COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **18**, Number **1**, January **2019**

doi:10.3934/cpaa.2019019

pp. **361–396**

SPREADING SPEEDS AND TRAVELING WAVES FOR SPACE-TIME PERIODIC NONLOCAL DISPERSAL COOPERATIVE SYSTEMS

Xiongxiong Bao*

School of Science, Chang'an University
Xi'an, Shaanxi 710064, China

Wenxian Shen

Department of Mathematics and Statistics
Auburn University, Auburn, AL 36849, USA

Zhongwei Shen

Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, AB T6G 2G1, Canada

(Communicated by Junping Shi)

Abstract. The present paper is concerned with the spatial spreading speeds and traveling wave solutions of cooperative systems in space-time periodic habitats with nonlocal dispersal. It is assumed that the trivial solution $\mathbf{u} = \mathbf{0}$ of such a system is unstable and the system has a stable space-time periodic positive solution $\mathbf{u}^*(t, x)$. We first show that in any direction $\xi \in \mathbb{S}^{N-1}$, such a system has a finite spreading speed interval, and under certain condition, the spreading speed interval is a singleton set, and hence, the system has a single spreading speed $c^*(\xi)$ in the direction of ξ. Next, we show that for any $c > c^*(\xi)$, there are space-time periodic traveling wave solutions of the form $\mathbf{u}(t, x) = \mathbf{\Phi}(x - ct\xi, t, ct\xi)$ connecting $\mathbf{u}^*$ and $\mathbf{0}$, and propagating in the direction of ξ with speed c, where $\Phi(x, t, y)$ is periodic in t and y, and there is no such solution for $c < c^*(\xi)$. We also prove the continuity and uniqueness of space-time periodic traveling wave solutions when the reaction term is strictly sub-homogeneous. Finally, we apply the above results to nonlocal monostable equations and two-species competitive systems with nonlocal dispersal and space-time periodicity.

1. Introduction.

The present paper is concerned with the spatial spreading speeds and traveling wave solutions of the following nonlocal dispersal cooperative system in space-time periodic habitats,

$$\frac{\partial \mathbf{u}}{\partial t}(t, x) = \int_{\mathbb{R}^N} k(y - x)\mathbf{u}(t, y)dy - \mathbf{u}(t, x) + \mathbf{F}(t, x, \mathbf{u}(t, x)), \quad x \in \mathbb{R}^N, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35C07, 45C05, 45G15, 45M20, 47G20, 92D25.

Key words and phrases. Cooperative system, nonlocal dispersal, space-time periodic habitat, spreading speed, linear determinacy, traveling wave solution.

X. Bao was partially supported by Natural Science Basic Research Plan in Shaanxi Province of China (2017JQ1014) and NSF of China (11701041). Z. Shen was supported by a start-up grant from the University of Alberta.

* Corresponding author.

where $k : \mathbb{R}^N \to \mathbb{R}$ is a C^1 nonnegative convolution kernel with compact support, and satisfies $k(0) > 0$ and $\int_{\mathbb{R}^N} k(z)dz = 1$; the vector-valued function $\mathbf{u}(t,x) = (u_1(t,x), \ldots, u_K(t,x))^\top$ represents the densities at the point $(t,x) \in \mathbb{R} \times \mathbb{R}^N$; and $\mathbf{F}(t,x,\mathbf{u}) = (F^1(t,x,\mathbf{u}), \ldots, F^K(t,x,\mathbf{u}))^\top$ is the reaction term. The following hypotheses are standard.

(H1): $\mathbf{F}(t,x,\mathbf{0}) = \mathbf{0}$ *for any* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. *For each* $i = 1, \ldots, K$, $F^i(t,x,\mathbf{u})$ *is* C^1 *in* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ *and* C^2 *in* $\mathbf{u} \in \mathbb{R}^K$, *and is periodic in* (t,x) *with period* $(T,P) := (T, p_1, p_2, \ldots, p_N)$, *that is,*

$$F^i(t+T,x,\cdot) = F^i(t,x,\cdot) = F^i(t,x+p_l\mathbf{e}_l,\cdot), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ l = 1, \ldots, N,$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ *is the standard base of* $\mathbb{R}^N$.

(H2): *Equation* (1.1) *admits a positive* (T,P)*-periodic solution* $\mathbf{u}^* : \mathbb{R} \times \mathbb{R}^N \to (0,\infty)^K$. *Moreover,* $\mathbf{u}^*$ *is stable in the sense that for any* $\mathbf{u}_0 \in X_p^{++}$ *with* $\mathbf{u}_0 \le \mathbf{u}^*(0,\cdot)$ *(see* (1.3) *for the definition of* X_p^{++} *and* (1.4) *for the meaning of "*$\le$*"), the solution* $\mathbf{u}(t,\cdot;\mathbf{u}_0)$ *of* (1.1) *with initial data* $\mathbf{u}_0$ *satisfies*

$$\sup_{x \in \mathbb{R}^N} |\mathbf{u}(t,x;\mathbf{u}_0) - \mathbf{u}^*(t,x)| \to 0 \quad \textit{as} \quad t \to \infty.$$

(H3): $\mathbf{F}(t,x,\mathbf{u})$ *is cooperative in the sense that for any* $1 \le i \ne j \le K$,

$$\frac{\partial F^i}{\partial u_j}(t,x,\mathbf{u}) \ge 0 \quad \textit{for all } \mathbf{u} \in [\mathbf{0}, \mathbf{u}^*(t,x)] \textit{ and } (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $[\mathbf{0}, \mathbf{u}^*(t,x)] := [0, u_1^*(t,x)] \times [0, u_2^*(t,x)] \times \cdots \times [0, u_K^*(t,x)]$.

Here are some biological interpretations of the hypotheses **(H1)-(H3)**. The space-time periodicity of $\mathbf{F}(\cdot,\cdot,\mathbf{u})$ in **(H1)** indicates that the underlying environment of (1.1) is subject to space-time periodic variations. Note that a (T,P)-periodic solution $\mathbf{u}^* : \mathbb{R} \times \mathbb{R}^N \to (0,\infty)^K$ of (1.1) is referred to as a *coexistence state* in literature. The hypothesis **(H2)** means that (1.1) has a unique coexistence state which is globally stable with respect to strictly positive perturbations. The hypothesis **(H3)** indicates that the K species described by system (1.1) are cooperative.

System (1.1) is a nonlocal dispersal counterpart of the following system with random dispersal,

$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} + \mathbf{F}(t,x,\mathbf{u}(t,x)), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Systems (1.1) and (1.2) model the population dynamics of a family of species with internal interaction or dispersal between individuals. Note that (1.2) is often used to model the evolution of population densities of cooperative species in which the internal interaction or movement of the individuals occurs randomly between adjacent spatial locations and is described by the differential operator $u \mapsto \Delta u$, referred to as the *random dispersal operator*. System (1.1) arises in modelling the evolution of population densities of cooperative species in which the internal interaction or movement of the individuals occurs between non-adjacent spatial locations and is described by the integral operator $u \mapsto \int_{\mathbb{R}^N} k(y-x)u_i(t,y)dy - u_i(t,x)$, referred to as the *nonlocal dispersal operator*.

Among central dynamical issues in (1.1) and (1.2) are spatial spreading speeds and traveling wave solutions. A huge amount research has been carried out toward the spatial spreading speeds and traveling wave solutions of (1.2). See, for example, [4, 8, 11, 12, 13, 16, 17, 19, 20, 23, 24, 25, 27, 28, 29, 46] for the study of (1.2) in space-time independent habitats, and [1, 9, 30, 31, 32, 43, 48, 49] for the study of (1.2) in time periodic or space-time periodic habitats. We point out that, very recently, Fang, Yu and Zhao in [10] established the existence of spreading speeds and traveling wave solutions for abstract space-time periodic monotone semiflows with monostable structures. The abstract results in [10] can be applied to two species competitive reaction-advection-diffusion system with space-time periodic coefficients.

There are also several studies on the spatial spreading speeds and traveling wave solutions of some special cases of (1.1). See, for example, [14, 18, 26, 33, 34, 35] and references therein for the study of the existence of traveling wave solutions of (1.1) in the space-time independent case, and [5, 6, 7, 15, 37, 38, 40, 41, 42] and references therein for the study of spectral theory of nonlocal dispersal operators and traveling wave solutions of nonlocal dispersal equations in space periodic habitats. Very recently, in [2], the authors established the existence, uniqueness and stability of periodic traveling wave solutions to nonlocal dispersal two species competitive systems with space periodic coefficients. Kong et al. [22] studied the spreading speeds of two species competitive systems with nonlocal dispersal and space-time periodic coefficients. However, these results can not be applied to the coupled system (1.1) with multiple variables directly. Due to the lack of compactness of solutions of (1.1), the abstract results on spreading speeds and traveling wave solutions established in [10] also can not be applied to (1.1). It is the objective of the present paper to carry out a study on spreading speeds and space-time periodic traveling wave solutions of (1.1).

To describe the problems studied and state the results obtained in the present paper, we let

$$\widetilde{X} = \big\{\mathbf{u} = (u_1, u_2, \cdots, u_K)^\top : \mathbb{R}^N \to \mathbb{R}^K : u_i \text{ is measurable and bounded}, \quad \forall i = 1, \dots, K\big\}$$

equipped with the supremum norm $\|\mathbf{u}\| := \sup_{x\in\mathbb{R}^N} |\mathbf{u}(x)|$ and

$$\widetilde{X}^+ = \Big\{\mathbf{u} \in \widetilde{X} = (u_1, u_2, \cdots, u_K)^\top : u_i(x) \geq 0,\ \forall x \in \mathbb{R}^N,\ i = 1, \dots, K\Big\}.$$

For given $d \in \mathbb{N}$, let

$$X(d) = \big\{\mathbf{u} = (u_1, \cdots, u_d)^\top \in C(\mathbb{R}^N, \mathbb{R}^d) : u_i \text{ is uniformly continuous and bounded},\ \ \forall i = 1, \dots, d\big\}$$

equipped with the norm $\|\mathbf{u}\| = \sup_{x\in\mathbb{R}^N} |\mathbf{u}(x)|$, and

$$X^+(d) = \big\{\mathbf{u} = (u_1, \cdots, u_d)^\top \in X(d) : u_i(x) \geq 0,\ \forall x \in \mathbb{R}^N,\ i = 1, \dots, d\big\}.$$

Let

$$\begin{aligned} X_p(d) &= \big\{\mathbf{u} \in C(\mathbb{R}^N, \mathbb{R}^d) : u_i(\cdot + p_l \mathbf{e}_l) = u_i(\cdot),\ l = 1, \dots, N, i = 1, \dots, d\big\}, \\ X_p^+(d) &= \big\{\mathbf{u} \in X_p : u_i(x) \geq 0,\ \forall x \in \mathbb{R}^N,\ i = 1, \dots, d\big\}, \\ X_p^{++}(d) &= \big\{\mathbf{u} \in X_p^+ : u_i(x) > 0,\ \forall x \in \mathbb{R}^N,\ i = 1, \dots, d\big\}. \end{aligned}$$

When $d = K$, we write

$$X = X(K), \quad X^+ = X^+(K), \quad X_p = X_p(K), \quad X_p^+ = X_p^+(K), \quad X_p^{++} = X_p^{++}(K). \tag{1.3}$$

For $\mathbf{u}, \mathbf{v} \in \widetilde{X}$, we define

$$\mathbf{u} \geq (\gg)\ \mathbf{v} \ \text{ if } \ u_i \geq (\gg)\ v_i \ \text{ for all } \ i = 1, ..., K, \tag{1.4}$$

where $u_i \geq (\gg)\ v_i$ if $\inf_{x\in\mathbb{R}^N}[u_i(x) - v_i(x)] \geq (>)0$.

By the general semigroup theory (see [36]), for any $\mathbf{u}_0 \in \tilde{X}$, (1.1) has a unique (local) solution $\mathbf{u}(t,x;\mathbf{u}_0)$ with $\mathbf{u}(0,x;\mathbf{u}_0) = \mathbf{u}_0$. By the comparison principle (see Proposition 2.1), if $\mathbf{u}_0 \in X^+$, then $\mathbf{u}(t,x;\mathbf{u}_0)$ exists for all $t \geq 0$ and $\mathbf{u}(t,\cdot;\mathbf{u}_0) \in X^+$. We remark that $\mathbf{u}(t,\cdot;\mathbf{u}_0) \in X$ if $\mathbf{u}_0 \in X$ and $\mathbf{u}(t,\cdot;\mathbf{u}_0) \in X_p$ if $\mathbf{u}_0 \in X_p$.

Throughout the rest of this paper, we assume $\mathbf{F}(t,x,\mathbf{u})$ satisfies **(H1)-(H3)**. Let

$$\mathcal{X}_p(d) = \left\{\mathbf{u} \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^d) : \mathbf{u}(\cdot + T, \cdot) = \mathbf{u}(\cdot, \cdot + p_l\mathbf{e}_l) = \mathbf{u}(\cdot,\cdot),\ l = 1,\dots,N\right\}$$

with the norm $\|\mathbf{u}\|_{\mathcal{X}_p(d)} := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} |\mathbf{u}(t,x)|$, and set

$$\mathcal{X}_p^+(d) = \left\{\mathbf{u} \in \mathcal{X}_p : u_i(t,x) \geq 0,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ i = 1,\dots,d\right\},$$
$$\mathcal{X}_p^{++}(d) = \left\{\mathbf{u} \in \mathcal{X}_p : u_i(t,x) > 0,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ i = 1,\dots,d\right\}.$$

When $d = K$, we write $\mathcal{X}_p = \mathcal{X}_p(K)$, $\quad \mathcal{X}_p^+ = \mathcal{X}_p^+(K) \quad$ and $\quad \mathcal{X}_p^{++} = \mathcal{X}_p^{++}(K)$.

Consider the linearization of (1.1) at the zero solution $\mathbf{0}$, namely,

$$\frac{\partial \mathbf{v}}{\partial t}(t,x) = \int_{\mathbb{R}^N} k(y-x)\mathbf{v}(t,y)dy - \mathbf{v}(t,x) + A_0(t,x)\mathbf{v}(t,x), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where

$$A_0(t,x) = \left(\frac{\partial F^i}{\partial u_j}(t,x,\mathbf{0})\right)_{K\times K}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{1.6}$$

Note that, for given $\mu \in \mathbb{R}$ and $\xi \in \mathbb{S}^{N-1} := \left\{\xi \in \mathbb{R}^N : |\xi| = 1\right\}$, solutions of (1.5) of the form $\mathbf{v}(t,x) = e^{-\mu(x\cdot\xi - ct)}\phi(t,x)$ with $\phi \in \mathcal{X}_p^{++}$ (if exist) play an important role in the study of spreading speeds and traveling wave solutions of (1.1) in the direction of ξ. Note also that, for such solutions (if exist), $\phi(t,x)$ and $\lambda = \mu c$ satisfy

$$\begin{cases} -\frac{\partial\phi}{\partial t}(t,x) + \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\phi(t,y)dy - \phi(t,x) + A_0(t,x)\phi(t,x) \\ = \lambda\phi(t,x), \quad x \in \mathbb{R}^N \\ \phi(\cdot+T,\cdot) = \phi(\cdot,\cdot+p_l\mathbf{e}_l) = \phi(\cdot,\cdot), \quad l = 1,2,\cdots,N. \end{cases} \tag{1.7}$$

Let $\mathbf{I}$ be the identity map. Define the map $\mathcal{K}_{\xi,\mu}$ by setting

$$(\mathcal{K}_{\xi,\mu}\mathbf{v})(t,x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\mathbf{v}(t,y)dy, \tag{1.8}$$

where the kernel k is as in (1.1). The existence of $\phi \in \mathcal{X}_p^{++}$ and $\lambda \in \mathbb{R}$ satisfying (1.7) is then related to the existence of the principal eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_0$ in $\mathcal{X}_p$ (see Definition 2.2 for the definition of the principal eigenvalue). Note that we do not specify the spaces on which the operators $\mathbf{I}$ and $\mathcal{K}_{\xi,\mu}$ are defined, but this should not cause any trouble.

Throughout this paper, we also assume that

(H4): *Let $A_0(t,x)$ be as in (1.6).*

(a) *For any $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, the matrix $A_0(t,x)$ is quasi-positive (i.e., off-diagonal entries are nonnegative), and is in a block lower triangular* *form, namely,*

$$A_0(t,x) = \begin{pmatrix} A_{01}(t,x) & 0 & \cdots & \cdots & \cdots & 0 \\ A_{21}(t,x) & A_{02}(t,x) & 0 & \cdots & \cdots & 0 \\ A_{31}(t,x) & A_{32}(t,x) & A_{03}(t,x) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ A_{N_b1}(t,x) & \cdots & \cdots & \cdots & \cdots & A_{0N_b}(t,x) \end{pmatrix}$$

for some $N_b \in \mathbb{N}$ independent of $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, and A_{01} is strongly irreducible.

(b) *For given $1 \le k \le N_b$, $\mu \ge 0$, and $\xi \in S^{N-1}$, let $\lambda_1(\xi,\mu,A_{0k})$ be the spectral bound (i.e., the largest real part of the spectrum) of $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_{0k}$ acting on $\mathcal{X}_p(d_k)$, where d_k is the dimension of the matrix A_{0k}. $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of the operator $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_{01}$ acting on $\mathcal{X}_p(d_1)$ with a positive eigenfunction $\boldsymbol{\phi}_1(t,x;\xi,\mu) \in \mathcal{X}_p^{++}(d_1)$.*

(c) *For any $\mu \ge 0$ and every $k > 1$, $\lambda_1(\xi,\mu,A_{01}) > 0$ and $\lambda_1(\xi,\mu,A_{01}) > \lambda_1(\xi,\mu,A_{0k})$. Moreover, for any $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, there is at least one nonzero entry to the left of each diagonal block of $A_0(t,x)$ other than the first block.*

Remark 1.1. (1) Let $\lambda_1(A_{0k}) := \lambda_1(\xi,0,A_{0k})$ for every $k \ge 1$. By **(H4)**(c), $\lambda_1(A_{01}) > 0$ means the equilibrium $\mathbf{u} = \mathbf{0}$ of (1.1) is unstable, i.e, the populations corresponding to the first block grow when all populations are sufficiently small. The inequality $\lambda_1(A_{01}) > \lambda_1(A_{0k})$ means that the growth rate of populations corresponding to the first block is larger than that corresponding to the k-th diagonal block.

(2) Recall that an $n \times n$ constant matrix $B = (b_{ij})$ is called *irreducible* if two nonempty subsets S, S' of $\{1,2,\cdots,n\}$ form a partition of $\{1,2,\cdots,n\}$, then there exist $i \in S$ and $k \in S'$ such that $b_{ik} \neq 0$. An $n \times n$ matrix $B(t,x) = (b_{ij}(t,x))$ on $\mathbb{R} \times \mathbb{R}^N$ is called *irreducible* if for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, $B(t,x)$ is irreducible. $B(t,x) = (b_{ij}(t,x))$ is called *strongly irreducible* on $\mathbb{R} \times \mathbb{R}^N$ if there is $\delta_0 > 0$ such that if two nonempty subsets S, S' of $\{1,2,\cdots,n\}$ form a partition of $\{1,2,\cdots,n\}$, then for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, there exist $i \in S$ and $k \in S'$ such that

$$|b_{ik}(t,x)| \ge \delta_0.$$

The strong irreducibility of $B(t,x)$ implies that any limiting matrix of $B(t,x)$ is irreducible, that is, if $B^* = \lim_{n\to\infty} B(t_n,x_n)$, then B^* is irreducible. In [3], under the assumption that $A_{01}(t,x)$ is cooperative and strongly irreducible on $\mathbb{R} \times \mathbb{R}^N$, some criteria for the existence of the principle eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_{01}$ were established (also see Proposition 2.3). The strong irreducibility of A_{01} is implicitly used in the assumption (H4)(2) since if A_{01} is not strongly irreducible, it may not make sense to assume $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of the operator $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_{01}$.

(3) When $A_0(t,x)$ is independent of t and x, it is called the Frobenius form, in which all the diagonal blocks are irreducible, see [46]. Note that Weinberger et al. studied in [46] the spreading speeds and linear determinacy of the discrete-time recursion system $\mathbf{u}_{n+1} = Q[\mathbf{u}_n]$, while Hu et al. studied in [18]

the spreading speeds and traveling wave solutions of the cooperative system (1.1) when $A_0(t,x)$ is independent of t and x. It should be pointed out that (H4) is much more general than that in [46, Hypotheses 2.1(v)] and [18, Hypotheses 2.2].

Regarding the existence of the principal eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - \mathbf{I} + A_0$, we prove

- (H4) implies that $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-\mathbf{I}+A_0$ acting on $\mathcal{X}_p$, that is, $-\partial_t+\mathcal{K}_{\xi,\mu}-\mathbf{I}+A_0$ has an eigenfunction $\boldsymbol{\phi}(t,x;\mu,\xi) \gg \mathbf{0}$ corresponding to $\lambda_1(\xi,\mu,A_{01})$ (see Proposition 2.4 for more details).

Spatial spreading speeds from $\mathbf{u}^*$ to $\mathbf{0}$ and traveling wave solutions connecting $\mathbf{u}^*$ and $\mathbf{0}$ are among the most interesting dynamics of (1.1). Roughly, for any given $\xi \in \mathbb{S}^{N-1}$, a finite interval $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is called the *spreading speed interval* of (1.1) from $\mathbf{u}^*$ to $\mathbf{0}$ in the direction of ξ if for any $\mathbf{u}_0 \in X^+$ satisfying $\mathbf{0} \le \mathbf{u}_0 \ll \mathbf{u}^*(0,\cdot)$, $\mathbf{u}_0(x) = 0$ for $x\cdot\xi \gg 1$ and $\liminf_{x\cdot\xi\to-\infty} \mathbf{u}_0(x) \gg \mathbf{0}$, there holds

$$\limsup_{t\to\infty} \sup_{x\cdot\xi\le ct} |\mathbf{u}(t,x;\mathbf{u}_0) - \mathbf{u}^*(t,x)| = 0, \quad \forall c < c^*_{\inf}(\xi),$$

$$\limsup_{t\to\infty} \sup_{x\cdot\xi\ge ct} |\mathbf{u}(t,x;\mathbf{u}_0)| = 0, \quad \forall c > c^*_{\sup}(\xi),$$

(see Definition 3.1 for details).

About spreading speeds of (1.1), among others, we prove

- (Finiteness of spreading speeds) Assume (H1)-(H3). For any $\xi \in \mathbb{S}^{N-1}$, (1.1) has a finite spreading speed interval $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ in the direction of ξ (see Theorem 3.1 for details).
- (Linear determinacy) Assume (H1)-(H4). For given $\xi \in \mathbb{S}^{N-1}$, if

$$\mathbf{F}(t,x,\rho\boldsymbol{\phi}^*(t,x)) \le \rho\mathbf{F}_{\mathbf{u}}(t,x,\mathbf{0})\boldsymbol{\phi}^*(t,x), \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad \rho>0, \tag{1.9}$$

where $\boldsymbol{\phi}^*(t,x) = \boldsymbol{\phi}(t,x;\mu^*,\xi)$ with μ^* satisfying

$$\frac{\lambda_1(\xi,\mu^*,A_{01})}{\mu^*} = \inf_{\mu>0} \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}.$$

Then,

$$c^*(\xi) := c^*_{\inf}(\xi) = c^*_{\sup}(\xi) = \inf_{\mu>0} \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$$

is the spreading speed of (1.1) in the direction of ξ (see Theorem 3.2 for details).

Let $\xi \in \mathbb{S}^{N-1}$. Roughly, an entire positive solution $\mathbf{u}(t,x)$ of (1.1) is called *a traveling wave solution of* (1.1) *connecting* $\mathbf{u}^*$ *and* $\mathbf{0}$ *propagating in the direction of* ξ *with speed* c if there is bounded measurable function $\boldsymbol{\Phi} : \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^K$ such that, for any $i = 1,\dots,K$,

$$u_i(t,x) = \Phi_i(x - ct\xi, t, ct\xi), \quad \forall t\in\mathbb{R},\ x\in\mathbb{R}^N,$$

$$\Phi_i(x,t+T,z) = \Phi_i(x,t,z+p_l\mathbf{e}_l) = \Phi_i(x,t,z), \quad \forall x,z\in\mathbb{R}^N,$$

and

$$\lim_{x\cdot\xi\to-\infty} \boldsymbol{\Phi}(x,t,z) = \mathbf{u}^*(t,x+z), \qquad \lim_{x\cdot\xi\to\infty} \boldsymbol{\Phi}(x,t,z) = \mathbf{0}$$

uniformly in $(t,z)\in\mathbb{R}\times\mathbb{R}^N$ (see Definition 4.1 for details).

Among others, assume (H1)-(H4), then for given $\xi\in\mathbb{S}^{N-1}$, we prove the following results about traveling wave solutions of (1.1) (see Theorem 4.1 for details).

- (Existence of traveling waves) Assume (1.9). For any $c > c^*(\xi)$, (1.1) admits periodic traveling wave solutions of the form $u(t,x) = \mathbf{\Phi}(x - ct\xi, t, ct\xi)$ connecting $\mathbf{u}^*$ and $\mathbf{0}$ that propagate in the direction of ξ with speed c, and for $c < c^*(\xi)$, no such solution of (1.1) exists.
- (Continuity and uniqueness of traveling waves) Assume (1.9) and $\mathbf{F}(t,x,\alpha\mathbf{u}) > \alpha\mathbf{F}(t,x,\mathbf{u})$ for $\mathbf{u} \in (\mathbf{0},\mathbf{u}^*]$, $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $\alpha \in (0,1)$. The periodic traveling wave solutions of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$ that propagate in the direction of ξ with speed $c > c^*(\xi)$ are continuous and unique.

We point out the followings. First, for (1.2) in the case $\mathbf{F}(t,x,\mathbf{u}) \equiv \mathbf{F}(\mathbf{u})$, Weinberger et al. studied in [46] the weakly coupled reaction-diffusion system and provided conditions ensuring that the reaction-diffusion system has a spreading speed and is linearly determined, see [46, Theorem 4.2]. Our assumptions for the linear determinacy of spreading speeds of (1.1) in the case that the coupled term $\mathbf{F}(t,x,\mathbf{u})$ is independent of t and x are the same as those in [46, 18]. Following from the assumptions in [22], it is easy to verify that our assumptions **(H1)-(H4)** and (1.9) hold for two species competitive system with nonlocal dispersal and the results on spreading speeds and linear determinacy in our work can also be applied to two species competitive system with nonlocal dispersal in space periodic habitats (see Subsection 5.2 for more details).

Second, in this work we establish the existence of space-time periodic traveling wave solutions of (1.1) for $c > c^*(\xi)$. It remains open, which remains open even for scalar nonlocal dispersal equations in space-time periodic habitats, whether there are traveling wave solutions propagating in the direction of $\xi \in \mathbb{S}^{N-1}$ with speed $c = c^*(\xi)$ for (1.1).

Third, we prove the uniqueness and continuity of traveling wave solutions in the case that $\mathbf{F}(t,x,\mathbf{u})$ is strictly sub-homogeneous, that is, $\mathbf{F}(t,x,\alpha\mathbf{u}) > \alpha\mathbf{F}(t,x,\mathbf{u})$ for $\mathbf{u} \in (\mathbf{0},\mathbf{u}^*]$, $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $\alpha \in (0,1)$. It remains open whether space-time periodic traveling wave solutions of (1.1) without the strictly sub-homogeneous condition are continuous and unique. We will further study the spreading speeds and traveling wave solutions of some epidemic models by the results obtained in this paper somewhere else.

The rest of this paper is organized as follows. In Section 2, we state the definition of the principal eigenvalue for space-time periodic nonlocal dispersal operators, establish some useful properties for the principal eigenvalue, and present a comparison principle for (1.1) and some related linear cooperative systems with nonlocal dispersal. In Section 3, we investigate the existence of spreading speed intervals and linear determinacy of spreading speeds. The existence, nonexistence and uniqueness of space-time periodic traveling wave solutions of (1.1) is established in Section 4. In Section 5, we discuss the applications of the above results to nonlocal monostable equations and two-species competitive systems with nonlocal dispersal and space-time periodicity.

2. **Preliminary.**

In this section, we introduce some principal eigenvalue theory for space-time periodic linear cooperative systems with nonlocal dispersal, and present a comparison principle for (1.1) and some related linear cooperative systems with nonlocal dispersal.

2.1. **Comparison principle.** In this subsection, we present a comparison principle for system (1.1) and the following linear cooperative system with nonlocal dispersal,

$$\frac{\partial \mathbf{v}}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)\mathbf{v}(t,y)dy - \mathbf{v}(t,x) + A(t,x)\mathbf{v}(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.1}$$

where $\xi \in \mathbb{S}^{N-1}$, $\mu \in \mathbb{R}$ and A satisfies

(A1): *$A = (a_{ij}(t,x))_{d\times d} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{d\times d}$ is continuous and (T,P)-periodic with $a_{ij}(t,x) \geq 0$ for $i,j = 1,2,\cdots,d$, $i \neq j$, where d is a fixed positive integer.*

Definition 2.1. *A bounded measurable vector-valued function $\mathbf{u} = (u_1,\dots,u_K)^\top$ (resp. $\mathbf{v} = (v_1,\cdots,v_d)^\top$)) on $[0,T')\times\mathbb{R}^N$ is called a super-solution (or sub-solution) of (1.1) (resp. (2.1)) if for each $x \in \mathbb{R}^N$ and $i = 1,\dots,K$ (resp. $i = 1,\cdots,d$), $u_i(t,x)$ (resp. $v_i(t,x)$) is continuous in $t \in [0,T^{'})$ and*

$$u_i(t,x) \geq (\leq) u_{0i}(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)u_i(s,y)dy - u_i(s,x) + F^i(s,x,\mathbf{u}(s,x))\right] ds$$

$$(resp.\ \ v_i(t,x) \geq (\leq) v_{0i}(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)v_i(s,y)dy - v_i(s,x) + \sum_{j=1}^d a_{ij}(s,x)v_j(s,x))\right] ds)$$

holds for all $t \in [0,T')$.

Proposition 2.1 (Comparison Principle)**.** *Assume* **(H1)**-**(H3)** *and* **(A1)**. *The following statements hold.*

1. *Suppose that $\mathbf{u}^-$ (resp. $\mathbf{v}^-$) is a sub-solution of (1.1) (resp. (2.1)) on $[0,T')\times\mathbb{R}^N$ and $\mathbf{u}^+$ (resp. $\mathbf{v}^+$) is a super-solution of (1.1) (resp. (2.1)) on $[0,T')\times\mathbb{R}^N$, and $\mathbf{u}^\pm \in [\mathbf{0},\mathbf{u}^*]$. If $\mathbf{u}^-(0,\cdot) \leq \mathbf{u}^+(0,\cdot)$ (resp. $\mathbf{v}^-(0,\cdot) \leq \mathbf{v}^+(0,\cdot)$), then $\mathbf{u}^- \leq \mathbf{u}^+$ (resp. $\mathbf{v}^- \leq \mathbf{v}^+$) on $[0,T')\times\mathbb{R}^N$.*
2. *For every $\mathbf{u}_0 \in \widetilde{X}^+$ with $\mathbf{u}_0 \leq \mathbf{u}^*(0,\cdot)$, $\mathbf{u}(t,x;\mathbf{u}_0)$ exists for all $t \geq 0$, where $\mathbf{u}(t,x;\mathbf{u}_0)$ is the solution of (1.1) with initial data $\mathbf{u}_0$.*
3. *Assume* **(H1)**-**(H3)**. *Let $\mathbf{u}_1,\mathbf{u}_2 \in X^+$. If $\mathbf{u}_1 \leq \mathbf{u}_2 \leq \mathbf{u}^*(0,\cdot)$, then*

$$\mathbf{0} \leq \mathbf{u}(t,x;\mathbf{u}_1) \leq \mathbf{u}(t,x;\mathbf{u}_2) \leq \mathbf{u}^*(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}^N.$$

Proof. The proposition follows from the arguments in [41, Proposition 2.1]. □

2.2. **General principal eigenvalue theory.** Let $\mathcal{K}_{\xi,\mu}$ be as in (1.8). Consider the following eigenvalue problem

$$L_{\xi,\mu,A}[\mathbf{w}] := -\mathbf{w}_t + (\mathcal{K}_{\xi,\mu} - \mathbf{I})\mathbf{w} + A\mathbf{w} = \lambda\mathbf{w}, \quad \mathbf{w} \in \mathcal{X}_p(d), \tag{2.2}$$

where $\xi \in \mathbb{S}^{N-1}$, $\mu \in \mathbb{R}$ and A is as in (2.1).

In the sequel, results in this subsection with different d will be used. We point out that, if $\mathbf{u}(t,x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\boldsymbol{\phi}(t,x;\mu,\xi,A)$ with $\boldsymbol{\phi}(t,\cdot;\mu,\xi,A) \in \mathcal{X}_p(d)\backslash\{\mathbf{0}\}$ is a solution of (2.1) with $e^{-\mu(y-x)\cdot\xi}k(y-x)$ being replaced by $k(y-x)$, then λ is an eigenvalue of (2.2) and $\mathbf{w} = \boldsymbol{\phi}(t,x;\mu,\xi,A)$ is a corresponding eigenfunction.

Let $\sigma(L_{\xi,\mu,A})$ be the spectrum of $L_{\xi,\mu,A}$ on $\mathcal{X}_p(d)$. Set

$$\lambda(\xi,\mu,A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(L_{\xi,\mu,A})\},$$

which is spectral bound of the operator $L_{\xi,\mu,A}$. Observe that if $\mu = 0$, then (2.2) is independent of ξ, and hence we set

$$\tilde{\lambda}(A) := \lambda(\xi, 0, A), \quad \forall \xi \in \mathbb{S}^{N-1}.$$

Definition 2.2. *We call* $\lambda(\xi,\mu,A)$ *the* principal spectrum point *of* $L_{\xi,\mu,A}$. $\lambda(\xi,\mu,A)$ *is called the* principal eigenvalue *of* $L_{\xi,\mu,A}$ *or it is said that* $L_{\xi,\mu,A}$ has a principal eigenvalue *if* $\lambda(\xi,\mu,A)$ *is an isolated eigenvalue of* $L_{\xi,\mu,A}$ *with an eigenfunction* $\mathbf{w} \in \mathcal{X}_p^+(d)$.

Assume, in addition, that

(A2): $A(t,x)$ *is strongly irreducible for any* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Throughout the rest of this subsection, we assume **(A1)** and **(A2)**.

We note that the principal spectrum point $\lambda(\xi,\mu,A)$ of $L_{\xi,\mu,A}$ belongs to $\sigma(L_{\xi,\mu,A})$. In general, $\lambda(\xi,\mu,A)$ may not be the principal eigenvalue of $L_{\xi,\mu,A}$ and hence $L_{\xi,\mu,A}$ may not have a principal eigenvalue. The reader is referred to [5] and [41] for examples in the case $d = 1$. In the recent paper [3], the first two authors of the current paper established some useful criteria for the existence of a principal eigenvalue of $L_{\xi,\mu,A}$. For example, for any fixed $x \in \mathbb{R}^N$, let $\lambda(x,A)$ be the principal eigenvalue (i.e., the eigenvalue with largest real part and with a positive eigenfunction) of

$$\begin{cases} -\frac{d\boldsymbol{\phi}(t)}{dt} + A(t,x)\boldsymbol{\phi}(t) = \lambda(x,A)\boldsymbol{\phi}(t), \\ \boldsymbol{\phi}(t+T) = \boldsymbol{\phi}(t). \end{cases}$$

It is proved in [3] that, *if* $\lambda(x,A)$ *is* C^N *and there is some* $x_0 \in \mathbb{R}^N$ *such that* $\lambda(x_0,A) = \max_{x\in\mathbb{R}^N} \lambda(x,A)$*, and the partial derivatives of* $\lambda(x,A)$ *up to order* $N-1$ *at* x_0 *are zero, then the principal eigenvalue of* $L_{\xi,\mu,A}$ *exists* (see [3, Corollary 2.1]). Moreover, it is proved that, *if* $\lambda(\xi,\mu,A)$ *is the principal eigenvalue of* $L_{\xi,\mu,A}$*, then* $\lambda(\xi,\mu,A)$ *is algebraically simple* (see [3, Theorem 2.3]).

Consider the inhomogeneous linear system

$$\frac{\partial \mathbf{v}}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)\mathbf{v}(t,y)dy - \mathbf{v}(t,x) + A(t,x)\mathbf{v}(t,x) + B(t,x), \tag{2.3}$$

where $B : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^d$ is a continuous and (T,P)-periodic vector-valued function. We have the following proposition on the existence of bounded entire solutions.

Proposition 2.2. *If* $\lambda(\xi,\mu,A) < 0$*, then* (2.3) *has a unique bounded entire solution* $\mathbf{u}^{**} \in \mathcal{X}_p(d)$*, which is a globally stable solution of* (2.3) *with respect to perturbations in* $X_p(d)$*. Furthermore, if the components of* $B(t,x)$ *are nonnegative and* $B(t,x) \not\equiv 0$ *for any* $(t,x) \in \mathbb{R}\times\mathbb{R}^N$*, then* $\mathbf{u}^{**}(t,\cdot) \in X_p^{++}(d)$ *for all* $t \in \mathbb{R}$.

Proof. It can be proven by the arguments similar to those in [22, Proposition 2.4]. We omit it here. ☐

For given $\rho \geq 0$, let

$$X(d;\rho) = \left\{\mathbf{u} \in C(\mathbb{R}^N, \mathbb{R}^d) : \text{the function } x \mapsto e^{-\rho\|x\|}\mathbf{u}(x) \text{ belongs to } X(d)\right\}.$$

Denote by $\mathbf{v}(t,x;\mathbf{v}_0,\xi,\mu,A)$ the unique solution of (2.1) with $\mathbf{v}(0,\cdot;\mathbf{v}_0,\xi,\mu,A) = \mathbf{v}_0 \in X(d;\rho)$. Define the solution operator of (2.1) by

$$\boldsymbol{\Phi}(t;\xi,\mu,A)\mathbf{v}_0 = \mathbf{v}(t,\cdot;\mathbf{v}_0,\xi,\mu,A) \in X(d;\rho).$$

Note that $X(d;0) = X(d)$ and $\boldsymbol{\Phi}(t;\xi,\mu,A)\mathbf{v}_0 \in X_p(d)$ if $\mathbf{v}_0 \in X_p(d)$. Thus, the restriction $\boldsymbol{\Phi}^p(T;\xi,\mu,A) := \boldsymbol{\Phi}(T;\xi,\mu,A)|_{X_p(d)}$ is well-defined. Denote by

$r(\mathbf{\Phi}^p(T;\xi,\mu,A))$ the spectrum radius of $\mathbf{\Phi}^p(T;\xi,\mu,A)$. It follows from arguments as in [37, Proposition 3.10] that

$$\lambda(\xi,\mu,A) = \frac{\ln r(\mathbf{\Phi}^p(T;\xi,\mu,A))}{T}, \quad \forall \xi \in \mathbb{S}^{N-1},\ \mu \in \mathbb{R}. \tag{2.4}$$

Note that $\mathbf{\Phi}(t;\xi,0,A)$ is independent of $\xi \in \mathbb{S}^{N-1}$. Therefore, we put

$$\widetilde{\mathbf{\Phi}}(t;A) = \mathbf{\Phi}(t;\xi,0,A), \quad \forall \xi \in \mathbb{S}^{N-1}.$$

For $\mathbf{v}_0 \in X(d)$, $\xi \in \mathbb{S}^{N-1}$ and $\mu \in \mathbb{R}$, let $\mathbf{v}_0^{\xi,\mu} = e^{-\mu x\cdot\xi}\mathbf{v}_0$. Note that $\mathbf{v}_0^{\xi,\mu} \in X(d;\rho)$ with $\rho = |\mu|$. Then, the uniqueness of solutions of (2.1) with initial function in $X(d;\rho)$ ensures

$$\mathbf{\Phi}(t;\xi,\mu,A)\mathbf{v}_0 = e^{\mu x\cdot\xi}\widetilde{\mathbf{\Phi}}(t;A)\mathbf{v}_0^{\mu,\xi}. \tag{2.5}$$

For each $x \in \mathbb{R}^N$, there is a family of nonnegative bounded measures $m_{ij}(x;y,dy)$ $(i,j=1,\dots,d)$ such that

$$\big(\widetilde{\Phi}(T;A)\mathbf{v}_0\big)_i(x) = \sum_{j=1}^{d}\int_{\mathbb{R}^N} v_{0j}(y)m_{ij}(x;y,dy), \quad \forall 1 \le i \le d,$$

where $\mathbf{v}_0 = (v_{01}, v_{02}, \dots, v_{0d})^\top$. Note that

$$\big(\widetilde{\Phi}(T;A)\mathbf{v}_0(\cdot - p_l\mathbf{e}_l)\big)_i(x) = \big(\widetilde{\Phi}_i(T;A)\mathbf{v}_0\big)(x - p_l\mathbf{e}_l)$$

for all $x \in \mathbb{R}^N$, $l = 1,\dots,N$ and $i = 1,\dots,d$. Then, for any $i,j = 1,\dots,d$,

$$\begin{aligned}\int_{\mathbb{R}^N} v_{0j}(y)m_{ij}(x - p_l\mathbf{e}_l;y,dy) &= \int_{\mathbb{R}^N} v_{0j}(y - p_l\mathbf{e}_l)m_{ij}(x;y,dy)\\ &= \int_{\mathbb{R}^N} v_{0j}(y)m_{ij}(x;y + p_l\mathbf{e}_l,dy).\end{aligned}$$

Hence

$$m_{ij}(x - p_l\mathbf{e}_l;y,dy) = m_{ij}(x;y + p_l\mathbf{e}_l,dy), \quad \forall\ i,j = 1,\dots,d.$$

By (2.5), we have

$$(\Phi(T;\xi,\mu,A)\mathbf{v}_0)_i(x) = \sum_{j=1}^{d}\int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}v_{0j}(y)m_{ij}(x;y,dy), \quad \forall \mathbf{v}_0 \in X,\ 1 \le i \le d. \tag{2.6}$$

Proposition 2.3. *Let $\xi \in \mathbb{S}^{N-1}$. Suppose that $\lambda(\xi,\mu,A)$ is the principal eigenvalue of $L_{\xi,\mu,A}$ for any $\mu \ge 0$ and $\tilde{\lambda}(A) > 0$. Then, there is $\mu^* := \mu^*(\xi,A) \in (0,\infty)$ such that*

$$\frac{\lambda(\xi,\mu^*,A)}{\mu^*} = \inf_{\mu>0}\frac{\lambda(\xi,\mu,A)}{\mu}.$$

Proof. Let $\xi \in \mathbb{S}^{N-1}$ and A be fixed. By the perturbation theory of isolated eigenvalue of closed operators (see [21]), $\lambda(\mu) := \lambda(\xi,\mu,A)$ is twice continuously differentiable in μ.

For any $\mu > 0$, let

$$\Psi(\mu) = \frac{\lambda(\mu)}{\mu}.$$

Clearly, $(\mu\Psi(\mu))' = \lambda'(\mu)$. Similar to that in [27, Lemma 3.7], we have that $\lambda(\xi,\mu,A)$ is convex in μ, that is, $\lambda''(\mu) \ge 0$. We see

$$\Psi'(\mu) = \frac{1}{\mu}[\lambda'(\mu) - \Psi(\mu)] \quad \text{and} \quad \big(\mu^2\Psi'(\mu)\big)' = \mu\lambda''(\mu) \ge 0. \tag{2.7}$$

By the definition of $\Psi(\mu)$ and the assumption $\tilde{\lambda}(A) > 0$, we have $\Psi(\mu) \to \infty$ as $\mu \to 0^+$. Thus, (2.7) implies that $\Psi(\mu)$ is decreasing near $\mu = 0^+$.

Note that there are $k_0 > 0$ and $r_0 > 0$ such that $k(z) \geq k_0$ for $\|z\| \leq \frac{r_0}{2}$. Let $m_n(\xi) = k_0 \int_{\|z\| \leq \frac{r_0}{2}} \frac{(-z\cdot\xi)^n}{n!} dz$. Then for any $\mu > 0$ and $i = 1, ..., d$,

$$\begin{aligned}\int_{\mathbb{R}^N} k(z) e^{-\mu z\cdot\xi} dz - 1 + a_{ii}(t,x) \geq & k_0 \int_{\|z\| \leq \frac{r_0}{2}} e^{-\mu z\cdot\xi} dz - 1 + a_{ii}(t,x) \\ \geq & m_0 + m_2(\xi)\mu^2 + \sum_{m=2}^{\infty} m_{2n}(\xi)\mu^{2n} - 1 + a_{ii}(t,x).\end{aligned}$$

Let $m := \inf_{\xi \in S^{N-1}} m_2(\xi) (> 0)$ and $a_{ii}^{\mu,\xi}(t,x) = \int_{\mathbb{R}^N} k(z) e^{-\mu z\cdot\xi} dz - 1 + a_{ii}(t,x)$ for $i = 1, ..., d$. Then we have

$$a_{ii}^{\mu,\xi}(t,x) \geq m_0 + m\mu^2 - 1 + a_{ii}(t,x)$$

for $\mu > 0$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Hence there is $\beta > 0$ such that $a_{ii}^{\mu,\xi}(t,x) \geq \beta\mu^2$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $\xi \in S^{N-1}$ and $\mu \gg 1$. Recall that $\mathbf{\Phi}(t;\xi,\mu,A)\mathbf{v}_0$ is the solution operator of (2.1). Note that $A(t,x)$ is cooperative for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Define

$$A^{\mu,\xi}(t,x)\mathbf{v}(t,x) = \int_{\mathbb{R}^N} k(z) e^{-\mu z\cdot\xi} \mathbf{v}(t,z) dz - \mathbf{v}(t,x) + A(t,x)\mathbf{v}(t,x).$$

Then for any $\mathbf{v} \gg \mathbf{0}$,

$$A^{\mu,\xi}(t,x)\mathbf{v} \geq \mathrm{diag}(\beta\mu^2, ..., \beta\mu^2)\mathbf{v}, \quad \forall t \in \mathbb{R},\ \mu \gg 1.$$

By the comparison principle for linear cooperative systems,

$$\big(\mathbf{\Phi}(t;\xi,\mu,A)\boldsymbol{\phi}(t,x;\mu,\xi,A)\big)_i \geq e^{\beta\mu^2 t}\phi_i(t,x;\mu,\xi,A), \quad 1 \leq i \leq d$$

for $\mu \gg 1$ and $t > 0$, which implies that

$$\mathbf{\Phi}(T;\xi,\mu,A)\boldsymbol{\phi}(T,x;\mu,\xi,A) \geq e^{\beta\mu^2 T}\boldsymbol{\phi}(T,x;\mu,\xi,A).$$

Then by (2.4), we can obtain that $\frac{\lambda(\mu)}{\mu} \geq \beta\mu$ and $\frac{\lambda(\mu)}{\mu} \to \infty$ as $\mu \to \infty$.

Therefore there exists $\mu^* \in (0,\infty)$ such that $\frac{\lambda(\mu^*)}{\mu^*} = \inf_{\mu > 0} \frac{\lambda(\mu)}{\mu}$, which implies that $\frac{\lambda(\mu^*)}{\mu^*} < \frac{\lambda(\mu)}{\mu}$ for any $\forall \mu \in (0,\mu^*)$. This completes the proof. □

The following corollary follows from the convexity of $\lambda(\xi,\mu,A)$ in μ and Proposition 2.3.

Corollary 2.1. *Let $\xi \in \mathbb{S}^{N-1}$. Suppose that $\lambda(\xi,\mu,A)$ is the principal eigenvalue of $L_{\xi,\mu,A}$ for any $\mu > 0$ and $\tilde{\lambda}(A) > 0$. Let $\mu^*(\xi,A) \in (0,\infty)$ is as in Proposition 2.3. Then,*

$$\frac{\partial \lambda}{\partial \mu}(\xi,\mu,A) < \frac{\lambda(\xi,\mu,A)}{\mu}, \quad \forall 0 < \mu < \mu^*(\xi,A).$$

Moreover, for any $\epsilon > 0$, there exists some $\mu_\epsilon = \mu_\epsilon(\xi,A) > 0$ such that for $\mu_\epsilon < \mu < \mu^*(\xi,A)$,

$$-\frac{\partial \lambda}{\partial \mu}(\xi,\mu,A) < -\frac{\lambda(\xi,\mu^*(\xi,A),A)}{\mu^*(\xi,A)} + \epsilon.$$

2.3. **Principal eigenvalue theory for L_{ξ,μ,A_0}.** In this subsection, we study the principal eigenvalue theory of the operator L_{ξ,μ,A_0}, where $A_0(t,x)=\mathbf{F}_u(t,x,\mathbf{0})$ is assumed to satisfy **(H4)**. Recall that $\lambda(\xi,\mu,A_0)$ is the principal spectrum point of L_{ξ,μ,A_0} given in Definition 2.2, and that $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of $L_{\xi,\mu,A_{01}}$ with a positive principal eigenfunction $\boldsymbol{\phi}_1(t,x;\mu,\xi)$ for any $\mu>0$ and $\xi\in\mathbb{S}^{N-1}$.

Proposition 2.4. *Assume* **(H1)**-**(H4)**. *Then for each $\mu>0$ and $\xi\in\mathbb{S}^{N-1}$, $\lambda(\xi,\mu,A_0)=\lambda_1(\xi,\mu,A_{01})$ and $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of L_{ξ,μ,A_0}. Moreover, $\lambda_1(\xi,\mu,A_{01})$ is continuous in A_{01}.*

Proof. By **(H4)**, $L_{\xi,\mu,A_{01}}$ has the principal eigenvalue $\lambda_1(\xi,\mu,A_{01})$ for any $\mu>0$ and $\xi\in\mathbb{S}^{N-1}$, where A_{01} is the first diagonal block of A_0. Moreover, by **(H4)**(c), there holds $\lambda_1(\xi,\mu,A_{01})>\lambda_1(\xi,\mu,A_{0k})$ for every $k>1$, where $\lambda_1(\xi,\mu,A_{0k})$ is the principal spectrum point of the k-th diagonal block of $-\partial_t+\mathcal{K}_{\xi,\mu}-\mathbf{I}+A_0$.

Let us consider

$$A_0'(t,x)=\begin{pmatrix}A_{01}(t,x) & 0\\ A_{12}(t,x) & A_{02}(t,x)\end{pmatrix}.$$

Note that the principal spectrum point $\lambda_1(\xi,\mu,A_{02})$ of $-\partial_t+\mathcal{K}_{\xi,\mu}-\mathbf{I}+A_{02}$ satisfies $\lambda_1(\xi,\mu,A_{02})<\lambda_1(\xi,\mu,A_{01})$. Let λ be the principal spectrum point of

$$L_{\xi,\mu,A_{02}}[\mathbf{v}]-\lambda_1(\xi,\mu,A_{01})\mathbf{v}=\lambda\mathbf{v}.$$

Clearly, $\lambda=\lambda_1(\xi,\mu,A_{02})-\lambda_1(\xi,\mu,A_{01})<0$. By Proposition 2.2, there is a unique space-time periodic positive solution $\boldsymbol{\phi}_2(t,x;\mu,\xi)$ of

$$\mathbf{v}_t=\mathcal{K}_{\xi,\mu}\mathbf{v}-\mathbf{v}+A_{02}(t,x)\mathbf{v}-\lambda_1(\xi,\mu,A_{01})\mathbf{v}+A_{12}(t,x)\boldsymbol{\phi}_1,\quad x\in\mathbb{R}^N. \tag{2.8}$$

Moreover, $\boldsymbol{\phi}_2(t,x;\mu,\xi)$ is a globally asymptotically stable solution of (2.8). It then follows that $(\boldsymbol{\phi}_1(t,x;\mu,\xi),\ \boldsymbol{\phi}_2(t,x;\mu,\xi))^\top$ is a principal eigenfunction of $L_{\xi,\mu,A_0'}$ corresponding to the principal eigenvalue $\lambda_1(\xi,\mu,A_{01})$.

Now, let us consider

$$A_0''(t,x)=\begin{pmatrix}A_{01}(t,x) & 0 & 0\\ A_{21}(t,x) & A_{02}(t,x) & 0\\ A_{31}(t,x) & A_{32}(t,x) & A_{03}(t,x)\end{pmatrix}=\begin{pmatrix}A_0'(t,x) & 0\\ * & A_{03}(t,x)\end{pmatrix}.$$

Since $\lambda_1(\xi,\mu,A_{01})>\lambda_1(\xi,\mu,A_{03})$, similar arguments as above ensure that $\lambda_1(\xi,\mu,A_{01})$ is the principal eigenvalue of $L_{\xi,\mu,A_0''}$.

Thus, by induction, we obtain that L_{ξ,μ,A_0} has the principal eigenvalue $\lambda(\xi,\mu,A_0)=\lambda_1(\xi,\mu,A_{01})$ with a principal eigenfunction given by

$$\boldsymbol{\phi}(t,x;\mu,\xi):=\big(\boldsymbol{\phi}_1(t,x;\mu,\xi),\boldsymbol{\phi}_2(t,x;\mu,\xi),\cdots,\boldsymbol{\phi}_{N_b}(t,x;\mu,\xi)\big)^\top\in\mathcal{X}_p^{++}. \tag{2.9}$$

The continuity of $\lambda_1(\xi,\mu,A_{01})$ in A_{01} follows from [3, Theorem 2.1] and perturbation theory of isolated eigenvalues for closed linear operators. This completes the proof. □

We remark that, by Proposition 2.4, results proven in Subsection 2.2 apply to the operator L_{ξ,μ,A_0} in Proposition 2.4.

3. **Spreading speeds and linear determinacy.** In this section, we investigate the spreading speeds of (1.1) and explore the linear determinacy for the spreading speeds. Recall that $\mathbf{u}(t,x;\mathbf{u}_0)$ is the unique solution of (1.1) with $\mathbf{u}(0,\cdot;\mathbf{u}_0)=\mathbf{u}_0\in X$.

We first introduce the notion of spreading speed intervals and spreading speeds for the cooperative system (1.1). For $\xi\in\mathbb{S}^{N-1}$, let

$$X^+(\xi):=\big\{\mathbf{u}\in X^+:\mathbf{u}\ll\mathbf{u}^*(0,\cdot),\ \mathbf{u}(x)=\mathbf{0}\text{ for }x\cdot\xi\gg1\text{ and }\liminf_{x\cdot\xi\to-\infty}\mathbf{u}(x)\gg\mathbf{0}\big\}.$$

Definition 3.1. *For $\xi\in\mathbb{S}^{N-1}$, let*

$$C_{\sup}(\xi)=\left\{c\in\mathbb{R}:\limsup_{t\to\infty,x\cdot\xi\geq ct}|\mathbf{u}(t,x;\mathbf{u}_0)|=0,\ \forall\mathbf{u}_0\in X^+(\xi)\right\},$$

$$C_{\inf}(\xi)=\left\{c\in\mathbb{R}:\limsup_{t\to\infty,x\cdot\xi\leq ct}|\mathbf{u}(t,x;\mathbf{u}_0)-\mathbf{u}^*(t,x)|=0,\ \forall\mathbf{u}_0\in X^+(\xi)\right\}.$$

Define

$$c^*_{\inf}(\xi)=\sup\{c:c\in C_{\inf}(\xi)\}\quad\text{and}\quad c^*_{\sup}(\xi)=\inf\{c:c\in C_{\sup}(\xi)\}.$$

We call $[c^*_{\inf}(\xi),c^*_{\sup}(\xi)]$ the spreading speed interval of (1.1) in the direction of ξ. If $c^*_{\inf}(\xi)=c^*_{\sup}(\xi)$, we call $c^*(\xi):=c^*_{\inf}(\xi)=c^*_{\sup}(\xi)$ the spreading speed of (1.1) in the direction of ξ.

Observe that if $c_1\in C_{\inf}(\xi)$ and $c_2\in C_{\sup}(\xi)$, then $c_1<c_2$, and hence, $c^*_{\inf}(\xi)\leq c^*_{\sup}(\xi)$. Therefore, the interval $[c^*_{\inf}(\xi),c^*_{\sup}(\xi)]$ is well-defined.

The main results of this section are stated in the following theorems.

Theorem 3.1. *Suppose* **(H1)**-**(H3)**. *For any $\xi\in\mathbb{S}^{N-1}$, $[c^*_{\inf}(\xi),c^*_{\sup}(\xi)]$ is a finite interval.*

Next, we try to explore conditions such that $c^*_{\inf}(\xi)=c^*_{\sup}(\xi)$. Fix $\xi\in\mathbb{S}^{N-1}$. Let

$$\mu^*:=\mu^*(\xi,A_{01})\tag{3.1}$$

be as in Proposition 2.3 with $A=A_{01}$, and set

$$\lambda_1(\mu^*):=\lambda_1(\xi,\mu^*,A_{01})\quad\text{and}\quad\boldsymbol{\phi}^*(t,x):=\boldsymbol{\phi}(t,x;\mu^*,\xi),\tag{3.2}$$

where $\phi(t,x;\mu^*,\xi)$ is as in (2.9) with $\mu=\mu^*$.

We introduce the following additional assumption.

(H5): *For all $\rho>0$, $\mathbf{F}(t,x,\rho\boldsymbol{\phi}^*)\leq\rho\mathbf{F}_{\mathbf{u}}(t,x,\mathbf{0})\boldsymbol{\phi}^*$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$.*

Theorem 3.2 (Linear determinacy)**.** *Suppose* **(H1)**-**(H5)**. *For any $\xi\in\mathbb{S}^{N-1}$, there holds*

$$c^*_{\inf}(\xi)=c^*_{\sup}(\xi)=\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,A_{01})}{\mu}=\frac{\lambda_1(\mu^*)}{\mu^*}.$$

That is, for any $\xi\in\mathbb{S}^{N-1}$, the spreading speed $c^*(\xi)=\frac{\lambda_1(\mu^*)}{\mu^*}$ exists and it is linearly determined.

Remark 3.1. (1) Under the assumptions **(H1)**-**(H5)**, we have shown that (1.1) has a spreading speed and it is linearly determined. In particular, our assumptions and results extend the results in Weinberger et al. [46] for discrete-time recursion system $\mathbf{u}_{n+1}=Q[\mathbf{u}_n]$ to more general nonlocal dispersal cooperative system (1.1) in space-time periodic habitats.

(2) We will apply the results of the present paper on spreading speeds, that is, Theorems 3.1 and 3.2 to two species competitive system with nonlocal dispersal and space-time periodic coefficients.

3.1. **Proof of Theorem 3.1.** In this subsection, we prove Theorem 3.1. We first present some lemmas. Let us consider the space shifted systems of (1.1), i.e.,

$$\frac{\partial \mathbf{u}}{\partial t}(t,x)=\int_{\mathbb{R}^N}k(y-x)\mathbf{u}(t,y)dy-\mathbf{u}(t,x)+\mathbf{F}(t,x+z,\mathbf{u}(t,x)),\quad x\in\mathbb{R}^N,\quad (3.3)$$

where $z\in\mathbb{R}^N$. Let $\mathbf{u}(t,x;\mathbf{u}_0,z)$ be the unique solution of (3.3) with $\mathbf{u}(0,\cdot;\mathbf{u}_0,z)=\mathbf{u}_0\in X$. In particular, $\mathbf{u}(t,x;\mathbf{u}_0,0)=\mathbf{u}(t,x;\mathbf{u}_0)$.

By assumptions **(H1)-(H3)** and Proposition 2.1, we have the following lemma. Let K_1 be the dimension of the first diagonal block of A_0, that is, K_1 is the dimension of A_{01}.

Lemma 3.1. *Let $\xi\in\mathbb{S}^{N-1}$, $c\in\mathbb{R}$ and $\mathbf{u}_0\in X^+(\xi)$. If there is $\delta_0>0$ such that*

$$\inf_{z\in\mathbb{R}^N}\min_{i=1,\dots,K_1}\left\{\liminf_{x\cdot\xi\le cnT,n\to\infty}u_i(nT,x;\mathbf{u}_0,z)\right\}\ge\delta_0,$$

then for any $c'<c$, there holds

$$\limsup_{x\cdot\xi\le c't,t\to\infty}|\mathbf{u}(t,x;\mathbf{u}_0,z)-\mathbf{u}^*(t,x+z)|=0\quad \textit{uniformly in } z\in\mathbb{R}^N.$$

Proof. Due to the stability of $\mathbf{u}^*$, for each $\mathbf{u}_0\in X^+(\xi)$ there holds the convergence $|\mathbf{u}(t,x;\mathbf{u}_0,z)-\mathbf{u}^*(t,x+z)|\to0$ as $t\to\infty$ for $x\in\mathbb{R}^N$ and $z\in\mathbb{R}^N$. By **(H4)**, we know that an increase in the first K_1 components will increase all components as time elapses. Hence, the conclusion of the lemma can be shown using the comparison principle and arguments similar to those in [22, Lemma 3.4]. Here we omit the details. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For $\mathbf{u}=(u_1,\dots,u_K)^\top$ satisfying $u_j=0$ for all $j\ne i$, we write $f_i(t,x,u_i):=F^i(t,x,\mathbf{u})$. Since $\mathbf{F}(t,x,\mathbf{u})$ is cooperative, we have for any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and $i=1,\dots,K$

$$F^i(t,x,\mathbf{u})\ge f_i(t,x,u_i),\quad \forall\mathbf{u}\in[\mathbf{0},\mathbf{u}^*(t,x)].\quad (3.4)$$

Let us consider the decoupled equations

$$\frac{\partial u_i}{\partial t}=\int_{\mathbb{R}^N}k(y-x)u_i(t,y)dy-u_i(t,x)+f_i(t,x,u_i),\quad i=1,\dots,K.\quad (3.5)$$

To state the spreading properties of (3.5), we set

$$X_i^+(\xi)=\left\{u\in X^+(1):0\le u<u_i^*(0,\cdot),\ \liminf_{x\cdot\xi\to-\infty}u(x)>0\,\text{and}\,u(x)=0,\ \forall x\cdot\xi\gg1\right\}.$$

For any $u_0\in X_i^+(\xi)$, let $\overline{u}_i(t,x;u_0)$ be the solution of (3.5) with $\overline{u}_i(0,\cdot;u_0)=u_0$. By arguments as in [38], there is a finite spreading speed interval $[\overline{c}^*_{i,\inf}(\xi),\overline{c}^*_{i,\sup}(\xi)]$ of (3.5) for each $i=1,\dots,K$.

By (3.4) and Proposition 2.1, for any $\mathbf{u}_0=(u_{01},\dots,u_{0K})^\top\in X^+(\xi)$ and $i=1,\dots,K$,

$$u_i(t,x;\mathbf{u}_0)\ge\overline{u}_i(t,x;u_{0i}),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

It then follows from [38] that for each $i = 1, \dots, K$, $c < \overline{c}^*_{i,\inf}(\xi)$ implies that $\liminf_{x\cdot\xi\le ct, t\to\infty} \overline{u}_i(t,x;u_{0i}) > 0$. Let $c_0 = \min_{i=1,\dots,K}\{\overline{c}^*_{i,\inf}(\xi)\}$. Then, $c < c_0$ yields

$$\liminf_{x\cdot\xi\le ct, t\to\infty} \overline{u}_i(t,x;u_{0i}) > 0, \quad \forall i = 1,\dots,K,$$

and hence,

$$\liminf_{x\cdot\xi\le ct, t\to\infty} u_i(t,x;\mathbf{u_0}) > 0, \quad \forall i = 1,\dots,K.$$

Note that $\mathbf{u}(t,x;\mathbf{u}_0,0) = \mathbf{u}(t,x;\mathbf{u}_0)$ and Lemma 3.1 holds uniformly for $z \in \mathbb{R}^N$. We then conclude from Lemma 3.1 that for any $c < c_0$, there holds

$$\limsup_{x\cdot\xi\le ct, t\to\infty} |\mathbf{u}(t,x;\mathbf{u}_0) - \mathbf{u}^*(t,x)| = 0.$$

From this, we see $c^*_{\inf}(\xi) \ge c_0$.

Next, we show that there exists an upper bound for the spreading speed interval. Define

$$\hat{u}_i(t,x;C) = u^*_i(t,x)\left(1 - \eta(x\cdot\xi - Ct)\right), \quad i = 1,\dots,K,$$

where $\eta(s) = \frac{1}{2}\left(1 + \tanh\frac{s}{2}\right)$ and C is a positive constant. Then there is a constant $C_0 > 0$ such that $\hat{\mathbf{u}}(t,x;C) := (\hat{u}_1(t,x;C),\dots,\hat{u}_K(t,x;C))^\top$ is a super-solution of (1.1) on $[0,\infty)$ for every $C \ge C_0$. Hence, for any given $\mathbf{u}_0 \in X^+(\xi)$, there is $N > 0$ such that $\mathbf{u}_0 \le \mathbf{u}^+(NT,\cdot;C_0)$. It then follows from Proposition 2.1 that

$$\mathbf{u}(t,x;\mathbf{u}_0) \le \hat{\mathbf{u}}(NT + t, x; C_0), \quad t \ge 0,\ x \in \mathbb{R}^N,$$

which means that $\limsup_{x\cdot\xi\ge ct, t\to\infty} |\mathbf{u}(t,x;\mathbf{u}_0)| = 0$ for any $c > C_0$. Hence, $c^*_{\sup}(\xi) \le C_0$.

Therefore, $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a bounded interval. □

3.2. **Proof of Theorem 3.2.** We prove Theorem 3.2 in this subsection. Throughout this subsection, we assume **(H1)-(H5)**.

First, we present the following lemma, which provides a lower bound for $c^*_{\inf}(\xi)$ and plays a crucial role in the proof of Theorem 3.2.

Lemma 3.2. *Assume that there exists a space-time periodic $K \times K$ matrix $A(t,x)$ satisfying* **(H4)** *with $A_0(t,x)$ being replaced by $A(t,x)$ such that $\mathbf{F}(t,x,\mathbf{u}) \ge A(t,x)\mathbf{u}$ for $\mathbf{0} \le \mathbf{u} \le \boldsymbol{\beta}$ with $\boldsymbol{\beta} \in \mathbb{R}^K$ and $0 < \beta_i \ll 1$, $i = 1,\dots,K$. Let $A_1(t,x)$ be the first diagonal block of $A(t,x)$. For any $\xi \in \mathbb{S}^{N-1}$, let $\lambda_1(\xi,\mu,A_1)$ be the principal eigenvalue of L_{ξ,μ,A_1} for any $\mu > 0$. Then*

$$c^*_{\inf}(\xi) \ge \inf_{\mu>0} \frac{\lambda_1(\xi,\mu,A_1)}{\mu}. \tag{3.6}$$

Lemma 3.2 can be proven by the strategy which has been used in several papers (see [27, Proposition 3.9], [38, Lemma 4.4], [44, Lemma 9.1]). We provide the proof of Lemma 3.2 in Appendix A for interested readers.

Next, we present a lemma, which will be used in the proof of Theorem of 3.2 to get an upper bound for $c^*_{\sup}(\xi)$. To this end, for given $M > 0$, let

$$\widetilde{\mathbf{u}}(t,x;M) := Me^{-\mu^*\left(x\cdot\xi - \frac{\lambda_1(\mu^*)}{\mu^*}t\right)}\boldsymbol{\phi}^*(t,x),$$

where μ^* is as in (3.1) and $\lambda_1(\mu^*)$ and $\phi^*(t,x)$ are as in (3.2). Let

$$\begin{aligned}\tilde{u}^+_i(t,x;M) &= \min\{\tilde{u}_i(t,x;M), u^*_i(t,x)\},\\ \tilde{\mathbf{u}}^+(t,x;M) &= (\tilde{u}^+_1(t,x;M),\dots,\tilde{u}^+_K(t,x;M))^\top.\end{aligned}$$

Lemma 3.3. *For any* $\mathbf{u}_0 \in X^+(\xi)$ *with* $\mathbf{u}_0 \le \tilde{\mathbf{u}}^+(0,\cdot;M)$*, there holds*

$$\mathbf{u}(t,x;\mathbf{u}_0) \le \tilde{\mathbf{u}}^+(t,x;M), \quad t \ge 0, \; x \in \mathbb{R}^N. \tag{3.7}$$

Proof. Define

$$\overline{u}_i(t,x;M,\overline{M}) := e^{\overline{M}t}\widetilde{u}_i(t,x;M), \; \overline{u}_i^*(t,x;\overline{M}) := e^{\overline{M}t}u_i^*(t,x)$$

and

$$\overline{u}_i^+(t,x;M,\overline{M}) = e^{\overline{M}t}\widetilde{u}_i^+(t,x;M), \; \overline{u}_i(t,x;\mathbf{u}_0,\overline{M}) = e^{\overline{M}t}u_i(t,x;\mathbf{u}_0)$$

for $i = 1,\dots,K$, where $\overline{M}$ is some positive constant to be determined later. It suffices to prove that, for any $\mathbf{u}_0 \in X^+(\xi)$, if $\mathbf{u}_0(x) \le \overline{\mathbf{u}}^+(0,x;M,\overline{M})$ for $x \in \mathbb{R}^N$, then

$$\overline{\mathbf{u}}(t,x;\mathbf{u}_0,\overline{M}) \le \overline{\mathbf{u}}^+(t,x;M,\overline{M}), \quad 0 \le t \le T, \; x \in \mathbb{R}^N. \tag{3.8}$$

In the rest of the proof, if no confusion occurs, for each $i = 1,\dots,K$, we write $\overline{u}_i(t,x;M,\overline{M})$, $\overline{u}_i^*(t,x;\overline{M})$ and $\overline{u}_i^+(t,x;M,\overline{M})$ as $\overline{u}_i(t,x)$, $\overline{u}_i^*(t,x)$ and $\overline{u}_i^+(t,x)$, respectively. Let

$$\overline{F}^i(t,x,\mathbf{u}) = (\overline{M}-1)u_i + e^{\overline{M}t}F^i(t,x,e^{-\overline{M}t}\mathbf{u}), \quad i = 1,\dots,K.$$

Note that, for any $1 \le i \ne j \le K$,

$$\frac{\partial \overline{F}^i}{\partial u_j}(t,x,\mathbf{u}) \ge 0 \quad \text{for } \mathbf{u} \in [\mathbf{0},\overline{\mathbf{u}}^*(t,x)] \text{ and } (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.9}$$

Choose $\overline{M} > 0$ larger enough such that

$$\frac{\partial \overline{F}^i}{\partial u_i}(t,x,\mathbf{u}) := -1 + \overline{M} + \frac{\partial F^i}{\partial u_i}(t,x,e^{-\overline{M}t}\mathbf{u}) \ge 0, \quad i = 1,\dots,K \tag{3.10}$$

for $\mathbf{u} \in [\mathbf{0},\overline{\mathbf{u}}^*(t,x)]$ and $(t,x) \in \mathbb{R}^+\times\mathbb{R}^N$.

For given $t \in [0,T]$, $x \in \mathbb{R}^N$, and $i \in \{1,2,\cdots,K\}$, if $\overline{u}_i^+(t,x) = \overline{u}_i^*(t,x)$, then we have

$$\begin{aligned}\overline{u}_i^+(t,x) = \overline{u}_i^*(t,x) &= \overline{u}_i^*(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i^*(\tau,y)dy + \overline{F}^i(\tau,x,\overline{\mathbf{u}}^*(\tau,x))\right]d\tau \\ &\ge \overline{u}_i^+(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i^+(\tau,y)dy + \overline{F}^i(\tau,x,\overline{\mathbf{u}}^+(\tau,x))\right]d\tau. \end{aligned} \tag{3.11}$$

Similarly, for given $t \in [0,T]$, $x \in \mathbb{R}^N$, and $i \in \{1,2,\cdots,K\}$, if $\overline{u}_i^+(t,x) = \overline{u}_i(t,x)$, then

$$\begin{aligned}\overline{u}_i^+(t,x) &= \overline{u}_i(0,x) + \int_0^t \Bigg[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i(\tau,y)dy + (\overline{M}-1)\overline{u}_i(\tau,x) \\ &\qquad + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(\tau,x,\mathbf{0})\overline{u}_j(\tau,x)\Bigg]d\tau \\ &\ge \overline{u}_i(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i(\tau,y)dy + \overline{F}^i(\tau,x,\overline{\mathbf{u}}(\tau,x))\right]d\tau \quad (by\,(H5)) \\ &\ge \overline{u}_i^+(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i^+(\tau,y)dy + \overline{F}^i(\tau,x,\overline{\mathbf{u}}^+(\tau,x))\right]d\tau. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), for any $(t,x) \in [0,T]\times\mathbb{R}^N$ and $i \in \{1,2,\cdots,K\}$,

$$\overline{u}_i^+(t,x) \ge \overline{u}_i^+(0,x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y-x)\overline{u}_i^+(\tau,y)dy + \overline{F}^i(\tau,x,\overline{\mathbf{u}}^+(\tau,x))\right]d\tau. \tag{3.13}$$

Therefore, $\overline{\mathbf{u}}^+(t,x;M,\overline{M})$ is a super-solution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x,\mathbf{u})+\mathbf{u}=(\bar{F}^1(t,x,\mathbf{u})+u_1,\cdots,\bar{F}^K(t,x,\mathbf{u})+u_k)^\top$ on $[0,T]$.

Let $\overline{u}_i(t,x)=e^{\overline{M}t}u_i(t,x;\mathbf{u}_0)$ for $i=1,\dots,K$. Then, for all $t\geq 0$ and $x\in\mathbb{R}^N$,

$$\overline{u}_i(t,x)=\overline{u}_{i0}(x)+\int_0^t\left[\int_{\mathbb{R}^N}k(y-x)\overline{u}_i(\tau,y)dy+\overline{F}^i(\tau,x,\overline{\mathbf{u}}(\tau,x;\mathbf{u_0}))\right]d\tau. \quad (3.14)$$

Hence $\overline{\mathbf{u}}(t,x;\mathbf{u_0})=e^{\overline{M}t}\mathbf{u}(t,x;\mathbf{u_0})$ is a subsolution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x,\mathbf{u})+\mathbf{u}=(\bar{F}^1(t,x,\mathbf{u})+u_1,\cdots,\bar{F}^K(t,x,\mathbf{u})+u_k)^\top$ on $[0,T]$. (3.8) then follows from Proposition 2.1.

By (3.8), we have

$$\mathbf{u}(t,x;\mathbf{u_0})\leq\tilde{\mathbf{u}}^+(t,x;M)\quad\forall\, x\in\mathbb{R}^N,\ t\in[0,T].$$

Repeating the above arguments, we have

$$\mathbf{u}(t,x;\mathbf{u_0})\leq\tilde{\mathbf{u}}^+(t,x;M)\quad\forall\, x\in\mathbb{R}^N,\ t\in[T,2T].$$

Then by induction, (3.7) holds. □

Now, we prove Theorem 3.2.

Proof of Theorem 3.2. For any $\mathbf{u}_0\in X^+(\xi)$, let $M_0>0$ be such that $\mathbf{u}_0\leq\tilde{\mathbf{u}}^+(0,\cdot;M_0)$. By Lemma 3.3,

$$\mathbf{u}(t,x;\mathbf{u}_0)\leq\tilde{\mathbf{u}}^+(t,x;M_0),\quad t\geq 0,\ x\in\mathbb{R}^N.$$

This implies $\limsup_{x\cdot\xi\geq ct,t\to\infty}|\mathbf{u}(t,x;\mathbf{u}_0)|=0$ for any $c>\frac{\lambda_1(\mu^*)}{\mu^*}$. Hence,

$$c^*_{\sup}(\xi)\leq\frac{\lambda_1(\mu^*)}{\mu^*}=\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,A_{01})}{\mu}. \quad (3.15)$$

On the other hand, for any given $\tau_2>\tau_1>0$, there exists $\boldsymbol{\beta}\in\mathbb{R}^K$ with $0<\beta_i\ll 1$ for each $i=1,...,K$ such that

$$\mathbf{F}(t,x,\mathbf{u})\geq(1-\tau_1)A_0(t,x)\mathbf{u}-\tau_2\mathbf{u},\quad\forall\mathbf{u}\in[\mathbf{0},\boldsymbol{\beta}].$$

Note that for any small enough $\tau_1>0$ and $\tau_2>0$, we also have matrix $(1-\tau_1)A_0-\tau_2\mathbf{I}$ satisfies (H4) and matrix $(1-\tau_1)A_0-\tau_2\mathbf{I}$ converges to the matrix A_0 as $\tau_1,\tau_2\to 0$.

By Lemma 3.2, we must have

$$c^*_{\inf}(\xi)\geq\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,(1-\tau_1)A_{01}-\tau_2\mathbf{I})}{\mu}.$$

Letting $\tau_1,\tau_2\to 0$, by Proposition 2.4, we have $c^*_{\inf}(\xi)\geq\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$. This together with (3.15) yields

$$c^*_{\inf}(\xi)=c^*_{\sup}(\xi)=\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,A_{01})}{\mu}.$$

Hence, $c^*(\xi)=\inf_{\mu>0}\frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$ is the spreading speed of (1.1). This completes the proof. □

4. **Traveling wave solutions.** In this section, we investigate the space-time periodic traveling wave solutions of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$. We first introduce the concept of space-time periodic traveling wave solutions of (1.1).

Definition 4.1 (Traveling wave solution)**.** *Let $\xi \in \mathbb{S}^{N-1}$.*

(1) An entire solution $\mathbf{u}(t,x)$ of (1.1) is called a traveling wave solution of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$ propagating in the direction of ξ with speed c *if there is bounded measurable function $\mathbf{\Phi} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to (\mathbb{R}^K)^+$ such that $\mathbf{u}(t,x;\mathbf{\Phi}(\cdot,0,z),z)$ exists for all $t \in \mathbb{R}$ and it satisfies, for any $i = 1,\dots,K$,*

$$u_i(t,x) = u_i(t,x;\mathbf{\Phi}(\cdot,0,0),0) = \Phi_i(x-ct\xi,t,ct\xi), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N,$$

$$u_i(t,x;\mathbf{\Phi}(\cdot,0,z),z) = \Phi_i(x-ct\xi,t,z+ct\xi), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{4.1}$$

$$\Phi_i(x,t,z-x) = \Phi_i(x',t,z-x'), \quad \forall x,x' \in \mathbb{R}^N \text{ with } x\cdot\xi = x'\cdot\xi, \tag{4.2}$$

$$\Phi_i(x,t+T,z) = \Phi_i(x,t,z+p_l\mathbf{e}_l) = \Phi_i(x,t,z), \quad \forall x,z \in \mathbb{R}^N \tag{4.3}$$

and

$$\lim_{x\cdot\xi\to-\infty} \mathbf{\Phi}(x,t,z) = \mathbf{u}^*(t,x+z), \qquad \lim_{x\cdot\xi\to\infty} \mathbf{\Phi}(x,t,z) = \mathbf{0} \tag{4.4}$$

uniformly in $(t,z) \in \mathbb{R}\times\mathbb{R}^N$.

(2) A bounded measurable function $\mathbf{\Phi} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to (\mathbb{R}^K)^+$ is said to generate a traveling wave solution of (1.1) in the direction of ξ with speed c if it satisfies (4.1)-(4.4).

The proof of the existence of traveling wave solutions of (1.1) is based on the idea of constructing appropriate sub- and super-solutions of (1.1). For convenience, we set for $\mathbf{u} = (u_1,\dots,u_K)^\top$ and $\mathbf{v} = (v_1,\dots,v_K)^\top$,

$$\mathbf{u} \vee \mathbf{v} = (\max\{u_1,v_1\},...,\max\{u_K,v_K\})^\top.$$

Under the assumptions **(H1)-(H5)**, Theorem 3.2 says that $c^*(\xi) = \inf_{\mu>0} \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$ is the spreading speed of (1.1) in the direction of ξ. Moreover, by Proposition 2.3, there is $\mu^*(\xi) := \mu^*(\xi,A_{01}) \in (0,\infty)$ such that $\frac{\lambda_1(\xi,\mu^*(\xi),A_{01})}{\mu^*(\xi)} = \inf_{\mu>0} \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$. For any given $\xi \in \mathbb{S}^{N-1}$ and $c > c^*(\xi)$, let $\mu \in (0,\mu^*(\xi))$ be such that

$$c = \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}.$$

Let $\boldsymbol{\phi}(\cdot,\cdot;\mu,\xi) \in \mathcal{X}_p^{++}$ be the eigenfunction of L_{ξ,μ,A_0} corresponding to the principal eigenvalue $\lambda(\xi,\mu,A_0) = \lambda_1(\xi,\mu,A_{01})$ with $\|\boldsymbol{\phi}(\cdot,\cdot;\mu,\xi)\| = 1$.

The main results of this section are stated in the following theorem.

Theorem 4.1. *Assume* **(H1)**-**(H5)**. *Let $\xi \in \mathbb{S}^{N-1}$.*

(1) (Existence) For any $c > c^*(\xi)$, let $0 < \mu < \mu^*(\xi)$ be such that $c = \frac{\lambda_1(\xi,\mu,A_{01})}{\mu}$. Then, there is a bounded measurable function $\mathbf{\Phi} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^K \to (\mathbb{R}^K)^+$ satisfying the following properties:

(a) it generates a traveling wave solution $\mathbf{u}(t,x) := \mathbf{\Phi}(x-ct\xi,t,ct\xi)$ of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$ and propagating in the direction of ξ with speed c;

(b) there holds

$$\lim_{x\cdot\xi\to-\infty} \frac{\Phi_i(x,t,z)}{e^{-\mu x\cdot\xi}\phi_i(t,x+z;\mu,\xi)} = 1 \quad \text{uniformly in } z \in \mathbb{R}^N,\ t \in \mathbb{R} \text{ and } i = 1,\dots,K.$$

(2) *(Nonexistence) For $c < c^*(\xi)$, there is no such solution of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$ and propagating in the direction of ξ with speed c.*
(3) *(Continuity and uniqueness) If $\mathbf{F}(t,x,\mathbf{u})$ is strictly sub-homogeneous, that is, $\mathbf{F}(t,x,\alpha\mathbf{u}) > \alpha\mathbf{F}(t,x,\mathbf{u})$ for $\mathbf{u} \in (\mathbf{0},\mathbf{u}^*]$, $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $\alpha \in (0,1)$. Then, $\mathbf{\Phi}$ is unique and continuous.*

Remark 4.1. We refer to [47, Definition 2.3.1] for more information about strict sub-homogeneity. When $\mathbf{F}(t,x,\mathbf{u}) = uf(t,x,u)$, if $\frac{\partial f}{\partial u}(t,x,u) < 0$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $f(t,x,u) < 0$ for $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $u \gg 1$, we know from [38] that the strictly sub-homogeneous condition holds for the following nonlocal monostable equation

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + uf(t,x,u)$$

and our results can be applied to the above nonlocal dispersal equation (see Subsection 5.1).

4.1. **Sub- and super-solutions.** In this subsection, we construct sub- and super-solutions of some equations related to (3.3) that are used in the proof of Theorem 4.1. Throughout the rest of this section, we assume **(H1)-(H5)**. Recall $A_0(t,x) = \mathbf{F}_{\mathbf{u}}(t,x,\mathbf{0})$.

First, we construct sub-solutions of some equations related to (3.3). Note that there are positive constants ϖ and γ such that for any $i = 1,\dots,K$,

$$F^i(t,x,\mathbf{u}) \geq \sum_{j=1}^{K} \frac{\partial F^i}{\partial u_j}(t,x,\mathbf{0})u_j - \varpi\left(\sum_{j=1}^{K}|u_j|\right)^{1+\gamma} \tag{4.5}$$

for $\mathbf{0} \leq \mathbf{u} \leq \mathbf{u}^*(t,x)$ and $(t,x) \in \mathbb{R}\times\mathbb{R}^N$.

Fix $\xi \in \mathbb{S}^{N-1}$ and $c > c^*(\xi)$. Let $0 < \mu < \mu_1 < \min\{2\mu, \mu^*(\xi)\}$ be such that

$$c = \frac{\lambda_1(\xi,\mu,A_{01})}{\mu} \quad \text{and} \quad \frac{\lambda_1(\xi,\mu,A_{01})}{\mu} > \frac{\lambda_1(\xi,\mu_1,A_{01})}{\mu_1} > c^*(\xi).$$

Set

$$\boldsymbol{\phi}_0(\cdot,\cdot) = \boldsymbol{\phi}(\cdot,\cdot;0,\xi), \quad \boldsymbol{\phi}(\cdot,\cdot) = \boldsymbol{\phi}(\cdot,\cdot;\mu,\xi) \quad \text{and} \quad \boldsymbol{\psi}(\cdot,\cdot) = \boldsymbol{\phi}(\cdot,\cdot;\mu_1,\xi).$$

When no confusion occurs, we write $\lambda(\xi,\mu,A_0) = \lambda_1(\xi,\mu,A_{01})$ as $\lambda_1(\mu)$. Set $\epsilon = \mu_1 - \mu$ and $\Lambda = c\mu_1 - \lambda_1(\mu_1)$. Then, $\Lambda > 0$ and $\mu\lambda - \epsilon > 0$. For given $d > 0$, let

$$\underline{\mathbf{u}}(t,x;z,d) = e^{-\mu(x\cdot\xi - ct)}\left(\boldsymbol{\phi}(t,x+z) - de^{-\epsilon(x\cdot\xi-ct)}\boldsymbol{\psi}(t,x+z)\right).$$

Note that $\boldsymbol{\phi}(t,x)$ satisfies

$$L_{\xi,\mu,A_0}[\boldsymbol{\phi}] = -\partial_t\boldsymbol{\phi} + \mathcal{K}_{\xi,\mu}\boldsymbol{\phi} - \boldsymbol{\phi} + A_0(t,x)\boldsymbol{\phi} = \lambda_1(\mu)\boldsymbol{\phi}. \tag{4.6}$$

Let

$$d^* = \max_{1\leq i\leq K}\left[\max_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{\varpi\left(\sum_{j=1}^{K}|\phi_j(t,x)|\right)^{1+\gamma}}{\psi_i(t,x)\Lambda}\right]$$

and

$$\zeta^* = \max\left\{0, \max_{1\leq i\leq K}\left[\max_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{1}{\mu}\ln\frac{\phi_i(t,x)}{u_i^*(t,x)}\right]\right\}.$$

Proposition 4.1. *For any $d > d^*$, $\underline{\mathbf{u}}(t,x;z,d) = (\underline{u}_1,\ldots,\underline{u}_K)^\top$ satisfies*

$$\frac{\partial \underline{u}_i}{\partial t} \le \mathcal{K}\underline{u}_i - \underline{u}_i + F^i(t,x+z,\underline{\mathbf{u}}), \quad \forall i = 1,\ldots,K \tag{4.7}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ satisfying $x\cdot\xi - ct > \zeta^*$ and $\min_{1\le i\le K} \underline{u}_i(t,x;z,d) > 0$.

Proof. Let $\zeta = x\cdot\xi - ct$ and

$$\mathcal{N}_i[\underline{\mathbf{u}}] := \frac{\partial \underline{u}_i}{\partial t} - \Big[\int_{\mathbb{R}^N} k(y-x)\underline{u}_i(t,y)dy - \underline{u}_i(t,x)\Big] - F^i(t,x+z,\underline{\mathbf{u}}), \quad i = 1,\ldots,K.$$

Then,

$$\underline{u}_i(t,x;z,d) = e^{-\mu\zeta}\left(\phi_i(t,x+z) - de^{-\epsilon\zeta}\psi_i(t,x+z)\right), \quad \forall i = 1,...,K.$$

Fix $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ satisfying $x\cdot\xi - ct > \zeta^*$ and $\min_{1\le i\le K}\underline{u}_i(t,x;z,d) > 0$. For any $i = 1,\ldots,K$, it is easy to see that $\underline{u}_i(t,x;z,d) \le u_i^*(t,x+z)$ and $\phi_i(t,x+z) - de^{-\epsilon\zeta}\psi_i(t,x+z) > 0$. Hence for any $i = 1,\ldots,K$,

$$\begin{aligned}
\mathcal{N}_i[\underline{\mathbf{u}}] =& c\mu e^{-\mu\zeta}\phi_i(t,x+z) + e^{-\mu\zeta}\frac{\partial\phi_i}{\partial t} - dc\mu_1 e^{-\mu_1\zeta}\psi_i(t,x+z) - de^{-\mu_1\zeta}\frac{\partial\psi_i}{\partial t}\\
&- \int_{\mathbb{R}^N} k(y-x)\Big[e^{-\mu(y\cdot\zeta-ct)}\phi_i(t,y+z) - de^{-\mu_1(y\cdot\zeta-ct)}\psi_i(t,y+z)\Big]dy\\
&+ e^{-\mu\zeta}\Big(\phi_i(t,x+z) - de^{-\epsilon\zeta}\psi_i(t,x+z)\Big) - F^i(t,x+z,\underline{\mathbf{u}})\\
=& e^{-\mu\zeta}\Big[c\mu\phi_i + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t,x+z,\mathbf{0})\phi_j - \lambda_1(\mu)\phi_i\Big]\\
&- de^{-\mu_1\zeta}\left[c\mu_1\psi_i + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t,x+z,\mathbf{0})\psi_j - \lambda_1(\mu_1)\psi_i\right] - F^i(t,x+z,\underline{\mathbf{u}})\\
\le& \varpi\left(\sum_{j=1}^K |\phi_j - e^{-\epsilon\zeta}\psi_j|\right)^{1+\gamma} - de^{-\mu_1\zeta}\psi_i\left[c\mu_1 - \lambda_1(\mu_1)\right] \ \text{(by (4.5))}\\
\le& - de^{-\mu_1\zeta}\psi_i\Lambda + \varpi e^{-\mu(1+\gamma)\zeta}\left(\sum_{j=1}^K|\phi_j|\right)^{1+\gamma} \qquad \text{(because } \phi_j - e^{-\epsilon\zeta}\psi_j > 0)\\
\le& e^{-\mu_1\zeta}\left\{-d\psi_i\Lambda + \varpi\left(\sum_{j=1}^K|\phi_j|\right)^{1+\gamma}\right\}\\
\le& 0 \quad (\text{because } d > d^*).
\end{aligned}$$

Thus, (4.7) holds and the proposition follows. □

Proposition 4.2. *Let $\lambda_1(A_{01}) := \lambda_1(\xi,0,A_{01})$ and $\boldsymbol{\phi}_0$ be the positive eigenfunction of $L_{\xi,0,A_0}$ corresponding to $\lambda_1(A_{01})$ with $\|\boldsymbol{\phi}_0\| = 1$. Then, for any $z\in\mathbb{R}^N$, $\underline{\mathbf{u}}'(t,x;z,\rho_1) := \rho_1\boldsymbol{\phi}_0(t,x+z)$ is a sub-solution of (3.3), where ρ_1 satisfies*

$$0 < \rho_1 \ll \min_{1\le i\le K}\left\{1, \min_{\mathbb{R}\times\mathbb{R}^N}\frac{u_i^*}{\phi_{i0}}, \left(\min_{\mathbb{R}\times\mathbb{R}^N}\frac{\lambda_1(A_{01})\phi_{i0}}{\varpi C_\gamma\sum_{j=1}^K(\phi_{j0})^{1+\gamma}}\right)^{\frac{1}{\gamma}}\right\}.$$

Proof. By (4.6) and (4.5), we have

$$\begin{aligned}\mathcal{N}_i[\underline{\mathbf{u}}'] =&\frac{\partial \underline{u}_i'}{\partial t} - \left[\int_{\mathbb{R}^N} k(y-x)\underline{u}_i'(t,y)dy - \underline{u}_i'(t,x)\right] - F^i(t,x+z,\underline{\mathbf{u}}')\\ =&-\lambda_1(A_{01})\rho_1\phi_{i0} + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t,x+z,\mathbf{0})\rho_1\phi_{j0} - F^i(t,x+z,\rho_1\boldsymbol{\phi}_0)\\ \le& -\lambda_1(A_{01})\rho_1\phi_{i0} + \varpi\left(\sum_{j=1}^K |\rho_1\phi_{j0}|\right)^{1+\gamma}\\ =&\rho_1\left[-\lambda_1(A_{01})\phi_{i0} + \varpi\rho_1^\gamma C_\gamma\sum_{j=1}^K(\phi_{j0})^{1+\gamma}\right] \le 0\end{aligned}$$

for any $i = 1, ..., K$, which implies $\underline{\mathbf{u}}'(t,x;z,\rho_1)$ is a sub-solution of (3.3) for any $z \in \mathbb{R}^N$. This completes the proof. □

Observe that for ζ^* in Proposition 4.1, $\underline{u}(t,x;z,d) \le 0$ for $x\cdot\xi - ct \le \zeta^*$ and sufficiently large d. Observe also that for ρ_1 as in Proposition 4.2, there holds $\rho_1\boldsymbol{\phi}_0(t,x+z) \le \mathbf{u}^*(t,x+z)$ for $x, z \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Let r_0 be such that $\text{supp}(k) \subset \left\{z \in \mathbb{R}^N : |z| \le r_0\right\}$.

Fix $d \gg d^*$ such that $\underline{u}(t,x;z,d) \le 0$ for $x\cdot\xi - ct \le \zeta^*$. Then, for small ρ_1 there exists $M > 2r_0 + \zeta^*$ such that

$$\underline{\mathbf{u}}(t,x;z,d) \ge \begin{cases}\rho_1\boldsymbol{\phi}_0(t,x+z), & M - 2r_0 \le x\cdot\xi - ct \le M\\ 0, & x\cdot\xi - ct \ge M.\end{cases}$$

For such ρ_1, we define

$$\mathbf{u}^-(t,x;z,d,\rho_1) := \begin{cases}\rho_1\boldsymbol{\phi}_0(t,x+z) \vee \underline{\mathbf{u}}(t,x;z,d), & x\cdot\xi - ct < M,\\ \underline{\mathbf{u}}(t,x;z,d), & x\cdot\xi - ct \ge M.\end{cases}$$

For given $M_1 > 0$, let

$$\overline{F}^i(t,x,\mathbf{u}) = (M_1 - 1)u_i + e^{M_1 t}F^i(t,x,e^{-M_1 t}\mathbf{u}), \quad i = 1,\dots,K. \tag{4.8}$$

Fix $M_1 > 0$ larger enough such that

$$\frac{\partial \overline{F}^i}{\partial u_i}(t,x,e^{-M_1 t}\mathbf{u}) = -1 + M_1 + \frac{\partial F^i}{\partial u_i}(t,x,\mathbf{u}) \ge 0 \tag{4.9}$$

for $t \ge 0$, $x \in \mathbb{R}^N$, $\mathbf{u} \in [\mathbf{0}, \mathbf{u}^*(t,x)]$, and $i = 1, 2, \cdots, K$.

Recall that $\mathbf{u}(t,x;\mathbf{u}_{0,z}^-,z)$ is the solution of (3.3) with the initial value $\mathbf{u}_{0,z}^- = \mathbf{u}^-(0,\cdot;z,d,\rho_1)$. Let

$$u_i(t,x;M_1) = e^{M_1 t}u_i(t,x;\mathbf{u}_{0,z}^-,z) \quad \text{and} \quad u_i^-(t,x;z,d,\rho_1,M_1) = e^{M_1 t}u_i^-(t,x;z,d,\rho_1)$$

for $i = 1,\dots,K$.

Proposition 4.3. *Let d and ρ_1 be chosen as in the above. Then, $\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$ and $\mathbf{u}(t,x;M_1)$ are sub-solution and super-solution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x+z,\mathbf{u})+\mathbf{u} = (\bar{F}^1(t,x+z,\mathbf{u})+u_1,\cdots,\bar{F}^K(t,x+z,\mathbf{u})+u_k)^\top$ on $[0,\infty)$, respectively, and hence there holds*

$$\mathbf{u}(t,x;\mathbf{u}_{0,z}^-,z) \ge \mathbf{u}^-(t,x;z,d,\rho_1), \quad t \ge 0.$$

Proof. First, let $\underline{\mathbf{u}}(t,x;z,d,M_1)=e^{M_1 t}\underline{\mathbf{u}}(t,x;z,d)$ and $\underline{\mathbf{u}}'(t,x;z,\rho_1,M_1)=e^{M_1 t}\rho_1\boldsymbol{\phi}_0(t,x+z)$. Fix an $x\in\mathbb{R}^N$. Note that $\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$ is absolutely continuous in t on any bounded interval of $[0,\infty)$. Hence, $\frac{\partial}{\partial t}\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$ exists for a.e. $t\ge 0$ and

$$\mathbf{u}^-(t,x;z,d,\rho_1,M_1)=\mathbf{u}^-(0,x;z,d,\rho_1,M_1)+\int_0^t\frac{\partial}{\partial t}\mathbf{u}^-(s,x;z,d,\rho_1,M_1)ds,\quad \forall\ t\ge 0.$$

Note also that

$$\frac{\partial}{\partial t}\mathbf{u}^-(t,x;z,d,\rho_1,M_1)=\frac{\partial}{\partial t}\underline{u}(t,x;z,d,M_1),\quad \forall\ t<\frac{x\cdot\xi-M}{c},\tag{4.10}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\mathbf{u}^-(t,x;z,d,\rho_1,M_1)\\&=\begin{cases}\frac{\partial}{\partial t}\underline{u}(t,x;z,d,M_1)\\ a.e.\ t\in\left(\frac{x\cdot\xi-M}{c},\infty\right)\cap\{t:\underline{u}(t,x;z,d,M_1)\ge\underline{u}'(t,x;z,\rho_1,M_1)\}\\ \frac{\partial}{\partial t}\underline{u}'(t,x;z,\rho_1,M_1)\\ a.e.\ t\in\left(\frac{x\cdot\xi-M}{c},\infty\right)\cap\{t:\underline{u}(t,x;z,d,M_1)\le\underline{u}'(t,x;z,\rho_1,M_1)\}.\end{cases}\end{aligned}\tag{4.11}$$

Next, we fix $x\in\mathbb{R}^N$. For any $t\ge 0$ and $i\in\{1,2,\cdots,K\}$, if $u_i^-(t,x;z,d,\rho_1,M_1)=\underline{u}_i(t,x;z,d,M_1)$, then by Proposition 4.1 and the definition of $\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$, we have that $x\cdot\xi-ct>\zeta^*$ and

$$\begin{aligned}&\frac{\partial}{\partial t}\underline{u}_i(t,y;z,d,\rho_1,M_1)\\&\le\int_{\mathbb{R}^N}k(y-x)\underline{u}_i(t,x;z,d,M_1)dy+\overline{F}^i(t,x+z,\underline{\mathbf{u}}(t,x;z,d,M_1))\\&\le\int_{\mathbb{R}^N}k(y-x)u_i^-(t,y;z,d,\rho_1,M_1)dy+\overline{F}^i(t,x+z,\mathbf{u}^-(t,x;z,d,\rho_1,M_1)).\end{aligned}\tag{4.12}$$

If $u_i^-(t,x;z,d,\rho_1,M_1)=\underline{u}_i'(t,x;z,d,M_1)$, then by Proposition 4.2 and the definition of $\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$, we have that $x\cdot\xi-ct\le M-2r_0$ and

$$\begin{aligned}&\frac{\partial}{\partial t}\underline{u}_i'(t,x;z,\rho_1,M_1)\\&\le\int_{\mathbb{R}^N}k(y-x)\underline{u}_i'(t,y;z,\rho_1,M_1)dy+\overline{F}^i(t,x+z,\underline{\mathbf{u}}'(t,x;z,\rho_1,M_1))\\&\le\int_{\mathbb{R}^N}k(y-x)u_i^-(t,y;z,d,\rho_1,M_1)dy+\overline{F}^i(t,x+z,\mathbf{u}^-(t,x;z,d,\rho_1,M_1)).\end{aligned}\tag{4.13}$$

By (4.10)-(4.13), we find

$$\begin{aligned}&\frac{\partial}{\partial t}\underline{u}_i^-(t,x;z,\rho_1,M_1)\\&\le\int_{\mathbb{R}^N}k(y-x)u_i^-(t,y;z,d,\rho_1,M_1)dy+\overline{F}^i(t,x+z,\mathbf{u}^-(t,x;z,d,\rho_1,M_1))\end{aligned}\tag{4.14}$$

for a.e. $t\ge 0$.

Therefore, for any $t\ge 0$, $x\in\mathbb{R}^N$, and $i\in\{1,2,\cdots,K\}$,

$$u_i^-(t,x;z,d,\rho_1,M_1)\le u_i^-(0,x;z,d,\rho_1,M_1)$$

$$+\int_0^t \Big[\int_{\mathbb{R}^N} k(y-x)u_i^-(s,y;z,d,\rho_1,M_1)dy + \overline{F}^i(s,x+z,\mathbf{u}^-(s,x;z,d,\rho_1,M_1))\Big]ds. \tag{4.15}$$

Hence, $\mathbf{u}^-(t,x;z,d,\rho_1,M_1)$ is a sub-solution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x+z,\mathbf{u})+\mathbf{u} = (\bar{F}^1(t,x+z,\mathbf{u})+u_1,\cdots,F^K(t,x+z,\mathbf{u})+u_k)^\top$ on $[0,\infty)$.

Note that for any $t \ge 0$, $x \in \mathbb{R}^N$, and $i \in \{1,2,\cdots,K\}$,

$$\begin{aligned} u_i(t,x;M_1) &= u_i(0,x;M_1) \\ &\quad + \int_0^t \Big[\int_{\mathbb{R}^N} k(y-x)u_i(s,x;M_1)dy + \overline{F}^i(s,x+z,\mathbf{u}(s,x;M_1))\Big]ds. \end{aligned} \tag{4.16}$$

Hence, $\mathbf{u}(t,x;M_1)$ is a solution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x+z,\mathbf{u})+\mathbf{u} = (\bar{F}^1(t,x+z,\mathbf{u})+u_1,\cdots,F^K(t,x+z,\mathbf{u})+u_k)^\top$ on $[0,\infty)$. The proposition then follows from Proposition 2.1. □

Next, we construct super-solutions of some equations related to (3.3). Let

$$\mathbf{u}^+(t,x;z) = (u_1^+(t,x;z),\dots,u_K^+(t,x;z))^\top$$

be defined by

$$u_i^+(t,x;z) = \min\{e^{-\mu(x\cdot\xi - ct)}\phi_i(t,x+z), u_i^*(t,x+z)\}, \quad i=1,...,K.$$

Fix M_1 such that (4.9) holds. Let

$$\mathbf{u}^+(t,x;z,M_1) = e^{M_1 t}\mathbf{u}^+(t,x;z), \quad \mathbf{u}(t,x;\mathbf{u}_{0,z}^+(x),z,M_1) = e^{M_1 t}\mathbf{u}(t,x;\mathbf{u}_{0,z}^+(x),z),$$

where $\mathbf{u}_{0,z}^+ = \mathbf{u}^+(0,\cdot;z)$.

Proposition 4.4. *The following statements hold.*

(1) For any given $z \in \mathbb{R}^N$, $\mathbf{u}^+(t,x;z,M_1)$ and $\mathbf{u}(t,x;\mathbf{u}_{0,z}^+(x),z,M_1)$ are super-solution and sub-solution of (1.1) with $\mathbf{F}(t,x,\mathbf{u})$ being replaced by $\overline{\mathbf{F}}(t,x+z,\mathbf{u})+\mathbf{u} = (\bar{F}^1(t,x+z,\mathbf{u})+u_1,\cdots,\bar{F}^K(t,x+z,\mathbf{u})+u_k)^\top$ on $[0,\infty)$, respectively, and hence

$$\mathbf{u}(t,x;\mathbf{u}_{0,z}^+(x),z) \le \mathbf{u}^+(t,x;z), \quad t \ge 0.$$

(2) For any $i=1,\dots,K$, there is a constant C such that

$$\inf_{x\cdot\xi\le C, t\ge 0, z\in\mathbb{R}^N} u_i(t,x+ct\xi;\mathbf{u}_{0,z}^+,z) \ge \inf_{x\cdot\xi\le C, t\ge 0, z\in\mathbb{R}^N} u_i(t,x+ct\xi;\mathbf{u}_{0,z}^-,z) > 0. \tag{4.17}$$

Proof. (1) It follows from similar arguments as in Lemma 3.3.

(2) By Proposition 4.3 and (1), for any $t \ge 0$ and $i=1,\dots,K$, we have

$$u_i^-(t,x;z,d,\rho_1) \le u_i(t,x;\mathbf{u}_{0,z}^-,z) \le u_i(t,x;\mathbf{u}_{0,z}^+,z) \le u_i^+(t,x;z). \tag{4.18}$$

Observe that, for $x\cdot\xi \le M$ and $i=1,\dots,K$,

$$u_i^-(t,x+ct\xi;z,d,\rho_1) \ge \rho_1\phi_{i0}(t,x+ct\xi+z) \ge \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \rho_1\phi_{i0}(t,x) > 0.$$

This together with (4.18) implies (4.17). □

4.2. **Proof of Theorem 4.1.** In this subsection, we investigate the existence and uniqueness of traveling wave solutions of (1.1), that is, we prove Theorem 4.1. For convenience, we set $\mathbf{u}^-(t,x;z) = \mathbf{u}^-(t,x;z,d,\rho_1)$. Let $\mathbf{u}^{\pm}_{0,z}$ be given in Proposition 4.3 and Proposition 4.4.

Lemma 4.1. *Let $\mathbf{u}^n(t,x,z) = (u^n_1(t,x,z),\dots,u^n_K(t,x,z))^\top$ and $\mathbf{u}_n(t,x,z) = (u_{n1}(t,x,z),\dots,u_{nK}(t,x,z))^\top$ be defined by*

$$u^n_i(t,x,z) = u_i\left(t+nT, x+cnT\xi; \mathbf{u}^+_{0,z-cnT\xi}(x), z-cnT\xi\right), \quad i=1,\dots,K$$

and

$$u_{ni}(t,x,z) = u_i\left(t+nT, x+cnT\xi; \mathbf{u}^-_{0,z-cnT\xi}(x), z-cnT\xi\right), \quad i=1,\dots,K.$$

Then, for any given bounded interval $I\subset\mathbb{R}$, there is $N_0\in\mathbb{N}$ such that $\mathbf{u}^n(t,x,z)$ is non-increasing in n for $n\ge N_0$ and $\mathbf{u}_n(t,x,z)$ is non-decreasing in n for $n\ge N_0$, $t\in I$, $x\in\mathbb{R}^N$ and $z\in\mathbb{R}^N$.

Proof. It follows from straightforward calculations by using Proposition 4.3 and Proposition 4.4. □

Let

$$\mathbf{U}^+(t,x,z) = \lim_{n\to\infty}\mathbf{u}^n(t,x,z), \quad \mathbf{U}^-(t,x,z) = \lim_{n\to\infty}\mathbf{u}_n(t,x,z)$$

and

$$\mathbf{\Phi}^{\pm}_0(x,z) = \mathbf{U}^{\pm}(0,x,z).$$

Then, $\mathbf{U}^+(t,x,z)$ and $\mathbf{\Phi}^+_0(x,z)$ are upper semi-continuous in $(t,x,z)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, and $\mathbf{U}^-(t,x,z)$ and $\mathbf{\Phi}^-_0(x,z)$ are lower semi-continuous in $(t,x,z)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$. By the similar arguments as those in [2, Lemma 4.2], we have

$$\mathbf{U}^{\pm}(t,x,z) = \mathbf{u}(t,x;\mathbf{\Phi}^{\pm}_0(\cdot,z),z), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Hence $u=\mathbf{U}^{\pm}(t,x,z)$ are entire solution of (3.3).

Let

$$\mathbf{\Phi}^{\pm}(x,t,z) = \mathbf{U}^{\pm}(t,x+ct\xi,z-ct\xi) = \mathbf{u}(t,x+ct\xi;\mathbf{\Phi}^{\pm}_0(\cdot,z-ct\xi),z-ct\xi).$$

Lemma 4.2. *The following statements hold.*

(1) For given $z\in\mathbb{R}^N$,

$$\mathbf{u}(t,x;\mathbf{\Phi}^{\pm}(\cdot,0,z),z) = \mathbf{\Phi}^{\pm}(x-ct\xi,t,z+ct\xi).$$

(2) There holds

$$\lim_{x\cdot\xi\to\infty}\frac{\Phi^{\pm}_i(x,t,z)}{e^{-\mu x\cdot\xi}\phi_i(t,x+z)} = 1, \quad \forall i=1,\dots,K$$

uniformly in $t\in\mathbb{R}$ and $z\in\mathbb{R}^N$.

Proof. (1) follows directly from the definition of $\mathbf{\Phi}^{\pm}$ and (2) follows from Propositions 4.1 and 4.4. □

Next, we prove the main results of Theorem 4.1.

Proof of Theorem 4.1. (1) Let $\mathbf{\Phi}=\mathbf{\Phi}^+$. It suffices to prove that $\mathbf{\Phi}$ generates a traveling wave solution of (1.1) with speed c in the direction of ξ. First, it follows from Lemma 4.2 that Φ_i satisfies (4.1) for any $i=1,\dots,K$. On the other hand, $\mathbf{\Phi}(x,t,z)$ is periodic in space x and time t, that is,

$$\mathbf{\Phi}^+(x,T+t,z) = \mathbf{\Phi}^+(x,t,z) = \mathbf{\Phi}^+(x,t,z+p_i\mathbf{e}_i)$$

and $\mathbf{\Phi}^+(x,t,z-x)=\mathbf{\Phi}^+(x',t,z-x')$ for any $x,x'\in\mathbb{R}^N$ with $x\cdot\xi=x'\cdot\xi$ (see also [38, Theorem 5.1]), which imply (4.2) and (4.3) hold true.

Next, we prove that

$$\lim_{x\cdot\xi\to-\infty}\mathbf{\Phi}(x,t,z)=\mathbf{u}^*(t,x+z)\quad\text{uniformly in } t\in\mathbb{R},\ z\in\mathbb{R}^N. \tag{4.19}$$

Note that there is $N_0\in\mathbb{N}$ such that for $t\in[0,T]$ and $n\geq N_0$,

$$\begin{aligned} u_i^*(t,x+z) &\geq u_i\left(t+nT,x+cnT\xi+ct\xi;\mathbf{u}^+_{0,z-cnT\xi-ct\xi},z-cnT\xi-ct\xi\right)\\ &\geq \Phi_i(x,t,z)\\ &\geq u_i\left(t+nT,x+cnT\xi+ct\xi;\mathbf{u}^-_{0,z-cnT\xi-ct\xi},z-cnT\xi-ct\xi\right). \end{aligned} \tag{4.20}$$

for $i=1,\ldots,K$. By Proposition 4.4, there are $\sigma>0$ and $C\in\mathbb{R}$ such that

$$\begin{aligned} &u_i\left(t+nT,x;\mathbf{u}^-_{0,z-cnT\xi-ct\xi}(\cdot+cnT\xi+ct\xi),z\right)\\ &\quad=u_i\left(t+nT,x+cnT\xi+ct\xi;\mathbf{u}^-_{0,z-cnT\xi-ct\xi},z-cnT\xi-ct\xi\right)\geq\sigma \end{aligned}$$

for $t\in[0,T]$, $n\geq1$ and $x\cdot\xi\leq C$. By Proposition 4.3, we have that $\Phi_i(x,t,z)\leq u_i^*(t,x+z)$ for $t\in\mathbb{R}$ and $x,z\in\mathbb{R}^N$, $i=1,\ldots,K$. Then, by Lemma 3.1, for any $\epsilon>0$ and $c'<0$, there is $N^*\in\mathbb{N}$ with $N^*\geq N_0$ such that

$$\begin{aligned} &u_i(t+N^*T,x+cN^*T\xi+ct\xi;\mathbf{u}^-_{0,z-cN^*T\xi-ct\xi},z-cN^*T\xi-ct\xi)\\ &\quad=u_i(t+N^*T,x;\mathbf{u}^-_{0,z-cN^*T\xi-ct\xi}(\cdot+cN^*T\xi-ct\xi),z)\geq u_i^*(t+N^*T,x+z)-\epsilon \end{aligned}$$

for $t\in[0,T]$ and $x\cdot\xi\leq c'(N^*+1)T$. Then, (4.19) follows from (4.20) and (4.2), and hence, $\mathbf{\Phi}$ generates a traveling wave solution of (1.1) in the direction of ξ with speed c.

(2) Assume by contraction that there exists a traveling wave solution $\mathbf{u}(t,x)$ of (1.1) with speed $c_1\in(0,c^*(\xi))$ connecting $\mathbf{u}^*$ and $\mathbf{0}$ in the direction of ξ. Then, for any $t\in\mathbb{R}$, $\liminf_{x\cdot\xi\to-\infty}u_i(t,x)>0$ for $i=1,\ldots,K$. Let $\mathbf{u}_0\in X^+(\xi)$ be such that $u_{i0}(x)\leq u_i(0,x)$ for $x\cdot\xi\leq0$ and $u_{i0}(x)=0$ for $x\cdot\xi\geq0$, $i=1,\ldots,K$. Then, Proposition 2.1 implies that $u_i(t,x;\mathbf{u}_0)\leq u_i(t,x)$ for $x\in\mathbb{R}^N, t\geq0$ and $i=1,\ldots,K$.

Let $c',c''\in(c_1,c^*(\xi))$ with $c'>c''$. From Lemma 3.1, we know that

$$\limsup_{x\cdot\xi\leq c't,t\to\infty}|\mathbf{u}(t,x;\mathbf{u}_0)-\mathbf{u}^*(t,x)|=0.$$

Since $\mathbf{u}(t,x)$ is the solution of (1.1) with speed $c_1\in(0,c^*(\xi))$ connecting $\mathbf{u}^*$ and $\mathbf{0}$, the sub- and super-solutions in Propositions 4.3 and Proposition 4.4 imply that

$$\limsup_{x\cdot\xi\geq c''t,t\to\infty}|\mathbf{u}(t,x)|=0.$$

Hence,

$$\limsup_{x\cdot\xi\geq c''t,t\to\infty}|\mathbf{u}(t,x;\mathbf{u}_0)|\leq\limsup_{x\cdot\xi\geq c''t,t\to\infty}|\mathbf{u}(t,x)|=0,$$

which leads to a contradiction. Hence, there is no traveling wave solution of (1.1) connecting $\mathbf{u}^*$ and $\mathbf{0}$ in the direction of ξ with speed $c<c^*(\xi)$.

(3) It follows from standard arguments using strict sub-homogeneity and some trick using the decay rate of $\mathbf{\Phi}$ as $x\to\infty$ given in (1). We refer the read to the proof in [39, Theorem 2.2] for more details. □

5. **Application.** In this section, we discuss the applications of the results obtained in Sections 2-4 to a nonlocal KPP equation and a two-species competitive system with nonlocal dispersal.

5.1. **A nonlocal KPP equation.** In this subsection, we consider the applications of the results obtained in Sections 2-4 to the following nonlocal monostable equation in space-time periodic habitats,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(t,x,u(t,x)), \quad x \in \mathbb{R}^N, \tag{5.1}$$

where $k(\cdot)$ is the same as in (1.1).

Assume that

- **(B1):** *$f(t,x,u)$ is C^1 in $(t,x,u) \in \mathbb{R}\times\mathbb{R}^N\times[0,\infty)$, and $f(\cdot+T,\cdot,\cdot) = f(\cdot,\cdot,\cdot) = f(\cdot,\cdot+p_l\mathbf{e}_l,\cdot)$ for each $l = 1,\dots,N$, where $\{\mathbf{e}_1,\dots,\mathbf{e}_N\}$ is the standard base of $\mathbb{R}^N$.*
- **(B2):** *$\frac{\partial f(t,x,u)}{\partial u} < 0$ for $(t,x,u) \in \mathbb{R}\times\mathbb{R}^N\times[0,\infty)$, and $f(t,x,u) < 0$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $u \gg 1$,*
- **(B3):** *For any $\mu \ge 0$, $\lambda(\xi,\mu,a_0)$ is the principal eigenvalue of the operator $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot,\cdot)I$ and $\lambda(\xi,\mu,a_0) > 0$, where $a_0(t,x) = f(t,x,0)$.*

From **(B3)**, we have $\lambda(a_0) := \lambda(\xi,0,a_0) > 0$ and then $u \equiv 0$ is linearly unstable in X_p. The assumptions **(B2)** and **(B3)** imply that (5.1) has exactly two time periodic solution in $\mathcal{X}_p^+$, that is, $u = 0$ and $u = u^*(t,x)$, where $u = 0$ is linearly unstable and $u^*(t,x)$ is asymptotically stable with respect to positive perturbation in X_p^+ (see [37]). We remark that the existence of spreading speeds of (5.1) does not require the existence of principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a_0(\cdot,\cdot)I$, that is, **(B3)**. However, operator $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot,\cdot)I$ may not have a principal eigenvalue for $N \ge 3$ (see [37] for an example). In order to establish the traveling wave solution of (5.1), we need **(B3)** (also see [37]). Note that assumptions **(B1)-(B3)** imply that **(H1)-(H3)** in Section 1 hold for equation (5.1).

Next, we show **(H4)** and **(H5)** also hold for (5.1). Consider the linearization of (5.1) at $u = 0$,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + a_0(t,x)u(t,x), \tag{5.2}$$

where $a_0(t,x) = f(t,x,0)$. Then, we have $A_0(t,x) = a_0(t,x)$. For any $\xi \in \mathbb{S}^{N-1}$ and $\mu > 0$, $\lambda(\xi,\mu,a_0)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a_0(\cdot,\cdot)I$ and $\phi(\cdot,\cdot;\xi,\mu)$ is the positive eigenfunction corresponding to $\lambda(\xi,\mu,a_0)$ with $\|\phi(\cdot,\cdot;\xi,\mu)\| = 1$, which implies **(H4)**. We refer to see [37] for the principal eigenvalue theory and criteria for the existence of principal eigenvalue for nonlocal dispersal operators in space-time periodic habitats.

Note that for any $\rho > 0$, $\rho\phi f(t,x,\rho\phi) \le \rho\phi a_0(t,x) = \rho\phi f(t,x,0)$. Together with **(B2)**, we see that **(H5)** holds for (5.1). We also see that $uf(t,x,u)$ is strictly sub-homogeneous, that is,

$$\alpha uf(t,x,\alpha u) > \alpha uf(t,x,u), \quad \forall \alpha \in (0,1).$$

Then, the following theorem, which recovers the results of [38] for spreading speeds and traveling wave solutions of (5.1), holds.

Theorem 5.1. *Assume* **(B1)-(B3)**. *Then, $c^*(\xi) := \inf_{\mu>0} \frac{\lambda(\xi,\mu,a_0)}{\mu}$ is the spreading speed of (5.1) in the direction of ξ, and for any $c > c^*(\xi)$, (5.1) has a continuous* *periodic traveling wave solution* $u(t,x) = \Phi(x - ct\xi, t, ct\xi)$ *connecting* $u^*(t,x)$ *and* 0 *in the direction of* ξ.

5.2. **Two species competitive system.** In this subsection, we consider the applications of the results obtained in Sections 2-4 to the following two species competitive system with nonlocal dispersal,

$$\begin{cases} \frac{\partial u_1}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_1(t,y)dy - u_1(t,x) + u_1(a_1(t,x) - b_1(t,x)u_1 - c_1(t,x)u_2), \\ \frac{\partial u_2}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_2(t,y)dy - u_2(t,x) + u_2(a_2(t,x) - b_2(t,x)u_1 - c_2(t,x)u_2), \end{cases} \tag{5.3}$$

where $a_i(\cdot,\cdot)$, $b_i(\cdot,\cdot)$, $c_i(\cdot,\cdot)$ are C^0 in $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, T-periodic in t and p_j-period in x_j, and $b_i(t,x) > 0$, $c_i(t,x) > 0$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where $i = 1,2$, $j = 1,\dots,N$.

Let $\sigma(-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I)$ be the spectrum of $-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I$ acting on $\mathcal{X}_p(1)$ and

$$\lambda_0(a) := \sup \left\{ \mathrm{Re}\lambda : \lambda \in \sigma(-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I) \right\}.$$

We call $\lambda_0(a)$ is the principal spectrum point of $-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I$ acting on $\mathcal{X}_p(1)$ (see Definition 2.2). Assume that

(C1): $\lambda_0(a_i) > 0$, $i = 1,2$.

Note that **(C1)** implies that the trivial solution $(0,0)^\top$ of (5.3) is unstable with respect to perturbations in X_p^+ and (5.3) has two semi-trivial time periodic solutions $(u_1^*,0)^\top$ and $(0,u_2^*)^\top$ in $\mathcal{X}_p^+(2)$, see [22, Proposition 2.8]. We also assume that

(C2): $\lambda_0(a_1 - c_1u_2^*) > 0$, $\lambda_0(a_2 - b_2u_1^*) < 0$, *and for any* $(u_{10},u_{20})^\top \in X_p^+(2)$ *with* $u_{10} \not\equiv 0$, $(u_1(t,x;u_{10},u_{20}),\ u_2(t,x;u_{10},u_{20}))^\top \to (u_1^*(t,x),0)^\top$ *as* $t \to \infty$ *uniformly in* $x \in \mathbb{R}^N$.

Note that **(C2)** indicates that the species u_1 can completely invade the species u_2, that is, $(0,u_2^*)^\top$ is linearly unstable and $(u_1^*,0)^\top$ is linearly and globally asymptotically stable with respect to perturbations in $X_p^+(2)$.

Assume **(C1)** and **(C2)**. It is interesting to know how fast the species u_1 invades the species u_2 and whether there are traveling wave solutions of (5.3) connecting $(u_1^*,0)^\top$ and $(0,u_2^*)^\top$.

In [22], the authors studied the existence of spreading speeds and linear determinacy for two species competitive system (5.3). We will show that our results can also be applied to (5.3). To see this, first of all, as in [22], we transform (5.3) to a cooperative system via the following standard change of variable,

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2^* - u_2$$

and obtain the following cooperative system

$$\begin{cases} \frac{\partial u_1}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_1(t,y)dy - u_1(t,x) \\ \qquad\qquad + u_1(a_1(t,x) - b_1(t,x)u_1 - c_1(t,x)(u_2^* - u_2)), \\ \frac{\partial u_2}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_2(t,y)dy - u_2(t,x) + b_2(t,x)(u_2^* - u_2)u_1 \\ \qquad\qquad + u_2(a_2(t,x) - 2c_2(t,x)u_2^* + c_2(t,x)u_2). \end{cases} \tag{5.4}$$

Observe that the trivial solution $(0,0)^\top$ of (5.1) becomes $(0,u_2^*)^\top$, the semi-trivial solution $(0,u_2^*)^\top$ of (5.1) becomes $(0,0)^\top$, and $(u^*,0)^\top$ becomes $(u_1^*,u_2^*)^\top$. Under the assumptions **(C1)-(C2)**, we know that $(0,0)^\top$ is a unstable solution of (5.4) and $(u_1^*,u_2^*)^\top$ is a linearly and globally stable solution of (5.4). Hence, **(C1)-(C2)** together with the periodicity of $a_i(t,x)$, $b_i(t,x)$, and $c_i(t,x)$ $(i=1,2)$ imply that **(H1)-(H3)** hold for (5.4).

Consider the linearization of (5.4) at $(0,0)^\top$,

$$\begin{cases} \frac{\partial u_1}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_1(t,y)dy - u_1(t,x) + (a_1(t,x)-c_1(t,x)u_2^*(t,x))u_1(t,x), \\ \frac{\partial u_2}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u_2(t,y)dy - u_2(t,x) + b_2(t,x)u_2^*(t,x)u_1 \\ \qquad\qquad +(a_2(t,x)-2c_2(t,x)u_2^*(t,x))u_2, \end{cases} \tag{5.5}$$

as well as the following eigenvalue problem associated to (5.5)

$$\begin{cases} -\frac{\partial u_1}{\partial t} + \mathcal{K}_{\xi,\mu}u_1 - u_1(t,x) + (a_1(t,x)-c_1(t,x)u_2^*(t,x))u_1(t,x) = \lambda u_1, \\ -\frac{\partial u_2}{\partial t} + \mathcal{K}_{\xi,\mu}u_2 - u_2(t,x) + b_2(t,x)u_2^*(t,x)u_1 + (a_2(t,x) \\ -2c_2(t,x)u_2^*(t,x))u_2 = \lambda u_2, \end{cases} \tag{5.6}$$

where $\mathcal{K}_{\xi,\mu}$ is as in (1.8). Let

$$A_0(t,x) = \begin{pmatrix} a_1(t,x)-c_1(t,x)u_2^*(t,x) & 0 \\ b_2(t,x)u_2^*(t,x) & a_2(t,x)-2c_2(t,x)u_2^*(t,x) \end{pmatrix}$$

Clearly, $A_0(t,x)$ is in a block lower triangular form.

Let **(C3)-(C5)** be the following assumptions.

(C3): *For any $\xi \in \mathbb{S}^{N-1}$ and $\mu \geq 0$, $\lambda_1(\xi,\mu)$ is the principal eigenfunction of*

$$-\frac{\partial u_1}{\partial t} + \mathcal{K}_{\xi,\mu}u_1 - u_1(t,x) + (a_1(t,x)-c_1(t,x)u_2^*(t,x))u_1(t,x) = \lambda u_1.$$

(C4): *For all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,*

$$a_1(t,x)-c_1(t,x)u_2^*(t,x)-a_2(t,x)+2c_2(t,x)u_2^*(t,x)-b_2(t,x)u_2^*(t,x) \geq 0,$$
$$b_1(t,x) \geq c_1(t,x), \quad \text{and} \quad b_2(t,x) \geq c_2(t,x).$$

(C5): *For all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,*

$$a_1(t,x)-c_1(t,x)u_2^*(t,x)-a_2(t,x)+2c_2(t,x)u_2^*(t,x)-b_2(t,x)u_2^*(t,x)\frac{c_{1M}}{b_{1L}} \geq 0,$$

and

$$a_1(t,x)-c_1(t,x)u_2^*(t,x)-a_2(t,x)+2c_2(t,x)u_2^*(t,x)-b_2(t,x)u_2^*(t,x)\frac{c_{2M}}{b_{2L}} \geq 0,$$

where $b_{iL} = \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} b_i(t,x)$ and $c_{iM} = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} c_i(t,x)$, $i=1,2$.

We remark that **(C3)-(C5)** are related to the linear determinacy for the spreading speeds of (5.4). In fact, **(C3)** and **(C4)** (resp. **(C3)** and **(C5)**) imply **(H4)** and **(H5)**. To see this, let $\lambda_2(\mu,\xi)$ be the principle spectrum point of

$$\begin{cases} -\frac{\partial u_2}{\partial t} + \mathcal{K}_{\xi,\mu}u_2 - u_2 + (a_2(t,x)-2c_2(t,x)u_2^*(t,x))u_2 = \lambda u_2, \\ u_2(t,x) \in \mathcal{X}_p^+(1). \end{cases}$$

Since $(0,0)^\top$ is a unstable solution of (5.4), $\lambda_1(\mu,\xi) > 0$ for any $\mu \geq 0$ and $\xi \in \mathbb{S}^{N-1}$. By **(C3)**,

$$\lambda(\mu,\xi,a_1(t,x)-c_1(t,x)u_2^*(t,x)-\lambda_1(\mu,\xi)) = 0,$$

and by **(C4)** or **(C5)**, $a_1(t,x)-c_1(t,x)u_2^*(t,x) > a_2(t,x)-2c_2(t,x)u_2^*(t,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Hence, we have from [22, Lemma 4.1(4)] that $\lambda_2(\mu,\xi) < \lambda_1(\mu,\xi)$. Then, **(C3)** and **(C4)** (resp. **(C3)** and **(C5)**) imply that $A_0(t,x)$ satisfies

(H4) and there is the positive eigenfunction $(\phi_1(t,x;\mu,\xi),\phi_2(t,x;\mu,\xi))^\top$ of (5.6) corresponding to the principal eigenvalue $\lambda_1(\mu,\xi)$, see also [22].

Moreover, assume that **(C3)** and **(C4)** (resp. **(C3)** and **(C5)**) hold. Then by [22, Lemma 4.2],

$$c_1(t,x)\phi_2 \le b_1(t,x)\phi_1 \quad \text{and} \quad c_2(t,x)\phi_2 \le b_2(t,x)\phi_1. \tag{5.7}$$

Let

$$\begin{cases} f(t,x,u_1,u_2) = u_1(a_1(t,x) - b_1(t,x)u_1 - c_1(t,x)(u_2^*(t,x) - u_2)) \\ g(t,x,u_1,u_2) = b_2(t,x)(u_2^*(t,x) - u_2)u_1 + u_2(a_2(t,x) \\ \qquad\qquad\qquad -2c_2(t,x)u_2^*(t,x) + c_2(t,x)u_2). \end{cases}$$

By (5.7), for any $\rho > 0$, we have

$$\begin{cases} f(t,x,\rho\phi_1,\rho\phi_2) \le (a_1(t,x) - c_1(t,x)u_2^*(t,x))\rho\phi_1 \\ g(t,x,\rho\phi_1,\rho\phi_2) \le b_2(t,x)u_2^*(t,x)\rho\phi_1 + (a_2(t,x) - 2c_2(t,x)u_2^*(t,x))\rho\phi_2. \end{cases}$$

Hence **(C3)** and **(C4)** (resp. **(C3)** and **(C5)**) also imply that **(H5)** holds for (5.4). The reader is referred to [22] for the special forms of **(C4)-(C5)** when a_i, b_i, and c_2 $(i = 1, 2)$ are constants.

By Theorem 4.1, we then have the following theorem for (5.1).

Theorem 5.2. *Suppose* **(C1)**-**(C3)**, **(C4)** *or* **(C5)**. *For any* $\xi \in \mathbb{S}^{N-1}$, (5.1) *has a finite spreading speed interval* $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ *and*

$$c^*(\xi) := c^*_{\inf}(\xi) = c^*_{\sup}(\xi) = \inf_{\mu>0} \frac{\lambda(\xi,\mu,a_1(t,x) - c_1(t,x)u_2^*(t,x))}{\mu}$$

is the spreading speed of (5.1). *Moreover, for any* $c > c^*(\xi)$, (5.1) *has a space-time periodic traveling wave solution* $(\Phi_1(x - ct\xi, t, ct\xi), \Phi_2(x - ct\xi, t, ct\xi))^\top$ *connecting* $(0, u_2^*)^\top$ *and* $(u_1^*, 0)^\top$ *in the direction* ξ *with speed* c, *and for* $c < c^*(\xi)$ *there is no such solution of* (5.1).

Remark 5.1. When (5.3) is space periodic but independent of t, it has been proven in [2] that for any $c > c^*(\xi)$, (5.3) has space periodic traveling wave solutions. Hence, the results of the current paper also recover the results on the existence of traveling wave solutions and spreading speeds in [2]. It is easy to verify that two species competitive system does not satisfy the sub-homogeneous condition, and therefore, we can not obtain the continuity and uniqueness of traveling wave solutions of (5.3). Moreover, we will further apply our results to more epidemic models with nonlocal dispersal in periodic media somewhere else.

Appendix A. **Proof of Lemma 3.2.** This appendix is devoted to the proof of Lemma 3.2.

Proof of Lemma 3.2. For given $\xi \in \mathbb{S}^{N-1}$, set $\lambda_1(\mu) := \lambda_1(\xi,\mu,A_1)$. For any $\mu > 0$, let $\boldsymbol{\phi}(\cdot,\cdot;\xi,\mu,A_1) \in \mathcal{X}_p^{++}(K_1)$ be the eigenfunction of L_{ξ,μ,A_1} with $\|\boldsymbol{\phi}(\cdot,\cdot;\xi,\mu,A_1)\| = 1$, where K_1 is the dimension of A_1. Then

$$L_{\xi,\mu,A_1}[\boldsymbol{\phi}(\cdot,\cdot;\xi,\mu,A_1)] = \lambda_1(\mu)\boldsymbol{\phi}(\cdot,\cdot;\xi,\mu,A_1).$$

Let $r(\xi,\mu,A_1)$ be the spectral radius of $\mathbf{\Phi}^p(T;\xi,\mu,A_1)$, where $\mathbf{\Phi}^p(t;\xi,\mu,A_1)$ is the solution operator of (2.1) with A being replaced by A_1 in the space $X_p(K_1)$. Then, $\lambda_1(\mu) = \frac{1}{T}\ln r(\xi,\mu,A_1)$ and

$$\mathbf{\Phi}^p(T;\xi,\mu,A_1)\boldsymbol{\phi}(\mu,\cdot) = r(\xi,\mu,A_1)\boldsymbol{\phi}(\mu,\cdot) = e^{\lambda_1(\mu)T}\boldsymbol{\phi}(\mu,\cdot), \tag{A.1}$$

where $\boldsymbol{\phi}(\mu, x) = \boldsymbol{\phi}(T, x; \xi, \mu)$. By Proposition 2.3, there exists $\mu^* \in (0, \infty)$ such that $\frac{\lambda_1(\mu^*)}{\mu^*} = \inf_{\mu>0} \frac{\lambda_1(\mu)}{\mu}$. Then, by Corollary 2.1, for any $\epsilon > 0$, there is $\mu_\epsilon = \mu_\epsilon(\xi, A_1)$ such that $-\lambda_1'(\mu) < -\frac{\lambda_1(\mu^*)}{\mu^*} + \epsilon$ for $\mu_\epsilon < \mu < \mu^*(\xi)$.

In the rest of the proof, we fix any $\epsilon > 0$ and $\mu \in (\mu_\epsilon, \mu^*(\xi))$. Let $\zeta : \mathbb{R} \to [0, 1]$ be a smooth function satisfying that

$$\zeta(s) = \begin{cases} 1, & |s| \le 1, \\ 0, & |s| \ge 2. \end{cases}$$

Write $\boldsymbol{\phi}(\mu, x) = (\phi_1(\mu, x), \dots, \phi_{K_1}(\mu, x))^\top$ and define

$$k^i(\mu, z) := \frac{\partial \phi_i(\mu, z)}{\partial \mu} \frac{1}{\phi_i(\mu, z)}, \quad i = 1, \dots, K_1.$$

Motivated by the works [44, 45], we define for given $\gamma > 0$, $B > 0$ and $z \in \mathbb{R}^N$,

$$\tau_i(\mu, \gamma, z, B) = \frac{1}{\gamma} \tan^{-1} \frac{\sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \sin \gamma(-(y-z)\cdot\xi + k^j(\mu, y))\zeta(|y-z|/B) m_{ij}(z; y, dy)}{\sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \cos \gamma(-(y-z)\cdot\xi + k^j(\mu, y))\zeta(|y-z|/B) m_{ij}(z; y, dy)}.$$

In the following, we write $\tau_i = \tau_i(\mu, \gamma, z, B)$ and $\zeta_B = \zeta(|y-z|/B)$ for simplicity. It follows from the arguments of [38, Lemma 4.2] that the measures m_{ij}, $i, j = 1, \dots, K_1$, satisfy

$$\int_{|y-x|\ge B} e^{\mu|y-x|} m_{ij}(x; y, dy) \to 0 \quad \text{as} \quad B \to \infty \tag{A.2}$$

locally uniformly in $\mu \in \mathbb{R}$ and uniformly $x \in \mathbb{R}^N$.

It is easy to see that $\tau_i(\mu, \gamma, z, B)$ is a family of equicontinuous and uniformly bounded function of z and μ for any $B > 0$ and $0 < \gamma \ll 1$. The fact $\frac{d}{dt} \tan^{-1} t = 1$ at $t = 0$ yields

$$\lim_{\gamma\to 0} \tau_i(\mu, \gamma, z, B) = \frac{\sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} (-(y-z)\cdot\xi + k^j(\mu, y)) \zeta_B m_{ij}(z; y, dy)}{\sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \zeta_B m_{ij}(z; y, dy)}$$

uniformly in $z \in \mathbb{R}^N$ and $B > 0$. By (A.1) and (2.6),

$$e^{\lambda_1(\mu)T} \phi_i(\mu, \cdot) = (\boldsymbol{\Phi}^p(T; \xi, \mu, A_1)\boldsymbol{\phi}(\mu, \cdot))_i = \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} e^{-\mu(y-\cdot)\cdot\xi} \phi_j(\mu, y) m_{ij}(\cdot; y, dy), \quad i = 1, \dots, K_1,$$

which together with (A.2) leads to

$$\lim_{B\to+\infty} \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \zeta_B m_{ij}(z; y, dy) = e^{\lambda_1(\mu)T} \phi_i(\mu, z) \tag{A.3}$$

uniformly in $z \in \mathbb{R}^N$ for any $i = 1, \ldots, K_1$. Taking the derivative with respect to μ, we arrive at

$$\begin{aligned}
&\lim_{B\to+\infty} \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} (-(y-z)\cdot\xi + k^j(\mu, y)) \zeta_B m_{ij}(z; y, dy) \\
&\quad = \lambda_1'(\mu) T e^{\lambda_1(\mu)T} \phi_i(\mu, z) + e^{\lambda_1(\mu)T} \frac{\partial \phi_i(\mu, z)}{\partial \mu}
\end{aligned} \tag{A.4}$$

uniformly in $z \in \mathbb{R}^N$ for any $i = 1, \ldots, K_1$. Then,

$$\lim_{B\to+\infty} \lim_{\gamma\to 0} \tau_i = T\lambda_1'(\mu) + k^i(\mu, z), \quad \forall i = 1, \ldots, K_1. \tag{A.5}$$

By (A.3) and (A.4), we can choose $B \gg 1$ and fix it so that $0 < \gamma \ll 1$, and for $z, z' \in \mathbb{R}^N$,

$$\gamma \left(2B + |\tau_i(\mu, \gamma, z, B)| + |k^j(\mu, z')|\right) < \pi, \quad 1 \le i, j \le K_1 \tag{A.6}$$

and

$$k^i(\mu, z) - \tau_i(\mu, \gamma, z, B) < -T(\lambda_1'(\mu) - \epsilon). \tag{A.7}$$

Let $\epsilon_1 > 0$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_K)$ be such that

$$\lambda_1(\mu) - \lambda_1^{'}(\mu)\mu > \epsilon_1 \tag{A.8}$$

(such ϵ_1 exists because of Corollary 2.1) and $F(t, x, \mathbf{u}) \ge A(t, x)\mathbf{u} - \epsilon_1 \mathbf{u}$ for all $\mathbf{u} \in [\mathbf{0}, \beta]$.

Define $\mathbf{v}(s,z) = (v_1(s,z), ..., v_K(s,z))^\top$ and $\widetilde{\mathbf{v}}(x;s,z) = (\widetilde{v}_1(x;s,z), ..., \widetilde{v}_K(x;s,z))^\top$ by setting

$$v_i(s, z) = \begin{cases} \begin{cases} \epsilon_2 \phi_i(\mu, z) e^{-\mu s} \sin \gamma(s - k^i(\mu, z)), & 0 \le s - k^i(\mu, z) \le \frac{\pi}{\gamma}, \\ 0, & \text{otherwise}, \end{cases} \\ \hfill i = 1, \ldots, K_1, \\ 0, \hfill i = K_1 + 1, \ldots, K \end{cases} \tag{A.9}$$

and

$$\widetilde{v}_i(x; s, z) = v_i(x \cdot \xi + s, x + z), \quad i = 1, \ldots, K, \tag{A.10}$$

where $\epsilon_2 > 0$ is sufficiently small so that $0 \le u_i(t, x; \widetilde{\mathbf{v}}, z) \le \beta_i$ for $t \in [0, T]$ and $x, z \in \mathbb{R}^N$, and $\mathbf{u}(t, x; \mathbf{u}_0, z) = (u_1(t, x; \mathbf{u}_0, z), \ldots, u_K(t, x; \mathbf{u}_0, z))^\top$ is the solution of (3.3) with initial data $\mathbf{u}_0$.

Let $\mathbf{\Phi}(t; A(\cdot, \cdot + z))$ be the solution operator of the equation

$$\mathbf{u}_t(t, x) = \int_{\mathbb{R}^N} k(y - x)\mathbf{u}(t, y) dy - \mathbf{u}(t, x) + A(t, x + z)\mathbf{u}(t, x), \quad x \in \mathbb{R}^N.$$

Then, by comparison principle, we have

$$\mathbf{u}(t, x; \widetilde{\mathbf{v}}, z) \ge e^{-\epsilon_1 t} \left(\mathbf{\Phi}(t; A(\cdot, \cdot + z))\widetilde{\mathbf{v}}\right)(x), \quad 0 \le t \le T, \ x, z \in \mathbb{R}^N. \tag{A.11}$$

Since $\widetilde{v}_i = 0$ for all $i = K_1 + 1, \ldots, K$, we obtain from (2.6) that

$$\left(\mathbf{\Phi}(T; A(\cdot, \cdot + z))\widetilde{\mathbf{v}}\right)_i(x) = \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \widetilde{v}_j(y - z; s, z) m_{ij}(x + z; y, dy), \quad i = 1, \ldots, K_1. \tag{A.12}$$

Set $\eta^i := -k^i(\mu, z) + \tau_i(\mu, \gamma, z, B)$ for $i = 1, \dots, K_1$. For any given $i = 1, \dots, K_1$, we define $\mathbf{v}^i(x; s, z) = (v_1^i(x; s, z), \dots, v_K^i(x; s, z))^\top$ by setting

$$v_j^i(x; s, z) = \begin{cases} \widetilde{v}_j(x; s + \eta^i, z), & j = 1, \dots, K_1, \\ 0, & j = K_1 + 1, \dots, K. \end{cases}$$

Clearly, (A.11) and (A.12) hold with $\widetilde{\mathbf{v}}$ replaced by $\mathbf{v}^i$ for each $i = 1, \dots, K_1$, that is,

$$\mathbf{u}(t, x; \mathbf{v}^i, z) \geq e^{-\epsilon_1 t} \left(\mathbf{\Phi}(t; A(\cdot, \cdot + z)) \mathbf{v}^i \right)(x), \quad 0 \leq t \leq T, \ x, z \in \mathbb{R}^N$$

for each $i = 1, \dots, K_1$. Note that if $0 \leq s - k^i(\mu, z) \leq \frac{\pi}{\gamma}$ and $\|y - z\| \leq 2B$, then (A.6) ensures

$$-\frac{\pi}{\gamma} \leq (y - z) \cdot \xi + s - k^i(\mu, z) + \tau_i(\mu, \gamma, z, B) - k^j(\mu, y) \leq \frac{2\pi}{\gamma}, \quad \forall j = 1, \dots, K_1,$$

which implies that

$$\begin{aligned} v_j^i(y - z; s, z) &= v_j((y - z) \cdot \xi + s + \eta^i, y) \\ &\geq \epsilon_2 \phi_1^j(\mu, y) e^{-\mu((y-z)\cdot\xi + s + \eta^i)} \sin \gamma((y - z) \cdot \xi + s + \eta^i - k^j(\mu, y)), \\ &\qquad \forall j = 1, \dots, K_1. \end{aligned}$$

We claim that, for $i = 1, \dots, K_1$

$$u_i\left(T, 0; \mathbf{v}^i, z\right) \geq v_i(s, z) = v_i^i\left((k^i(\mu, z) - \tau_i)\xi; s, (-k^i(\mu, z) + \tau_i(\mu, \gamma, z, B))\xi + z\right). \tag{A.13}$$

To show (A.13), we consider two cases. Let us fix any $i = 1, \dots, K_1$.

Case 1.: If $s \leq k^i(\mu, z)$ or $s \geq k^i(\mu, z) + \frac{\pi}{\gamma}$, then $v_i(s, z) = 0$ by (A.9). Thus, (A.13) is trivial.

Case 2.: If $0 \le s - k^i(\mu, z) \le \frac{\pi}{\gamma}$, we have

$$\begin{aligned}
&\left(\Phi_i(T; A(\cdot, \cdot + z))\mathbf{v}^i\right)(0) \\
\ge&\epsilon_2 \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu((y-z)\cdot\xi + s + \eta^i)} \sin\gamma((y-z)\cdot\xi + s + \eta^i \\
&\qquad - k^j(\mu, y))\zeta_B m_{ij}(z; y, dy) \\
=&\epsilon_2 e^{-\mu(s+\eta^i)} \sum_{j=1}^{K} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \cos\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy) \\
&\cdot \Bigg[\sin\gamma(s+\eta^i) - \cos\gamma(s+\eta^i) \\
&\cdot \frac{\sum_{j=1}^{K} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \sin\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy)}{\sum_{j=1}^{K} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \cos\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy)}\Bigg] \\
=&\epsilon_2 e^{-\mu(s+\eta^i)} \sum_{j=1}^{K} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \cos\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy) \\
&\cdot \left[\sin\gamma(s+\eta^i) - \cos\gamma(s+\eta^i)\tan\gamma\tau_i\right] \\
=&v_i(s, z) e^{-\mu\eta^i} \frac{\sec\gamma\tau_i}{\phi_i(\mu, z)} \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \\
&\qquad \cos\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy).
\end{aligned} \tag{A.14}$$

By (A.3) and (A.5),

$$\lim_{B\to\infty} \lim_{\gamma\to 0} e^{-\mu\eta^i} \frac{\sec\gamma\tau_i}{\phi_i(\mu, z)} \sum_{j=1}^{K_1} \int_{\mathbb{R}^N} \phi_j(\mu, y) e^{-\mu(y-z)\cdot\xi} \tag{A.15}$$

$$\cos\gamma(k^j(\mu, y) - (y-z)\cdot\xi)\zeta_B m_{ij}(z; y, dy)$$

$$= e^{-\mu T \lambda_1'(\mu)} e^{\lambda_1(\mu) T}. \tag{A.16}$$

We then conclude (A.13) from (A.8), (A.10), (A.11), (A.14) and (A.15).

For each $1 \le i \le K_1$ and $z \in \mathbb{R}^N$, let $\overline{s}_i(z)$ be the unique real number such that $v_i(\overline{s}_i(z), z) = \max_{s\in\mathbb{R}} v_i(s, z)$. Set $\widetilde{k} := \max_{z\in\mathbb{R}^N, 1\le i\le K_1} k^i(\mu, z)$, and define $\overline{\mathbf{v}}(s, z) = (\overline{v}_1(s, z), \ldots, \overline{v}_K(s, z))^\top$ by

$$\overline{v}_i(s, z) = \begin{cases} \begin{cases} v_i(\overline{s}_i(z), z), & s \le \overline{s}_i(z) - \frac{\pi}{\gamma} - \widetilde{k}, \\ v_i(s + \frac{\pi}{\gamma} + \widetilde{k}, z), & s \ge \overline{s}_i(z) - \frac{\pi}{\gamma} - \widetilde{k}, \end{cases} & i = 1, \ldots, K_1, \\ 0, & i = K_1 + 1, \ldots, K. \end{cases}$$

It is easy to see that $\overline{\mathbf{v}}(s, z)$ is continuous in (s, z) and each of its component is nonincreasing in s and vanishes for $s \ge 0$. Let $\eta := \min_{z\in\mathbb{R}, 1\le i\le K_1} \left[-k^i(\mu, z) + \tau_i\right]$. Define $\overline{\mathbf{v}}^*(x; s, z) := \overline{\mathbf{v}}(x\cdot\xi + s + \eta, x + z)$. Then,

$$\overline{\mathbf{v}}^*(x; s, z) \ge \mathbf{v}^i(x; s + \frac{\pi}{\gamma} + \widetilde{k}, z), \quad i = 1, \ldots, K_1. \tag{A.17}$$

We see that

$$u_i(T,0;\overline{\mathbf{v}}^*,z) \geq \overline{v}_i^*\left((k^i(\mu,z)-\tau_i)\xi; s, (-k^i(\mu,z)+\tau_i)\xi+z\right), \quad \forall i=1,\dots,K_1. \tag{A.18}$$

To see this, we fix $i=1,\dots,K_1$. If $s \geq \overline{s}_i(z) - \frac{\pi}{\gamma} - \widetilde{k}$, we have from (A.13) and (A.17) that

$$\begin{aligned}
u_i(T,0;\overline{\mathbf{v}}^*,z) &\geq u_i(T,0;\mathbf{v}^i(\cdot;s+\frac{\pi}{\gamma}+\widetilde{k},z),z) \geq v_i(s+\frac{\pi}{\gamma}+\widetilde{k},z)\\
&= \overline{v}_i(s,z)\\
&\geq \overline{v}_i^*\left((k^i(\mu,z)-\tau_i)\xi; s, (-k^i(\mu,z)+\tau_i)\xi+z\right).
\end{aligned}$$

If $s < \overline{s}_i(z) - \frac{\pi}{\gamma} - \widetilde{k}$, there holds $\overline{\mathbf{v}}^*(x;s,z) \geq \overline{\mathbf{v}}^*(x;\overline{s}_i(z)-\frac{\pi}{\gamma}-\widetilde{k},z) \geq \mathbf{v}^i(x;\overline{s}_i(z),z)$, which implies that

$$\begin{aligned}
u_i(T,0;\overline{\mathbf{v}}^*,z) &\geq u_i(T,0;\mathbf{v}^i(\cdot;\overline{s}_i(z),z),z) \geq v_i(\overline{s}_i(z),z)\\
&= \overline{v}_i(s,z)\\
&\geq \overline{v}_i^*\left((k^i(\mu,z)-\tau_i)\xi; s, (-k^i(\mu,z)+\tau_i)\xi+z\right).
\end{aligned}$$

Hence, (A.18) follows.

Let $\mathbf{v}_0(x,z) = \overline{\mathbf{v}}(x\cdot\xi, x+z)$. Note that each component of $\overline{\mathbf{v}}(s,z)$ is non-increasing in s. Hence, for any $i=1,\dots,K_1$,

$$\begin{aligned}
u_i(T,x;\mathbf{v}_0,z) &= u_i(T,0;\mathbf{v}_0(\cdot+x,z),x+z)\\
&= u_i\left(T,0;\overline{\mathbf{v}}^*(\cdot;x\cdot\xi+k^i(\mu,x+z)-\tau_i,x+z),x+z\right)\\
&\geq \overline{v}_i(x\cdot\xi+k^i(\mu,x+z)-\tau_i,x+z) \quad (\text{by (A.18)})\\
&\geq \overline{v}_i(x\cdot\xi - T\lambda_1'(\mu)+T\epsilon,x+z) \quad (\text{by(A.7)})\\
&\geq \overline{v}_i(x\cdot\xi - \frac{\lambda_1(\mu^*(\xi))}{\mu^*(\xi)}T+2T\epsilon,x+z)\\
&= v_{0i}(x-\widetilde{c}^*(\xi)T\xi, \widetilde{c}^*(\xi)T\xi+z),
\end{aligned}$$

where $\widetilde{c}^*(\xi) = \frac{\lambda_1(\mu^*(\xi))}{\mu^*(\xi)} - 2\epsilon$ and $\mathbf{v}_0(x,z) = (v_{01}(x,z),\dots,v_{02}(x,z))^\top$. Then,

$$u_i(T,x;\mathbf{v}_0,z) \geq v_{0i}(x-\widetilde{c}^*(\xi)T\xi, \widetilde{c}^*(\xi)T\xi+z), \quad \forall i=1,\dots,K_1$$

By induction, for any $n\geq 1$,

$$u_i(nT,x;\mathbf{v}_0,z) \geq v_{0i}(x-n\widetilde{c}^*(\xi)T\xi, n\widetilde{c}^*(\xi)T\xi+z), \quad \forall i=1,\dots,K_1$$

Note that $v_{0i}(x-ncT\xi, ncT\xi+z) = \overline{v}_i(x\cdot\xi - ncT\xi, x+z)$ is nondecreasing in c, as $\overline{v}_i(s,z)$ is non-increasing in s. Hence, for any $c<\widetilde{c}^*(\xi)$ and $i=1,...,K_1$

$$\begin{aligned}
\liminf_{x\cdot\xi\leq cn, n\to\infty} u_i(nT,x;\mathbf{v}_0,z) &\geq \liminf_{x\cdot\xi\leq cn, n\to\infty} v_{0i}(x-n\widetilde{c}^*(\xi)T\xi, n\widetilde{c}^*(\xi)T\xi+z)\\
&\geq \liminf_{x\cdot\xi\leq cn, n\to\infty} v_{0i}(x-cnT\xi, cnT\xi+z) > 0,
\end{aligned}$$

which together with Lemma 3.1 imply $\widetilde{c}^*(\xi)\in C_{\inf}(\xi)$, and hence, $c^*_{\inf}(\xi) \geq \widetilde{c}^*(\xi) = \frac{\lambda_1(\mu^*(\xi))}{\mu^*(\xi)} - 2\epsilon$. Due to the arbitrariness of $\epsilon>0$, (3.6) holds. This completes the proof of Lemma 3.2. □

Acknowledgments. The authors would like to thank the referee for valuable comments and suggestions which improved the presentation of this paper considerably.

REFERENCES

[1] X. Bao and Z.-C. Wang, Existence and stability of time periodic traveling waves for a periodic bistable Lotka-Volterra competition system, *J. Differential Equations*, **255** (2013), 2402–2435.

[2] X. Bao, W.-T. Li and W. Shen, Traveling wave solutions of Lotka-Volterra competition systems with nonlocal dispersal in periodic habitats, *J. Differential Equations*, **260** (2016), 8590–8637.

[3] X. Bao and W. Shen, Criteria for the existence of principal eigenvalue of time periodic cooperative linear system with nonlocal dispersal, *Proc. Amer. Math. Soc.*, **145** (2017), 2881–2894.

[4] C. Conley and R. Gardner, An application of the generalized Morse index to traveling wave solutions of a competitive reaction diffusion model, *Indiana Univ. Math. J.*, **33** (1984), 319–343.

[5] J. Coville, On a simple criterion for the existtence of a principal eigenfucntion of some nonlocal operators, *J. Differential Equations*, **249** (2010), 2921–2953.

[6] J. Coville, J. Dávila and S. Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **30** (2013), 179–223.

[7] W. Ding and X. Liang, Principal eigenvalues of generalized convolution operators on the circle and spreading speeds of noncompact evolution systems in periodic media, *SIAM J. Math. Anal.*, **47** (2015), 855–896.

[8] S. R. Dunbar, Traveling wave solutions of diffusive Lotka-Volterra equations, *J. Math. Biol.*, **17** (1983), 11–32.

[9] J. Fang and X.-Q. Zhao, Traveling waves for monotone semiflows with weak compactness, *SIAM J. Math. Anal.*, **46** (2014), 3678–3704.

[10] J. Fang, X. Yu and X.-Q. Zhao, Traveling waves and spreading speeds for time-space periodic monotone systems, *J. Functional Analysis*, **272** (2017), 4222–4262.

[11] J.-S. Guo and C.-H. Wu, Wave propagation for a two-component lattice dynamical system arising in strong competition models, *J. Differential Equations*, **250** (2011), 3504–3533.

[12] J.-S. Guo and C.-H. Wu, Traveling wave front for a two-component lattice dynamical system arising in competition models, *J. Differential Eqnuations*, **252** (2012), 4357–4391.

[13] J.-S. Guo and X. Liang, The minimal speed of traveling fronts for Lotka-volterra competition system, *J. Dynam. Differential Equations*, **23** (2011), 353–363.

[14] G. Hetzer, T. Nguyen and W. Shen, Coexistence and extinction in the Lotka-Volterra competition model with nonlocal dispersal, *Commun. Pure Appl. Anal.*, **11** (2012), 1699–1722.

[15] G. Hetzer, W. Shen and A. Zhang, Effects of spatial variatious and dispersal strategies on principal eigenvalues of dispersal operators and spreading speeds of monostable equations, *Rocky Mountain Journal of Mathematics*, **43** (2013), 489–513.

[16] Y. Hosono, The minimal spread of traveling fronts for a diffusive Lotka-Volterra competition model, *Bull. Math. Biol.*, **66** (1998), 435–448.

[17] W. Huang, Problem on minimum wave speed for a Lotka-Volterra reaction diffusion competition model, *J. Dynam. Differential Equations,* **22** (2010), 285–297.

[18] C. Hu, Y. Kuang, B. Li and H. Liu, Spreading speeds and traveling wave solutions in cooperative intergal-differential system, *Discrete Contin. Dyn. Syst. B*, **20** (2015), 1663–1684.

[19] Y. Kan-on, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, *SIAM J. Math. Anal.*, **26** (1995), 340–363.

[20] Y. Kan-on, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, *Nonlinear Anal.*, **28** (1997), 145–164.

[21] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin-Heidelberg, 1976.

[22] L. Kong, Nar Rawal and W. Shen, Spreading speeds and linear determinacy for two species competition systems with nonlocal dispersal in periodic habitats, *Math. Model. Nat. Phenom.*, **10** (2015), 113–141.

[23] M. Lewis, H. Weinberger and B. Li, Spreading speed and linear determincay for two species competition models, *J. Math. Biol.*, **45** (2002), 219–233.

[24] B. Li, H. F. Weinberger and M. Lewis, Spreading speeds as slowest wave speeds for cooperative system, *Math. Biosci.*, **196** (2005), 82–98.

[25] W.-T. Li, G. Lin and S. Ruan, Existence of traveling wave solutions in delayed reaction diffusion systems with applications to diffusion competition system, *Nonlinearity*, **19** (2006), 1253–1273.

[26] W.-T. Li, L, Zhang and G.-B. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1531–1560.

[27] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), 1–40.
[28] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *J. Functional Analysis*, **259** (2012), 857–903.
[29] R. Lui, Biological growth and spread modeled by system of recursions. I. Mathematical theory, *Math. Biosci.*,0 **93** (1989) 269–295.
[30] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Appl.*, **92** (2009), 232–262.
[31] J. Nolen, M. Rudd and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dyn. Partial Differ. Equ.*, **2** (2005), 1–24.
[32] J. Nolen and J. Xin, Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle, *Discrete Contin. Dyn. Syst.*, **13** (2005), 1217–1234.
[33] S. Pan, W.-T. Li and G. Lin, Travelling wave fronts in nonlocal delayed reaction-diffsion systems and applications, *Z. Angew. Math. Phys.*, **60** (2009), 377–392.
[34] S. Pan, W.-T. Li and G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, *Nonliner Anal. Theory, Methods, Applications*, **72** (2010), 3150–3158.
[35] S. Pan and G. Lin, Invasion traveling wave solutions of a competition system with dispersal, *Boundary Value Problems* 2012, 2012:120.
[36] A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, New York, Berlin Heidelberg Tokyo, 1983.
[37] Nar Rawal and W. Shen, Criteria for the existence and lower bounds of principal eigenvalues of time periodic nonlocal dispersal operators and applications, *J. Dynamics and Differential Equations*, **24** (2012), 927–954.
[38] N. Rawal, W. Shen and A. Zhang, Spreading speeds and traveling waves of nonlocal monostable equations in time and space periodic habitats, *Discrete and Continuous Dynamical System A*, **35** (2015), 1609–1640.
[39] W. Shen, Stability of transition waves and positive entire solutions of Fisher-KPP equations with time and space dependence, *Nonlinearity*, **30** (2017), 3466–3491.
[40] W. Shen and A. Zhang, Traveling wave solutions of monostable equations with nonlocal dispersal in space periodic habitats, *Communications on Applied Nonlinear Analysis*, **19** (2012), 73–101.
[41] W. Shen and A. Zhang, Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitates, *J. Differential Equations*, **249** (2010), 747–795.
[42] W. Shen and A. Zhang, Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats, *Proc. Amer. Math. Soc.*, **140** (2012), 1681–1696.
[43] X. Yu and X.-Q. Zhao, Propagation phenomena for a reaction advection diffusion competition model in a periodic habitat, *J. Dynam. Differential Equations*, **29** (2017), 41–66.
[44] H. F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.*, **13** (1982), 353–396.
[45] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Biol.*, **45** (2002), 511–548.
[46] H. F. Weinberger, M. Lewis and B. Li, Analysis of linear determinacy for speed in cooperative models, *J. Math. Biol.*, **45** (2002), 183–218.
[47] X.-Q. Zhao, *Dynamical Systems in Population Biology*, CMS Books in Mathematics, Ser. 16, Springer-Verlag, NewYork, 2003.
[48] G. Zhao and S. Ruan, Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka-Volterra competition system with diffusion, *J. Math. Pures Appl.*, **95** (2011), 627–671.
[49] G. Zhao and S. Ruan, Time periodic traveling wave solutions for periodic advection reaction diffusion systems, *J. Differential Equations*, **257** (2014), 1078–1147.

Received January 2018; revised January 2017.

E-mail address: baoxx2016@chd.edu.cn