CONCENTRATION OF QUASI-STATIONARY DISTRIBUTIONS FOR ONE-DIMENSIONAL DIFFUSIONS WITH APPLICATIONS

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ABSTRACT. We consider small noise perturbations to an ordinary differential equation (ODE) that have a uniform absorbing state and exhibit transient dynamics in the sense that interesting dynamical behaviors governed by transient states display over finite time intervals and the eventual dynamics is simply controlled by the absorbing state. To capture the transient states, we study the noise-vanishing concentration of the so-called quasi-stationary distributions (QSDs) that describe the dynamics before reaching the absorbing state. By establishing concentration estimates based on constructed uniform-in-noises Lyapunov functions, we show that QSDs tend to concentrate on the global attractor of the ODE as noises vanish, and that each limiting measure of QSDs, if exists, must be an invariant measure of the ODE. Overcoming difficulties caused by the degeneracy and singularity of noises at the absorbing state, we further show the tightness of the family of QSDs under additional assumptions motivated by applications, that not only validates a priori information on the concentration of QSDs, but also asserts the reasonability of using QSDs in the mathematical modeling of transient states. Our approaches to the concentration and tightness of QSDs are purely analytic without probabilistic heuristics. Applications to diffusion approximations of chemical reactions and birth-and-death processes of logistic type are also discussed. Rigorously studying the transient dynamics and characterizing the transient states, our study is of both theoretical and practical significance.

CONTENTS

1. **Introduction**

The present paper aims at making some theoretical understanding of an important class of transient dynamics described by quasi-stationary distributions (QSDs) [12] in randomly perturbed dynamical systems. While QSDs have been widely observed as transient states controlling long transient dynamical behaviors in many applications such as chemical oscillations [63, 61], ecology [56], etc., there have not been many rigorous studies of the phenomena perhaps due to technical difficulties caused by the degeneracy and singularity of noises at the absorbing state in a typical model. In the present paper, we study the concentration of QSDs for a commonly adopted one-dimensional model, namely, the following stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \epsilon\sqrt{a(X_t)}dW_t, \quad X_0 \in [0, \infty), \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small parameter, $b : [0, \infty) \to \mathbb{R}$, $a : [0, \infty) \to [0, \infty)$ and W_t is the standard one-dimensional Wiener process on some probability space. SDEs of the form (1.1) arise in many scientific areas such as ecology, epidemiology, chemical kinetics and population genetics, in which solutions of (1.1) describe the evolution of certain species. They are also derived as diffusion approximations of scaled Markov jump processes that model the evolution of species of large numbers [44, 19, 1]. Following the pioneering work of Feller [21, 22], the investigation of the SDE (1.1) from either a mathematical or scientific viewpoint has attracted a vast amount of attention that results in a huge number of literature.

Our main interest in (1.1) lies in the situations where small noises cause dramatic effects on the deterministic dynamics. Not only does our study serve as a significant step to rigorously understand *transient dynamics* that commonly exist and play important roles in multi-scale systems, but also it is well-motivated by applications in population dynamics. To be more specific, let us make the following assumptions on the coefficients. Throughout this paper, for a (not necessarily open) interval $I \subset [0, \infty)$, we denote by $C(I)$ the space of continuous functions on I, and by $C^k(I)$, $k \in \mathbb{N}$ the space of k times continuously differentiable functions on the interior $\mathring{I}$ with all derivatives of order $\leq k$ having continuous extensions to I.

(A1) The functions $b : [0, \infty) \to \mathbb{R}$ and $a : [0, \infty) \to [0, \infty)$ are assumed to satisfy the following conditions:
    (1) $b \in C([0, \infty)) \cap C^1((0, \infty))$, $b(0) = 0$, $b(x) > 0$ for all $0 < x \ll 1$, and $b(x) < 0$ for all $x \gg 1$;
    (2) $a \in C([0, \infty)) \cap C^2((0, \infty))$, $a(0) = 0$, and $a > 0$ on $(0, \infty)$.

By **(A1)**, the state 0, often referred to as the *extinction state*, is an absorbing state of (1.1) for each $0 < \epsilon \ll 1$. Additional assumptions can be easily imposed on a and b to ensure that the processes generated by the solutions of (1.1) are absorbed at the extinction state in finite time almost surely. This is in sharp contrast to the asymptotic dynamics of the following unperturbed ordinary differential

equation (ODE):

$$\dot{x} = b(x), \quad x \in [0, \infty), \tag{1.2}$$

whose solutions with positive initial data are attracted by the attractor in $(0, \infty)$, and hence, remain bounded and away from the extinction state over an infinite time period. In other words, the unstable equilibrium 0 and the attractor of (1.2) are respectively stabilized and de-stabilized by the small noises. Such a dynamical disagreement between a deterministic model and its stochastic counterpart is well-known as *Keizer's paradox*, which was originally raised in the modeling of chemical reactions (see e.g. [38, 39, 61, 10]). But, this does not mean the triviality of the dynamics of (1.1). In fact, due to the sample path large deviation theory [26, 15] guaranteeing the closeness between solutions of (1.1) and that of (1.2) over any given finite time interval, solutions of (1.1) typically first approach to the attractor of (1.2), then follow the dynamics of (1.2) on the attractor, and finally deviate from the deterministic trajectories and relax to the extinction state. That is, the equation (1.1) exhibits multi-scale dynamics, and the interesting ones display over finite time periods and can not be observed in the classical long-time limits. Such dynamics are often referred to as *transient dynamics*. The corresponding states, characterizing the interesting dynamics, are called *transient states*. Metastable states and quasi-stationary states are also used in literature. In terms of (1.1), transient states are closely related to the attractor of the unperturbed ODE (1.2).

Transient dynamics arise naturally in population dynamics (see e.g. [29, 30, 56]) and chemical reactions (see e.g. [59, 63]). Modeling the evolution of species by the ODE (1.2), species are forced to survive over an infinite time horizon. However, the extinction of species in finite time is inevitable due to limited resources, finite population sizes, mortality, etc. This makes (1.1) a more appropriate model than (1.2). Nevertheless, species typically persist for a long period before eventually going to extinction, and thus, their evolution is transient in nature. Moreover, the states of species that are observed in practice are actually the transient states rather than the extinction state. Hence, transient states are the true physical states.

Consequently, the investigation of transient dynamics is of both theoretical and practical importance. A key step towards the understanding of transient dynamics is to properly define, study and interpret the corresponding transient states. The notion, quasi-stationary distribution, has been used to capture transient states from a distributional viewpoint. Recall that

$$L_\epsilon u = \frac{\epsilon^2}{2}(au)'' - (bu)' \quad \text{on} \quad (0, \infty) \tag{1.3}$$

is the Fokker-Planck operator associated to (1.1), where $'$ stands for the spatial derivative. Denote by

$$\mathcal{L}_\epsilon u = \frac{\epsilon^2}{2}au'' + bu' \quad \text{on} \quad (0, \infty) \tag{1.4}$$

the formal L^2-adjoint operator of L_ϵ. It is the generator of (1.1). Quasi-stationary distributions of (1.1) can be defined using the "principal eigenfunction" of the operator L_ϵ as follows. Denote by $C_0^2((0, \infty))$ the space of twice continuously differentiable functions on $(0, \infty)$ with compact support.

Definition 1.1 (Quasi-stationary distribution)**.** A Borel probability measure ν_ϵ on $(0, \infty)$ is called a *quasi-stationary distribution (QSD)* of (1.1) if there exists a (unique) $\lambda_\epsilon > 0$ such that $L_\epsilon \nu_\epsilon = -\lambda_\epsilon \nu_\epsilon$ in the sense that

$$\int_0^\infty (\mathcal{L}_\epsilon + \lambda_\epsilon)\phi d\nu_\epsilon = 0, \quad \forall \phi \in C_0^2((0, \infty)).$$

We remark that in the case that the diffusion process $\{X_t^\epsilon\}_{t\geq 0}$ generated by the solutions of (1.1) is well-defined, a commonly used but equivalent probabilistic definition of QSD reads as follows (see [53, Proposition 4]): a Borel probability measure ν_ϵ on $(0,\infty)$ is called a QSD of (1.1) if

$$\mathbb{P}_{\nu_\epsilon}^\epsilon \{X_t^\epsilon \in A | T_0^\epsilon > t\} = \nu_\epsilon(A), \quad \forall t \geq 0, \quad A \in \mathcal{B}((0,\infty)),$$

where $T_0^\epsilon = \inf\{t > 0 : X_t^\epsilon = 0\}$, $\mathbb{P}_\nu^\epsilon$ is the law of the process $\{X_t^\epsilon\}_{t\geq 0}$ with initial distribution ν, and $\mathcal{B}((0,\infty))$ is the Borel σ-algebra of $(0,\infty)$. Moreover, there is a unique $\lambda_\epsilon > 0$ such that if $X_0^\epsilon \sim \nu_\epsilon$, then $T_0^\epsilon \sim \exp(\lambda_\epsilon)$. For this reason, λ_ϵ is often called the *extinction rate* associated with the QSD ν_ϵ. This definition clearly says that QSDs of (1.1) are indeed initial distributions on $(0,\infty)$ such that X_t^ϵ conditioned on the non-extinction or survival up to time t is independent of $t \geq 0$. In light of such time variance, QSDs are expected to describe the asymptotic dynamics of (1.1) in the distributional sense before the extinction happens. It is certainly the case under additional assumptions [12]. Hence, QSDs have the capability of capturing the transient states.

The existence, uniqueness of QSDs and convergence to QSDs for one-dimensional diffusion processes of the form (1.1) with fixed ϵ have been extensively studied. Existing literatures usually focus on the canonical form:

$$dX_t = -q(X_t)dt + dW_t, \quad X_0 \in [0,\infty), \tag{1.5}$$

when 0 is an accessible (i.e., regular or exit) boundary and ∞ is an inaccessible (i.e., natural or entrance) boundary according to Feller's classification [22], and distinguish between the regular case $q \in C^1([0,\infty))$ and the singular case $q \in C^1((0,\infty))$. Transforming (1.1) into the form of (1.5) with $q = q_\epsilon$, it is not hard to see from the assumption **(A1)** that 0 is an exit boundary and $q_\epsilon \in C^1((0,\infty))$ is singular near 0 as $q_\epsilon(x) \sim \frac{C_\epsilon}{x}$ as $x \to 0^+$ for some $C_\epsilon > 0$. We refer the reader to [50, 11, 52, 60, 42, 65, 5, 6, 7] and references therein for studies in the regular case. The theory in the singular case is only recently developed in [2] and subsequent works [48, 55, 31]. It is referred to the survey article [53] and the book [12] for related topics and extensive remarks. Similar studies of QSDs in the singular case for higher dimensional systems have attracted a lot of attention in recent years (see [3, 5, 7, 8, 32] and references therein).

The present paper aims at rigorously analyzing the connection between the QSDs of (1.1) and the deterministic attractor of (1.2). This amounts to investigating the concentration of the QSDs as $\epsilon \to 0^+$, which also provides information about the distributions of the QSDs when $0 < \epsilon \ll 1$. By the assumption **(A1)**(1), the ODE (1.2) is indeed dissipative in $(0,\infty)$, and therefore, it admits a *global attractor* $\mathcal{A}$ in $(0,\infty)$, which is the largest compact invariant set of the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ generated by the solutions of (1.2) with initial data in $(0,\infty)$ and has bounded dissipation property in the sense that

$$\lim_{t\to\infty} \text{dist}_H\left(\varphi^t(B), \mathcal{A}\right) = 0, \quad \forall B \subset\subset (0,\infty),$$

where dist_H denotes the Hausdorff semi-distance on $(0,\infty)$, that is, $\text{dist}_H(E,F) := \sup_{x\in E} \text{dist}(x,F)$ for $E, F \subset (0,\infty)$, and $\subset\subset$ denotes the compact inclusion, that is, $E \subset\subset F$ if $\overline{E}$ is compact, F is open and $\overline{E} \subset F$.

Our results concerning the concentration of QSDs state as follows.

Theorem A. *Assume* **(A1)** *and let* $\{\nu_\epsilon\}_\epsilon$ *be a family of QSDs of* (1.1). *Then the following statements hold.*

(1) *For each $0 < \epsilon \ll 1$, ν_ϵ admits a positive density $v_\epsilon \in C^2((0,\infty))$. Moreover, for any open set $\mathcal{O} \subset\subset (0,\infty) \setminus \mathcal{A}$, there exist constants $\gamma_\mathcal{O} > 0$ and $0 < \epsilon_\mathcal{O} \ll 1$ such that*

$$\sup_\mathcal{O} v_\epsilon \le e^{-\frac{\gamma_\mathcal{O}}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\mathcal{O}].$$

(2) *Each limiting measure of $\{\nu_\epsilon\}_\epsilon$ as $\epsilon \to 0$, if exists, is supported on $\mathcal{A}$ and is an invariant measure of $\{\varphi^t\}_{t\in\mathbb{R}}$.*

Remark 1.1. *(i) The assumption* **(A1)** *guarantees neither the existence nor uniqueness of QSDs of* (1.1). *In the absence of coming down from infinity* [2] *(or equivalently, ∞ being an entrance boundary), for each $0 < \epsilon \ll 1$,* (1.1) *could admit infinitely many QSDs that can be described as follows:*

- *if $\lambda_\epsilon^* := \inf \sigma(-\mathcal{L}_\epsilon) > 0$, where the spectral theory of $\mathcal{L}_\epsilon$ is considered in the usual weighted L^2 space, then for any $\lambda_\epsilon \in (0, \lambda_\epsilon^*]$, there is a unique QSD ν_{λ_ϵ} having λ_ϵ as the extinction rate;*
- *the QSDs $\{\nu_{\lambda_\epsilon} : \lambda_\epsilon \in (0, \lambda_\epsilon^*]\}$ are partially ordered in the sense that $0 < \lambda_\epsilon^1 < \lambda_\epsilon^2 \le \lambda_\epsilon^*$ implies $\nu_{\lambda_\epsilon^1}((0,x]) \le \nu_{\lambda_\epsilon^2}((0,x])$ or $\nu_{\lambda_\epsilon^1}((x,\infty)) \ge \nu_{\lambda_\epsilon^2}((x,\infty))$ for all $x \in (0,\infty)$. For this reason, $\nu_{\lambda_\epsilon^*}$ is often called the* minimal *QSD.*

Such a scenario of infinitely many QSDs is known in many situations (see e.g. [51, 49, 12, 64] for one-dimensional diffusion processes, and [4, 23, 24] for jump processes). We show in Proposition 2.1 that $\lim_{\epsilon\to 0} \lambda_\epsilon^* = 0$. It can be easily seen from the proof that the constant Λ appearing in Proposition 2.1 is independent of QSDs.

(ii) In Theorem A, we assume that QSDs of (1.1) *exist for each $0 < \epsilon \ll 1$, and ν_ϵ is any one of them if there are indeed multiple ones. Then the theorem asserts that for any open set $\mathcal{O} \subset\subset (0,\infty)\setminus\mathcal{A}$, there are constants $\gamma_\mathcal{O} > 0$ and $0 < \epsilon_\mathcal{O} \ll 1$, independent of the particular choice of QSDs, such that*

$$\sup_{\lambda_\epsilon \in (0, \lambda_\epsilon^*]} \sup_\mathcal{O} v_{\lambda_\epsilon} \le e^{-\frac{\gamma_\mathcal{O}}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\mathcal{O}], \tag{1.6}$$

where v_{λ_ϵ} is the density of ν_{λ_ϵ}.

We refer the readers to Remark 2.3 for the reasoning of the independence of $\gamma_\mathcal{O}$ and $\epsilon_\mathcal{O}$ on the particular choice of QSDs.

(iii) Observe that if for each $0 < \epsilon \ll 1$, (1.1) *admits infinitely many QSDs $\{\nu_{\lambda_\epsilon} : \lambda_\epsilon \in (0, \lambda_\epsilon^*]\}$, then tails of QSDs can become flatter as $\lambda_\epsilon \to 0$. Our concentration estimates* (1.6) *does not contradict such an observation. Indeed, the condition that $\mathcal{O}$ stays away from 0 and ∞ can not be dropped in general. As $\inf \mathcal{O}$ approaches 0, or $\sup \mathcal{O}$ approaches ∞, $\gamma_\mathcal{O}$ and $\epsilon_\mathcal{O}$ may decay to 0 due to the possible concentration of QSDs near 0 and ∞. Though* (1.6) *implies that, for any $\mathcal{O} \subset\subset (0,\infty) \setminus \mathcal{A}$ whose left or right boundary can be sufficiently close to 0 or ∞, we always have $\lim_{\epsilon\to 0} \nu_{\lambda_\epsilon}(\mathcal{O}) = 0$ for any QSD ν_{λ_ϵ}, this limit as $\epsilon \to 0$ is however not uniform in $\mathcal{O}$. In general, it is well possible to have QSDs $\{\nu_{\lambda_\epsilon}\}_\epsilon$ and large numbers $M_\epsilon \to \infty$ such that*

$$\liminf_{\epsilon\to 0} \nu_{\lambda_\epsilon}([M_\epsilon, \infty)) > 0.$$

In consideration of the partial order of QSDs, it is also possible to have small numbers $\delta_\epsilon \to 0$ such that

$$\liminf_{\epsilon\to 0} \nu_{\lambda_\epsilon^*}((0, \delta_\epsilon]) > 0,$$

that is, the minimal QSDs $\{\nu_{\lambda_\epsilon^*}\}_\epsilon$ have positive concentration near 0.

In fact, the general non-uniformity with respect to $\mathcal{O}$ suggested in our concentration estimate (1.6) *would provide a mechanism of possible flatter tails of multiple QSDs. We note that if conditions* **(A2)**-**(A4)** *below hold, then this estimate becomes uniform in $\mathcal{O}$ away from $\mathcal{A}$ (see the paragraph*

below Theorem B for details), implying that, under these conditions, for any $0 < \epsilon \ll 1$, no QSD (though there can be infinitely many of them) can have a flat tail.

Note that for each fixed QSD ν_ϵ, there is a unique λ_ϵ as in Definition 1.1. In what follows, λ_ϵ is always the unique number associated with a given QSD ν_ϵ.

As the ODE (1.2) is one-dimensional, the global attractor $\mathcal{A}$ is either a single equilibrium, or a compact interval consisting of equilibria, or a compact interval consisting of equilibria and their connecting orbits. As invariant measures of $\{\varphi^t\}_{t\in\mathbb{R}}$ can not have positive measure on any connecting orbit, any limiting measure of $\{\nu_\epsilon\}_\epsilon$ as $\epsilon \to 0$, if exists, must be supported on the set of non-zero equilibria of (1.2).

The noise-vanishing concentration of QSDs has been investigated in several situations. The first work dates back to [33], where the author studied the stochastic Ricker model. This work was generalized in [41, 58] to randomly perturbed interval maps that apply to density-dependent branching processes. Further generalizations were considered in [37, 20], where the authors studied general randomly perturbed dynamical systems, and applied their abstract results to various population models. The approaches taken in the aforementioned works are probabilistic and based on large deviation arguments. A detailed analysis of the QSDs has been done in [9] for scaled one-dimensional birth-and-death processes of logistic type, in which the concentration of QSDs on the Dirac delta measure at the stable equilibrium of the mean field ODE in the Gaussian fashion is directly shown.

Our approach to the concentration of QSDs of (1.1) is based on the construction of Lyapunov-type functions. More precisely, we construct uniform-in-ϵ Lyapunov-type functions using the dynamics of (1.2) and prove pointwise estimates for $\{v_\epsilon\}_\epsilon$ as stated in Theorem A (1) on regions where Lyapunov-type conditions are satisfied. It turns out that such regions can be constructed to exhaust $(0,\infty) \setminus \mathcal{A}$. As a result, locally uniform pointwise estimates of $\{v_\epsilon\}_\epsilon$ are established on $(0,\infty) \setminus \mathcal{A}$. Our approach has at least two features. (i) It gives quantitative estimates of QSDs that are not restricted to the domain $(0,\infty) \setminus \mathcal{A}$. In fact, if $\mathcal{J}$ is a local attractor and Ω is its domain of attraction, then locally uniform pointwise estimates for $\{v_\epsilon\}_\epsilon$ as stated in Theorem A (1) can be established on $\Omega \setminus \mathcal{J}$. By constructing anti-Lyapunov-type functions, similar estimates could be established for local repellers. (ii) Since the construction of Lyapunov-type functions only utilizes the dynamics of the ODE (1.2), our approach has the potential to be generalized to treat problems in higher dimensions that even exhibit chaotic behaviors.

We remark that the constant $\gamma_\mathcal{O}$ appearing in Theorem A (1) depends in particular on constructed uniform-in-ϵ Lyapunov-type functions. Therefore, it is unclear how to determine the minimal one. A promising approach to this issue is to establish the large deviation principle [15] for the QSDs $\{\nu_\epsilon\}_\epsilon$, that is, to study the limit $\lim_{\epsilon\to 0^+} \epsilon^2 \ln \nu_\epsilon(\cdot)$ or $\lim_{\epsilon\to 0^+} \epsilon^2 \ln v_\epsilon$. This interesting problem is left for future work.

Theorem A only provides a priori information on the concentration of the QSDs $\{\nu_\epsilon\}_\epsilon$ as $\epsilon \to 0$. A valid description of the concentration requires the tightness of the family $\{\nu_\epsilon\}_\epsilon$. This is a challenging problem as we have to handle the troubles caused by the behaviors of coefficients near 0 and ∞. To address this issue, we make the following additional assumptions.

(A2) There exists a C^2 function U_∞ defined on $[x_\infty, \infty)$ for some $x_\infty \gg 1$ such that
    (1) $0 < \inf U_\infty < \sup U_\infty < \infty$;
    (2) $\lim_{x\to\infty} U_\infty(x) = \sup U_\infty$;
    (3) there is $\gamma_\infty > 0$ such that $\mathcal{L}_\epsilon U_\infty \leq -\gamma_\infty$ on $[x_\infty, \infty)$ for all $0 < \epsilon \ll 1$.

(A3) $a \in C^2([0, \infty))$ and $\int_0^1 \frac{1}{\sqrt{a(s)}} ds < \infty$.

(A4) $b \in C^1([0, \infty))$ and $b'(0) > 0$.

The assumption **(A2)** is a uniform-in-ϵ Lyapunov-type condition for the SDE (1.1) near ∞, which ensures the dissipativity of its solutions near ∞ that leads to concentration estimates of $\{\nu_\epsilon\}_\epsilon$ near ∞. The assumptions **(A3)** and **(A4)** are motivated by applications (see Section 4), and guarantee the existence of uniform-in-ϵ integrable upper bounds near 0 for the densities $\{v_\epsilon\}_\epsilon$ of the QSDs $\{\nu_\epsilon\}_\epsilon$. This is far-reaching due to $a(0) = 0$ and the singularity of $x \mapsto \sqrt{a(x)}$ at 0. We would like to point out that Lyapunov-type conditions near 0 do NOT exist under these assumptions. Indeed, if U satisfies the Lyapunov-type condition near 0, then it is necessary that $U'(x) \to -\infty$ and $U''(x) \to \infty$ as $x \to 0^+$. In this case, it is generally true that $\lim_{x\to 0^+} \frac{U'(x)}{U''(x)} = 0$. As the limit $\lim_{x\to 0^+} \frac{a(x)}{b(x)}$ exists and is a positive number due to **(A3)** and **(A4)**, there holds $\liminf_{x\to 0^+} \mathcal{L}_\epsilon U(x) \geq 0$, which contradicts the Lyapunov-type condition. Hence, a new approach independent of Lyapunov-type functions has to be developed to treat $\{\nu_\epsilon\}_\epsilon$ near 0.

Our result concerning the tightness of the family of QSDs $\{v_\epsilon\}_\epsilon$ is stated in the following theorem.

Theorem B. *Assume* **(A1)**-**(A4)** *and let* $\{\nu_\epsilon\}_\epsilon$ *be a family of QSDs of* (1.1). *Then, any sequence* $\{\nu_{\epsilon_n}\}_n$ *of* $\{\nu_\epsilon\}_\epsilon$ *with* $\epsilon_n \to 0$ *as* $n \to \infty$ *is tight. Consequently, as* $\epsilon \to 0$, *any sequence of* $\{\nu_\epsilon\}_\epsilon$ *has a convergent subsequence in the topology of weak convergence whose limit is an invariant measure of* $\{\varphi^t\}_{t\in\mathbb{R}}$ *and is supported in* $\mathcal{A}$. *In particular, if* $\mathcal{A} = \{x_e\}$ *is a singleton set, then*

$$\lim_{\epsilon \to 0^+} \nu_\epsilon = \delta_{x_e} \quad \textit{in the topology of weak convergence,}$$

where δ_{x_e} *is the Dirac delta measure at* x_e.

The tightness in Theorem B follows from studying concentration estimates of $\{v_\epsilon\}_\epsilon$ near 0 and ∞. Using the Lyapunov-type condition **(A2)**, we show in Proposition 3.1 the existence of some $\gamma > 0$ such that

$$\nu_\epsilon((x_\infty + 1, \infty)) \leq e^{-\frac{\gamma}{\epsilon^2}}, \quad \forall 0 < \epsilon \ll 1,$$

where x_∞ is given in **(A2)**. Establishing the concentration estimates of $\{\nu_\epsilon\}_\epsilon$ near 0 is a tricky task due to the degeneracy and singularity of the noises at the absorbing state 0. By examining the densities $\{v_\epsilon\}_\epsilon$ of the QSDs $\{\nu_\epsilon\}_\epsilon$ in the new coordinate system obtained from applying the standard change of variables for one-dimensional diffusion processes (see the proof of Lemma 3.1 for more details), we show in Corollary 3.1 the boundedness of v_ϵ near 0 with an ϵ-dependent upper bound. Arguments in the spirit of proving a maximum principle then allow us to establish in Proposition 3.2 the following uniform-in-ϵ integrable upper bounds near 0: for each $\kappa \in (0, 1)$ there are $0 < x_* \ll 1$ and $0 < \epsilon_* \ll 1$ such that

$$v_\epsilon(x) \leq \frac{1}{x^\kappa}, \quad \forall x \in (0, x_*), \quad \epsilon \in (0, \epsilon_*).$$

Assumptions **(A3)** and **(A4)** ensure the successfulness of such arguments.

As mentioned earlier, QSDs describe the asymptotic dynamics of the SDE (1.1) before extinction for each fixed ϵ. Theorem B yields that such asymptotic dynamics have uniform-in-ϵ properties that are governed by the dynamics of the ODE (1.2) on the global attractor $\mathcal{A}$. Therefore, information about the transient states is essentially contained in $\mathcal{A}$. Moreover, Theorem B implies the compatibility, up to observables, of the ODE (1.2) and the SDE (1.1), as models for the evolution of the same species. This validates the ODE model (1.2) if the global attractor $\mathcal{A}$ is interpreted as observed states instead of eventual states of the species. A clearer picture can be drawn in the case that $\mathcal{A} = \{x_e\}$ is a

singleton set as follows: if for each $0 < \epsilon \ll 1$, X_t^ϵ conditioned on the survival up to time t (denoted by $\tilde{X}_t^\epsilon$) converges to the QSD ν_ϵ as $t \to \infty$, then the diagram

$$
\begin{array}{ccc}
\tilde{X}_t^\epsilon & \xrightarrow{t \to \infty} & \nu_\epsilon \\
{\scriptstyle \epsilon \to 0} \downarrow & & \downarrow {\scriptstyle \epsilon \to 0} \\
x_t & \xrightarrow[t \to \infty]{} & \delta_{x_e}
\end{array}
$$

commutes, where the solution $\{x_t\}_{t \geq 0}$ of (1.2) is considered in the distributional sense.

As applications of Theorem A and Theorem B, we study diffusion processes related to chemical reactions and birth-and-death processes (BDPs) of logistic type. In particular, we give a rigorous justification of Keizer's paradox [38]. The corresponding results are presented in Section 4.

We would like to highlight some novelties of the present paper. (i) Inspired by the work [34] in which measure estimates of stationary measures under Lyapunov conditions are treated, we establish the framework for estimating QSDs against Lyapunov functions (see Subsection 2.2). To apply such a framework, we develop a way of constructing uniform-in-ϵ Lyapunov-type functions in domains of interest using the dynamical properties of the unperturbed ODE (see Subsection 2.3). Both methods are not tailored for one-dimensional problems, and have potential analogues in higher-dimensions. (ii) The most challenging issue is to deal with difficulties caused by the degeneracy and singularity of noises at the absorbing state in order to derive an uniform-in-ϵ integrable upper bound of QSDs near 0, leading to the tightness of the QSDs. This is accomplished by a two-step approach: a careful analysis of eigen-equations satisfied by QSDs in new coordinates produces ϵ-dependent upper bounds (see Lemma 3.1 and Corollary 3.1); it is followed by a result of maximum principle type giving rise to the expected upper bound. As it is indicated, our approaches are purely analytic without probabilistic heuristics.

To end this section, we comment on the possibilities and difficulties in the extension to higher-dimensions. For clarity, we focus on the following stochastic system:

$$
dX_{i,t} = X_{i,t} f_i(X_t) dt + \epsilon \sqrt{X_{i,t}} dW_t^i, \quad i \in \{1, \ldots, d\}, \quad X_0 \in [0, \infty)^d, \tag{1.7}
$$

where $X_t = (X_{1,t}, \ldots, X_{d,t})$, $0 < \epsilon \ll 1$, $d \in \mathbb{N}$ and W_t^i, $i \in \{1, \ldots, d\}$ are independent standard one-dimensional Wiener processes on some probability space. Such a system, often arising from biology and ecology, is used to model the evolution of interacting species in a community. The functions $f_i : [0, \infty)^d \to \mathbb{R}$, $i \in \{1, \ldots, d\}$, often called *per-capita growth rates*, are sufficiently regular and ensure that the system of ODEs:

$$
\dot{x}_i = x_i f_i(x), \quad i \in \{1, \ldots, d\}, \quad x = (x_1, \ldots, x_d) \in (0, \infty)^d \tag{1.8}
$$

is dissipative, that is, there exists a compact set $K \subset (0, \infty)^d$ such that for any solution $x(t)$ of (1.8) with initial condition $x(0) = x_0 \in (0, \infty)^d$, there exists $t_{x_0} > 0$ such that $x(t) \in K$ for all $t \geq t_{x_0}$. Then, (1.8) admits a global attractor $\mathcal{A}_{sys}$ contained in K.

Suppose now the stochastic system (1.7) admits QSDs $\{\tilde{\nu}_\epsilon\}_\epsilon$ on $(0, \infty)^d$ with extinction rates $\{\lambda_\epsilon\}_\epsilon$. If we can show $\lambda_\epsilon \to 0$ as $\epsilon \to 0^+$, then it is possible to construct uniform-in-ϵ Lyapunov-type functions, using the dynamical properties of (1.8), to derive the concentration estimates: for any $\mathcal{O} \subset\subset (0, \infty)^d \setminus \mathcal{A}_{sys}$, there are $\gamma_\mathcal{O} > 0$ and $0 < \epsilon_\mathcal{O} \ll 1$ such that

$$
\tilde{\nu}_\epsilon(\mathcal{O}) \leq e^{-\frac{\gamma_\mathcal{O}}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\mathcal{O}].
$$

Pointwise estimates for the densities of $\{\tilde{\nu}_\epsilon\}_\epsilon$ are also possible. Each limit measure of $\{\tilde{\nu}_\epsilon\}_\epsilon$, if exists, must be an invariant measure of (1.8). However, our approach can be barely adapted to show the tightness of the QSDs $\{\tilde{\nu}_\epsilon\}_\epsilon$, which could be an extremely difficult problem. The reason is that our approach more or less relies on the fact that the one-dimensional diffusion (1.1) is reversible, or the generator $\mathcal{L}_\epsilon$ given in (1.4) extends to a self-adjoint operator in $L^2((0,\infty), u_\epsilon(x)dx)$, where

$$u_\epsilon(x) = \frac{1}{a(x)} \exp\left\{ \frac{2}{\epsilon^2} \int_0^x \frac{b(s)}{a(s)} ds \right\}, \quad x \in (0,\infty).$$

But, the stochastic system (1.7) is irreversible in general.

To have a better understanding of the system (1.7) in regard to the extinction or the loss of the population diversity, we need to acquire relevant knowledge about the noise-vanishing asymptotic of the extinction event or exit event

$$(T_\Gamma^\epsilon, X_{T_\Gamma^\epsilon}^\epsilon),$$

where Γ is the boundary of $[0,\infty)^d$, $\{X_t^\epsilon\}_{t\geq 0}$ is the diffusion process generated by solutions of (1.7), and T_Γ^ϵ is the first time that $\{X_t^\epsilon\}_{t\geq 0}$ reaches Γ. The noise-vanishing asymptotic of the exit event for randomly perturbed dynamical systems is a classical topic, has attracted a lot of attention and finds many applications (see e.g. [13, 14, 26]). Recently, exploring very fine asymptotic of the QSD and its associated extinction rate, the authors obtain in [46, 17, 18, 45] very fine asymptotic of the exit event from a smooth bounded open domain Ω for the overdamped Langevin equation

$$dX_t = -\nabla f(X_t)dt + \epsilon dW_t, \quad X_0 \in \mathbb{R}^d, \tag{1.9}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is sufficiently smooth and W_t is the standard d-dimensional Wiener process. A typical case treated in these works assumes that f has a unique critical point x_* in Ω and satisfies $\frac{\partial f}{\partial n}|_{\partial \Omega} > 0$. These conditions imply that $\overline{\Omega}$ is positively invariant under the flow generated by

$$\dot{x} = -\nabla f(x), \tag{1.10}$$

and contained in the basin of attraction of x_*.

When the global attractor $\mathcal{A}_{sys}$ is a singleton set, the dynamics of (1.8) is as simple as that of (1.10) in Ω under the aforementioned conditions on f. However, (1.8) generally has equilibria on the boundary Γ. In addition, the stochastic system (1.7) is degenerate on the boundary Γ and irreversible in general. In consideration of these essential differences, results and approaches for (1.9) do not apply to (1.7). Nonetheless, the works [46, 17, 18, 45] ignite the light of hope and would for sure guide the study of the asymptotic of $(T_\Gamma^\epsilon, X_{T_\Gamma^\epsilon}^\epsilon)$, especially when tackling the problem from the perspective of the asymptotic of the QSD and its associated extinction rate.

The rest of the paper is organized as follows. In Section 2, we study the concentration of QSDs and prove Theorem A. In Section 3, we investigate the tightness of QSDs and prove Theorem B. Applications of Theorem A and Theorem B are discussed in Section 4. We include a Harnack's inequality in Appendix A.

2. Concentration estimates of QSDs

Throughout this section, let $\{\nu_\epsilon\}_\epsilon$ be the QSDs of (1.1) and $\{\lambda_\epsilon\}_\epsilon$ be the unique numbers corresponding to $\{\nu_\epsilon\}_\epsilon$ as given in Definition 1.1. The purpose of this section is to derive concentration estimates of $\{\nu_\epsilon\}_\epsilon$ and prove Theorem A. We first study the asymptotics of λ_ϵ as $\epsilon \to 0^+$ in Subsection 2.1. In Subsection 2.2, we provide concentration estimates of the QSDs $\{\nu_\epsilon\}_\epsilon$ assuming the existence of Lyapunov-type functions. The proof of Theorem A is given in Subsection 2.3.

2.1. **Asymptotic of λ_ϵ as $\epsilon \to 0^+$.** The following result gives the regularity of the QSDs $\{\nu_\epsilon\}_\epsilon$. Its proof follows from the standard regularity theory of second-order ODEs.

Lemma 2.1. *Assume* **(A1)**. *For each $0 < \epsilon \ll 1$, ν_ϵ admits a positive density $v_\epsilon \in C^2((0, \infty))$. In particular, v_ϵ satisfies*

$$L_\epsilon v_\epsilon = -\lambda_\epsilon v_\epsilon \quad on \quad (0, \infty). \tag{2.1}$$

The asymptotic behaviors of λ_ϵ as $\epsilon \to 0$ is stated in the following result.

Proposition 2.1. *If* **(A1)** *holds, then there is $\Lambda > 0$ such that $\lambda_\epsilon \leq e^{-\frac{\Lambda}{\epsilon^2}}$ for all $0 < \epsilon \ll 1$.*

Proof. Set $v_\epsilon^G(x) := \frac{1}{a(x)} e^{-\frac{2}{\epsilon^2} \int_1^x \frac{b(s)}{a(s)} ds}$ for $x \in (0, \infty)$. It is just the Gibbs density of (1.1). Given (2.1), it is straightforward to check that $\phi_\epsilon := \frac{v_\epsilon}{v_\epsilon^G}$ satisfies

$$\mathcal{L}_\epsilon \phi_\epsilon = -\lambda_\epsilon \phi_\epsilon \quad on \quad (0, \infty).$$

By **(A1)**, there are $0 < x_1 < x_2 < \infty$ such that $b(x_1) > 0$ and $b(x_2) < 0$. Note that $\mathcal{L}_\epsilon$ is uniformly elliptic on $[x_1, x_2]$. Denote by $\Lambda_\epsilon > 0$ the principal eigenvalue of $-\mathcal{L}_\epsilon$ restricted on (x_1, x_2) with zero Dirichlet boundary conditions on x_1 and x_2. It is well known (see e.g. [25]) that there is $\Lambda > 0$ such that $\Lambda_\epsilon \leq e^{-\frac{\Lambda}{\epsilon^2}}$ for all $0 < \epsilon \ll 1$. The conclusion then follows from the standard comparison principle (see e.g. [16, Lemma 4.1]) that ensures $\lambda_\epsilon \leq \Lambda_\epsilon$. $\square$

We point out that the fact $\lambda_\epsilon \to 0$ as $\epsilon \to 0^+$ implied by Proposition 2.1 plays important roles in the sequel.

2.2. **Abstract concentration estimates.** In this subsection, we derive abstract concentration estimates of QSDs $\{\nu_\epsilon\}_\epsilon$ assuming the existence of Lyapunov-type functions.

We start with the definition of compact functions and uniform Lyapunov functions. Let $0 \leq \alpha < \beta \leq \infty$ be given.

Definition 2.1 (Compact function). A continuous, non-negative and non-zero function U on (α, β) is called a *compact function* if

$$\rho_M := \sup_{(\alpha, \beta)} U = \lim_{x \to \alpha^+} U(x) = \lim_{x \to \beta^-} U(x).$$

We call ρ_M the *essential upper bound* of U.

For a compact function U on (α, β), we define its sub-level sets:

$$\Omega_\rho := \{x \in (\alpha, \beta) : U(x) < \rho\}, \quad \rho \in (0, \rho_M).$$

Clearly, $\Omega_{\rho_M} = (\alpha, \beta)$. For each $\rho \in (0, \rho_M]$, we denote

$$\alpha_\rho := \inf \Omega_\rho \quad \text{and} \quad \beta_\rho := \sup \Omega_\rho.$$

Note that $(\alpha_{\rho_M}, \beta_{\rho_M}) = (\alpha, \beta)$.

For each $0 < \epsilon \ll 1$, we define the operator $\mathcal{M}_\epsilon$ by setting

$$\mathcal{M}_\epsilon u := \mathcal{L}_\epsilon u + \lambda_\epsilon u = \frac{\epsilon^2}{2} a u'' + b u' + \lambda_\epsilon u.$$

Definition 2.2 (uniform Lyapunov function)**.** A C^2 compact function U on (α, β) is called a *uniform Lyapunov function* with respect to the family $\{\mathcal{M}_\epsilon\}_\epsilon$ if there exist $\rho_m \in (0, \rho_M)$ and $\gamma > 0$ such that

$$\mathcal{M}_\epsilon U \leq -\gamma \quad \text{on} \quad \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m} \tag{2.2}$$

for all $0 < \epsilon \ll 1$. We call ρ_m the *essential lower bound* of U, γ the *Lyapunov constant* of U and $\Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$ the essential domain of U.

Remark 2.1. We comment on Definition 2.2.

- Ignoring λ_ϵ in the operator $\mathcal{M}_\epsilon$, such defined uniform Lyapunov functions have been used in [35] to treat stationary measures for general diffusion processes. As we are dealing with QSDs, it is natural to include the extinction rate λ_ϵ in the definition.
- The definition of uniform Lyapunov functions more or less follows those used in [40] to study the existence of stationary distributions for SDEs (also see [54]). Except, we work with compact functions that appropriately generalize continuous and non-negative functions on $\mathbb{R}$ with pre-compact sub-level sets. This plays an important role in deriving concentration estimates for the QSDs.
- Since $\lambda_\epsilon \to 0$ as $\epsilon \to 0^+$ by Proposition 2.1, passing to the limit $\epsilon \to 0^+$ in (2.2) yields $bU' \leq -\gamma$ on $\Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$. In particular, U is a Lyapunov function of the ODE (1.2) in (α, β). This is the starting point of the construction of uniform Lyapunov functions (see Subsection 2.3).

We make the following assumption.

(A5) There exists a uniform Lyapunov function U on (α, β) with respect to $\{\mathcal{M}_\epsilon\}_\epsilon$ with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant γ.

Our results on the abstract concentration estimates of the QSDs $\{\nu_\epsilon\}_\epsilon$ are stated as follows.

Proposition 2.2. *Assume* **(A1)** *and* **(A5)**.

(1) *There hold*

$$\nu_\epsilon(\Omega_{\rho_M} \setminus \Omega_\rho) \leq e^{-\frac{2\gamma}{\epsilon^2} \int_{\rho_m}^{\rho} \frac{1}{H(t)} dt} \nu_\epsilon(\Omega_{\rho_M} \setminus \Omega_{\rho_m}), \quad \forall \rho \in (\rho_m, \rho_M], \quad 0 < \epsilon \ll 1,$$

where

$$H(\rho) = \max \left\{ a(\alpha_\rho) \left(U'(\alpha_\rho) \right)^2, a(\beta_\rho) \left(U'(\beta_\rho) \right)^2 \right\} > 0, \quad \rho \in (\rho_m, \rho_M). \tag{2.3}$$

(2) *For each open set $\mathcal{O} \subset\subset \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$, there exist $\gamma_{\mathcal{O}} > 0$ and $0 < \epsilon_{\mathcal{O}} \ll 1$ such that*

$$\sup_{\mathcal{O}} v_\epsilon \leq e^{-\frac{\gamma_{\mathcal{O}}}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_{\mathcal{O}}].$$

Before proving Proposition 2.2, we first prove two lemmas. The first one gives an integral identity.

Lemma 2.2. *Assume* **(A1)** *and* **(A5)**.

(1) $\Omega_\rho = (\alpha_\rho, \beta_\rho)$ *for all $\rho \in (\rho_m, \rho_M]$. Moreover, if $\rho \in (\rho_m, \rho_M)$, then $U'(\alpha_\rho) < 0$ and $U'(\beta_\rho) > 0$.*

(2) *For each $\rho \in (\rho_m, \rho_M]$, there holds*

$$\int_{\alpha_\rho}^{\beta_\rho} v_\epsilon \mathcal{M}_\epsilon U dx = \rho \lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx + \frac{\epsilon^2}{2} \left(a v_\epsilon U' \right) \Big|_{\alpha_\rho}^{\beta_\rho}.$$

Proof. (1) Since

$$\frac{\epsilon^2}{2}aU'' + bU' + \lambda_\epsilon u = \mathcal{M}_\epsilon U \le -\gamma \quad \text{on} \quad \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$$

for all $0 < \epsilon \ll 1$, we conclude from Proposition 2.1 that $bU' \le -\frac{\gamma}{2}$ on $\Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$, which implies that U' admits a constant sign on each component of $\Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$. As $\Omega_{\rho_M} = (\alpha, \beta) = (\alpha_{\rho_M}, \beta_{\rho_M})$ and $\Omega_{\rho_m} \subset (\alpha_m, \beta_m)$, there holds

$$(\alpha_{\rho_M}, \alpha_{\rho_m}) \cup (\beta_{\rho_m}, \beta_{\rho_M}) \subset \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}.$$

It then follows from the definition of U that $U' < 0$ on $(\alpha_{\rho_M}, \alpha_{\rho_m})$ and $U' > 0$ on $(\beta_{\rho_m}, \beta_{\rho_M})$. In particular, $U'(\alpha_\rho) < 0$ and $U'(\beta_\rho) > 0$ for each $\rho \in (\rho_m, \rho_M)$.

Let $\rho \in (\rho_m, \rho_M)$. If $\Omega_\rho \subsetneqq (\alpha_\rho, \beta_\rho)$, then U admits a local maximum value at some point $x_0 \in (\alpha_\rho, \beta_\rho) \setminus \Omega_\rho$. Hence, $U'(x_0) = 0$. But, $(\alpha_\rho, \beta_\rho) \setminus \Omega_\rho \subset \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$ yields that $U'(x_0) \ne 0$, which leads to a contradiction. Consequently, $\Omega_\rho = (\alpha_\rho, \beta_\rho)$.

(2) Multiplying (2.1) by U and integrating the resulting equation over Ω_ρ, we obtain from integration by parts that

$$\frac{\epsilon^2}{2}\left[(av_\epsilon)'U\right]\Big|_{\alpha_\rho}^{\beta_\rho} - \frac{\epsilon^2}{2}(av_\epsilon U')\Big|_{\alpha_\rho}^{\beta_\rho} + \frac{\epsilon^2}{2}\int_{\alpha_\rho}^{\beta_\rho} av_\epsilon U'' dx - (bv_\epsilon U)\Big|_{\alpha_\rho}^{\beta_\rho} + \int_{\alpha_\rho}^{\beta_\rho} bv_\epsilon U' dx = -\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon U dx. \quad (2.4)$$

Since $U(\alpha_\rho) = \rho = U(\beta_\rho)$, we find

$$\frac{\epsilon^2}{2}\left[(av_\epsilon)'U\right]\Big|_{\alpha_\rho}^{\beta_\rho} - (bv_\epsilon U)\Big|_{\alpha_\rho}^{\beta_\rho} = \rho\left[\frac{\epsilon^2}{2}(av_\epsilon)'\Big|_{\alpha_\rho}^{\beta_\rho} - (bv_\epsilon)\Big|_{\alpha_\rho}^{\beta_\rho}\right].$$

Moreover, integrating (2.1) over Ω_ρ, we obtain

$$\frac{\epsilon^2}{2}(av_\epsilon)'\Big|_{\alpha_\rho}^{\beta_\rho} - (bv_\epsilon)\Big|_{\alpha_\rho}^{\beta_\rho} = -\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx.$$

It follows that

$$\frac{\epsilon^2}{2}\left[(av_\epsilon)'U\right]\Big|_{\alpha_\rho}^{\beta_\rho} - (bv_\epsilon U)\Big|_{\alpha_\rho}^{\beta_\rho} = -\rho\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx.$$

Substituting the above equality into (2.4), we arrive at

$$-\rho\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx - \frac{\epsilon^2}{2}(av_\epsilon U')\Big|_{\alpha_\rho}^{\beta_\rho} + \frac{\epsilon^2}{2}\int_{\alpha_\rho}^{\beta_\rho} av_\epsilon U'' dx + \int_{\alpha_\rho}^{\beta_\rho} bv_\epsilon U' dx = -\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon U dx,$$

which leads to the result. $\qquad\square$

In the second lemma, we use the Harnack's inequality, Lemma A.1, to derive some estimates of $\{v_\epsilon\}_\epsilon$.

Lemma 2.3. *Assume* **(A1)**. *For any open interval $\mathcal{I} \subset\subset (0, \infty)$, if $x, R > 0$ are such that $(x - 4R, x + 4R) \subset \mathcal{I}$, then*

$$\sup_{(x-R, x+R)} v_\epsilon \le C_0^{\frac{\sup_{\mathcal{I}} a}{\inf_{\mathcal{I}} a} + \frac{\epsilon^2 C_1 R}{\inf_{\mathcal{I}} a}} \inf_{(x-R, x+R)} v_\epsilon, \quad \forall 0 < \epsilon \ll 1,$$

where $C_0 > 0$ is a universal constant and $C_1 = C_1(\epsilon) = \sqrt{\sup_{\mathcal{I}}\left(\frac{\epsilon^2}{2}a' - b\right)^2 + \frac{\epsilon^2 \lambda_\epsilon}{2}\inf_{\mathcal{I}} a}$.

Proof. We apply Lemma A.1 in the case $\alpha = \frac{\epsilon^2}{2}a$, $\beta = \frac{\epsilon^2}{2}a' - b$ and $\kappa = \lambda_\epsilon$ so that $Lu = \frac{\epsilon^2}{2}(au)'' - (bu)' + \lambda_\epsilon u$ is in the form of interest.

Let

$$\lambda = \frac{\epsilon^2}{2}\inf_{\mathcal{I}} a, \quad \Lambda = \frac{\epsilon^2}{2}\sup_{\mathcal{I}} a, \quad \text{and} \quad \nu = \sqrt{\frac{\sup_{\mathcal{I}}\left(\frac{\epsilon^2}{2}a' - b\right)^2}{\lambda^2} + \frac{\lambda_\epsilon}{\lambda}}.$$

Then, for any $R > 0$,

$$\frac{\Lambda}{\lambda} + \nu R = \frac{\sup_{\mathcal{I}} a}{\inf_{\mathcal{I}} a} + \frac{\epsilon^2 C_1 R}{\inf_{\mathcal{I}} a},$$

where $C_1 = C_1(\epsilon)$ is as in the statement of the lemma. Applying Lemma A.1, we find the result. $\square$

Now, we prove Proposition 2.2.

Proof of Proposition 2.2. (1) Let $\rho_* \in (\rho_m, \rho_M)$. Applying Lemma 2.2 (2), we find for any $\rho \in (\rho_m, \rho_*]$ that

$$-\int_{(\alpha_{\rho_*}, \beta_{\rho_*}) \backslash (\alpha_\rho, \beta_\rho)} v_\epsilon \mathcal{M}_\epsilon U\, dx$$

$$= \rho\lambda_\epsilon \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx - \rho_*\lambda_\epsilon \int_{\alpha_{\rho_*}}^{\beta_{\rho_*}} v_\epsilon dx + \frac{\epsilon^2}{2}\left(av_\epsilon U'\right)\Big|_{\alpha_\rho}^{\beta_\rho} - \frac{\epsilon^2}{2}\left(av_\epsilon U'\right)\Big|_{\alpha_{\rho_*}}^{\beta_{\rho_*}}$$

$$\leq \frac{\epsilon^2}{2}\left(av_\epsilon U'\right)\Big|_{\alpha_\rho}^{\beta_\rho},$$

where we used the facts $\rho \int_{\alpha_\rho}^{\beta_\rho} v_\epsilon dx \leq \rho_* \int_{\alpha_{\rho_*}}^{\beta_{\rho_*}} v_\epsilon dx$ and $(av_\epsilon U')|_{\alpha_\rho}^{\beta_\rho} \geq 0$ due to the positivity of a and v_ϵ on $(0, \infty)$

Denote

$$f(\rho) := \int_{(\alpha_{\rho_*}, \beta_{\rho_*}) \backslash (\alpha_\rho, \beta_\rho)} v_\epsilon dx.$$

As $\mathcal{M}_\epsilon U \leq -\gamma$ on $\Omega_{\rho_M} \backslash \overline{\Omega}_{\rho_m}$ for all $0 < \epsilon \ll 1$, we have

$$f(\rho) \leq \frac{\epsilon^2}{2\gamma}\left(av_\epsilon U'\right)\Big|_{\alpha_\rho}^{\beta_\rho}, \quad \forall \rho \in (\rho_m, \rho_*], \quad 0 < \epsilon \ll 1. \tag{2.5}$$

Since $U(\alpha_\rho) = \rho = U(\beta_\rho)$, we have from Lemma 2.2 (1) that

$$\frac{d\alpha_\rho}{d\rho} = \frac{1}{U'(\alpha_\rho)} \quad \text{and} \quad \frac{d\beta_\rho}{d\rho} = \frac{1}{U'(\beta_\rho)}.$$

It follows that

$$f'(\rho) = \frac{d}{d\rho}\int_{(\alpha_{\rho_*}, \beta_{\rho_*}) \backslash (\alpha_\rho, \beta_\rho)} v_\epsilon dx = \frac{v_\epsilon(\alpha_\rho)}{U'(\alpha_\rho)} - \frac{v_\epsilon(\beta_\rho)}{U'(\beta_\rho)}.$$

For the term on the right-hand side of (2.5), we have

$$\left(av_\epsilon U'\right)\Big|_{\alpha_\rho}^{\beta_\rho} = a(\beta_\rho)\left(U'(\beta_\rho)\right)^2\frac{v_\epsilon(\beta_\rho)}{U'(\beta_\rho)} - a(\alpha_\rho)\left(U'(\alpha_\rho)\right)^2\frac{v_\epsilon(\alpha_\rho)}{U'(\alpha_\rho)}$$

$$\leq -H(\rho)\left[\frac{v_\epsilon(\alpha_\rho)}{U'(\alpha_\rho)} - \frac{v_\epsilon(\beta_\rho)}{U'(\beta_\rho)}\right]$$

$$= -H(\rho)f'(\rho),$$

where $H(\rho)$ is as in (2.3) and is positive due to **(A1)**(2) and Lemma 2.2 (1). Hence, we have from (2.5) that

$$\frac{\epsilon^2 H(\rho)}{2\gamma} f'(\rho) + f(\rho) \le 0, \quad \forall \rho \in (\rho_m, \rho_*], \quad 0 < \epsilon \ll 1.$$

It follows that

$$f(\rho) \le e^{-\frac{2\gamma}{\epsilon^2} \int_{\rho_m}^{\rho} \frac{1}{H(t)} dt} f(\rho_m), \quad \forall \rho \in (\rho_m, \rho_*], \quad 0 < \epsilon \ll 1,$$

that is,

$$\int_{(\alpha_{\rho_*}, \beta_{\rho_*}) \setminus (\alpha_\rho, \beta_\rho)} v_\epsilon dx \le e^{-\frac{2\gamma}{\epsilon^2} \int_{\rho_m}^{\rho} \frac{1}{H(t)} dt} \int_{(\alpha_{\rho_*}, \beta_{\rho_*}) \setminus (\alpha_{\rho_m}, \beta_{\rho_m})} v_\epsilon dx, \quad \forall \rho \in (\rho_m, \rho_*], \quad 0 < \epsilon \ll 1.$$

Letting $\rho_* \to \rho_M$, we prove (1).

(2) We assume, without loss of generality, that $\mathcal{O}$ is an interval. Let $\delta > 0$ be such that $\mathcal{O}_\delta \subset\subset \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$, where $\mathcal{O}_\delta$ is the δ-neighborhood of $\mathcal{O}$. We see from (1) that there exist $\gamma_\delta > 0$ and $\epsilon_\delta > 0$ such that

$$\nu_\epsilon(\mathcal{O}_\delta) \le e^{-\frac{\gamma_\delta}{\epsilon^2}}, \quad \forall \epsilon \in (0, \epsilon_\delta]. \tag{2.6}$$

Suppose for contradiction that the conclusion fails. Then there exist sequences $\{x_n\}_n \subset \mathcal{O}$ and $\{\epsilon_n\}_n \subset (0, \infty)$ satisfying $x_n \to x_\infty$ for some $x_\infty \in \overline{\mathcal{O}}$ and $\epsilon_n \to 0$ as $n \to \infty$ such that

$$v_{\epsilon_n}(x_n) > e^{-\frac{\gamma_\delta}{2\epsilon_n^2}}, \quad \forall n.$$

Applying Lemma 2.3 with $\mathcal{I} = \mathcal{O}_\delta$, we have

$$\sup_{(x-R, x+R)} v_\epsilon \le C_0^{\frac{\sup_{\mathcal{O}_\delta} a}{\inf_{\mathcal{O}_\delta} a} + \frac{C_1(\epsilon) R}{\inf_{\mathcal{O}_\delta} a} \frac{1}{\epsilon^2}} \inf_{(x-R, x+R)} v_\epsilon, \quad \forall x \in \mathcal{O}, \quad R \in \left(0, \frac{\delta}{4}\right), \quad 0 < \epsilon \ll 1, \tag{2.7}$$

where $C_0 > 0$ is a universal constant and

$$C_1(\epsilon) = \sqrt{\sup_{\mathcal{O}_\delta} \left(\frac{\epsilon^2 a'}{2} - b\right)^2 + \frac{\epsilon^2 \lambda_\epsilon \inf_{\mathcal{O}_\delta} a}{2}}.$$

Setting $x = x_n$, $R = \epsilon_n^2$ and $\epsilon = \epsilon_n$ in (2.7) yields

$$\sup_{(x_n - \epsilon_n^2, x_n + \epsilon_n^2)} v_{\epsilon_n} \le C_0^{\frac{\sup_{\mathcal{O}_\delta} a}{\inf_{\mathcal{O}_\delta} a} + \frac{C_1(\epsilon_n)}{\inf_{\mathcal{O}_\delta} a}} \inf_{(x_n - \epsilon_n^2, x_n + \epsilon_n^2)} v_{\epsilon_n}, \quad \forall n.$$

Denote $C_2 := \sup_n C_1(\epsilon_n) < \infty$. Then

$$\sup_{(x_n - \epsilon_n^2, x_n + \epsilon_n^2)} v_{\epsilon_n} \le C_2 \inf_{(x_n - \epsilon_n^2, x_n + \epsilon_n^2)} v_{\epsilon_n}, \quad \forall n.$$

It follows that

$$\nu_{\epsilon_n}((x_n - \epsilon_n^2, x_n + \epsilon_n^2)) = \int_{x_n - \epsilon_n^2}^{x_n + \epsilon_n^2} v_{\epsilon_n} dx \ge \frac{2\epsilon_n^2}{C_2} \sup_{(x_n - \epsilon_n^2, x_n + \epsilon_n^2)} v_{\epsilon_n} \ge \frac{2\epsilon_n^2}{C_2} e^{-\frac{\gamma_\delta}{2\epsilon_n^2}}, \quad \forall n.$$

However, the inclusion $(x_n - \epsilon_n^2, x_n + \epsilon_n^2) \subset \mathcal{O}_\delta$ and (2.6) yield that

$$\nu_{\epsilon_n}((x_n - \epsilon_n^2, x_n + \epsilon_n^2)) \le e^{-\frac{\gamma_\delta}{\epsilon_n^2}}, \quad \forall n \gg 1,$$

which leads to a contradiction. This proves (2). $\qquad\square$

2.3. **Proof of Theorem A.** We derive concentration estimates for the QSDs $\{\nu_\epsilon\}_\epsilon$ by constructing uniform Lyapunov functions and applying Proposition 2.2. In particular, we prove Theorem A in this subsection.

We recall the definition of limiting measures.

Definition 2.3 (Limiting measure). A Borel probability measure ν on $(0, \infty)$ is called a *limiting measure* of $\{\nu_\epsilon\}_\epsilon$ as $\epsilon \to 0$ if there exists a sequence $\{\epsilon_n\}_n$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \nu_{\epsilon_n} = \nu \text{ in the topology of weak convergence,}$$

i.e., $\lim_{n\to\infty} \int_0^\infty \phi d\nu_{\epsilon_n} = \int_0^\infty \phi d\nu$ for any bounded and continuous function $\phi : (0, \infty) \to \mathbb{R}$.

Recall that the global attractor $\mathcal{A}$ of (1.2) in $(0, \infty)$ is a compact closed interval. More precisely, if we set

$$\alpha_\mathcal{A} := \min \left\{ x \in (0, \infty) : b(x) = 0 \right\},$$
$$\beta_\mathcal{A} := \max \left\{ x \in (0, \infty) : b(x) = 0 \right\},$$

then $\mathcal{A} = [\alpha_\mathcal{A}, \beta_\mathcal{A}]$.

We first prove Theorem A (1).

Proof of Theorem A (1). The regularity of ν_ϵ follows from Lemma 2.1. We prove the concentration estimates.

For clarity, we first give the proof under the stronger condition that $b \in C^2((0, \infty))$. It is followed by a sketch on how to proceed under the conditions on b using approximation techniques.

We first construct a candidate of uniform Lyapunov functions. Recall that $\{\varphi^t\}_{t\in\mathbb{R}}$ is the flow generated by the solutions of (1.2) in $(0, \infty)$. Let $\theta : (0, \infty) \to [0, \infty)$ be a smooth function satisfying the following conditions:

- θ vanishes exactly on $\mathcal{A}$;
- $\liminf_{x\to 0^+} \theta(x) > 0$ and $\liminf_{x\to\infty} \theta(x) > 0$;
- $\theta(x) \to 0$ so fast as $x \to \alpha_\mathcal{A}^-$ and $x \to \beta_\mathcal{A}^+$ that the integral $\int_0^\infty \theta(\varphi^t(x))dt$ is well-defined for any $x \in (0, \infty) \setminus \mathcal{A}$, and the function $x \mapsto \int_0^\infty \theta(\varphi^t(x))dt$ is C^2 on $(0, \infty)$.

The existence of such a function θ with the focus on the fast convergence $\theta(x) \to 0$ as $x \to \alpha_\mathcal{A}^-$ or $\beta_\mathcal{A}^+$ can be seen as follows. We note that

- if $x \in \mathcal{A} = [\alpha_\mathcal{A}, \beta_\mathcal{A}]$, then $\varphi^t(x) \in \mathcal{A}$ and hence $\theta(\varphi^t(x)) = 0$ for all $t \geq 0$, leading to $\int_0^\infty \theta(\varphi^t(x))dt = 0$;
- if $x \in (0, \alpha_\mathcal{A})$, then $\varphi^t(x)$ is increasing in t and $\lim_{t\to\infty} \varphi^t(x) = \alpha_\mathcal{A}$; if the convergence $\theta(x) \to 0$ as $x \to \alpha_\mathcal{A}^-$ is sufficiently fast, then the integral $\int_0^\infty \theta(\varphi^t(x))dt$ is well-defined;
- if $x \in (\beta_\mathcal{A}, \infty)$, then $\varphi^t(x)$ is decreasing in t and $\lim_{t\to\infty} \varphi^t(x) = \beta_\mathcal{A}$; if the convergence $\theta(x) \to 0$ as $x \to \beta_\mathcal{A}^+$ is sufficiently fast, then the integral $\int_0^\infty \theta(\varphi^t(x))dt$ is well-defined.

The required fast convergence $\theta(x) \to 0$ as $x \to \alpha_\mathcal{A}^-$ or $\beta_\mathcal{A}^+$ in general depends on how fast $\varphi^t(x)$ converges to $\alpha_\mathcal{A}$ or $\beta_\mathcal{A}$ as $t \to \infty$ for $x \in (0, \infty) \setminus \mathcal{A}$. Given the regularity of $x \mapsto \varphi^t(x)$, we can certainly make the convergence $\theta(x) \to 0$ as $x \to \alpha_\mathcal{A}^-$ or $\beta_\mathcal{A}^+$ faster if necessary to ensure the C^2 regularity of $x \mapsto \int_0^\infty \theta(\varphi^t(x))dt$.

Consider the function $U : (0, \infty) \to [0, \infty)$ defined by

$$U(x) = \int_0^\infty \theta(\varphi^t(x))dt, \quad x \in (0, \infty). \tag{2.8}$$

By the choice of θ, the function U is well-defined and C^2. Moreover, it satisfies the following properties:

  (i) U vanishes exactly on $\mathcal{A}$;
  (ii) U is decreasing on $(0, \alpha_\mathcal{A})$ and satisfies $U(x) \to \infty$ as $x \to 0^+$;
  (iii) U is increasing on $(\beta_\mathcal{A}, \infty)$ and satisfies $U(x) \to \infty$ as $x \to \infty$;
  (iv) $bU' = -\theta$ on $(0, \infty)$. In particular, $U' < 0$ on $(0, \alpha_\mathcal{A})$ and $U' > 0$ on $(\beta_\mathcal{A}, \infty)$.

Properties (i)-(iii) are obvious. The property (iv) holds because

$$
\begin{aligned}
b(x)U'(x) &= \frac{d}{dt}U(\varphi^t(x))\Big|_{t=0} \\
&= \lim_{h \to 0} \frac{1}{h}\left[\int_0^\infty \theta(\varphi^{t+h}(x))dt - \int_0^\infty \theta(\varphi^t(x))dt\right] \\
&= \lim_{h \to 0} \frac{1}{h}\left[\int_h^\infty \theta(\varphi^t(x))dt - \int_0^\infty \theta(\varphi^t(x))dt\right] \\
&= -\lim_{h \to 0} \frac{1}{h}\int_0^h \theta(\varphi^t(x))dt = -\theta(x).
\end{aligned}
$$

Next, we *claim* that for any $\rho_* < \rho^*$, there exist $\gamma_* > 0$ and $0 < \epsilon_* \ll 1$ such that U is a uniform Lyapunov function in Ω_{ρ^*} with respect to $\{\mathcal{M}_\epsilon\}_{\epsilon \in (0, \epsilon_*]}$ with essential upper bound ρ^*, essential lower bound ρ_* and Lyapunov constant γ_*.

Note that U is a C^2 compact function on Ω_{ρ^*}. By the construction of U, we see that

$$
\mathcal{M}_\epsilon U = \frac{\epsilon^2}{2}aU'' - \theta + \lambda_\epsilon U.
$$

Clearly, there hold $\inf_{\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_*}} \theta > 0$ and $\frac{\epsilon^2}{2}\sup_{\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_*}} (a|U''|) \to 0$ as $\epsilon \to 0$. Moreover, Proposition 2.1 ensures that $\lambda_\epsilon \sup_{\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_*}} U \to 0$ as $\epsilon \to 0$. Thus, setting $\gamma_* := \frac{1}{2}\inf_{\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_*}} \theta$, we find some $0 < \epsilon_* \ll 1$ such that

$$
\mathcal{M}_\epsilon U \leq -\gamma_* \quad \text{on} \quad \Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_*}
$$

for all $\epsilon \in (0, \epsilon_*]$. This proves the *claim*.

To finish the proof, we let $\{\rho_{*,n}\}_n$ and $\{\rho_n^*\}_n$ be two sequences such that $0 < \rho_{*,n} < \rho_n^* < \infty$, and $\rho_{*,n} \to 0$ and $\rho_n^* \to \infty$ as $n \to \infty$. Obviously,

$$
\bigcup_n \left(\Omega_{\rho_n^*} \setminus \overline{\Omega}_{\rho_{*,n}}\right) = (0, \infty) \setminus \mathcal{A}.
$$

The proof in the case $b \in C^2((0, \infty))$ is now completed by applying Proposition 2.2 with the uniform Lyapunov function U in $\Omega_{\rho_n^*}$ for each n.

In the case that b only satisfies the regularity condition stated in **(A1)** (1), we still let U be defined as in (2.8). Again, U satisfies properties (i)-(iv), but now it is only a C^1 function. Let $\{U_\delta\}_{0 < \delta \ll 1}$ be a family of C^2 functions on $(0, \infty)$ such that $U_\delta \to U$ and $U_\delta' \to U'$ locally uniformly in $(0, \infty)$ as $\delta \to 0^+$. Then,

$$
bU_\delta' = -\theta + b(U_\delta' - U') \quad \text{on} \quad (0, \infty).
$$

As a result, we obtain a family of uniform Lyapunov functions. The union of their essential domains exhaust $(0, \infty) \setminus \mathcal{A}$. This completes the proof of Theorem A (1). $\qquad\square$

Remark 2.2. *The construction of the function U in the proof of Theorem A (1) is motivated by the converse Lyapunov theorem (see e.g. [62, Theorem 5.7.24] and [47, Theorem 4.3.1]) for a system of ODEs in the basin of an attracting equilibrium, which says that a smooth, strict, entire Lyapunov*

function exists in the basin of such an equilibrium, and can be constructed to have an integral form similar to (2.8).

The main idea of the integral construction of U is to integrate solutions against the function θ so that U must be non-increasing along solutions. The main feature of θ is to make sure that U is well-defined and has required regularity.

Recall that a Borel probability measure μ on $(0, \infty)$ is called an *invariant measure* of (1.2) if

$$\mu(\varphi^t B) = \mu(B), \quad \forall t \in \mathbb{R}, \quad B \in \mathcal{B}((0, \infty)),$$

where $\mathcal{B}((0, \infty))$ is the Borel σ-algebra of $(0, \infty)$. Under the condition **(A1)**(1), it follows from [35, Proposition 2.8] that μ is an invariant measure of (1.2) if and only if

$$\int_0^\infty b\phi' d\mu = 0, \quad \forall \phi \in C_0^1((0, \infty)). \tag{2.9}$$

Now, we prove Theorem A (2).

Proof of Theorem A (2). Let ν be a limiting measure of $\{\nu_\epsilon\}_\epsilon$ as $\epsilon \to 0$, that is, ν is a Borel probability measure on $(0, \infty)$ and there exists a sequence $\{\epsilon_n\}_n$ with $\epsilon_n \to 0$ as $n \to 0$ such that $\nu_{\epsilon_n} \to \nu$ in the topology of weak convergence as $n \to \infty$.

We see from Theorem A (1) that $\nu(\mathcal{A}) = 1$, that is, ν is supported on $\mathcal{A}$. Indeed, for any open set $\mathcal{O} \subset\subset (0, \infty) \setminus \mathcal{A}$, let $\tilde{\mathcal{O}}$ be an open set satisfying $\mathcal{O} \subset\subset \tilde{\mathcal{O}} \subset\subset (0, \infty) \setminus \mathcal{A}$ and $f : (0, \infty) \to [0, 1]$ be a continuous function satisfying $\mathbf{1}_\mathcal{O} \leq f \leq \mathbf{1}_{\tilde{\mathcal{O}}}$. Then

$$\nu(\mathcal{O}) \leq \int_0^\infty f d\nu = \lim_{n \to \infty} \int_0^\infty f d\nu_{\epsilon_n} \leq \lim_{n \to \infty} \nu_{\epsilon_n}(\tilde{\mathcal{O}}) = 0.$$

If $\{\mathcal{O}_k\}_k$ are open sets satisfying $\mathcal{O}_k \subset\subset (0, \infty) \setminus \mathcal{A}$ for all k and $\cup_k \mathcal{O}_k = (0, \infty) \setminus \mathcal{A}$, then

$$\nu((0, \infty) \setminus \mathcal{A}) \leq \sum_k \nu(\mathcal{O}_k) = 0,$$

which leads to $\nu(\mathcal{A}) = 1$.

It remains to show that ν is an invariant measure of the ODE (1.2). Passing to the limit $n \to \infty$ in

$$\int_0^\infty \left(\frac{\epsilon_n^2}{2} a\phi'' + b\phi' + \lambda_{\epsilon_n}\phi \right) d\nu_{\epsilon_n} = 0, \quad \forall \phi \in C_0^2((0, \infty)),$$

we conclude that

$$\int_0^\infty b\phi' d\nu = 0, \quad \forall \phi \in C_0^2((0, \infty)).$$

Indeed, since $\nu_{\epsilon_n} \to \nu$ in the topology of weak convergence as $n \to \infty$ and $\phi \in C_0^2((0, \infty))$, we find

$$\lim_{n \to \infty} \int_0^\infty a\phi'' d\nu_{\epsilon_n} = \int_0^\infty a\phi'' d\nu, \quad \lim_{n \to \infty} \int_0^\infty b\phi' d\nu_{\epsilon_n} = \int_0^\infty b\phi' d\nu \quad \text{and} \quad \lim_{n \to \infty} \int_0^\infty \phi d\nu_{\epsilon_n} = \int_0^\infty \phi d\nu.$$

It follows from $\lim_{n \to \infty} \epsilon_n = 0$ and $\lim_{n \to \infty} \lambda_{\epsilon_n} = 0$ that

$$\begin{aligned}
0 &= \int_0^\infty \left(\frac{\epsilon_n^2}{2} a\phi'' + b\phi' + \lambda_{\epsilon_n}\phi \right) d\nu_{\epsilon_n} \\
&= \frac{\epsilon_n^2}{2} \int_0^\infty a\phi'' d\nu_{\epsilon_n} + \int_0^\infty b\phi' d\nu_{\epsilon_n} + \lambda_{\epsilon_n} \int_0^\infty \phi d\nu_{\epsilon_n} \\
&\to \int_0^\infty b\phi' d\nu \quad \text{as} \quad n \to \infty.
\end{aligned}$$

A density argument then ensures that

$$\int_0^\infty b\phi' d\nu = 0, \quad \forall \phi \in C_0^1((0,\infty)),$$

that is, ν satisfies (2.9). Hence, ν is an invariant measure of (1.2). $\qquad\square$

This section is ended up with the following remark.

Remark 2.3. *We point out that the constants $\gamma_{\mathcal{O}}$ and $\epsilon_{\mathcal{O}}$ appearing in the statement of Theorem A are independent of the particular choice of the QSDs $\{\nu_\epsilon\}_\epsilon$ for the following reasons.*

(i) *The upper bound of λ_ϵ given in Proposition 2.1 is independent of the particular choice of the QSDs $\{\nu_\epsilon\}_\epsilon$. This is clear from its proof.*

(ii) *The abstract concentration estimates of the QSDs $\{\nu_\epsilon\}_\epsilon$ presented in Proposition 2.2 depend on the assumed uniform Lyapunov function U. In the proof of Theorem A, the uniform Lyapunov function U, which is constructed in (2.8), uses only the dynamical properties of the ODE (1.2) in $(0,\infty)$, and hence, is independent of the QSDs $\{\nu_\epsilon\}_\epsilon$.*

3. **Tightness of QSDs**

Throughout the section, let $\{\nu_\epsilon\}_\epsilon$ and $\{\lambda_\epsilon\}_\epsilon$ be as in Section 2. In this section, we study the tightness of the QSDs $\{\nu_\epsilon\}_\epsilon$ and prove Theorem B. In Subsection 3.1, we study concentration estimates of $\{\nu_\epsilon\}_\epsilon$ near 0 and ∞ that yield Theorem B. In Subsection 3.2, we provide the proof of a technical and key result that is used in the proof of concentration estimates of $\{\nu_\epsilon\}_\epsilon$ near 0.

3.1. **Proof of Theorem B.** We first study the concentration estimates of the QSDs $\{\nu_\epsilon\}_\epsilon$ near ∞.

Proposition 3.1. *Assume* **(A1)** *and* **(A2)**. *Then there exists $\gamma > 0$ such that*

$$\nu_\epsilon((x_\infty + 1, \infty)) \le e^{-\frac{\gamma}{\epsilon^2}}, \quad \forall 0 < \epsilon \ll 1.$$

Proof. It follows from **(A2)**(3) that

$$bU_\infty' \le -\frac{\gamma_\infty}{2} \quad \text{on} \quad [x_\infty, \infty).$$

This together with **(A1)**(1) and **(A2)**(1)(2) implies that $U_\infty' > 0$ on $[x_\infty, \infty)$. In particular, U_∞ is increasing on $[x_\infty, \infty)$.

Set $\rho_m := U_\infty(x_\infty)$ and $\rho_M := \lim_{x\to\infty} U_\infty(x) < \infty$. Let U be defined as in (2.8). Then there exists a unique $x_0 \in (0, \alpha_{\mathcal{A}})$ such that $U(x_0) = \rho_M$. Let $W : (x_0, \infty) \to [0, \rho_M)$ be a C^2 function satisfying

$$
\begin{aligned}
W(x) &= U(x), \quad x \in (x_0, \alpha_{\mathcal{A}}), \\
W(x) &\in [0, \rho_m), \quad x \in [\alpha_{\mathcal{A}}, x_\infty), \\
W(x) &= U_\infty(x), \quad x \in [x_\infty, \infty).
\end{aligned}
$$

We *claim* that there exist $\gamma_W > 0$ and $0 < \epsilon_W \ll 1$ such that W is a uniform Lyapunov function in (x_0, ∞) with respect to $\{\mathcal{M}_\epsilon\}_{\epsilon \in (0, \epsilon_W]}$ with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant γ_W.

Indeed, it is easy to see that W is a compact function on (x_0, ∞). To verify the Lyapunov condition, we let $x_1 \in (x_0, \alpha_{\mathcal{A}})$ be the unique point satisfying $U(x_1) = \rho_m$. Then an argument similar to that in the proof of Theorem A (1) yields the existence of some $\gamma_1 > 0$ and $0 < \epsilon_1 \ll 1$ such that

$$\mathcal{M}_\epsilon W = \mathcal{M}_\epsilon U \le -\gamma_1 \quad \text{on} \quad (x_0, x_1), \quad \forall \epsilon \in (0, \epsilon_1].$$

Moreover, the boundedness of U_∞, **(A2)**(3) and Proposition 2.1 imply the existence of $0 < \epsilon_2 \ll 1$ such that

$$\mathcal{M}_\epsilon W = \mathcal{M}_\epsilon U_\infty = \mathcal{L}_\epsilon U_\infty + \lambda_\epsilon U_\infty \le -\frac{\gamma_\infty}{2} \quad \text{on} \quad (x_\infty, \infty), \quad \forall \epsilon \in (0, \epsilon_2].$$

Thus, setting $\gamma_W := \min\left\{\gamma_1, \frac{\gamma_\infty}{2}\right\}$ and $\epsilon_W := \min\{\epsilon_1, \epsilon_2\}$, we find that

$$\mathcal{M}_\epsilon W \le -\gamma_W \quad \text{on} \quad (x_0, x_1) \cup (x_\infty, \infty), \quad \forall \epsilon \in (0, \epsilon_W].$$

Since $(x_0, x_1) \cup (x_\infty, \infty) = \Omega_{\rho_M} \setminus \overline{\Omega}_{\rho_m}$, where

$$\Omega_\rho := \{x \in (x_0, \infty) : W(x) < \rho\}, \quad \rho > 0$$

are sub-level sets of W, the *claim* follows.

Now, it is easy to see that for each $\rho \in [\rho_m, \rho_M]$, Ω_ρ is an open interval, denoted by $(\alpha_\rho, \beta_\rho)$. Using Proposition 2.2 (1), we conclude that

$$\nu_\epsilon(\Omega_{\rho_M} \setminus \Omega_\rho) \le e^{-\frac{2\gamma_W}{\epsilon^2} \int_{\rho_m}^{\rho} \frac{1}{H(t)} dt}, \quad \forall \rho \in (\rho_m, \rho_M], \quad \epsilon \in (0, \epsilon_W],$$

where

$$H(\rho) = \max\left\{a(\alpha_\rho)\left(U'(\beta_\rho)\right)^2, a(\alpha_\rho)\left(U'(\beta_\rho)\right)^2\right\} > 0, \quad \rho \in (\rho_m, \rho_M).$$

This completes the proof. $\qquad\square$

Next, we study the concentration estimates of the QSDs $\{\nu_\epsilon\}_\epsilon$ near 0, which is rather tricky thanks to the degeneracy of the diffusion (namely, $a(0) = 0$) and the singularity of $x \mapsto \sqrt{a(x)}$ at $x = 0$. Let $\{v_\epsilon\}_\epsilon$ be the densities of $\{\nu_\epsilon\}_\epsilon$.

Proposition 3.2. *Assume* **(A1)**, **(A3)** *and* **(A4)**. *Then for each* $\kappa \in (0, 1)$, *there exist* $0 < x_* \ll 1$ *and* $0 < \epsilon_* \ll 1$ *such that*

$$v_\epsilon(x) \le \frac{1}{x^\kappa}, \quad \forall x \in (0, x_*]$$

for all $\epsilon \in (0, \epsilon_*]$. *Consequently, for each* $0 < \delta \ll 1$, *there exists* $0 < x_\delta \ll 1$ *such that*

$$\nu_\epsilon((0, x_\delta)) \le \delta, \quad \forall 0 < \epsilon \ll 1.$$

The next technical lemma is the key to the proof of Proposition 3.2. We have stated it in the form that is more general than needed.

Lemma 3.1. *Assume* **(A1)**. *Suppose in addition that* $a \in C^1([0, 1))$ *satisfies*

$$a(x) \ge C_* x^{1+\kappa_*}, \quad \forall x \in (0, 1)$$

for some $C_* > 0$ *and* $\kappa_* \in [0, 1)$. *Then, for each* $0 < \epsilon \ll 1$, *there exists* $C_\epsilon > 0$ *such that*

$$v_\epsilon(x) \le \frac{C_\epsilon}{x^{\kappa_*}}, \quad \forall x \in (0, 1).$$

As the proof of Lemma 3.1 is long, we defer it to Subsection 3.2.

Corollary 3.1. *Assume* **(A1)** *and* **(A3)**. *Then, for each* $0 < \epsilon \ll 1$, *there exists* $C_\epsilon > 0$ *such that*

$$v_\epsilon(x) \le C_\epsilon, \quad \forall x \in (0, 1).$$

Proof. By **(A1)**(2), **(A3)** and the second-order Taylor expansion, we find

$$a(x) = a'(0)x + \frac{a''(0)}{2}x^2 + h(x)x^2,$$

where h satisfies $h(x) \to 0$ as $x \to 0^+$. It follows from the integral condition in **(A3)** that $a'(0) > 0$, which implies the existence of some $C_* > 0$ such that $a(x) \geq C_* x$ for all $x \in (0,1)$. The corollary then follows from applying Lemma 3.1 with $\kappa_* = 0$. $\square$

Now, we prove Proposition 3.2. The arguments are in the spirit of proving a maximum principle.

Proof of Proposition 3.2. Let $\kappa \in (0,1)$ be fixed. We *claim* that there exist $0 < x_* \ll 1$ and $0 < \epsilon_* \ll 1$ such that

$$(L_\epsilon + \lambda_\epsilon)\left[\frac{1}{x^\kappa}\right] < 0 \quad \text{on} \quad (0, x_*), \quad \forall \epsilon \in (0, \epsilon_*]. \tag{3.1}$$

In fact, direct calculations yield that

$$\begin{aligned}
(L_\epsilon + \lambda_\epsilon)\left[\frac{1}{x^\kappa}\right] &= \frac{\epsilon^2}{2}\left(\frac{a}{x^\kappa}\right)'' - \left(\frac{b}{x^\kappa}\right)' + \frac{\lambda_\epsilon}{x^\kappa} \\
&= \frac{\epsilon^2}{2}\left[\frac{a''}{x^\kappa} - \frac{2\kappa a'}{x^{\kappa+1}} + \frac{\kappa(\kappa+1)a}{x^{\kappa+2}}\right] - \left[\frac{b'}{x^\kappa} - \frac{\kappa b}{x^{\kappa+1}}\right] + \frac{\lambda_\epsilon}{x^\kappa} \\
&= \frac{\epsilon^2 a'' - 2b' + 2\lambda_\epsilon}{2x^\kappa} + \frac{\kappa b - \epsilon^2 \kappa a'}{x^{\kappa+1}} + \frac{\epsilon^2 \kappa(\kappa+1)a}{2x^{\kappa+2}}.
\end{aligned}$$

By **(A1)**, **(A3)**, **(A4)** and Taylor expansions,

$$\begin{aligned}
a(x) &= a'(0)x + \frac{a''(0)}{2}x^2 + h_1(x)x^2, \quad x \in (0,1), \\
a'(x) &= a'(0) + a''(0)x + h_2(x)x, \quad x \in (0,1), \\
b(x) &= b'(0)x + h_3(x)x,
\end{aligned}$$

where $h_i(x) \to 0$ as $x \to 0^+$ for $i = 1, 2, 3$. It follows from the fact $a'(0) > 0$ (see the proof of Corollary 3.1) that

$$\begin{aligned}
&\frac{\kappa b - \epsilon^2 \kappa a'}{x^{\kappa+1}} + \frac{\epsilon^2 \kappa(\kappa+1)a}{2x^{\kappa+2}} \\
&= \frac{4\kappa[b'(0) + h_3(x)] - 4\epsilon^2\kappa[a''(0) + h_2(x)] + \epsilon^2\kappa(\kappa+1)[a''(0) + 2h_1(x)]}{4x^\kappa} \\
&\quad - \frac{\epsilon^2[2\kappa - \kappa(\kappa+1)]a'(0)}{2x^{\kappa+1}} \\
&\leq \frac{4\kappa[b'(0) + h_3(x)] - 4\epsilon^2\kappa[a''(0) + h_2(x)] + \epsilon^2\kappa(\kappa+1)[a''(0) + 2h_1(x)]}{4x^\kappa} \quad \text{on} \quad (0,1).
\end{aligned}$$

Hence,

$$(L_\epsilon + \lambda_\epsilon)\left[\frac{1}{x^\kappa}\right] \leq \frac{\epsilon^2 h_4 + 4\lambda_\epsilon - 4b' + 4\kappa[b'(0) + h_3]}{4x^\kappa} \quad \text{on} \quad (0,1),$$

where

$$h_4(x) = 2a''(x) - 4\kappa[a''(0) + h_2(x)] + \kappa(\kappa+1)[a''(0) + 2h_1(x)].$$

As $b'(x) = b'(0) + h_5(x)$ with $h_5(x) \to 0$ as $x \to 0^+$, we find $0 < x_* \ll 1$ and $0 < \epsilon_* \ll 1$ such that the *claim* (3.1) holds.

For each $\epsilon \in (0, \epsilon_*]$, consider

$$w_\epsilon(x) = v_\epsilon(x) - \frac{1}{x^\kappa}, \quad x \in (0, x_*].$$

We *claim* that

$$w_\epsilon \leq 0 \quad \text{on} \quad (0, x_*], \quad \forall \epsilon \in (0, \epsilon_*]. \tag{3.2}$$

Indeed, making ϵ_* smaller if necessary, we can apply Corollary 3.1 and Theorem A (1) to conclude that

$$\limsup_{x \to 0^+} w_\epsilon(x) < 0, \quad w_\epsilon(x_*) < 0, \quad \forall \epsilon \in (0, \epsilon_*].$$

Thus, if (3.2) fails, then there is some $\epsilon_0 \in (0, \epsilon_*]$ such that $w := w_{\epsilon_0}$ has a positive maximum attained at some point in $(0, x_*)$. Let $x_0 \in (0, x_*)$ be such that

$$w(x_0) = \max_{(0, x_*]} w > 0.$$

Then, $w'(x_0) = 0$ and $w''(x_0) \leq 0$.

As $(L_{\epsilon_0} + \lambda_{\epsilon_0})v_{\epsilon_0} = 0$, (3.1) yields that

$$(L_{\epsilon_0} + \lambda_{\epsilon_0})w > 0 \quad \text{on} \quad (0, x_*).$$

But, making ϵ_* and x_* smaller if necessary, we have

$$\begin{aligned}
(L_{\epsilon_0} + \lambda_{\epsilon_0})w(x_0) &= \frac{\epsilon_0^2}{2} \left[a(x_0)w''(x_0) + 2a'(x_0)w'(x_0) + a''(x_0)w(x_0) \right] \\
&\quad - \left[b(x_0)w'(x_0) + b'(x_0)w(x_0) \right] + \lambda_{\epsilon_0}w(x_0) \\
&\leq \left[\frac{\epsilon_0^2}{2}a''(x_0) - b'(x_0) + \lambda_{\epsilon_0} \right] w(x_0) < 0,
\end{aligned}$$

which leads to a contradiction. Thus, the *claim* (3.2) follows. This completes the proof. $\qquad\square$

In Proposition 3.1 and Proposition 3.2, concentration estimates of $\{\nu_\epsilon\}_\epsilon$ near ∞ and 0 are respectively established. The proof of Theorem B now follows readily.

Proof of Theorem B. Let $\{\nu_{\epsilon_n}\}_n$ be a sequence of $\{\nu_\epsilon\}_\epsilon$, where $\epsilon_n \to 0$ as $n \to \infty$. For any $\delta > 0$, we see from Proposition 3.1 and Proposition 3.2 that there exists a compact set $K_\delta^1 \subset (0, \infty)$ and an integer $n_\delta \gg 1$ such that

$$\nu_{\epsilon_n}(K_\delta^1) \geq 1 - \delta, \quad \forall n \geq n_\delta.$$

Clearly, there exists a compact set $K_\delta^2 \subset (0, \infty)$ such that

$$\nu_{\epsilon_n}(K_\delta^2) \geq 1 - \delta, \quad \forall n \in \{1, \ldots, n_\delta\}.$$

Setting $K_\delta := K_\delta^1 \cup K_\delta^2$, we find

$$\nu_{\epsilon_n}(K_\delta) \geq 1 - \delta, \quad \forall n.$$

Hence, the family of QSDs $\{\nu_{\epsilon_n}\}_n$ is tight.

The rest of the conclusions in the statement follow from Theorem A. $\qquad\square$

3.2. **Proof of Lemma 3.1.** The idea is to use the standard change of variable for one-dimensional problems (see e.g. [21]) to rewrite the densities $\{v_\epsilon\}_\epsilon$ of the QSDs $\{\nu_\epsilon\}_\epsilon$ in the new coordinate system. A delicate analysis of $\{v_\epsilon\}_\epsilon$ in the new coordinate system near 0 then gives required information about $\{v_\epsilon\}_\epsilon$ near 0.

Throughout the rest of this subsection, the conditions in the statement of Lemma 3.1 are assumed, that is, we assume **(A1)** and $a \in C^1([0,1))$ with

$$a(x) \geq C_* x^{1+\kappa_*}, \quad \forall x \in (0,1) \tag{3.3}$$

for some $C_* > 0$ and $\kappa_* \in [0,1)$.

We see from (3.3) that $\int_0^1 \frac{1}{\sqrt{a(x)}}dx < \infty$. This together with **(A1)**(2) implies that for each $0 < \epsilon \ll 1$, the function $\xi_\epsilon : [0,\infty) \to [0,\infty)$, defined by

$$\xi_\epsilon(x) = \int_0^x \frac{1}{\epsilon\sqrt{a(s)}}ds, \quad x \in [0,\infty),$$

is well-defined and increasing on $[0,\infty)$ with range $[0,\Xi_\epsilon)$, where

$$\Xi_\epsilon = \int_0^\infty \frac{1}{\epsilon\sqrt{a(s)}}ds.$$

It is worthwhile to mention that ξ_ϵ introduces the canonical transform converting the SDE satisfied by X_t^ϵ (namely, the SDE (1.1)) to the one with the simplest noise coefficient. More precisely, $Y_t^\epsilon := \xi_\epsilon(X_t^\epsilon)$ satisfies

$$dY_t^\epsilon = \mathcal{L}_\epsilon \xi_\epsilon(\xi_\epsilon^{-1}(Y_t^\epsilon))dt + dW_t, \quad Y_0^\epsilon \in [0,\Xi_\epsilon).$$

Set

$$q_\epsilon(x) = -\mathcal{L}_\epsilon \xi_\epsilon(\xi_\epsilon^{-1}(x)), \quad x \in (0,\Xi_\epsilon)$$

and

$$Q_\epsilon(x) = \int_1^x 2q_\epsilon(s)ds, \quad x \in (0,\Xi_\epsilon).$$

Lemma 3.2. *The following hold.*

(1) *For each $0 < \epsilon \ll 1$,*

$$e^{-Q_\epsilon(x)} = \frac{\eta_\epsilon(\xi_\epsilon^{-1}(x))}{\sqrt{a(\xi_\epsilon^{-1}(x))}}, \quad x \in (0,\Xi_\epsilon),$$

where $\eta_\epsilon : (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$0 < \inf_{(0,A)} \eta_\epsilon \leq \sup_{(0,A)} \eta_\epsilon < \infty, \quad \forall A > 0.$$

(2) *For each $0 < \epsilon \ll 1$,*

$$e^{Q_\epsilon(x)} = \zeta_\epsilon(\xi_\epsilon^{-1}(x))\sqrt{a(\xi_\epsilon^{-1}(x))}, \quad x \in (0,\Xi_\epsilon),$$

where $\zeta_\epsilon : (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$0 < \inf_{(0,A)} \zeta_\epsilon \leq \sup_{(0,A)} \zeta_\epsilon < \infty, \quad \forall A > 0.$$

(3) *For each $0 < \epsilon \ll 1$, $\int_0^1 e^{-Q_\epsilon(x)}dx = \infty$.*

Proof. (1) A straightforward calculation yields

$$\mathcal{L}_\epsilon \xi_\epsilon = -\frac{\epsilon}{4}\frac{a'}{\sqrt{a}} + \frac{b}{\epsilon\sqrt{a}}.$$

Thus, for any $x \in (0, \Xi_\epsilon)$,

$$\begin{aligned}
Q_\epsilon(x) &= -2\int_1^x \mathcal{L}_\epsilon \xi_\epsilon(\xi_\epsilon^{-1}(s))ds = -2\int_{\xi_\epsilon^{-1}(1)}^{\xi_\epsilon^{-1}(x)} \mathcal{L}_\epsilon \xi_\epsilon(s)\xi_\epsilon'(s)ds \\
&= \int_{\xi_\epsilon^{-1}(1)}^{\xi_\epsilon^{-1}(x)} \left[\frac{1}{2}\frac{a'(s)}{a(s)} - \frac{2}{\epsilon^2}\frac{b(s)}{a(s)}\right]ds \\
&= \frac{1}{2}\ln a(\xi_\epsilon^{-1}(x)) - \frac{1}{2}\ln a(\xi_\epsilon^{-1}(1)) - \frac{2}{\epsilon^2}\int_{\xi_\epsilon^{-1}(1)}^{\xi_\epsilon^{-1}(x)} \frac{b(s)}{a(s)}ds.
\end{aligned}$$

Note that we may assume, without loss of generality, that $\Xi_\epsilon > 1$ for all $0 < \epsilon \ll 1$ so that $\xi_\epsilon^{-1}(1)$ is well-defined. Setting

$$\eta_\epsilon(x) := \exp\left\{\frac{1}{2}\ln a(\xi_\epsilon^{-1}(1)) + \frac{2}{\epsilon^2}\int_{\xi_\epsilon^{-1}(1)}^x \frac{b(s)}{a(s)}ds\right\}, \quad x \in (0, \infty),$$

we find the desired expression for e^{-Q_ϵ}. By **(A1)** and (3.3), the integral $\int_{\xi_\epsilon^{-1}(1)}^x \frac{b(s)}{a(s)}ds$ is finite for all $x > 0$. This yields the desired properties of η_ϵ.

(2) Setting

$$\zeta_\epsilon(x) := \exp\left\{-\frac{1}{2}\ln a(\xi_\epsilon^{-1}(1)) - \frac{2}{\epsilon^2}\int_{\xi_\epsilon^{-1}(1)}^x \frac{b(s)}{a(s)}ds\right\}, \quad x \in (0, \infty),$$

the results follow from arguments as in the proof of (1).

(3) We see from (1) that

$$\int_0^1 e^{-Q_\epsilon(x)}dx = \int_0^{\xi_\epsilon^{-1}(1)} \frac{\eta_\epsilon(x)}{\sqrt{a(x)}}\xi_\epsilon'(x)dx = \frac{1}{\epsilon}\int_0^{\xi_\epsilon^{-1}(1)} \frac{\eta_\epsilon(x)}{a(x)}dx \geq \frac{c_\epsilon}{\epsilon}\int_0^{\xi_\epsilon^{-1}(1)} \frac{1}{a(x)}dx,$$

where $c_\epsilon = \inf_{(0,\xi_\epsilon^{-1}(1))} \eta_\epsilon > 0$. The result then follows from **(A1)**(2) and $a \in C^1([0,1))$. In fact, Taylor expansion gives $a(x) = a'(0)x + h(x)x$ with $h(x) \to 0$ as $x \to 0^+$. If $a'(0) > 0$, $\int_0^{\xi_\epsilon^{-1}(1)} \frac{1}{a(x)}dx = \infty$ follows readily. If $a'(0) = 0$, then $h(x) > 0$ near $x = 0$, which implies $\int_0^{\xi_\epsilon^{-1}(1)} \frac{1}{a(x)}dx = \infty$ as well. $\square$

For each $0 < \epsilon \ll 1$, consider the function

$$u_\epsilon(x) = \frac{v_\epsilon(\xi_\epsilon^{-1}(x))}{\xi_\epsilon'(\xi_\epsilon^{-1}(x))}e^{Q_\epsilon(x)}, \quad x \in (0, \Xi_\epsilon). \tag{3.4}$$

Lemma 3.3. *For each $0 < \epsilon \ll 1$, u_ϵ satisfies the following properties:*

(1) $\int_0^{\Xi_\epsilon} u_\epsilon(x)e^{-Q_\epsilon(x)}dx = 1$;

(2) *there holds*

$$\frac{1}{2}u_\epsilon'' - q_\epsilon u_\epsilon' = -\lambda_\epsilon u_\epsilon \quad in \quad (0, \Xi_\epsilon); \tag{3.5}$$

(3) *the function $x \mapsto u_\epsilon'(x)e^{-Q_\epsilon(x)}$ is decreasing on $(0, \Xi_\epsilon)$, and has finite limit as $x \to 0^+$.*

Proof. (1) The property holds because

$$\int_0^{\Xi_\epsilon} u_\epsilon(x)e^{-Q_\epsilon(x)}dx = \int_0^{\Xi_\epsilon} \frac{v_\epsilon(\xi_\epsilon^{-1}(x))}{\xi_\epsilon'(\xi_\epsilon^{-1}(x))}dx = \int_0^\infty v_\epsilon(x)dx = 1.$$

(2) Since $\xi_\epsilon' = \frac{1}{\epsilon\sqrt{a}}$ on $(0,\infty)$, we have

$$w_\epsilon(x) := \frac{v_\epsilon(\xi_\epsilon^{-1}(x))}{\xi_\epsilon'(\xi_\epsilon^{-1}(x))} = \epsilon\sqrt{a(\xi_\epsilon^{-1}(x))}v_\epsilon(\xi_\epsilon^{-1}(x)), \quad x \in (0,\Xi_\epsilon).$$

We *claim* that w_ϵ satisfies

$$\frac{1}{2}w_\epsilon'' + (q_\epsilon w_\epsilon)' = -\lambda_\epsilon w_\epsilon \quad \text{on} \quad (0,\Xi_\epsilon). \tag{3.6}$$

Let $y = \xi_\epsilon^{-1}(x)$, or $x = \xi_\epsilon(y)$. Then,

$$w_\epsilon(x)\xi_\epsilon'(y) = v_\epsilon(y).$$

Since

$$\begin{aligned}
[a(y)v_\epsilon(y)]_y &= [a(y)w_\epsilon(x)\xi_\epsilon'(y)]_y \\
&= w_\epsilon'(x)\xi_\epsilon'(y)a(y)\xi_\epsilon'(y) + w_\epsilon(x)[a(y)\xi_\epsilon'(y)]_y \\
&= \frac{w_\epsilon'(x)}{\epsilon^2} - w_\epsilon(x)a(y)\xi_\epsilon''(y),
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{\epsilon^2}{2}[a(y)v_\epsilon(y)]_{yy} - [b(y)v_\epsilon(y)]_y &= \frac{1}{2}w_\epsilon''(x)\xi_\epsilon'(y) - \frac{\epsilon^2}{2}[w_\epsilon(x)a(y)\xi_\epsilon''(y)]_y - [b(y)w_\epsilon(x)\xi_\epsilon'(y)]_y \\
&= \frac{1}{2}w_\epsilon''(x)\xi_\epsilon'(y) - \left[\left(\frac{\epsilon^2}{2}a(y)\xi_\epsilon''(y) + b(y)\xi_\epsilon'(y)\right)w_\epsilon(x)\right]_y \\
&= \frac{1}{2}w_\epsilon''(x)\xi_\epsilon'(y) - [\mathcal{L}_\epsilon\xi_\epsilon(y)w_\epsilon(x)]_y \\
&= \frac{1}{2}w_\epsilon''(x)\xi_\epsilon'(y) + [q_\epsilon(x)w_\epsilon(x)]_y \\
&= \frac{1}{2}w_\epsilon''(x)\xi_\epsilon'(y) + [q_\epsilon(x)w_\epsilon(x)]_x\,\xi_\epsilon'(y).
\end{aligned}$$

Hence by (2.1), we deduce

$$\frac{1}{2}w_\epsilon''(x) + [q_\epsilon(x)w_\epsilon(x)]_x = \frac{1}{\xi_\epsilon'(y)}\left\{\frac{\epsilon^2}{2}[a(y)v_\epsilon(y)]_{yy} - [b(y)v_\epsilon(y)]_y\right\} = -\lambda_\epsilon\frac{v_\epsilon(y)}{\xi_\epsilon'(y)} = -\lambda_\epsilon w_\epsilon(x),$$

that is, the *claim* holds.

Now, we find from $w_\epsilon = u_\epsilon e^{-Q_\epsilon}$ that $w_\epsilon' = u_\epsilon' e^{-Q_\epsilon} - 2q_\epsilon w_\epsilon$ and

$$w_\epsilon'' = u_\epsilon'' e^{-Q_\epsilon} - 2q_\epsilon u_\epsilon' e^{-Q_\epsilon} - 2(q_\epsilon w_\epsilon)',$$

that is,

$$\frac{1}{2}w_\epsilon'' + (q_\epsilon w_\epsilon)' = \frac{1}{2}u_\epsilon'' e^{-Q_\epsilon} - q_\epsilon u_\epsilon' e^{-Q_\epsilon}.$$

This together with (3.6) gives

$$-\lambda_\epsilon w_\epsilon = \frac{1}{2}w_\epsilon'' + (q_\epsilon w_\epsilon)' = \frac{1}{2}u_\epsilon'' e^{-Q_\epsilon} - q_\epsilon u_\epsilon' e^{-Q_\epsilon},$$

which leads to the desired equality for u_ϵ.

(3) Multiplying (3.5) by e^{-Q_ϵ}, we find

$$\left(u'_\epsilon e^{-Q_\epsilon}\right)' = -2\lambda_\epsilon u_\epsilon e^{-Q_\epsilon},$$

which leads to

$$u'_\epsilon(x)e^{-Q_\epsilon(x)} = u'_\epsilon(x_0)e^{-Q_\epsilon(x_0)} - 2\lambda_\epsilon \int_{x_0}^x u_\epsilon(s)e^{-Q_\epsilon(s)}ds, \quad \forall x_0, x \in (0, \Xi_\epsilon). \tag{3.7}$$

Note that if $x > x_0$, then $u'_\epsilon(x)e^{-Q_\epsilon(x)} < u'_\epsilon(x_0)e^{-Q_\epsilon(x_0)}$, that is, the function $x \mapsto u'_\epsilon(x)e^{-Q_\epsilon(x)}$ is decreasing on $(0, \Xi_\epsilon)$. Letting $x_0 \to 0^+$ in (3.7), we conclude from (1) that the limit $\lim_{x \to 0^+} u'_\epsilon(x)e^{-Q_\epsilon(x)}$ exists and is finite. $\qquad\square$

We are ready to prove Lemma 3.1.

Proof of Lemma 3.1. According to (3.4), the definition of ξ_ϵ, and Lemma 3.2 (1), we have

$$v_\epsilon(x) = u_\epsilon(\xi_\epsilon(x))e^{-Q_\epsilon(\xi_\epsilon(x))}\xi'_\epsilon(x) = \frac{u_\epsilon(\xi_\epsilon(x))}{\epsilon}\frac{\eta_\epsilon(x)}{a(x)}, \quad x \in (0, \infty).$$

To examine the behaviors of $u_\epsilon(\xi_\epsilon(x))$ as $x \to 0^+$, we first note that

$$\lim_{x \to 0^+} u_\epsilon(x) = 0. \tag{3.8}$$

In fact, Lemma 3.2 (3) and Lemma 3.3 (1) imply that $\liminf_{x \to 0^+} u_\epsilon(x) = 0$. If $\limsup_{x \to 0^+} u_\epsilon(x) > 0$, then u'_ϵ is unbounded on $(0, 1)$. But Lemma 3.2 (1) implies that $\lim_{x \to 0^+} e^{-Q_\epsilon(x)} = \infty$, which together with Lemma 3.3 (3) implies that $\lim_{x \to 0^+} u'_\epsilon(x) = 0$, a contradiction. Hence, (3.8) holds.

The limit (3.8) ensures that

$$u_\epsilon(x) = \int_0^x u'_\epsilon(s)ds, \quad x \in (0, \Xi_\epsilon).$$

Setting $\ell_\epsilon := \lim_{x \to 0^+} u'_\epsilon(x)e^{-Q_\epsilon(x)}$, we find from Lemma 3.3 (3) and Lemma 3.2 (2) that

$$\begin{aligned}
u_\epsilon(\xi_\epsilon(x)) &= \int_0^{\xi_\epsilon(x)} u'_\epsilon(s)ds \\
&= \int_0^{\xi_\epsilon(x)} u'_\epsilon(s)e^{-Q_\epsilon(s)}e^{Q_\epsilon(s)}ds \\
&\le \ell_\epsilon \int_0^{\xi_\epsilon(x)} e^{Q_\epsilon(s)}ds \\
&= \ell_\epsilon \int_0^{\xi_\epsilon(x)} \zeta_\epsilon(\xi_\epsilon^{-1}(s))\sqrt{a(\xi_\epsilon^{-1}(s))}ds \\
&= \ell_\epsilon \int_0^x \zeta_\epsilon(s)\sqrt{a(s)}\xi'_\epsilon(s)ds \\
&= \frac{\ell_\epsilon}{\epsilon} \int_0^x \zeta_\epsilon(s)ds, \quad \forall x \in (0, \infty).
\end{aligned}$$

It follows that

$$v_\epsilon(x) \le \frac{\ell_\epsilon}{\epsilon^2}\frac{\eta_\epsilon(x)}{a(x)} \int_0^x \zeta_\epsilon(s)ds \le \left[\frac{\ell_\epsilon}{\epsilon^2}\eta_\epsilon(x)\sup_{(0,1)}\zeta_\epsilon\right]\frac{x}{a(x)}, \quad \forall x \in (0, 1).$$

The lemma then follows from Lemma 3.2 (1)(2) and (3.3). $\qquad\square$

4. **Applications**

In this section, we discuss applications of Theorem A and Theorem B to models arising from chemical reactions and population dynamics. Due to the similarity, we study a class of chemical reactions in details in Subsection 4.1 and roughly discuss the application to diffusion approximations of birth-and-death processes in Subsection 4.2.

4.1. **Keizer's paradox.** We consider the following simple chemical reactions:

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \quad X \xrightarrow{k_2} C, \tag{4.1}$$

where the positive numbers k_1, k_{-1} and k_2 are reaction rate constants. The concentration of the chemical substance A is assumed to remain constant, and is denoted by x_A. We consider in particular the case with $k_1 x_A > k_2$.

Let V be the total volume of the system and $\{X_t^V\}_{t\geq 0}$ be the continuous-time Markov jump process counting the number of the substance X. By the law of large numbers [19, 1], the scaled process $\left\{\frac{X_t^V}{V}\right\}_{t\geq 0}$ converges to the solutions of the following ODE:

$$\dot{x} = b(x), \quad x \in [0, \infty), \tag{4.2}$$

on any given finite time interval as the volume V grows to infinity, where

$$b(x) = -k_{-1}x^2 + k_1 x_A x - k_2 x = k_{-1}x \left(\frac{k_1 x_A - k_2}{k_{-1}} - x\right).$$

The ODE (4.2) is the classical mean field model for the concentration of the substance X. It is clear that (4.2) admits two equilibria: an unstable one at 0 and a globally asymptotically stable one at

$$x_e := \frac{k_1 x_A - k_2}{k_{-1}}.$$

The fluctuation of $\left\{\frac{X_t^V}{V}\right\}_{t\geq 0}$ around the solutions of (4.2) for $V \gg 1$ is captured by the central limit theorem [44, 19, 1]. More precisely, for sufficiently large V, the process $\left\{\frac{X_t^V}{V}\right\}_{t\geq 0}$ stays close to the solutions of the following SDE

$$dX_t = b(X_t)dt + \epsilon\sqrt{a(X_t)}dW_t, \quad X_0 \in [0, \infty), \tag{4.3}$$

on any given finite time interval, where $\epsilon = \frac{1}{\sqrt{V}}$ and

$$a(x) = k_{-1}x^2 + k_1 x_A x + k_2 x.$$

The SDE (4.3) is often referred to as the *diffusion approximation* of the process $\left\{\frac{X_t^V}{V}\right\}_{t\geq 0}$. The following result, being indeed a special case of those contained in [43, 44], provides a rigorous justification of the diffusion approximation.

Theorem 4.1 ([43, 44])**.** *Let $\{Z_t^V\}_{t\geq 0}$ be the diffusion process given by*

$$Z_t^V = Z_0^V + \int_0^t b(Z_s^V)ds + \frac{1}{V}\left[W^1\left(V\int_0^t k_1 x_A Z_s^V ds\right) - W^{-1}\left(V\int_0^t \left(k_{-1}(Z_s^V)^2 + k_2 Z_s^V\right)ds\right)\right],$$

where $W^1(t)$ and $W^{-1}(t)$ are independent standard one-dimensional Wiener processes. Then, the following hold.

(1) $\{Z_t^V\}_{t\geq 0}$ has the same distribution as the diffusion process generated by solutions of (4.3).

(2) *For any $L > 0$ and $T > 0$, there is a random variable K satisfying $\mathbb{E}\{e^{\lambda K}\} < \infty$ for some $\lambda > 0$ such that if $Z_0^V = \frac{X_0^V}{V} \to x_0 \in (0, L)$ as $V \to \infty$, then*

$$\sup_{0 \leq t \leq T \wedge \tau_L^V} \left| \frac{X_t^V}{V} - Z_t^V \right| \leq K \frac{\ln V}{V} \quad a.s \quad \text{for all } V \gg 1,$$

where $\tau_L^V = \inf \left\{ t \geq 0 : \frac{X_t^V}{V} \notin (0, L) \text{ or } Z_t^V \notin (0, L) \right\}$. Moreover, if $L > x_e$, then $\lim_{V \to \infty} \mathbb{P}\{\tau_L^V > T\} = 1$.

It is not hard to show that the solutions of (4.3) almost surely reach 0 in finite time, while the solutions of (4.2) with positive initial data converge to the equilibrium x_e. This gives rise to the so-called *Keizer's paradox* [38], which is often formulated in terms of the chemical master equation (CME) satisfied by the distributions of $\{X_t^V\}_{t\geq 0}$ or $\left\{\frac{X_t^V}{V}\right\}_{t\geq 0}$ (see e.g. [39, 61, 10]). In [61], the authors numerically showed that the QSD of the CME tends to the Dirac delta measure at x_e as the volume $V \to \infty$, and calculated the first passage time to the extinction as well as to the QSD to exhibit the multi-scale nature. The multi-scale nature of Keizer's paradox was further investigated in [10], where the authors used a slow manifold reduction method to estimate the spectral gaps of relevant operators that quantify the first passage time to the extinction and to the QSD.

The purpose of this subsection is to give a rigorous justification of Keizer's paradox in term of the diffusion approximation (4.3). More precisely, we prove the following theorem.

Theorem 4.2. *Assume $k_1 x_A > k_2$. Then the following hold.*

(1) *For each $0 < \epsilon \ll 1$, the SDE (4.3) admits a unique QSD ν_ϵ on $(0, \infty)$.*

(2) *$\lim_{\epsilon \to 0^+} \nu_\epsilon = \delta_{x_e}$ in the topology of weak convergence, where δ_{x_e} is the Dirac delta measure at x_e.*

The rest of this subsection is devoted to the proof of Theorem 4.2. We first apply the theory developed in [2] (also see [48, 31]) to study the existence and uniqueness of QSDs for (4.3) for fixed ϵ. In particular, we prove Theorem 4.2 (1).

For each $0 < \epsilon \ll 1$, consider the function $\xi_\epsilon : [0, \infty) \to [0, \infty)$:

$$\xi_\epsilon(x) := \int_0^x \frac{1}{\epsilon \sqrt{a(s)}} ds$$

$$= \frac{1}{\epsilon \sqrt{k_{-1}}} \int_0^x \frac{1}{\sqrt{s^2 + 2c_0 s}} ds$$

$$= \frac{1}{\epsilon \sqrt{k_{-1}}} \int_0^x \frac{1}{\sqrt{(s + c_0)^2 - c_0^2}} ds$$

$$= \frac{1}{\epsilon \sqrt{k_{-1}}} \cosh^{-1}\left(\frac{x + c_0}{c_0}\right),$$

where $c_0 = \dfrac{k_1 x_A + k_2}{2k_{-1}}$. Clearly, ξ_ϵ is increasing with range $[0, \infty)$. Simple calculations lead to

$$\xi_\epsilon^{-1}(x) = c_0 \left[\cosh\left(\frac{x}{\epsilon \sqrt{k_{-1}}}\right) - 1 \right] = \frac{c_0}{2} \left(e^{\frac{\epsilon \sqrt{k_{-1}}}{2}x} - e^{-\frac{\epsilon \sqrt{k_{-1}}}{2}x} \right)^2.$$

Define $Y_t = \xi_\epsilon(X_t)$. Itô's formula yields

$$dY_t = \mathcal{L}_\epsilon^X \xi_\epsilon(\xi_\epsilon^{-1}(Y_t))dt + dW_t, \tag{4.4}$$

where $\mathcal{L}_\epsilon^X$ is the generator of (4.3), namely,

$$\mathcal{L}_\epsilon^X = \frac{\epsilon^2}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

Hence,

$$\mathcal{L}_\epsilon^X \xi_\epsilon(x) = \frac{\epsilon^2}{2}a(x)\xi_\epsilon''(x) + b(x)\xi_\epsilon'(x) = \frac{\frac{b(x)}{\epsilon} - \frac{\epsilon k_{-1}}{2}(x + c_0)}{\sqrt{k_{-1}x^2 + k_1 x_A x + k_2 x}}.$$

Let

$$q_\epsilon(y) = -\mathcal{L}_\epsilon^X \xi_\epsilon(\xi_\epsilon^{-1}(y)), \quad y \in [0, \infty).$$

The SDE (4.4) can be rewritten as

$$dY_t = -q_\epsilon(Y_t)dt + dW_t. \tag{4.5}$$

Lemma 4.1. *The following hold for each $0 < \epsilon \ll 1$:*

$$y \to 0^+: \quad \xi_\epsilon^{-1}(y) \sim \frac{\epsilon^2 c_0 k_{-1}}{2}y^2, \qquad q_\epsilon(y) \sim \frac{1}{2y},$$

$$y \to \infty: \quad \xi_\epsilon^{-1}(y) \sim \frac{c_0}{2}e^{\epsilon\sqrt{k_{-1}}y}, \qquad q_\epsilon(y) \sim \frac{c_0\sqrt{k_{-1}}}{2\epsilon}e^{\epsilon\sqrt{k_{-1}}y}.$$

Proof. The asymptotic behaviors of $\xi_\epsilon^{-1}(y)$ as $y \to 0^+$ and $y \to \infty$ follow readily. The asymptotic behaviors of $q_\epsilon(y)$ as $y \to 0^+$ and $y \to \infty$ follow from those of $\xi_\epsilon^{-1}(y)$ and

$$x \to 0^+: \quad \mathcal{L}_\epsilon^X \xi(x) \sim -\frac{\epsilon\sqrt{c_0 k_{-1}}}{2\sqrt{2}}\frac{1}{\sqrt{x}},$$

$$x \to \infty: \quad \mathcal{L}_\epsilon^X \xi(x) \sim -\frac{\sqrt{k_{-1}}}{\epsilon}x.$$

$\square$

Let

$$Q_\epsilon(y) = \int_1^y 2q_\epsilon(s)ds, \quad y \in (0, \infty).$$

Lemma 4.2. *The following hold for each $0 < \epsilon \ll 1$:*

$$y \to 0^+: \quad Q_\epsilon(y) \sim \ln y,$$

$$y \to \infty: \quad Q_\epsilon(y) \sim \frac{c_0}{\epsilon^2}e^{\epsilon\sqrt{k_{-1}}y}.$$

Proof. It follows immediately from Lemma 4.1. $\square$

Denote by τ^ϵ the explosion time for the solution of (4.3). Let

$$T_0^\epsilon = \lim_{n \to \infty} \inf\left\{0 \le t < \tau^\epsilon : y^\epsilon(t) = \frac{1}{n}\right\},$$

$$T_\infty^\epsilon = \lim_{n \to \infty} \inf\{0 \le t < \tau^\epsilon : y^\epsilon(t) = n\}.$$

Since q_ϵ is regular on $(0, \infty)$, there holds $\tau^\epsilon = \min\{T_0^\epsilon, T_\infty^\epsilon\}$.

It is not hard to check that assumptions (H1), (H2), (H4) and (H5) in [2] hold. More precisely, we have the following result.

Lemma 4.3. *The following statements hold for each $0 < \epsilon \ll 1$.*

(1) *For all $y > 0$, $\mathbb{P}_y^\epsilon(T_0^\epsilon < T_\infty^\epsilon) = 1$.*

(2) $\inf_{y>0}\left[q_\epsilon^2(y) - q_\epsilon'(y)\right] > -\infty$ and $\lim_{y\to\infty}\left[q_\epsilon^2(y) - q_\epsilon'(y)\right] = \infty$.

(3) $\int_1^\infty e^{-Q_\epsilon(s)}ds < \infty$ and $\int_0^1 se^{-\frac{Q_\epsilon(s)}{2}}ds < \infty$.

(4) $\int_1^\infty e^{-Q_\epsilon(s)}\int_1^s e^{Q_\epsilon(t)}dtds < \infty$, or equivalently, $\int_1^\infty e^{Q_\epsilon(s)}\int_s^\infty e^{-Q_\epsilon(t)}dtds < \infty$.

Proof. It follows from Lemma 4.1 and Lemma 4.2. $\qquad\square$

It is well-known (see e.g. [36, Chapter VI, Theorem 3.2]) that Lemma 4.3 (1) is equivalent to the following elementary conditions:

$$\Lambda_\epsilon(\infty) = \infty \quad \text{and} \quad \kappa_\epsilon(0^+) < \infty,$$

where $\Lambda_\epsilon(y) = \int_1^y e^{Q_\epsilon(s)}ds$ and $\kappa_\epsilon(y) = \int_1^y e^{Q_\epsilon(s)}\int_1^s e^{-Q_\epsilon(t)}dtds$.

Let

$$d\mu_\epsilon^Y(y) = e^{-Q_\epsilon(y)}dy \quad \text{on} \quad (0,\infty),$$

and

$$\mathcal{L}_\epsilon^Y := \frac{1}{2}\frac{d^2}{dy^2} - q_\epsilon(y)\frac{d}{dy}$$

be the generator of (4.5). It is straightforward to check that the operator $\mathcal{L}_\epsilon^Y$ is formally self-adjoint in $L^2(\mu_\epsilon^Y)$. We can construct a self-adjoint extension of $\mathcal{L}_\epsilon^Y$ in $L^2(\mu_\epsilon^Y)$, still denoted by $\mathcal{L}_\epsilon^Y$, as follows. Consider the symmetric form $\mathcal{E}_\epsilon^Y : C_0^\infty((0,\infty)) \times C_0^\infty((0,\infty)) \to [0,\infty)$ defined by

$$\mathcal{E}_\epsilon^Y(\phi,\psi) = \int_0^\infty \phi'\psi' d\mu_\epsilon^Y, \quad \forall\phi,\psi \in C_0^\infty((0,\infty)).$$

It can be shown that $\mathcal{E}_\epsilon^Y$ is Markovian and closable following the proof of [27, Theorem 2.1.4]. Its smallest closed extension, still denoted by $\mathcal{E}_\epsilon^Y$, is a Dirichlet form with domain $D(\mathcal{E}_\epsilon^Y)$ being the closure of $C_0^\infty((0,\infty))$ under the norm $\|\phi\|_{D(\mathcal{E}_\epsilon^Y)}^2 := \|\phi\|_{L^2(\mu_\epsilon^Y)}^2 + \mathcal{E}_\epsilon^Y(\phi,\phi)$. Then, the non-negative self-adjoint operator associated with $\mathcal{E}_\epsilon^Y$ is a self-adjoint extension of $-\mathcal{L}_\epsilon^Y$.

Proposition 4.1. *For each $0 < \epsilon \ll 1$, the SDE (4.5) admits a unique QSD on $(0,\infty)$ given by $\eta_\epsilon^Y d\mu_\epsilon^Y$, where $\eta_\epsilon^Y \in C^2((0,\infty)) \cap L^2(\mu_\epsilon^Y) \cap L^1(\mu_\epsilon^Y)$ is positive on $(0,\infty)$ and is the unique (up to constant multiplication) eigenfunction associated to the first eigenvalue λ_ϵ of $-\mathcal{L}_\epsilon^Y$, which has a purely discrete spectrum contained in $(0,\infty)$ with the only accumulation point ∞.*

Proof. It follows from Lemma 4.3 and [2]. $\qquad\square$

Let

$$v_\epsilon := (\eta_\epsilon^Y \circ \xi_\epsilon)e^{-Q_\epsilon\circ\xi_\epsilon}\xi_\epsilon' \quad \text{and} \quad d\nu_\epsilon(x) := v_\epsilon(x)dx. \tag{4.6}$$

Clearly, v_ϵ is positive and C^2 on $(0,\infty)$. Moreover,

$$\int_0^\infty v_\epsilon dx = \int_0^\infty \eta_\epsilon^Y(y)e^{-Q_\epsilon(y)}dy = \int_0^\infty \eta_\epsilon^Y d\mu_\epsilon^Y = 1.$$

Theorem 4.2 (1) is a special case of the following corollary.

Corollary 4.1. *For each $0 < \epsilon \ll 1$, the SDE (4.3) admits a unique QSD given by ν_ϵ. Moreover, its density v_ϵ satisfies*

$$\frac{\epsilon^2}{2}(av_\epsilon)'' - (bv_\epsilon)' = -\lambda_\epsilon v_\epsilon \quad \text{on} \quad (0,\infty).$$

Proof. As the transformation ξ_ϵ is invertible, the QSDs of (4.3) and that of (4.5) are in one-to-one correspondence. Hence, applying Proposition 4.1, we conclude that (4.3) admits a unique QSD for each $0 < \epsilon \ll 1$. Arguments as in the proof of Lemma 3.3 (2) ensure that v_ϵ satisfies the desired equation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

It remains to prove Theorem 4.2 (2).

Proof of Theorem 4.2 (2). The proof amounts to the verification of conditions **(A1)**-**(A4)**. It is trivial to see that conditions **(A1)**, **(A3)**, and **(A4)** are satisfied.

To verify **(A2)**, we need to construct a C^2 function U_∞ defined on $[x_\infty, \infty)$ for some $x_\infty \gg 1$ such that

(1) $0 < \inf_{[x_\infty, \infty)} U_\infty < \sup_{[x_\infty, \infty)} U_\infty < \infty$;
(2) $\lim_{x \to \infty} U_\infty(x) = \sup_{[x_\infty, \infty)} U_\infty$;
(3) there is $\gamma_\infty > 0$ such that $\mathcal{L}_\epsilon^X U_\infty \leq -\gamma_\infty$ on $[x_\infty, \infty)$ for all $0 < \epsilon \ll 1$.

To do so, we first consider function $w : (x_e, \infty) \to [0, \infty)$:

$$w(x) = -\int_{x_*}^x b(s)ds, \quad x \in (x_*, \infty),$$

It is easy to see that $w \in C^2((x_e, \infty))$ and satisfies

$$w' > 0 \quad \text{on} \quad (x_e, \infty), \quad \lim_{x \to x_e^-} w(x) = 0, \quad \text{and} \quad \lim_{x \to \infty} w(x) = \infty.$$

Next, we modify w to obtain U_∞. For fixed $\rho_M \gg 1$, consider

$$\phi(x) = \rho_M - \frac{2\rho_M}{3 \ln x}, \quad x > 0.$$

Obviously, ϕ is monotonically increasing, smooth, and concave on $(0, \infty)$. We now fix $x_\infty \gg 1$ and define $U_\infty : [x_\infty, \infty) \to [0, \infty)$ by setting

$$U_\infty(x) = \phi(w(x)), \quad x \in [x_\infty, \infty).$$

Clearly, U_∞ is C^2, monotonically increasing, positive, and bounded. In particular, U_∞ satisfies (1) and (2).

It remains to verify (3). By using the concavity of ϕ on $[x_\infty, \infty)$ and the fact that $\phi'(w)ab' < 0$ on $[x_\infty, \infty)$, we have

$$\begin{aligned}
\mathcal{L}_\epsilon^X U_\infty &= \frac{\epsilon^2}{2} a U_\infty'' + b U_\infty' \\
&= \phi'(w)\left(\frac{\epsilon^2}{2} a w'' + b w'\right) + \frac{1}{2V} a \phi''(w)(w')^2 \\
&\leq \phi'(w)\left(\frac{\epsilon^2}{2} a w'' + b w'\right) \\
&= \phi'(w)\left(\frac{\epsilon^2}{2} ab' - b^2\right) \leq -\phi'(w)b^2 \quad \text{on} \quad [x_\infty, \infty).
\end{aligned}$$

By the definitions of ϕ and w, it is straightforward to check that $\phi'(w(x))b^2(x) \to \infty$ as $x \to \infty$. As a result, there is $\gamma_\infty > 0$ such that

$$-\phi'(w)b^2 \leq -\gamma_\infty \quad \text{on} \quad [x_\infty, \infty).$$

Hence, U_∞ satisfies (3), and **(A2)** holds.

As x_e is the globally asymptotically stable equilibrium of the ODE (4.2) in $(0, \infty)$, the result follows from Theorem B. $\qquad\square$

4.2. **BDPs and their diffusion approximations.** In this subsection, we roughly discuss the application of our main results on the concentration of QSDs to diffusion approximations of birth-and-death processes (BDPs). Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider a continuous-time birth-and-death process $\{Z_t^K\}_{t \geq 0}$ on the state space $\mathbb{N}_0$ with birth rates

$$\lambda_0^K = 0, \quad \lambda_n^K = n\tilde{\lambda}\left(\frac{n}{K}\right), \quad n \in \mathbb{N},$$

and death rates

$$\mu_n^K = n\tilde{\mu}\left(\frac{n}{K}\right), \quad n \in \mathbb{N},$$

where $K > 1$ is the scaling parameter, often referred to as the carrying capacity, and $\tilde{\lambda}$ and $\tilde{\mu}$ are positive functions on $[0, \infty)$, and satisfy $\tilde{\lambda}(0) > \tilde{\mu}(0)$. A typical example is the logistic BDP whose birth and death rates are respectively given by

$$\lambda_0^K = 0, \quad \lambda_n^K = \tilde{\lambda}n \quad \text{and} \quad \mu_n^K = n\left(\tilde{\mu} + \frac{n}{K}\right), \quad n \in \mathbb{N},$$

where $\tilde{\lambda} > \tilde{\mu} > 0$.

Let

$$\lambda(x) = x\tilde{\lambda}(x), \quad \mu(x) = x\tilde{\mu}(x), \quad x \in [0, \infty).$$

By the central limit theorem (see e.g. [43, 44, 19]), for sufficiently large K, the process $\left\{\frac{Z_t^K}{K}\right\}_{t \geq 0}$ stays close to solutions of the following SDE

$$dX_t = [\lambda(X_t) - \mu(X_t)]\, dt + \sqrt{\frac{\lambda(X_t) + \mu(X_t)}{K}}\, dW_t, \quad X_0 \in [0, \infty) \tag{4.7}$$

on any given finite time interval. The SDE (4.7) is the diffusion approximation of $\left\{\frac{Z_t^K}{K}\right\}_{t \geq 0}$. Appropriate conditions on $\tilde{\lambda}(x)$ and $\tilde{\mu}(x)$ can be imposed to ensure that for each $K > 1$, the SDE (4.7) admits a unique QSD ν_K on $(0, \infty)$, and assumptions **(A1)-(A4)** are satisfied, and hence, conclusions of Theorem A and Theorem B hold for ν_K. For the logistic case,

$$\lim_{K \to \infty} \nu_K = \delta_{\tilde{\lambda} - \tilde{\mu}} \text{ in the topology of weak convergence.}$$

APPENDIX A. **Harnack's inequality**

Let $\mathcal{I} \subset \mathbb{R}$ be an open interval and consider the differential operator

$$Lu := (\alpha u' + \beta u)' + \kappa u \quad \text{on} \quad \mathcal{I},$$

where α, β and κ are measurable and bounded functions on $\mathcal{I}$. The following result is a special case of [28, Theorem 8.20].

Lemma A.1 (Harnack's inequality). *Suppose that there are constants $\lambda > 0$, $\Lambda > 0$ and $\nu \geq 0$ such that for any $x \in \mathcal{I}$,*

$$\lambda \leq \alpha(x) \leq \Lambda \quad and \quad \frac{\beta^2(x)}{\lambda^2} + \frac{\kappa(x)}{\lambda} \leq \nu^2.$$

If $u \in W^{1,2}(\mathcal{I})$ be a nonnegative solution of $Lu = 0$ on $\mathcal{I}$, then for any interval $(y - 4R, y + 4R) \subset \mathcal{I}$, there holds

$$\sup_{(y-R,y+R)} u \leq C_0^{(\frac{\Lambda}{\lambda}+\nu R)} \inf_{(y-R,y+R)} u,$$

where $C_0 > 0$ is a universal constant.

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