Euro. Jnl of Applied Mathematics (2020), *vol.* 31, *pp.* 601–645
doi:10.1017/S0956792519000202

Existence, uniqueness and stability of transition fronts of non-local equations in time heterogeneous bistable media†

WENXIAN SHEN[1] and ZHONGWEI SHEN[2]

[1] *Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA*
e-mail: wenxish@auburn.edu
[2] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada*
e-mail: zhongwei@ualberta.ca

(*Received 20 February 2018*; *revised 22 November 2018*; *accepted 5 August 2019*;
first published online 28 August 2019)

The present paper is devoted to the study of the existence, the uniqueness and the stability of transition fronts of non-local dispersal equations in time heterogeneous media of bistable type under the unbalanced condition. We first study space non-increasing transition fronts and prove various important qualitative properties, including uniform steepness, stability, uniform stability and exponential decaying estimates. Then, we show that any transition front, after certain space shift, coincides with a space non-increasing transition front (if it exists), which implies the uniqueness, up-to-space shifts and monotonicity of transition fronts provided that a space non-increasing transition front exists. Moreover, we show that a transition front must be a periodic travelling front in periodic media and asymptotic speeds of transition fronts exist in uniquely ergodic media. Finally, we prove the existence of space non-increasing transition fronts, whose proof does not need the unbalanced condition.

Key words: transition front, non-local dispersal equation, bistable, time heterogeneous media

2010 Mathematics Subject Classification: Primary: 35C07, 35K55, 35K57; Secondary: 92D25

1 Introduction

This present paper is devoted to the investigation of transition fronts of the following non-local dispersal equation in time heterogeneous media

$$\partial_t u = J * u - u + f(t, u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $u(t, x)$ is an unknown density function, J is a symmetric dispersal kernel function, $[J * u](t, x) = \int_{\mathbb{R}} J(x - y)u(t, y)dy$ and f is a bistable-type non-linearity. More precisely, we assume that J and f satisfy (H1)–(H3) stated in the following.

(H1) $J : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $J \not\equiv 0$, $J(x) = J(-x) \geq 0$ for $x \in \mathbb{R}$, $\int_{\mathbb{R}} J(x)dx = 1$ and $\int_{\mathbb{R}} J(x)e^{\gamma x}dx < \infty$ for some $\gamma > 0$.

† Z. Shen is supported by a start-up grant from the University of Alberta and an NSERC discovery grant.

(H2) There exist C^2 functions $f_B : \mathbb{R} \to \mathbb{R}$ and $f_{\tilde{B}} : \mathbb{R} \to \mathbb{R}$ such that

$$f_B(u) \le f(t,u) \le f_{\tilde{B}}(u), \quad (t,u) \in \mathbb{R} \times [0,1].$$

Moreover, the following conditions hold:
- $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and satisfies

$$\sup_{(t,u)\in\mathbb{R}\times[-1,2]} (|\partial_t f(t,u)| + |\partial_u f(t,u)|) < \infty;$$

- f_B is of standard bistable type, that is, $f_B(0) = f_B(\theta) = f_B(1) = 0$ for some $\theta \in (0,1)$, $f_B(u) < 0$ for $u \in (0,\theta)$, $f_B(u) > 0$ for $u \in (\theta,1)$ and satisfies the unbalanced condition

$$\text{the speed of travelling fronts of } \partial_t u = J * u - u + f_B(u) \text{ is positive;} \tag{1.2}$$

- $f_{\tilde{B}}$ is also of standard bistable type, that is, $f_{\tilde{B}}(0) = f_{\tilde{B}}(\tilde{\theta}) = f_{\tilde{B}}(1) = 0$ for some $\tilde{\theta} \in (0,1)$, $f_{\tilde{B}}(u) < 0$ for $u \in (0,\tilde{\theta})$ and $f_{\tilde{B}}(u) > 0$ for $u \in (\tilde{\theta},1)$.

We remark that (H2) implies that $f(t,0) = 0 = f(t,1)$ for all $t \in \mathbb{R}$; that is, $u \equiv 0$ and $u \equiv 1$ are two constant solutions of (1.1), and the speed of travelling fronts of

$$\partial_t u = J * u - u + f_B(u) \tag{1.3}$$

is unique, and travelling fronts of (1.3) are unique up to shifts (see [8]). Here, by travelling fronts of (1.3), we mean global-in-time solutions of the form $\phi_B(x - c_B t)$ with $\phi_B(-\infty) = 1$ and $\phi_B(\infty) = 0$. Moreover, the unbalanced condition (1.2) is equivalent to the speed of travelling fronts of (1.3) being non-zero and $\int_0^1 f_B(u)du > 0$. We point out that the condition $\int_0^1 f_B(u)du > 0$ alone does not ensure the positivity of the speed of travelling fronts of (1.3) (see [8] for a necessary and sufficient condition). This is different from that in the classical case, where the condition $\int_0^1 f_B(u)du > 0$ is equivalent to the positivity of the speed of travelling fronts of $\partial_t u = \partial_{xx} u + f_B(u)$ (see, e.g., [4]).

The next assumption makes sure the uniform stability of $u \equiv 0$ and $u \equiv 1$.

(H3) There exist θ_0, θ_1 with $0 < \theta_0 < \tilde{\theta} \le \theta < \theta_1 < 1$ and $\beta_0 > 0$, $\beta_1 > 0$ such that

$$\partial_u f(t,u) \le -\beta_0, \quad u \in [-1,\theta_0] \quad \text{and} \quad \partial_u f(t,u) \le -\beta_1, \quad u \in [\theta_1, 2]$$

for all $t \in \mathbb{R}$.

Sometimes, we also assume that f satisfies the following stronger bistable assumption.

(H4) The ordinary differential equation (ODE)

$$\dot{u} = f(t,u) \tag{1.4}$$

has an entire solution $u_0 : \mathbb{R} \to \mathbb{R}$ satisfying
- $0 < \inf_{t\in\mathbb{R}} u_0(t) \le \sup_{t\in\mathbb{R}} u_0(t) < 1$;
- there exists $0 < \delta_0 \ll 1$ such that

$$\inf_{t\in\mathbb{R}} \inf_{u\in[u_0(t)-\delta_0, u_0(t)+\delta_0]} \partial_u f(t,u) > 0, \tag{1.5}$$

– for any $t_0 \in \mathbb{R}$, $u_1 \in (0, u_0(t_0))$ and $u_2 \in (u_0(t_0), 1)$, there holds

$$u(t; t_0, u_1) \to 0, \quad u(t; t_0, u_2) \to 1 \quad \text{as} \quad t - t_0 \to \infty,$$

where $u(t; t_0, u_i)$ are the solution of (1.4) with $u(t_0; t_0, u_i) = u_i$ $(i = 1, 2)$.

Equation (1.1) is used to model the evolution of the population density $u(t, x)$ of an invasive or a spreading species. The quantity $J(x - y)$, depending only on the displacement $x - y$, represents the probability that an individual moves from the location y to the location x, and, therefore, $[J * u](t, x)$ is the rate of individuals arriving at the location x, and $u(t, x) = \int_{\mathbb{R}} J(y - x)u(t, x)dy$ is the rate of individuals leaving the location x. With the operator $u \mapsto J * u - u$ describing the dispersal of the species, it is assumed that the dispersal of the species happens over long distances and follows the distribution J. Such a species is often called a *non-local disperser* (see, e.g., [35]), which includes diseases [38, 40] and seeds [31]. The symmetry of J in (H1) indicates that the species has no preference or is not forced to move to any particular direction, or that the probability an individual moves from the location y to the location x is the same as that an individual moves from x to y. In contrast, a stream-dwelling species is often forced to disperse along the stream, and therefore, the dispersal kernel is asymmetric (see, e.g., [32]). The decaying conditions on J in (H1) is satisfied by all compactly supported dispersal kernels, which are the biologically realistic ones, and by all Gaussian probability density functions, which are often used in literature. The growth rate function $f(t, u)$ is assumed to exhibit Allee effect [20] meaning the increase in the per capita growth rate $\frac{f(t,u)}{u}$ at low densities. The time dependence of the growth rate function takes into consideration the temporally varying environments, which are known to have great influences on the invasion or spread of species [61]. A typical example of the growth rate function is $f(t, u) = u(u - \theta(t))(1 - u)$ for an appropriate $\theta(t)$.

Solutions to (1.1) of particular interest are transition fronts connecting 0 and 1 due to their importance in describing extinction and persistence of the population. In the case that $f(t, u) \equiv f(u)$ is independent of t, transition fronts are strongly related to travelling fronts, that is, solutions of the form $u(t, x) = \phi(x - ct)$ for some $\phi : \mathbb{R} \to (0, 1)$ and $c \in \mathbb{R}$ with $\phi(-\infty) = 1$ and $\phi(\infty) = 0$. The reader is referred to [8, 17] for the study of the existence, the uniqueness and the stability of travelling fronts of (1.1) in time-independent bistable media. Also, see [1, 6, 7, 67] and references therein for more related works. In [16], time almost-periodic travelling fronts of (1.1) in the present of random diffusion are studied when $f(t, u)$ is almost periodic in t. As far as general time heterogeneity is concerned, there is little study on transition fronts of (1.1) with bistable non-linearity.

The objective of this paper is to study the existence, the uniqueness and the stability of transition fronts of (1.1) when f is a general time-dependent function satisfying (H2) and (H3). We recall the definition of transition fronts.

Definition 1.1 Suppose $f(t, 0) = 0 = f(t, 1)$ for all $t \in \mathbb{R}$. A global-in-time solution $u(t, x)$ of (1.1) is called a (right-moving) transition front (connecting 0 and 1) in the sense of Berestycki–Hamel (see [11, 12], also see [48, 49]) if $u(t, x) \in (0, 1)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ and there exists a function $X : \mathbb{R} \to \mathbb{R}$, called interface location function, such that

$$\lim_{x \to -\infty} u(t, x + X(t)) = 1 \text{ and } \lim_{x \to \infty} u(t, x + X(t)) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

The notion of a transition front is a proper generalisation of a travelling front in homogeneous media or a periodic (or pulsating) travelling front in periodic media (see, e.g., [10, 63, 62]). The interface location function $X(t)$ tells the position of the transition front $u(t,x)$ as time t elapses. Note that if $\xi(t)$ is a bounded function, then $X(t)+\xi(t)$ is also an interface location function. Thus, interface location function is not unique. But, it is easy to check that if $Y(t)$ is another interface location function, then $X(t)-Y(t)$ is a bounded function. Hence, interface location functions are unique up to addition by bounded functions. The uniform-in-t limits (the essential property in the definition) show the *bounded interface width*, that is,

$$\forall\ 0<\epsilon_1\leq\epsilon_2<1,\quad \sup_{t\in\mathbb{R}}\operatorname{diam}\{x\in\mathbb{R}:\epsilon_1\leq u(t,x)\leq\epsilon_2\}<\infty. \tag{1.6}$$

This actually gives an equivalent definition of transition fronts; that is, a global-in-time solution $u(t,x)$ of (1.1) is called a transition front if $u(t,x)\in(0,1)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, $u(t,x)\to 1$ as $x\to-\infty$ and $u(t,x)\to 0$ as $x\to\infty$ for all $t\in\mathbb{R}$, and (1.6) holds.

In the study of the existence, the stability and the uniqueness of transition fronts of (1.1), sub- and super-solutions and comparison principles play crucial roles. We remark that showing a function constructed from a transition front is a sub-solution or a super-solution usually involves the space derivative of the transition front. However, neither the definition nor equation (1.1) guarantees any space regularity of transition fronts. In [57], we studied the space regularity of transition fronts of non-local dispersal equations in general heterogeneous media. The following proposition follows from [57, Theorems 1.1 and Corollary 1.6].

Proposition 1.2 *Assume (H1)–(H3). Let $u(t,x)$ be an arbitrary transition front of* (1.1) *and $X(t)$ be its interface location function. Then,*

(i) *there exists a continuously differentiable function $\tilde{X}:\mathbb{R}\to\mathbb{R}$ satisfying*

$$c_{\min}\leq\dot{\tilde{X}}(t)\leq c_{\max},\quad \forall t\in\mathbb{R},$$

for some $0<c_{\min}\leq c_{\max}<\infty$ such that

$$\sup_{t\in\mathbb{R}}\left|X(t)-\tilde{X}(t)\right|<\infty;$$

in particular, $\tilde{X}(t)$ is also an interface location function of $u(t,x)$;

(ii) *$u(t,x)$ is regular in space, that is, $u(t,x)$ is continuously differentiable in x for any $t\in\mathbb{R}$ and satisfies*

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|\partial_x u(t,x)|<\infty.$$

We point out that Proposition 1.2 highly relies on the unbalanced condition (1.2). Replacing (1.2) by the speed of travelling fronts of (1.3) being non-negative, Proposition 1.2 fails when (1.3) admits discontinuous travelling fronts with zero speed (see [8] for the sufficient and necessary condition). Whether Proposition 1.2 holds when (1.3) admits continuous travelling fronts with zero speed leaves an interesting open question.

By Proposition 1.2, without loss of generality, we may then assume that the interface location function $X(t)$ of a transition front $u(t,x)$ is continuously differentiable and satisfies

$$c_{\min}\leq\dot{X}(t)\leq c_{\max},\quad \forall t\in\mathbb{R}, \tag{1.7}$$

for some $0 < c_{\min} \le c_{\max} < \infty$. This shows the rightward propagation nature of transition fronts in the sense of Definition 1.1.

By general semigroup theory (see, e.g., [43]) and comparison principles, for any $u_0 \in C^b_{\rm unif}(\mathbb{R},\mathbb{R})$ and $t_0 \in \mathbb{R}$, (1.1) has a unique global solution $u(t,\cdot;t_0,u_0) \in C^b_{\rm unif}(\mathbb{R},\mathbb{R})$ with $u(t_0,\cdot;t_0,u_0) = u_0$, where

$$C^b_{\rm unif}(\mathbb{R},\mathbb{R}) = \left\{ u \in C(\mathbb{R},\mathbb{R}) \,:\, u \text{ is uniformly continuous on } \mathbb{R} \text{ and } \sup_{x\in\mathbb{R}} |u(x)| < \infty \right\}$$

equipped with the norm $\|u\| := \sup_{x\in\mathbb{R}} |u(x)|$.

Throughout this paper, we assume (H1)–(H3). Among others, we prove in this paper the following results:

(i) (Uniform steepness) Assume that $u(t,x)$ is a space non-increasing transition front of (1.1) with $X(t)$ being its interface location function. For any $M > 0$, there holds

$$\sup_{t\in\mathbb{R}} \sup_{|x-X(t)|\le M} \partial_x u(t,x) < 0$$

(see Theorem 2.1).

(ii) (Uniform exponential stability) Assume that $u(t,x)$ is a space non-increasing transition front of (1.1). Let $\{u_{t_0}\}_{t_0\in\mathbb{R}}$ be a family of uniformly continuous initial data satisfying

$$u(t_0, x-\xi_0^-) - \mu_0 \le u_{t_0}(x) \le u(t_0, x-\xi_0^+) + \mu_0, \quad x\in\mathbb{R},\ t_0\in\mathbb{R},$$

for $\xi_0^\pm \in \mathbb{R}$ and $\mu_0 \in (0, \min\{\theta_0, 1-\theta_1\})$ being independent of $t_0 \in \mathbb{R}$. Then, there exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and a family of shifts $\{\xi_{t_0}\}_{t_0\in\mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}| < \infty$ such that

$$\sup_{x\in\mathbb{R}} \left| u(t,x;t_0,u_{t_0}) - u(t, x-\xi_{t_0}) \right| \le C e^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$ and $t_0 \in \mathbb{R}$ (see Theorem 3.1).

(iii) (Exponential decaying estimates) Assume that $u(t,x)$ is a space non-increasing transition front of (1.1) with $X(t)$ being its interface location function. There exist two exponents $c_\pm > 0$ and two shifts $h_\pm > 0$ such that

$$u(t, x+X(t)+h_+) \le e^{-c_+x} \quad \text{and} \quad u(t, x+X(t)-h_-) \ge 1 - e^{c_-x}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ (see Theorem 4.1).

(iv) (Uniqueness and monotonicity) If $u(t,x)$ and $v(t,x)$ are two transition fronts of (1.1) with $u(t,x)$ being non-increasing in x, then there exists a shift $\xi \in \mathbb{R}$ such that

$$v(t,x) = u(t, x+\xi), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R},$$

and hence $v(t,x)$ is also non-increasing in x (see Theorem 5.1).

(v) (Periodicity) If $u(t,x)$ is a space non-increasing transition front of (1.1) and, in addition, $f(t,u)$ is periodic in t, then $u(t,x)$ is a periodic travelling front (see Theorem 6.1(i)).

(vi) (Asymptotic speeds) If $u(t,x)$ is a space non-increasing transition front of (1.1) and, in addition, $f(t,u)$ is uniquely ergodic, then $\lim_{t\to\pm\infty} \frac{X(t)}{t}$ exists (see Theorem 6.1(ii)).

(vii) (Existence) Assume, in addition, (H4). There is a space non-increasing transition front of (1.1) (see Theorem 7.1).

We make some remarks about the above results.

(1) Using (1.1) to study the invasion or spread of a species, the most important problem is to understand how fast-propagating solutions, representing the invasive or spreading species, could spread. The most straightforward way is to calculate or estimate the spreading speed of propagating solutions. However, spreading speed in general does not exist in heterogeneous environments [56] and there exists no good approach to estimate it. An alternative and powerful way is to find special solutions such as travelling fronts in homogeneous environments and transition fronts in heterogeneous environments and study their stability. In the present paper, we take the latter way to study the invasion or spread of species modelled by (1.1). Our results on the existence, stability and uniqueness of transition fronts indicate that the dynamics of propagating solutions are slaved by the (unique) transition front $u(t,x)$, and therefore, how fast they could spread are described by the interface location function $X(t)$, more precisely, by the speed $\dot{X}(t)$ or the average speed $\frac{X(t)}{t}$. In the almost periodic case, we show the existence of the asymptotic spreading speed $c^* := \lim_{t\to\infty}\frac{X(t)}{t}$, which says that propagating solutions eventually spread (to the right) with speed c^*. More precisely, if $u(t,x;u_0)$ is a solution of (1.1) with appropriate initial data u_0 representing the initial distribution of the species, then

$$\lim_{t\to\infty}\inf_{x\le ct} u(t,x;u_0)=1, \quad \forall c\in(0,c^*),$$
$$\lim_{t\to\infty}\sup_{x\ge ct} u(t,x;u_0)=0, \quad \forall c\in(c^*,\infty).$$

Similar conclusions can be drawn for appropriate compactly supported initial data. The periodic dependence of $f(t,u)$ on t takes the seasonal effects on the invasion or spread of species into consideration. In this case, transition fronts admit special profiles and the asymptotic spreading speed c^* can be calculated in terms of the profiles (see Theorem 6.1 for more details).

(2) From (i)–(iv), we see that if (1.1) admit a space non-increasing transition front under assumptions (H1)–(H3), then transition fronts of (1.1) are non-increasing in space, exponentially stable, exponentially decaying and unique up to space shifts. We point out that it can be shown that any transition front of corresponding reaction–diffusion equations in time heterogeneous media is non-increasing in space (see, e.g., [48, 53]), while it is not an easy job for non-local dispersal equations partly due to the lack of Harnack's inequality.

(3) Note that the non-linearity $f(t,u)$ satisfying (H2) and (H3) are bistable only in the general sense. For each $t\in\mathbb{R}$, $f(t,\cdot)$ may not be of bistable type. In particular, multiple zeros between 0 and 1 are allowed. It is not known (even in the reaction–diffusion equation case) whether the assumptions (H2) and (H3) on $f(t,u)$ are sufficient for the existence of space non-increasing transition fronts, which is guaranteed under the additional assumption (H4). This is given in (vii). We further point out that (H4) implies the non-existence of stable periodic solutions between 0 and 1, which in light of the theory developed in [28] is expected to be a sharp sufficient condition for the existence of space non-increasing transition fronts. In fact, the authors considered in [28] time-periodic travelling fronts for abstract monotone systems of bistable type, and proved that the non-existence of stable periodic solutions between two stable periodic solutions is a sharp sufficient condition for the existence of time-periodic travelling fronts connecting two stable ones. Moreover, examples

of time-periodic reaction–diffusion equations of bistable type admitting no transition front connecting two stable equilibria were constructed in [72].

(4) The establishment of the uniform exponential stability in (ii) in this general form is the most important result of the present paper, and the applicability of the uniform exponential stability to arbitrary transition fronts and other families of initial data is the key to the proof of (iii) and (iv), and then to that of (v) and (vi). We remark that for reaction–diffusion equations, the usual exponential stability instead of the uniform exponential stability, together with standard arguments using parabolic regularity, comparison principles and Harnack's inequality, is sufficient for various qualitative properties such as exponential decaying estimate and uniqueness (see, e.g., [39, 53]). But for non-local equations, the standard arguments do not work very well, since we lack enough space regularity, the use of comparison principles are not as flexible as that for reaction–diffusion equations and Harnack's inequality is not known in the non-local case.

(5) The proof of (vii) actually does not need the unbalanced condition (1.2). This is because we take a perturbation approach, that is, we consider the perturbed equation

$$\partial_t u = J * u - u + \epsilon \partial_{xx} u + f(t,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.8}$$

and take advantage of the fact that the existence of transition fronts of (1.8) does not need (1.2). Of course, without (1.2), constructed transition fronts of (1.1) may not be continuous in space as mentioned after Proposition 1.2. It would be interesting to study qualitative properties of transition fronts in the absence of (1.2).

(6) We point out that travelling fronts for equation (1.3) were studied in [8]. In [8], the authors assumed the continuous differentiability and the symmetry of J as in (H1). Besides, they only need J to satisfy $\int_{\mathbb{R}} |x|J(x)dx < \infty$ and J' to be integrable rather than the much stronger exponential tail assumption on J as in (H1). We would expect to assume the weaker assumptions on J as in [8], but we are unable to do so due to some technical reasons including the following: (i) we need the exponential tail of J to ensure the finiteness of J^N in the proof of Lemma 2.4; (ii) we need the exponential tail of J in the proof of Lemma 4.4 to ensure the finiteness of $\int_{\mathbb{R}} J(y)e^{cy}dy$ for all small $c > 0$.

(7) It should be pointed out that (H2) can also be applied to a general bistable non-linearity $f(t,u)$ with the speed of travelling fronts of $\partial_t u = J * u - u + f_{\tilde{B}}(u)$ being negative. In fact, let $v(t,x) = 1 - u(t,x)$. Then, $v(t,x)$ satisfies

$$\partial_t v = J * v - v + \tilde{f}(t,v), \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where $\tilde{f}(t,v) = -f(t, 1-v)$. Hence

$$\tilde{f}_B(v) \le \tilde{f}(t,v) \le \tilde{f}_{\tilde{B}}(v), \quad (t,v) \in \mathbb{R} \times \mathbb{R} \times [0,1],$$

where $\tilde{f}_B(v) = -f_{\tilde{B}}(1-v)$ and $\tilde{f}_{\tilde{B}}(v) = -f_B(1-v)$. Clearly, $\tilde{f}_B(\cdot)$ and $\tilde{f}_{\tilde{B}}(\cdot)$ are two standard bistable non-linearities and the speed of travelling fronts of $\partial_t u = J * u - u + \tilde{f}_B(u)$ is positive.

We remark that transition fronts can be defined in the same way for more general equations, say,

$$\partial_t u = J * u - u + f(t,x,u). \tag{1.9}$$

Equation (1.9) in various homogeneous media, that is, $f(t,x,u)=f(u)$ with various types of non-linearity $f(\cdot)$, is well studied. We refer to [8, 15, 17, 21, 22, 47] and references therein for results concerning travelling fronts. There are also some results concerning periodic travelling fronts in periodic media of monostable type (see, e.g., [23, 26, 46, 58, 59, 60]). The study of (1.9) in general heterogeneous media is very recent and results concerning front propagation are very limited. In [9], Berestycki, Coville and Vo studied principal eigenvalue, positive solution and long-time behaviour of solutions of (1.9) in the space heterogeneous monostable media. In [36], Lim and Zlatoš also studied (1.9) in the space heterogeneous monostable media, but with different settings, and proved the existence of transition fronts. In [13], Berestycki and Rodriguez studied propagation and blocking phenomenon of (1.9) with barrier non-linearities in space heterogeneous media of bistable type. In [54, 55], the authors of the present paper studied (1.9) in time heterogeneous media of ignition type and proved the existence, regularity and stability of transition fronts.

We end the introduction by mentioning some relevant results about reaction–diffusion equations and discrete equations in bistable media, that is,

$$\partial_t u = \partial_{xx} u + f(t,x,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.10}$$

and

$$\dot{u}_i = u_{i+1} - 2u_i + u_{i-1} + f(t,i,u), \quad (t,i) \in \mathbb{R} \times \mathbb{Z}, \tag{1.11}$$

where f in both cases is of bistable type. As a classical model, (1.10) has been attracting extensive studies and results concerning front propagation are quite complete except in the most general case; that is, $f(t,x,u)$ depends generally on both t and x. See [4, 29, 5, 33, 64] for travelling fronts, [3] for time-periodic travelling fronts, [48, 49, 51] for time-almost-periodic travelling fronts, [10, 62, 25] for pulsating fronts, [52, 30, 42, 24, 72] for transition fronts, [34, 72] for the wave-blocking phenomenon and [70, 27, 44, 41, 45] for the sharp transition phenomenon. We also refer the reader to surveys [65, 66] for related works and extensive remarks. As (1.9) in the bistable case, not a lot is known about (1.11). We refer the readers to [14, 19, 37, 69, 68] and references therein for works in homogeneous media, and to [18, 50] for works in periodic media.

The main difference between (1.9) and (1.10) lies in the dispersal, which however results in fundamental difference between them, that is, solutions of (1.10) gain space regularity right after the initial moment, while solutions of (1.9) are lack of space regularity in the sense that they do not become smoother in space as time elapses and are only as smooth in space as their initial data. The lack of space regularity causes one of the main difficulties in studying (1.9). Equation (1.11) is often considered as a space discrete version of (1.10). It can also be considered as a non-local equation but with a singular dispersal kernel, and therefore, the ideas and methods for (1.9) cannot be simply adapted to treat (1.11).

The rest of the paper is organised as follows. In Section 2, we focus our study on uniform steepness of space non-increasing transition fronts of (1.1). We investigate uniform exponential stability and exponential decaying estimates of space non-increasing transition fronts of (1.1) in Sections 3 and 4, respectively. In Section 5, we show that any transition front of equation (1.1), after certain space shift, coincides with a space non-increasing transition front (if it exists). In Section 6, under the additional time-periodic assumption on the non-linearity, we show that any transition front must be a periodic travelling front. Under the assumption that the non-linearity

$f(t,u)$ is uniquely ergodic, we show that asymptotic speeds of transition fronts exist. In Section 7, we prove the existence of space non-increasing transition fronts of (1.1). In Section 8, we conclude the paper with some discussions. In Appendix A, we state some comparison principles.

2 Uniform steepness of space non-increasing transition fronts

In this section, we study the uniform steepness of space non-increasing transition fronts of (1.1). Throughout this section, we assume (H1)–(H3).

Suppose that $u(t,x)$ is a transition front. For $\lambda\in(0,1)$, let $X_\lambda^-(t)$ and $X_\lambda^+(t)$ be the leftmost and rightmost interface locations at λ, that is,

$$X_\lambda^-(t)=\inf\{x\in\mathbb{R}:u(t,x)\le\lambda\}\quad\text{and}\quad X_\lambda^+(t)=\sup\{x\in\mathbb{R}:u(t,x)\ge\lambda\}. \tag{2.1}$$

Trivially, $X_\lambda^-(t)\le X_\lambda^+(t)$ and $X_\lambda^\pm(t)$ are non-increasing in λ. We see that it may happen that $u(t,X_\lambda^-(t))>\lambda$ or $u(t,X_\lambda^+(t))<\lambda$ due to possible jumps. But, it is clear that $u(t,x)>\lambda$ for $x<X_\lambda^-(t)$ and $u(t,x)<\lambda$ for $x>X_\lambda^+(t)$.

In what follows in this section, $u(t,x)$ will be an arbitrary transition front of (1.1) that is non-increasing in space, that is, $\partial_x u(t,x)\le 0$ for $(t,x)\in\mathbb{R}\times\mathbb{R}$ (recall that by Proposition 1.2 any transition front is continuously differentiable in space). By the strong maximum principle, $u(t,x)$ is decreasing in x for any $t\in\mathbb{R}$. As a result, for any $\lambda\in(0,1)$, the leftmost and rightmost interface locations coincide, that is, $X_\lambda^+(t)=X_\lambda^-(t)$, which will be denoted by $X_\lambda(t)$. In particular, $u(t,X_\lambda(t))=\lambda$. Let $X(t)$ be the interface location function corresponding to $u(t,x)$. Without loss of generality, we assume that $X(t)$ satisfies (1.7).

The main result in this section is given in the following.

Theorem 2.1 *For any $M>0$, there holds*

$$\sup_{t\in\mathbb{R}}\sup_{|x-X(t)|\le M}\partial_x u(t,x)<0.$$

To prove Theorem 2.1, we first prove two lemmas. The first lemma follows directly from the definition of transition fronts.

Lemma 2.2 *For any $\lambda\in(0,1)$, there holds*

$$\sup_{t\in\mathbb{R}}\left|X(t)-X_\lambda^\pm(t)\right|<\infty.$$

Proof By the uniform-in-t limits $\lim_{x\to-\infty}u(t,x+X(t))=1$ and $\lim_{x\to\infty}u(t,x+X(t))=0$, there exist x_1 and x_2 such that $u(t,x+X(t))>\lambda$ for all $x\le x_1$ and $t\in\mathbb{R}$, and $u(t,x+X(t))<\lambda$ for all $x\ge x_2$ and $t\in\mathbb{R}$. It then follows from the definition of $X_\lambda^\pm(t)$ that $x_1+X(t)\le X_\lambda^-(t)$ and $x_2+X(t)\ge X_\lambda^+(t)$ for all $t\in\mathbb{R}$. In particular,

$$x_1+X(t)\le X_\lambda^-(t)\le X_\lambda^+(t)\le x_2+X(t),\qquad t\in\mathbb{R}.$$

This completes the proof. □

We remark that the monotonicity of $u(t,x)$ in x is not required in the above lemma, which is true for an arbitrary transition front. That is why we used $X_\lambda^\pm(t)$ instead of $X_\lambda(t)$.

As a simple consequence of implicit function theorem, the equation $u(t, X_\lambda(t)) = \lambda$, Lemma 2.2 and Theorem 2.1, we find the following.

Corollary 2.3 *For any $\lambda \in (0, 1)$, $X_\lambda(t)$ is continuously differentiable and satisfies*

$$\dot{X}_\lambda(t) = -\frac{\partial_t u(t, X_\lambda(t))}{\partial_x u(t, X_\lambda(t))}, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \sup_{t\in\mathbb{R}} |\dot{X}_\lambda(t)| < \infty.$$

Since $\partial_t u(t, x)$ changes signs in general due to the time dependence of $f(t, u)$, $\dot{X}_\lambda(t)$ changes its signs. Thus, in general, transition fronts in the present case move to the right with oscillations.

The next lemma inspired by [17, Theorem 5.1] and [48, Lemma 3.2] is crucial to uniform steepness. We refer the reader to Appendix A for comparison principles.

Lemma 2.4 *Let $u_1(t, x; \tau)$ and $u_2(t, x; \tau)$ be sub-solution and super-solution of (1.1), respectively, and satisfy*

$$-1 \le u_1(t, x; \tau) \le u_2(t, x; \tau) \le 2, \quad x \in \mathbb{R}, \quad t \ge \tau.$$

Then, for any $t > t_0 \ge \tau$, $h > 0$ and $z \in \mathbb{R}$, there holds

$$u_1(t, x; \tau) - u_2(t, x; \tau) \le C \int_{z-h}^{z+h} [u_1(t_0, y; \tau) - u_2(t_0, y; \tau)]dy, \quad x \in \mathbb{R},$$

where $C = C(t - t_0, |x - z|, h) > 0$ satisfies

(i) *$C \to 0$ polynomially as $t - t_0 \to 0$ and $C \to 0$ exponentially as $t - t_0 \to \infty$;*
(ii) *$C : (0, \infty) \times [0, \infty) \times (0, \infty) \to (0, \infty)$ is locally uniformly positive in the sense that for any $0 < t_1 < t_2 < \infty$, $M_1 > 0$ and $h_1 > 0$, there holds*

$$\inf_{t\in[t_1,t_2], M\in[0,M_1], h\in(0,h_1]} C(t, M, h) > 0.$$

Proof Let $t > t_0 \ge \tau$. Set $v_1(t, x) := u_1(t, x; \tau)$ and $v_2(t, x) := u_2(t, x; \tau)$. By assumptions, $v(t, x) := v_1(t, x) - v_2(t, x) \le 0$ and satisfies

$$\partial_t v \le J * v - v + f(t, v_1) - f(t, v_2).$$

By (H2), we can find $K > 0$ such that $f(t, v_1) - f(t, v_2) \le -K(v_1 - v_2)$, which implies that

$$\partial_t v \le J * v - v - Kv.$$

Setting $\tilde{v}(t, x) := e^{(1+K)(t-t_0)} v(t, x) \le 0$, we see

$$\partial_t \tilde{v} \le J * \tilde{v} \le 0. \tag{2.2}$$

In particular, $\tilde{v}(t, x) \le \tilde{v}(t_0, x)$. It then follows that

$$\partial_t \tilde{v}(t, x) \le [J * \tilde{v}(t, \cdot)](x) \le [J * \tilde{v}(t_0, \cdot)](x).$$

Integrating over $[t_0, t]$ with respect to the time variable, we find from $\tilde{v}(t_0, x) \le 0$ that

$$\tilde{v}(t, x) \le (t - t_0) [J * \tilde{v}(t_0, \cdot)](x) + \tilde{v}(t_0, x) \le (t - t_0) [J * \tilde{v}(t_0, \cdot)](x).$$

In particular, for any $T > 0$, we have

$$\tilde{v}(t_0 + T, x) \leq T\,[J * \tilde{v}(t_0, \cdot)]\,(x). \tag{2.3}$$

Then, considering (2.2) with initial time at $t_0 + T$ and repeating the above arguments, we find

$$\tilde{v}(t_0 + T + T, x) \leq T\,[J * \tilde{v}(t_0 + T, \cdot)]\,(x) \leq T^2\,[J * J * \tilde{v}(t_0, \cdot)]\,(x),$$

where we used (2.3) in the second inequality. Repeating this, we conclude that for any $T > 0$ and any $N = 1, 2, 3, \ldots$, there holds

$$\tilde{v}(t_0 + NT, x) \leq T^N \left[J^N * \tilde{v}(t_0, \cdot)\right](x), \tag{2.4}$$

where $J^N = \underbrace{J * J * \cdots * J}_{N \text{ times}}$. Note that J^N is non-negative, and if J is compactly supported, then J^N is not everywhere positive no matter how large N is. But, since J is non-negative and positive on some open interval, J^N can be positive on any fixed bounded interval if N is large. Moreover, since J is symmetric, so is J^N.

Now, let $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $h > 0$, and let $N := N(|x - z|, h)$ be large enough so that

$$\tilde{C} = \tilde{C}(|x - z|, h) := \inf_{y \in [x - z - h, x - z + h]} J^N(y) > 0.$$

Note that the dependence of N on $x - z$ through $|x - z|$ is due to the symmetry of J^N. Moreover, the positivity of $\tilde{C} : [0, \infty) \times (0, \infty) \to (0, \infty)$ is uniform on compacts sets, which is because N can be chosen to be non-decreasing in $|x - z|$ and in h.

Then, for $t > t_0$, we see from (2.4) with $T = \frac{t - t_0}{N}$ that

$$\begin{aligned}
\tilde{v}(t, x) &\leq \left(\frac{t - t_0}{N}\right)^N \int_{\mathbb{R}} J^N(x - y)\tilde{v}(t_0, y)dy \\
&\leq \left(\frac{t - t_0}{N}\right)^N \int_{z-h}^{z+h} J^N(x - y)\tilde{v}(t_0, y)dy \\
&\leq \tilde{C}\left(\frac{t - t_0}{N}\right)^N \int_{z-h}^{z+h} \tilde{v}(t_0, y)dy,
\end{aligned}$$

since $x - y \in [x - z - h, x - z + h]$ when $y \in [z - h, z + h]$. Going back to $v(t, x)$, we find

$$u_1(t, x; \tau) - u_2(t, x; \tau) \leq \tilde{C} e^{-(1+K)(t - t_0)} \left(\frac{t - t_0}{N}\right)^N \int_{z-h}^{z+h} [u_1(t_0, y; \tau) - u_2(t_0, y; \tau)]\, dy.$$

The result then follows with $C = \tilde{C} e^{-(1+K)(t - t_0)} \left(\frac{t - t_0}{N}\right)^N$. □

As a simple consequence of Lemma 2.4, we have the following.

Corollary 2.5 *For any $t > t_0 \geq \tau$, $h > 0$ and $z \in \mathbb{R}$, there holds*

$$\partial_x u(t, x) \leq C \int_{z-h}^{z+h} \partial_x u(t_0, y)dy, \quad x \in \mathbb{R},$$

where $C > 0$ is as in Lemma 2.4.

Proof Applying Lemma 2.4 with $u_1 = u(t, x + \epsilon)$ and $u_2 = u(t, x)$, dividing the result by ϵ and passing to the limit $\epsilon \to 0^+$, we conclude the lemma. □

Now, we prove Theorem 2.1.

Proof of Theorem 2.1 Recall $X_\lambda(t) := X_\lambda^\pm(t)$ for $\lambda \in (0, 1)$. By Lemma 2.2, $\sup_{t\in\mathbb{R}} |X(t) - X_\lambda(t)| < \infty$.

Fix any $\lambda_0 \in (0, 1)$ and set

$$h_{\lambda_0} := \max\left\{ \sup_{t\in\mathbb{R}} |X(t) - X_{\frac{\lambda_0}{2}}(t)|, \quad \sup_{t\in\mathbb{R}} |X(t) - X_{\frac{1+\lambda_0}{2}}(t)| \right\}.$$

Then, $h_{\lambda_0} < \infty$ and

$$X(t) + h_{\lambda_0} \geq X_{\frac{\lambda_0}{2}}(t), \quad X(t) - h_{\lambda_0} \leq X_{\frac{1+\lambda_0}{2}}(t), \tag{2.5}$$

for all $t \in \mathbb{R}$. Now, fix $\tau > 0$. For $t \in \mathbb{R}$, we apply Lemma 2.4 with $z = X(t)$ and $h = h_{\lambda_0}$ to see that if $|x - X(t)| \leq M$, then

$$\begin{aligned} \partial_x u(\tau + t, x) &\leq \tilde{C}(\tau, M, h_{\lambda_0}) \int_{X(t)-h_{\lambda_0}}^{X(t)+h_{\lambda_0}} \partial_x u(t, y) dy \\ &= \tilde{C}(\tau, M, h_{\lambda_0}) \left[u(t, X(t) + h_{\lambda_0}) - u(t, X(t) - h_{\lambda_0})\right] \\ &\leq \tilde{C}(\tau, M, h_{\lambda_0}) \left[u(t, X_{\frac{\lambda_0}{2}}(t)) - u(t, X_{\frac{1+\lambda_0}{2}}(t))\right] \\ &= -\frac{\tilde{C}(\tau, M, h_{\lambda_0})}{2}, \end{aligned} \tag{2.6}$$

where we used (2.5) and the monotonicity in the second inequality, and $\tilde{C}(\tau, M, h_{\lambda_0}) = \inf_{K\in[0,M]} C(\tau, K, h_{\lambda_0})$. To apply (2.6), we see that if $|x - X(t + 1)| \leq M$, then

$$|x - X(t)| \leq |x - X(t + 1)| + |X(t + 1) - X(t)| \leq M + c_{\max},$$

where we used (1.7). We then apply (2.6) with M replaced by $M + c_{\max}$ and τ replaced by 1 to conclude that

$$\partial_x u(t + 1, x) \leq -\frac{1}{2} \inf_{K\in[0,M+c_{\max}]} C(1, K, h_{\lambda_0}).$$

Since $t \in \mathbb{R}$ is arbitrary, we arrive at the result. □

3 Uniform exponential stability of space non-increasing transition fronts

In this section, we study the stability of space non-increasing transition fronts of (1.1). Throughout this section, we assume (H1)–(H3) and assume that $u(t, x)$ is a transition front of (1.1) with interface location function $X(t)$ and $\partial_x u(t, x) \leq 0$.

The main results in this section are stated in the following theorem.

Theorem 3.1 (i) *Let $u_0 : \mathbb{R} \to [0, 1]$ be uniformly continuous and satisfy*

$$\liminf_{x\to-\infty} u_0(x) > \theta_1 \quad \textit{and} \quad \limsup_{x\to\infty} u_0(x) < \theta_0,$$

where θ_0 and θ_1 are as in (H3). Then, for any $t_0 \in \mathbb{R}$, there exist $\xi = \xi(t_0, u_0) \in \mathbb{R}$, $C = C(u_0) > 0$ (independent of t_0) and $\omega_* > 0$ (independent of t_0 and u_0) such that

$$\sup_{x\in\mathbb{R}} |u(t, x; t_0, u_0) - u(t, x - \xi)| \le Ce^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$.

(ii) *Let $\{u_{t_0}\}_{t_0\in\mathbb{R}}$ be a family of uniformly continuous initial data satisfying*

$$u(t_0, x - \xi_0^-) - \mu_0 \le u_{t_0}(x) \le u(t_0, x - \xi_0^+) + \mu_0, \quad x \in \mathbb{R}, \ t_0 \in \mathbb{R},$$

for $\xi_0^\pm \in \mathbb{R}$ and $\mu_0 \in (0, \min\{\theta_0, 1 - \theta_1\})$ being independent of $t_0 \in \mathbb{R}$. Then, there exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and a family of shifts $\{\xi_{t_0}\}_{t_0\in\mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}| < \infty$ such that

$$\sup_{x\in\mathbb{R}} \left|u(t, x; t_0, u_{t_0}) - u(t, x - \xi_{t_0})\right| \le Ce^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$ and $t_0 \in \mathbb{R}$.

To prove Theorem 3.1, we first show two lemmas. Let

$$\omega = \min\{\beta_0, \beta_1\} > 0,$$

where β_0 and β_1 are as in (H3).

Lemma 3.2 *Let u_0 be as in Theorem 3.1. Let $\mu = \max\left\{\mu_0^-, \mu_0^+\right\}$, where $\mu_0^\pm = \mu_0^\pm(u_0)$ are such that*

$$\theta_1 < 1 - \mu_0^- < \liminf_{x\to-\infty} u_0(x) \quad \text{and} \quad \limsup_{x\to\infty} u_0(x) < \mu_0^+ < \theta_0.$$

Then, for any $t_0 \in \mathbb{R}$, there exist $\xi_0^\pm = \xi_0^\pm(t_0, u_0) \in \mathbb{R}$ such that

$$u(t, x - \xi^-(t)) - \mu e^{-\omega(t-t_0)} \le u(t, x; t_0, u_0) \le u(t, x - \xi^+(t)) + \mu e^{-\omega(t-t_0)}, \quad x \in \mathbb{R}, \tag{3.1}$$

for $t \ge t_0$, where

$$\xi^\pm(t) = \xi_0^\pm \pm \frac{A\mu}{\omega}\left(1 - e^{-\omega(t-t_0)}\right), \quad t \ge t_0,$$

for some universal constant $A > 0$. In particular, there holds

$$u(t, x - \xi^-) - \mu e^{-\omega(t-t_0)} \le u(t, x; t_0, u_0) \le u(t, x - \xi^+) + \mu e^{-\omega(t-t_0)}, \quad x \in \mathbb{R},$$

for $t \ge t_0$, where $\xi^\pm = \xi_0^\pm \pm \frac{A\mu}{\omega}$.

Proof Let u_0 be as in the statement of Theorem 3.1(i). Let $\mu_0^\pm = \mu_0^\pm(u_0)$ be as in the statement. Then, for any $t_0 \in \mathbb{R}$, we can find $\xi_0^\pm = \xi_0^\pm(t_0, u_0)$ such that

$$u(t_0, x - \xi_0^-) - \mu_0^- \le u_0(x) \le u(t_0, x - \xi_0^+) + \mu_0^+, \quad x \in \mathbb{R}. \tag{3.2}$$

To show the lemma, we then construct appropriate sub- and super-solutions and apply comparison principle. We here only prove the first inequality in (3.1); the second one can be proved along the same line. To do so, we fix $A>0$ (to be chosen) and set

$$u^-(t,x)=u(t,x-\xi(t))-\mu_0^- e^{-\omega(t-t_0)},$$

where $\xi(t)=\xi_0^- - \frac{A\mu_0^-}{\omega}\left(1-e^{-\omega(t-t_0)}\right)$. We then compute

$$\begin{aligned}&\partial_t u^- - [J*u^- - u^-] - f(t,u^-)\\ &\quad = f(t,u(t,x-\xi(t))) - f(t,u^-(t,x)) + A\mu_0^- e^{-\omega(t-t_0)}\partial_x u(t,x-\xi(t)) + \omega\mu_0^- e^{-\omega(t-t_0)}.\end{aligned}$$

Now, we let $M>0$ be so large that

$$\forall t\in\mathbb{R},\quad \begin{cases} u(t,x)\le\theta_0 & \text{if} \quad x-X(t)\ge M,\\ u(t,x)\ge\theta_1+\mu_0^- & \text{if} \quad x-X(t)\le -M.\end{cases}$$

Notice such an M exists due to Lemma 2.2. Then, we see

- if $x-\xi(t)-X(t)\ge M$, then $u^-(t,x)\le u(t,x-\xi(t))\le\theta_0$, and then by (H3),

$$\begin{aligned} f(t,u(t,x-\xi(t))) - f(t,u^-(t,x)) &\le -\beta_0\left[u(t,x-\xi(t)) - u^-(t,x)\right]\\ &= -\beta_0\mu_0^- e^{-\omega(t-t_0)}.\end{aligned}$$

  Since $A\mu_0^- e^{-\omega(t-t_0)}\partial_x u(t,x-\xi(t))\le 0$, we find

$$\partial_t u^- - [J*u^- - u^-] - f(t,u^-) \le -\beta_0\mu_0^- e^{-\omega(t-t_0)} + \omega\mu_0^- e^{-\omega(t-t_0)} \le 0$$

  if $\omega\le\beta_0$;
- if $x-\xi(t)-X(t)\le -M$, then

$$u(t,x-\xi(t))\ge u^-(t,x)=u(t,x-\xi(t))-\mu_0^- e^{-\omega(t-t_0)}\ge\theta_1+\mu_0^- - \mu_0^- = \theta_1,$$

  and then by (H3),

$$f(t,u(t,x-\xi(t))) - f(t,u^-(t,x)) \le -\beta_1\mu_0^- e^{-\omega(t-t_0)}.$$

  Hence, $\partial_t u^- - [J*u^- - u^-] - f(t,u^-)\le 0$ if $\omega\le\beta_1$;
- if $|x-\xi(t)-X(t)|\le M$, then by Theorem 2.1,

$$C_M := \sup_{t\in\mathbb{R}}\ \sup_{|x-\xi(t)-X(t)|\le M}\partial_x u(t,x-\xi(t)) = \sup_{t\in\mathbb{R}}\ \sup_{|x-X(t)|\le M}\partial_x u(t,x) < 0.$$

  Since

$$\left|f(t,u(t,x-\xi(t))) - f(t,u^-(t,x))\right| \le C_*\mu_0^- e^{-\omega(t-t_0)}$$

  for some $C_*>0$, we find

$$\partial_t u^- - [J*u^- - u^-] - f(t,u^-) \le \left(C_*\mu_0^- + A\mu_0^- C_M + \omega\mu_0^-\right)e^{-\omega(t-t_0)} \le 0$$

  if $A\ge\frac{C_*+\omega}{-C_M}>0$.

Hence, if we choose $A=\frac{2C_*}{-C_M}$ (note $\omega=\min\{\beta_0,\beta_1\}\le C_*$), we find

$$\partial_t u^- \le J*u^- - u^- + f(t,u^-),\quad x\in\mathbb{R},\quad t>t_0,$$

that is, $u^-(t,x)$ is a sub-solution on (t_0,∞). Since

$$u^-(t_0,x)=u(t_0,x-\xi_0^-)-\mu_0^-\le u_0(x)$$

due to (3.2), we conclude from comparison principle that

$$u(t,x-\xi(t))-\mu_0^-e^{-\omega(t-t_0)}=u^-(t,x)\le u(t,x;t_0,u_0),\quad x\in\mathbb{R},\quad t\ge t_0.$$

This completes the proof. □

The proof of Lemma 3.2 gives the following result.

Corollary 3.3 *Suppose that $\tilde{u}_0:\mathbb{R}\to[0,1]$ is uniformly continuous and satisfies*

$$u(t_0,x-\tilde{\xi}_0^-)-\tilde{\mu}_0^-\le\tilde{u}_0(x)\le u(t_0,x-\tilde{\xi}_0^+)+\tilde{\mu}_0^+,\quad x\in\mathbb{R},$$

for $t_0\in\mathbb{R}$, $\tilde{\xi}_0^\pm\in\mathbb{R}$ and $\tilde{\mu}_0^\pm>0$ satisfying $\theta_1<1-\tilde{\mu}_0^-$ and $\tilde{\mu}_0^+<\theta_0$, where θ_0 and θ_1 are as in (H3). Then, there exists $\tilde{\mu}:=\max\{\tilde{\mu}_0^-,\tilde{\mu}_0^+\}>0$ such that

$$u(t,x-\tilde{\xi}^-(t))-\tilde{\mu}e^{-\omega(t-t_0)}\le u(t,x;t_0,\tilde{u}_0)\le u(t,x-\tilde{\xi}^+(t))+\tilde{\mu}e^{-\omega(t-t_0)},\quad x\in\mathbb{R},$$

for $t\ge t_0$, where

$$\tilde{\xi}^\pm(t)=\tilde{\xi}_0^\pm\pm\frac{A\tilde{\mu}}{\omega}\left(1-e^{-\omega(t-t_0)}\right),\quad t\ge t_0,$$

for some universal constant $A>0$. In particular, we have

$$u(t,x-\tilde{\xi}^-)-\tilde{\mu}e^{-\omega(t-t_0)}\le u(t,x;t_0,\tilde{u}_0)\le u(t,x-\tilde{\xi}^+)+\tilde{\mu}e^{-\omega(t-t_0)},\quad x\in\mathbb{R},$$

for $t\ge t_0$, where $\tilde{\xi}^\pm=\tilde{\xi}_0^\pm\pm\frac{A\tilde{\mu}}{\omega}$.

The next lemma is the key to the proof of Theorem 3.1. We will let $u(t,x;t_0)$, $t\ge t_0$ be a solution with initial data u_0 at time $t_0\in\mathbb{R}$.

Lemma 3.4 *There exists $\epsilon^*\in(0,1)$ such that if there holds*

$$u(\tau,x-\hat{\xi})-\hat{\delta}\le u(\tau,x;t_0)\le u(\tau,x-\hat{\xi}-\hat{h})+\hat{\delta},\quad x\in\mathbb{R},\tag{3.3}$$

for some $\tau\ge t_0$, $\hat{\xi}\in\mathbb{R}$, $\hat{h}>0$ and $\hat{\delta}\in(0,\min\{\theta_0,1-\theta_1\})$, then there exist $\hat{\xi}(t)$, $\hat{h}(t)$ and $\hat{\delta}(t)$ satisfying

$$\begin{aligned}\hat{\xi}(t)&\in\left[\hat{\xi}-\frac{2A\hat{\delta}}{\omega},\hat{\xi}+\epsilon^*\min\{1,\hat{h}\}\right],\\ 0\le&\hat{h}(t)\le\hat{h}-\epsilon^*\min\left\{1,\hat{h}\right\}+\frac{4A\hat{\delta}}{\omega},\\ 0\le&\hat{\delta}(t)\le\left[\hat{\delta}e^{-\omega}+C^*\epsilon^*\min\{1,\hat{h}\}\right]e^{-\omega(t-\tau-1)},\end{aligned}$$

such that

$$u(t,x-\hat{\xi}(t))-\hat{\delta}(t)\le u(t,x;t_0)\le u(t,x-\hat{\xi}(t)-\hat{h}(t))+\hat{\delta}(t),\quad x\in\mathbb{R},$$

for $t\ge\tau+1$, where $A>0$ is some universal constant and $C^*=\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|\partial_x u(t,x)|$.

Proof Applying Corollary 3.3 to (3.3), we find

$$u(t,x-\hat{\xi}^-(t))-\hat{\delta}e^{-\omega(t-\tau)} \le u(t,x;t_0) \le u(t,x-\hat{\xi}^+(t)-\hat{h})+\hat{\delta}e^{-\omega(t-\tau)}, \quad x\in\mathbb{R}, \tag{3.4}$$

for $t\ge\tau$, where $\omega=\min\{\beta_0,\beta_1\}$ and $\hat{\xi}^{\pm}(t)=\hat{\xi}\pm\frac{A\hat{\delta}}{\omega}\left(1-e^{-\omega(t-\tau)}\right)$.

We now modify (3.4) at $t=\tau+1$ to get a new estimate for $u(\tau+1,x;t_0)$, and then apply Corollary 3.3 to this new estimate to conclude the result. To this end, we set

$$h=\min\left\{\hat{h},1\right\} \quad \text{and} \quad C_{\text{steep}}=\frac{1}{2}\sup_{t\in\mathbb{R}}\sup_{|x-X(t)|\le 2}\partial_x u(t,x).$$

By Theorem 2.1, $C_{\text{steep}}<0$. Taylor expansion then yields

$$\int_{X(t)-\frac{1}{2}}^{X(t)+\frac{1}{2}}[u(t,x-h)-u(t,x)]\,dx \ge -2C_{\text{steep}}h, \quad \forall t\in\mathbb{R}.$$

In particular, at $t=\tau$, either

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}}\left[u(\tau,x-h)-u(\tau,x+\hat{\xi};t_0)\right]dx \ge -C_{\text{steep}}h \tag{3.5}$$

or

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}}\left[u(\tau,x+\hat{\xi};t_0)-u(\tau,x)\right]dx \ge -C_{\text{steep}}h \tag{3.6}$$

must be the case.

Suppose first that (3.6) holds. We estimate the following term:

$$u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^* h)$$

from below, where $\epsilon^*>0$ is to be chosen. To do so, let $M>0$ and consider two cases: (i) $|x-\hat{\xi}-X(\tau)|\le M$; (ii) $|x-\hat{\xi}-X(\tau)|\ge M$.

(i) $|x-\hat{\xi}-X(\tau)|\le M$ In this case, we write

$$\begin{aligned}
&u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^* h)\\
&\quad=\left[u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^-(\tau+1))\right]\\
&\qquad+\left[u(\tau+1,x-\hat{\xi}^-(\tau+1))-u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^* h)\right]\\
&\quad=:(\mathrm{I})+(\mathrm{II}).
\end{aligned}$$

For (I), we argue

$$\begin{aligned}
(\mathrm{I})+\hat{\delta}e^{-\omega}&=u(\tau+1,x;t_0)-\left[u(\tau+1,x-\hat{\xi}+\frac{A\hat{\delta}}{\omega}(1-e^{-\omega}))-\hat{\delta}e^{-\omega}\right]\\
&=u(\tau+1,y+\hat{\xi};t_0)-\left[u(\tau+1,y+\frac{A\hat{\delta}}{\omega}(1-e^{-\omega}))-\hat{\delta}e^{-\omega}\right]\\
&\quad\left(\text{by } y=x-\hat{\xi}\in X(\tau)+[-M,M]\right)
\end{aligned}$$

$$
\begin{aligned}
&= u(\tau+1, y+\hat{\xi}; t_0) - \hat{u}(\tau+1, y) \\
&\quad \left(\text{where } \hat{u}(t,y) = u(t, y + \frac{A\hat{\delta}}{\omega}(1-e^{-\omega(t-\tau)})) - \hat{\delta}e^{-\omega(t-\tau)}\right) \\
&\geq C(M)\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} \left[u(\tau, y+\hat{\xi}; t_0) - \hat{u}(\tau, y)\right] dy \\
&\geq C(M)\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} \left[u(\tau, y+\hat{\xi}; t_0) - u(\tau, y)\right] dy \\
&\geq -C(M)C_{\text{steep}}h,
\end{aligned}
$$

where the first inequality follows from Lemma 2.4. In fact, we know $u(t, y+\hat{\xi}; t_0)$ is a solution of $\partial_t v = J * v - v + f(t, v)$, while $\hat{u}(t, y)$ is a sub-solution by the proof of Lemma 3.2. Moreover, $u(t, y+\hat{\xi}; t_0) \geq \hat{u}(t, y)$ by (3.4). Then, we apply Lemma 2.4 with $u_1 = \hat{u}(t, y)$ and $u_2 = u(t, y+\hat{\xi}; t_0)$ to conclude the inequality. Hence, (I) $\geq -\hat{\delta}e^{-\omega} - C(M)C_{\text{steep}}h$.

For (II), Taylor expansion yields for some $x_* \in (0, \epsilon^* h)$

$$
\text{(II)} = \partial_x u(\tau+1, x - \hat{\xi}^-(\tau+1) - x_*)\epsilon^* h \geq -\epsilon^* h \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |\partial_x u(t,x)| \geq C(M)C_{\text{steep}}h
$$

if we choose $\epsilon^* = \min\left\{1, \frac{-C(M)C_{\text{steep}}}{\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|\partial_x u(t,x)|}\right\}$. It then follows that

$$
u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h) \geq -\hat{\delta}e^{-\omega}. \tag{3.7}
$$

(ii) $|x - \hat{\xi} - X(\tau)| \geq M$. In this case, we have

$$
\begin{aligned}
&u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h) \\
&\quad = [u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^-(\tau+1))] \\
&\qquad + [u(\tau+1, x - \hat{\xi}^-(\tau+1)) - u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h)] \\
&\quad \geq -\hat{\delta}e^{-\omega} - \epsilon^* h \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |\partial_x u(t,x)|,
\end{aligned}
\tag{3.8}
$$

where we used the first inequality in (3.4) and Taylor expansion.

Hence, by (3.7), (3.8) and the second inequality in (3.4), we find

$$
\begin{aligned}
&u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h) - \hat{\delta}e^{-\omega} - C^*\epsilon^* h \\
&\quad \leq u(\tau+1, x; t_0) \leq u(\tau+1, x - \hat{\xi}^+(\tau+1) - \hat{h}) + \hat{\delta}e^{-\omega},
\end{aligned}
\tag{3.9}
$$

where $C^* = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|\partial_x u(t,x)|$. Taking ϵ^* smaller, if necessary, so that $\hat{\delta}e^{-\omega} + C^*\epsilon^* h < 1 - \theta_1$, and applying Corollary 3.3 to (3.9), we conclude

$$
u(t, x - \tilde{\xi}^-(t)) - \tilde{\delta}e^{-\omega(t-\tau-1)} \leq u(t, x; t_0) \leq u(t, x - \tilde{\xi}^+(t)) + \tilde{\delta}e^{-\omega(t-\tau-1)} \tag{3.10}
$$

for $t \geq \tau+1$, where $\omega = \min\{\beta_0, \beta_1\}$, $\tilde{\delta} = \max\left\{\hat{\delta}e^{-\omega} + C^*\epsilon^* h, \hat{\delta}e^{-\omega}\right\} = \hat{\delta}e^{-\omega} + C^*\epsilon^* h$ and

$$
\begin{aligned}
\tilde{\xi}^-(t) &= \hat{\xi}^-(\tau+1) + \epsilon^* h - \frac{A\hat{\delta}}{\omega}\left(1 - e^{-\omega(t-\tau-1)}\right) \\
&= \hat{\xi} - \frac{2A\hat{\delta}}{\omega} + \epsilon^* h + \frac{A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right],
\end{aligned}
$$

$$\begin{aligned}\tilde{\xi}^{+}(t) &= \hat{\xi}^{+}(\tau+1) + \hat{h} + \frac{A\hat{\delta}}{\omega}\left(1 - e^{-\omega(t-\tau-1)}\right)\\ &= \hat{\xi} + \frac{2A\hat{\delta}}{\omega} + \hat{h} - \frac{A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right].\end{aligned}$$

Setting

$$\begin{aligned}\hat{\xi}(t) &= \tilde{\xi}^{-}(t) = \hat{\xi} - \frac{2A\hat{\delta}}{\omega} + \epsilon^{*}h + \frac{A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right],\\ \hat{h}(t) &= \tilde{\xi}^{+}(t) - \tilde{\xi}^{-}(t) = \hat{h} - \epsilon^{*}h + \frac{4A\hat{\delta}}{\omega} - \frac{2A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right],\\ \hat{\delta}(t) &= \tilde{\delta}e^{-\omega(t-\tau-1)} = \left[\hat{\delta}e^{-\omega} + C^{*}\epsilon^{*}h\right]e^{-\omega(t-\tau-1)},\end{aligned}$$

the estimate (3.10) can be written as

$$u(t, x - \hat{\xi}(t)) - \hat{\delta}(t) \le u(t, x; t_0) \le u(t, x - \hat{\xi}(t) - \hat{h}(t)) + \hat{\delta}(t), \quad x \in \mathbb{R}, \ t \ge \tau + 1. \tag{3.11}$$

Note that (3.11) is obtained under the assumption (3.6).

Now, we assume (3.5) and estimate the following term:

$$u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^{+}(\tau+1) - \hat{h} + \epsilon^{*}h)$$

from above. Arguing as before and replacing $\hat{h}$ by h at appropriate steps lead to

$$u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^{+}(\tau+1) - \hat{h} + \epsilon^{*}h) \le \hat{\delta}e^{-\omega} + C^{*}\epsilon^{*}h,$$

where $C^{*} = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)|$. This, together with the first inequality in (3.4), yields

$$\begin{aligned}&u(\tau+1, x - \hat{\xi}^{-}(\tau+1)) - \hat{\delta}e^{-\omega}\\ &\quad \le u(\tau+1, x; t_0) \le u(\tau+1, x - \hat{\xi}^{+}(\tau+1) - \hat{h} + \epsilon^{*}h) + \hat{\delta}e^{-\omega} + C^{*}\epsilon^{*}h.\end{aligned} \tag{3.12}$$

Then, applying Corollary 3.3 to (3.12), we find (3.10) again with

$$\begin{aligned}\hat{\xi}(t) &= \hat{\xi} - \frac{2A\hat{\delta}}{\omega} + \frac{A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right],\\ \hat{h}(t) &= \hat{h} - \epsilon^{*}h + \frac{4A\hat{\delta}}{\omega} - \frac{2A\hat{\delta}}{\omega}\left[e^{-\omega} + e^{-\omega(t-\tau-1)}\right],\\ \hat{\delta}(t) &= \left[\hat{\delta}e^{-\omega} + C^{*}\epsilon^{*}h\right]e^{-\omega(t-\tau-1)}.\end{aligned}$$

This completes the proof. □

Now, we use the 'squeezing technique' (see, e.g., [17, 48, 53]) to prove Theorem 3.1.

Proof of Theorem 3.1 (i) Let u_0 be the initial data as in the statement of the theorem. For any $t_0 \in \mathbb{R}$, Lemma 3.2 ensures the existence of $\xi^{\pm} = \xi^{\pm}(t_0, u_0) \in \mathbb{R}$ and $\mu = \mu(u_0)$ (independent of t_0) such that

$$u(t, x - \xi^{-}) - \mu e^{-\omega(t-t_0)} \le u(t, x; t_0, u_0) \le u(t, x - \xi^{+}) + \mu e^{-\omega(t-t_0)}$$

for $t \geq t_0$, where $\omega = \min\{\beta_0, \beta_1\}$. Let ϵ^* and C^* be as in the statement of Lemma 3.4. Choosing $T_0 = T_0(u_0) > 0$ such that

$$\delta_0 := \mu e^{-\omega T_0} < \delta_* := \min\left\{\theta_0, 1-\theta_1, \frac{\epsilon^*\omega}{8A}\right\} < 1,$$

we find

$$u(t_0+T_0, x-\xi_0) - \delta_0 \leq u(t_0+T_0, x; t_0, u_0) \leq u(t_0+T_0, x-\xi_0-h_0) + \delta_0, \tag{3.13}$$

where $\xi_0 = \xi^-$ and $h_0 = \xi^+ - \xi^-$. Notice, we may assume, without loss of generality, that $\xi^+ > \xi^-$, so $h_0 > 0$. But h_0 depends on u_0, so we may assume, without loss of generality, that $h_0 > 1$. Let $T > 1$ be such that

$$\left[e^{-\omega} + C^*\epsilon^*\right] e^{-\omega(T-1)} \leq \delta_* := \min\left\{\theta_0, 1-\theta_1, \frac{\epsilon^*\omega}{8A}\right\}.$$

We are going to reduce h_0.

Applying Lemma 3.4 to (3.13), we find

$$\begin{aligned} &u(t_0+T_0+T, x-\xi_1) - \delta_1 \\ &\quad \leq u(t_0+T_0+T, x; t_0, u_0) \leq u(t_0+T_0+T, x-\xi_1-h_1) + \delta_1, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \xi_1 \in &\left[\xi_0 - \frac{2A\delta_0}{\omega}, \xi_0 + \epsilon^* \min\{1, h_0\}\right] = \left[\xi_0 - \frac{2A\delta_0}{\omega}, \xi_0 + \epsilon^*\right] \subset \left[\xi_0 - \frac{\epsilon^*}{4}, \xi_0 + \epsilon^*\right], \\ 0 \leq &h_1 \leq h_0 - \epsilon^* \min\{1, h_0\} + \frac{4A\delta_0}{\omega} = h_0 - \epsilon^* + \frac{4A\delta_0}{\omega} \leq h_0 - \frac{\epsilon^*}{2}, \\ 0 \leq &\delta_1 \leq \left[\delta_0 e^{-\omega} + C^*\epsilon^* \min\{1, h_0\}\right] e^{-\omega(T-1)} = \left[\delta_0 e^{-\omega} + C^*\epsilon^*\right] e^{-\omega(T-1)} \leq \delta_*. \end{aligned}$$

If $h_1 \leq 1$, we stop. Otherwise, we apply Lemma 3.4 to (3.14) to find

$$\begin{aligned} &u(t_0+T_0+2T, x-\xi_2) - \delta_2 \\ &\quad \leq u(t_0+T_0+2T, x; t_0, u_0) \leq u(t_0+T_0+2T, x-\xi_2-h_2) + \delta_2, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \xi_2 \in &\left[\xi_1 - \frac{2A\delta_1}{\omega}, \xi_1 + \epsilon^* \min\{1, h_1\}\right] = \left[\xi_1 - \frac{2A\delta_1}{\omega}, \xi_1 + \epsilon^*\right] \subset \left[\xi_1 - \frac{\epsilon^*}{4}, \xi_1 + \epsilon^*\right], \\ 0 \leq &h_2 \leq h_1 - \epsilon^* \min\{1, h_1\} + \frac{4A\delta_1}{\omega} = h_1 - \epsilon^* + \frac{4A\delta_1}{\omega} \leq h_0 - 2\left(\frac{\epsilon^*}{2}\right), \\ 0 \leq &\delta_2 \leq \left[\delta_1 e^{-\omega} + C^*\epsilon^* \min\{1, h_1\}\right] e^{-\omega(T-1)} = \left[\delta_1 e^{-\omega} + C^*\epsilon^*\right] e^{-\omega(T-1)} \leq \delta_*. \end{aligned}$$

If $h_2 \leq 1$, we stop. Otherwise, we apply Lemma 3.4 to (3.15), and repeat this. Suppose $h_i > 1$ for all $i = 0, 1, 2, \ldots n-1$, we then have

$$\begin{aligned} &u(t_0+T_0+nT, x-\xi_n) - \delta_n \\ &\quad \leq u(t_0+T_0+nT, x; t_0, u_0) \leq u(t_0+T_0+nT, x-\xi_n-h_n) + \delta_n, \end{aligned} \tag{3.16}$$

where

$$\xi_n \in \left[\xi_{n-1} - \frac{2A\delta_{n-1}}{\omega}, \xi_{n-1} + \epsilon^* \min\{1, h_{n-1}\}\right] \subset \left[\xi_{n-1} - \frac{\epsilon^*}{4}, \xi_{n-1} + \epsilon^*\right],$$
$$0 \le h_n \le h_{n-1} - \epsilon^* \min\{1, h_{n-1}\} + \frac{4A\delta_{n-1}}{\omega} = h_{n-1} - \epsilon^* + \frac{4A\delta_{n-1}}{\omega} \le h_0 - n\left(\frac{\epsilon^*}{2}\right),$$
$$0 \le \delta_n \le \left[\delta_{n-1}e^{-\omega} + C^*\epsilon^* \min\{1, h_{n-1}\}\right] e^{-\omega(T-1)} = \left[\delta_{n-1}e^{-\omega} + C^*\epsilon^*\right] e^{-\omega(T-1)} \le \delta_*.$$

Note that since $h_0 > 1$ and $\frac{\epsilon^*}{2} \in (0, 1)$, there must exist some $N = N(u_0) > 0$ such that $h_i > 1$ for $i = 0, 1, 2, \ldots, N-1$ and $0 < h_0 - N(\frac{\epsilon^*}{2}) \le 1$. In particular, $h_N \le 1$. Then, we stop and obtain from (3.16) that

$$u(\tilde{t}_0, x - \tilde{\xi}_0) - \tilde{\delta}_0 \le u(\tilde{t}_0, x; t_0, u_0) \le u(\tilde{t}_0, x - \tilde{\xi}_0 - \tilde{h}_0) + \tilde{\delta}_0, \tag{3.17}$$

where $\tilde{t}_0 = t_0 + T_0 + NT$, $\tilde{\xi}_0 = \xi_N$, $\tilde{\delta}_0 = \delta_N \le \delta_*$ and $\tilde{h}_0 = h_N \le 1$.

Now, we treat (3.17) as the new initial estimate and run the iteration argument again. Let $\tilde{T} > 1$ be such that

$$\left[e^{-\omega} + C^*\epsilon^*\right] e^{-\omega(\tilde{T}-1)} \le \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon^*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\}.$$

Applying Lemma 3.4 to (3.17), we find

$$u(\tilde{t}_0 + \tilde{T}, x - \tilde{\xi}_1) - \tilde{\delta}_1 \le u(\tilde{t}_0 + \tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + \tilde{T}, x - \tilde{\xi}_1 - \tilde{h}_1) + \tilde{\delta}_1, \tag{3.18}$$

where

$$\tilde{\xi}_1 \in \left[\tilde{\xi}_0 - \frac{2A\tilde{\delta}_0}{\omega}, \tilde{\xi}_0 + \epsilon^*\tilde{h}_0\right],$$
$$0 \le \tilde{h}_1 \le \tilde{h}_0 - \epsilon^*\tilde{h}_0 + \frac{4A\tilde{\delta}_0}{\omega} \le 1 - \frac{\epsilon^*}{2},$$
$$0 \le \tilde{\delta}_1 \le \left[\tilde{\delta}_0 e^{-\omega} + C^*\epsilon^*\tilde{h}_0\right] e^{-\omega(\tilde{T}-1)} \le \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon^*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\}.$$

Applying Lemma 3.4 to (3.18), we find

$$u(\tilde{t}_0 + 2\tilde{T}, x - \tilde{\xi}_2) - \tilde{\delta}_2 \le u(\tilde{t}_0 + 2\tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + 2\tilde{T}, x - \tilde{\xi}_2 - \tilde{h}_2) + \tilde{\delta}_2,$$

where

$$\tilde{\xi}_2 \in \left[\tilde{\xi}_1 - \frac{2A\tilde{\delta}_1}{\omega}, \tilde{\xi}_1 + \epsilon^*\tilde{h}_1\right],$$
$$0 \le \tilde{h}_2 \le \tilde{h}_1 - \epsilon^*\tilde{h}_1 + \frac{4A\tilde{\delta}_1}{\omega} \le \left(1 - \frac{\epsilon^*}{2}\right)(1 - \epsilon^*) + \frac{\epsilon_*}{2}\left(1 - \frac{\epsilon^*}{2}\right) = \left(1 - \frac{\epsilon^*}{2}\right)^2,$$
$$0 \le \tilde{\delta}_2 \le \left[\tilde{\delta}_1 e^{-\omega} + C^*\epsilon^*\tilde{h}_1\right] e^{-\omega(\tilde{T}-1)} \le \left(1 - \frac{\epsilon^*}{2}\right) \times \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon_*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\}.$$

Applying Lemma 3.4 repeatedly, we find for $n \ge 3$

$$u(\tilde{t}_0 + n\tilde{T}, x - \tilde{\xi}_n) - \tilde{\delta}_n \le u(\tilde{t}_0 + n\tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + n\tilde{T}, x - \tilde{\xi}_n - \tilde{h}_n) + \tilde{\delta}_n, \tag{3.19}$$

where

$$
\begin{aligned}
&\tilde{\xi}_n \in \left[\tilde{\xi}_{n-1} - \frac{2A\tilde{\delta}_{n-1}}{\omega}, \tilde{\xi}_{n-1} + \epsilon^* \tilde{h}_{n-1}\right],\\
&0 \leq \tilde{h}_n \leq \tilde{h}_{n-1} - \epsilon^* \tilde{h}_{n-1} + \frac{4A\tilde{\delta}_{n-1}}{\omega} \leq \left(1 - \frac{\epsilon^*}{2}\right)^{n-1} (1-\epsilon^*) + \frac{\epsilon_*}{2}\left(1 - \frac{\epsilon^*}{2}\right)^{n-1} = \left(1 - \frac{\epsilon^*}{2}\right)^{n},\\
&0 \leq \tilde{\delta}_n \leq \left[\tilde{\delta}_{n-1} e^{-\omega} + C^* \epsilon^* \tilde{h}_{n-1}\right] e^{-\omega(\tilde{T}-1)} \leq \left(1 - \frac{\epsilon^*}{2}\right)^{n-1} \times \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon^*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\}.
\end{aligned}
\tag{3.20}
$$

The result then follows readily. In fact, applying Corollary 3.3 to (3.19) for $n \geq 0$, we find, in particular,

$$
u(t, x - \tilde{\xi}_n + \frac{A}{\omega}\tilde{\delta}_n) - \tilde{\delta}_n e^{-\omega(t-\tilde{t}_0-n\tilde{T})} \leq u(t, x; t_0, u_0) \leq u(t, x - \tilde{\xi}_n - \tilde{h}_n - \frac{A}{\omega}\tilde{\delta}_n) + \tilde{\delta}_n e^{-\omega(t-\tilde{t}_0-n\tilde{T})}
$$

for all $t \geq \tilde{t}_0 + n\tilde{T}$. Therefore, setting $\tilde{\delta}(t) = \tilde{\delta}_n e^{-\omega(t-\tilde{t}_0-n\tilde{T})}$, $\tilde{\xi}(t) = \tilde{\xi}_n - \frac{A}{\omega}\tilde{\delta}_n$ and $\tilde{h}(t) = \tilde{h}_n + \frac{2A}{\omega}\tilde{\delta}_n$ for $t \in [\tilde{t}_0 + n\tilde{T}, \tilde{t}_0 + (n+1)\tilde{T})$ and $n = 1, 2, 3 \ldots$, we arrive at

$$
u(t, x - \tilde{\xi}(t)) - \tilde{\delta}(t) \leq u(t, x; t_0, u_0) \leq u(t, x - \tilde{\xi}(t) - \tilde{h}(t)) + \tilde{\delta}(t).
$$

Hence, we find

$$
u(t, x - \tilde{\xi}(t)) - \tilde{\delta}(t) \leq u(t, x; t_0, u_0) \leq u(t, x - \tilde{\xi}(t) - \tilde{h}(t)) + \tilde{\delta}(t), \quad t \geq \tilde{t}_0.
$$

To finish the proof, it suffices to show that $\tilde{\xi}(t) \to \tilde{\xi}(\infty)$, $\tilde{\delta}(t) \to 0$ and $\tilde{h}(t) \to 0$ exponentially as $h \to \infty$ for some $\tilde{\xi}(\infty) \in \mathbb{R}$. We see from (3.20) and the definitions of $\tilde{\delta}(t)$ and $\tilde{h}(t)$ that

$$
\begin{aligned}
&0 \leq \tilde{\delta}(t) \leq \left(1 - \frac{\epsilon^*}{2}\right)^{n-1} \times \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon^*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\} \times e^{-\omega\tilde{T}},\\
&0 \leq \tilde{h}(t) \leq \left(1 - \frac{\epsilon^*}{2}\right)^{n} + \frac{2A}{\omega} \times \left(1 - \frac{\epsilon^*}{2}\right)^{n-1} \times \min\left\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon^*}{2}\left(1 - \frac{\epsilon^*}{2}\right)\right\}
\end{aligned}
$$

for $t \in [\tilde{t}_0 + n\tilde{T}, \tilde{t}_0 + (n+1)\tilde{T})$ and $n = 1, 2, 3 \ldots$. Therefore, $\tilde{\delta}(t) \to 0$ and $\tilde{h}(t) \to 0$ exponentially as $t \to \infty$. For $\tilde{\xi}(t)$, we first see from (3.20) that the sequence $\{\tilde{\xi}_n\}_n$ is Cauchy, and $\tilde{\xi}_n \to \tilde{\xi}_\infty$ exponentially as $n \to \infty$ for some $\tilde{\xi}_\infty \in \mathbb{R}$. Then, $\tilde{\xi}(t) \to \tilde{\xi}(\infty) := \tilde{\xi}_\infty$ exponentially as $t \to \infty$. This completes the result.

We remark that the dependence of C on u_0 in the statement of the theorem is due to the dependence of T_0 on u_0.

(ii) By Corollary 3.3, we see

$$
u(t, x - \xi^-) - \mu_0 e^{-\omega(t-t_0)} \leq u(t, x; t_0, u_{t_0}) \leq u(t, x - \xi^+) + \mu_0 e^{-\omega(t-t_0)}, \quad x \in \mathbb{R},
$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$, where $\omega = \min\{\beta_0, \beta_1\}$ and $\xi^\pm = \xi_0^\pm \pm \frac{A\mu_0}{\omega}$. Then, by the arguments as in (i), there exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and a family of shifts $\{\xi_{t_0}\}_{t_0 \in \mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0 \in \mathbb{R}} |\xi_{t_0}| < \infty$ such that

$$
\sup_{x \in \mathbb{R}} \left| u(t, x; t_0, u_{t_0}) - u(t, x - \xi_{t_0}) \right| \leq C e^{-\omega_*(t-t_0)}
$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$. $\square$

4 Exponential decaying estimates of space non-increasing transition fronts

In this section, we prove exponential decaying estimates of space non-increasing transition fronts of (1.1). Throughout this section, we assume (H1)–(H3) and assume that $u(t,x)$ is a transition front of (1.1) with interface location functions $X(t)$ and $X_\lambda(t)$ and $u_x(t,x) \le 0$.

The main results in this section are stated in the following theorem.

Theorem 4.1 *There exist $c^\pm > 0$ and $h^\pm > 0$ such that*

$$u(t,x) \le e^{-c^+(x-X(t)-h^+)} \quad \text{and} \quad 1-u(t,x) \le e^{c^-(x-X(t)+h^-)}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. In particular, for any $\lambda \in (0,1)$, there exist $h^\pm_\lambda > 0$ such that

$$u(t,x) \le e^{-c^+(x-X_\lambda(t)-h^+_\lambda)} \quad \text{and} \quad 1-u(t,x) \le e^{c^-(x-X_\lambda(t)+h^-_\lambda)}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}$.

To prove Theorem 4.1, we first prove several lemmas. Let $\theta_2 \in (0, \min\{\frac{1}{4}, \theta_0, 1-\theta_1\})$ be small and $h > 0$, and define $u_0^\pm : \mathbb{R} \to [0,1]$ to be continuously differentiable and non-increasing functions satisfying

$$u_0^+(x) = \begin{cases} 1-\theta_2, & x \le -h, \\ 0, & x \ge 0, \end{cases} \quad \text{and} \quad u_0^-(x) = \begin{cases} 1, & x \le 0, \\ \theta_2, & x \ge h. \end{cases} \tag{4.1}$$

Moreover, we can make u_0^+ decreasing on $(-h,0)$ and u_0^- decreasing on $(0,h)$. For $t_0 \in \mathbb{R}$, we define

$$\begin{aligned} u^+(t,x;t_0) &:= u(t,x;t_0,u_0^+(\cdot - X_{1-\theta_2}(t_0))), \\ u^-(t,x;t_0) &:= u(t,x;t_0,u_0^-(\cdot - X_{\theta_2}(t_0))) \end{aligned}$$

for $t \ge t_0$.

Lemma 4.2 *$u^\pm(t,x;t_0)$ satisfy the following properties:*

(i) *$u^\pm(t,x;t_0)$ are decreasing in x for any $t > t_0$;*
(ii) *for any $t > t_0$, we have*

$$\begin{aligned} &\lim_{x\to-\infty} u^+(t,x;t_0) > 1-\theta_2, \quad \lim_{x\to\infty} u^+(t,x;t_0) = 0, \\ &\lim_{x\to-\infty} u^-(t,x;t_0) = 1 \quad \text{and} \quad \lim_{x\to\infty} u^-(t,x;t_0) < \theta_2. \end{aligned}$$

Proof (i) It follows from the fact that $u_0^\pm$ are non-increasing and the 'Moreover' part in Proposition A.1(iii) or Proposition A.3(ii), which is applicable here due to the assumption (H2) and the continuous differentiability of $u^\pm$.

(ii) By (i), the limits $\lim_{x\to\pm\infty} u^+(t,x;t_0)$ are well defined. We show $\lim_{x\to\infty} u^+(t,x;t_0) = 0$. Let $\phi_{\tilde{B}}(x - c_{\tilde{B}}t)$ be a travelling front of $\partial_t u = J*u - u + f_{\tilde{B}}(u)$ such that $\phi_{\tilde{B}}(-\infty) = 1$ and $\phi_{\tilde{B}}(\infty) = 0$. By the definition of u_0^+, we can find a shift $x_1 \gg 1$ such that

$$u_0^+(\cdot - X_{1-\theta_2}(t_0)) \le \phi_{\tilde{B}}(\cdot - x_1 - c_{\tilde{B}}t_0).$$

It then follows from comparison principle that $u^+(t,\cdot;t_0)\le\phi_{\tilde{B}}(\cdot-x_1-c_{\tilde{B}}t)$ for any $t>0$. From which, we conclude $\lim_{x\to\infty}u^+(t,x;t_0)=0$.

We show $\lim_{x\to-\infty}u^+(t,x;t_0)>1-\theta_2$. Note that $u^+(t,x;t_0)$ satisfies

$$\partial_t u^+(t,x;t_0)=\int_{\mathbb{R}}J(x-y)u^+(t,y;t_0)dy-u^+(t,x;t_0)+f(t,u^+(t,x;t_0)),\quad t>t_0, \tag{4.2}$$

and

$$\begin{aligned}\partial_{tt}u^+(t,x;t_0)=&\int_{\mathbb{R}}J(x-y)\partial_t u^+(t,y;t_0)dy-\partial_t u^+(t,x;t_0)\\&+\partial_t f(t,u^+(t,x;t_0))+\partial_u f(t,u^+(t,x;t_0))\partial_t u^+(t,x;t_0),\quad t>t_0.\end{aligned}$$

Pick an arbitrary sequence $\{x_n\}$ with $x_n\to-\infty$ as $n\to\infty$. We see that there is an $M>0$ such that

$$\max\left\{\left|\partial_t u^+(t,x_n;t_0)\right|,\left|\partial_{tt}u^+(t,x_n;t_0)\right|\right\}\le M\ \text{ for all }\ t>t_0\ \text{ and }\ n\ge 1.$$

Since $u^+(t,x_n;t_0)\in[0,1]$ for all $t\ge t_0$ and $n\ge1$, we conclude from the Arzelà–Ascoli theorem that there exists a continuous function $w:[t_0,\infty)\to[0,1]$, differentiable on (t_0,∞) such that

$$\begin{aligned}&u^+(t,x_n;t_0)\to w(t)\ \text{locally uniformly in } t\in[t_0,\infty) \text{ as } n\to\infty,\quad\text{and}\\&\partial_t u^+(t,x_n;t_0)\to w_t(t)\ \text{locally uniformly in } t\in(t_0,\infty) \text{ as } n\to\infty.\end{aligned}$$

As a consequence, letting $x\to-\infty$ along the sequence $\{x_n\}$ in (4.2), we find that $w(t)$ is the unique solution of

$$\begin{cases}\dot{w}(t)=f(t,w(t)), & t>t_0,\\ w(t_0)=1-\theta_2.\end{cases}$$

Now, comparing $f(t,u)$ with $f_B(u)$, we conclude from the comparison principle for ODEs that $w(t)>1-\theta_2$ for all $t>t_0$. But the monotonicity of $u^+(t,x;t_0)$ in x from (i) yields

$$\lim_{x\to-\infty}u^+(t,x;t_0)=\lim_{n\to\infty}u^+(t,x_n;t_0)=w(t)>1-\theta_2,\quad t>t_0.$$

The limits $\lim_{x\to-\infty}u^-(t,x;t_0)=1$ and $\lim_{x\to\infty}u^-(t,x;t_0)<\theta_2$ follow from similar arguments, and therefore, we omit the proof. □

By Lemma 4.2, for any $\lambda\in(\theta_2,1-\theta_2)$, the interface locations $X^{\pm}_{\lambda}(t;t_0)\in\mathbb{R}$ such that

$$u^{\pm}(t,X^{\pm}_{\lambda}(t;t_0);t_0)=\lambda$$

are well defined for all $t\ge t_0$.

The first lemma gives the uniform boundedness of the gap between the interface locations of $u^{\pm}(t,x;t_0)$ and $u(t,x)$.

Lemma 4.3 *For any $\lambda\in(\theta_2,1-\theta_2)$, there hold*

$$\sup_{t_0\in\mathbb{R}}\sup_{t\ge t_0}\left|X^{\pm}_{\lambda}(t;t_0)-X(t)\right|<\infty.$$

Proof Let $\lambda \in (\theta_2, 1-\theta_2)$. By the definition of u_0^+, we see that $u_0^+(x - X_{1-\theta_2}(t_0)) \le u(t_0, x)$ for $x \in \mathbb{R}$. Comparison principle then yields $u^+(t,x;t_0) \le u(t,x)$ for $x \in \mathbb{R}$ and $t \ge t_0$. In particular, $X_\lambda^+(t;t_0) \le X_\lambda(t)$ for all $t \ge t_0$.

Moreover, we readily check that

$$u_0^+(x - X_{\theta_2}(t_0) - h) + \theta_2 \ge u(t_0, x),$$

which is equivalent to

$$u(t_0, x + X_{\theta_2}(t_0) + h - X_{1-\theta_2}(t_0)) - \theta_2 \le u_0^+(x - X_{1-\theta_2}(t_0)) = u^+(t_0, x; t_0).$$

Setting $L := \sup_{t_0 \in \mathbb{R}} |X_{\theta_2}(t_0) + h - X_{1-\theta_2}(t_0)| < \infty$, we see from the monotonicity of $u(t,x)$ in x that

$$u(t_0, x - (-L)) - \theta_2 \le u^+(t_0, x; t_0) \text{ for all } t_0 \in \mathbb{R}.$$

We apply the 'In particular' part in Corollary 3.3 to conclude that

$$u(t, x - (-L - \frac{A\theta_2}{\omega})) - \theta_2 \le u(t, x - (-L - \frac{A\theta_2}{\omega})) - \theta_2 e^{-\omega(t-t_0)} \le u^+(t,x;t_0), \quad x \in \mathbb{R},$$

for all $t \ge t_0$ and $t_0 \in \mathbb{R}$. Setting $x = -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t)$, we find

$$\lambda \le u^+(t, -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t); t_0),$$

which implies by monotonicity that

$$X_\lambda^+(t;t_0) \ge -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t) \quad \text{for all} \quad t \ge t_0.$$

Hence, we have shown that

$$X_\lambda^+(t;t_0) \le X_\lambda(t) \quad \text{and} \quad X_\lambda^+(t;t_0) \ge -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t)$$

for all $t \ge t_0$ and $t_0 \in \mathbb{R}$. Since $\sup_{t\in\mathbb{R}} |X_\lambda(t) - X_{\lambda+\theta_2}(t)| < \infty$ and L, θ_2 and A are t_0-independent, we arrive at

$$\sup_{t_0 \in \mathbb{R}} \sup_{t \ge t_0} |X_\lambda^+(t;t_0) - X_\lambda(t)| < \infty,$$

which is clearly equivalent to $\sup_{t_0\in\mathbb{R}} \sup_{t\ge t_0} |X_\lambda^+(t;t_0) - X(t)| < \infty$.

The another result $\sup_{t_0\in\mathbb{R}} \sup_{t\ge t_0} |X_\lambda^-(t;t_0) - X(t)| < \infty$ follows along the same line. □

Next, we prove the uniform exponential decaying estimates of $u^\pm(t,x;t_0)$.

Lemma 4.4 *There exist $c^\pm > 0$ and $h^\pm > 0$ such that*

$$u^+(t,x;t_0) \le e^{-c^+(x - X(t) - h^+)} \quad \textit{and} \quad u^-(t,x;t_0) \ge 1 - e^{c^-(x - X(t) + h^-)}$$

for all $x \in \mathbb{R}$, $t \ge t_0$ and $t_0 \in \mathbb{R}$.

Proof We prove the first estimate; the second one can be proved in a similar way. Note first that $f(t,u) \le -\beta_0 u$ for $u \in [0,\theta_0]$. Let

$$h := \max\left\{\sup_{t\ge t_0}|X^+_{\theta_0}(t;t_0) - X(t)|, \quad \sup_{t\in\mathbb{R}}|X_{1-\theta_2}(t) - X(t)|\right\} < \infty$$

by Lemma 4.3, since $\theta_0 \in (\theta_2, 1-\theta_2)$. We consider

$$N[u] = \partial_t u - [J * u - u] + \beta_0 u.$$

Since $u^+(t,x;t_0) \le \theta_0$ for $x \ge X^+_{\theta_0}(t;t_0)$, we find

$$N[u^+] = \beta_0 u^+ + f(t,u^+) \le 0 \quad \text{for} \quad x \ge X^+_{\theta_0}(t;t_0).$$

In particular, $N[u^+] \le 0$ for $x \ge X(t) + h$.

Now, let $c > 0$. We see

$$N[e^{-c(x-X(t)-h)}] = \left[c\dot{X}(t) - \int_{\mathbb{R}} J(y)e^{cy}dy + 1 + \beta_0\right] e^{-c(x-X(t)-h)}.$$

Since $\dot{X}(t) \ge c_{\min} > 0$ by (1.7) and $\int_{\mathbb{R}} J(y)e^{cy}dy \to 1$ as $c \to 0$, we can find some $c^+ > 0$ such that $N[e^{-c^+(x-X(t)-h)}] \ge 0$. Thus, we have

- $N[u^+(t,x;t_0)] \le 0 \le N[e^{-c^+(x-X(t)-h)}]$ for $x \ge X(t)+h$ and $t > t_0$,
- $u^+(t,x;t_0) < 1 \le e^{-c^+(x-X(t)-h)}$ for $x \le X(t)+h$ and $t > t_0$,
- $u^+(t_0,x;t_0) = u_0^+(x - X_{1-\theta_2}(t_0)) \le e^{-c^+(x-X(t_0)-h)}$ for $x \in \mathbb{R}$.

We then conclude from Proposition A.1(i) that $u^+(t,x;t_0) \le e^{-c^+(x-X(t)-h)}$ for all $x \in \mathbb{R}$, $t \ge t_0$ and $t_0 \in \mathbb{R}$. This completes the proof. □

We also need the uniform-in-t_0 exponential convergence of $u^\pm(t,x;t_0)$ to $u(t,x)$.

Lemma 4.5 *There exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and two families of shifts $\{\xi^\pm_{t_0}\}_{t_0\in\mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}}|\xi^\pm_{t_0}| < \infty$ such that*

$$\sup_{x\in\mathbb{R}}\left|u^\pm(t,x;t_0) - u(t,x-\xi^\pm_{t_0})\right| \le Ce^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$ and $t_0 \in \mathbb{R}$.

Proof Let $C_2 = \sup_{t\in\mathbb{R}}|X_{\theta_2}(t) - X_{1-\theta_2}(t)| < \infty$. Then, it is easy to see that for any $t_0 \in \mathbb{R}$,

$$u(t_0, x + C_2 + h) - \theta_2 \le u_0^+(x - X_{1-\theta_2}(t_0)) \le u(t_0,x) + \epsilon_0, \quad x \in \mathbb{R},$$
$$u(t_0,x) - \epsilon_0 \le u_0^-(x - X_{\theta_2}(t_0)) \le u(t_0, x - C_2 - h) + \theta_2, \quad x \in \mathbb{R},$$

for arbitrary fixed $\epsilon_0 \in (0, \min\{\frac{1}{4}, \theta_0, 1-\theta_1\})$, that is,

$$u(t_0, x + C_2 + h) - \mu_0 \le u^+(t_0,x;t_0) \le u(t_0,x) + \mu_0, \quad x \in \mathbb{R},$$
$$u(t_0,x) - \mu_0 \le u^-(t_0,x;t_0) \le u(t_0, x - C_2 - h) + \mu_0, \quad x \in \mathbb{R},$$

where $\mu_0 = \max\{\theta_2, \epsilon_0\}$ and h is as in (4.1). Since C_2, h and μ_0 are independent of $t_0 \in \mathbb{R}$, we apply Theorem 3.1 to conclude the result. □

Finally, we prove Theorem 4.1.

Proof of Theorem 4.1 By Lemmas 4.4 and 4.5, we have

$$u(t,x-\xi_{t_0}^+)\le u^+(t,x;t_0)+Ce^{-\omega_*(t-t_0)}\le e^{-c^+(x-X(t)-h^+)}+Ce^{-\omega_*(t-t_0)}$$

for all $x\in\mathbb{R}$ and $t\ge t_0$. Since $\sup_{t_0\in\mathbb{R}}|\xi_{t_0}^+|<\infty$, there exists $\xi^+\in\mathbb{R}$ such that $\xi_{t_0}^+\to\xi^+$ as $t_0\to-\infty$ along some subsequence. Thus, for any $(t,x)\in\mathbb{R}\times\mathbb{R}$, letting $t_0\to-\infty$ along this subsequence, we find $u(t,x-\xi^+)\le e^{-c^+(x-X(t)-h^+)}$. The lower bound for $u(t,x)$ follows similarly. The 'in particular' part then is a simple consequence of the fact that $\sup_{t\in\mathbb{R}}|X_\lambda(t)-X(t)|<\infty$ for any $\lambda\in(0,1)$. □

5 Uniqueness and monotonicity of transition fronts

In this section, we study the uniqueness and monotonicity of transition fronts of (1.1) under the assumptions Hypotheses (H1)–(H3) and the assumption that (1.1) has a space non-increasing transition front $u(t,x)$.

Let $v(t,x)$ be an arbitrary transition front (not necessarily non-increasing in space), and $u(t,x)$ be an arbitrary space non-increasing transition front of (1.1). Let $Y(t)$, $Y_\lambda^\pm(t)$ be the interface location functions of $v(t,x)$, and $X(t)$, $X_\lambda(t)=X_\lambda^\pm(t)$ be the interface location functions of $u(t,x)$. By Proposition 1.2, we may assume that both $X(t)$ and $Y(t)$ are continuously differentiable and satisfy (1.7). By Corollary 2.3, $X_\lambda(t)$ is continuously differentiable. But, $Y_\lambda^\pm(t)$ may have a jump.

We prove the following theorem.

Theorem 5.1 *There exists some $\xi\in\mathbb{R}$ such that $v(t,x)=u(t,x+\xi)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$. In particular, $v(t,x)$ is non-increasing in x.*

To show Theorem 5.1, we first prove the following lemma.

Lemma 5.2 *There holds* $\sup_{t\in\mathbb{R}}|X(t)-Y(t)|<\infty$.

Proof Since $\sup_{t\in\mathbb{R}}|X_{\frac12}(t)-X(t)|<\infty$, it suffices to show: (i) $\sup_{t\ge0}|Y(t)-X_{\frac12}(t)|<\infty$; (ii) $\sup_{t\le0}|Y(t)-X_{\frac12}(t)|<\infty$.

(i) Let $\mu\in(0,\min\{\frac14,\theta_0,1-\theta_1\})$ be small. We first see that

$$u(0,x-Y_{1-\mu}^-(0)+X_\mu(0))-\mu\le v(0,x)\le u(0,x-Y_\mu^+(0)+X_{1-\mu}(0))+\mu,\quad x\in\mathbb{R}. \tag{5.1}$$

In fact, if $x\ge Y_{1-\mu}^-(0)$, then by the monotonicity of $u(t,x)$ in x, we have

$$u(0,x-Y_{1-\mu}^-(0)+X_\mu(0))-\mu\le u(0,X_\mu(0))-\mu=0<v(0,x).$$

If $x<Y_{1-\mu}^-(0)$, then

$$v(0,x)\ge1-\mu>u(0,x-Y_{1-\mu}^-(0)+X_\mu(0))-\mu.$$

This proves the first inequality. The second one is checked similarly.

Setting $\xi_0^-=Y_{1-\mu}^-(0)-X_\mu(0)$ and $\xi_0^+=Y_\mu^+(0)-X_{1-\mu}(0)$ in (5.1), and then, applying Corollary 3.3 to (5.1), we find

$$u(t,x-\xi^-)-\mu\le v(t,x)\le u(t,x-\xi^+)+\mu,\quad x\in\mathbb{R}, \tag{5.2}$$

for all $t \geq 0$, where $\xi^{\pm} = \xi_0^{\pm} \pm \frac{A\mu}{\omega}$. It then follows from the first inequality in (5.2) and the monotonicity of $u(t,x)$ in x that

$$\frac{1}{2} - \mu = u(t, X_{\frac{1}{2}}(t)) - \mu < u(t, x - \xi^-) - \mu \leq v(t,x) \quad \text{for all} \quad x < \xi^- + X_{\frac{1}{2}}(t),$$

which implies that $\xi^- + X_{\frac{1}{2}}(t) \leq Y^-_{\frac{1}{2}-\mu}(t)$ for $t \geq 0$. Similarly, the second inequality in (5.2) and the monotonicity of $u(t,x)$ in x implies that

$$v(t,x) \leq u(t, x - \xi^+) + \mu < u(t, X_{\frac{1}{2}}(t)) + \mu = \frac{1}{2} + \mu \quad \text{for all} \quad x > \xi^+ + X_{\frac{1}{2}}(t),$$

which leads to $Y^+_{\frac{1}{2}+\mu}(t) \leq \xi^+ + X_{\frac{1}{2}}(t)$ for $t \geq 0$. Since $\sup_{t\in\mathbb{R}} |Y^-_{\frac{1}{2}-\mu}(t) - Y(t)| < \infty$ and $\sup_{t\in\mathbb{R}} |Y(t) - Y^+_{\frac{1}{2}+\mu}(t)| < \infty$ by Lemma 2.2, we conclude that $\sup_{t\geq 0} |X_{\frac{1}{2}}(t) - Y(t)| < \infty$.

(ii) Suppose on the contrary that $\sup_{t\leq 0} |Y(t) - X_{\frac{1}{2}}(t)| = \infty$. Since both $Y(t)$ and $X_{\frac{1}{2}}(t)$ are continuous, there exists a sequence $t_n \to -\infty$ as $n \to \infty$ such that either $Y(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ or $Y(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n \to \infty$.

Suppose first that $Y(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ as $n \to \infty$. Since $\sup_{t\in\mathbb{R}} |Y(t) - Y^-_{\frac{1}{2}}(t)| < \infty$, we in particular have $Y^-_{\frac{1}{2}}(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ as $n \to \infty$. Then, for any $\mu > 0$ and $\xi_0 \in \mathbb{R}$, we can find an $N = N(\mu, \xi_0) > 0$ such that $t_N < 0$ and $u(t_N, x - \xi_0) - \mu \leq v(t_N, x)$ for $x \in \mathbb{R}$. We then apply Corollary 3.3 to conclude that

$$u(t, x - \xi_0 + \frac{A\mu}{\omega}) - \mu \leq v(t,x), \quad x \in \mathbb{R}, \quad t \geq t_N.$$

Then, setting $t = 0$ in the above estimate, we find from the monotonicity of $u(t,x)$ in x that

$$\frac{1}{2} - \mu = u(0, X_{\frac{1}{2}}(0)) - \mu < u(0, x - \xi_0 + \frac{A\mu}{\omega}) - \mu \leq v(0,x), \quad \forall x < \xi_0 - \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0),$$

which implies that $\xi_0 - \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0) \leq Y^-_{\frac{1}{2}-\mu}(0)$. Letting $\xi_0 \to \infty$, we arrive at a contradiction.

Now, suppose $Y(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n \to \infty$, which implies $Y^+_{\frac{1}{2}}(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n \to \infty$. Then, for any $\mu > 0$ and $\xi_0 \in \mathbb{R}$, we can find some $N = N(\mu, \xi_0) > 0$ such that $t_N < 0$ and $v(t_N, x) \leq u(t_N, x - \xi_0) + \mu$ for $x \in \mathbb{R}$. Applying Corollary 3.3, we find

$$v(t,x) \leq u(t, x - \xi_0 - \frac{A\mu}{\omega}) + \mu, \quad x \in \mathbb{R}, \quad t \geq t_N.$$

Setting $t = 0$ in the above estimate, we find

$$v(0,x) \leq u(0, x - \xi_0 - \frac{A\mu}{\omega}) + \mu < u(0, X_{\frac{1}{2}}(0)) + \mu = \frac{1}{2} + \mu, \quad \forall x > \xi_0 + \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0),$$

which implies that $Y^+_{\frac{1}{2}+\mu}(0) \leq \xi_0 + \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0)$. This leads to a contradiction if we let $\xi_0 \to -\infty$. Hence, we have $\sup_{t\leq 0} |Y(t) - X_{\frac{1}{2}}(t)| < \infty$. This completes the proof. □

Now, we prove Theorem 5.1.

Proof of Theorem 5.1 Let $\theta_3 \in (0, \min\{\theta_0, 1 - \theta_1\})$. For $t_0 \in \mathbb{R}$, we define

$$u^-(t_0, x) = u(t_0, x - Y^-_{1-\theta_3}(t_0) + X_{\theta_3}(t_0)) - \theta_3,$$
$$u^+(t_0, x) = u(t_0, x - Y^+_{\theta_3}(t_0) + X_{1-\theta_3}(t_0)) + \theta_3.$$

We claim

$$u^-(t_0,x) \le v(t_0,x) \le u^+(t_0,x), \quad x \in \mathbb{R}.$$

In fact, if $x \ge Y^-_{1-\theta_3}(t_0)$, then by monotonicity,

$$u^-(t_0,x) \le u(t_0, X_{\theta_3}(t_0)) - \theta_3 = 0 < v(t_0,x).$$

If $x < Y^-_{1-\theta_3}(t_0)$, then by the definition of $Y^-_{1-\theta_3}(t_0)$,

$$v(t_0,x) > 1-\theta_3 > u^-(t_0,x).$$

Hence, $u^-(t_0,x) \le v(t_0,x)$. The inequality $v(t_0,x) \le u^+(t_0,x)$ is checked similarly.

By Lemmas 2.2 and 5.2, we have

$$L := \max\left\{ \sup_{t_0\in\mathbb{R}} |Y^-_{1-\theta_3}(t_0) - X_{\theta_3}(t_0)|, \sup_{t_0\in\mathbb{R}} |Y^+_{\theta_3}(t_0) - X_{1-\theta_3}(t_0)| \right\} < \infty.$$

Then, shifting $u^-(t_0,x)$ to the left and $u^+(t_0,x)$ to the right, we conclude from the monotonicity of $u(t,x)$ in x that for all $t_0 \in \mathbb{R}$, there holds

$$u(t_0,x+L) - \theta_3 \le u^-(t_0,x) \le v(t_0,x) \le u^+(t_0,x) \le u(t_0,x-L) + \theta_3. \tag{5.3}$$

That is, we are in the position to apply Theorem 3.1. So, we apply Theorem 3.1 to (5.3) to conclude that there exist t_0-independent constants $C>0$ and $\omega_*>0$, and a family of shifts $\{\xi_{t_0}\}_{t_0\in\mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}| < \infty$ such that

$$\sup_{x\in\mathbb{R}} |v(t,x) - u(t,x-\xi_{t_0})| \le Ce^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$. We now pass to the limit $t_0 \to -\infty$ along some subsequence to conclude $\xi_{t_0} \to \xi$ for some $\xi \in \mathbb{R}$, and then conclude that $v(t,x) = u(t,x-\xi)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. This completes the proof. □

6 Periodicity and asymptotic speeds of transition fronts

In this section, we study the periodicity of transition fronts of (1.1) under the additional time periodic assumption on f; that is, there exists $T>0$ such that $f(t+T,u) = f(t,u)$ for all $t \in \mathbb{R}$ and $u \in [0,1]$. We also study asymptotic speeds of transition fronts of (1.1) under the additional *uniquely ergodic* assumption on f, that is, the dynamical system $\{\sigma_t\}_{t\in\mathbb{R}}$ defined by

$$\sigma_t : H(f) \to H(f), \quad f \mapsto f(\cdot + t, \cdot) \tag{6.1}$$

is compact (i.e. $H(f)$ is compact and metrisable) and uniquely ergodic, that is, $\{\sigma_t\}_{t\in\mathbb{R}}$ admits one and only one invariant measure, where

$$H(f) = \overline{\{f(\cdot+t,\cdot) : t \in \mathbb{R}\}}$$

with the closure taken under the open-compact topology (which is equivalent to locally uniform convergence in our case). Throughout this section, we assume (H1)–(H3).

Let $u(t,x)$ be a space non-increasing transition front of (1.1) with interface $X(t)$. The main results of this section are stated in the following theorem.

Theorem 6.1 (i) *Assume that $f(t,u)$ is T-periodic in t. Then, $u(t,x)$ is a T-periodic travelling front; that is, there are a constant $c>0$ and a function $\psi:\mathbb{R}\times\mathbb{R}\to(0,1)$ satisfying*

$$\begin{cases}\partial_t\psi=J*\psi-\psi+c\partial_x\psi+f(t,\psi),\\ \lim_{x\to-\infty}\psi(t,x)=1,\ \ \lim_{x\to\infty}\psi(t,x)=0 \ \textit{uniformly in}\ t\in\mathbb{R},\\ \psi(t,\cdot)=\psi(t+T,\cdot)\ \textit{for all}\ t\in\mathbb{R},\end{cases}\tag{6.2}$$

such that $u(t,x)=\psi(t,x-ct)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$. In particular,

$$c=-\frac{\int_0^T\int_{\mathbb{R}}f(t,\psi(t,x))\partial_x\psi(t,x)dxdt}{\int_0^T\int_{\mathbb{R}}[\partial_x\psi(t,x)]^2\,dxdt}.$$

(ii) *Assume that $f(t,u)$ is uniquely ergodic in t, and, in addition, twice continuously differentiable with*

$$\sup_{(t,u)\in\mathbb{R}\times[-1,2]}[|\partial_{tt}f(t,u)|+|\partial_{tu}f(t,u)|+|\partial_{uu}f(t,u)|]<\infty.$$

Then, the asymptotic speeds $\lim_{t\to\pm\infty}\frac{X(t)}{t}$ exist.

To prove Theorem 6.1, let us first do some preparation. Note that if f is periodic in t, then it is uniquely ergodic. In the rest of this section, we assume that $f(t,u)$ satisfies the assumptions in Theorem 6.1(ii).

Observe that any $g\in H(f)$ satisfies (H2)–(H3) due to the regularity assumptions on $f(t,u)$. For any $g\in H(f)$, there is a sequence $\{t_n\}$ such that $f(t+t_n,u)\to g(t,u)$ as $n\to\infty$ in open-compact topology. By the regularity, without loss of generality, we may assume that there is $u^g(t,x)$ such that $u(t+t_n,x+X(t_n))\to u^g(t,x)$ as $n\to\infty$ in open-compact topology. It is not difficult to see that $u^g(t,x)$ is a space non-increasing transition front of

$$\partial_t u=J*u-u+g(t,u).\tag{6.3}$$

Then, $u^g(t,x)$ is the unique transition front of (6.3) satisfying the normalisation $X^g_{\frac12}(0)=0$, where $X^g_{\frac12}(t)$ is the interface location function of $u^g(t,x)$ at $\frac12$, that is, $u^g(t,X^g_{\frac12}(t))=\frac12$ for all $t\in\mathbb{R}$.

Let

$$\psi^g(t,x)=u^g(t,x+X^g_{\frac12}(t)),\quad \forall(t,x)\in\mathbb{R}\times\mathbb{R},\tag{6.4}$$

be the profile function of $u^g(t,x)$. Then, $\psi^g(t,0)=\frac12$ for all $t\in\mathbb{R}$.

We prove the following.

Lemma 6.2 *There hold the following statements:*

(i) *for any $g\in H(f)$, there holds*

$$\psi^g(t+\tau,x)=\psi^{g\cdot\tau}(t,x),\quad \forall(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},$$

where $g\cdot\tau=g(\cdot+\tau,\cdot)$;

(ii) *there holds $\sup_{(t,\tau)\in\mathbb{R}\times\mathbb{R}}|\dot X^{f\cdot\tau}_{\frac12}(t)|<\infty$;*

 https://doi.org/10.1017/S0956792519000202

(iii) *if, in addition, f is twice continuously differentiable and satisfies*

$$\sup_{(t,u)\in\mathbb{R}\times[-1,2]}\big(|\partial_{tt}f(t,u)|+|\partial_{tu}f(t,u)|+|\partial_{uu}f(t,u)|\big)<\infty,$$

then there hold

$$\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}}\big[u^{f\cdot\tau}(t,x)+|\partial_t u^{f\cdot\tau}(t,x)|+|\partial_x u^{f\cdot\tau}(t,x)|+|\partial_{tt}u^{f\cdot\tau}(t,x)|$$
$$+|\partial_{tx}u^{f\cdot\tau}(t,x)|+|\partial_{xx}u^{f\cdot\tau}(t,x)|\big]<\infty,$$

and

$$\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}}\big[\psi^{f\cdot\tau}(t,x)+|\partial_t \psi^{f\cdot\tau}(t,x)|+|\partial_x \psi^{f\cdot\tau}(t,x)|+|\partial_{tt}\psi^{f\cdot\tau}(t,x)|$$
$$+|\partial_{tx}\psi^{f\cdot\tau}(t,x)|+|\partial_{xx}\psi^{f\cdot\tau}(t,x)|\big]<\infty.$$

(iv) *the limits*

$$\lim_{x\to-\infty}\psi^g(t,x)=1 \quad \textit{and} \quad \lim_{x\to\infty}\psi^g(t,x)=0$$

are uniformly in $t\in\mathbb{R}$ and $g\in H(f)$;

(v) *there holds $\sup_{g\in H(f)}\sup_{t\in\mathbb{R}}|\dot{X}^g_{\frac{1}{2}}(t)|<\infty$;*

We remark that (ii) is a special case of (v), but it plays an important role in proving the lemma, so we state it explicitly.

Proof of Lemma 6.2 For simplicity, we write $X^g(t)=X^g_{\frac{1}{2}}(t)$. Therefore, $u^g(t,X^g(t))=\frac{1}{2}$ and $X^g(0)=0$.

(i) Fix any $\tau\in\mathbb{R}$. We see that both

$$u_1(t,x)=\psi^{g\cdot\tau}(t,x-X^{g\cdot\tau}(t)) \quad \text{and} \quad u_2(t,x)=\psi^g(t+\tau,x-X^g(t+\tau))$$

are transition fronts of $\partial_t u=J*u-u+g(t+\tau,x)$. Then, by uniqueness, that is, Theorem 5.1, there exists $\xi\in\mathbb{R}$ such that $u_1(t,x)=u_2(t,x+\xi)$. Moreover, since

$$u_1(t,X^{g\cdot\tau}(t))=\psi^{g\cdot\tau}(t,0)=\frac{1}{2} \quad \text{and} \quad u_2(t,X^g(t+\tau))=\psi^g(t+\tau,0)=\frac{1}{2},$$

we find

$$u_1(t,X^g(t+\tau)-\xi)=u_2(t,X^g(t+\tau))=\frac{1}{2},$$

and hence,

$$X^{g\cdot\tau}(t)=X^g(t+\tau)-\xi \tag{6.5}$$

by monotonicity. It then follows that

$$\psi^{g\cdot\tau}(t,x)=u_1(t,x+X^{g\cdot\tau}(t))=u_2(t,x+X^{g\cdot\tau}(t)+\xi)$$
$$=u_2(t,x+X^g(t+\tau))=\psi^g(t+\tau,x).$$

(ii) For each $\tau \in \mathbb{R}$, we obtain from (6.5) that

$$X^{f\cdot\tau}(t) = X^f(t+\tau) - \xi, \quad \forall t \in \mathbb{R},$$

for some $\xi = \xi(\tau)$. The result then follows from Corollary 2.3.

(iii) By (6.4), (ii) and Proposition 1.2 (ii), there hold

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} \left[u^f(t,x) + |\partial_t u^f(t,x)| + |\partial_x u^f(t,x)|\right] < \infty,$$
$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} \left[\psi^f(t,x) + |\partial_t \psi^f(t,x)| + |\partial_x \psi^f(t,x)|\right] < \infty,$$

which together with (i) give

$$\begin{aligned} &\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}} \left[u^{f\cdot\tau}(t,x) + |\partial_t u^{f\cdot\tau}(t,x)| + |\partial_x u^{f\cdot\tau}(t,x)|\right] < \infty, \\ &\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}} \left[\psi^{f\cdot\tau}(t,x) + |\partial_t \psi^{f\cdot\tau}(t,x)| + |\partial_x \psi^{f\cdot\tau}(t,x)|\right] < \infty. \end{aligned} \tag{6.6}$$

Since

$$\partial_t u^{f\cdot\tau} = J * u^{f\cdot\tau} - u^{f\cdot\tau} + f(t+\tau, u^{f\cdot\tau}),$$

we can take the partial derivatives to conclude from (6.4), (6.6) and the assumptions on f that

$$\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}} \left[|\partial_{tt} u^{f\cdot\tau}(t,x)| + |\partial_{tx} u^{f\cdot\tau}(t,x)|\right] < \infty \quad \text{and} \quad \sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}} |\partial_{tx}\psi^{f\cdot\tau}(t,x)| < \infty.$$

It remains to show

$$\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}} |\partial_{xx} u^{f\cdot\tau}(t,x)| < \infty \quad \text{and} \quad \sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}} |\partial_{xx}\psi^{f\cdot\tau}(t,x)| < \infty \tag{6.7}$$

and

$$\sup_{(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}} |\partial_{tt}\psi^{f\cdot\tau}(t,x)| < \infty. \tag{6.8}$$

Due to (6.4) and (i), (6.7) follows if we can prove

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}} |\partial_{xx} u^f(t,x)| < \infty. \tag{6.9}$$

Note that $\partial_x u^f(t,x)$ satisfies the equation

$$\partial_t(\partial_x u^f) = J * \partial_x u^f - \partial_x u^f + \partial_u f(t, u^f)\partial_x u^f.$$

The uniform bound (6.9) can be proved by directly adapting the arguments leading to Proposition 1.2 (ii) from the equation $\partial_t u^f = J * u^f - u + f(t, u^f)$. We refer the reader to [57, Section 3] for detailed arguments, which require the propagating properties of u^f and the uniform bound on $\partial_{tu} f$ and $\partial_{uu} f$.

By Corollary 2.3, we find $\dot{X}^{f\cdot\tau}(t) = -\frac{\partial_t u(t, X^{f\cdot\tau}(t))}{\partial_x u(t, X^{f\cdot\tau}(t))}$. Taking the time derivative and using (ii), (6.6) and (6.7) lead to

$$\sup_{(t,\tau)\in\mathbb{R}\times\mathbb{R}} \left|\ddot{X}^{f\cdot\tau}(t)\right| < \infty.$$

Taking the second time derivative of $\psi^{f\cdot\tau}(t,x)=u^{f\cdot\tau}(t,x+X^{f\cdot\tau}(t))$, we deduce (6.8)

(iv) By (i), we in particular have

$$\psi^{f\cdot\tau}(t,x)=\psi^{f}(t+\tau,x),\quad \forall(t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Since the limits $\psi^f(t,x)\to 1$ as $x\to-\infty$ and $\psi^f(t,x)\to 0$ as $x\to\infty$ are uniform in $t\in\mathbb{R}$, we find

$$\lim_{x\to-\infty}\psi^{f\cdot\tau}(t,x)=1\quad\text{and}\quad\lim_{x\to\infty}\psi^{f\cdot\tau}(t,x)=0\quad\text{uniformly in}\quad(t,\tau)\in\mathbb{R}\times\mathbb{R}.\tag{6.10}$$

For any $g\in H(f)$, there is a sequence $\{t_n\}$ such that $g_n:=f\cdot t_n\to g$ in $H(f)$. By (iii) and Arzelà–Ascoli theorem, there exists a continuous function $\psi(\cdot,\cdot;g):\mathbb{R}\times\mathbb{R}\to[0,1]$ such that $\lim_{n\to\infty}\psi^{g_n}(t,x)=\psi(t,x;g)$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$. We then conclude from (6.10) that

$$\lim_{x\to-\infty}\psi(t,x;g)=1\quad\text{and}\quad\lim_{x\to\infty}\psi(t,x;g)=0\quad\text{uniformly in }t\in\mathbb{R}\text{ and }g\in H(f).\tag{6.11}$$

It remains to show $\psi^g(t,x)=\psi(t,x;g)$. Fix any $g\in H(f)$. By (ii), there exists a continuous function $X(\cdot;g):\mathbb{R}\to\mathbb{R}$ such that, up to a subsequence,

$$X^{g_n}(t)\to X(t;g)\quad\text{and}\quad\psi^{g_n}(t,x-X^{g_n}(t))\to\psi(t,x-X(t;g);g)\tag{6.12}$$

as $n\to\infty$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$. By (iii), we also have

$$\frac{d}{dt}\psi^{g_n}(t,x-X^{g_n}(t))\to\frac{d}{dt}\psi(t,x-X(t;g);g)\tag{6.13}$$

as $n\to\infty$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$. Thus, $\psi(t,x-X(t;g);g)$ is a global-in-time solution of (6.3), and hence, it is a transition front due to (6.11). Uniqueness of transition fronts and the normalisation $X^{g_n}(0)=0$ then imply that $\psi^g(t,x)=\psi(t,x;g)$.

(v) It is a simple consequence of (ii), (iii) and the proof of (iv). □

Now, we prove Theorem 6.1.

Proof of Theorem 6.1 (i) By periodicity, $u(t+T,x)$ is also a transition front of (1.1). Theorem 5.1 then yields the existence of some $\xi\in\mathbb{R}$ such that

$$u(t+T,x)=u(t,x+\xi),\quad\forall(t,x)\in\mathbb{R}\times\mathbb{R}.\tag{6.14}$$

Fix some $\theta_*\in(0,1)$. Setting $t=0$ and $x=X_{\theta_*}(T)$ in (6.14), we find

$$\theta_*=u(T,X_{\theta_*}(T))=u(0,X_{\theta_*}(T)+\xi),$$

which leads to $X_{\theta_*}(0)=X_{\theta_*}(T)+\xi$ by monotonicity. It then follows from (6.14) that

$$u(t+T,x)=u(t,x+X_{\theta_*}(0)-X_{\theta_*}(T)),\quad\forall(t,x)\in\mathbb{R}\times\mathbb{R}.$$

Setting $c=\frac{X_{\theta_*}(T)-X_{\theta_*}(0)}{T}$ and $\psi(t,x)=u(t,x+ct)$ for $(t,x)\in\mathbb{R}\times\mathbb{R}$, we readily verify that (c,ψ) satisfies (6.2). The fact that $c>0$ follows from the fact $u(t,x)$ moves to the right.

To find the formula for c, we multiply the equation in (6.2) by $\partial_x\psi(t,x)$, and then take the integral over $\mathbb{R}$ with respect to dx to find

$$\int_{\mathbb{R}}\partial_t\psi\,\partial_x\psi\,dx=\int_{\mathbb{R}}(J*\psi)\partial_x\psi\,dx-\int_{\mathbb{R}}\psi\,\partial_x\psi\,dx+c\int_{\mathbb{R}}(\partial_x\psi)^2dx+\int_{\mathbb{R}}f(t,\psi)\partial_x\psi\,dx.$$

Clearly, $\int_{\mathbb{R}}\psi\,\partial_x\psi\,dx=\frac{1}{2}\int_{\mathbb{R}}\partial_x(\psi^2)dx=-\frac{1}{2}$. We find from integration by parts that

$$\begin{aligned}\int_{\mathbb{R}}(J*\psi)\partial_x\psi\,dx&=-\lim_{x\to-\infty}\left[\int_{-\infty}^{\infty}J(x-y)\psi(t,y)dy\psi(t,x)\right]-\int_{\mathbb{R}}(J*\partial_x\psi)\psi\,dx\\&=-1-\int_{\mathbb{R}}(J*\psi)\partial_x\psi\,dx,\end{aligned}$$

where we used the symmetry of J. Thus, $\int_{\mathbb{R}}(J*\psi)\partial_x\psi\,dx=-\frac{1}{2}$. Hence,

$$\int_{\mathbb{R}}\partial_t\psi\,\partial_x\psi\,dx=c\int_{\mathbb{R}}(\partial_x\psi)^2dx+\int_{\mathbb{R}}f(t,\psi)\partial_x\psi\,dx.$$

Now, integrating the above equality over $[0,T]$ with respect to dt, we conclude from the T-periodicity of $\psi(t,x)$ that $\int_0^T\int_{\mathbb{R}}\partial_t\psi\,\partial_x\psi\,dxdt=0$, which leads to

$$0=c\int_0^T\int_{\mathbb{R}}(\partial_x\psi)^2dxdt+\int_0^T\int_{\mathbb{R}}f(t,\psi)\partial_x\psi\,dxdt.$$

The formula for c follows.

(ii) Write $X^g(t)=X^g_{\frac{1}{2}}(t)$. Since $\sup_{t\in\mathbb{R}}|X^f(t)-X(t)|<\infty$, it suffices to show the existence of the limits $\lim_{t\to\pm\infty}\frac{X^f(t)}{t}$. Since

$$\lim_{t\to\pm\infty}\frac{X^f(t)}{t}=\lim_{t\to\pm\infty}\frac{X^f(t)-X^f(0)}{t}=\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\dot{X}^f(s)ds,$$

we only need to show the dynamical system (i.e. the shift operators) generated by $\dot{X}^f(t)$ is compact and uniquely ergodic.

To this end, we first derive a formula for $\dot{X}^g(t)$. We claim

$$\dot{X}^g(t)=-\frac{\int_{\mathbb{R}}J(y)\psi^g(t,-y)dy-\frac{1}{2}+g(t,\frac{1}{2})}{\partial_x\psi^g(t,0)},\quad\forall t\in\mathbb{R}.\tag{6.15}$$

In fact, differentiating $u^g(t,X^g(t))=\frac{1}{2}$, we find

$$\dot{X}^g(t)=-\frac{\partial_t u^g(t,X^g(t))}{\partial_x u^g(t,X^g(t))}=-\frac{[J*u^g(t,\cdot)]\,(X^g(t))-u^g(t,X^g(t))+g(t,u^g(t,X^g(t)))}{\partial_x u^g(t,X^g(t))}.$$

The equality (6.15) then follows from $u^g(t,x+X^g(t))=\psi^g(t,x)$ and $u^g(t,X^g(t))=\frac{1}{2}$. Note that due to (i) in Lemma 6.2 and (6.15), there holds $\dot{X}^{g\cdot\tau}(t)=\dot{X}^g(t+\tau)$ for all $t,\tau\in\mathbb{R}$.

Next, we define

- the phase space $\tilde{H}=\{(\psi^g,\dot{X}^g):g\in H(f)\}$, equipped with the topology of locally uniform convergence in $t\in\mathbb{R}$ and uniform convergence in $x\in\mathbb{R}$;

- the shift operators $\{\tilde{\sigma}\}_{t\in\mathbb{R}}$, that is, the dynamical system on $\tilde{H}$,

$$\tilde{\sigma}_t : \tilde{H} \to \tilde{H}, \quad (\psi^g, \dot{X}^g) \mapsto (\psi^{g\cdot t}, \dot{X}^{g\cdot t}) = (\psi^g(\cdot + t, \cdot), \dot{X}^g(\cdot + t));$$

- an operator $\Omega : H(f) \to \tilde{H}$, $g \mapsto (\psi^g, \dot{X}^g)$.

Clearly,

$$\tilde{\sigma}_t \circ \Omega = \Omega \circ \sigma_t, \quad \forall t \in \mathbb{R}, \tag{6.16}$$

where $\{\sigma_t\}_{t\in\mathbb{R}}$ is given in (6.1).

We show that Ω is a homeomorphism. We first claim that Ω is continuous. To do so, let $\{g_n\} \subset \{f \cdot \tau : \tau \in \mathbb{R}\}$ (since $\{f \cdot \tau : \tau \in \mathbb{R}\}$ is dense in $H(f)$) and $g_* \in H(f)$, and assume $g_n \to g_*$ in $H(f)$ as $n \to \infty$. We show $\Omega g_n \to \Omega g$ in $H(f)$ as $n \to \infty$. By (6.15), this is the case if we can show

$$\psi^{g_n}(t, x) \to \psi^{g_*}(t, x) \text{ locally uniform in } t \in \mathbb{R} \text{ and uniformly in } x \in \mathbb{R} \tag{6.17}$$

and

$$\partial_x \psi^{g_n}(t, 0) \to \partial_x \psi^{g_*}(t, 0) \text{ locally uniform in } t \in \mathbb{R} \tag{6.18}$$

as $n \to \infty$.

As in the proof of (iv) in Lemma 6.2, there exist continuous functions $X^* : \mathbb{R} \to \mathbb{R}$ and $\psi^* : \mathbb{R} \times \mathbb{R} \to [0, 1]$ such that

$$X^{g_n}(t) \to X^*(t) \text{ and } \psi^{g_n}(t, x - X^{g_n}(t)) \to \psi^*(t, x - X^*(t)) \text{ locally uniformly in } (t, x) \in \mathbb{R} \times \mathbb{R}$$

as $n \to \infty$. As in (6.13), we also have

$$\frac{d}{dt}\psi^{g_n}(t, x - X^{g_n}(t)) \to \frac{d}{dt}\psi^*(t, x - X^*(t)) \text{ locally uniformly in } (t, x) \in \mathbb{R} \times \mathbb{R}$$

as $n \to \infty$. In particular, $\psi^*(t, x - X^*(t))$ is global-in-time solution of (6.3) with g replaced by g^*. Moreover, (iv) in Lemma 6.2 forces $\psi^*(t, x - X^*(t))$ to be a transition front, and hence, $\psi^*(t, x) = \psi^{g_*}(t, x)$ by uniqueness and normalisation. It then follows that $\psi^{g_n}(t, x) \to \psi^{g_*}(t, x)$ locally uniform in $(t, x) \in \mathbb{R} \times \mathbb{R}$ as $n \to \infty$. But, this actually leads to (6.17) due to the uniform limits as $x \to \pm\infty$ as in (iv) in Lemma 6.2.

The convergence (6.18) follows from Lemma 6.2 (iii). This proves the convergence of Ωg_n to Ωg in $H(f)$ as $n \to \infty$. Hence, Ω is continuous.

Clearly, from the continuity of Ω and the compactness of $H(f)$, $\tilde{H} = \Omega(H(f))$ is compact, and hence, $\tilde{H} = \overline{\{(\psi^{f\cdot t}, \dot{X}^{f\cdot t}) : t \in \mathbb{R}\}}$. Thus, if we can show that Ω is one-to-one, then its inverse Ω^{-1} exists and must be continuous, and hence, Ω is a homeomorphism.

We show Ω is one-to-one. Assume $\Omega g_1 = \Omega g_2$, that is, $(\psi^{g_1}, \dot{X}^{g_1}) = (\psi^{g_2}, \dot{X}^{g_2})$. In particular, $\dot{X}^{g_1} = \dot{X}^{g_2}$, which together with the normalisation $X^{g_1}(0) = 0 = X^{g_2}(0)$ gives $X^{g_1} = X^{g_2}$. It then follows from (6.4) that

$$u^{g_1}(t, x) = u^{g_2}(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

which leads to $g_1(t, u(t, x)) = g_2(t, u(t, x))$, where $u = u^{g_1} = u^{g_2}$. Since $u(t, x)$ is continuous and connects 0 and 1 for any $t \in \mathbb{R}$, we conclude that $g_1 = g_2$ on $\mathbb{R} \times [0, 1]$. Hence, Ω is one-to-one, and therefore, Ω is a homeomorphism.

Since Ω is a homeomorphism, invariant measures on $H(f)$ and $\tilde{H}$ are related by Ω. We then conclude from (6.16) and the fact $\{\sigma_t\}_{t\in\mathbb{R}}$ is compact and uniquely ergodic that $\{\tilde{\sigma}_t\}_{t\in\mathbb{R}}$ is compact and uniquely ergodic. Now, define $\Phi : \tilde{H} \to \mathbb{R}$ by setting $\Phi(\psi^g, \dot{X}^g) = \dot{X}^g(0)$. Clearly, Φ is continuous. We then conclude from the unique ergodicity of $\{\tilde{\sigma}_t\}_{t\in\mathbb{R}}$ that there exist constants $c^\pm = c^\pm(\Phi) \in \mathbb{R}$ such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \Phi(\tilde{\sigma}_s(\psi^g, \dot{X}^g))ds = c^\pm$$

uniformly in $g \in H(f)$. In particular, $\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \Phi(\tilde{\sigma}_s(\psi^f, \dot{X}^f))ds = c^\pm$. But

$$\Phi(\tilde{\sigma}_s(\psi^f, \dot{X}^f)) = \Phi((\psi^f(\cdot + s, \cdot), \dot{X}^f(\cdot + s))) = \dot{X}^f(s).$$

This completes the proof. □

7 Existence of space non-increasing transition fronts

In this section, we investigate the existence of space non-increasing transition fronts of equation (1.1). Throughout this section, we assume (H1)–(H4).

The main result of this section is stated in the following theorem.

Theorem 7.1 *Equation* (1.1) *admits a transition front* $u(t,x)$ *that is non-increasing in space.*

Proof We use the perturbation method. Fix $0 < \epsilon_0 \ll 1$. For $\epsilon \in (0, \epsilon_0]$, we consider the following perturbation of (1.1):

$$\partial_t u = J * u - u + \epsilon \partial_{xx} u + f(t,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{7.1}$$

The advantage of considering the above perturbed equation is that we are able to apply the methods in [49] (also see [16, 17, 51]) to construct transition fronts of (7.1). Here, we are not going to repeat the construction since it is lengthy. We just point out that the construction highly relies on the regularity of solutions of (7.1) coming from the regular perturbation ϵu_{xx}, and the instability of the solution $u_0(t)$ of the ODE (1.4), which forces approximating solutions to stay steep in the course of propagation so that any limiting point of approximating solutions is a transition front.

Thus, for each $\epsilon \in (0, \epsilon_0]$, equation (7.1) admits space decreasing transition fronts. Moreover, direct adaption of the proof of Theorem 5.1 yields the uniqueness, up to space shifts, of transition fronts of (7.1). For each $\epsilon \in (0, \epsilon_0]$, let $u^\epsilon(t,x)$ be the unique transition front of (7.1) satisfying the normalisation $u^\epsilon(0,0) = \frac{1}{2}$. Also, from the construction, there also holds the uniform bounded interface width for $\{u^\epsilon(t,x)\}_{\epsilon\in(0,\epsilon_0]}$, that is,

$$\forall\ 0 < \lambda_1 \le \lambda_2 < 1, \qquad \sup_{\epsilon\in(0,\epsilon_0]} \sup_{t\in\mathbb{R}} \operatorname{diam}\{x \in \mathbb{R} : \lambda_1 \le u^\epsilon(t,x) \le \lambda_2\} < \infty. \tag{7.2}$$

Note that if the sequence $\{u^\epsilon(t,x)\}$ converges to some solution of (1.1), then this solution must be a transition front of (1.1) due to (7.2). However, the convergence of $\{u^\epsilon(t,x)\}_{\epsilon\in(0,\epsilon_0]}$ to some solution of (1.1) is far from being clear, since we have no idea whether $\partial_x u^\epsilon(t,x)$ and $\partial_{xx} u^\epsilon(t,x)$ are locally bounded in (t,x) and uniformly in ϵ, that means, we cannot simply pass to the limit $\epsilon \to 0$ in (7.1). To circumvent this difficulty, we first consider solutions in some weak sense.

We see that for any $\phi \in C_0^\infty(\mathbb{R})$, there holds

$$\begin{aligned}\int_{\mathbb{R}} u^\epsilon(t,x)\phi(x)dx &= \int_{\mathbb{R}} u^\epsilon(0,x)\phi(x)dx + \int_0^t \int_{\mathbb{R}} [J*u^\epsilon - u^\epsilon]\phi(x)dxd\tau \\ &\quad + \epsilon \int_0^t \int_{\mathbb{R}} u^\epsilon(\tau,x)\phi''(x)dxd\tau + \int_0^t \int_{\mathbb{R}} f(\tau,u^\epsilon(\tau,x))\phi(x)dxd\tau.\end{aligned} \tag{7.3}$$

To pass to the limit $\epsilon \to 0$ in (7.3), we derive some convergence properties of $u^\epsilon(t,x)$.

Since $\{u^\epsilon(t,x)\}$ is pre-compact in $L^1_{loc}(\mathbb{R}\times\mathbb{R})$ (see Lemma 7.2), we can use the diagonal argument to find some $u \in L^1_{loc}(\mathbb{R}\times\mathbb{R})$ and a sequence $\{\epsilon_n\}$ such that

$$u^{\epsilon_n}(t,x) \to u(t,x) \text{ for a.e. } (t,x)\in\mathbb{R}\times\mathbb{R} \text{ as } n\to\infty. \tag{7.4}$$

Let $\Omega \subset \mathbb{R}\times\mathbb{R}$ be a measurable set with Lebesgue measure zero such that

$$u^{\epsilon_n}(t,x) \to u(t,x) \text{ pointwise in } (t,x)\in(\mathbb{R}\times\mathbb{R})\backslash\Omega \text{ as } n\to\infty.$$

Since the functions $\{u^{\epsilon_n}(0,x)\}$ are decreasing in x and uniformly bounded, Helly's selection theorem implies that there exists a subsequence, still denoted by $\{\epsilon_n\}$, and a non-increasing function $v(0,\cdot)$ such that

$$u^{\epsilon_n}(0,x) \to v(0,x) \text{ pointwise in } x\in\mathbb{R} \text{ as } n\to\infty. \tag{7.5}$$

Fix $t\in\mathbb{R}\backslash\{0\}$. Again, by Helly's selection theorem, there exists a subsequence $\{\epsilon^t_{n_k}\}\subset\{\epsilon_n\}$ and a non-increasing function $v(t,\cdot)$ such that

$$u^{\epsilon^t_{n_k}}(t,x) \to v(t,x) \text{ pointwise in } x\in\mathbb{R} \text{ as } k\to\infty.$$

Clearly, $u(t,x)=v(t,x)$ for $(t,x)\in(\mathbb{R}\times\mathbb{R})\backslash\Omega$. We then redefine $u(t,x)$ on Ω to be $v(t,x)$. Hence, (7.4) is still true, and, moreover, for any $t\in\mathbb{R}$, we have

$$u^{\epsilon^t_{n_k}}(t,x) \to u(t,x) \text{ pointwise in } x\in\mathbb{R} \text{ as } k\to\infty, \tag{7.6}$$

where $\{\epsilon^0_{n_k}\}=\{\epsilon_n\}$. Also, $u(t,x)$ is non-increasing in x.

Now, for fixed $t\in\mathbb{R}$, using (7.4), (7.5) and (7.6), we pass to the limit $\epsilon\to0$ along the subsequence $\{\epsilon^t_{n_k}\}$ as $k\to\infty$ in (7.3) to obtain

$$\begin{aligned}&\int_{\mathbb{R}} u(t,x)\phi(x)dx \\ &\quad= \int_{\mathbb{R}} u(0,x)\phi(x)dx + \int_0^t\int_{\mathbb{R}}[J*u-u]\phi(x)dxd\tau + \int_0^t\int_{\mathbb{R}} f(\tau,u(\tau,x))\phi(x)dxd\tau \\ &\quad= \int_{\mathbb{R}}\left\{u(0,x)+\int_0^t[J*u-u]d\tau+\int_0^t f(\tau,u(\tau,x))d\tau\right\}\phi(x)dx\end{aligned}$$

for any $\phi\in C_0^\infty(\mathbb{R})$. In particular, for any fixed $t\in\mathbb{R}$,

$$u(t,x)=u(0,x)+\int_0^t [J*u-u+f(\tau,u(\tau,x))]\,d\tau \quad \text{for} \quad a.e.\ x\in\mathbb{R}. \tag{7.7}$$

For $t\in\mathbb{R}$, let $\Omega_t\subset\mathbb{R}$ be the measurable set with measure zero such that (7.7) is true for any $x\in\mathbb{R}\backslash\Omega_t$. Note that $\mathbb{R}\backslash\Omega_t$ is dense in $\mathbb{R}$, otherwise Ω_t contains an open interval, which is impossible.

For $(t,x) \in \mathbb{R}\times\mathbb{R}$, define

$$u^*(t,x) = \lim_{y\in\mathbb{R}\backslash\Omega_t, y\downarrow x} u(t,y).$$

This is well defined, since $u(t,x)$ is non-increasing in x and $\mathbb{R}\backslash\Omega_t$ is dense in $\mathbb{R}$. By (7.7), we have that for any $t\in\mathbb{R}$,

$$u^*(t,x) = u^*(0,x) + \int_0^t [J * u^* - u^* + f(\tau, u^*(\tau,x))]\, d\tau \quad \text{for all } x\in\mathbb{R}.$$

This implies that $u^*(t,x)$ is continuous in t uniformly with respect to $x\in\mathbb{R}$ and then

$$\partial_t u^* = J * u^* - u^* + f(t,u^*) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}.$$

We then conclude from (7.2) that $u^*(t,x)$ is a transition front. Moreover, $u^*(t,x)$ is non-increasing in x, since $u(t,x)$ is non-increasing in x. □

To finish the proof of Theorem 7.1 under assumption (H4), we prove the following lemma.

Lemma 7.2 *$\{u^\epsilon(\cdot,\cdot)\}_{\epsilon\in(0,\epsilon_0]}$ is pre-compact in $L^1_{loc}(\mathbb{R}\times\mathbb{R})$.*

Proof We first show that

$$\{u^\epsilon(t,\cdot)\}_{\epsilon\in(0,\epsilon_0],t\in\mathbb{R}} \text{ is pre-compact in } L^1_{loc}(\mathbb{R}). \tag{7.8}$$

Since $u^\epsilon(t,\cdot)$ is a decreasing function for any $\epsilon\in(0,\epsilon_0]$ and $t\in\mathbb{R}$ and $\{u^\epsilon(t,\cdot)\}_{\epsilon\in(0,\epsilon_0],t\in\mathbb{R}}$ is uniformly bounded, Helly's selection theorem yields that for any sequence (ϵ_n,t_n) there exists a subsequence $(\epsilon_{n_k},t_{n_k})\subset(\epsilon_n,t_n)$ and a non-increasing function $v:\mathbb{R}\to[0,1]$ such that

$$u^{\epsilon_{n_k}}(t_{n_k},x)\to v(x) \text{ pointwise in } x\in\mathbb{R} \text{ as } k\to\infty,$$

which, together with dominated convergence theorem and boundedness, imply that

$$u^{\epsilon_{n_k}}(t_{n_k},\cdot)\to v \text{ in } L^1_{loc}(\mathbb{R}) \text{ as } k\to\infty.$$

This verifies (7.8).

Fix $r>0$ and let $B_r=(-r,r)\times(-r,r)$. It remains to show that $\{u^\epsilon(t,x)\}_{\epsilon\in(0,\epsilon_0]}$ restricted to B_r is pre-compact in $L^1(B_r)$. Applying the classical compactness criterion in $L^1(B_r)$ (see, e.g., [2, Theorem 2.32]), it suffices to verify

(i) for any $\bar\epsilon>0$, there exists $O\subset B_r$ such that

$$\int_{B_r\backslash O} u^\epsilon(t,x)dtdx \le |B_r\backslash O| \le \bar\epsilon$$

for all $\epsilon\in(0,\epsilon_0]$;

(ii) for any $\bar\epsilon>0$, there exists $\delta>0$ such that

$$\int_{B_r} |u^\epsilon(t+\Delta t, x+\Delta x) - u^\epsilon(t,x)|\, dtdx \le \bar\epsilon$$

for all $\epsilon\in(0,\epsilon_0]$ and $(\Delta t,\Delta x)\in\mathbb{R}\times\mathbb{R}$ with $|\Delta t|+|\Delta x|\le\delta$.

Clearly, (i) follows from the uniform boundedness of $\{u^\epsilon(t,x)\}_{\epsilon\in(0,\epsilon_0]}$. It remains to prove (ii). To this end, let $\bar\epsilon>0$. For $(\Delta t,\Delta x)\in\mathbb{R}\times\mathbb{R}$, we have

$$\begin{aligned}\int_{B_r}&|u^\epsilon(t+\Delta t,x+\Delta x)-u^\epsilon(t,x)|dtdx\\&\le\int_{B_r}|u^\epsilon(t+\Delta t,x+\Delta x)-u^\epsilon(t+\Delta t,x)|dtdx+\int_{B_r}|u^\epsilon(t+\Delta t,x)-u^\epsilon(t,x)|dtdx.\end{aligned}\tag{7.9}$$

We use (7.8) to control the first integral on the right-hand side of (7.9). In fact, by (7.8) and [2, Theorem 2.32], for any $\bar\epsilon>0$, there exists $\delta_1>0$ such that

$$\int_{-r}^{r}|u^\epsilon(t,x+\Delta x)-u^\epsilon(t,x)|dx\le\frac{\bar\epsilon}{4r}$$

for all $\epsilon\in(0,\epsilon_0]$, $t\in\mathbb{R}$ and $\Delta x\in\mathbb{R}$ with $|\Delta x|\le\delta_1$. It then follows that

$$\int_{B_r}|u^\epsilon(t+\Delta t,x+\Delta x)-u^\epsilon(t+\Delta t,x)|dtdx\le\frac{\bar\epsilon}{2}$$

for all $\epsilon\in(0,\epsilon_0]$ and $(\Delta t,\Delta x)\in\mathbb{R}\times\mathbb{R}$ with $|\Delta x|\le\delta_1$.

For the second integral on the right-hand side of (7.9), we claim the existence of some continuous and non-decreasing function $\alpha_r(\cdot)$ satisfying $\alpha_r(0)=0$ such that

$$\int_{-r}^{r}|u^\epsilon(t+\Delta t,x)-u^\epsilon(t,x)|dx\le\alpha_r(|\Delta t|)\tag{7.10}$$

for all $\epsilon\in(0,\epsilon_0]$ and $t\in\mathbb{R}$. Assuming (7.10) for the moment, we see that there exists $\delta_2>0$ such that

$$\int_{B_r}|u^\epsilon(t+\Delta t,x)-u^\epsilon(t,x)|dtdx\le\frac{\bar\epsilon}{2}$$

for all $\epsilon\in(0,\epsilon_0]$ and $\Delta t\in\mathbb{R}$ with $|\Delta t|\le\delta_2$. Setting $\delta=\min\{\delta_1,\delta_2\}$, we verify (ii) and then complete the proof.

It remain to show (7.10). We follow the arguments as in the proof of [50, Lemma 2.5]. Let $v(x)=u^\epsilon(t+\Delta t,x)-u^\epsilon(t,x)$. Then, for any $s\ge0$

$$\begin{aligned}\int_{-s}^{s}|v(x+\Delta x)-v(x)|dx&\le\int_{-s}^{s}|u^\epsilon(t+\Delta t,x+\Delta x)-u^\epsilon(t+\Delta t,x)|dx\\&\quad+\int_{-s}^{s}|u^\epsilon(t,x+\Delta x)-u^\epsilon(t,x)|dx\\&\le\tilde\alpha_s(|\Delta x|),\end{aligned}$$

for all $t\in\mathbb{R}$, $\Delta t\in\mathbb{R}$ and $|\Delta x|\le\delta_1$, where $\tilde\alpha_s(\cdot)$ is a continuous and non-decreasing function satisfying $\tilde\alpha_s(0)=0$. The existence of such a function $\tilde\alpha_s(\cdot)$ follows from (7.8) and [2, Theorem 2.32]. We then apply [50, Lemma 2.1] to conclude that

$$\int_{-s}^{s}\Big||v(x)|-v(x)g^h(x)\Big|\,dx\le2\tilde\alpha_s(h),\qquad\forall0<h\ll s,\tag{7.11}$$

where $g^h := \frac{1}{h}\eta(\frac{\cdot}{h}) * \mathrm{sign}(v)$ for some fixed non-negative $\eta \in C_0^\infty(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \eta(x)dx = 1$. It follows that for any $0 < h \ll 1$,

$$\int_{-r}^{r} |u^\epsilon(t+\Delta t, x) - u^\epsilon(t,x)|\, dx = \int_{-r}^{r} |v(x)| dx \le \int_{-r}^{r} v(x) g^h(x) dx + 2\tilde{\alpha}_r(h). \tag{7.12}$$

Note that $0 \le g^h \le 1$, $\left|\frac{d}{dx} g^h\right| \le \frac{C_1}{h}$ and $\left|\frac{d^2}{dx^2} g^h\right| \le \frac{C_1}{h^2}$ for some universal constant $C_1 > 0$. Then, a twice continuously differentiable function $\tilde{g}^h : [-r-2h, r+2h]$ satisfying

$$\tilde{g}^h = g^h \quad \text{on} \quad [-r, r]$$

and

$$\begin{aligned} &\tilde{g}^h(-r-2h) = 0 = \tilde{g}^h(r+2h), \quad \frac{d}{dx}\tilde{g}^h(-r-2h) = 0 = \frac{d}{dx}\tilde{g}^h(r+2h), \\ &0 \le \tilde{g}^h \le 2, \quad \left|\frac{d}{dx}\tilde{g}^h\right| \le \frac{C_1}{h^2} \quad \text{and} \quad \left|\frac{d^2}{dx^2}\tilde{g}^h\right| \le \frac{C_1}{h^4} \end{aligned}$$

can be constructed. It follows that

$$\begin{aligned} \int_{-r}^{r} v(x) g^h(x) dx &= \int_{-r}^{r} [u^\epsilon(t+\Delta t, x) - u^\epsilon(t,x)]\, g^h(x) dx \\ &\le \int_{-r-2h}^{r+2h} [u^\epsilon(t+\Delta t, x) - u^\epsilon(t,x)]\, \tilde{g}^h(x) dx + C_2 h \\ &= \int_t^{t+\Delta t} \int_{-r-2h}^{r+2h} \partial_t u^\epsilon(\tau, x) \tilde{g}^h(x) dx d\tau + C_2 h \\ &= \int_t^{t+\Delta t} \int_{-r-2h}^{r+2h} [(J * u^\epsilon)(\tau, x) - u^\epsilon(\tau, x) + \epsilon \partial_{xx} u^\epsilon(\tau, x) + f(t, u^\epsilon(\tau, x))]\, \tilde{g}^h(x) dx d\tau + C_2 h \\ &= \int_t^{t+\Delta t} \int_{-r-2h}^{r+2h} [(J * u^\epsilon)(\tau, x) - u^\epsilon(\tau, x) + f(t, u^\epsilon(\tau, x))]\, \tilde{g}^h(x) dx d\tau \\ &\quad + \epsilon \int_t^{t+\Delta t} \int_{-r-2h}^{r+2h} u^\epsilon(\tau, x) \frac{d^2}{dx^2}\tilde{g}^h(x) dx d\tau + C_2 h \\ &\le C_r |\Delta t| + \epsilon C_r \frac{|\Delta t|}{h^4} + C_2 h, \end{aligned}$$

where $C_2 > 0$ is universal and $C_r > 0$ depends on r. This together with (7.12) implies that

$$\int_{-r}^{r} |u^\epsilon(t+\Delta t, x) - u^\epsilon(t,x)|\, dx \le C_r |\Delta t| + \epsilon C_r \frac{|\Delta t|}{h^4} + C_2 h + 2\tilde{\alpha}_r(h), \quad \forall 0 < h \ll 1.$$

The claim (7.10) follows from setting $h = |\Delta t|^{\frac{1}{8}}$ in the above estimate. □

Remark 7.3 *We remark that transition fronts of* (7.1) *can be constructed without the unbalanced condition* (1.2). *Hence, Theorem 7.1 is true if we drop* (1.2). *But, in the absence of* (1.2), *the constructed transition front may not be continuous in space. We refer the reader to* [8] *for a sufficient and necessary condition for the existence of discontinuous travelling fronts of* $\partial_t u = J * u - u + f_B(u)$. *It would be interesting and important to study the stability and uniqueness of transition fronts in the absence of* (1.2).

We end this paper by mentioning a variation on (H4). The point is that we allow the failure of (1.5). But, then, we need an additional assumption, that is, $u_0(t) \equiv \theta_*$ for some $\theta_* \in [\tilde{\theta}, \theta]$. We assume

(H5) There exists $\theta_* \in [\tilde{\theta}, \theta]$ such that

$$f(t,u) < 0, \ \ u \in (0, \theta_*) \quad \text{and} \quad f(t,u) > 0, \ \ u \in (\theta_*, 1)$$

for all $t \in \mathbb{R}$.

Using different techniques, we are able to prove Theorem 7.1 under assumptions (H1)–(H3) and (H5). But, in this case, we cannot drop the unbalanced condition (1.2).

Proof of Theorem 7.1 under assumptions. (H1)–(H3) and (H5) The proof can be done along the same line as that in the ignition case (see [54]), so we here only outline the strategies within the following four steps.

Step 1. Approximating front-like solutions. Let ϕ_B is the decreasing profile of bistable travelling fronts of $\partial_t u = J * u - u + f_B(u)$ with the normalisation $\phi_B(0) = \theta_*$. For $s < 0$ and $y \in \mathbb{R}$, denote by $u(t,x;s,\phi_B(\cdot - y))$ the classical solution of (1.1) with initial data $u(s,x;s,\phi_B(\cdot - y)) = \phi_B(x-y)$. Then, it can be shown that for any $s < 0$, there exists a unique $y_s \in \mathbb{R}$ such that $u(0,0;s,\phi_B(\cdot - y_s)) = \theta_*$. Moreover, $y_s \to -\infty$ as $s \to -\infty$.

Set $u(t,x;s) := u(t,x;s,\phi_B(\cdot - y_s))$. We see that $u(t,x;s)$ is decreasing in x. The functions $\{u(t,x;s)\}_{s<0, t\geq s}$ are the approximating front-like solutions.

Step 2. Bounded interface width-I. For $s < 0$, $t \geq s$ and $\lambda \in (0,1)$, let $X_\lambda(t;s)$ be such that $u(t, X_\lambda(t;s);s) = \lambda$. It is well defined and continuous in t.

Then, there exists $\lambda_* \in (\theta_*, 1)$ such that for any $\lambda_1, \lambda_2 \in (0, \lambda_*]$, there holds

$$\sup_{s<0, t\geq s} \left| X_{\lambda_1}(t;s) - X_{\lambda_2}(t;s) \right| < \infty.$$

This is the difficult part in constructing transition fronts. Its proof is based on the rightward propagation estimate of $X_\lambda(t;s)$ and an idea of Zlatoš (see [71, Lemma 2.5]). It is important that $\lambda_* > \theta_*$, and it is the reason why we need $f(t,\theta_*) = 0$ for all $t \in \mathbb{R}$.

Step 3. Bounded interface width-II. We extend the result in Step 2 to

$$\forall \lambda_1, \lambda_2 \in (0,1), \quad \sup_{s<0, t\geq s} \left| X_{\lambda_1}(t;s) - X_{\lambda_2}(t;s) \right| < \infty.$$

It is done through the following two steps:

(i) there are $c_{\min} > 0$, $c_{\max} > 0$, $\tilde{c}_{\max} > 0$ and $d_{\max} > 0$ such that for any $s < 0$, there exists a continuously differentiable function $X(t;s) : [s, \infty) \to \mathbb{R}$ satisfying

$$c_{\min} \leq \dot{X}(t;s) \leq c_{\max} \quad \text{and} \quad |\ddot{X}(t;s)| \leq \tilde{c}_{\max} \quad \text{for} \quad t \geq s$$

such that $0 \leq X(t;s) - X_{\lambda_*}(t;s) \leq d_{\max}$ for $t \geq s$; moreover, $\{\dot{X}(\cdot, s)\}_{s<0}$ and $\{\ddot{X}(\cdot, s)\}_{s<0}$ are uniformly bounded and uniformly Lipschitz continuous;

(ii) using (i), we can find $c_\pm > 0$ and $h_\pm > 0$ such that

$$u(t,x;s) \geq 1 - e^{c_-(x - X(t;s) + h_-)} \quad \text{and} \quad u(t,x;s) \leq e^{-c_+(x - X(t;s) - h_+)}$$

for all $x \in \mathbb{R}$, $s < 0$ and $t \geq s$.

Clearly, the bounded interface width follows.

Step 4. Construction of transition fronts. The approximating solutions $\{u(t,x;s)\}_{s<0,t\geq s}$ converge to some transition front (as in Theorem 7.1) as $s \to -\infty$ along some subsequence due to the properties in Step 3 and the following: there holds

$$\sup_{s<0,t\geq s} \sup_{x\neq y} \left| \frac{u(t,y;s)-u(t,x;s)}{y-x} \right| < \infty,$$

whose proof relies on the observation that for fixed x, the term $\frac{u(t,x+\eta;s)-u(t,x;s)}{\eta}$ for $0<|\eta|\leq \eta_0 \ll 1$ can only grow for a period of time that is independent of x. □

8 Conclusion

In this paper, we study a class of non-local dispersal equations (1.1) modelling the invasion of species in temporally heterogeneous environments. We focus on the existence and qualitative properties of transition fronts, which are proper generalisations of travelling fronts in homogeneous environments and periodic (or pulsating) travelling fronts in periodic environments, and hence, are appropriate to describe the invasion or spread of species in heterogeneous environments.

The study of transition fronts begins with the investigation of qualitative properties such as stability and decaying estimates of space non-increasing transition fronts (assuming the existence). The most important property of space non-increasing transition fronts obtained is the uniform exponential stability, which allows us to overcome the difficulties caused by the following facts:

1. the solutions are not regular enough to support standard regularity arguments for reaction–diffusion equations;
2. the use of comparison principles is not as flexible as that for reaction–diffusion equations;
3. Harnack's inequality is not known;

and show that any transition front of (1.1) must coincide with a space non-increasing transition front, and transition fronts are unique up to space shifts. These results are established under very general bistable assumptions. The existence of space non-increasing transition fronts are proved under fairly standard bistable assumptions.

Besides the existence and qualitative properties of transition fronts in heterogeneous environments, we show that transition fronts must be periodic travelling fronts in periodic environments whose front speed can be calculated in terms of the profile function and the growth rate function. Moreover, in almost-periodic environments, we show that transition fronts admit an asymptotic spreading speed, and therefore, all propagating solutions of (1.1) spread with this speed.

Appendix A Comparison principles

We state comparison principles used in the previous sections.

Proposition A.1 *Let $K:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be continuous and satisfy $\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}K(x,y)dy<\infty$. Let $a:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous and uniformly bounded.*

(i) *Suppose that* $X : [0,\infty) \to \mathbb{R}$ *is continuous and that* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following:* $u, \partial_t u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous, the limit* $\lim_{x\to\infty} u(t,x) = 0$ *is locally uniformly in t and*

$$\begin{cases} \partial_t u(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), & x > X(t), \ t > 0, \\ u(t,x) \geq 0, \quad x \leq X(t), \ t > 0, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}. \end{cases}$$

Then $u(t,x) \geq 0$ *for* $(t,x) \in (0,\infty) \times \mathbb{R}$.

(ii) *Suppose that* $X : [0,\infty) \to \mathbb{R}$ *is continuous and that* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following:* $u, \partial_t u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous, the limit* $\lim_{x\to-\infty} u(t,x) = 0$ *is locally uniformly in t and*

$$\begin{cases} \partial_t u(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), & x < X(t), \ t > 0, \\ u(t,x) \geq 0, \quad x \geq X(t), \ t > 0, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}. \end{cases}$$

Then $u(t,x) \geq 0$ *for* $(t,x) \in (0,\infty) \times \mathbb{R}$.

(iii) *Suppose that* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following:* $u, \partial_t u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous,* $\inf_{t\geq 0, x\in\mathbb{R}} u(t,x) > -\infty$ *and*

$$\begin{cases} \partial_t u(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), & x \in \mathbb{R}, \ t > 0, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}. \end{cases}$$

Then $u(t,x) \geq 0$ *for* $(t,x) \in (0,\infty) \times \mathbb{R}$. *Moreover, if* $u_0(x) \not\equiv 0$, *then* $u(t,x) > 0$ *for* $(t,x) \in (0,\infty) \times \mathbb{R}$.

Proof See [54, Proposition A.1] for the proof. □

Definition A.2 Let $t_0 \in \mathbb{R}$ and $T > 0$. A continuous function $u : [t_0, t_0 + T) \times \mathbb{R} \to \mathbb{R}$ is called a super-solution (or sub-solution) of (1.1) on $[t_0, t_0 + T)$ if $u(t,x)$ is differentiable in t on $(t_0, t_0 + T)$ for any $x \in \mathbb{R}$ and satisfies

$$\partial_t u(t,x) \geq (\text{or} \leq) \int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x) + f(t,u(t,x)), \quad (t,x) \in (t_0, t_0+T) \times \mathbb{R}.$$

Proposition A.1(iii) gives the following comparison principal for (1.1).

Proposition A.3 *Let* $t_0 \in \mathbb{R}$ *and* $T > 0$. *Suppose* $u^+(t,x)$ *and* $u^-(t,x)$ *are super- and sub-solutions of* (1.1) *on* $[t_0, t_0+T)$, *respectively.*

(i) *If* $u^+(t_0,\cdot) \geq u^-(t_0,\cdot)$, *then* $u^+(t,x) \geq u^-(t,x)$ *for* $(t,x) \in (t_0, t_0+T) \times \mathbb{R}$.
(ii) *If* $u^+(t_0,\cdot) \geq u^-(t_0,\cdot)$ *and* $u^+(t_0,\cdot) \not\equiv u^-(t_0,\cdot)$, *then* $u^+(t,x) > u^-(t,x)$ *for* $(t,x) \in (t_0, t_0+T) \times \mathbb{R}$.

Acknowledgements

The authors would like to thank three anonymous referees for carefully reading the manuscript and providing invaluable suggestions, which greatly improved the presentation of this article.

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