RESPONSIVE TORI IN HAMILTONIAN SYSTEMS OF HIGH ORDER DEGENERACY - THE SUPER-CRITICAL CASE

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ABSTRACT. Consider the quasi-periodically forced, 2nd order differential equations

$$\ddot{x} + \lambda x^l + \varepsilon f(\omega t, x) = 0, \qquad x \in \mathbb{R}^1,$$

where $\lambda \neq 0$ is a constant, $l \geq 2$ is an integer, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector, ε is a small parameter, and $f : \mathbb{T}^d \times \mathbb{R}^1 \to \mathbb{R}^1$ is real analytic. It is shown in [31] that if the leading order p of non-degeneracy of $[f(\cdot, x)]$ satisfies $0 \leq p < l/2$, then response solutions of the equation exist under some minor conditions. Indeed, $l/2$ is the critical order of non-degeneracy of $[f(\cdot, x)]$ such that relative equilibria of the equation can be solved from its averaged equation - a typical mechanism for the existence of response solutions in perturbed, quasi-periodically forced, 2nd order nonlinear equations. In this paper, we consider the existence of response solutions of the equation for the super-critical case, i.e., $[f(\cdot, x)]$ is degenerate at least up to an order $l/2 \leq p < l$. We will show in this case that response solutions can still exist by perturbing relative equilibria of the normalized equation by considering non-degeneracy of the new perturbation after the normalization that is of at least $O(\varepsilon^2)$ order. This reveals a mechanism for the existence of response solutions of the equation in the super-critical case.

For the sake of generality, we will actually consider a general Hamiltonian normal form containing the normalized equation as a particular case. We will prove a general theorem concerning the existence of response tori of the normal form through averaging, finding relative equilibria, improving the order of perturbations, KAM iterations, and measure estimates. The general result will then be applied to the problem of the existence of response solutions of the above equation in the super-critical case. In several special situations, we will give explicitly checkable conditions for such existence in terms of Fourier coefficients of lower order terms of f.

1. INTRODUCTION

This paper is motivated by studying response solutions of the quasi-periodically forced, 2nd order differential equations

$$\ddot{x} + \lambda x^l + \varepsilon f(\omega t, x) = 0, \qquad x \in \mathbb{R}^1, \tag{1.1}$$

where $\lambda \neq 0$ is a constant, $l \geq 2$ is an integer, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector, ε is a small parameter, and $f : \mathbb{T}^d \times \mathbb{R}^1 \to \mathbb{R}^1$ is real analytic. For a quasi-periodically forced differential equation, *response solutions* are quasi-periodic ones whose frequencies coincide with that of the forcing function. These solutions, by being the most robust ones, play an important role in understanding the stability, harmonic response, oscillatory properties, and related synchronization behaviors of quasi-periodically forced differential equations, especially when they are nonlinear oscillators. The study of the subject also brings in certain mathematically challenges in the application of KAM theory. It is due to such physically and mathematical importance that response solutions have been extensively studied for a broader class of quasi-periodically forced 2nd order differential equations

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including damped or undamped, non-degenerate or completely degenerate nonlinear oscillators, even with Brjuno or Liouvillean forcing frequencies. We refer the readers to [6], [8]-[12], [18] - [21], [30], [38, 43] for classical and recent studies of the subject.

With $l \geq 2$ arbitrarily given, the origin of the unperturbed system corresponding to (1.1) is degenerate with an arbitrarily high order of degeneracy. Still, as a typical mechanism, response solutions can exist in (1.1) as perturbations to certain relative equilibria near the origin. Indeed, it is shown in [31, 32] that if

$$[f(\cdot,0)] = [\frac{\partial f(\cdot,0)}{\partial x}] = \cdots = [\frac{\partial^{p-1} f(\cdot,0)}{\partial x^{p-1}}] = 0, \quad [\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] \neq 0 \tag{1.2}$$

for some $0 \leq p < l/2$, then relative equilibria of (1.1) can be solved from its averaged equation if either $l-p$ is odd or if $\lambda[\frac{\partial^p f}{\partial x^p}(\cdot,0,0)] < 0$ when $l-p$ is even, yielding the existence of response solutions of (1.1) for ε sufficiently small. In fact, $p = l/2$ is the critical order of degeneracy of $[f(\cdot,x)]$ for the existence of relative equilibria that aare solved from the averaged equation.

The present work focuses on the existence of response solutions in the *super-critical case* that

$$[f(\cdot,0)] = [\frac{\partial f(\cdot,0)}{\partial x}] = \cdots = [\frac{\partial^p f(\cdot,0)}{\partial x^p}] = 0 \tag{1.3}$$

for some $l/2 \leq p < l$. In this case, relative equilibria of (1.1) can still exist but they are only solvable from higher order ε terms of the new perturbation in the normalized equation in general. This thus suggests a mechanism for the existence of response solutions of (1.1) in the super-critical case by perturbing relative equilibria through higher order ε terms of the perturbation in the normalized equation. For instance, in the super-critical case, we are able to show that its ε^2-order normalized equation has the form

$$\ddot{x} + \lambda(1+\varepsilon R_l)x^l + \varepsilon(f_1(x,\varepsilon) + f_2(\omega t, x, \epsilon)) + \varepsilon^2 F(\omega t, x, \epsilon) = 0, \qquad x \in \mathbb{R}^1, \tag{1.4}$$

where R_l is a constant, f_1 is a polynomial in x consisting of terms x^i, $i = p, \cdots, l$, which is smooth in ε, f_2 and F are real analytic in x and smooth in ε, and $f_2 = O(x^{l+1})$. With certain non-degenerate conditions imposing on the average of the ε^2 order term F, we are able to show the existence of relative equilibria of (1.4) and further obtain its response solutions by perturbing these relative equilibria. Of course, a general normal form reduction can be both implicit and complicated. However, as what we will show, such a normal form procedure for the equation (1.1) can be described precisely by recursive homological equations, leading to explicitly verifiable conditions in applications.

For the sake of generality, we will actually consider a Hamiltonian normal form up to any given ε^α order of perturbation for some fixed integer $\alpha \geq 2$ whose equation of motions is also more general than (1.1). More precisely, we consider a C^2 family of parametrized, real analytic Hamiltonian systems in the vicinity of the origin of $\mathbb{T}^{d+2} \times \mathbb{R}^{d+2}$, with respect to the standard symplectic structure, having Hamiltonians of the following form:

$$H = e(\varepsilon) + \langle \omega, I \rangle + \frac{\lambda}{l+1}(1+\varepsilon R_l)x^{l+1} + \frac{1}{2}(1+\varepsilon R_2)y^2 + \varepsilon G(\theta,z,\varepsilon) + \varepsilon^\alpha P(\theta,z,\varepsilon), \tag{1.5}$$

$(\theta, I) \in \mathbb{T}^d \times \mathbb{R}^d$, $z = (x,y) \in \mathbb{R}^2$, where $0 \leq \varepsilon \ll 1$ is a parameter, $\omega \in \mathbb{R}^d$ is a given Diophantine frequency vector, $l, \alpha \geq 2$ are given integers, R_l, R_2 are constants, and the function G has the form

$$G(z,\varepsilon) = G_1(x,\varepsilon) + G_2(x,y,\varepsilon) + G_3(\theta,x,y,\varepsilon),$$

for a polynomial G_2 in x, y consisting of terms $x^i y^j$, $i \geq 1$, $j \geq 2$ and $i + j \leq l+1$, a function $G_3 = O(|z|^{l+2})$, and a polynomial $G_1 = G^1(x,\varepsilon) + \varepsilon G^2(x,\varepsilon) + \cdots + \varepsilon^{\alpha-2} G^\alpha(x,\varepsilon)$ in which each term G^i, $i = 1, 2, \cdots, \alpha$, is a polynomial of x consisting of terms x^j, $j = p+1, \cdots, l+1$, for an integer $\frac{\alpha-1}{\alpha} l \leq p < l$. We refer a quasi-periodic, invariant d-torus of the Hamiltonian system corresponding to H as a *response torus* if it has the frequency vector ω.

Assume the following conditions:

(A1) There exists an integer $0 \le \hat{p} < \min\{\alpha p + l - \alpha l, \frac{l}{\alpha+1}\}$ such that

$$[P(\cdot,0,0)] = [\frac{\partial P(\cdot,0,0)}{\partial x}] = \cdots = [\frac{\partial^{\hat{p}} P(\cdot,0,0)}{\partial x^{\hat{p}}}] = 0, \quad \text{and} \quad a_{\hat{p}} =: [\frac{\partial^{\hat{p}+1} P(\cdot,0,0)}{\partial x^{\hat{p}+1}}] \neq 0;$$

(A2) $a_{\hat{p}}/\lambda < 0$ when $l - \hat{p}$ is even.

Let

$$\tilde{\lambda} := \begin{cases} \lambda, & a_{\hat{p}}/\lambda < 0, \\ \lambda, & a_{\hat{p}}/\lambda > 0 \ \text{ and } \hat{p} \ \text{ is odd}, \\ -\lambda, & a_{\hat{p}}/\lambda > 0 \ \text{ and } \hat{p} \ \text{ is even}. \end{cases} \tag{1.6}$$

Then we have the following result.

Main Theorem. *Consider Hamiltonian systems corresponding to* (1.5) *and assume conditions* **(A1)***,* **(A2)***. Then the followings hold.*

(1) *If $\tilde{\lambda} < 0$, then there exists an $0 < \varepsilon_* \ll 1$ such that, as $0 < \varepsilon \le \varepsilon_*$, the Hamiltonian systems admit a C^1 smooth family of real analytic, hyperbolic responsive tori around a family of saddle type of relative equilibria of the size $O(\varepsilon_*^{\frac{\alpha}{l+1-\hat{p}}})$.*
(2) *If $\tilde{\lambda} > 0$, then there exists a $0 < \varepsilon_* \ll$ and a Cantor set $\mathcal{D}_* \in (0, \varepsilon_*)$ of Lebesgue measure $\frac{|\text{meas } \mathcal{D}_*|}{\varepsilon^*} \sim 1 - O(\varepsilon_*^{\frac{l-1}{l+1-\hat{p}}})$ such that, as $\varepsilon \in \mathcal{D}_*$, the Hamiltonian systems admit a C^1 Whitey smooth family of real analytic responsive tori around a family of elliptic type of relative equilibria of the size $O(\varepsilon_*^{\frac{\alpha}{l+1-\hat{p}}})$.*

The above theorem will be proved using KAM iterations. While the problem is in line with the general study of the existence of lower dimensional, quasi-periodic, invariant tori in Hamiltonian systems, the normal degeneracy of the Hamiltonian (1.5), especially in higher orders, brings in substantial challenges into the study (see e.g., [31, 32, 44]), in addition to the well-known difficulty in preserving the fixed tangential frequencies during KAM iterations. While the essential idea of overcoming the problem of normal degeneracy is to compensate it with certain non-degeneracy among the perturbation as suggested by the Poincaré mechanism of the existence of lower dimensional tori in the resonance zones (see e.g., [15, 37]), the compensated non-degeneracy in the present situation however lies in a higher order term of ε, resulting in multi-scale Hamiltonian systems which are more complicated than those previously considered (e.g., in [40]).

When restricting the Hamiltonian (1.5) to (1.4), we have $\alpha = 2$, $R_1 = 0$, $G_3 \equiv 0$,

$$G_1 = \int_0^x f_1(t,\varepsilon)dt, \ G_2(\theta,x,\varepsilon) = \int_0^x f_2(\theta,t,\varepsilon)dt, \ \text{ and } P(\theta,x,\varepsilon) = \int_0^x F(\theta,t,\varepsilon)dt.$$

By applying the Main Theorem to (1.4) with small $\hat{p}$, e.g., $\hat{p} = 0$ or 1, we immediately obtain the following corollaries concerning the existence of response solutions of (1.4) under simpler conditions.

Corollary 1 ($\hat{p} = 0$). *Consider* (1.4) *with $l/2 \le p < l$. Assume that $a_0 =: [F(\cdot,0,0)] \neq 0$ and that λ and a_0 have the opposite sign when l is even. Then the following hold.*

(1) *If $\lambda < 0$, then there exists an $0 < \varepsilon_* \ll 1$ such that, as $0 < \varepsilon < \varepsilon_*$,* (1.4) *admits a smooth family of real analytic response solutions of hyperbolic type.*

(2) *If $\lambda > 0$, then there is an $0 < \varepsilon_* \ll 1$ and a Cantor set $\mathcal{D}_* \subset (0, \varepsilon_*)$ with almost full Lebesgue measure such that, as $\varepsilon \in \mathcal{D}_*$,* (1.4) *admits a Whitney smooth family of real analytic response solutions of of elliptic type.*

In both cases, the amplitudes of response solutions are of the scale of $O(\varepsilon^{\frac{2}{l}})$.

Corollary 2 ($\hat{p} = 1$). *Consider* (1.4) *with $l/2 \le p < l$. Assume that $a_0 =: [F(\cdot, 0, 0)] = 0$, $a_1 =: [\frac{\partial F(\cdot,0,0)}{\partial x}] \ne 0$, and that λ and a_1 have the opposite sign when l is odd. Then the following hold.*

(1) *If $\lambda > 0$, then there exists an $0 < \varepsilon_* \ll 1$ such that, as $0 < \varepsilon < \varepsilon_*$,* (1.4) *admits a smooth family of small amplitude, real analytic response solutions of hyperbolic type.*
(2) *If $\lambda < 0$, then there is an $0 < \varepsilon_* \ll 1$ and a Cantor set $\mathcal{D}_* \subset (0, \varepsilon_*)$ with almost full Lebesgue measure such that, as $\varepsilon \in \mathcal{D}_*$,* (1.4) *admits a Whitney smooth family of small amplitude, real analytic response solutions of of elliptic type.*

In both cases, the amplitudes of response solutions are of the scale of $O(\varepsilon^{\frac{2}{l-1}})$.

With normal form reductions, the above results can be applied to analyze the existence of response solution to the original 2nd order differential equations (1.1) in the super-critical case. By explicitly solving the homological equations involved in the normal form reductions, we are able to give explicit conditions for the existence of response solutions by using Corollaries 1,2 above. More precisely, consider (1.1) in the super-critical case that (1.3) holds for some $l/2 \le p < l$. Write f into Taylor-Fourier series:

$$f(\theta, x) = \sum_{i=0}^{\infty} f_i(\theta) x^i, \quad f_i(\theta) = \sum_{k \in \mathbb{T}^d} f_{ki} e^{\sqrt{-1}\langle k, \theta \rangle}, \quad i = 0, 1, \cdots . \tag{1.7}$$

Let

$$a_0 = \begin{cases} 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{f_{k0} f_{-k1}}{|\langle k,\omega\rangle|^2} + 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{\lambda |f_{k0}|^2}{|\langle k,\omega\rangle|^4}, & \text{when } l = 2, \\ 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{f_{k0} f_{-k1}}{|\langle k,\omega\rangle|^2}, & \text{when } l \ge 3; \end{cases} \tag{1.8}$$

$$a_1 = 2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{f_{k0} f_{-k2}}{|\langle k,\omega\rangle|^2} + 2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{|f_{k1}|}{|\langle k,\omega\rangle|^2}, \text{ when } l \ge 4. \tag{1.9}$$

Then we have the following result.

Corollary 3. *Consider* (1.1) *in the super-critical case. Then conclusions of Corollaries 1,2 hold with a_0, a_1 given in the formulas* (1.8), (1.9) *to yield the existence of response solutions of* (1.1).

We note that, in applications, only the signs of a_0, a_1, instead of their actual values, are needed in determining the existence and nature of response solutions. These signs, in many situations, may be simply determined by examining the evenness or oddness of the Fourier coefficients of f_0, f_1, f_2. For instance, consider (1.1) with $l = 4$ and $\lambda < 0$, and assume that f_0 is even with $[f_0] = 0$, f_1, f_2 are odd (hence $[f_1] = [f_2] = 0$), and $f_1(\theta) \not\equiv 0$. Then one can show that $a_0 = 0, a_1 > 0$ so that Corollary 3 (in particular Corollary 2 (2)) are applicable to yield a Whitney smooth family of response solutions of elliptic type as ε lying in an almost full Lebesgue measure set in $(0, \varepsilon_*)$ as $0 < \varepsilon_* \ll 1$ (see Section 4.3 for details). We note that if $f = f_0 + f_1 x + f_2 x^2$, then $[f] \equiv 0$ which is completely degenerate.

The rest of the paper is organized as follows. In Section 1, we re-normalize the the Hamiltonian (1.5) into a new normal form by performing averaging, finding relative equilibria, and improving the order of perturbations. We perform KAM iterations to the re-normalized Hamiltonian in Section 3 along with measure estimates. This will prove the Main Theorem and Corollaries 1, 2. Section 4 is devoted to the study of response solutions of (1.1) in the super-critical case. We will show Corollary 3 by performing normal form reduction and deriving formulas (1.8) and (1.9) by explicitly solving homological equations in the averaging process. A concrete example of $l = 4$ and $\lambda = -1$ will also be discussed.

2. Renormalization

In this section, we will renormalize the Hamiltonian normal form (1.5), under the conditions **(A1) - (A2)**, into a new one in order to perform KAM iterations. The renormalization procedure includes averaging, finding relative equilibria, and verifying their non-degeneracy and types. As the transformed Hamiltonians in the vicinity of relative equilibria are of multi-scale in ε, their order of perturbations also need to be improved via one step of KAM iteration in order to perform infinite steps of KAM iterations.

2.1. **Weighted norms.** We first introduce some weighted norms of functions to be used in the rest of the paper. For given $r, s > 0$, denote

$$D(r,s) = \mathbb{T}^d_r \times \mathbf{B}_s,$$

where

$$\mathbf{B}_s := \{z = (x,y) \in \mathbb{C}^2 : \ |z| \le s\}$$

is the ball of radius r in $\mathbb{C}^2$ and

$$\mathbb{T}^d_r := \{\theta = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : \ |\ \mathrm{Im}\ \theta_j| \le r, \quad j = 1, 2, \dots, d\}$$

is the strip neighborhood of size r of the d-torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ in $\mathbb{C}^d$. Let $\varepsilon_* > 0$ be given and $\mathcal{D} \subset (0, \varepsilon_*)$ be a given set. For any function $f : \ D(r,s) \times \mathcal{D} \to \mathbb{C}$:

$$f(\theta, z, \varepsilon) = \sum_{\imath \in \mathbb{Z}^2_+, \ k \in \mathbb{Z}^d} f_k(\varepsilon) z^\imath \mathrm{e}^{\sqrt{-1}\langle k, \theta\rangle},$$

which is real analytic in $(\theta, z) \in D(r,s)$ and C^N-Whitney smooth in $\varepsilon \in \mathcal{D}$ for some integer $N \ge 1$, we define its $\|\cdot\|_{D(r,s)\times\mathcal{D}}$ norm by

$$\|f\|_{D(r,s)\times\mathcal{D}} = \sum_{k\in\mathbb{Z}^d, \imath\in\mathbb{Z}^2_+} \sup_{\varepsilon\in\mathcal{D}} (|f_k(\varepsilon)| + \varepsilon|\frac{\partial f_k(\varepsilon)}{\partial\varepsilon}| \cdots + \varepsilon^N |\frac{\partial^N f_k(\varepsilon)}{\partial\varepsilon^N}|) s^{|\imath|} \mathrm{e}^{r|k|},$$

where $|k| = \sum_{j=1}^d |k_j|$ for $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$. If the function f is independent of ε, or z, or both, we define its $\|\cdot\|_{D(r,s)}$ norm, $\|\cdot\|_{D(r)\times\mathcal{D}}$ norm, or $\|\cdot\|_{D(r)}$ norm, where $D(r) = \mathbb{T}^d_r$, simply by taking $\varepsilon = 0$, or $s = 0$, or $\varepsilon = 0$ and $s = 0$ in the above. If f is independent of (θ, z) then we define its $\|\cdot\|_{\mathcal{D}}$ norm simply by taking $k = 0, s = 0$ in the above. The space of functions with finite $\|\cdot\|_{D(r,s)\times\mathcal{D}}$ norm, $\|\cdot\|_{D(r,s)}$ norm, $\|\cdot\|_{D(r)\times\mathcal{D}}$ norm, $\|\cdot\|_{D(r)}$, and $\|\cdot\|_{\mathcal{D}}$ norm are all Banach algebras under the respective norms, which are denoted by $C^{\varpi,N}(D(r,s)\times\mathcal{D})$, $C^{\varpi}(D(r,s))$, $C^{\varpi,N}(D(r)\times\mathcal{D})$, $C^{\varpi}(D(r))$, and $C^N(\mathcal{D})$, respectively.

With respect to Hamiltonians of the form (1.5), a transformation $\Phi(\theta, z)$ in some domain $D(r,s)$ is a called *canonical* if (id, Φ) is a symplectic transformation on $\mathbb{C}^d \times D(r,s)$ with respect to the standard symplectic structure $dI \wedge d\theta \wedge dz$.

For any function $f : \mathbb{T}^d \times D \to \mathbb{C}$, where D is a domain in an Euclidean space, we denote

$$[f(\cdot, z)] = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta, z)\ \mathrm{d}\theta, \qquad z \in D,$$

as its average on $\mathbb{T}^d$. The following lemma, whose proof is straightforward, will be frequently used in the paper.

Lemma 2.1. *Let $r > 0$, $N \geq 1$ and consider the homological equation*

$$\partial_\omega V(\theta, \varepsilon) = P(\theta, \varepsilon), \qquad \theta \in \mathbb{T}^d, \quad \varepsilon \in \mathcal{D},$$

where $P \in C^{\varpi, N}(D(r) \times \mathcal{D})$ with $[P(\cdot, \varepsilon)] = 0$ for all $\varepsilon \in \mathcal{D}$ and $\partial_\omega = \omega \cdot \frac{\partial}{\partial \theta}$ for some Diophantine vector $\omega \in \mathbb{R}^d$ of Diophantine constants $\gamma > 0, \tau > d - 1$, i.e.,

$$|\langle k, \omega \rangle| > \frac{\gamma}{|k|^\tau}, \qquad k \in \mathbb{Z}^d.$$

Then for any $0 < \eta < r$, the equation admits a unique solution $V \in C^\varpi(D(r - \eta) \times \mathcal{D})$ with zero-average on $\mathbb{T}^d$, satisfying

$$\|V\|_{D(r-\eta)\times\mathcal{D}} < C_0 \|F\|_{D(r)\times\mathcal{D}}$$

for some constant $C_0 > 0$ depending only on γ, τ, r and η.

Through the rest of the paper, we fix a Diophantine frequency vector $\omega \in \mathbb{R}^d$ with given Diophantine constants $\gamma > 0, \tau > d - 1$.

2.2. **Averaging.** In the Hamiltonian normal form H in (1.5), we express the perturbation term into the following Taylor series

$$P = \sum_{\imath=1}^{\infty} P_\imath = \sum_{\imath=1}^{\infty} \sum_{i,j \in\ \mathbb{Z}_+,\, i+j=\imath} P_{i,j}(\theta, \varepsilon) x^i y^j.$$

We note that $H \in C^{\varpi,2}(D(r, s) \times [0, \varepsilon_*))$ for some $r, s, \varepsilon_* > 0$ sufficiently small. To solve for relative equilibria of H, we need to average the coefficients P up to order $l + 1$. Below, for simplicity, we omit the explicit dependence of all averaged terms on ε.

Proposition 2.1. *Consider the family of parametrized Hamiltonians H in (1.5). Then for any given $0 < \eta < \min\{r/8,\ s/8\}$, there exists an $0 < \varepsilon_* \ll 1$ and a C^1 family of nearly identity, canonical transformations $\Phi_\varepsilon : D(r - \eta, s - \eta) \to D(r, s)$, $\varepsilon \in (0, \varepsilon_*)$, such that*

$$\begin{aligned} \mathcal{H} \quad &= \quad H \circ \Phi = \mathfrak{e}(\varepsilon) + \langle \omega, I \rangle + \frac{\lambda}{l+1}(1 + \varepsilon \mathcal{R}_l) x^{l+1} + \frac{1}{2}(1 + \varepsilon \mathcal{R}_2) y^2 \\ &+ \quad \varepsilon \mathcal{G}(\theta, z, \varepsilon) + \varepsilon^{\alpha+1} \mathcal{P}(\theta, z, \varepsilon), \end{aligned} \tag{2.1}$$

where $\mathfrak{e}(\varepsilon) = e(\varepsilon) + \varepsilon^\alpha [P_{0,0}]$, $\mathcal{R}_l = R_l + (l+1)\varepsilon^{\alpha-1}[P_{l+1,0}]$, $\mathcal{R}_2 = R_2 + 2\varepsilon^{\alpha-1}[P_{0,2}]$, and

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$$

with

$$\begin{aligned} \mathcal{G}_1 &= G_1 + \varepsilon^{\alpha-1} \sum_{1 \leq i \leq l+1} [P_{i,0}] x^i, \\ \mathcal{G}_2 &= G_2 + \varepsilon^{\alpha-1} \sum_{\substack{i+j \leq l+1 \\ i \geq 1,\ j \geq 2}} [P_{i,j}] x^i y^j, \\ \mathcal{G}_3 &= O(|z|^{l+2}), \\ \mathcal{G}_4 &= \varepsilon^{\alpha-1} \sum_{0 \leq i \leq l+1} [P_{i,1}] x^i y. \end{aligned}$$

Moreover,

(2.2) $\|\mathrm{D}^j(\mathcal{G}_m - G_m)\|_{D(r-\eta,s-\eta)\times(0,\varepsilon_*)}$, $\|\mathrm{D}^j\mathcal{P}\|_{D(r-\eta,s-\eta)\times(0,\varepsilon_*)} \le c, \quad \forall\ m = 1,\ 2,\ 3,\ j = 0,1,2,$

where $c > 0$ *is a constant depending only on* $\gamma,\ \tau,\ r,\ s,\ |R_2|,\ |R_l|$, *and* $\|P\|_{D(r,s)\times(0,\varepsilon_*)}$.

Proof. Denote

$$N = \langle \omega, I\rangle + \frac{1}{2}(1+\varepsilon R_{0,2})y^2, \quad h = \frac{\lambda}{l+1}(1+\varepsilon R_{l,0})x^{l+1}.$$

For each ε, we would like to construct the transformation Φ as the time-1 map $\phi^1_{\varepsilon^\alpha V}$ of the Hamiltonian flow generated by a generating function $\varepsilon^\alpha V$ with $V = V(\theta, x, y)$ having the form

$$V(\theta,x,y) = \sum_{\imath=0}^{l+1} V_\imath = \sum_{\imath=0}^{l+1}\sum_{i+j=\imath} V_{i,j}(\theta,\varepsilon)x^i y^j, \quad [V_{ij}] = 0. \tag{2.3}$$

Consider the homological equations

$$\{N, V_\imath\} + P_\imath - [P_\imath] = 0, \qquad \imath = 0,\ 1,\ \cdots,\ l-1, \tag{2.4}$$

$$\{N, V_l\} + P_l - [P_l] + \{h, V_1\} = 0, \tag{2.5}$$

$$\{N, V_{l+1}\} + P_{l+1} - [P_{l+1}] + \{h, V_2\} = 0. \tag{2.6}$$

We note for each $\imath = 0,\ 1, \cdots,\ l+1$ that

$$\begin{aligned}
\{N, V_\imath\} &= \frac{\partial N}{\partial \theta}\frac{\partial V_\imath}{\partial I} - \frac{\partial N}{\partial I}\frac{\partial V_\imath}{\partial \theta} + \frac{\partial N}{\partial x}\frac{\partial V_\imath}{\partial y} - \frac{\partial N}{\partial y}\frac{\partial V_\imath}{\partial x},\\
&= -\partial_\omega V_{\imath,0}x^\imath - \partial_\omega V_{\imath-1,1}x^{\imath-1}y - \cdots - \partial_\omega V_{1,\imath-1}xy^{\imath-1} - \partial_\omega V_{0,\imath}y^\imath,\\
&\quad -\imath(1+\varepsilon R_2)V_{\imath,0}x^{\imath-1}y - (\imath-1)(1+\varepsilon R_2)V_{\imath-1,1}x^{\imath-2}y^2\\
&\quad - \cdots - 2(1+\varepsilon R_2)V_{2,\imath-2}xy^{\imath-1} - (1+\varepsilon R_2)V_{1,\imath-1}y^\imath.
\end{aligned}$$

For each $\imath = 0,\ 1, \cdots,\ l-1$, using the above and comparing the coefficients of the same orders in (2.4), we see that (2.4) is equivalent to

$$\begin{cases}
\partial_\omega V_{\imath,0} - P_{\imath,0} + [P_{\imath,0}] = 0,\\
\partial_\omega V_{\imath-1,1} + \imath(1+\varepsilon R_2)V_{\imath,0} - P_{\imath-1,1} + [P_{\imath-1,1}] = 0,\\
\quad\cdots\\
\partial_\omega V_{0,\imath} + (1+\varepsilon R_2)V_{1,\imath-1} - P_{0,\imath} + [P_{0,\imath}] = 0.
\end{cases} \tag{2.7}$$

For given $0 < \eta < r$, by applying Lemma 2.1, each equation in (2.7) can be uniquely solved inductively to yield solutions $V_{\imath-j,j}$, $j = 0, 1, \cdots, \imath$, satisfying

$$\|V_{\imath-j,j}\|_{D(r-\frac{j}{2(\imath+1)}\eta)\times(0,\varepsilon_*)} \le c\|P\|_{D(r)\times(0,\varepsilon_*)}, \quad j = 0,1,\cdots,\imath, \tag{2.8}$$

where $c > 0$ is a constant depending only on γ, τ, r, η and R_2. This solves (2.4) for all $V_\imath$, $\imath = 0, 1, \cdots, l-1$ on $D(r-\frac{\eta}{2}, s-\frac{\eta}{2})\times(0,\varepsilon_*)$. Similarly, (2.5), (2.6) can be solved uniquely on $D(r-\frac{\eta}{2}, s-\frac{\eta}{2})\times(0,\varepsilon_*)$ whose coefficients satisfy estimates similar to (2.8) with a constant $c > 0$ also depending on R_l. Thus, we obtain the function V from the homological equations (2.4) - (2.6), which, by Cauchy estimates, satisfies

$$\|\mathrm{D}^j V\|_{D(r-\eta,s-\eta)\times(0,\varepsilon_*} \le c_0,\ j = 0,1,2, \tag{2.9}$$

where $c_0 > 0$ is a constant depending only on $\gamma,\ \tau,\ r,\ s,\ |R_2|,\ |R_l|$, and $\|P\|_{D(r,s)\times(0,\varepsilon_*)}$. It follows that the time-1 map Φ, generated by the Hamiltonian flow associated with the generating function $\varepsilon^\alpha V$, is a well defined, near identity transformation as long as ε_* is sufficiently small. Now,

$$\begin{aligned}
H\circ\Phi &= (N + h(x) + \varepsilon G)\circ\phi^1_{\varepsilon^\alpha V} + \varepsilon^\alpha P\circ\phi^1_{\varepsilon^\alpha V}\\
&= N + h(x) + \varepsilon G + \varepsilon^\alpha P + \varepsilon^\alpha\{N+h, V\} + \varepsilon^{\alpha+1}\{G, V\}\\
&\quad + \varepsilon^{2\alpha}\{\frac{1}{2}\{N+h,V\} + P, V\} + \varepsilon^{2\alpha}\int_0^1 \{\{\frac{1}{2}(1-t)(\varepsilon^\alpha P + \varepsilon G), V\}, V\}\circ\phi^t_{\varepsilon^\alpha V}\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&+\varepsilon^{3\alpha}\int_0^1\{\{\{\frac{1}{2}(1-t)^2(N+h),V\},V\},V\}\circ\phi^t_{\varepsilon^\alpha V}\mathrm{d}t\\
=\ &\mathfrak{e}(\varepsilon)+\langle\omega,I\rangle+\frac{\lambda}{l+1}(1+\varepsilon\mathcal{R}_l)x^{l+1}+\frac{1}{2}(1+\varepsilon\mathcal{R}_2)y^2\\
+\ &\varepsilon(\mathcal{G}_1(x,\varepsilon)+\mathcal{G}_2(z,\varepsilon)+\mathcal{G}_3(\theta,z,\varepsilon)+\mathcal{G}_4(x,\varepsilon))+\varepsilon^{\alpha+1}\mathcal{P}(\theta,z,\varepsilon),
\end{aligned}$$

where $\mathfrak{e}$, $\mathcal{R}_l$, $\mathcal{R}_2$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_4$ are as in (2.1) and

$$\begin{aligned}
\mathcal{G}_3 &= G_3+\sum_{\imath\geq l+2}P_\imath(\theta,z,\varepsilon)+\sum_{\imath=3}^{l+1}\{h,V_\imath\}=O(|z|^{l+2}),\\
\mathcal{P} &= \{G,V\}+\varepsilon^{\alpha-1}\{\frac{1}{2}\{N+h,V\}+P,V\}\\
&\quad+\varepsilon^{\alpha-1}\int_0^1\{\{\frac{1}{2}(1-t)(\varepsilon^\alpha P+\varepsilon G),V\},V\}\circ\phi^t_{\varepsilon^\alpha V}\mathrm{d}t\\
&\quad+\varepsilon^{2\alpha-1}\int_0^1\{\{\{\frac{1}{2}(1-t)^2(N+h),V\},V\},V\}\circ\phi^t_{\varepsilon V}\mathrm{d}t,
\end{aligned}$$

from which we also see that (2.2) holds. □

Remark 2.1. (1) *We note that conditions* **(A1)**, **(A2)** *are not needed in* Proposition 2.1.

(2) *However, as to be shown below, the condition* **(A1)** *is crucial for the existence of relative equilibria associated with* (2.1). *In the case that the condition fails for the Hamiltonian* (1.5), *one can use the following quasi-linear transformation*

$$\mathcal{L}:\quad x\to x,\qquad y\to y-\varepsilon^\alpha\sum_{i=0}^{l}[P_{i1}]x^i$$

to (2.1) *to eliminate the $\varepsilon\mathcal{G}_4$ term, so that the transformed Hamiltonian has the same form as that of* (1.5). *It is then hopeful that the condition* **(A1)** *holds for the transformed Hamiltonian. In case it does not, then one can repeatedly apply the averaging procedure described in the proof of* Proposition 2.1, *followed by a quasi-linear transformation as in the above, to obtain new Hamiltonians of the form* (1.5), *until the condition* **(A1)** *is satisfied. This is one of the reasons for us to consider* (1.5) *with an $\alpha\geq 2$ because it made be derived from the perturbation order $O(\varepsilon^2)$ to some $O(\varepsilon^\alpha)$ by repeatedly applying such reduction procedures until the condition* **(A1)** *holds.*

2.3. **Relative equilibria.** Relative equilibria corresponding to (2.1) are referred to as equilibria of the averaged part of the Hamiltonian vector fields in the normal direction. We have the following result.

Lemma 2.2. *Under the conditions* **(A1)**, **(A2)**, *the Hamiltonians* (2.1) *admit a smooth family of relative equilibria of the form*

$$z_\varepsilon=(x_\varepsilon,\ y_\varepsilon)=(\varepsilon^{\frac{\alpha}{l-\hat p}}x_*+O(\varepsilon^{\frac{\alpha}{l-\hat p}+\delta}),\ \varepsilon^\alpha y_*+O(\varepsilon^{\alpha+\delta})),$$

as $0<\varepsilon\ll 1$, where $\delta>0$ and $x_*\neq 0$.

Proof. Let

$$\mathcal{H}_*(x,y,\varepsilon)=\frac{\lambda}{l+1}(1+\varepsilon\mathcal{R}_l)x^{l+1}+\frac{1}{2}(1+\varepsilon\mathcal{R}_2)y^2+\varepsilon(\mathcal{G}_1+\mathcal{G}_2+[\mathcal{G}_3]+\mathcal{G}_4)+\varepsilon^{\alpha+1}[\mathcal{P}].$$

Then relative equilibria corresponding to (2.1) are roots of

$$\left\{\begin{array}{l}\dfrac{\partial \mathcal{H}_*(x,y,\varepsilon)}{\partial x} = \lambda x^l + +\varepsilon^\alpha a_{\hat{p}} x^{\hat{p}} + O(\varepsilon)(O(x^p) + O(y^2)) + O(\varepsilon^\alpha)O(y) + O(\varepsilon^{\alpha+1}) = 0,\\ \dfrac{\partial \mathcal{H}_*(x,y,\varepsilon)}{\partial y} = y + \varepsilon^\alpha b_1 + O(\varepsilon^\alpha)O(x) + O(\varepsilon)(O(x^{l+1}) + O(y)) + O(\varepsilon^{\alpha+1}) = 0.\end{array}\right. \tag{2.10}$$

where

$$b_1 = [\frac{\partial P(\cdot,0,0)}{\partial y}].$$

Under the re-scaling

$$x \to \varepsilon^{\frac{\alpha}{l-\hat{p}}} x, \qquad y \to \varepsilon^\alpha y,$$

(2.10) becomes

$$\left\{\begin{array}{l}\lambda x^l + a_{\hat{p}} x^{\hat{p}} + O(\varepsilon^\delta)(O(x^p) + O(y) + O(1)) = 0,\\ y + b_1 + O(\varepsilon^\delta)(O(x) + O(y) + O(1)) = 0,\end{array}\right. \tag{2.11}$$

where

$$\delta = \min\{\frac{\alpha p + l - \alpha l - \hat{p}}{l-\hat{p}}, \frac{l - (\alpha+1)\hat{p}}{l - \hat{p}}\}.$$

Conditions **(A1)**, **(A2)** together with the implicit function theorem clearly imply that (2.11) admits a smooth family of solutions

$$x_* = (-\frac{a_{\hat{p}}}{\lambda})^{\frac{1}{l-\hat{p}}} + O(\varepsilon^\delta), \quad y_* = -b_1 + O(\varepsilon^\delta),$$

which, by tracing back to the rescaling, yields desired solutions of (2.10). □

By translating the Hamiltonian (2.1) in the vicinity of the relative equilibria given in Lemma 2.2, we obtain a new normal form as follows.

Lemma 2.3. *Assume conditions* **(A1)**, **(A2)** *and let z_ε be the relative equilibria defined in Lemma 2.2. Then, with the translation $L: z \to z + z_\varepsilon$, the Hamiltonian* (2.1) *becomes*

$$\tilde{H} = \mathcal{H} \circ L = \tilde{e} + \langle \omega, I\rangle + \frac{1}{2}\langle \tilde{M} z, z\rangle + \tilde{h}(z,\varepsilon) + \varepsilon \tilde{G}(\theta, z, \varepsilon) + \varepsilon^{\alpha_1+\sigma}\tilde{E}(\theta,z,\varepsilon), \tag{2.12}$$

where $\tilde{e}$ is a smooth function of ε,

$$\tilde{M} = \begin{pmatrix} \varepsilon^{\alpha_1}\tilde{m}_{11} & \varepsilon^\alpha \tilde{m}_{12}\\ \varepsilon^\alpha \tilde{m}_{12} & \tilde{m}_{22}\end{pmatrix} \tag{2.13}$$

with $\alpha_1 = \frac{\alpha(l-1)}{l-\hat{p}}$, $\tilde{m}_{11} = \frac{\partial^2 \mathcal{H}_*(x_*,y_*,0)}{\partial x^2} + O(\varepsilon^\delta)$, $\tilde{m}_{22} = 1 + O(\varepsilon)$, $\tilde{m}_{12} = \tilde{m}_{21}$, *and* $\tilde{h} = O(|z|^3)$, $\tilde{G} = O(|z|^3)$, $\sigma = \frac{l+\alpha-(\alpha+1)\hat{p}}{l-\hat{p}}$, *$\tilde{E}$ is a quadratic polynomial in z with*

$$[\frac{\partial^j \tilde{E}(\cdot,0,\varepsilon)}{\partial z^j}] = 0, \tag{2.14}$$

$$\|\mathrm{D}^j \tilde{E}\|_{D(r-\eta,s-\eta)\times(0,\varepsilon_*)} \le 1, \tag{2.15}$$

for all $j = 0,1,2$.

Proof. After the translation of relative equilibria, we let

$$\begin{aligned}\tilde{M} &= \frac{\partial^2 \mathcal{H}_*(z_\varepsilon,\varepsilon)}{\partial z^2},\\ \tilde{h} &= \mathcal{H}_*(z+z_\varepsilon,\varepsilon) - \mathcal{H}_*(z_\varepsilon,\varepsilon) - \frac{1}{2}\langle \tilde{M} z, z\rangle,\\ \tilde{e} &= \mathfrak{e}(\varepsilon) + \mathcal{H}_*(z_\varepsilon,\varepsilon).\end{aligned}$$

Consider the Taylor expansion of

$$\varepsilon(\mathcal{G}_3(\theta, z+z_\varepsilon, \varepsilon) - [\mathcal{G}_3(\cdot, z+z_\varepsilon, \varepsilon)]) + \varepsilon^{\alpha+1}(\mathcal{P}(\theta, z+z_\varepsilon, \varepsilon) - [\mathcal{P}(\cdot, z+z_\varepsilon, \varepsilon)])$$

in z. Then $\varepsilon^{\alpha_1+\sigma}\tilde{E}$ consists of up to quadratic order terms of the Taylor series and $\tilde{G}$ consists of the remaining terms. It is clear that (2.14) holds and (2.15) follows from (2.2). □

2.4. **Improving order of pertuebation.** Since $|\tilde{M}^{-1}| = O(\varepsilon^{-\alpha_1})$, the perturbation order $O(\varepsilon^{\alpha_1+\sigma})$ in (2.12) is not small enough for performing infinite steps of KAM iterations. To overcome this obstacle, the final step of the renormalization procedure is to improve the order of the perturbation in (2.12) to $O(\varepsilon^{2\alpha_1+\sigma})$ which at least doubles that in the quadratic part. We will do so by performing one step of KAM iteration using a scheme that cannot be carried over to infinite steps.

Proposition 2.2. *Consider the Hamiltonian* (2.12) *in* $D(r-2\eta, s-2\eta) \times \mathcal{D}$ *for fixed* $0 < \eta \leq \min\{r, s\}/8$ *and* $\mathcal{D} = (0, \varepsilon_*)$, *where* ε_* *is sufficiently small. Then there exists a* C^1 *family of real analytic, canonical transformations*

$$\Phi_{0,\varepsilon} : D(r-2\eta,\ s-2\eta) \to D(r-3\eta,\ s-3\eta), \qquad \varepsilon \in \mathcal{D},$$

under which the Hamiltonian is transformed to

$$H_0 := \tilde{H} \circ \Phi_{0,\varepsilon} = e_0(\varepsilon) + \langle \omega, I\rangle + \langle M_0(\varepsilon)z, z\rangle + h_0(z, \varepsilon) + \varepsilon G_0(\theta, z, \varepsilon) + \varepsilon^{2\alpha_1+\sigma} P_0(\theta, z, \varepsilon),$$

where M_0 *is nonsingular with* $|M_0^{-1}| = O(\varepsilon^{-\alpha_1})$ *for each* $\varepsilon \in \mathcal{D}$, $h_0(z, \varepsilon)$, $G_0(\theta, z, \varepsilon) = O(|z|^3)$, *and*

$$\|\mathrm{D}^j G_0\|_{D(r-3\eta,\ s-3\eta)\times\mathcal{D}} \leq c\varepsilon_*^{\alpha_1+\sigma-1}, \tag{2.16}$$

$$\|\mathrm{D}^j P_0\|_{D(r-3\eta,\ s-3\eta)\times\mathcal{D}} \leq c\varepsilon_*^{\sigma}, \tag{2.17}$$

$j = 0, 1, 2$, *for some constant* $c > 0$.

Proof. For simplicity, we use c to denote any intermediate positive constants depending only on $\gamma,\ \tau,\ \eta,\ r,\ s$. Let

$$K = \min\{\kappa > 0 : \int_\kappa^\infty t^{d+1} \mathrm{e}^{-\frac{\eta t}{4}} \mathrm{d}t \leq \varepsilon_*^{\alpha_1+\sigma}\}. \tag{2.18}$$

We will eliminate terms up to order K in the Fourier expansion of the perturbation $\tilde{E}$ in (2.12), i.e., the truncation of the Fourier series of $\tilde{E}$ of the form

$$\begin{aligned}\bar{E}(\theta, z, \varepsilon) \quad &= \quad \sum_{0<|k|\leq K} E_{k0}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\leq K} \langle E_{k1}, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\leq K} \langle E_{k2}z, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} \\ &=: \quad \bar{E}_0 + \bar{E}_1 + \bar{E}_2,\end{aligned}$$

by a canonical transformation ϕ_F^1 which is the time-1 map of the Hamiltonian flow ϕ_F^t generated by a generating function F of the form

$$F(\theta, z, \varepsilon) = \sum_{0<|k|\leq K} f_{k0}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\leq K} \langle f_{k1}, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\leq K} \langle f_{k2}z, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} =: F_0 + F_1 + F_2,$$

where for each k, i, f_{ki} only depend on ε and E_{k2}, f_{k2} are symmetric for each ε. Let $\tilde{h}_3$, $\tilde{G}_3$ consist of all cubic order terms in the Taylor expansion of $\tilde{h}$, $\tilde{G}$, respectively, and denote

$$R_2 := \{\tilde{h}_3, F_1\}, \quad Q_2 := \{\tilde{G}_3, F_1\} - [\{\tilde{G}_3, F_1\}].$$

it is clear that both $R_2, \tilde{G}$ are quadratic monomials in z and $[R_2] = 0,\ [Q_2] = 0$. Thus, their truncated Fourier series up to order K read

$$\bar{R}_2(\theta, z, \varepsilon) \quad = \quad \sum_{0<|k|\leq K} \langle R_{k2}z, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle},$$

$$\bar{Q}_2(\theta, z, \varepsilon) \quad = \quad \sum_{0<|k|\le K} \langle Q_{k2} z, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle},$$

where for each k,

$$R_{k2} = \begin{pmatrix} 3x_\varepsilon^{l-3} f_{k01} & 0 \\ 0 & 0 \end{pmatrix}$$

with x_ε being the first component of the relative equilibria and f_{k01} being the second component of f_{k1}, G_{k2} is an ε-dependent symmetric matrix. We will determine F by the following homological equations

$$\{N, F_0\} + \varepsilon^{\alpha_1+\sigma}\bar{E}_0 = 0, \tag{2.19}$$

$$\{N, F_1\} + \varepsilon^{\alpha_1+\sigma}\bar{E}_1 = 0, \tag{2.20}$$

$$\{N, F_2\} + \varepsilon^{\alpha_1+\sigma}\bar{E}_2 + \bar{R}_2 + \varepsilon\bar{Q}_2 = 0, \tag{2.21}$$

where $N = \tilde{e} + \langle \omega, I\rangle + \frac{1}{2}\langle \tilde{M} z, z\rangle$. For each k, let

$$\begin{aligned} L_{k0} &= \sqrt{-1}\langle k,\omega\rangle, \\ L_{k1} &= \sqrt{-1}\langle k,\omega\rangle I_2 - \tilde{M}(\varepsilon) J, \\ L_{k2} &= \sqrt{-1}\langle k,\omega\rangle I_4 - \tilde{M}(\varepsilon) J\otimes I_2 - I_2 \otimes J\tilde{M}(\varepsilon), \end{aligned}$$

where I_2 and I_4 are 2 and 4 dimensional identity matrixes, respectively, J is the standard 2-dimensional symplectic matrix, and $\otimes$ denotes the tensor product of matrixes. Then simple calculations using the comparison of coefficients show that the equations (2.19)-(2.21) are equivalent to

$$L_{k0} f_{k0} = -\varepsilon^{\alpha_1+\sigma} E_{k0}, \qquad 0 < |k| \le K, \tag{2.22}$$

$$L_{k1} f_{k1} = -\varepsilon^{\alpha_1+\sigma} E_{k1}, \qquad 0 < |k| \le K, \tag{2.23}$$

$$L_{k2} F_{k2} = -U_{k2}, \qquad 0 < |k| \le K, \tag{2.24}$$

where F_{k2} and U_{k2} are column 4-vectors deriving from matrixes f_{k2} and $\varepsilon^{\alpha+\sigma}E_{k2} + \bar{R}_{k2} + \varepsilon\bar{Q}_{k2}$, respectively, by ordering their entries in the first row followed by the second row from the left to the right. Let $0 < \varepsilon_* \ll 1$ be such that

$$\varepsilon_*^{\frac{5\alpha_1}{6}} |K|^{2\tau} \le \gamma. \tag{2.25}$$

Then for each $0 < |k| \le K$,

$$\begin{aligned} |\det L_{k1}|_{\mathcal{D}} &= |(\sqrt{-1}\langle k,\omega\rangle)^2 + \varepsilon^{\alpha_1}(\tilde{m}_{11}\tilde{m}_{22} - \varepsilon^{2\alpha-\alpha_1}\tilde{m}_{12}\tilde{m}_{21})|_{\mathcal{D}} \\ &\ge |\langle k,\omega\rangle|^2 |1 - O(\varepsilon^{\alpha_1})|k|^{2\tau}\gamma^{-2}| \ge \frac{\gamma^2}{2|k|^{2\tau}}, \\ |\det L_{k2}|_{\mathcal{D}} &= |(\sqrt{-1}\langle k,\omega\rangle)^4 + \varepsilon^{\alpha_1}\tilde{m}_{11}\tilde{m}_{22}(\sqrt{-1}\langle k,\omega\rangle)^2 - \varepsilon^{2\alpha}(\sqrt{-1}\langle k,\omega\rangle)^2\tilde{m}_{12}\tilde{m}_{21}|_{\mathcal{D}} \\ &\ge |(\sqrt{-1}\langle k,\omega\rangle)|^4 (1 - O(\varepsilon_*^{\alpha_1})|K|^2\gamma^{-2}) \ge |\langle k,\omega\rangle|^4 \ge \frac{\gamma^4}{2|k|^{4\tau}}, \end{aligned}$$

as $0 < \varepsilon_* \ll 1$. It follows that (2.19)-(2.21) are uniquely solvable to yield the generating function F which is real analytic on $D(r-2\eta, s-2\eta)$ for each $\varepsilon \in \mathcal{D}$ and depend on ε C^1 smoothly. To estimate F, we note by straightforward calculations that

$$|L_{k1}^{-1}|_{\mathcal{D}} \le c|k|^{4\tau+4}\gamma^{-4}, \quad |L_{k2}^{-1}|_{\mathcal{D}} \le c|k|^{8\tau+12}\gamma^{-8}.$$

It follows from (2.22), (2.23) that

$$\|\mathrm{D}^j F_0\|_{D(r-\frac{5\eta}{2},\ s-\frac{5\eta}{2})\times\mathcal{D}} \le \varepsilon^{\alpha_1+\sigma} c C_\eta, \tag{2.26}$$

$$\|\mathrm{D}^j F_1\|_{D(r-\frac{5\eta}{2},\ s-\frac{5\eta}{2})\times\mathcal{D}} \le \varepsilon^{\alpha_1+\sigma} c C_\eta, \tag{2.27}$$

$j=0,1,2$, where

$$C_\eta=\sum_{0<|k|\le K}\mathrm{e}^{-\frac{\eta}{4}}|k|^{8\tau+12}. \tag{2.28}$$

Tracing back to the definitions of $\bar R_2$, $\bar Q_2$, we have by (2.26) and Cauchy estimates that

$$|R_{k2}|\le\varepsilon^{\alpha_1+\sigma}cC_\eta\mathrm{e}^{-(r-\frac{5\eta}{2})},\qquad |Q_{k2}|\le\varepsilon^{\alpha_1+\sigma}cC_\eta\mathrm{e}^{-(r-\frac{5\eta}{2})},$$

$0<|k|\le K$. It follows from (2.24) that

$$\|\mathrm{D}^jF_2\|_{D(r-\frac{11\eta}{4},\ s-\frac{11\eta}{4})\times\mathcal{D}}\le\varepsilon^{\alpha_1+\sigma}cC_\eta^2,\quad j=0,1,2.$$

Using standard arguments in KAM theory, we see that there exists a constant $c_0>0$ such that

$$\begin{aligned}&\|\mathrm{D}^jF\|_{D(r-3\eta,s-3\eta)\times\mathcal{D}}\le c_0\varepsilon^{\alpha_1+\sigma}(C_\eta+C_\eta^2),\ j=0,1,2,\\&|\phi_F^t-\mathrm{id}|_{D(s-3\eta,r-3\eta)\times\mathcal{D}}\le c_0\varepsilon^{\alpha_1+\sigma}(C_\eta+C_\eta^2),\ t\in[0,1].\\&\|\mathrm{D}\phi_F^1-Id\|_{D(s-3\eta,r-3\eta)\times\mathcal{D}}\le c_0\varepsilon^{\alpha_1+\sigma}(C_\eta+C_\eta^2).\end{aligned}\tag{2.29}$$

By making ε_* further small if necessary such that

$$c_0\varepsilon_*^{\alpha_1+\sigma}(C_\eta+C_\eta^2)\le\eta,$$

we see that the transformations $\Phi_{0,\varepsilon}:=\phi_F^1:D(r-3\eta,s-3\eta)\to D(r-2\eta,s-2\eta)$, $\varepsilon\in\mathcal{D}$, are well defined. Define

$$\begin{aligned}M_0&=\tilde M+\varepsilon[\{\tilde G_3,F_1\}],\\h_0&=\tilde h,\\G_0&=\tilde G+\varepsilon^{-1}\{\tilde h_3+\varepsilon\tilde G_3,F_2\}+\varepsilon^{-1}\{\tilde h-\tilde h_3+\tilde G-\tilde G_3,F_1+F_2\},\\P_0&=\varepsilon^{-\alpha_1}(\tilde E-\bar E)+\varepsilon^{-2\alpha_1-\sigma}(Q_2-\bar Q_2)+\varepsilon^{-\alpha_1}\int_0^1\{\tilde E-\bar E,F\}\circ\phi_F^t\mathrm{d}t\\&\quad+\varepsilon^{-2\alpha-\sigma}\int_0^1\{\{(1-t)(N+\tilde h+\varepsilon\tilde G),F\},F\}\circ\phi_F^t\mathrm{d}t.\end{aligned}$$

Then straightforward calculations show that

$$\tilde H\circ\Phi_{0,\varepsilon}=e_0(\varepsilon)+\langle\omega,I\rangle+\langle M_0(\varepsilon)z,z\rangle+h_0(z,\varepsilon)+\varepsilon G_0(\theta,z,\varepsilon)+\varepsilon^{2\alpha_1+\sigma}P_0(\theta,z,\varepsilon),$$

for an appropriate term $e_0(\varepsilon)$ depending only on ε. Since

$$|\tilde M-M_0|\le\varepsilon^{1+\alpha_1+\sigma}cC_\eta\le\varepsilon^{\alpha_1+1}\tag{2.30}$$

as $0<\varepsilon_*\ll1$, we have

$$\begin{aligned}|M_0^{-1}|&\le|(\tilde M+(M_0-\tilde M))^{-1}|\le\frac{|\tilde M^{-1}|}{1-|\tilde M^{-1}||M_0-\tilde M|}\\&\le\frac{\varepsilon^{-\alpha_1}}{1-O(\varepsilon^{1+\sigma})}\le c\varepsilon^{-\alpha_1},\end{aligned}$$

as $0<\varepsilon_*\ll1$. By definition of K, we have

$$\begin{aligned}\|D^j(\tilde E-\bar E)\|_{D(r-9\eta/4,s-2\eta)\times\mathcal{D}}&\le\sum_{|k|>K}c\mathrm{e}^{-\frac{\eta|k|}{4}}\le c\int_K^\infty t^{d+1}\mathrm{e}^{-\frac{\eta t}{4}}\mathrm{d}t\\&\le c\varepsilon_*^{\alpha_1+\sigma},\quad j=0,\ 1,\ 2.\end{aligned}$$

Similarly,

$$\|D^j(Q_2-\bar Q_2)\|_{D(r-11\eta/4-,s-11\eta/4)}\times\mathcal{D}\le c\varepsilon_*^{2\alpha_1+2\sigma},\quad j=0,1,2.$$

These, together with estimates on F_1,F_2,F above, imply (2.16) and (2.17). □

Remark 2.2. *We note that the above reduction procedure only holds for a finite step of iterations. For instance, the condition* (2.25) *does not hold for an infinite step of iterations.*

3. KAM iteration

With the Hamiltonian normal form (2.16), the Main Theorem can be shown using standard KAM iterations. We sketch in this section the main steps of the KAM scheme and refer the readers to [31, 41] for more details.

Denote $r_0 = r - 3\eta$, $s_0 = \varepsilon_*^{\frac{\sigma}{6}}$, $\mu_0 = \varepsilon_*^{\frac{\sigma}{3}}$, $\gamma_0 = \min\{\gamma^2,\ \gamma^4\}$, and $\mathcal{D}_0 := (0, \varepsilon_*)$, $\beta = 2\alpha_1 + \frac{\sigma}{4}$. As $0 < \varepsilon_* \ll 1$, we clearly have $s_0 \le s - 3\eta$ and $\gamma_0 \ll \varepsilon_*^{\frac{\sigma}{12}}$. Then Hamiltonian (2.16) can be re-written as

$$H_0 = e_0(\varepsilon) + \langle \omega, I\rangle + \langle M_0(\varepsilon)z, z\rangle + h_0(z,\varepsilon) + \varepsilon G_0(\theta, z, \varepsilon) + \varepsilon^\beta P_0(\theta, z, \varepsilon),$$

$(\theta, z) \in D(r_0, s_0)$, $I \in \mathbb{C}^d$, $\varepsilon \in \mathcal{D}_0$, such that $\|M_0^{-1}\|_{\mathcal{D}_0} = O(\varepsilon_*^{-\alpha_1})$ and

$$\|\mathrm{D}^j G_0\|_{D(r_0,s_0)\times\mathcal{D}_0} \le \varepsilon_*^{\alpha_1 - 1}\gamma_0^2 s_0^{2-j}\mu_0,$$
$$\|\mathrm{D}^j P_0\|_{D(r_0,s_0)\times\mathcal{D}_0} \le \gamma_0^2 s_0^{2-j}\mu_0,$$

$j = 0, 1, 2$.

3.1. **One cycle of KAM iteration.**

Suppose that we have arrived at a νth KAM step and obtained the following C^1 family of real analytic Hamiltonians

$$H = e(\varepsilon) + \langle \omega, I\rangle + \langle M(\varepsilon)z, z\rangle + h(z,\varepsilon) + \varepsilon G(\theta, z, \varepsilon) + \varepsilon^\beta P(\theta, z, \varepsilon), \tag{3.1}$$

on a phase domain $D(r, s) \subset D(r_0, s_0)$ and a parameter domain $\mathcal{D} \subset \mathcal{D}_0$, where M is nonsingular, symmetry matrix for each $\varepsilon \in \mathcal{D}$, $h(z, \varepsilon)$, $G(\theta, z, \varepsilon) = O(|z|^3)$, and

$$\|\mathrm{D}^j P\|_{D(r,s)\times\mathcal{D}} \le \gamma^2 s^{2-j}\mu,\ \ j = 0, 1, 2,$$

for some $0 < \mu \le \mu_0$ and $0 < \gamma \le \gamma_0$. Let

$$\begin{aligned}
r_+ &= \frac{r}{2} + \frac{r_0}{4},\\
s_+ &= \frac{1}{4}\hat\alpha s,\ \ \hat\alpha = \mu^{\frac{1}{3}},\\
\mu_+ &= c_0\mu^{\frac{7}{6}},\\
\gamma_+ &= \frac{\gamma}{2} + \frac{\gamma_0}{4},\\
\mathcal{D}_+ &= \{\varepsilon \in \mathcal{D} :\ |\det(\sqrt{-1}\langle k, \omega\rangle I_2 - MJ)| > \frac{\gamma}{|k|^{5\tau}},\\
&\qquad |\det(\sqrt{-1}\langle k, \omega\rangle I_4 - MJ\otimes I_2 - I_2 \otimes MJ)| > \frac{\gamma}{|k|^{5\tau}},\ 0 < |k| \le K_+\},
\end{aligned}$$

where c_0 is a constant to be determined later and

$$K_+ = ([\log\frac{1}{\mu}] + 1)^3.$$

The KAM iteration at this cycle amounts to find a canonical transformation $\Phi_+ :\ D(r_+, s_+) \times \mathcal{D}_+ \to D(r, s) \times \mathcal{D}$, which is C^1 in $\varepsilon \in \mathcal{D}$, real analytic and canonical in (θ, z), and transforms (3.1) to a new Hamiltonian

$$H_+ := H \circ \Phi_+ \ = \ e_+(\varepsilon) + \langle \omega, I\rangle + \langle M_+(\varepsilon)z, z\rangle + h_+(z, \varepsilon) + \varepsilon G_+(z, \theta, \varepsilon) + \varepsilon^\beta P_+(\theta, z, \varepsilon),$$

where M_+ is nonsingular, symmetry matrix for each $\varepsilon \in \mathcal{D}_+$, $h_+(z, \varepsilon)$, $G_+(\theta, z, \varepsilon) = O(|z|^3)$, and

$$\|\mathrm{D}^j P_+\|_{D(r_+,s_+)\times\mathcal{D}_+} \le \gamma_+^2 s_+^{2-j}\mu_+,\ j = 0, 1, 2.$$

Below, we denote

$$\begin{aligned} N &= e + \langle \omega, I\rangle + \langle Mz, z\rangle. \\ D_i &= D(r_+ + \frac{i-1}{4}(r - r_+), \frac{i}{4}\hat{\alpha}s), \qquad i = 1,2,3,4 \\ \Gamma(r - r_+) &= \sum_{0<|k|\le K_+} |k|^{10\tau+12}\mathrm{e}^{-|k|\frac{r-r_+}{4}}, \\ L_{k0} &= \sqrt{-1}\langle k, \omega\rangle, \\ L_{k1} &= \sqrt{-1}\langle k, \omega\rangle I_2 - MJ, \\ L_{k2} &= \sqrt{-1}\langle k, \omega\rangle I_4 - MJ\otimes I_2 - I_2 \otimes MJ, \qquad 0 < |k| \le K_+ \end{aligned}$$

and use c and c_i, $i = 1, 2, \cdots, 5$ to denote the constants that are independent of the iteration process.

Write P into Taylor-Fourier series:

$$P = \sum_{k\in\mathbb{Z}^d, \imath\in\mathbb{Z}^2_+} p_{k\imath} z^\imath \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle}$$

and consider its truncation

$$R = \sum_{|k|\le K_+} p_{k00}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{|k|\le K_+} \langle p_{k01}, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{|k|\le K_+} \langle z, p_{k02} z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} := R_0 + R_1 + R_2.$$

The transformation Φ_+ will be constructed, according to the form of R, as the time-1 map ϕ^1_F of the Hamiltonian flow ϕ^t_F of a generating function

$$\begin{aligned} F &= F_0 + F_1 + F_2 + \bar{F}_1 \\ =: & \sum_{0<|k|\le K_+} f_{k0}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\le K_+} \langle f_{k1}, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \sum_{0<|k|\le K_+} \langle f_{k2} z, z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle} + \langle f_{01}, z\rangle. \end{aligned}$$

Let h_3, G_3 be cubic order monomials in h, G, respectively, and denote

$$Q = \{h_3,\ F_1\} + \varepsilon\{G_3, \bar{F}_1\} = \sum_{k\in\mathbb{Z}^d} \langle z, q_{k2} z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle},$$

$$\bar{Q} = \sum_{|k|\le K_+} \langle z, q_{k2} z\rangle \mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle}.$$

We determine F from the following homological equations:

$$Mf_{01} + \varepsilon^\beta p_{01} = 0, \tag{3.2}$$

$$\{N, F_0\} + \varepsilon^\beta R_0 - \varepsilon^\beta [R_0] = 0, \tag{3.3}$$

$$\{N, F_1\} + \varepsilon^\beta R_1 = 0, \tag{3.4}$$

$$\{N, F_2\} + \varepsilon^\beta (R_2 - [R_2]) + (\bar{Q} - [\bar{Q}]) = 0, \tag{3.5}$$

which are equivalent to

$$Mf_{01} = -\varepsilon^\beta p_{01}, \tag{3.6}$$

$$L_{k0} f_{k0} = -\varepsilon^\beta p_{k1}, \qquad 0 < |k| < K_+, \tag{3.7}$$

$$L_{k1} f_{k1} = -\varepsilon^\beta p_{k1}, \qquad 0 < |k| < K_+, \tag{3.8}$$

$$L_{k2} F_{k2} = -\varepsilon^\beta U_{k2}, \qquad 0 < |k| < K_+, \tag{3.9}$$

where F_{k2}, U_{k2} are column 4-vectors deriving from matrixes f_{k2} and $R_{k2} + Q_{k2}$, respectively, by ordering their entries in the first row followed by the second row from the left to the right. In vital of the definition of $\mathcal{D}_+$ and the invertibility of $M(\varepsilon)$ for each $\varepsilon \in \mathcal{D}$, it is clear that (3.6)-(3.9) are uniquely solvable on $\mathcal{D}_+$ for each $0 < |k| < K_+$. Hence F is uniquely determined.

To estimate F and the associated transformation Φ_+, we first assume that

H1) $\int_{K_+}^{\infty} t^{d+1} e^{-t\frac{r-r_+}{4}} \mathrm{d}t \le \mu.$

Then it is easy to see that there is a constant c_1 such that

$$\begin{aligned}\|\mathrm{D}^j(P-R)\|_{D_4\times\mathcal{D}} &\le c_1\varepsilon_*^{\beta}\gamma^2 s^{2-j}\mu^2,\\ \|\mathrm{D}^j R\|_{D_4\times\mathcal{D}} &\le c_1\varepsilon_*^{\beta}\gamma^2 s^{2-j}\mu,\end{aligned}$$

for all $j=0,1,2$. Using Cauchy estimates, we have

$$|p_{k\imath}| \le \gamma^2 s^{2-|\imath|}\mathrm{e}^{-|k|r}, \qquad |\imath| = 0,1,2. \tag{3.10}$$

Assuming that

H2) $\|M-M_0\|_{\mathcal{D}} \le \varepsilon_*^{\alpha_1}\mu^{\frac{3}{4}}$,

we have by properties of M_0 that

$$\|M^{-1}\|_{\mathcal{D}} = O(\varepsilon_*^{-\alpha_1}). \tag{3.11}$$

We note by direct calculation that there exists a positive constants c_2 such that

$$\|L_{ki}^{-1}\|_{\mathcal{D}_+} \le c_2\frac{|k|^{12+10\tau}}{\gamma^2}, \quad i=1,2. \tag{3.12}$$

This, together with **H2)**, (3.10), and (3.11), implies that solutions of (3.6)-(3.8) satisfy

$$\begin{aligned}&\|f_{01}\|_{\mathcal{D}_+} \le c\varepsilon_*^{\beta-\alpha_1}\gamma^2 s\mu,\\ &\|f_{k\imath}\|_{\mathcal{D}_+} \le c\varepsilon_*^{\beta}\gamma^2 s^{2-|\imath|}\mu\mathrm{e}^{-|k|r}, \quad |\imath|=0,1,\ 0<|k|\le K_+,\\ &\|\mathrm{D}^j F_1\|_{D_4\times\mathcal{D}_+} \le c\varepsilon_*^{\beta}s^{2-j}\mu\Gamma_+(r-r_+), \quad j=0,1,\\ &\|\mathrm{D}^j \bar F_1\|_{D_4\times\mathcal{D}_+} \le c\varepsilon_*^{\beta-\alpha_1}s^{2-j}\mu\Gamma_+(r-r_+), \quad j=0,1,\end{aligned}$$

for some constant $c>0$. Assuming that

H3) $\|\mathrm{D}^j G - \mathrm{D}^j G_0\|_{D(r,s)\times\mathcal{D}} \le \varepsilon_*^{\alpha_1-1}\mu$, $j=0,1,2$,

we have by properties of G_0 that $\|D^j G_3\|_{D(r,s)\times\mathcal{D}} = O(\varepsilon_*^{\alpha_1-1})$, $j=0,1,2$. It follows from the above estimate for F_1 that

$$\|\mathrm{D}^j Q\|_{D_3\times\mathcal{D}_+} \le c\varepsilon_*^{\beta}s^{2-j}\mu\Gamma_+(r-r_+), \qquad \|\mathrm{D}^j(Q-\bar Q)\|_{D_3\times\mathcal{D}_+} \le c\varepsilon_*^{\beta}s^{2-j}\mu^2\Gamma_+(r-r_+), \qquad j=0,1,2.$$

This yields an estimate on F_2, and consequently, there is a positive constant c_3 such that

$$\|\mathrm{D}^j F\|_{D_{2\alpha}\times\mathcal{D}_+} \le c_3(\varepsilon_*^{\beta}s^{2-j}\mu\Gamma_+^2(r-r_+) + \varepsilon_*^{\beta-\alpha_1}s^{1-j}\mu),\ j=0,1,2.$$

Assuming that

H4) $c_3\mu\Gamma(r-r_+) + c_3\mu < \frac{1}{4}(r-r_+)$;
H5) $c_3 s\mu\Gamma(r-r_+) + c_3 s\mu < s_+$,

standard arguments show that for all $0 \le t \le 1$, $\phi_F^t : D_2 \to D_4$ are well defined for $\varepsilon \in \mathcal{D}$, and there is a constant c_4 such that for all $t \in [0,1]$,

$$|\phi_F^1 - id|_{D_2 \times \mathcal{D}_+} \le c_4(\varepsilon_*^{\beta} s^2 \mu \Gamma_+(r - r_+) + \varepsilon_*^{\beta-\alpha_1} s\mu),$$
$$|\mathrm{D}\phi_F^1 - Id|_{D_2 \times \mathcal{D}_+} \le c_4 \mu \Gamma_+(r - r_+).$$

Let $\Phi_+ = \phi_F^1$. Then $\Phi_+ : D(r_+, s_+) \to D(r, s)$, $\varepsilon \in \mathcal{D}_+$, is a smooth family of well-defined, near identity canonical transformations. Now,

$$\begin{aligned} H_+ &:= H \circ \phi_F^1 = (N + h + \varepsilon G + \varepsilon^{\beta} R) \circ \phi_F^1 + \varepsilon^{\beta}(P - R) \circ \phi_F^1 \\ &= N + h + \varepsilon G + \varepsilon^{\beta} R + \{N, F\} + \{h, F\} + \varepsilon\{G, F\} \\ &\quad + \int_0^1 \{\varepsilon^{\beta} R, F\} \circ \phi_F^t \mathrm{d}t + \int_0^1 \{\{(1-t)(N + h + \varepsilon G), F\}, F\} \circ \phi_F^t \mathrm{d}t + \varepsilon^{\beta}(P - R) \circ \phi_F^1. \\ &= e_+ + \langle \omega, I \rangle + \langle M_+ z, z \rangle + h_+ + \varepsilon G_+ + \varepsilon^{\beta} P_+, \end{aligned}$$

where

$$\begin{aligned} e_+ &= e + \varepsilon^{\beta}[R_0], \\ M_+ &= M + \varepsilon^{\beta}\mathrm{D}^2[R_2] + \mathrm{D}^2\{h_3, \bar{F}_1\} + \mathrm{D}^2[\varepsilon G_3, \bar{F}_1], \\ h_+ &= h + \{h - h_3, \bar{F}_1\}, \\ G_+ &= G + \varepsilon^{-1}\{h,\ F_2\} + \{G - G_3, \bar{F}_1\}, \\ P_+ &= \int_0^1 \{R, F\} \circ \phi_F^t \mathrm{d}t + \varepsilon^{-\beta} \int_0^1 \{\{(1-t)(N + h + \varepsilon G), F\}, F\} \circ \phi_F^t \mathrm{d}t \\ &\quad + (P - R) \circ \phi_F^1 + \varepsilon^{-\beta+1}\{G, F - \bar{F}_1\}. \end{aligned}$$

By the estimates above, we obtain that

$$\|e_+ - e\|_{\mathcal{D}_+} \le c\varepsilon_*^{\alpha_1} s\mu,$$
$$\|M_+ - M\|_{\mathcal{D}_+} \le c(\varepsilon_*^{\beta}\gamma s^2 \mu + \varepsilon_*^{\beta-\alpha_1} s\mu) \le \varepsilon_*^{\alpha_1} \mu^{3/4}, \tag{3.13}$$
$$\|h_+ - h\|_{D_2 \times \mathcal{D}_+} \le c_3 \varepsilon_*^{\alpha_1 + \sigma/4} s^3 \mu,$$
$$\|G_+ - G\|_{D_2 \times \mathcal{D}_+} \le c_3 \varepsilon_*^{\beta-1} s^3 \mu + \varepsilon_*^{\alpha_1 + \sigma/4} s^3 \mu, \tag{3.14}$$

for some constant $c > 0$. By (3.13), we see that, for each $\varepsilon \in \mathcal{D}_+$, M_+ is invertible, and,

$$|\det(\sqrt{-1}\langle k, \omega \rangle I_2 - M_+ J)| \ge \frac{\gamma_+}{|k|^{5\tau}},$$
$$|\det(\sqrt{-1}\langle k, \omega \rangle I_4 - M_+ J \otimes I_2 - I_2 \otimes M_+ J)| \ge \frac{\gamma_+}{|k|^{5\tau}}$$

hold for all $0 < |k| \le K_+$, if we further assume

H6) $\mu |k|^{5\tau} < \gamma - \gamma_+, \quad 0 < |k| \le K_+.$

From the form of P_+ and relevant estimates above, we also see that there exists a constant $c_5 > 0$ such that

$$\|\mathrm{D}^j P_+\|_{D(r_+, s_+) \times \mathcal{D}_+} \le c_5(\gamma^2 s_+^2 s^2 \mu \Gamma_+ + \gamma^4 s^4 \mu^2 \Gamma_+^4), \qquad j = 0, 1, 2. \tag{3.15}$$

Consequently, if

H7) $c_5(\gamma^2 s_+^2 s^2 \mu \Gamma_+ + \gamma^4 s^4 \mu^2 \Gamma_+^4) \le \gamma_+^2 s_+^2 \mu_+,$

then

$$\|\mathrm{D}^j P_+\|_{D(r_+,s_+)\times\mathcal{D}_+} \le \gamma_+^2 s_+^{2-j}\mu_+,\ j=0,1,2. \tag{3.16}$$

We now define the constant c_0 in the definition of μ_+ as

$$c_0 = \max\{c_1, c_2, \cdots, c_5\}.$$

This completes one cycle of KAM iteration.

3.2. **Iteration and convergence.** Recursively applying the definitions of quantities in the previous sub-section with $\nu+1$ in place of $+$ for $\nu = 0, 1, \cdots$, we obtain the following iterative sequences

$$\begin{aligned}
& r_\nu = r_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
& s_\nu = \frac{1}{4}\alpha_{\nu-1}s_{\nu-1},\\
& \alpha_\nu = \mu_\nu^{\frac{1}{3}},\\
& \mu_\nu = c_0\mu_{\nu-1}^{\frac{7}{6}},\\
& \gamma_\nu = \gamma_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
& K_\nu = ([\log(\frac{1}{\mu_{\nu-1}})]+1)^3,\\
& L_{1k,\nu-1} = \sqrt{-1}\langle k,\omega\rangle I_2 - M_{\nu-1}J, \quad 0<|k|\le K_\nu,\\
& L_{2k,\nu-1} = \sqrt{-1}\langle k,\omega\rangle I_4 - (M_{\nu-1}J)\otimes I_2 - I_2\otimes(JM_{\nu-1}), \quad 0<|k|\le K_\nu,\\
(3.17)\qquad & \mathcal{D}_\nu = \{\xi\in\mathcal{D}_{\nu-1}:\ |\det L_{1k,\nu-1}| > \frac{\gamma_{\nu-1}}{|k|^{5\tau}},\ |\det L_{2k,\nu-1}| > \frac{\gamma_{\nu-1}}{|k|^{5\tau}},\ 0<|k|\le K_\nu\},\\
& H_\nu = e_\nu(\varepsilon) + \langle\omega, I\rangle + \langle z, M_\nu(\varepsilon)z\rangle + h_\nu(z,\varepsilon) + \varepsilon G_\nu(\theta,z,\varepsilon) + \varepsilon^\beta P_\nu(\theta,z,\varepsilon),\\
(3.18)\qquad & \Phi_\nu : D(r_\nu,s_\nu)\to D(r_{\nu-1},s_{\nu-1}),\ \ \varepsilon\in\mathcal{D}_\nu \text{ such that } H_\nu = H_{\nu-1}\circ\Phi_\nu,
\end{aligned}$$

$\nu = 1, 2, \cdots$.

Then we have the following iteration and convergence result.

Lemma 3.1. *As ε_* sufficiently small, the above sequences are well-defined and satisfy the following properties for all $\nu = 1, 2, \cdots$:*

1) $\|M_\nu - M_0\|_{\mathcal{D}_\nu} \le \varepsilon_*^{\alpha_1}\mu_0^{3/5}$.
2) $\|\mathrm{D}^j P_\nu\|_{D(r_\nu,s_\nu)\times\mathcal{D}_\nu} \le \gamma_\nu^2 s_\nu^{2-j}\mu_\nu,\ j=0,1,2.$
3) $\mathcal{D}_\nu = \{\omega\in\mathcal{D}_{\nu-1}:\ |\det L_{1k,\nu-1}| > \frac{\gamma_{\nu-1}}{|k|^{5\tau}},\ |\det L_{2k,\nu-1}| > \frac{\gamma_{\nu-1}}{|k|^{5\tau}},\ K_{\nu-1}<|k|\le K_\nu\}$.
4) *The Whitney extensions of H_ν and $\Psi_\nu =: \Phi_1\circ\Phi_2\circ\cdots\circ\Phi_\nu$ converge C^1 uniformly on $D(\frac{r_0}{2},\frac{s_0}{2})\times\mathcal{D}_*$, where $\mathcal{D}_* = \bigcap_{\nu\ge0}\mathcal{D}_\nu$.*

Proof. The proof amounts to the verification of assumptions **H1)-H7**), following standard arguments, see e.g., [7, 16, 42]. □

Let Ψ_∞, $e_\infty = \lim_{\nu\to\infty} e_\nu$, $M_\infty = \lim_{\nu\to\infty} M_\nu$, $h_\infty = \lim_{\nu\to\infty} h_\nu$, $G_\infty = \lim_{\nu\to\infty} G_\nu$, and $P_\infty = \lim_{\nu\to\infty} P_\nu$. Then $\Psi_\infty : D(\frac{r_0}{2}, \frac{s_0}{2}) \to D(r_0, s_0)$, $\varepsilon \in \mathcal{D}_*$, is a Whitney smooth family of canonical transformations such that

$$H_0 \circ \Psi_\infty = H_\infty =: e_\infty + \langle \omega, I\rangle + \langle z, M_\infty z\rangle + h_\infty + \varepsilon G_\infty + \varepsilon^\beta P_\infty.$$

Since $|h_\infty|, |G_\infty| = O(|z|^3)$ and

$$\|\mathrm{D}^j P_\infty\|_{D(\frac{r_0}{2},\frac{s_0}{2})\times\mathcal{D}_*} = 0, \quad j = 0, 1, 2$$

by Lemma 3.1 2), we see that, for each $\varepsilon \in \mathcal{D}_*$, $\mathbb{T}^d \times \{I = 0\} \times \{z = 0\}$ is a quasi-periodic invariant torus of H_∞ with the frequency vector ω, which corresponds to a response torus of the Hamiltonian H_0.

3.3. **Measure estimate.** For each $\nu = 1,\ 2, \cdots$ and $k \in \mathbb{Z}^d \setminus \{0\}$, denote

$$R_k^\nu(\varepsilon) = R_{k,1}^\nu \bigcup R_{k,2}^\nu,$$

where

$$\begin{aligned} R_{k,1}^\nu &= \{\varepsilon \in \mathcal{D}_{\nu-1} :\ |\det L_{1k,\nu-1}| \le \frac{\gamma_{\nu-1}}{|k|^{5\tau}},\ K_{\nu-1} < |k| \le K_\nu\}, \\ R_{k,2}^\nu &= \{\varepsilon \in \mathcal{D}_{\nu-1} :\ |\det L_{2k,\nu-1}| \le \frac{\gamma_{\nu-1}}{|k|^{5\tau}},,\ K_{\nu-1} < |k| \le K_\nu\}. \end{aligned}$$

It follows that

$$\mathcal{D}_0 \setminus \mathcal{D}_* = \bigcup_{\nu=0}^{\infty} \bigcup_{K_{\nu-1}<|k|\le K_\nu} (R_{k,1}^\nu + R_{k,2}^\nu). \tag{3.19}$$

By Lemma 3.1 1) and (2.30), we have

$$\|M_\nu - \tilde{M}\|_{\mathcal{D}_\nu} \le \|M_\nu - M_0\|_{\mathcal{D}_\nu} + \|M_0 - \tilde{M}\|_{\mathcal{D}_\nu} \le \varepsilon_*^{\alpha_1 + \frac{\sigma}{5}}.$$

Since ε_* is arbitrary, we can obtain the same estimate for any $\varepsilon \in \mathcal{D}_\nu$ to conclude $|M_\nu(\varepsilon) - \tilde{M}(\varepsilon)| = O(\varepsilon^{\alpha_1+\frac{\sigma}{5}})$, $\varepsilon \in \mathcal{D}_\nu$. It follows from straightforward calculations that

$$\det L_{1k,\nu-1} = (\sqrt{-}\langle k, \omega\rangle)^2 + \varepsilon^{\alpha_1}\tilde{m}_{11}\tilde{m}_{22} + O(\varepsilon^{2\alpha}), \quad \varepsilon \in \mathcal{D}_\nu \tag{3.20}$$

$$\det L_{2k,\nu-1} = (\sqrt{-1}\langle k, \omega\rangle)^2((\sqrt{-1}\langle k, \omega\rangle)^2 + \varepsilon^{\alpha_1}\tilde{m}_{11}\tilde{m}_{22} + O(\varepsilon^{2\alpha})), \quad \varepsilon \in \mathcal{D}_\nu. \tag{3.21}$$

Let $\tilde{\lambda}$ be as in (1.6). It is clear that $\tilde{\lambda} < 0$ implies $\tilde{m}_{11}\tilde{m}_{22} < 0$, which further implies that each $M_0 J$ is hyperbolic. In this case, we clearly have $\varepsilon^{\alpha_1}\tilde{m}_{11}\tilde{m}_{22} + O(\varepsilon^{2\alpha}) < 0$, $\varepsilon \in \mathcal{D}_\nu$. It follows that, for any $k \in \mathbb{Z}^d$ and $\varepsilon \in \mathcal{D}_\nu$, we have

$$\begin{aligned} |\det L_{1k,\nu-1}| &> |\sqrt{-1}\langle k, \omega\rangle|^2 > \frac{\gamma^2}{|k|^{2\tau}} \ge \frac{\gamma_0}{|k|^{2\tau}} > \frac{\gamma_\nu}{|k|^{5\tau}}, \\ |\det L_{2k,\nu-1}| &> |\sqrt{-1}\langle k, \omega\rangle|^4 > \frac{\gamma^4}{|k|^{4\tau}} \ge \frac{\gamma_0}{|k|^{4\tau}} > \frac{\gamma_\nu}{|k|^{5\tau}}, \end{aligned}$$

implying $R_k^\nu = \emptyset$. Sice ν and k are arbitrarily, we have $\mathcal{D}_* = (0, \varepsilon_*)$ in this case.

We now consider the case $\tilde{\lambda} > 0$, i.e., $\tilde{m}_{11}\tilde{m}_{22} > 0$, which implies that each $M_0 J$ is elliptic. When $\tilde{m}_{11}\tilde{m}_{22} > 0$, we have to eliminate the sets R_{ki}^ν at each iterative step. By (3.20) and (3.21), there is a positive constant C independent of ν, ε such that

$$\begin{aligned} &|\det L_{\nu-1,k1}| > \frac{\gamma^2}{|k|^{2\tau}} - C\varepsilon^{\alpha_1}, \\ &|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\det L_{\nu-1,k1|} = |\varepsilon^{\alpha_1-1}(\tilde{m}_{11}\tilde{m}_{22} + \frac{\mathrm{d}\tilde{m}_{11}}{\mathrm{d}\varepsilon} + \frac{\mathrm{d}\tilde{m}_{22}}{\mathrm{d}\varepsilon}) + O(\varepsilon^{2\alpha-1})| > C\varepsilon^{\alpha_1-1}. \end{aligned}$$

Let $K_* = \max\{|k| : \ k \in \mathbb{Z}^d, \ 2C\varepsilon_*^{\alpha_1}|k|^{2\tau} \le \gamma^2\}$. Without great loose of generality, we assume that there exists ν_* such that $K_{\nu_*} \le K_* < K_{\nu_*+1}$. We split $R_{k1}^{\nu_*+1}$ into the union of the following disjoint sets:

$$
\begin{aligned}
R_{k,1}^{\nu_*+1,1} &= \{\varepsilon \in \mathcal{D}_{\nu_*} : \ |\det L_{1k,\nu-1}| \le \frac{\gamma_{\nu-1}}{|k|^{5\tau}}, \ K_{\nu_*} < |k| \le K_*\},\\
R_{k,1}^{\nu_*+1,2} &= \{\varepsilon \in \mathcal{D}_{\nu_*} : \ |\det L_{1k,\nu-1}| \le \frac{\gamma_{\nu-1}}{|k|^{5\tau}}, \ K_* < |k| \le K_{\nu_*+1}\}.
\end{aligned}
$$

Then for any $|k| \le K_*$, we have

$$
|\mathrm{det} L_{k1,\nu-1}| \ge |\frac{\gamma^2}{|k|^{2\tau}} - C\varepsilon^{\alpha_1}| \ge \frac{\gamma^2}{|k|^{2\tau}} - C\varepsilon_*^{\alpha_1} > \frac{\gamma^2}{2|k|^{2\tau}} \ge \frac{\gamma_{\nu-1}}{|k|^{5\tau}},
$$

implying $R_{k,1}^i = \emptyset$ for $i = 1, \cdots, \nu_*$ and $R_{k,1}^{\nu_*+1,1} = \emptyset$. For $|k| > K_*$, i.e., $\frac{\gamma^2}{|k|^{2\tau}} < 2C\varepsilon_*^{\alpha_1}$, we have by standard measure estimate that

$$
\begin{aligned}
|R_{k,1}^{\nu_*+1,2}| &\le \frac{\gamma_{\nu-1}}{C\varepsilon_*^{\alpha_1-1}|k|^{5\tau}} \le c\varepsilon_*^{1+\frac{\alpha_1}{2}}\frac{\gamma}{|k|^{2\tau}},\\
|R_{k,1}^i| &\le \frac{\gamma_{\nu-1}}{C\varepsilon_*^{\alpha_1-1}|k|^{5\tau}} \le c\varepsilon_*^{1+\frac{\alpha_1}{2}}\frac{\gamma}{|k|^{2\tau}}, \quad i > \nu_*,
\end{aligned}
$$

where $c := \sqrt{2C}$. Therefore,

$$
|\sum_{\nu=1}^{\infty} \sum_{K_{\nu-1}<|k|\le K_\nu} R_{k,1}^\nu| \le \sum_{\nu=\nu_*+1}^{\infty} \sum_{K_{\nu-1}<|k|\le K_\nu} |R_{k,1}^\nu| = c\varepsilon_*^{1+\frac{\alpha_1}{2}} \sum_{k\in\mathbb{T}^d} \frac{\gamma}{|k|^{2\tau}} = O(\varepsilon_*^{1+\frac{\alpha_1}{2}}).
$$

Similarly,

$$
|\sum_{\nu=1}^{\infty} \sum_{K_{\nu-1}<|k|\le K_\nu} R_{k,2}^\nu| = O(\varepsilon_*^{1+\frac{\alpha_1}{2}}).
$$

It follows that

$$
\frac{|\mathcal{D}_*|}{\varepsilon_*} \le 1 - \frac{1}{\varepsilon_*}|\sum_{\nu=1}^{\infty} \sum_{K_{\nu-1}<|k|\le K_\nu} R_{k1}^\nu| - \frac{1}{\varepsilon_*}|\sum_{\nu=1}^{\infty} \sum_{K_{\nu-1}<|k|\le K_\nu} R_{k1}^\nu| = 1 - O(\varepsilon_*^{\frac{\alpha_1}{2}}),
$$

which implies that $\mathcal{D}_*$ is a Cantor set of almost full Lebesgue measure relative to $(0, \varepsilon_*)$ when $\tilde{\lambda} > 0$.

The proof of Main Theorem is now complete. Corollaries 1, 2 also follow immediately.

4. Response solutions of (1.1) in the super critical case

In this section, we consider the quasi-periodically forced, second order equation (1.1), i.e.,

$$
\ddot{x} + \lambda x^l + \varepsilon f(\omega t, x) = 0, \qquad x \in \mathbb{R}, \tag{4.1}
$$

where $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector, f is real analytic, and $\lambda \neq 0$ is a constant, ε is a small parameter, and $l \ge 2$ is an integer. We consider the super-critical case that

$$
[f(\cdot, 0)] = [\frac{\partial f(\cdot, 0)}{\partial x}] = \cdots = [\frac{\partial^p f(\cdot, 0)}{\partial x^p}] = 0 \tag{4.2}
$$

for some $l/2 \le p < l$. As remarked in Section 1, $p = l/2$ is a critical order for finding response solutions from the averaged equation of (4.1). In the super-critical case, response solutions of (4.1) can still exist but they need to be find through some ε^α-order of perturbations in some normalized equation of (4.1).

For the simplest case $\alpha = 2$, we now describe the procedure and give verifiable conditions for the existence of response solutions. This will particularly yield Corollary 3.

4.1. **O(ε^2)-order normal form.** Equation (4.1) is equivalent to the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{I} = 0, \\ \dot{x} = y, \\ \dot{y} = -\lambda x^l - \varepsilon f(\omega t, x), \end{cases} \tag{4.3}$$

$\theta \in \mathbb{T}^d$, $I \in \mathbb{R}^d$, $(x, y) \in \mathbb{R}^2$, which is a Hamiltonian system with respect to the standard symplectic structure $\mathrm{d}\theta \wedge \mathrm{d}I \wedge \mathrm{d}x \wedge \mathrm{d}y$, whose Hamiltonian function is simply

$$H = N + h + \varepsilon P(\theta, x), \tag{4.4}$$

where

$$N = \langle \omega, I \rangle + \frac{y^2}{2},$$
$$h = \lambda \frac{x^{l+1}}{l+1},$$
$$P(\theta, x) = \int_0^x f(\theta, t) dt.$$

Expanding f into Taylor-Fourier series, we have

$$f(\theta, x) = \sum_{i=0}^{\infty} f_i(\theta) x^i \text{ with } f_i(\theta) = \sum_{k \in \mathbb{T}^d} f_{ki} \mathrm{e}^{\sqrt{-1}\langle k, \theta \rangle}, \ i = 1, 2, \cdots,$$

$$P(\theta, x) = \sum_{i=1}^{\infty} P_i(\theta) x^i \text{ with } P_i(\theta) = \frac{f_{i-1}(\theta)}{i}, \ i = 1, 2, \cdots. \tag{4.5}$$

Then the condition (4.2) is equivalent to

$$[f_0] = [P_1] = [f_1] = [P_2] = \cdots = [f_{p-1}] = [P_p] = 0.$$

For each ε, we would like to construct the transformation Φ as the time-1 map $\phi^1_{\varepsilon^\alpha V}$ of the Hamiltonian flow generated by a generating function $\varepsilon^\alpha V$, with $V = V(\theta, x, y)$ having the form

$$V(\theta, x, y) = \sum_{\imath=1}^{l+1} V_\imath = \sum_{\imath=0}^{l+1} \sum_{i+j=\imath} V_{ij}(\theta, \varepsilon) x^i y^j, \quad [V_{ij}] = 0, \qquad i + j = 1, 2, \cdots, l+1 \tag{4.6}$$

which satisfies the homological equations

$$\{N, V_i\} + (P_i - [P_i]) x^i = 0, \qquad i = 1, 2, \ \cdots, \ l-1, \tag{4.7}$$
$$\{N, V_l\} + (P_l - [P_l]) x^l + \{h, V_1\} = 0, \tag{4.8}$$
$$\{N, V_{l+1}\} + (P_{l+1} - [P_{l+1}]) x^{l+1} + \{h, V_2\} = 0. \tag{4.9}$$

We have the following normal form reduction for (4.4).

Lemma 4.1. *Consider* (4.4) *in the super-critical case. Then the following holds.*

(1) *For given $r, s > 0$,* (4.7) - (4.9) *are uniquely solvable to yield a solution $V \in C^\varpi(D(r, s))$.*

(2) *Let $\Phi_\varepsilon = \phi^1_{\epsilon V}$ be the time-1 map of the Hamiltonian flow generated from εV. Then for given $0 < \eta < \min\{r, s\}$, there exists $0 < \varepsilon_* \ll 1$ such that $\Phi_\varepsilon : D(r - \eta, s - \eta) \to D(r, s)$, $\varepsilon \in (0, \varepsilon_*)$, is a smooth family of real analytic canonical transformations which transforms* (4.4) *to*

$$\mathcal{H} := H \circ \Phi_\varepsilon = \langle \omega, I \rangle + \lambda(1 + \varepsilon(l+1)[P_{l+1}]) \frac{x^{l+1}}{l+1} + \frac{y^2}{2} + \varepsilon(\mathcal{G}_1(x) + \mathcal{G}_2(\theta, x, \varepsilon)) + \varepsilon^2 \mathcal{P}(x, y, \theta, \varepsilon),$$

where

$$\begin{aligned}
\mathcal{G}_1 &= \sum_{i=p+1}^{l+1}[P_i]x^i, \\
\mathcal{G}_2 &= \sum_{i=l+2}^{\infty}P_i(\theta)x^i + \{h, \sum_{\imath=3}^{l+1}V_\imath\}, \\
\mathcal{P} &= \frac{1}{2}\{\{N+h(x),V\},V\} + \{P,V\} + \frac{\varepsilon}{2}\int_0^1\{\{\{(1-t)^2(N+h),V\},V\},V\}\circ\phi_{\varepsilon V}^t\mathrm{d}t \\
&\quad +\frac{\varepsilon}{2}\{\{(1-t)P,V\},V\}\circ\phi_{\varepsilon V}^t\mathrm{d}t. \qquad (4.10)
\end{aligned}$$

Proof. We note that Hamiltonian (4.4) is in the same form of (1.5) with $R_l = R_2 = G \equiv 0$. The proof simply follows from the argument of that of Proposition 2.1 by taking $\alpha = 1$ and (4.7) - (4.9) in place of (2.4) - (2.6). □

We note that when letting $R_l = (l+1)[P_{l+1}]$, $f_i = \frac{\partial \mathcal{G}_i}{\partial x}$, $i = 1, 2$, and $F = \frac{\partial \mathcal{P}}{\partial x}$, the equation of motion of $\mathcal{H}$ becomes the normalized equation (1.4) for which Corollaries 1,2 hold.

4.2. **Computing non-degeneracy.** With the normal form (4.10), the Main Theorem or Corollaries 1, 2 can now be applied to study response solutions of (4.1) if $a_{\hat{p}} =: [\frac{\partial^{\hat{p}+1}\mathcal{P}(\cdot,0)}{\partial x^{\hat{p}+1}}] = [\frac{\partial^{\hat{p}}F(\cdot,0)}{\partial x^{\hat{p}}}]$ can be computed in term of f. Let

$$\bar{\mathcal{P}} = \frac{1}{2}\{\{N+h,V\},V\} + \{P,V\},$$

where N, h, P are as in (4.4) and V solves (4.7) - (4.9). Then it is clear that

$$a_{\hat{p}} = [\frac{\partial^{\hat{p}+1}\bar{\mathcal{P}}(\cdot,0)}{\partial x^{\hat{p}+1}}], \qquad (4.11)$$

which is explicitly computable because, as shown in the proof of Proposition 2.1, V can be solved recursively from (4.7) - (4.9) in terms of the Taylor-Fourier series of P or f.

We now demonstrate such computations for cases $\hat{p} = 0, 1$. For each $i, j = 1, 2, \cdots, l+1$, consider the Fourier expansions

$$P_i = \sum_{k\in\mathbb{Z}^d} p_{ki}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle}, \qquad (4.12)$$

$$V_{ij} = \sum_{k\in\mathbb{Z}^d} v_{kij}\mathrm{e}^{\sqrt{-1}\langle k,\theta\rangle}. \qquad (4.13)$$

<u>**Computation of** a_0</u>: We only need to consider the first order terms of the Taylor series of $\bar{\mathcal{P}}$ which are simply

$$\bar{\mathcal{P}}_1 =: \{P_1x, V_2\} + \{P_2x^2, V_1\} = (P_1(\theta)V_{11}(\theta) + 2P_2(\theta)V_{01}(\theta))x + P_1(\theta)V_{02}(\theta)y.$$

Thus,

$$a_0 = [\frac{\partial\bar{\mathcal{P}}_1(\cdot,0,0)}{\partial x}] = [P_1V_{11}] + 2[P_2V_{01}]. \qquad (4.14)$$

When $l \geq 3$, V_1, V_2 are given by (4.7) with $[P_1] = [P_2] = 0$, i.e.,

$$\partial_\omega V_{10}x + \partial_\omega V_{01}y + V_{10}y - P_1x = 0,$$

$$\partial_\omega V_{20}x^2 + \partial_\omega V_{11}xy + \partial_\omega V_{02}y^2 + 2V_{20}xy + V_{11}y^2 - P_2x^2 = 0.$$

Equating coefficients of x, y of equal powers, we have

$$\begin{aligned}&\partial_\omega V_{10} - P_1 = 0,\ \partial_\omega V_{01} + V_{10} = 0,\\ &\partial_\omega V_{20} - P_2 = 0,\ \partial_\omega V_{11} + 2V_{20} = 0,\ \partial_\omega V_{02} + V_{11} = 0.\end{aligned}$$

Using the Fourier expansions (4.12), (4.13), we have by comparison of coefficients that

$$v_{k01} = \frac{p_{k1}}{|\langle k,\omega\rangle|^2},\ v_{k01} = \frac{p_{k1}}{|\langle k,\omega\rangle|^2},\ v_{k20} = \frac{p_{k2}}{\sqrt{-1}\langle k,\omega\rangle},\ v_{k11} = 2\frac{p_{k2}}{|\langle k,\omega\rangle|^2},\ k \in \mathbb{Z}^d\setminus\{0\}. \tag{4.15}$$

In particular, we have obtained V_{11}, V_{01}. Using (4.16), we then have

$$a_0 = 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k1}p_{-k2}}{|\langle k,\omega\rangle|^2} + 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k2}p_{-k1}}{|\langle k,\omega\rangle|^2} = 4\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k1}p_{-k2}}{|\langle k,\omega\rangle|^2} = 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{f_{k0}f_{-k1}}{|\langle k,\omega\rangle|^2}. \tag{4.16}$$

When $l = 2$, V_1 is still given (4.7) with $[P_1] = 0$ to yield V_{01} as in (4.15). But V_2 is given by (4.8) with $[P_2] = 0$. Solving (4.8) using the same argument as in the above, we have

$$v_{k11} = \frac{2p_{k2}}{|\langle k,\omega\rangle|^2} + \frac{2\lambda p_{k1}}{|\langle k,\omega\rangle|^4},\ k \in \mathbb{Z}^d\setminus\{0\},$$

and consequently,

$$a_0 = [P_1V_{11} + 2P_2V_{01}] = 4\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k1}p_{-k2}}{|\langle k,\omega\rangle|^2} + 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{\lambda|p_{k1}|^2}{|\langle k,\omega\rangle|^4} = 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{f_{k0}f_{-k1}}{|\langle k,\omega\rangle|^2} + 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{\lambda|f_{k0}|^2}{|\langle k,\omega\rangle|^4}.$$

Computation of a_1: To simplify the computation, we only consider the case that $l \geq 4$ so that the second order terms of Taylor series of $\bar{\mathcal{P}}$ are simply

$$\begin{aligned}\bar{\mathcal{P}}_2 \ &=: \ \{P_1x, V_3\} + \{P_3x^3, V_1\} + \{P_2x^2, V_2\},\\ &= \ (3P_3(\theta)V_{01}(\theta) + 2P_2(\theta)V_{11}(\theta) + P_1(\theta)V_{21}(\theta))x^2\\ &\quad +2(P_1V_{12}(\theta) + 2P_2(\theta)V_{02}(\theta))xy + 3P_1(\theta)V_{03}(\theta)y^2.\end{aligned}$$

It follows that

$$a_1 = [\frac{\partial^2\bar{\mathcal{P}}_2(\cdot,0,0)}{\partial x^2}] = 3[P_3V_{01}] + 2[P_2V_{11}] + [P_1V_{21}]. \tag{4.17}$$

We note that V_{01}, V_{11} are the same as that given in (4.15), but V_{21} needs to be solved from (4.7) for $i = 3$. Using Fourier expansions and the arguments as in the above, we obtain

$$v_{k21} = 3\frac{p_{k3}}{|\langle k,\omega\rangle|^2},\ k \in \mathbb{Z}^d\setminus\{0\}.$$

It now follows (4.17) that

$$\begin{aligned}a_1 \ &= \ 3\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k3}p_{-k1}}{|\langle k,\omega\rangle|^2} + 3\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k1}p_{-k3}}{|\langle k,\omega\rangle|^2} + 4\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k2}p_{-k2}}{|\langle k,\omega\rangle|^2},\\ &= \ 6\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{p_{k1}p_{-k3}}{|\langle k,\omega\rangle|^2} + 4\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{|p_{k2}|^2}{|\langle k,\omega\rangle|^2} = 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{f_{k0}f_{-k2}}{|\langle k,\omega\rangle|^2} + 2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{|f_{k1}|^2}{|\langle k,\omega\rangle|^2}.\end{aligned} \tag{4.18}$$

The quantities a_0, a_1 are precisely those stated in (1.8), (1.9). We note that since ω is diophantine, f is real analytic, both a_1 and a_2 are well defined real numbers. Moreover, with the normal form in Lemma 4.1, Corollaries 1, 2 are clearly applicable to yield Corollary 3 with the current a_0, a_1.

Remark 4.1. (1) *In applications, lower order terms of f are often a triangle functions in θ, e.g., in some models of point vertex problem* ([26]) *and celestial mechanics* ([25], [27])*. In these applications, a_0, a_1 become finite sums which can be easily computed.*

(2) *In application of* Corollary 3*, only signs of a_0 or a_1 matters instead of their actual values. These signs can be often determined by certain symmetric properties of lower order terms of f. For instance, when $l \geq 4$, let f_0, f_1, and f_2 be the zero-th, first, and second order terms, respectively, in the Taylor expansion of f and assume that f_0 is even, f_1 and f_2 are odd, and $f_1 \not\equiv 0$. Then it is easy to see from* (4.16) *and* (4.18) *that*

$$a_0 = 0, \ a_1 = 2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{|f_{k1}|^2}{|\langle k, \omega \rangle|^2} > 0.$$

4.3. **An example.** As an example in applying Corollary 3, we consider

$$\ddot{x} - x^4 + \varepsilon f(\omega t, x) = 0, \qquad x \in \mathbb{R}^1, \tag{4.19}$$

where ε is small parameter and $f(\theta, x) = \sum_{i=0} f_i(\theta) x^i$ is real analytic. We assume that f_0 is even, f_1 and f_2 are odd, and $f_1(\theta) \not\equiv 0$. Then we clear have $[f_1] = [f_2] = 0$. Since $l/2 = 2$, the equation lies in the super-critical case if $[f_0] = 0$. We have the following three cases.

Case I: $[f_0] < 0$. In this case, (4.19) admits no small amplitude responsive solution. This follows from [32, Proposition 4.1].

Case II: $[f_0] > 0$. In this case, there exists a sufficiently small ε_*, such that (4.19) admits a smooth family of response solutions of saddle type having amplitude $\sim \varepsilon^{1/4}$, as $\varepsilon \in (0, \varepsilon_*)$. This follows from the main result of [32].

Case III: $[f_0] = 0$. In this case, there exists a sufficiently small ε_*, such that (4.19) admits a smooth family of response solutions of elliptic type having amplitude $\sim \varepsilon^{2/3}$, as ε lying in a Cantor set of almost full Lebesgue measure in $(0, \varepsilon_*)$. We note that Remark 4.1 (2) now applies, so $a_0 = 0$ and $a_1 > 0$. The conclusion then follows from Corollary 3 (in fact, Corollary 2 (2)). We note that if $f = f_0 + f_1 x + f_2 x^2$, then $[f(\cdot, x)] \equiv 0$ which is completely degenerate.

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