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An improved Combes–Thomas estimate of magnetic Schrödinger operators

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Abstract. In the present paper, we prove an improved Combes–Thomas estimate, viz. the Combes–Thomas estimate in trace-class norms, for magnetic Schrödinger operators under general assumptions. In particular, we allow for unbounded potentials. We also show that for any function in the Schwartz space on the reals the operator kernel decays, in trace-class norms, faster than any polynomial.

1. Introduction

The present paper is concerned with the so-called Combes–Thomas estimate of the following Schrödinger operator with magnetic field

$$H_\Lambda(A,V)=\tfrac{1}{2}(-i\nabla-A(x))^2+V(x) \quad \text{on } \Lambda, \tag{1.1}$$

where i is the imaginary unit, $\nabla=(\partial_{x_1},\partial_{x_2},...,\partial_{x_d})$ is the gradient, A is the vector potential giving rise to the magnetic field $\nabla\times A$, V is the electric potential and $\Lambda\subset\mathbb{R}^d$ is the configuration space with dimension d. This operator is used to characterize a spinless particle subject to a scalar potential and a magnetic field in nonrelativistic quantum physics (see [21], [22] and [46]).

As is known, the Combes–Thomas estimate plays an important role in the theory of Schrödinger operators, magnetic Schrödinger operators, classical wave operators, etc. in random media. It was invented by Combes and Thomas [11] to study the asymptotic behavior of eigenfunctions for multiparticle Schrödinger operators. Later, Fröhlich and Spencer [20] used it to study the localization for the multidimensional discrete Anderson model. Meanwhile, the Combes–Thomas estimate, as well as Wegner estimate [48] and Lifshitz tail [37], became important ingredients in multiscale analysis. Specifically, the initial scale estimate in multiscale analysis for localization near the bottom of the spectrum is successful because of the Combes–Thomas estimate. See [1], [5], [9], [15], [17], [18], [23], [24], [25], [29], [31], [32], [33], [41], [44] and references therein for further applications. Moreover, a stronger version of the Combes–Thomas estimate, viz. the estimate in trace-class norms, is also very useful. In [10] and [30], such estimates have been applied to study the regularity of the integrated density of states, a concept of great physical significance [38]. There are also applications in quantum statistical mechanics (see, e.g., [12]). See [3] and [34] for other applications.

Since the pioneering work of Combes and Thomas [11], the Combes–Thomas estimate in operator norm has been well studied (see [1], [17], [18], [32], [41], [44] and references therein). We point out the work of Germinet and Klein [24]. They proved a Combes–Thomas estimate, in operator norm, with explicit bounds for general Schrödinger operators including the Schrödinger operator, the magnetic Schrödinger operator, the acoustic operator, the Maxwell operator and so on. For the Combes–Thomas estimate in trace-class norms, existing results are scattered throughout the literature (see e.g. [6], [9], [10] and [34]) and most of them were proven (for special purposes), more or less, under additional assumptions. For instance, Klopp proved in [34] the estimate for Schrödinger operators with bounded potentials. There are also related work in [35] and [36]. Barbaroux, Combes and Hislop's result, proven in [3] with an open spectrum gap assumption, works for a broad class of magnetic Schrödinger operators, but was only proven for infinite-volume operators. Therefore, it is expected that one can obtain unified results for both finite-volume and infinite-volume magnetic Schrödinger operators under general assumptions.

The main goal of the current paper is to obtain the Combes–Thomas estimate of (1.1) and the associated operator kernel estimate in trace-class norms under general assumptions, which allow for unbounded potentials. We first prove an improved Combes–Thomas estimate, viz. the Combes–Thomas estimate in trace-class norms, for the magnetic Schrödinger operator (1.1) under general assumptions. Based on the improved Combes–Thomas estimate, we then show that for any function in the Schwartz space on the reals the operator kernel decays, in trace-class norms, faster than any polynomial.

To be more specific, we assume that the magnetic vector potential $A\in\mathcal{H}_{\rm loc}(\mathbb{R}^d)$ is $\mathbb{R}^d$-valued, the electric potential $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ is real-valued and the dimension $d\geq 2$. The notation $\mathcal{H}_{\rm loc}(\mathbb{R}^d)$ and $\mathcal{K}_\pm(\mathbb{R}^d)$ for spaces are explained in Section 2. Let $\Lambda\subset\mathbb{R}^d$ be an open set. We assume that Λ is bounded with sufficiently smooth boundary if it is not the whole space. The self-adjoint realization of $H_\Lambda(A,V)$ on $L^2(\Lambda)$ is still denoted by $H_\Lambda(A,V)$. If $\Lambda\neq\mathbb{R}^d$, then $H_\Lambda(A,V)$ is nothing but the localized operator with homogeneous Dirichlet boundary on $\partial\Lambda$. These self-adjoint operators are constructed via sesquilinear forms. In Section 3, we will recall the constructions done in [7].

Our first purpose is to study the Combes–Thomas estimate in trace class norms, i.e., the trace ideal estimate of the operators

$$\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma, \quad \beta,\gamma\in\mathbb{R}^d,$$

where χ_β is the characteristic function of the unit cube centered at $\beta\in\mathbb{R}^d$ and $z\in\rho(H_\Lambda(A,V))$, the resolvent set of $H_\Lambda(A,V)$. More precisely, we want to obtain the exponential decay of $\|\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma\|_{\mathcal{J}_p}$ in terms of $|\beta-\gamma|$ for suitable n and p, where $\|\cdot\|_{\mathcal{J}_p}$ is the pth von Neumann–Schatten norm reviewed in Section 2. Following the definition in [24], the family of operators

$$\{\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma\}_{\beta,\gamma\in\mathbb{R}^d}$$

is also called the operator kernel of the bounded operator $(H_\Lambda(A,V)-z)^{-n}$. In general, if f is a bounded Borel function on $\sigma(H_\Lambda(A,V))$, the spectrum of $H_\Lambda(A,V)$, then the family $\{\chi_\beta f(H_\Lambda(A,V))\chi_\gamma\}_{\beta,\gamma\in\mathbb{R}^d}$ is called the operator kernel of the bounded linear operator $f(H_\Lambda(A,V))$. Our first main result regarding the Combes–Thomas estimate is roughly stated as follows (see Theorems 4.6 and 4.7 for details).

Theorem 1.1. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $p>d/2n$ with $n\in\mathbb{N}$ and $n\geq 1$. For any $z\in\rho(H_\Lambda(A,V))$, the resolvent set of $H_\Lambda(A,V)$, there exist constants $C=C(p,z,n)>0$ and $a_0=a_0(z)>0$ such that*

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma\|_{\mathcal{J}_p}\leq Ce^{-a_0|\beta-\gamma|} \quad \textit{for all } \beta,\gamma\in\mathbb{R}^d.$$

In this paper, we also study operator kernel estimates in trace-class norms. That is, we prove the polynomial decay of the operators

$$\chi_\beta f(H_\Lambda(A,V))\chi_\gamma, \quad \beta,\gamma\in\mathbb{R}^d,$$

in trace-class norms in terms of $|\beta-\gamma|$, where f belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ reviewed in Section 2. The main result related to operator kernel estimates is roughly stated as follows (see Theorem 5.2 for details).

Theorem 1.2. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $p>d/2$. Then, for any $f\in\mathcal{S}(\mathbb{R})$ and any $k\in\mathbb{N}$, there exists a constant $C=C(p,k,f)>0$ such that*

$$\|\chi_\beta f(H_\Lambda(A,V))\chi_\gamma\|_{\mathcal{J}_p}\leq C|\beta-\gamma|^{-k} \quad \textit{for all } \beta,\gamma\in\mathbb{R}^d. \tag{1.2}$$

Estimates like (1.2), with A being $\mathbb{Z}^d$-periodic, V being bounded and f being a smooth function with compact support, have been used, as a technical tool, to study the regularity of integrated density of states. For instance, Combes, Hislop and Klopp [10, equation (2.30)] utilize the polynomial decay of any order to prove the convergence of some series, which leads to an expected estimate. It should be pointed out that Germinet and Klein proved in [24], for slowly decreasing smooth functions (see Appendix B for the definition), that the operator kernels, for general Schrödinger operators, decay, in the operator norm, faster than any polynomial. Their result was then used as a crucial ingredient in their following paper [25]. Later, sub-exponential decay for functions in Gevrey classes and exponential decay for real-analytic functions were obtained in [4] by Bouclet, Germinet and Klein.

The rest of the paper is organized as follows. In Section 2, we collect the notation used in this paper. In Section 3, we study trace ideal estimates of operators of the form $gf(H_\Lambda(A,V))$ for suitable f and g. Such estimates, with g being characteristic functions of unit cubes and f being integer powers of the resolvent of $H_\Lambda(A,V)$, are used as technical tools in the proof of Theorem 1.1. Section 4 is devoted to the study of the Combes–Thomas estimate in trace-class norms. That is, we prove Theorem 1.1. In Section 5, we study the operator kernel estimates in trace-class norms and prove Theorem 1.2.

2. Standing notation

In this section, we collect the notation which will be used in the sequel.

The configuration space Λ is an open set of $\mathbb{R}^d$. We assume that Λ is bounded with sufficiently smooth boundary unless it is the whole space. We also assume that the dimension $d\geq 2$ since, by gauge transform, vector potentials in one spatial dimension are of no physical interest.

We denote by χ_β the characteristic function of the unit cube centered at $\beta\in\mathbb{R}^d$. If the configuration space in question is Λ ($\neq\mathbb{R}^d$), then χ_β should be understood as $\chi_\beta\chi_\Lambda$, where χ_Λ is the characteristic function of Λ. Generally speaking, if a function is defined on Λ, then we consider it as a function defined on $\mathbb{R}^d$ by zero extension on $\mathbb{R}^d\backslash\Lambda$.

The Banach space of pth Lebesgue integrable functions on Λ is

$$L^p(\Lambda)=\{\phi \text{ measurable on } \Lambda \mid \|\phi\|_p<\infty\},$$

where $\|\phi\|_p = (\int_\Lambda |\phi(x)|^p\,dx)^{1/p}$, if $p\in[1,\infty)$, and $\|\phi\|_\infty = \operatorname{ess\,sup}_{x\in\Lambda}|\phi(x)|$. When $p=2$, $L^2(\Lambda)$ is a Hilbert space with inner product

$$\langle\phi,\psi\rangle = \int_\Lambda \bar{\phi}(x)\psi(x)\,dx.$$

Moreover, $\|\phi\|_2 = \sqrt{\langle\phi,\phi\rangle}$. As a convention, we simply write $\|\cdot\|_2$ as $\|\cdot\|$.

If $L\colon L^p(\Lambda)\to L^q(\Lambda)$ is a bounded linear operator, the operator norm is defined by

$$\|L\|_{p,q} := \sup_{\|\phi\|_p=1} \|L\phi\|_q.$$

If $p=q=2$, we simply write $\|\cdot\|_{2,2}$ as $\|\cdot\|$.

Although we use the same notation $\|\cdot\|$ for both the norm of a function in $L^2(\Lambda)$ and the norm of an operator on $L^2(\Lambda)$, it should not give rise to any confusion. Similarly, we do not distinguish the notation for norms corresponding to different configuration spaces.

For any $p\in[1,\infty)$, the Banach space $\mathcal{J}_p$ (also an operator ideal) is defined by

$$\mathcal{J}_p = \{C\colon L^2(\Lambda)\to L^2(\Lambda) \text{ linear and bounded} \mid \|C\|_{\mathcal{J}_p} < \infty\},$$

where $\|C\|_{\mathcal{J}_p} = (\operatorname{Tr}|C|^p)^{1/p} < \infty$ is the pth von Neumann–Schatten norm of C. See [39] and [42] for more details. We here single out the space $\mathcal{J}_2$ (also called the space of Hilbert–Schmidt operators) for the following important property (see [40, Theorem VI.23]): A bounded linear operator K on $L^2(\Lambda)$ belongs to $\mathcal{J}_2$ if and only if it is an integral operator with some integral kernel $k(x,y)$ in $L^2(\Lambda\times\Lambda)$. In this case, $\|K\|_{\mathcal{J}_2} = (\int_{\Lambda\times\Lambda}|k(x,y)|^2\,dx\,dy)^{1/2}$. We will use this property in Section 3.

Let $g(x) = -\log|x|$ if $d=2$ and $g(x)=|x|^{2-d}$ if $d\geq 3$. We say that a function $V\in\mathcal{K}(\mathbb{R}^d)$ is in the *Kato class* if

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq\varepsilon} g(x-y)|V(y)|\,dy = 0.$$

A function V is said to be in the *local Kato class* $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ if $V\chi_K\in\mathcal{K}(\mathbb{R}^d)$ for all compact sets $K\subset\mathbb{R}^d$, where χ_K is the characteristic function of K. We refer to [45] for equivalent definitions from the viewpoint of probability theory.

Let V defined on $\mathbb{R}^d$ be real-valued. We say that V is *Kato decomposable*, in symbols $V\in\mathcal{K}_\pm(\mathbb{R}^d)$, if the positive part V_+ is in $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ and the negative part V_- is in $\mathcal{K}(\mathbb{R}^d)$.

A $\mathbb{C}^d$-valued function A is said to be in the class $\mathcal{H}(\mathbb{R}^d)$ if its squared norm $A\cdot A$ and its divergence $\nabla\cdot A$, considered as a distribution on $C_0^\infty(\mathbb{R}^d)$, are both in the Kato class $\mathcal{K}(\mathbb{R}^d)$. It is said to be in the class $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ if both $A\cdot A$ and $\nabla\cdot A$

are in the local Kato class $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$. We refer the reader to [2], [7], [8] and [13] for further remarks about these spaces.

The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of those $C^\infty(\mathbb{R})$ functions which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. More precisely, for any $N\in\mathbb{Z}$, $N\geq 0$, and any $r\in\mathbb{Z}$, $r\geq 0$, we define for $f\in C^\infty(\mathbb{R})$

$$\|f\|_{N,r}=\sup_{x\in\mathbb{R}}(1+|x|)^N|f^{(r)}(x)|.$$

Then

$$\mathcal{S}(\mathbb{R})=\{f\in C^\infty(\mathbb{R})\,|\,\|f\|_{N,r}<\infty \text{ for all } N \text{ and } r\}.$$

See Folland [19] for more discussions about the Schwartz space.

3. Semigroup and trace ideal estimates

In this section, as a preparation for proving Theorems 1.1 and 1.2, we study estimates of operators of the form $gf(H_\Lambda(A,V))$ in trace-class norms for suitable f and g.

The self-adjoint realization of $H_\Lambda(A,V)$ on $L^2(\Lambda)$, still denoted by $H_\Lambda(A,V)$, is defined via sesquilinear forms as follows (see [7]): the sesquilinear form

$$h_\Lambda^{A,V_+}: C_0^\infty(\Lambda)\times C_0^\infty(\Lambda)\longrightarrow\mathbb{C},$$
$$(\psi,\phi)\longmapsto h_\Lambda^{A,V_+}(\psi,\phi),$$

where

$$h_\Lambda^{A,V_+}(\psi,\phi)=\big\langle\sqrt{V_+}\psi,\sqrt{V_+}\phi\big\rangle+\frac{1}{2}\sum_{j=1}^d\langle(-i\partial_j-A_j)\psi,(-i\partial_j-A_j)\phi\rangle,$$

is densely defined in $L^2(\Lambda)$, nonnegative and closable, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L^2(\Lambda)$. Its closure is still denoted by h_Λ^{A,V_+} with form domain $\mathcal{Q}(h_\Lambda^{A,V_+})$, which is the completion of $C_0^\infty(\Lambda)$ with respect to the norm

$$\|\phi\|_{h_\Lambda^{A,V_+}}=\sqrt{\|\phi\|^2+h_\Lambda^{A,V_+}(\phi,\phi)},$$

where $\|\cdot\|=\|\cdot\|_2$ is the norm on $L^2(\Lambda)$ associated with $\langle\cdot,\cdot\rangle$ as mentioned in Section 2. We denote by $H_\Lambda(A,V_+)$ the associated self-adjoint operator. Since $V_-\in\mathcal{K}(\mathbb{R}^d)$ is infinitesimally form-bounded with respect to $H_\Lambda(A,0)$ ($\leq H_\Lambda(A,V_+)$), i.e., there exist $\Theta_1\in(0,1)$ (which can be taken to be arbitrarily small) and $\Theta_2\geq 0$ depending on Θ_1 so that

$$\langle\phi,V_-\phi\rangle\leq\Theta_1 h_\Lambda^{A,0}(\phi,\phi)+\Theta_2\|\phi\|^2,\quad \phi\in\mathcal{Q}(h_\Lambda^{A,0}). \tag{3.1}$$

The KLMN (Kato–Lax–Lions–Milgram–Nelson) theorem (see [40, Theorem X.17]) yields that, with $\mathcal{Q}(h_\Lambda^{A,V})=\mathcal{Q}(h_\Lambda^{A,V_+})$, the sesquilinear form

$$\begin{aligned} h_\Lambda^{A,V}: \mathcal{Q}(h_\Lambda^{A,V})\times\mathcal{Q}(h_\Lambda^{A,V}) &\longrightarrow \mathbb{C}, \\ (\psi,\phi) &\longmapsto h_\Lambda^{A,V}(\psi,\phi), \end{aligned} \tag{3.2}$$

where

$$h_\Lambda^{A,V}(\psi,\phi)=h_\Lambda^{A,V_+}(\psi,\phi)-\langle\sqrt{V_-}\psi,\sqrt{V_-}\phi\rangle,$$

is closed and bounded from below and has $C_0^\infty(\Lambda)$ as a form core. The associated semibounded selfadjoint operator is denoted by $H_\Lambda(A,V)$.

The main result of this section is stated as follows. Let

$$E_0=\text{the infimum of the } L^2(\mathbb{R}^d)\text{-spectrum of } H_{\mathbb{R}^d}(0,V). \tag{3.3}$$

Theorem 3.1. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose $p\geq 2$. Let f be a Borel function satisfying*

$$|f(\lambda)|\leq C(1+|\lambda|)^{-\alpha}, \quad \lambda\in\sigma(H_\Lambda(A,V)), \tag{3.4}$$

for $\alpha>d/2p$. Then $gf(H_\Lambda(A,V))$ is in $\mathcal{J}_p$ with

$$\|gf(H_\Lambda(A,V))\|_{\mathcal{J}_p}\leq C_{\alpha,p,\lambda_0}\|g\|_p\|(H_\Lambda(A,V)-\lambda_0)^\alpha f(H_\Lambda(A,V))\|$$

whenever $g\in L^p(\Lambda)$, where $\lambda_0<E_0$ and $C_{\alpha,p,\lambda_0}>0$ depends only on α, p and λ_0.

To prove the above theorem, we first present some lemmas. We begin with the celebrated Feynman–Kac–Itô formula proven by Broderix, Hundertmark and Leschke (see [28], [43], [45] and references therein for earlier versions).

Lemma 3.2. ([7]) *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. For any $\phi\in L^2(\Lambda)$ and $t\geq 0$, we have*

$$(e^{-tH_\Lambda(A,V)}\phi)(x)=\mathbb{E}_x\{e^{-S_t^\omega(A,V)}\Xi_{\Lambda,t}(\omega)\phi(\omega(t))\} \quad \text{for a.e. } x\in\Lambda,$$

where

$$S_t^\omega(A,V)=i\int_0^t A(\omega(s))\,d\omega(s)+\frac{i}{2}\int_0^t(\nabla\cdot A)(\omega(s))\,ds+\int_0^t V(\omega(s))\,ds,$$

$\mathbb{E}_x\{\cdot\}$ denotes the expectation for the Brownian motion starting at x and $\Xi_{\Lambda,t}$ is the characteristic function of the set $\{\omega|\omega(s)\in\Lambda$ for all $s\in[0,t]\}$.

As consequences of Lemma 3.2, we get the so-called *diamagnetic inequality*

$$|e^{-tH_\Lambda(A,V)}\phi| \le e^{-tH_\Lambda(0,V)}|\phi|, \quad t \ge 0,$$

the monotonicity of the semigroup for vanishing magnetic field in the sense that for $\Lambda \subset \Lambda'$,

$$e^{-tH_\Lambda(0,V)}\chi_\Lambda \phi \le e^{-tH_{\Lambda'}(0,V)}\phi, \quad \phi \ge 0 \text{ and } t \ge 0,$$

and then the L^p-smoothing of semigroups: For $1 \le p \le q \le \infty$, there exist a constant $C>0$ and E such that

$$\|e^{-tH_\Lambda(A,V)}\|_{p,q} \le \|e^{-tH_\Lambda(0,V)}\|_{p,q} \le \|e^{-tH_{\mathbb{R}^d}(0,V)}\|_{p,q} \le Ct^{-\gamma}e^{Et}, \tag{3.5}$$

where $\gamma = \frac{1}{2}d(1/p-1/q)$. We remark that E can be chosen such that $-E<E_0$ (see, e.g., [7] and [41]).

We extend [41, Theorem B.2.1] to the magnetic case.

Lemma 3.3. *Let $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ be open. Let $\alpha>0$ and $1 \le p \le q \le \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{d}. \tag{3.6}$$

Then $(H_\Lambda(A,V)-z)^{-\alpha}$ is bounded from $L^p(\Lambda)$ to $L^q(\Lambda)$ whenever $\operatorname{Re} z < E_0$.

Proof. This follows from the formula

$$(H_\Lambda(A,V)-z)^{-\alpha} = c_\alpha \int_0^\infty e^{-tH_\Lambda(A,V)}e^{tz}t^{\alpha-1}\,dt$$

and (3.5), where the assumption (3.6) is applied to ensure the convergence of the above integral. □

As a consequence of Lemma 3.3, we have the following result.

Lemma 3.4. *Let $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ be open. Let $\alpha>0$ and $1 \le p \le 2 \le q \le \infty$ satisfy (3.6). For any Borel function f satisfying (3.4), the operator $f(H_\Lambda(A,V))$ is bounded from $L^p(\Lambda)$ to $L^q(\Lambda)$ with*

$$\|f(H_\Lambda(A,V))\|_{p,q} \le C_{p,q,\alpha,\lambda_0}\|(H_\Lambda(A,V)-\lambda_0)^\alpha f(H_\Lambda(A,V))\|,$$

where $\lambda_0 < E_0$ and $C_{p,q,\alpha,\lambda_0}>0$ depends only on p, q, α and λ_0.

Proof. This follows from the arguments in [41, Theorem B.2.3]. □

We next discuss the trace ideal estimate of operators of the form $gf(H_\Lambda(A,V))$ for suitable f and g. We start with recalling a result of Dunford and Pettis (see [13], [41] and [47] for abstract versions).

Lemma 3.5. *Let (M,μ) be a separable measurable space. If L is a bounded linear operator from $L^p(M)$ to $L^\infty(M)$ with $1\le p<\infty$, then there is a measurable function $k(\cdot,\cdot)$ on $M\times M$ such that L is an integral operator with integral kernel $k(\cdot,\cdot)$ and*

$$\sup_{x\in M}\left(\int_M |k(x,y)|^{p'}\,d\mu(y)\right)^{1/p'} = \|L\|_{p,\infty} < \infty,$$

where $p'=p/(p-1)$ is the conjugate exponent of p.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By complex interpolation (see [42, Theorem 2.9]), it suffices to prove the result in the case $p=2$, which we show now. For $p=2$ and $q=\infty$, we have $\frac{1}{2}d(1/p-1/q)=\frac{1}{4}d<\alpha$ by assumption, i.e., (3.6) is satisfied, and thus, Lemma 3.4 implies that $f(H_\Lambda(A,V))$ is bounded from $L^2(\Lambda)$ to $L^\infty(\Lambda)$. By Lemma 3.5, $f(H_\Lambda(A,V))$ is an integral operator with kernel $k_\Lambda^{A,V}(x,y)$ satisfying

$$\sup_{x\in\Lambda}\int_\Lambda |k_\Lambda^{A,V}(x,y)|^2\,dy = \|f(H_\Lambda(A,V))\|_{2,\infty}^2 < \infty.$$

Thus, $gf(H_\Lambda(A,V))$ is an integral operator on $L^2(\Lambda)$ with kernel $g(x)k_\Lambda^{A,V}(x,y)$. Moreover,

$$\begin{aligned}\iint_{\Lambda\times\Lambda} |g(x)k_\Lambda^{A,V}(x,y)|^2\,dx\,dy &\le \|g\|_2^2 \sup_{x\in\Lambda}\int_\Lambda |k_\Lambda^{A,V}(x,y)|^2\,dy\\ &= \|g\|_2^2\|f(H_\Lambda(A,V))\|_{2,\infty}^2,\end{aligned}$$

which implies that $gf(H_\Lambda(A,V))$ is a Hilbert–Schmidt operator as mentioned in Section 2, i.e., in $\mathcal{J}_2$, with $\mathcal{J}_2$-norm bounded by $\|g\|_2\|f(H_\Lambda(A,V))\|_{2,\infty}$. The expected bound is given by Lemma 3.4. This completes the proof. □

We remark that results obtained in this section are well known for Schrödinger operators without magnetic fields. See [2], [41] and references therein. It should be pointed out that the result of Theorem 3.1 in the case $H_{\mathbb{R}^d}(0,V)$ was proven in [41, Theorem B.9.3] for any $p\ge 1$. To prove the result for $p\in[1,2)$, it was first shown that $gf(H_{\mathbb{R}^d}(0,V))\in\mathcal{J}_1$ for $g\in\ell^1(L^2(\mathbb{R}^d))$, the Birman–Solomyak space, and then complex interpolation was used. The proof relies on the translation invariance of the free Laplacian (see [41, Theorem B.9.2] and [42, Theorem 4.5] for instance), which, however, is not true for magnetic Schrödinger operators. This prevents us from obtaining the result for $p\in[1,2)$.

4. The Combes–Thomas estimate in trace ideals

In this section, we study the improved Combes–Thomas estimate, i.e., the trace ideal estimate of the operators

$$\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma \quad \text{for } \beta,\gamma\in\mathbb{R}^d,$$

where χ_β is the characteristic function of the unit cube centered at β. More precisely, we want to obtain the exponential decay of $\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}$ in terms of $|\beta-\gamma|$. The main result is stated in Theorem 1.1. Since we also consider localized operator, χ_β should be understood as $\chi_\beta\chi_\Lambda$ if the operator is restricted to Λ as is mentioned in Section 2, where χ_Λ is the characteristic function of the domain Λ. The basic tools we use here are sectorial forms and m-sectorial operators, which are reviewed in Appendix A. We also employ the classical argument of Combes and Thomas developed in [11].

First of all, we establish some results by applying the theory of sectorial forms and m-sectorial operators. For this purpose, we first define auxiliary sesquilinear forms with associated operators formally given by

$$H^a_\Lambda(A,V)=e^{a\cdot x}H_\Lambda(A,V)e^{-a\cdot x},\quad a\in\mathbb{R}^d, \tag{4.1}$$

where $e^{a\cdot x}$ and $e^{-a\cdot x}$ are multiplicative operators. Note that the operator $H^a_\Lambda(A,V)$ is not self-adjoint unless $a=0$. First, we denote by $D_{A,\Lambda}$ the closure of the operator $\frac{1}{2}\sqrt{2}(-i\nabla-A)$ on $C_0^\infty(\Lambda)$, so $H_\Lambda(A,0)=D^*_{A,\Lambda}D_{A,\Lambda}$. This can be seen by sesquilinear forms. Moreover, the domain of $D_{A,\Lambda}$, denoted by $\mathcal{D}(D_{A,\Lambda})$, is the form domain, denoted by $\mathcal{Q}(h^{A,0}_\Lambda)$, of the sesquilinear form associated with the lower bounded self-adjoint operator $H_\Lambda(A,0)$. For $a\in\Lambda$, we define

$$D_{A,\Lambda}(a)=e^{a\cdot x}D_{A,\Lambda}e^{-a\cdot x}\quad\text{and}\quad D^*_{A,\Lambda}(a)=e^{a\cdot x}D^*_{A,\Lambda}e^{-a\cdot x}.$$

It is easy to see that

$$\begin{aligned} D_{A,\Lambda}(a)&=D_{A,\Lambda}+i\frac{\sqrt{2}}{2}a\quad\text{on }\mathcal{D}(D_{A,\Lambda}),\\ D^*_{A,\Lambda}(a)&=D^*_{A,\Lambda}+i\frac{\sqrt{2}}{2}a\quad\text{on }\mathcal{D}(D^*_{A,\Lambda}) \end{aligned} \tag{4.2}$$

and that they are closed, densely defined operators. Note $(D_{A,\Lambda}(a))^* \neq D^*_{A,\Lambda}(a)$ for $a \neq 0$. Next, we define the sesquilinear form $h^{A,0}_\Lambda(a)$ on $\mathcal{D}(D_{A,\Lambda}) = \mathcal{Q}(h^{A,0}_\Lambda)$ by

$$h^{A,0}_\Lambda(a)(\psi,\phi) = \langle (D^*_{A,\Lambda}(a))^*\psi, D_{A,\Lambda}(a)\phi\rangle. \tag{4.3}$$

We obviously have $h^{A,0}_\Lambda(0) \equiv h^{A,0}_\Lambda$. Finally, we define the sesquilinear form $h^{A,V}_\Lambda(a)$ on $\mathcal{Q}(h^{A,V_+}_\Lambda)$ by

$$h^{A,V}_\Lambda(a)(\psi,\phi) = h^{A,0}_\Lambda(a)(\psi,\phi) + \big\langle \sqrt{V_+}\psi, \sqrt{V_+}\phi\big\rangle - \big\langle \sqrt{V_-}\psi, \sqrt{V_-}\phi\big\rangle. \tag{4.4}$$

For $a_0>0$, let

$$\Xi_1(s) = \frac{2s}{1-\Theta_1} \quad \text{and} \quad \Xi_2(s,a_0) = \frac{2s\Theta_2}{1-\Theta_1} + \left(\frac{1}{2s}+\frac{s}{4}\right)a_0^2, \tag{4.5}$$

where Θ_1 and Θ_2 are given in (3.1). We will write $\Xi_1(s)$ and $\Xi_2(s,a_0)$ as Ξ_1 and Ξ_2, respectively, in the sequel.

We next prove several lemmas related to $H^a_\Lambda(A,V)$. Our first lemma is about the relation between $h^{A,V}_\Lambda(a)$ and $H^a_\Lambda(A,V)$.

Lemma 4.1. *Let $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ be open. The sesquilinear form $h^{A,V}_\Lambda(a)$ defined in (4.4) is a closed sectorial form associated with the unique m-sectorial operator $H^a_\Lambda(A,V)$ given by (4.1).*

Proof. By (3.2), (4.2), (4.3) and (4.4), we have for any $\phi \in \mathcal{Q}(h^{A,V}_\Lambda)$,

$$\begin{aligned} |h^{A,V}_\Lambda(a)(\phi,\phi) - h^{A,V}_\Lambda(\phi,\phi)| &= |h^{A,0}_\Lambda(a)(\phi,\phi) - h^{A,0}_\Lambda(\phi,\phi)| \\ &\le \sqrt{2}|\mathrm{Re}\langle \phi, a\cdot D_{A,\Lambda}\phi\rangle| + \tfrac12 |a|^2\|\phi\|^2 \end{aligned}$$

so that

$$|h^{A,V}_\Lambda(a)(\phi,\phi) - h^{A,V}_\Lambda(\phi,\phi)|^2 \le 4|a|^2\|\phi\|^2\|D_{A,\Lambda}\phi\|^2 + \tfrac12|a|^4\|\phi\|^4,$$

which implies that for any $s>0$,

$$\begin{aligned} |h^{A,V}_\Lambda(a)(\phi,\phi) - h^{A,V}_\Lambda(\phi,\phi)| &\le |a|\,\|\phi\|\left(4\|D_{A,\Lambda}\phi\|^2 + \frac12|a|^2\|\phi\|^2\right)^{1/2} \\ &\le \frac{1}{2s}|a|^2\|\phi\|^2 + \frac{s}{2}\left(4\|D_{A,\Lambda}\phi\|^2 + \frac12 |a|^2\|\phi\|^2\right) \\ &= 2sh^{A,0}_\Lambda(\phi,\phi) + \left(\frac{1}{2s}+\frac{s}{4}\right)|a|^2\|\phi\|^2, \end{aligned} \tag{4.6}$$

since $h_\Lambda^{A,0}(\phi,\phi)=\|D_{A,\Lambda}\phi\|^2$. Due to (3.1) and (3.2),

$$h_\Lambda^{A,V}\geq(1-\Theta_1)h_\Lambda^{A,0}-\Theta_2\quad\text{on } \mathcal{Q}(h_\Lambda^{A,V})(\subset\mathcal{Q}(h_\Lambda^{A,0})).$$

This, together with (4.6), implies that

$$|h_\Lambda^{A,V}(a)(\phi,\phi)-h_\Lambda^{A,V}(\phi,\phi)|\leq\Xi_1 h_\Lambda^{A,V}(\phi,\phi)+\Xi_2\|\phi\|^2,\quad \phi\in\mathcal{Q}(h_\Lambda^{A,V}),\tag{4.7}$$

where Ξ_1 and Ξ_2 are given in (4.5) with a_0 replaced by $|a|$.

To apply Theorem A.1, we choose $s\in\big(0,\frac{1}{2}(1-\Theta_1)\big)$ so that $\Xi_1=2s/(1-\Theta_1)<1$. Since $h_\Lambda^{A,V}$ is symmetric, closed and bounded from below, Theorem A.1 says that $h_\Lambda^{A,V}(a)$ is a closed sectorial form defined on $\mathcal{Q}(h_\Lambda^{A,V})$. Theorem A.2 then guarantees that there exists a unique m-sectorial operator, denoted by $H_\Lambda^a(A,V)$, associated with $h_\Lambda^{A,V}(a)$. □

The next lemma gives an operator equality connecting $H_\Lambda^a(A,V)$ and $H_\Lambda(A,V)$.

Lemma 4.2. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $s\in\big(0,\frac{1}{2}(1-\Theta_1)\big)$ so that $\Xi_1<1$. Set*

$$\widetilde{H}_\Lambda(A,V)=H_\Lambda(A,V)+\Xi_1^{-1}\Xi_2,\tag{4.8}$$

where Ξ_1 and Ξ_2 are given in (4.5) with a_0 replaced by $|a|$. Then $\widetilde{H}_\Lambda(A,V)$ is nonnegative and there exists a bounded linear operator B from $L^2(\Lambda)$ to itself with $\|B\|\leq 2\Xi_1$ such that

$$H_\Lambda^a(A,V)=H_\Lambda(A,V)+\sqrt{\widetilde{H}_\Lambda(A,V)}\,B\sqrt{\widetilde{H}_\Lambda(A,V)},\tag{4.9}$$

where $H_\Lambda^a(A,V)$ is the m-sectorial operator in Lemma 4.1.

Proof. Set

$$\begin{aligned}\bar{h}_\Lambda^{A,V}(a)&=h_\Lambda^{A,V}(a)-h_\Lambda^{A,V}\quad\text{on } \mathcal{Q}(h_\Lambda^{A,V}),\\ \tilde{h}_\Lambda^{A,V}&=h_\Lambda^{A,V}+\Xi_1^{-1}\Xi_2\quad\text{on } \mathcal{Q}(h_\Lambda^{A,V}).\end{aligned}\tag{4.10}$$

Then (4.7) can be rewritten as

$$|\bar{h}_\Lambda^{A,V}(a)(\phi,\phi)|\leq\Xi_1\tilde{h}_\Lambda^{A,V}(\phi,\phi),\quad \phi\in\mathcal{Q}(h_\Lambda^{A,V}),$$

which implies that $\tilde{h}_\Lambda^{A,V}$ is a densely defined, symmetric, nonnegative, closed sesquilinear form with the associated nonnegative self-adjoint operator $\widetilde{H}_\Lambda(A,V)$ defined in (4.8).

Theorem A.3 then ensures that there exists a bounded linear operator B from $L^2(\Lambda)$ to itself with $\|B\|\le 2\Xi_1$ so that

$$\bar{h}_\Lambda^{A,V}(a)(\psi,\phi)=\Big\langle\sqrt{\widetilde{H}_\Lambda(A,V)}\psi, B\sqrt{\widetilde{H}_\Lambda(A,V)}\phi\Big\rangle \tag{4.11}$$

for $\psi,\phi\in\mathcal{Q}(h_\Lambda^{A,V})=\mathcal{D}\Big(\sqrt{\widetilde{H}_\Lambda(A,V)}\Big)$. Let

$$\tilde{h}_\Lambda^{A,V}(a)=h_\Lambda^{A,V}(a)+\Xi_1^{-1}\Xi_2 \quad \text{on } \mathcal{Q}(h_\Lambda^{A,V}). \tag{4.12}$$

Since $h_\Lambda^{A,V}(a)$ is a densely defined closed sectorial form, so is $\tilde{h}_\Lambda^{A,V}(a)$ and the associated m-sectorial operator is given by

$$\widetilde{H}_\Lambda^a(A,V)=H_\Lambda^a(A,V)+\Xi_1^{-1}\Xi_2. \tag{4.13}$$

Considering (4.10) and (4.11), we also have

$$\begin{aligned}\tilde{h}_\Lambda^{A,V}(a)(\psi,\phi)&=\tilde{h}_\Lambda^{A,V}(\psi,\phi)+\bar{h}_\Lambda^{A,V}(a)(\psi,\phi)\\&=\Big\langle\sqrt{\widetilde{H}_\Lambda(A,V)}\psi,\sqrt{\widetilde{H}_\Lambda(A,V)}\phi\Big\rangle\\&\quad+\Big\langle\sqrt{\widetilde{H}_\Lambda(A,V)}\psi,B\sqrt{\widetilde{H}_\Lambda(A,V)}\phi\Big\rangle\\&=\Big\langle\sqrt{\widetilde{H}_\Lambda(A,V)}\psi,(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}\phi\Big\rangle,\quad \psi,\phi\in\mathcal{Q}(h_\Lambda^{A,V}).\end{aligned} \tag{4.14}$$

We claim that

$$\widetilde{H}_\Lambda^a(A,V)=\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}. \tag{4.15}$$

Let $\phi\in\mathcal{D}(\widetilde{H}_\Lambda^a(A,V))\subset\mathcal{Q}(\tilde{h}_\Lambda^{A,V}(a))=\mathcal{Q}(h_\Lambda^{A,V})$. We have

$$\tilde{h}_\Lambda^{A,V}(a)(\psi,\phi)=\langle\psi,\widetilde{H}_\Lambda^a(A,V)\phi\rangle \quad \text{for all } \psi\in\mathcal{Q}(\tilde{h}_\Lambda^{A,V}(a))=\mathcal{Q}(h_\Lambda^{A,V}).$$

Comparing this with (4.14) and recalling the definition of the adjoint of an operator, we see that $\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}\phi$ exists and is equal to $\widetilde{H}_\Lambda^a(A,V)\phi$, which implies that

$$\widetilde{H}_\Lambda^a(A,V)\subset\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)},$$

i.e., $\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}$ extends $\widetilde{H}_\Lambda^a(A,V)$. To show (4.15), it now suffices to show that $\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}$ is accretive since $\widetilde{H}_\Lambda^a(A,V)$ is m-sectorial, and thus has no proper accretive extension. For any

$$\psi\in\mathcal{D}\Big(\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}\Big)\subset\mathcal{Q}(h_\Lambda^{A,V}),$$

(4.14) and (4.12) give

$$\begin{aligned}\left\langle\psi, \sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}\psi\right\rangle &= \tilde{h}_\Lambda^{A,V}(a)(\psi,\psi)\\ &= h_\Lambda^{A,V}(a)(\psi,\psi)+\Xi_1^{-1}\Xi_2\|\psi\|^2\\ &= h_\Lambda^{A,V}(a)(\psi,\psi)-h_\Lambda^{A,V}(\psi,\psi)\\ &\quad +h_\Lambda^{A,V}(\psi,\psi)+\Xi_1^{-1}\Xi_2\|\psi\|^2.\end{aligned}$$

It then follows from

$$|\mathrm{Re}(h_\Lambda^{A,V}(a)(\psi,\psi)-h_\Lambda^{A,V}(\psi,\psi))|\le |h_\Lambda^{A,V}(a)(\psi,\psi)-h_\Lambda^{A,V}(\psi,\psi)|$$

and (4.7) that

$$\begin{aligned}&\mathrm{Re}\left\langle\psi, \sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}\psi\right\rangle\\ &\qquad = \mathrm{Re}(h_\Lambda^{A,V}(a)(\psi,\psi)-h_\Lambda^{A,V}(\psi,\psi))+h_\Lambda^{A,V}(\psi,\psi)+\Xi_1^{-1}\Xi_2\|\psi\|^2\\ &\qquad \ge -(\Xi_1 h_\Lambda^{A,V}(\psi,\psi)+\Xi_2\|\psi\|^2)+h_\Lambda^{A,V}(\psi,\psi)+\Xi_1^{-1}\Xi_2\|\psi\|^2\\ &\qquad = (1-\Xi_1)(h_\Lambda^{A,V}(\psi,\psi)+\Xi_1^{-1}\Xi_2\|\psi\|^2)\\ &\qquad \ge 0,\end{aligned}$$

since Ξ_1 is taken to be less than 1 and $h_\Lambda^{A,V}+\Xi_1^{-1}\Xi_2$ is nonnegative by (4.7). This shows that $\sqrt{\widetilde{H}_\Lambda(A,V)}(1+B)\sqrt{\widetilde{H}_\Lambda(A,V)}$ is accretive and, thus, (4.15) holds. Obviously, (4.9) is equivalent to (4.15) due to (4.8) and (4.13). This completes the proof. □

The last lemma bridges the resolvent set of $H_\Lambda(A,V)$ and that of $H_\Lambda^a(A,V)$. Before stating the result, we make following assumptions.

Pick and fix $\lambda_0<\min\{-\Theta_2,E_0\}$, where E_0 was defined in (3.3). This number is picked to be of technical use. The main advantage is that $H_\Lambda(A,V)-\lambda_0$ is strictly positive so that $(H_\Lambda(A,V)-\lambda_0)^{1/2}$ is well defined and boundedly invertible, as opposed to the ill-posedness of the fractional power of $H_\Lambda(A,V)-z$, which may cause some troubles.

Let

$$c_{z,\lambda_0}=\left\|\frac{\lambda-\lambda_0}{\lambda-z}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}.$$

Suppose that $s>0$ and $a_0>0$ satisfy

$$s<\frac{1-\Theta_1}{4c_{z,\lambda_0}}\quad\text{and}\quad a_0^2\le\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1} \tag{4.16}$$

or

$$\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}\le a_0^2<\left(\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}+\frac{2s(\delta\lambda_0+\Theta_2)}{\Theta_1-1}\right)\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}, \tag{4.17}$$

where $\delta=\delta(\lambda_0)\in(0,1)$ is such that $\delta\lambda_0\in(\lambda_0,\min\{-\Theta_2,E_0\})$. We will show the derivation for the above two classes for conditions in Lemma 4.5 below. We point out that

$$\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}<\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}+\frac{2s(\delta\lambda_0+\Theta_2)}{\Theta_1-1}$$

is nothing but $s<(1-\Theta_1)/4c_{z,\lambda_0}$.

Remark 4.3. Note that assumptions (4.16) and (4.17) can be considered together to form a more general one, but we consider them separately anyway for the following two reasons.

(i) The first reason is about the conditions giving rise to (4.16) and the first inequality in (4.17). In the proof of Lemma 4.4 below, we need conditions on s and a_0 to ensure

$$2\Xi_1 c_{z,\lambda_0}\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}<1,$$

i.e., (4.21), where the quantity $\|(\lambda+\Xi_1^{-1}\Xi_2)/(\lambda-\lambda_0)\|_{L^\infty(\sigma(H_\Lambda(A,V)))}$ appears. It is easy to see that

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}=\begin{cases}1, & \text{if } \Xi_1^{-1}\Xi_2\le-\lambda_0,\\ \dfrac{\inf\sigma(H_\Lambda(A,V))+\Xi_1^{-1}\Xi_2}{\inf\sigma(H_\Lambda(A,V))-\lambda_0}, & \text{if } \Xi_1^{-1}\Xi_2\ge-\lambda_0.\end{cases}$$

Moreover, the second inequality in (4.16) and the first inequality in (4.17) correspond to $\Xi_1^{-1}\Xi_2\le-\lambda_0$ and $\Xi_1^{-1}\Xi_2\ge-\lambda_0$, respectively.

(ii) The second reason is that (4.17) provides a nonzero lower bound for a_0, and in turn, an upper bound for $e^{-a_0|\beta-\gamma|}$, which is important in Section 5 because we need such an upper bound (of course after being simplified) to estimate some integrals there.

Lemma 4.4. *Let $A\in\mathcal{H}_{\rm loc}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and let $\Lambda\subset\mathbb{R}^d$ be open. Let $z\in\rho(H_\Lambda(A,V))$, the resolvent set of $H_\Lambda(A,V)$. Suppose that $s>0$ and $a\in\mathbb{R}^d$ satisfying $|a|=a_0>0$ obey (4.16) or (4.17). Then $H_\Lambda^a(A,V)-z$ is invertible, i.e., $z\in\rho(H_\Lambda^a(A,V))$, the resolvent set of $H_\Lambda^a(A,V)$. In other words, $\rho(H_\Lambda(A,V))\subset\rho(H_\Lambda^a(A,V))$.*

Proof. By (4.9), we have

$$H^a_\Lambda(A,V)-z = H_\Lambda(A,V)-z+\sqrt{\widetilde{H}_\Lambda(A,V)}B\sqrt{\widetilde{H}_\Lambda(A,V)} \\ = (H_\Lambda(A,V)-\lambda_0)^{1/2}(U+V)(H_\Lambda(A,V)-\lambda_0)^{1/2}, \tag{4.18}$$

where

$$U=(H_\Lambda(A,V)-\lambda_0)^{-1/2}(H_\Lambda(A,V)-z)(H_\Lambda(A,V)-\lambda_0)^{-1/2} \\ =(H_\Lambda(A,V)-z)(H_\Lambda(A,V)-\lambda_0)^{-1}$$

and

$$V=(H_\Lambda(A,V)-\lambda_0)^{-1/2}\sqrt{\widetilde{H}_\Lambda(A,V)}B\sqrt{\widetilde{H}_\Lambda(A,V)}(H_\Lambda(A,V)-\lambda_0)^{-1/2}.$$

Since $(H_\Lambda(A,V)-\lambda_0)^{1/2}$ is invertible, invertibility of $H^a_\Lambda(A,V)-z$ is equivalent to that of $U+V$.

We claim that $U+V$ is invertible under the assumptions of the lemma with

$$\|(U+V)^{-1}\| \le \begin{cases} \dfrac{c_{z,\lambda_0}(1-\Theta_1)}{1-\Theta_1-4sc_{z,\lambda_0}}, & \text{if } a_0 \text{ satisfies (4.16)}, \\ \dfrac{(\delta-1)\lambda_0 c_{z,\lambda_0}}{(\delta-1)\lambda_0-2(\delta\lambda_0\Xi_1+\Xi_2)c_{z,\lambda_0}}, & \text{if } a_0 \text{ satisfies (4.17)}. \end{cases} \tag{4.19}$$

Obviously, U is bounded and invertible with

$$U^{-1}=(H_\Lambda(A,V)-\lambda_0)(H_\Lambda(A,V)-z)^{-1}.$$

Recall that $\widetilde{H}_\Lambda(A,V)=H_\Lambda(A,V)+\Xi_1^{-1}\Xi_2\ge 0$. Since $\sqrt{(\lambda+\Xi_1^{-1}\Xi_2)/(\lambda-\lambda_0)}$ (as a function of λ) is bounded on $\sigma(H_\Lambda(A,V))$, both

$$(H_\Lambda(A,V)-\lambda_0)^{-1/2}\sqrt{\widetilde{H}_\Lambda(A,V)} \quad \text{and} \quad \sqrt{\widetilde{H}_\Lambda(A,V)}(H_\Lambda(A,V)-\lambda_0)^{-1/2}$$

are bounded, which implies that V is bounded. Then, by stability of bounded invertibility (see [27, Theorem IV.1.16]), it suffices to require that $\|V\|\,\|U^{-1}\|<1$. In this case, $U+V$ is invertible with

$$\|(U+V)^{-1}\| \le \frac{\|U^{-1}\|}{1-\|V\|\,\|U^{-1}\|}. \tag{4.20}$$

Since $\|U^{-1}\|\le c_{z,\lambda_0}$ and

$$\|V\| \le \|B\| \left\| \sqrt{\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}} \right\|^2_{L^\infty(\sigma(H_\Lambda(A,V)))} \le 2\Xi_1 \left\| \frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A,V)))},$$

it suffices to require that

$$2\Xi_1 c_{z,\lambda_0}\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}<1. \tag{4.21}$$

It is justified in Lemma 4.5 below that if s and a are as in the assumptions of the current lemma, then (4.21) holds, which then implies that $U+V$, and hence $H_\Lambda^a(A,V)-z$, is invertible. Finally, (4.19) follows from (4.20) and Lemma 4.5 below. □

To finish the proof of Lemma 4.4, we show the following lemma.

Lemma 4.5. *Let $z\in\rho(H_\Lambda(A,V))$. If $s>0$ and $a_0>0$ satisfy (4.16) or (4.17), then (4.21) holds. Moreover, if (4.16) is satisfied, then*

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}=1,$$

and if (4.17) is satisfied, then

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\le\frac{\delta\lambda_0+\Xi_1^{-1}\Xi_2}{(\delta-1)\lambda_0}$$

for some $\delta=\delta(\lambda_0)\in(0,1)$ satisfying $\delta\lambda_0\in(\lambda_0,\min\{-\Theta_2,E_0\})$.

Proof. Instead of proving (4.21) directly, we show how to derive (4.16) or (4.17) so that (4.21) holds. Recall that $\lambda_0<\min\{-\Theta_2,E_0\}$,

$$\Xi_1=\frac{2s}{1-\Theta_1},\quad \Xi_2=\frac{2s\Theta_2}{1-\Theta_1}+\left(\frac{1}{2s}+\frac{s}{4}\right)a_0^2\quad\text{and}\quad c_{z,\lambda_0}=\left\|\frac{\lambda-\lambda_0}{\lambda-z}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}.$$

We here discuss two classes of conditions separated by $\Xi_1^{-1}\Xi_2=-\lambda_0$.

(i) Due to the fact that $\sigma(H_\Lambda(A,V))$ contains a sequence tending to infinity, it is true that

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\ge 1.$$

So the best choice is

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}=1,$$

which holds if and only if $\Xi_1^{-1}\Xi_2\le-\lambda_0$ since $\inf\sigma(H_\Lambda(A,V))+\Xi_1^{-1}\Xi_2\ge 0$. By making a_0 small enough, the condition $\Xi_1^{-1}\Xi_2\le-\lambda_0$ is readily satisfied. Thus (4.21) reduces to

$$2\Xi_1 c_{z,\lambda_0}<1. \tag{4.22}$$

Note $\lim_{\lambda\to\infty}|(\lambda-\lambda_0)/(\lambda-z)|=1$ pointwise for $z\in\rho(H_\Lambda(A,V))$ and $\lambda_0<\min\{-\Theta_2, E_0\}$, which implies that $c_{z,\lambda_0}\geq 1$. Hence, if (4.22) holds, then automatically, $\Xi_1<\frac{1}{2}<1$.

For any fixed $z\in\rho(H_\Lambda(A,V))$ and $\lambda_0<\min\{-\Theta_2,E_0\}$, there exists s such that (4.22) is satisfied. Moreover, s cannot be chosen to be independent of z or λ_0 because of the facts that

$$\lim_{\substack{z\in\rho(H_\Lambda(A,V))\\ \mathrm{dist}(z,\sigma(H_{\Lambda(A,V)}))\to 0}} c_{z,\lambda_0}=\infty \quad \text{pointwise when } \lambda_0<\min\{-\Theta_2,E_0\}.$$

or

$$\lim_{\lambda_0\to-\infty} c_{z,\lambda_0}=\infty \quad \text{pointwise when } z\in\rho(H_\Lambda(A,V)),$$

respectively.

Explicitly, we can choose s to be any number satisfying

$$s\in\left(0,\frac{1-\Theta_1}{4c_{z,\lambda_0}}\right)\subset\left(0,\frac{1-\Theta_1}{2}\right) \tag{4.23}$$

so that (4.22) is satisfied, and thus $\Xi_1<1$ holds. Then, by requiring a_0 to satisfy

$$a_0^2\leq\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}, \tag{4.24}$$

the condition $\Xi_1^{-1}\Xi_2\leq-\lambda_0$ holds. Consequently, any pair (s,a_0) satisfying (4.23) and (4.24) guarantees (4.21).

(ii) Now, we require $\Xi_1^{-1}\Xi_2\geq-\lambda_0$. Then, $(\lambda+\Xi_1^{-1}\Xi_2)/(\lambda-\lambda_0)$, as a function of λ, is decreasing on (λ_0,∞), which implies that

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\leq\frac{\lambda_*+\Xi_1^{-1}\Xi_2}{\lambda_*-\lambda_0}\quad\text{for all } \lambda_*\in(\lambda_0,\min\{-\Theta_2,E_0\}).$$

In particular, we can take $\lambda_*=\delta\lambda_0$ for some $\delta=\delta(\lambda_0)\in(0,1)$ and obtain

$$\left\|\frac{\lambda+\Xi_1^{-1}\Xi_2}{\lambda-\lambda_0}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\leq\frac{\delta\lambda_0+\Xi_1^{-1}\Xi_2}{(\delta-1)\lambda_0}.$$

Then,

$$2\Xi_1 c_{z,\lambda_0}\frac{\delta\lambda_0+\Xi_1^{-1}\Xi_2}{(\delta-1)\lambda_0}<1,\quad\text{i.e.,}\quad \Xi_2<\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}-\delta\lambda_0\Xi_1,$$

will ensure (4.21). Moreover, considering the assumption $\Xi_1^{-1}\Xi_2\geq-\lambda_0$, we deduce that $2\Xi_1c_{z,\lambda_0}<1$, which leads to $\Xi_1<\frac{1}{2}$ since $c_{z,\lambda_0}\geq 1$. In conclusion, to ensure (4.21), we only need to require that

$$-\lambda_0\Xi_1\leq\Xi_2<\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}-\delta\lambda_0\Xi_1.$$

Explicitly, if $s>0$ and $a_0>0$ satisfy

$$\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}\le a_0^2<\left(\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}+\frac{2s(\delta\lambda_0+\Theta_2)}{\Theta_1-1}\right)\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}$$

for some $\delta=\delta(\lambda_0,\Theta_2,E_0)\in(0,1)$ such that $\delta\lambda_0\in(\lambda_0,\min\{-\Theta_2,E_0\})$, then $\Xi_1<1$ and (4.21) holds. □

We proceed to prove Theorem 1.1. Since the proof in the case $n\geq 2$ is based on the proof in the case $n=1$, we divide Theorem 1.1 into two parts according to $n=1$ and $n\geq 2$. Moreover, we restate the theorem in the cases $n=1$ and $n\geq 2$ as Theorems 4.6 and 4.7 below, respectively. For notational simplicity, we set

$$C_*=\begin{cases}\dfrac{c_{z,\lambda_0}(1-\Theta_1)}{1-\Theta_1-4sc_{z,\lambda_0}}, & \text{if } a_0 \text{ satisfies (4.16)},\\[2ex] \dfrac{(\delta-1)\lambda_0 c_{z,\lambda_0}}{(\delta-1)\lambda_0-2(\delta\lambda_0\Xi_1+\Xi_2)c_{z,\lambda_0}}, & \text{if } a_0 \text{ satisfies (4.17)}.\end{cases}\tag{4.25}$$

Then, $\|(U+V)^{-1}\|\leq C_*$.

Theorem 4.6. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $p>d/2$. Let $z\in\rho(H_\Lambda(A,V))$, the resolvent set of $H_\Lambda(A,V)$. Suppose that $s>0$ and $a_0>0$ satisfy (4.16) or (4.17). Then, for any $\beta,\gamma\in\mathbb{R}^d$,*

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\le C_{p,\lambda_0}C_*e^{\sqrt{d}a_0}e^{-a_0|\beta-\gamma|},\tag{4.26}$$

where $C_{p,\lambda_0}>0$ depends only on p and λ_0.

Proof. By Lemma 4.4 and the operator equality (4.18), we have

$$\chi_\beta(H^a_\Lambda(A,V)-z)^{-1}\chi_\gamma=\chi_\beta(H_\Lambda(A,V)-\lambda_0)^{-1/2}(U+V)^{-1}(H_\Lambda(A,V)-\lambda_0)^{-1/2}\chi_\gamma.$$

Since the function $(\lambda-\lambda_0)^{-1/2}$ satisfies (3.4) with $\alpha=\frac{1}{2}$, $\frac{1}{2}>d/(2\cdot 2p)$ and $2p>d\geq 2$, Theorem 3.1 ensures that both $\chi_\beta(H_\Lambda(A,V)-\lambda_0)^{-1/2}$ and $(H_\Lambda(A,V)-\lambda_0)^{-1/2}\chi_\gamma$ are in $\mathcal{J}_{2p}$ with $\mathcal{J}_{2p}$-norm only depending on p and λ_0. It then follows that $\chi_\beta(H^a_\Lambda(A,V)-z)^{-1}\chi_\gamma\in\mathcal{J}_p$ with

$$\begin{aligned}\|\chi_\beta(H^a_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}&\\ &\le\|\chi_\beta(H_\Lambda(A,V)-\lambda_0)^{-1/2}\|_{\mathcal{J}_{2p}}\|(U+V)^{-1}\|\,\|(H_\Lambda(A,V)-\lambda_0)^{-1/2}\chi_\gamma\|_{\mathcal{J}_{2p}}\\ &\le C_{p,\lambda_0}C_*,\end{aligned}$$

where (4.19) and (4.25) are used and $C_{p,\lambda_0}>0$ only depends on p and λ_0. Considering (4.1), we obtain

$$\begin{aligned}
&\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\\
&\qquad=\|\chi_\beta e^{-a\cdot x}(H_\Lambda^a(A,V)-z)^{-1}e^{a\cdot x}\chi_\gamma\|_{\mathcal{J}_p}\\
&\qquad=\|e^{-a\cdot(\beta-\gamma)}(e^{-a\cdot(x-\beta)}\chi_\beta)(\chi_\beta(H_\Lambda^a(A,V)-z)^{-1}\chi_\gamma)(\chi_\gamma e^{a\cdot(x-\gamma)})\|_{\mathcal{J}_p}\\
&\qquad\le\|\chi_\beta(H_\Lambda^a(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\|e^{-a\cdot(x-\beta)}\chi_\beta\|\,\|\chi_\gamma e^{a\cdot(x-\gamma)}\|e^{-a\cdot(\beta-\gamma)}\\
&\qquad\le C_{p,\lambda_0}C_* e^{\sqrt{d}a_0}e^{-a_0|\beta-\gamma|},
\end{aligned}$$

where we used the fact that both $\|e^{-a\cdot(x-\beta)}\chi_\beta\|$ and $\|\chi_\gamma e^{a\cdot(x-\gamma)}\|$ are bounded by $e^{\sqrt{d}|a|/2}$. By choosing $a=a_0|\beta-\gamma|^{-1}(\beta-\gamma)$, we obtain (4.26). □

Theorem 4.7. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $p>d/2n$ with $n\in\mathbb{N}$ and $n\ge2$. Let $z\in\rho(H_\Lambda(A,V))$. Suppose that $s>0$ and $a_0>0$ satisfy (4.16) or (4.17). Then, for any $\delta_0\in(0,1)$ and any $\beta,\gamma\in\mathbb{R}^d$,*

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma\|_{\mathcal{J}_p}\le(C_{p,n,\lambda_0}c_{\delta_0,a_0}C_*)^{n-1}e^{(n-1)\sqrt{d}a_0}e^{-\delta_0a_0|\beta-\gamma|},$$

where $C_{p,n,\lambda_0}>0$ only depends on p, n and λ_0 and $c_{\delta_0,a_0}=\sum_{\alpha\in\mathbb{Z}^d}e^{-(1-\delta_0)a_0|\alpha|}<\infty$.

Proof. Write

$$\chi_\beta(H_\Lambda(A,V)-z)^{-n}\chi_\gamma=\sum_{\substack{\alpha_j\in\mathbb{Z}^d\\ j=1,\dots,n-1}}R_{\beta,\alpha_1}R_{\alpha_1,\alpha_2}\dots R_{\alpha_{n-2},\alpha_{n-1}}R_{\alpha_{n-1},\gamma},$$

where

$$\begin{aligned}
R_{\beta,\alpha_1}&=\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_{\alpha_1},\\
R_{\alpha_j,\alpha_{j+1}}&=\chi_{\alpha_j}(H_\Lambda(A,V)-z)^{-1}\chi_{\alpha_{j+1}},\quad j=1,\dots,n-2,\\
R_{\alpha_{n-1},\gamma}&=\chi_{\alpha_{n-1}}(H_\Lambda(A,V)-z)^{-1}\chi_\gamma.
\end{aligned}$$

Since $pn>d/2$ by assumption, Theorem 4.6 says that

$$\|\chi_x(H_\Lambda(A,V)-z)^{-1}\chi_y\|_{\mathcal{J}_{pn}}\le C_{p,n,\lambda_0}C_*e^{\sqrt{d}a_0}e^{-a_0|\beta-\gamma|}\quad\text{for all } x,y\in\mathbb{R}^d,$$

where $C_{p,n,\lambda_0}>0$ only depends on p, n and λ_0. By Hölder's inequality for trace ideals (see [42, Theorem 2.8]), the result of the corollary follows from

(4.27)
$$\sum_{\substack{\alpha_j\in\mathbb{Z}^d\\ j=1,\dots,n-1}} e^{-a_0|\beta-\alpha_1|}e^{-a_0|\alpha_1-\alpha_2|}\dots e^{-a_0|\alpha_{n-2}-\alpha_{n-1}|}e^{-a_0|\alpha_{n-1}-\beta|}\le c_{\delta_0,a_0}^{n-1}e^{-\delta_0 a_0|\beta-\gamma|}$$

for any $\delta_0\in(0,1)$, where $c_{\delta_0,a_0}=\sum_{\alpha\in\mathbb{Z}^d} e^{-(1-\delta_0)a_0|\alpha|}<\infty$.

To complete the proof, we show (4.27). Pick and fix any $\delta_0\in(0,1)$. First, we have from the triangular inequality and Cauchy's inequality

$$\begin{aligned}\sum_{\alpha_1\in\mathbb{Z}^d} & e^{-a_0|\beta-\alpha_1|}e^{-a_0|\alpha_1-\alpha_2|}\\ &=\sum_{\alpha_1\in\mathbb{Z}^d} e^{-(1-\delta_0)a_0|\beta-\alpha_1|}e^{-\delta_0 a_0(|\beta-\alpha_1|+|\alpha_1-\alpha_2|)}e^{-(1-\delta_0)a_0|\alpha_1-\alpha_2|}\\ &\le e^{-\delta_0 a_0|\beta-\alpha_2|}\sum_{\alpha_1\in\mathbb{Z}^d} e^{-(1-\delta_0)a_0|\beta-\alpha_1|}e^{-(1-\delta_0)a_0|\alpha_1-\alpha_2|}\\ &\le e^{-\delta_0 a_0|\beta-\alpha_2|}\Bigg(\sum_{\alpha_1\in\mathbb{Z}^d} e^{-2(1-\delta_0)a_0|\beta-\alpha_1|}\Bigg)^{1/2}\Bigg(\sum_{\alpha_1\in\mathbb{Z}^d} e^{-2(1-\delta_0)a_0|\alpha_1-\alpha_2|}\Bigg)^{1/2}\\ &\le c_{\delta_0,a_0}e^{-\delta_0 a_0|\beta-\alpha_2|},\end{aligned}$$

where $c_{\delta_0,a_0}=\sum_{\alpha_1\in\mathbb{Z}^d} e^{-(1-\delta_0)a_0|\alpha_1|}\ge\sum_{\alpha_1\in\mathbb{Z}^d} e^{-2(1-\delta_0)a_0|\alpha_1|}$. Next, by the above estimate and the triangular inequality,

$$\begin{aligned}\sum_{\alpha_2\in\mathbb{Z}^d}\sum_{\alpha_1\in\mathbb{Z}^d} & e^{-a_0|\beta-\alpha_1|}e^{-a_0|\alpha_1-\alpha_2|}e^{-a_0|\alpha_2-\alpha_3|}\\ &\le c_{\delta,a_0}\sum_{\alpha_2\in\mathbb{Z}^d} e^{-\delta a_0|\beta-\alpha_2|}e^{-a_0|\alpha_2-\alpha_3|}\\ &=c_{\delta_0,a_0}\sum_{\alpha_2\in\mathbb{Z}^d} e^{-\delta_0 a_0|\beta-\alpha_2|}e^{-\delta_0 a_0|\alpha_2-\alpha_3|}e^{-(1-\delta_0)a_0|\alpha_2-\alpha_3|}\\ &\le c_{\delta,a_0}e^{-\delta_0 a_0|\beta-\alpha_3|}\sum_{\alpha_2\in\mathbb{Z}^d} e^{-(1-\delta_0)a_0|\alpha_2-\alpha_3|}\\ &=c_{\delta_0,a_0}^2 e^{-\delta_0 a_0|\beta-\alpha_3|}.\end{aligned}$$

By induction, we find (4.27). This completes the proof. □

5. The operator kernel estimate in trace ideals

In this section, we study operator kernel estimates in trace-class norms. More precisely, we prove polynomial decay, in trace ideals, of operators

$$\chi_\beta f(H_\Lambda(A,V))\chi_\gamma, \quad \beta,\gamma\in\mathbb{R}^d,$$

in terms of $|\beta-\gamma|$ for $f\in\mathcal{S}(\mathbb{R})$, the Schwartz space. The main result in this section is stated in Theorem 1.2, whose proof is based on Theorem 1.1 (in fact on Theorem 4.6) and the Helffer–Sjöstrand formula (see [26]), which is defined for a much larger class of slowly decreasing smooth functions on $\mathbb{R}$, denoted by $\mathcal{A}$. See Appendix B for the definition of $\mathcal{A}$ and the Helffer–Sjöstrand formula.

Before proving Theorem 1.2, we first simplify the second estimate in (4.26) by adding more conditions so that this estimate can by easily used. Our idea is as follows: by the Helffer–Sjöstrand formula (B.2), we have for any $f\in\mathcal{S}(\mathbb{R})$,

$$\chi_\beta f(H_\Lambda(A,V))\chi_\gamma=\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\partial\tilde{f}_n(z)}{\partial\bar{z}}\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\,du\,dv \quad \text{for all } n\geq 1.$$

Therefore, by (B.1),

$$\begin{aligned}(5.1)\quad &\|\chi_\beta f(H_\Lambda(A,V))\chi_\gamma\|_{\mathcal{J}_p}\\ &\leq\frac{C}{\pi}\sum_{r=0}^{n}\frac{1}{r!}\int_U|f^{(r)}(u)|\frac{|v|^r}{\langle u\rangle}\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\,du\,dv\\ &\quad+\frac{1}{2\pi n!}\int_V|f^{(n+1)}(u)|\,|v|^n\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\,du\,dv\end{aligned}$$

for any $n\geq 1$. Clearly, in order to estimate the integrals on the right-hand side of (5.1), we need (4.26). More precisely, we need that

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\leq\frac{C_{p,\lambda_0}(\delta-1)\lambda_0c_{z,\lambda_0}}{(\delta-1)\lambda_0-2(\delta\lambda_0\Xi_1+\Xi_2)c_{z,\lambda_0}}e^{\sqrt{d}a_0}e^{-a_0|\beta-\gamma|}, \tag{5.2}$$

since the conditions ensuring it provide a nonzero lower bound for a_0, which in turn provide an upper bound for the exponential term. However, this estimate is too rough to deal with since many parameters in the upper bound depend on z. To simplify it, we put more conditions on s and a_0.

For $s>0$, we assume that

$$s<\frac{1}{2}\frac{1-\Theta_1}{4c_{z,\lambda_0}}\frac{1-\delta}{2-\delta} \tag{5.3}$$

is such that

$$\frac{2s(2\lambda_0+\Theta_2)}{\Theta_1-1}<\frac{(\delta-1)\lambda_0}{4c_{z,\lambda_0}}+\frac{2s(\delta\lambda_0+\Theta_2)}{\Theta_1-1}<\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}+\frac{2s(\delta\lambda_0+\Theta_2)}{\Theta_1-1}. \tag{5.4}$$

For $a_0>0$, we require that

$$\frac{2s(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}\le a_0^2\le\frac{2s(2\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s}+\frac{s}{4}\right)^{-1}. \tag{5.5}$$

The intuitive interpretation of these conditions is that we make $2\Xi_1 c_{z,\lambda_0}$ smaller and bound $\Xi_1^{-1}\Xi_2$ by $-2\lambda_0$ from above. Indeed, (5.3) is equivalent to

$$2\Xi_1 c_{z,\lambda_0}<\frac{1}{2}\frac{1-\delta}{2-\delta}$$

and, (5.4) and (5.5) are equivalent to

$$-\lambda_0\Xi_1\le\Xi_2\le-2\lambda_0\Xi_1<\frac{(\delta-1)\lambda_0}{4c_{z,\lambda_0}}-\delta\lambda_0\Xi_1<\frac{(\delta-1)\lambda_0}{2c_{z,\lambda_0}}-\delta\lambda_0\Xi_1.$$

Clearly, (5.3) and (5.5) are stronger than (4.17). Hence, under the assumptions of (5.3) and (5.5), (5.2) holds. Moreover,

$$\frac{(\delta-1)\lambda_0 c_{z,\lambda_0}}{(\delta-1)\lambda_0-2(\delta\lambda_0\Xi_1+\Xi_2)c_{z,\lambda_0}}=\frac{c_{z,\lambda_0}}{1-2\Xi_1 c_{z,\lambda_0}\dfrac{\delta\lambda_0+\Xi_1^{-1}\Xi_2}{(\delta-1)\lambda_0}}\le 2c_{z,\lambda_0}$$

and, therefore, (5.2) can be simplified to

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\le C_{p,\lambda_0}c_{z,\lambda_0}e^{\sqrt{d}a_0}e^{-a_0|\beta-\gamma|}, \tag{5.6}$$

where $C_{p,\lambda_0}>0$ depends only on p and λ_0. Further, we rewrite (5.5) as

$$\frac{2(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s^2}+\frac{1}{4}\right)^{-1}\le a_0^2\le\frac{2(2\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s^2}+\frac{1}{4}\right)^{-1}, \tag{5.7}$$

which in particular says that a_0 is bounded from above by a constant independent of z, which implies that $e^{\sqrt{d}a_0}$ is also bounded from above by a constant independent of z. Hence, (5.6) is further simplified to

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\le C_{p,\lambda_0}c_{z,\lambda_0}e^{-a_0|\beta-\gamma|}, \tag{5.8}$$

where $C_{p,\lambda_0}>0$ only depends on p and λ_0.

Note that the lower bound of a_0 is not very easy to handle because of the uncertainty of s and the quantity c_{z,λ_0}. To find a simpler lower bound for a_0, we first fix some s, say

$$s=\frac{1}{4}\frac{1-\Theta_1}{4c_{z,\lambda_0}}\frac{1-\delta}{2-\delta},$$

and then give an explicit bound for c_{z,λ_0} with $z=u+iv$ under the assumptions $(u,v)\in U$ and $(u,v)\in V$, respectively. Recall that

$$c_{z,\lambda_0}=\left\|\frac{\lambda-\lambda_0}{\lambda-z}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))},$$
$$U=\{(u,v)\in\mathbb{R}^2|\langle u\rangle<|v|<2\langle u\rangle\},$$
$$V=\{(u,v)\in\mathbb{R}^2|0<|v|<2\langle u\rangle\}.$$

Lemma 5.1. *Let $z\in\rho(H_\Lambda(A,V))$. Then, with $z=u+iv$,*

$$\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\le\begin{cases}C_{p,\lambda_0}|v|e^{-C_{\lambda_0}|\beta-\gamma|/|v|}, & \text{if } (u,v)\in U,\\ C_{p,\lambda_0}\dfrac{\langle u\rangle}{|v|}e^{-C_{\lambda_0}|v|\,|\beta-\gamma|/\langle u\rangle}, & \text{if } (u,v)\in V,\end{cases}\tag{5.9}$$

where $C_{\lambda_0}>0$ depends only on λ_0 and $C_{p,\lambda_0}>0$ depends only on p and λ_0.

Proof. For any $z\in\rho(H_\Lambda(A,V))$, we let

$$s=\frac{1}{4}\frac{1-\Theta_1}{4c_{z,\lambda_0}}\frac{1-\delta}{2-\delta}\tag{5.10}$$

and let $a_0>0$ satisfying (5.5) be such that (5.8) holds. By (5.10) and the first inequality in (5.7) we have

$$a_0^2\ge\frac{2(\lambda_0+\Theta_2)}{\Theta_1-1}\left(\frac{1}{2s^2}+\frac{1}{4}\right)^{-1}=-\frac{(\lambda_0+\Theta_2)C_2}{C_{\lambda_0}c_{z,\lambda_0}^2+C_1}\tag{5.11}$$

for some $C_1>0$ and $C_2>0$.

Let $(u,v)\in U$. For any $\lambda\in\sigma(H_\Lambda(A,V))$, $|\lambda-z|\ge\operatorname{dist}(z,\sigma(H_\Lambda(A,V)))\ge|v|>\langle u\rangle\ge1$, which implies that

$$c_{z,\lambda_0}\le\left\|1+\frac{|z-\lambda_0|}{|\lambda-z|}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\le1+|z|-\lambda_0\le1-\lambda_0+\sqrt{2}|v|\le C_{\lambda_0}|v|$$

and then

$$a_0\ge\sqrt{\frac{-(\lambda_0+\Theta_2)C_2}{C_{\lambda_0}c_{z,\lambda_0}^2+C_1}}\ge\frac{C_{\lambda_0}}{|v|},\tag{5.12}$$

where the fact $|v|\ge1$ is used.

Let $(u,v)\in V$. Then

$$c_{z,\lambda_0}\le\left\|1+\frac{|z-\lambda_0|}{|\lambda-z|}\right\|_{L^\infty(\sigma(H_\Lambda(A,V)))}\le 1+\frac{|z|-\lambda_0}{|v|}\le\frac{5\langle u\rangle-\lambda_0}{|v|}\le C_{\lambda_0}\frac{\langle u\rangle}{|v|},$$

which, together with (5.11), implies that

$$a_0\ge C_{\lambda_0}\frac{|v|}{\langle u\rangle}, \tag{5.13}$$

where the fact $\langle u\rangle/|v|>\frac{1}{2}$ is used.

By means of (5.8), (5.12) and (5.13), we obtain (5.9). □

We now restate and prove Theorem 1.2.

Theorem 5.2. *Let $A\in\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V\in\mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda\subset\mathbb{R}^d$ be open. Suppose that $p>d/2$. Then, for any $f\in\mathcal{S}(\mathbb{R})$ and any $k\in\mathbb{N}$,*

$$\|\chi_\beta f(H_\Lambda(A,V))\chi_\gamma\|_{\mathcal{J}_p}\le C_{p,\lambda_0,k,f}|\beta-\gamma|^{-k}\quad \text{for all } \beta,\gamma\in\mathbb{R}^d,$$

where $C_{p,\lambda_0,k,f}>0$ depends only on p, λ_0, k and f.

Proof. Fix any $k\in\mathbb{N}$ and let $n=k+1$ in (5.1). Since the function $\theta(t)=e^{-t}t^k$, $t\ge0$, attains its global maximum at $t=k$, we have

$$e^{-t}\le e^{-k}k^k t^{-k}. \tag{5.14}$$

Applying (5.14) to $t=C_{\lambda_0}/|v|\,|\beta-\gamma|$ and $t=C_{\lambda_0}|v|/\langle u\rangle|\beta-\gamma|$, respectively, we obtain

$$e^{-C_{\lambda_0}|\beta-\gamma|/|v|}\le\frac{e^{-k}k^k}{C_{\lambda_0}^k|\beta-\gamma|^k}|v|^k \tag{5.15}$$

and

$$e^{-C_{\lambda_0}|v|\,|\beta-\gamma|/\langle u\rangle}\le\frac{e^{-k}k^k}{C_{\lambda_0}^k|\beta-\gamma|^k}\frac{\langle u\rangle^k}{|v|^k}, \tag{5.16}$$

respectively.

We now use (5.15) and (5.16) to estimate the integrals in (5.1). By the first estimate in (5.9) and (5.15), we have for some $C_{p,\lambda_0,k,f}>0$,

$$\sum_{r=0}^{k+1}\frac{1}{r!}\int_U|f^{(r)}(u)|\frac{|v|^r}{\langle u\rangle}\|\chi_\beta(H_\Lambda(A,V)-z)^{-1}\chi_\gamma\|_{\mathcal{J}_p}\,du\,dv$$
$$\le\frac{C_{p,\lambda_0}e^{-k}k^k}{C_{\lambda_0}^k|\beta-\gamma|^k}\sum_{r=0}^{k+1}\frac{1}{r!}\int_U|f^{(r)}(u)|\frac{|v|^{r+k+1}}{\langle u\rangle}\,du\,dv$$

$$= \frac{C_{p,\lambda_0} e^{-k} k^k}{C_{\lambda_0}^k |\beta-\gamma|^k} \sum_{r=0}^{k+1} \frac{1}{r!} \frac{2^{r+k+3}-2}{r+k+2} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r+k+1} \, du$$
$$\leq C_{p,\lambda_0,k,f} |\beta-\gamma|^{-k},$$

where the fact that $f \in \mathcal{S}(\mathbb{R})$, and so the integrals are convergent, is used.

Similarly, by the second estimate in (5.9) and (5.16),

$$\frac{1}{2\pi(k+1)!} \int_V |f^{(n+1)}(u)| \, |v|^n \| \chi_\beta (H_\Lambda(A,V)-z)^{-1} \chi_\gamma \|_{\mathcal{J}_p} \, du \, dv$$
$$\leq \frac{C_{p,\lambda_0} e^{-k} k^k}{2\pi(k+1)! C_{\lambda_0}^k |\beta-\gamma|^k} \int_V |f^{(k+2)}(u)| \langle u \rangle^{k+1} \, du \, dv$$
$$= \frac{4 C_{p,\lambda_0} e^{-k} k^k}{2\pi(k+1)! C_{\lambda_0}^k |\beta-\gamma|^k} \int_{\mathbb{R}} |f^{(k+2)}(u)| \langle u \rangle^{k+2} \, du$$
$$\leq C_{p,\lambda_0,k,f} |\beta-\gamma|^{-k}.$$

Consequently, for any $f \in \mathcal{S}(\mathbb{R})$, there exists $C_{p,\lambda_0,k,f} > 0$ so that

$$\| \chi_\beta f(H_\Lambda(A,V)) \chi_\gamma \|_{\mathcal{J}_p} \leq C_{p,\lambda_0,k,f} |\beta-\gamma|^{-k} \quad \text{for all } \beta,\gamma \in \mathbb{R}^d.$$

This proves Theorem 5.2. □

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Appendix A. Sectorial form and m-sectorial operators

In this section, we review some results about sectorial form and m-sectorial operators used in the above sections. The material is chosen from [27]. See also [16].

Let $\mathcal{H}$ be a separable Hilbert space and $h(\cdot,\cdot)\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ be a sesquilinear form. The form h is called *sectorial* if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0,\pi/2)$ so that

$$h(u,u) \in \{ z \in \mathbb{C} \mid |\arg(z-\gamma)| \leq \theta \} \quad \text{for any } u \in \mathcal{Q}(h) \text{ with } \|u\| = 1,$$

where $\mathcal{Q}(h)$ is the form domain of h. In particular, any symmetric sesquilinear form bounded from below is sectorial. For relatively bounded perturbations, we have the following result (see [27, Theorem VI.1.33]).

Theorem A.1. *Let h be a sectorial form and h' be h-bounded, i.e., $\mathcal{Q}(h)\subset\mathcal{Q}(h')$ and there exist nonnegative constants a and b such that*

$$|h'(u,u)|\le ah(u,u)+b\|u\|^2 \quad \textit{for any } u\in\mathcal{Q}(h).$$

If $a<1$, then $h+h'$ is sectorial. Moreover, $h+h'$ is closable or closed if and only if h is closable or closed, respectively.

Let $H:\mathcal{H}\to\mathcal{H}$ be a linear operator with domain $\mathcal{D}(H)$. H is said to be *accretive* if $\operatorname{Re}\langle u,Hu\rangle\ge 0$ for all $u\in\mathcal{D}(H)$. It is said to be *m-accretive* if for any $z\in\mathbb{C}$ with $\operatorname{Re} z>0$, it is true that

$$(H+z)^{-1}\in\mathcal{L}(\mathcal{H}) \quad\text{and}\quad \|(H+z)^{-1}\|\le\frac{1}{\operatorname{Re} z},$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of all bounded linear operators on $\mathcal{H}$. It is not hard to see that an m-accretive operator is *maximal accretive* in the sense that it is accretive and has no proper accretive extension. If there are $\gamma\in\mathbb{R}$ and $\theta\in[0,\pi/2)$ so that

$$\langle u,Hu\rangle\in\{z\in\mathbb{C}\mid |\arg(z-\gamma)|\le\theta\} \quad\text{for any } u\in\mathcal{D}(H) \text{ with } \|u\|=1,$$

then H is said to be *sectorial*. H is *m-sectorial* if it is both m-accretive and sectorial.

If H is sectorial, then the sesquilinear form $h(\cdot,\cdot)$ on $\mathcal{Q}(h)=\mathcal{D}(H)$ defined by

$$h(u,v)=\langle u,Hv\rangle,\quad u,v\in\mathcal{Q}(h),$$

is sectorial and closable (see [27, Theorem VI.1.27]). In particular, any symmetric operator bounded from below defines a closable sectorial form. Conversely, we have the following result (see [27, Theorems VI.2.1 and V.2.6]).

Theorem A.2. *Let $h(\cdot,\cdot)$ be a densely defined and closed sectorial form in $\mathcal{H}$ with form domain $\mathcal{Q}(h)$. Then there exists a unique m-sectorial operator H such that $\mathcal{D}(H)\subset\mathcal{Q}(h)$ and*

$$h(u,v)=\langle u,Hv\rangle \quad \textit{for } u\in\mathcal{Q}(h) \textit{ and } v\in\mathcal{D}(H).$$

If, in addition, $h(\cdot,\cdot)$ is symmetric and bounded from below, then the associated m-sectorial operator H is self-adjoint with the same lower bound.

The second part of the above theorem is well known and widely used in the theory of Schrödinger operators. We have also used the following result (see [27, Lemma VI.3.1]).

Theorem A.3. *Let $h(\cdot,\cdot)$ be a densely defined, symmetric, nonnegative closed form with the associated nonnegative self-adjoint operator H. Let $q(\cdot,\cdot)$ be a form relatively bounded with respect to h so that*

$$|q(u,u)| \le Ch(u,u), \quad u \in \mathcal{Q}(h),$$

for some $C \ge 0$. Then there is $B \in \mathcal{L}(\mathcal{H})$ with $\|B\| \le \varepsilon C$ such that

$$q(u,v) = \big\langle \sqrt{H}u, B\sqrt{H}v \big\rangle, \quad u,v \in \mathcal{Q}(h) = \mathcal{D}\big(\sqrt{H}\big),$$

where $\varepsilon = 1$ or $\varepsilon = 2$ according to whether q is symmetric or not.

Appendix B. The Helffer–Sjöstrand formula

In this section, we define the class of slowly decreasing smooth functions and review the Helffer–Sjöstrand formula (see [26]), which provides an alternative approach to the spectral theory of self-adjoint operators. The material below is taken from [14].

Definition B.1. A function f is said to be in $\mathcal{A}$, the class of slowly decreasing smooth functions on $\mathbb{R}$, if $f \in C^\infty(\mathbb{R})$ and there exit $\mu > 0$ and a sequence of constants $c_n \ge 0$, $n \ge 1$, so that

$$|f^{(n)}(u)| \le c_n \langle u \rangle^{-n-\mu} \quad \text{for all } u \in \mathbb{R} \text{ and } n \ge 1,$$

where $\langle u \rangle \equiv \sqrt{1+|u|^2}$. We define the following norms on $\mathcal{A}$,

$$\|\!|f|\!\|_n = \sum_{r=0}^{n} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r-1}\, dx \quad \text{for } f \in \mathcal{A} \text{ and } n \ge 1.$$

Let $\tau \in C^\infty(\mathbb{R})$ with $\tau(u) = 1$ if $|u| < 1$ and $\tau(u) = 0$ if $|u| > 2$. For $f \in \mathcal{A}$, the smooth (nonanalytic) extensions $\tilde{f}_n : \mathbb{C} \to \mathbb{C}$ of f are defined by

$$\tilde{f}_n(z) = \left(\frac{1}{r!} \sum_{r=0}^{n} f^{(r)}(u)(iv) \right) \sigma(u,v), \quad n \ge 1,$$

where $z = u + iv$ and $\sigma(u,v) = \tau(v/\langle u \rangle)$. Define

$$\frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial \tilde{f}_n(z)}{\partial u} + i \frac{\partial \tilde{f}_n(z)}{\partial v} \right).$$

Direct calculation shows that

$$\frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} = \frac{1}{2}\left(\sum_{r=0}^{n} \frac{1}{r!} f^{(r)}(u)(iv)^r\right)(\sigma_u(u,v)+\sigma_v(u,v))+\frac{1}{2n!} f^{(n+1)}(u)(iv)^n \sigma(u,v).$$

Obviously, $\sigma(u,v)=0$ if $|v|\geq 2\langle u\rangle$ and both $\sigma_u(u,v)=0$ and $\sigma_v(u,v)=0$ if $|v|\leq\langle u\rangle$ or $|v|\geq 2\langle u\rangle$. Thus, by introducing the sets $U=\{(u,v)\in\mathbb{R}^2 | \langle u\rangle<|v|<2\langle u\rangle\}$ and $V=\{(u,v)\in\mathbb{R}^2 | 0<|v|<2\langle u\rangle\}$, we have

$$\text{(B.1)} \quad \left|\frac{\partial \tilde{f}_n(z)}{\partial \bar{z}}\right| \leq C\left(\sum_{r=0}^{n} \frac{1}{r!} |f^{(r)}(u)| \frac{|v|^r}{\langle u\rangle}\right)\chi_U(u,v)+\frac{1}{2n!}|f^{(n+1)}(u)|\,|v|^n \chi_V(u,v)$$

for some $C>0$ only depending on τ, where χ_U and χ_V are the characteristic functions of U and V, respectively.

Theorem B.2. ([14]) *Let $f\in\mathcal{A}$ and H be self-adjoint on separable Hilbert space. Then the integral*

$$\int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}}(H-z)^{-1}\,du\,dv$$

converges in operator norm and is independent of n and τ. Moreover,

$$\left\|\int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}}(H-z)^{-1}\,du\,dv\right\| \leq c|\!|\!|f|\!|\!|_{n+1}, \quad n\geq 1,$$

where $c>0$ is a constant independent of f and n.

It should be pointed out that the fact that the constant c is independent of n is due to Germinet and Klein [24]. This follows from the fact that $2^n/n!\to 0$ as $n\to\infty$.

We then define, for $f\in\mathcal{A}$,

$$\text{(B.2)} \qquad f(H)=\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}}(H-z)^{-1}\,du\,dv,$$

which is referred to as the Helffer–Sjöstrand formula. By Theorem B.2 we have $\|f(H)\|\leq c|\!|\!|f|\!|\!|_{n+1}$ for all $n\geq 1$.

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