Consider the following stochastic system:

$$\mathrm{d}Z_t^i = Z_t^i f_i(Z_t)\mathrm{d}t + \sqrt{\gamma_i Z_t^i}\mathrm{d}W_t^i, \quad i \in \{1, \dots, d\}, \tag{6.1}$$

where $Z_t = (Z_t^i) \in \overline{\mathcal{U}}$, $\{f_i\}_i$ belong to $C^1(\overline{\mathcal{U}})$, $\{\gamma_i\}_i$ are positive constants, and $\{W^i\}_i$ are independent standard one-dimensional Wiener processes on some probability space. We make the following assumption.

(A) There exist $m \geq 0$, $0 \leq n \leq m$, $C_1, C_2, C_3, C_4 > 0$ and $R > 0$ such that

$$-C_1\left(1+\sum_{j=1}^d z_j^m\right) \leq f_i(z) \leq C_2 \mathbb{1}_{[0,R]}(z_i) - C_3 z_i^m \mathbb{1}_{(R,\infty)}(z_i) + \delta \sum_{j\neq i} z_j^n, \quad \forall z \in \overline{\mathcal{U}}, \tag{6.2}$$

and

$$|\partial_{z_i} f_i(z)| \leq C_4 |z|^{m-1}, \quad \forall z \in \mathcal{U} \setminus B_R^+, \tag{6.3}$$

for $i \in \{1, \dots, d\}$ and $\delta = 0$ if $d = 1$, $\delta \geq 0$ if $d \geq 2$ and $n < m$, or $\delta \in \left[0, \frac{C_3}{d-1}\right)$ if $d \geq 2$ and $n = m$.

Remark 6.1 Conditions (6.2) and (6.3) say that f_i and $\partial_{z_i} f_i$ are bounded above and below by simple polynomials. Conditions in the case $n < m$ tells us that the intraspecific competition dominates the interactions among species. In the case $n = m$, we can only treat weakly cooperative interactions among species – this is reflected by the smallness of δ. These are natural assumptions that can be applied to many population dynamics models: competitive Lotka-Volterra, weakly cooperative Lotka-Volterra, predator-prey Lotka-Volterra as well as more complex systems modelled by Holling type-II/III functional responses. These assumptions also allow us to use a very simple Lyapunov function $V(z) = |z|^{m+1}$ (when $|z| \geq 1$) which satisfies **(H1)**-**(H3)** and sometimes **(H4)**.

Under the assumption **(A)**, the stochastic system (6.1) generates a diffusion process Z_t that has Γ as an absorbing set. Furthermore, Z_t hits Γ in finite time almost surely.

Theorem 6.1 *Assume* **(A)**.

(1) Z_t *admits a QSD* μ_1, *and there exists* $r_1 > 0$ *such that*

- *for any* $0 < \epsilon \ll 1$ *and* $\mu \in \mathcal{P}(\mathcal{U})$ *with compact support in* $\mathcal{U}$ *one has*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t} \left\|\mathbb{P}^\mu[Z_t \in \bullet | t < T_\Gamma] - \mu_1\right\|_{TV} = 0;$$

- *there exists* $f \in C_b(\mathcal{U})$ *such that for a.e.* $x \in \mathcal{U}$, *there is a family of sets* $\{\mathcal{K}_{x,\epsilon}\}_{0<\epsilon\ll 1}$ *in* $(0, \infty)$ *satisfying* $\mathcal{K}_{x,\epsilon_2} \subset \mathcal{K}_{x,\epsilon_1}$ *for* $0 < \epsilon_1 < \epsilon_2 \ll 1$ *and* $\lim_{\epsilon\to 0} \inf_{T>0} |\mathcal{K}_{x,\epsilon} \cap (T, T+1)| = 1$ *such that*

$$\lim_{\substack{t\in\mathcal{K}_{x,\epsilon}\\ t\to\infty}} e^{(r_1+\epsilon)t} \left|\mathbb{E}^x[f(X_t) \big| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\mu_1\right| = \infty, \quad \forall 0 < \epsilon \ll 1.$$

(2) *If, in addition,* **(A)** *holds with* $m > 0$, *then* Z_t *admits a unique QSD, and for any* $0 < \epsilon \ll 1$ *and* $\mu \in \mathcal{P}(\mathcal{U})$, *there holds*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t} \left\| \mathbb{P}^{\mu}[Z_t \in \bullet | t < T_\Gamma] - \mu_1 \right\|_{TV} = 0.$$

Proof Let $m,\ C_1,\ C_2,\ C_3,\ C_4,\ R, \delta$ be as in **(A)** and $\eta \in C^\infty([0,\infty)$ be such that $\eta(t) = 0$ if $t \in [0, \frac{1}{2}]$ and $\eta = 1$ if $t \geq 1$. Set $V(z) := \eta(|z|)|z|^{m+1}$ for $z \in \mathcal{U}$ which obviously belongs to $C^2(\overline{\mathcal{U}})$. Thanks to Theorems A and B, it suffices to verity **(H1)-(H3)** when $m \geq 0$ and **(H4)** when $m > 0$.

Since $\partial_i |z|^{m+1} = (m+1)|z|^{m-1} z_i$, we deduce from **(A)** that for $|z| \geq 1$,

$$\sum_{i=1}^{d} z_i f_i \partial_i V \leq (m+1)|z|^{m-1}\left[\left(C_2 R + C_3 R^{m+1}\right)\sum_{i=1}^{d} z_i - C_3 \sum_{i=1}^{d} z_i^{m+2} + \delta \sum_{i=1}^{d}\sum_{j\neq i} z_i^2 z_j^n\right]. \tag{6.4}$$

If $d = 1$, it follows from (6.4) and $\delta = 0$ that there exists $C_5 > 0$ such that

$$z_1 f(z_1) V'(z_1) \leq -C_5 z_1^{2m+1}, \quad z_1 \gg 1. \tag{6.5}$$

In the following, we focus on $d \geq 2$. In case $m = n = 0$, there holds

$$\sum_{i=1}^{d} z_i f_i \partial_i V \leq (C_2 + C_3)R\sqrt{d} + [-C_3 + \delta(d-1)]|z| \leq -C_6|z|, \quad \forall |z| \gg 1 \tag{6.6}$$

for some $C_6 > 0$.

Now, we consider the case when $m > 0$. An application of Young's inequality yields

$$z_i^2 z_j^n \leq \frac{2\alpha}{m+2} z_i^{m+2} + \frac{m\alpha^{-\frac{2}{m}}}{m+2} z_j^{\frac{n(m+2)}{m}},$$

where $\alpha > 0$ is a parameter to be determined. Then, it follows from (6.4) that

$$\begin{aligned} \sum_{i=1}^{d} z_i f_i \partial_i V \leq{} & (m+1)(C_2 R + C_3 R^{m+1})|z|^{m-1} \sum_{i=1}^{d} z_i \\ & + \frac{\delta m(m+1)\alpha^{-\frac{2}{m}}(d-1)}{m+2}|z|^{m-1}\sum_{i=1}^{d} z_i^{\frac{n(m+2)}{m}} \\ & - (m+1)\left(C_3 - \frac{2\delta\alpha(d-1)}{m+2}\right)|z|^{m-1}\sum_{i=1}^{d} z_i^{m+2}, \quad \forall |z| \geq 1. \end{aligned} \tag{6.7}$$

We consider two cases.

- If $n < m$, we set $\alpha = \frac{(m+2)C_3}{4\delta(d-1)}$ in (6.7) (so that $C_3 - \frac{2\delta\alpha(d-1)}{m+2} = \frac{1}{2}C_3 > 0$) to find the existence of $C_6' > 0$ such that

$$\sum_{i=1}^{d} z_i f_i \partial_i V \leq -C_6' |z|^{2m+1} \quad \text{in} \quad |z| \gg 1. \tag{6.8}$$

- If $n = m$, setting $\alpha = 1$ in (6.7) and using the fact $\delta \in [0, \frac{C_3}{d-1})$, we find (6.8) holds with a possibly larger C_6'.

Considering (6.5), (6.6) and (6.8), we no longer distinguish whether $d = 1$ or not, $m = 0$ or not and assume (6.8) always holds.

Now, we verify **(H1)-(H3)**. It is easy to check that **(H1)** and **(H2)** hold. As $V \geq C(d) \sum_{i=1}^{d} z_i^{m+1}$ in $\mathcal{U}$ for some $C(d) > 0$ and $\int_1^\infty \frac{1}{s} \exp\left\{-\beta s^{m+1}\right\} \mathrm{d}s < \infty$ for any $\beta > 0$, **(H3)** (1)(2) follow from (6.8). Since

$$\begin{aligned}
\partial_i(z_i f_i) &= f_i(z) + z_i \partial_{z_i} f_i(z),\\
\gamma_i z_i \partial^2_{z_i z_i} V &= \gamma_i(m+1)(m-1)|z|^{m-3} z_i^3 + \gamma_i(m+1)|z|^{m-1} z_i,\\
\gamma_i z_i |\partial_{z_i} V|^2 &= \gamma_i(m+1)^2 |z|^{2m-2} z_i^3,
\end{aligned}$$

it is straightforward to verify **(H3)** (3)(4) by applying (6.2), (6.3) and (6.8). Hence, an application of Theorem A gives the conclusions in (1).

If $m > 0$, **(H4)** holds with $\gamma := \frac{m}{m+1}$. The conclusion in (2) follows from Theorem B. □

In the following, we apply Theorem 6.1 to various important ecological models.

Example 6.1 [Lotka-Volterra systems] For each $i \in \{1, \dots, d\}$ let

$$f_i(z) = r_i - \sum_{j=1}^{d} c_{ij} z_j, \quad z \in \overline{\mathcal{U}},$$

where $r_i \in \mathbb{R}$, $c_{ii} > 0$ and $c_{ij} \in \mathbb{R}$ for $j \neq i$.

Corollary 6.1 *Consider (6.1) with f_i, $i \in \{1, \dots, d\}$ given in Example 6.1. If $d \geq 2$, we further assume*

$$-\min_{i \neq j} c_{ij} < \frac{1}{d-1} \min_i c_{ii}. \tag{6.9}$$

Then, there exists a unique QSD of (6.1) such that the conclusions of Theorem 6.1 hold.

Proof It is straightforward to check that the assumption **(A)** with $m = n = 1$, $C_3 = \min_i c_{ii}$ and $\delta = 0$ if $d = 1$ or $\delta = \max_{i \neq j}\{-c_{ij}, 0\}$ if $d \geq 2$ is satisfied. The corollary then follows from Theorem 6.1. □

Remark 6.2 If the system is competitive, namely, $c_{ij} \geq 0$ for all $i \neq j$, then (6.9) is trivially satisfied. If the Lotka-Volterra system has either cooperation or predation, the condition (6.9) says that the intraspecific competition terms have to dominate in some sense the cooperative and the predation terms. Note that cooperative systems are known to behave poorly: see [38, Example 2.3] for details as to how a two-species stochastic cooperative system can exhibit either blow-up in finite time or have no stationary distributions.

Example 6.2 [Holling type- II / III functional response] For each $i \in \{1, \dots, d\}$,

$$f_i(z) = r_i - \sum_{j=1}^{d} \frac{c_{ij} z_j^k}{1 + z_j^k}, \quad z \in \overline{\mathcal{U}},$$

where $k \in \{1, 2\}$, $r_i \in \mathbb{R}$, $c_{ii} > 0$ and $c_{ij} \in \mathbb{R}$ for $j \neq i$. In literature, $k = 1$ and $k = 2$ correspond to Holling type- II and - III functional responses, respectively.

Corollary 6.2 *Consider (6.1) with f_i, $i \in \{1, \dots, d\}$ given in Example 6.2. Assume*

$$c_{ii} > r_i, \quad \forall i \in \{1, \dots, d\} \quad \textit{and} \quad -\min_{i \neq j} c_{ij} < \frac{1}{d-1} \min_i (c_{ii} - r_i) \ \textit{if}\ d \geq 2.$$

Then, the conclusions of Theorem 6.1 (1) hold.

Proof We verify the assumption **(A)** with $m = n = 0$. The desired result then follows from Theorem 6.1. Clearly, f_i is lower bounded. Let $0 < \gamma \ll 1$. Then, there exists $R > 0$ such that $\frac{t^k}{1+t^k} \in (1-\gamma, 1)$ for $t > R$. We compute

$$f_i = r_i - \sum_{j=1}^{d} \frac{c_{ij} z_j^k}{1+z_j^k} \le \begin{cases} r_i + (d-1) \times \max_{i \neq j}\{-c_{ij}, 0\}, & \text{if } z_i \in [0, R], \\ r_i - (1-\gamma) c_{ii} + (d-1) \times \max_{i \neq j}\{-c_{ij}, 0\}, & \text{if } z_i \in (R, \infty). \end{cases}$$

Noting that $c_{ii} > r_i$ for $d \ge 1$ and $-\min_{i \neq j} c_{ij} < \frac{1}{d-1} \min_i (c_{ii} - r_i)$ for $d \ge 2$, we derive $\sup_{z_i > R} f_i(z) < 0$ for $0 < \gamma \ll 1$ and thus, (6.2). Straightforward calculations give (6.3). This verifies **(A)**. □

Remark 6.3 For the stochastic Lotka-Volterra system with Holling type- II / III functional response considered in Example 6.2 or Corollary 6.2, the existence of a unique QSD that attracts all initial distributions supported in $\mathcal{U}$ is not expected. This is essentially due to the weak dissipativity of the system. Indeed, in the case $d = 1$, these properties are equivalent to showing that the process comes down from infinity, and therefore, according to [8, Theorem 7.3 and Proposition 7.5], equivalent to Assumption (H5) in [8]. However, it is easy to check that (H5) in [8] is not satisfied for the Holling type-II/III functional responses.

The situation in higher dimensions is worse. Even in the competitive case, the dissipativity of the system is weaker than that of the system with $f_i(z) = r_* - c_* \sum_{j=1}^{d} \frac{z_j^k}{1+z_j^k}$ for all $i \in \{1, \ldots, d\}$, where $r_* = \min_{i \in \{1,\ldots,d\}} r_i$ and $c_* = \max_{i,j \in \{1,\ldots,d\}} c_{ij}$. This latter system does not come down from infinity as it is bounded from below by a decoupled system whose individual components do not come down from infinity. In fact, we have

$$r_* - c_* \sum_{j=1}^{d} \frac{z_j^k}{1+z_j^k} \ge r_* - c_*(d-1) - c_* \frac{z_i^k}{1+z_i^k}, \quad \forall i \in \{1, \ldots, d\} \text{ and } z \in \overline{\mathcal{U}}.$$

Hence, the stochastic system in Example 6.2 or Corollary 6.2 does not come down from infinity.

We exhibit below a few more types of functional responses that can be treated by our framework.

Example 6.3 Consider the functional response

$$f_i(z) = r_i - c_{ii} z_i - \sum_{j \neq i} \frac{c_{ij} z_j^k}{1+z_j^k}, \quad z \in \overline{\mathcal{U}},$$

where $k \in \{1, 2\}$, $r_i \in \mathbb{R}$, $c_{ii} > 0$ and $c_{ij} \in \mathbb{R}$ for $j \neq i$. This is a combination of the regular intraspecific competition of the form $-c_{ii} z_i$ and Holling type functional responses for the interspecific competition/predation.

Corollary 6.3 *Consider* (6.1) *with f_i, $i \in \{1, \ldots, d\}$ given in Example 6.3. Then, there exists a unique QSD of* (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof It is straightforward to check that Assumption **(A)** holds with $m = 1$ and $n = 0$. Then, the application of Theorem 6.1 yields the conclusion. □

Example 6.4 Consider the extensively used Beddington-DeAngelis predator-prey dynamics. For each $i \in \{1, \dots, d\}$, let

$$f_i(z) = r_i - c_{ii}z_i - \sum_{j \neq i} \frac{c_{ij}z_j}{1 + \sum_{l=1}^{d} z_l}, \quad z \in \overline{\mathcal{U}},$$

where $r_i \in \mathbb{R}$, $c_{ii} > 0$, and $c_{ij} \in \mathbb{R}$ for $j \neq i$. This system was first proposed in [3, 22] in order to better explain certain predator-prey interactions.

Corollary 6.4 *Consider* (6.1) *with* f_i, $i \in \{1, \dots, d\}$ *given in Example 6.4. Then, there exists a unique QSD of* (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof It is straightforward to check that Assumption **(A)** holds with $m = 1$ and $n = 0$. Then, the application of Theorem 6.1 yields the conclusion. □

Example 6.5 Let $d = 2$. Consider the Crowley-Martin dynamics. Let

$$f_1(z) = r_1 - c_{11}z_1 - z_2\frac{z_1}{\beta + \alpha z_1 + \alpha_2 z_2 + \alpha_3 z_1 z_2}, \quad z \in \overline{\mathcal{U}},$$

$$f_2(z) = -r_2 - c_{22}z_2 + z_1\frac{z_1}{\beta + \alpha z_1 + \alpha_2 z_2 + \alpha_3 z_1 z_2}, \quad z \in \overline{\mathcal{U}},$$

where $c_{11}, c_{22}, \beta > 0$ and all the other quantities are nonnegative. This system was first proposed in [21] to study dragonflies.

Corollary 6.5 *Consider* (6.1) *in the case* $d = 2$ *with* f_1 *and* f_2 *given in Example 6.5. Assume* $\alpha > \frac{2}{3\min\{2c_{11}, c_{22}\}}$. *Then,* (6.1) *admits a unique QSD such that the conclusions of Theorem 6.1 hold.*

Proof Note that $f_1(z) \leq r_1 - c_{11}z_1$ and $f_2(z) \leq -r_2 - c_{22}z_2 + \frac{z_1}{\alpha}$. Following the arguments as in the proof of Theorem 6.1, it is straightforward to see that $V(z) := |z|^2$ for $z \in \mathcal{U}$ is a Lyapunov function satisfying **(H1)**-**(H4)**. From which, the conclusions of Theorem 6.1 hold. □

Appendix A. Proof of technical lemmas

We prove technical lemmas in this appendix.

A.1. Proof of Lemma 3.2

We need the following result.

Lemma A.1 *Assume* **(H1)**. *For each* $i \in \{1, \dots, d\}$, $\lim_{x_i \to 0} x_i^2 \left[q_i^2(x_i) - q_i'(x_i)\right] = C_i > 0$.

Proof Recall that $q_i(x_i) = \frac{a_i'(\xi_i^{-1}(x_i))}{4\sqrt{a_i(\xi_i^{-1}(x_i))}}$. Then, $q_i'(x_i) = \frac{1}{4}a_i''(\xi_i^{-1}(x_i)) - \frac{|a_i'|^2(\xi_i^{-1}(x_i))}{8a_i(\xi_i^{-1}(x_i))}$, resulting in

$$(q_i^2 - q_i')(x_i) = \frac{3|a_i'|^2(\xi_i^{-1}(x_i))}{16a_i(\xi_i^{-1}(x_i))} - \frac{1}{4}a_i''(\xi_i^{-1}(x_i)). \tag{A.1}$$

Since $\xi_i^{-1} \in C([0,\infty))$ and $\xi_i^{-1}(0) = 0$, we see from **(H1)** that $\lim_{x_i\to 0} a_i'(\xi_i^{-1}(x_i)) = a_i'(0) > 0$ and $\lim_{x_i\to 0} a_i''(\xi_i^{-1}(x_i)) = a_i''(0)$. Hence, $\left(q_i^2 - q_i'\right)(x_i) \sim \frac{3|a_i'|^2(0)}{16a_i(\xi_i^{-1}(x_i))} - \frac{1}{4}a_i''(0)$ as $x_i \to 0$. The conclusion follows if there is $C > 0$ such that

$$a_i(\xi_i^{-1}(x_i)) \sim Cx_i^2 \quad \text{as} \quad x_i \to 0. \tag{A.2}$$

We show that (A.2) holds with $C = \frac{|a_i'(0)|^2}{4}$. The assumption **(H1)** and Taylor's expansion give

$$a_i(z_i) \sim a_i'(0)z_i + o(z_i^2) \quad \text{as} \quad x_i \to 0, \tag{A.3}$$

leading to $\xi_i(z_i) = \int_0^{z_i} \frac{ds}{\sqrt{a_i'(0)s+o(s^2)}} \sim \frac{2\sqrt{z_i}}{\sqrt{a_i'(0)}}$ as $z_i \to 0$. Thus, $\xi_i^{-1}(x_i) \sim \frac{a_i'(0)x_i^2}{4}$ as $x_i \to 0$. Inserting this into (A.3) yields (A.2) with $C = \frac{|a_i'(0)|^2}{4}$. This completes the proof. □

Remark A.1 Thanks to (A.2), it is easy to see from the definition of Q given in (2.6) that $Q(x)$ behaves like $\sum_{i=1}^d \ln x_i$ as x approaches to Γ. Hence, $e^{-\frac{Q}{2}}$ is as singular as $\prod_{i=1}^d \frac{1}{\sqrt{x_i}}$ near Γ.

Proof of Lemma 3.2 We first prove (1). Recall that U is given in (2.5). Clearly,

$$\partial_{x_i} U(x) = \partial_{z_i} V(\xi^{-1}(x))\sqrt{a_i(\xi_i^{-1}(x_i))}, \quad \forall x \in \mathcal{U}.$$

We derive from **(H3)** (4) the existence of $C_1 > 0$ and $R_1 > 0$ such that

$$\left(|\nabla U|^2 + |p|^2\right)(x) \le -C_1(b\cdot\nabla_z V)(\xi^{-1}(x)) \le C_1\alpha(x), \quad \forall x \in \mathcal{U}\setminus B_{R_1}^+.$$

Since $\sup_{B_{R_1}^+}\left(|\nabla U|^2 + |p|^2\right) < \infty$ due to **(H2)** and **(H3)**(1) and $\inf_{\mathcal{U}} \alpha > 0$, there must exist some $C_2 > 0$ such that $(|\nabla U|^2 + |p|^2) < C_2\alpha$ in $B_{R_1}^+$. Setting $C := \min\{C_1, C_2\}$ yields the result.

The rest of the proof is arranged as follows. In **Step 1**, we analyze the asymptotic behaviors of terms in $e_{\beta,N}$ near the boundary Γ and in the vicinity of infinity. Based on these, the asymptotic behaviors of $e_{\beta,N}$ are derived in **Step 2**. The proof of (2) and (3) are respectively given in **Step 3** and **Step 4**. Recall that R_0 and δ_0 are fixed in Subsection 3.1 when defining α.

Step 1 We analyze the asymptotic behaviors of terms in $e_{\beta,N}$.

- For $p\cdot\nabla U$, we see from **(H3)** (1) that

$$(p\cdot\nabla U)(x) = (b\cdot\nabla V)(\xi^{-1}(x)) \to -\infty \quad \text{as} \quad |x| \to \infty. \tag{A.4}$$

- For $\frac{1}{2}\sum_{i=1}^d (q_i^2 - q_i')$, Lemma A.1 ensures the existence of $\delta_* \in (0, \delta_0)$ and $C_3, C_4 > 0$ such that

$$\frac{C_3}{x_i^2} \le \frac{1}{2}(q_i^2 - q_i')(x_i) \le \frac{C_4}{x_i^2}, \quad \forall x_i \in (0, \delta_*] \text{ and } i \in \{1,\dots,d\}. \tag{A.5}$$

  Since **(H1)** gives $\limsup_{s\to\infty}\left(\frac{|a_i'(s)|^2}{a_i(s)} + a_i''(s)\right) < \infty$, we find from (A.1) and (A.4) that for any $0 < \epsilon_1 \ll 1$, there exists $R_2 = R_2(\epsilon_1) > 0$ such that

$$\frac{1}{2}|q_i^2 - q_i'|(x_i) \le -\frac{\epsilon_1}{d}(p\cdot\nabla U)(x), \quad \forall x \in \{x \in \mathcal{U} : x_i \in (R_2,\infty)\} \text{ and } i \in \{1,\dots,d\}. \tag{A.6}$$

- For ΔU, $p \cdot q$ and $\nabla \cdot p$, we calculate

$$\partial^2_{x_i x_i} U(x) = \left[\partial^2_{z_i z_i} V(\xi^{-1}(x)) a_i(\xi_i^{-1}(x_i)) + \frac{1}{2}\partial_{z_i} V(\xi^{-1}(x)) a_i'(\xi_i^{-1}(x_i))\right],$$

$$p_i(x) q_i(x_i) = \frac{b_i(\xi^{-1}(x)) a_i'(\xi_i^{-1}(x_i))}{4 a_i(\xi_i^{-1}(x_i))},$$

$$\partial_{x_i} p_i(x) = \partial_{z_i} b_i(\xi^{-1}(x)) - \frac{b_i(\xi^{-1}(x)) a_i'(\xi_i^{-1}(x_i))}{2 a_i(\xi_i^{-1}(x_i))}.$$

By **(H1)-(H3)**, we have $U \in C^2(\overline{\mathcal{U}})$, and $p \cdot q, \nabla \cdot p \in C(\overline{\mathcal{U}})$. Moreover, **(H3)**(3) and (A.4) guarantee that for any $0 < \epsilon_2 \ll 1$, there exists $R_3 = R_3(\epsilon_2) > 0$ such that

$$|\Delta U| + |p \cdot q| + |\nabla \cdot p| \le -\epsilon_2 p \cdot \nabla U \quad \text{in} \quad \mathcal{U} \setminus B^+_{R_3}. \tag{A.7}$$

- For $\frac{1}{2}|\nabla U|^2$, we find from $|\nabla U|^2(x) = \sum_{i=1}^d |\partial_{z_i} V|^2(\xi^{-1}(x)) a_i(\xi_i^{-1}(x_i))$, **(H3)**(4) and (A.4) that there are $C_5 > 0$ and $R_4 > 0$ such that

$$\frac{1}{2}|\nabla U|^2 \le -C_5(p \cdot \nabla U) \quad \text{in} \quad \mathcal{U} \setminus B^+_{R_4}. \tag{A.8}$$

Step 2 We analyze the asymptotic behaviors of $e_{\beta,N}$ near Γ and in the vicinity of infinity.

Set $R_* := \max\{R_0, R_2, R_3, R_4\}$ and $C_6 := \frac{1}{2}\max_i \max_{x_i \in [\delta_*, R_*]} |q_i^2 - q_i'|(x_i)$. Obviously, R_* and C_6 depend on ϵ_1 and ϵ_2, which are to be determined in the proof of (3). Since α is piecewise defined, we analyze $e_{\beta,N}$ in four subdomains: $\Gamma_{\delta_*} \cap B^+_{R_*}$, $\Gamma_{\delta_*} \cap (\mathcal{U} \setminus B^+_{R_*})$, $(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}$ and $(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap (\mathcal{U} \setminus B^+_{R_*})$ separately, where we recall $\Gamma_{\delta_*} := \{x \in \mathcal{U} : x_i \le \delta_* \text{ for some } i \in \{1, \dots, d\}\}$. For simplicity, we set

$$\Psi := \frac{\beta}{2}|\Delta U| + \frac{\beta^2}{2}|\nabla U|^2 + \beta|p \cdot \nabla U| + |p \cdot q| + |\nabla \cdot p|.$$

(a) In $\Gamma_{\delta_*} \cap B^+_{R_*}$. We see from $U \in C^2(\overline{\mathcal{U}})$ and $p \cdot \nabla U, p \cdot q, \nabla \cdot p \in C(\overline{\mathcal{U}})$ that $\max_{\Gamma_{\delta_*} \cap B^+_{R_*}} \Psi < \infty$. It follows from (A.5) that

$$|e_{\beta,N}| \le C_4 \sum_{i=1}^d \frac{1}{x_i^2} + dC_6 + \max_{\Gamma_{\delta_*} \cap B^+_{R_*}} \Psi,$$

$$e_{\beta,N} \ge C_3 \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - d\left(\frac{C_3}{\delta_*^2} + C_6\right) - \max_{\Gamma_{\delta_*} \cap B^+_{R_*}} \Psi.$$

(b) In $\Gamma_{\delta_*} \cap (\mathcal{U} \setminus B^+_{R_*})$. It follows from (A.5), (A.7) and (A.8) that

$$|e_{\beta,N}| \le C_4 \sum_{i=1}^d \frac{1}{x_i^2} + dC_6 - \left(\beta + \epsilon_2(1 + \frac{\beta}{2}) + C_5\beta^2\right) p \cdot \nabla U,$$

$$e_{\beta,N} \ge C_3 \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - d\left(\frac{C_3}{\delta_*^2} + C_6\right) - \left(\beta - \epsilon_2(1 + \frac{\beta}{2}) - C_5\beta^2\right) p \cdot \nabla U.$$

(c) In $(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}$. There hold

$$|e_{\beta,N}| \le \max_{(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}} \left[\Psi + \frac{1}{2}\sum_{i=1}^d |q_i^2 - q_i'|\right], \quad e_{\beta,N} \ge -\max_{(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}} \left[\Psi + \frac{1}{2}\sum_{i=1}^d |q_i^2 - q_i'|\right].$$

(d) In $(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap (\mathcal{U} \setminus B^+_{R_*})$. It follows from (A.6), (A.7) and (A.8) that

$$|e_{\beta,N}| \le dC_6 - \left(\beta + \epsilon_1 + \epsilon_2(1 + \frac{\beta}{2}) + C_5\beta^2\right) p \cdot \nabla U,$$
$$e_{\beta,N} \ge -dC_6 - \left(\beta - \epsilon_1 - \epsilon_2(1 + \frac{\beta}{2}) - C_5\beta^2\right) p \cdot \nabla U.$$

Step 3 We prove (2). As $\alpha \ge \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\}$ in Γ_{δ_*} and $\inf_{\mathcal{U}} \alpha > 0$, we deduce from **Step 2** (a) the existence of $D_1(\beta) > 0$ such that $e_{\beta,N} \le D_1(\beta)\alpha$ in $\Gamma_{\delta_*} \cap B^+_{R_*}$ for all $N \ge 1$.

Since $\inf_{\mathcal{U}} \alpha > 0$ and

$$\alpha = \begin{cases} \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - p \cdot \nabla U & \text{in } \Gamma_{\delta_*} \cap (\mathcal{U} \setminus B^+_{R_*}), \\ -p \cdot \nabla U & \text{in } (\mathcal{U} \setminus \Gamma_{\delta_*}) \cap (\mathcal{U} \setminus B^+_{R_*}), \end{cases}$$

Step 2 (b)(d) ensures the existence of $D_2(\beta) > 0$ such that $|e_{\beta,N}| \le D_2(\beta)\alpha$ in $\mathcal{U} \setminus B^+_{R_*}$ for all $N \ge 1$.

Thanks to $\inf_{\mathcal{U}} \alpha > 0$, it follows from **Step 2** (c) the existence of $D_3(\beta) > 0$ such that $|e_{\beta,N}| \le D_3(\beta)\alpha$ in $(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}$ for all $N \ge 1$. Setting $C(\beta) := \max\{D_1(\beta), D_2(\beta), D_3(\beta)\}$ yields (2).

Step 4 We show (3). Setting $\beta_0 := \frac{1}{2C_5}$, $\epsilon_1 := \min\left\{1, \frac{1}{16C_5}\right\}$ and $\epsilon_2 := \min\left\{1, \frac{1}{2+8C_5}\right\}$, we deduce from **Step 2** (b)(d) that

$$\begin{aligned} e_{\beta_0,N} &\ge C_3 \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - d\left(\frac{C_3}{\delta_*^2} + C_6\right) - \left(\beta_0 - \epsilon_2(1 + \frac{\beta_0}{2}) - C_5\beta_0^2\right) p \cdot \nabla U \\ &\ge \min\left\{C_3, \frac{1}{8C_5}\right\}\alpha - d\left(\frac{C_3}{\delta_*^2} + C_6\right) \quad \text{in } \Gamma_{\delta_*} \cap (\mathcal{U} \setminus B^+_{R_*}) \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} e_{\beta_0,N} &\ge -dC_6 - \left(\beta_0 - \epsilon_1 - \epsilon_2(1 + \frac{\beta_0}{2}) - C_5\beta_0^2\right) p \cdot \nabla U \\ &\ge \frac{1}{16C_5}\alpha - dC_6 \quad \text{in } (\mathcal{U} \setminus \Gamma_{\delta_*}) \cap (\mathcal{U} \setminus B^+_{R_*}). \end{aligned} \tag{A.10}$$

Since $\alpha \le \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} + \max_{\Gamma_{\delta_*} \cap B^+_{R_*}} |p \cdot \nabla U|$ in $\Gamma_{\delta_*} \cap B^+_{R_*}$ and $\sup_{(\mathcal{U} \setminus \Gamma_{\delta_*}) \cap B^+_{R_*}} \alpha < \infty$, we conclude from (a) and (c) the existence of positive constants C_7 and $M > d\left(\frac{C_3}{\delta_*^2} + C_6\right)$ such that $e_{\beta_0,N} + M \ge C_7\alpha$ in $B^+_{R_*}$ for all $N \ge 1$, which together with (A.9) and (A.10) implies that $e_{\beta_0,N} + M \ge C_*\alpha$ in $\mathcal{U}$ for all $N \ge 1$, where $C_* := \min\{C_3, \frac{1}{16C_5}, C_7\}$. This proves (3), and completes the proof.

A.2 Proof of Lemma 4.3

Suppose $\tilde{w} \in C(\mathcal{U} \times [0, \infty)) \cap L^2([0, \infty), \mathcal{H}^1(\mathcal{U}))$ is a weak solution of (4.6). The proof is broken into two steps.

Step 1 We show

$$\frac{1}{2}\int_{\mathcal{U}}\tilde{w}^2(\cdot,t)\mathrm{d}x+\frac{1}{2}\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathcal{U}}e_{\beta_0,2}\tilde{w}^2\mathrm{d}x\mathrm{d}s=\frac{1}{2}\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x,\quad\forall t\in[0,\infty). \tag{A.11}$$

The idea of proving (A.11) is based on the classical "energy method". But, we have to deal with the fact that $\tilde{w}$ lacks the differentiability in t. For each $0<h\ll 1$, we define

$$\tilde{w}_h(x,t):=\frac{1}{h}\int_t^{t+h}\tilde{w}(x,s)\mathrm{d}s,\quad (x,t)\in\mathcal{U}\times[0,\infty).$$

Obviously, $\tilde{w}_h\in C(\mathcal{U}\times[0,\infty))\cap L^2([0,\infty),\mathcal{H}^1(\mathcal{U}))$ and $\partial_t\tilde{w}_h\in L^2(\mathcal{U}\times[0,T])$ for each $T>0$. It is easy to verify that $\tilde{w}_h$ is a weak solution of (4.6) with $\tilde{f}$ replaced by $\tilde{f}_h:=\tilde{w}_h(\cdot,0)=\frac{1}{h}\int_0^h\tilde{w}_h(\cdot,s)\mathrm{d}s$. Namely, for each $t\in[0,\infty)$ and $\phi\in C_0^{1,1}(\mathcal{U}\times[0,\infty))$, one has

$$\begin{aligned}&\int_{\mathcal{U}}\tilde{w}_h(\cdot,t)\phi(\cdot,t)\mathrm{d}x-\int_{\mathcal{U}}\tilde{f}_h\phi(\cdot,0)\mathrm{d}x-\int_0^t\int_{\mathcal{U}}\tilde{w}_h\partial_t\phi\mathrm{d}x\mathrm{d}s\\&\quad=-\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}_h\cdot\nabla\phi\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}_h\nabla\phi\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}e_{\beta_0}\tilde{w}_h\phi\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{A.12}$$

Let $\{\eta_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathcal{U})$ be a sequence of functions taking values in $[0,1]$ and satisfying

$$\eta_n(x)=\begin{cases}1, & x\in\left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\cap B^+_{\frac{n}{2}},\\ 0, & x\in\Gamma_{\frac{1}{n}}\bigcup\left(\mathcal{U}\setminus B^+_n\right),\end{cases}\quad\text{and}\quad|\nabla\eta_n(x)|\le\begin{cases}2n, & x\in\Gamma_{\frac{2}{n}}\setminus\Gamma_{\frac{1}{n}},\\ 4, & x\in\left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\cap\left(B^+_n\setminus B^+_{\frac{n}{2}}\right).\end{cases}$$

By standard approximation arguments, we deduce that (A.12) holds with ϕ replaced by $\eta_n^2\tilde{w}_h$. Moreover, integration by parts shows that the left hand side of (A.12) with ϕ replaced by $\eta_n^2\tilde{w}_h$ equals $\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}_h^2)(\cdot,t)\mathrm{d}x-\frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}_h^2\mathrm{d}x$. Thus, we find for each $t\in[0,\infty)$, $n\in\mathbb{N}$ and $0<h\ll 1$,

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}_h^2)(\cdot,t)\mathrm{d}x-\frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}_h^2\mathrm{d}x\\&\quad=-\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}_h\cdot\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}_h\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s\\&\qquad-\int_0^t\int_{\mathcal{U}}e_{\beta_0}\eta_n^2\tilde{w}_h^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{A.13}$$

We claim that passing to the limit $h\to 0$ in (A.13) yields that for each $t\in[0,\infty)$ and $n\in\mathbb{N}$,

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}^2)(\cdot,t)\mathrm{d}x-\frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}^2\mathrm{d}x\\&\quad=-\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}\cdot\nabla(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}\nabla(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s\\&\qquad-\int_0^t\int_{\mathcal{U}}e_{\beta_0}\eta_n^2\tilde{w}^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{A.14}$$

Assuming (A.14), we conclude (A.11) from letting $n\to\infty$ in (A.14) and arguments as in the proof of Lemma 3.3 (2).

It remains to justify (A.14). For notational simplicity, we rewrite (A.13) and (A.14) as

$$I_0(h) = I_1(h) + I_2(h) + I_3(h) \quad \text{and} \quad I_0 = I_1 + I_2 + I_3,$$

respectively, and show that $\lim_{h\to 0} I_i(h) = I_i$ for $i = 0, 1, 2, 3$.

Fix $t \in [0, \infty)$ and $n \in \mathbb{N}$. Note for each $0 < h \ll 1$,

$$\tilde{w}_h(\cdot, t) - \tilde{w}(\cdot, t) = \int_0^1 [\tilde{w}(\cdot, t + hs) - \tilde{w}(\cdot, t)]\mathrm{d}s, \quad \tilde{f}_h - \tilde{f} = \int_0^1 [\tilde{w}(\cdot, hs) - \tilde{f}]\mathrm{d}s.$$

Since $\tilde{w} \in C(\mathcal{U} \times [0, \infty))$, we find for each compact set $K \subset \mathcal{U}$,

$$\sup_{K\times[0,t]} |\tilde{w}_h - \tilde{w}| \to 0 \quad \text{as} \quad h \to 0 \tag{A.15}$$

and $\sup_K |\tilde{f}_h - \tilde{f}| \to 0$ as $h \to 0$. It follows that $\lim_{h\to 0} I_0(h) = I_0$ and $\lim_{h\to 0} I_3(h) = I_3$.

We claim that

$$\lim_{h\to 0} \int_0^t \int_{\mathcal{U}} |\nabla \tilde{w}_h - \nabla \tilde{w}|^2 \mathrm{d}x\mathrm{d}s = 0. \tag{A.16}$$

Since $\nabla \tilde{w}_h(\cdot, t) - \nabla \tilde{w}(\cdot, t) = \int_0^1 [\nabla \tilde{w}(\cdot, t + hs) - \nabla \tilde{w}(\cdot, t)]\mathrm{d}s$, we find

$$\begin{aligned}\int_0^t \int_{\mathcal{U}} |\nabla \tilde{w}_h - \nabla \tilde{w}|^2 \mathrm{d}x\mathrm{d}t' &\le \int_0^t \int_{\mathcal{U}} \int_0^1 |\nabla \tilde{w}(x, t' + hs) - \nabla \tilde{w}(x, t')|^2 \mathrm{d}s\mathrm{d}x\mathrm{d}t' \\ &\le \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\nabla \tilde{w}(x, t' + s) - \nabla \tilde{w}(x, t')|^2 \mathrm{d}x\mathrm{d}t',\end{aligned} \tag{A.17}$$

where we used Fubini's theorem before taking the supremum.

Since $\nabla \tilde{w} \in L^2(\mathcal{U} \times [0, 2t])$ and $C_0(\mathcal{U} \times [0, 2t])$ is dense in $L^2(\mathcal{U} \times [0, 2t])$, for each $\epsilon > 0$, we could find some $\Phi \in C_0(\mathcal{U} \times [0, 2t])$ such that $\|\Phi - \nabla \tilde{w}\|_{L^2(\mathcal{U}\times[0,2t])} < \epsilon$. Obviously, Φ is uniformly continuous on $\mathcal{U} \times [0, 2t]$, resulting in $\sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\Phi(x, t' + s) - \Phi(x, t')|^2 \mathrm{d}x\mathrm{d}t' \to 0$ as $h \to 0$. Therefore,

$$\begin{aligned}&\frac{1}{3} \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\nabla \tilde{w}(x, t' + s) - \nabla \tilde{w}(x, t')|^2 \mathrm{d}x\mathrm{d}t' \\ &\quad \le \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\nabla \tilde{w}(x, t' + s) - \Phi(x, t' + s)|^2 \mathrm{d}x\mathrm{d}t' \\ &\quad + \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\Phi(x, t' + s) - \Phi(x, t')|^2 \mathrm{d}x\mathrm{d}t' \\ &\quad + \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\Phi(x, t') - \nabla \tilde{w}(x, t')|^2 \mathrm{d}x\mathrm{d}t' \\ &\le 2\epsilon + \sup_{s\in[0,h]} \int_0^t \int_{\mathcal{U}} |\Phi(x, t' + s) - \Phi(x, t')|^2 \mathrm{d}x\mathrm{d}t'.\end{aligned}$$

Letting $h \to 0$ in the above estimates, we find (A.16) from the arbitrariness of $\epsilon > 0$ and (A.17).

It follows readily from (A.16) that $\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}_h|^2\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s$. Since

$$\begin{aligned}&\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s\\&\quad=\int_0^t\int_{\mathcal{U}}\eta_n(\tilde{w}_h-\tilde{w})\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}(\nabla\tilde{w}_h-\nabla\tilde{w})\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s,\end{aligned}$$

we apply Hölder's inequality to deduce from (A.15) and (A.16) that

$$\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\lim_{h\to 0}-2I_1(h)&=\lim_{h\to 0}\left(\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}_h|^2\mathrm{d}x\mathrm{d}s+2\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s\right)\\&=\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s+2\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s=-2I_1.\end{aligned}$$

Similar arguments yield $\lim_{h\to 0}I_2(h)=I_2$. Hence, letting $h\to 0$ in (A.13) yields (A.14).

Step 2 We show that $\int_{\mathcal{U}}\tilde{w}^2(\cdot,t)\mathrm{d}x\le\frac{e^{2Mt}}{M}\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x$ for all $t\in[0,\infty)$. Hence, $\tilde{w}=0$ if $\tilde{f}=0$. This proves the lemma.

As $e_{\beta_0,2}+M\ge 0$ by Lemma 3.2 (3), we derive from (A.11) that

$$\frac{1}{2}\int_{\mathcal{U}}\tilde{w}^2(\cdot,t)\mathrm{d}x\le M\int_0^t\int_{\mathcal{U}}\tilde{w}^2\mathrm{d}x\mathrm{d}s+\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x,\quad\forall t\in[0,\infty).\tag{A.18}$$

Setting $g(t)=\int_0^t\int_{\mathcal{U}}\tilde{w}^2\mathrm{d}x\mathrm{d}s$ for $t\in[0,\infty)$, we arrive at $\frac{1}{2}g'(t)\le Mg(t)+\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x$ for all $t\in[0,\infty)$. The conclusion then follows from Gronwall's inequality. □

Appendix B. A formal derivation of the stochastic Lotka-Volterra competition system

In this section, we formally derive the stochastic Lotka-Volterra competition system (1.1) as the diffusion approximation of a class of multi-type birth-and-death processes. Denote by $\mathbb{N}_0$ the set of non-negative integers and by e_i, $i\in\{1,\dots,d\}$ the canonical basis of $\mathbb{R}^d$.

For each large $K\in\mathbb{N}_0$, let $X_t^{(K)}$ be a continuous-time birth-and-death process on $\mathbb{N}_0^d$ with birth and death rates as follows: for each $i\in\{1,\dots,d\}$,

$$\begin{aligned}&\text{(birth) } x\longrightarrow x+e_i\quad\text{at rate}\quad r_{i,+}^{(K)}(x):=\left(\gamma_i+\frac{\lambda_i}{K}\right)x_i,\\&\text{(death) } x\longrightarrow x-e_i\quad\text{at rate}\quad r_{i,-}^{(K)}(x):=\left(\gamma_i+\frac{\mu_i}{K}\right)x_i+\frac{x_i}{K^2}\sum_{j=1}^d c_{ij}x_j,\end{aligned}$$

where $\gamma_i>0$, $\lambda_i>0$, $\mu_i\ge 0$, $c_{ii}>0$, and $c_{ij}\ge 0$. The case $c_{ij}\le 0$ $(i\ne j)$ is also allowed provided it is sufficiently small relative to c_{ii}.

Consider the spatio-temporal rescaled process

$$Z_t^{(K)}:=\frac{X_{Kt}^{(K)}}{K}\quad\text{on}\quad\left(\frac{\mathbb{N}_0}{K}\right)^d.$$

Its generator $\mathcal{L}^{(K)}$ is given as follows: for $f \in C_c^2([0,\infty)^d)$ and $z = \frac{x}{K}$,

$$\mathcal{L}^{(K)} f(z) = K \sum_{i=1}^{d} r_{i,+}^{(K)}(Kz) \left[f\left(z + \frac{e_i}{K}\right) - f(z) \right] + K \sum_{i=1}^{d} r_{i,-}^{(K)}(Kz) \left[f\left(z - \frac{e_i}{K}\right) - f(z) \right].$$

Using the second-order Taylor expansion

$$f\left(z \pm \frac{e_i}{K}\right) = f(z) \pm \frac{1}{K} \partial_{z_i} f(z) + \frac{1}{2K^2} \partial^2_{z_i z_i} f(z) + o\left(\frac{1}{K^2}\right),$$

we arrive at

$$\mathcal{L}^{(K)} f(z) = \mathcal{L} f(z) + o(1),$$

where

$$\mathcal{L} f(z) = \sum_{i=1}^{d} \gamma_i z_i \partial^2_{z_i z_i} f(z) + \sum_{i=1}^{d} z_i \left(\lambda_i - \mu_i - \sum_{j=1}^{d} c_{ij} z_j \right) \partial_{z_i} f(z).$$

Clearly, the operator $\mathcal{L}$ is the generator of the following stochastic Lotka-Volterra system

$$\mathrm{d}Z_t^i = Z_t^i \left(\lambda_i - \mu_i - \sum_{j=1}^{d} c_{ij} Z_t^j \right) \mathrm{d}t + \sqrt{2\gamma_i Z_t^i} \mathrm{d}W_t^i, \quad i \in \{1, \ldots, d\}. \tag{B.1}$$

The following result is more or less standard.

Theorem B.1 ([1, 26]) *Let Z_0 be a $[0,\infty)^d$-valued square-integrable random variable and Z_t be the unique solution of (B.1) with initial condition Z_0. If $Z_0^{(K)}$ converges in law to Z_0, then $\left\{ Z_t^{(K)}, t \in [0,T] \right\}$ converges in $D([0,T]; [0,\infty)^d)$ to $\{Z_t, t \in [0,T]\}$.*

Author Contributions All the authors contribute equally.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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