LOWER DIMENSION TORI OF GENERAL TYPES IN MULTI-SCALE HAMILTONIAN SYSTEMS

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Abstract. We consider a general canonical form of a multi-scale Hamiltonian system near a family of unperturbed lower dimensional, quasi-periodic, invariant tori, in which tangential and normal frequencies can admit equal or different scales. Extending works of [2, 15], we show the persistence of the majority of these lower dimensional tori when the normal matrices are non-degenerate and a Melnikov non-resonant condition among certain tangential and normal frequencies are satisfied when they admit equal scales. Such a general persistence result of lower dimensional tori allows a broader range of applications to stability problems arising in celestial mechanics.

1. Introduction

The present work concerns the persistence of lower dimensional, quasi-periodic, invariant tori in general nearly integrable, multi-scale Hamiltonian systems in $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, where n, m are positive integers. More precisely, associated with the symplectic form

$$\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{j=1}^{m} \mathrm{d}z_j \wedge \mathrm{d}z_{j+1},$$

we consider the canonical Hamiltonian of the form

$$H(x, y, z, \omega, \varepsilon) = \langle \omega_\varepsilon(\omega), y \rangle + \frac{1}{2}\langle M_\varepsilon(\omega) z, z \rangle + \varepsilon^{\bar{m}+1} \bar{P}(x, y, z, \omega, \varepsilon), \tag{1.1}$$

where $x = (x_1, \cdots, x_n)^\top \in \mathbb{T}^n$, $y = (y_1, \cdots, y_n)^\top \in \mathbb{R}^n$, $z = (z_1, \cdots, z_{2m})^\top \in \mathbb{R}^{2m}$, $\varepsilon > 0$ is a small parameter, and $\bar{m} \geq 1$ is a positive integer. In the above, $\omega = (\hat{\omega}^0, \hat{\omega}^1, \cdots, \hat{\omega}^a)^\top$ is an n-dimensional parameter varying in a bounded closed region $\mathcal{O}$ with $\hat{\omega}^0 \in \mathbb{R}^{n_0}$, $\hat{\omega}^i \in \mathbb{R}^{n_i - n_{i-1}}$, $i = 1, \cdots, a$, where a is a positive integer and $\{n_0, \cdots, n_a\}$ is an increasing sequence of positive integers with $n_a = n$, the tangential frequency is of the form

$$\omega_\varepsilon = (\varepsilon^{\tau_0}\hat{\omega}^0, \ \varepsilon^{\tau_1}\hat{\omega}^1, \cdots, \varepsilon^{\tau_a}\hat{\omega}^a)^\top,$$

where $\{\tau_0, \cdots, \tau_a\}$ is a sequence of distinct, non-negative integers, the parameter-dependent normal matrix $M_\varepsilon(\omega) = (m_{ij})$ is non-singular for each $\omega \in \mathcal{O}$ whose entries m_{ij} are bounded in absolute

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values to some order $O(\varepsilon^{\bar{\chi}_{ij}})$, for $i,j=1,2,\cdots,2m$, and eigenvalues of $JM_\varepsilon(\omega)$, where J is the standard $2m\times 2m$ symplectic matrix, are of the form

$$\begin{aligned}&\varepsilon^{\chi_0}\lambda_1^0(\omega),\cdots,\varepsilon^{\chi_0}\lambda_{2m_0}^0(\omega),\\&\varepsilon^{\chi_1}\lambda_{2m_0+1}^1(\omega),\cdots,\varepsilon^{\chi_1}\lambda_{2m_1}^1(\omega),\\&\cdots,\\&\varepsilon^{\chi_\alpha}\lambda_{2m_{\alpha-1}+1}^\alpha(\omega),\cdots,\varepsilon^{\chi_\alpha}\lambda_{2m_\alpha}^\alpha(\omega),\end{aligned}$$

where α, m_j are positive integers, $m_0<m_1<\cdots<m_\alpha:=m$, $\{\chi_0,\cdots,\chi_\alpha\}$ is a sequence of distinct, non-negative integers. Moreover, the Hamiltonian (1.1) is real analytic in $(x,y,z)\in D(r,s)$ and C^N smoothly depends on parameters ω, ε for some integer $N\ge 1$, where

$$D(r,s)=\{(x,y,z)\in\mathbb{C}^n/\mathbb{Z}^n\times\mathbb{R}^n\times\mathbb{R}^{2m}|\ |\operatorname{Im}x|<r,\ |y|<s^2,\ |z|<s\} \tag{1.2}$$

is a complex neighborhood of $\mathbb{T}^n\times\{0\}\times\{0\}$.

When $\bar{P}=0$ in (1.1), the unperturbed Hamiltonian

$$N_\varepsilon:=N_\varepsilon(y,z,\omega)=\langle\omega_\varepsilon(\omega),y\rangle+\frac{1}{2}\langle M_\varepsilon(\omega)z,z\rangle$$

clearly admits a family of invariant n-tori $T_{\omega,\varepsilon}=\mathbb{T}^n\times\{0\}\times\{0\}$ with linear flows $\{x_0+\omega_\varepsilon(\omega)t\}$. Like the case for a standard nearly integrable Hamiltonian system, the problem of the persistence of lower dimensional, quasi-periodic, invariant tori of (1.1) then concerns the persistence of the majority of these n-tori $\{T_{\omega,\varepsilon}\}$ when $\bar{P}\neq 0$.

The Hamiltonian (1.1) is a canonical form in studying the existence and linear stability of lower dimensional, quasi-periodic, invariant tori in multi-scale Hamiltonian systems especially those arising in celestial mechanics applications. There are three cases in the normal scales of (1.1):

(i) (*lower scale*) There exists a $0\le j\le\alpha$ such that $\tau_i>\chi_j$ for all $i=0,1,\cdots,a$;
(ii) (*higher scale*) There exists a $0\le j\le\alpha$ such that $\tau_i<\chi_j$ for all $i=0,1,\cdots,a$;
(iii) (*equal scale*) There exists a $0\le j\le\alpha$ such that $\tau_i=\chi_j$ for some $i=0,1,\cdots,a$.

We say j is a *lower scale index* (resp. *higher scale index*) if it satisfies (i) (resp. (ii)), and $\{i,j\}$ is an *equal scale index pair* if it satisfies (iii). The higher scale case often occurs in the perturbed Kepler problems involving the coupling of massive bodies and bodies of very small masses (see e.g. [3, 11, 12, 13, 14]). After certain normal form reductions, the normal form (1.1) in this case can be obtained in the resonance zone by splitting the resonant tori according to the Poincaré mechanism, for which various persistence results of lower dimensional, quasi-periodic, invariant tori have been proved (see [15, 16]). For the equal scale case, an interesting five body problem, consisting of Jupiter and its Galilean satellites, is treated in [2] and a persistence result of lower dimensional, quasi-periodic, invariant 2-tori is proved when all normal frequencies are simple. Unlike the higher scale case, the persistence problem of lower dimensional, quasi-periodic, invariant in the equal scale case necessarily requires Melnikov conditions between tangential and normal frequencies (see [1, 2]). The lower scale case naturally arises when an nearly integrable, multi-scale Hamiltonian in $\mathbb{T}^n\times\mathbb{R}^n$ is coupled with a lower scale Hamiltonian in $\mathbb{R}^{2m}$, in which the normal form (1.1) can be derived in the vicinity of an equilibrium of the lower scale Hamiltonian.

In the present work, we show a general persistence theorem of lower dimensional, quasi-periodic, invariant tori for the Hamiltonian (1.1) by considering all three cases of scales. Besides including the higher scale case in a unified approach and providing new result for the lower scale case, our result in the equal scale case will assume a weaker second order Melnikov condition originally derived in [17] for standard Hamiltonian systems which particularly allows multiple eigenvalues of $JM_\varepsilon(\omega)$.

Our main result states as follows.

Main Theorem. *Consider the Hamiltonian* (1.1) *and assume the following conditions:*

A1) $\bar{m} \geq \max \mathcal{F}$, *where*

$$\mathcal{F} = \{\tau_0,\ \tau_1,\ \cdots, \tau_a,\ \chi_0,\ \chi_1,\ \cdots,\ \chi_\alpha,\ \bar{\chi}_{ij},\ 1 \leq i, j \leq 2m\};$$

A2) *If* $\{i, j\}$, *where* $0 \leq i \leq a$ *and* $0 \leq j \leq \alpha$, *is an equal scale index pair, then the set* $O_{i,j} = \{\omega \in \mathcal{O} : \sqrt{-1}\langle k, \hat{\omega}^i\rangle - \lambda_p^j(\omega) - \lambda_q^j(\omega) \neq 0,\ \forall k \in \mathbb{Z}^{n_i - n_{i-1}} \setminus \{0\},\ 2\ m_{j-1} + 1 \leq p,\ q \leq 2m_j\}$ *admits full Lebesgue measure relative to* $\mathcal{O}$, *where* $n_{-1} = m_{-1} = 0$.

Then, as ε *sufficiently small, there exist Cantor-like sets* $\mathcal{O}_\varepsilon$ *with* $|\mathcal{O} \setminus \mathcal{O}_\varepsilon| \to 0$, *as* $\varepsilon \to 0$, *such that the unperturbed tori* $T_{\omega,\varepsilon}$, $\omega \in \mathcal{O}_\varepsilon$, *persist and give rise to a Whitney smooth family of quasi-periodic, invariant* n*-tori with slightly deformed Diophantine frequencies.*

Remark.

1) If there is no equal scale among any τ_i and χ_j, then the measure estimate in the Main Theorem can be made concrete as $|\mathcal{O} \setminus \mathcal{O}_\varepsilon| = O(\varepsilon^{\frac{\iota}{8m^2 N}})$ for some fixed real number $0 < \iota < \frac{1}{3}$. The same holds in the presence of equal scale when the eigenvalues of $JM_\varepsilon(\omega)$ are C^N smooth in ω.

2) The Main Theorem can be easily generalized to a system whose Hamiltonian has a normal form similar to (1.1) but involving independent parameter $\xi \in O \subset \mathbb{R}^k$ instead of ω in which k can be different from n. In other word, the tangential frequencies now have the form

$$\omega_\varepsilon(\xi) = (\varepsilon^{\tau_0}\hat{\omega}^{n_0}(\xi),\ \varepsilon^{\tau_1}\hat{\omega}^{n_1}(\xi), \cdots, \varepsilon^{\tau_a}\hat{\omega}^{n_a}(\xi))^\top$$

and M_ε and $\bar{P}$ depend on ξ instead of ω. With the conditions A1) and A2), the Main Theorem remains valid under the additional Bruno-Rüssmann non-degenerate condition that there exists a positive integer N_0 such that

$$\mathrm{Rank}\{\partial_\xi^l \Omega(\xi);\quad 0 \leq |l| \leq N_0\} = n,\ \ \forall \xi \in O,$$

where $\Omega(\xi) = (\omega^0(\xi), \omega^1(\xi),\ \cdots, \omega^a(\xi))^\top$. The proof for such a result follows from the approach of the present paper together with measure estimates involving Bruno-Rüssmann non-degenerate conditions (see e.g., [7]).

3) Using the same proof, the Main Theorem can be also generalized to the case that the order sequences $\{\tau_i\}_{i=0}^a$, $\{\chi_j\}_{j=0}^\alpha$ are distinctive, non-negative real numbers if the following additional condition is satisfied: There exists some positive constant $d_0 > 0$ such that $d > d_0 > 0$, where

$$d = \min\{|\tau_p - \tau_q|,\ |\chi_{\bar{p}} - \chi_{\bar{q}}| :\ 0 \leq p,\ q \leq a,\ 0 \leq \bar{p},\ \bar{q} \leq \alpha\}.$$

4) With more derivative estimates involved in the proof, the Main Theorem actually holds when the Hamiltonian (1.1) is of the class C^∞ in (x, y, z).

5) The Main Theorem can also be applied to treat certain cases of fast frequencies, i.e., cases when some ε-scales in the tangential and/or normal frequencies frequencies of H admit negative powers. Indeed, with respect to the new time variable $\tau = \varepsilon^{-m^*}t$, where m^* is the maximum among all absolute values of negative exponents of ε in the tangential and normal frequencies, one can consider the re-scaled Hamiltonian $\tilde{H} = \varepsilon^{m^*}H$ which is in the form of (1.1). The Main

Theorem is then applicable to the re-scaled Hamiltonian $\tilde{H}$ if conditions A1), A2) are satisfied, though the verification of A1), A2) in $\tilde{H}$ for specific cases can be a non-trivial matter due to the possible complexity of the re-scaled normal matrix $\varepsilon^{m^*}M_\varepsilon(\omega)$. We note that when conditions A1), A2) do not meet for the re-scaled Hamiltonian $\tilde{H}$, new KAM method may need to be developed in order to show a similar persistence of lower dimensional tori when fast frequencies present. This is certainly an interesting problem worthy for further investigations.

The proof of the Main Theorem combines various techniques for the persistence of lower dimensional, quasi-periodic, invariant tori in both standard and multi-scale Hamiltonian systems. In particular, as in [7], finite steps of KAM iterations need to be performed to push the perturbation to a sufficiently high order in order to carry over infinite steps of KAM iterations. Moreover, our KAM scheme for infinite steps of iterations are benefitted from those for studying lower dimensional, quasi-periodic, invariant tori in standard Hamiltonian systems especially the ones involving weaker second order Melniknov conditions (see e.g. [4, 8, 17]). For other works in standard Hamiltonian systems involving Melnikov type of conditions, we refer the reader to [5, 6, 9, 10] and references therein.

We remark that, besides technical details mentioned in the above in treating the multi-scale Hamiltonian (1.1) differing from those considered in our previous works [15], [16], one novelty of the present work lies in the measure estimate in subsection 2.5 for the frequency domain (2.13) in all cases of lower, higher and equal scales.

The rest of the sections are organized as follows. In Section 2, we perform finite steps of KAM iterations to Hamiltonian (1.1) to obtain a new normal form with sufficiently small perturbation. In Section 3, we prove the Main Theorem by infinite steps of KAM iterations. In Section 4, we illustrate the application of our Main Theorem by considering a celestial mechanical system contained in [2].

Through the rest of the paper, unless specified otherwise, we will use the same symbol $|\cdot|$ to denote an equivalent (finite dimensional) vector norm and its induced matrix norm, absolute value, and measure of sets etc., and use $|\cdot|_{\mathcal{D}}$ to denote the sup-norm of functions on a domain $\mathcal{D}$ of a metric space.

2. Improving the Order of Perturbation

In this section, we will perform finite steps of KAM iterations to improve the Hamiltonian (1.1) by pushing its perturbation to a desired high order in order to apply the standard KAM iteration.

We rewrite the Hamiltonian (1.1) as

$$\begin{aligned} H^0(x,y,z,\omega,\varepsilon) &= e^0_\varepsilon(\omega)+\langle\omega^0_\varepsilon(\omega),y\rangle \\ &+ \frac{1}{2}\langle M^0_\varepsilon(\omega)z,z\rangle+\varepsilon^{\bar{m}+\frac{1}{2}}P^0(x,y,z,\omega,\varepsilon),\ \ (x,y,z,\omega)\in D(r_0,s_0)\times\mathcal{O}_0, \end{aligned} \tag{2.1}$$

where $e^0_\varepsilon(\omega):=0$, $\omega^0_\varepsilon=:\omega_\varepsilon$, $M^0_\varepsilon(\omega)=(m^0_{ij})=:M_\varepsilon(\omega)=(m_{ij})$, $P^0(x,y,z,\omega)=:\varepsilon^{\frac{1}{2}}\bar{P}(x,y,z,\omega)$, $r_0=:r$, $s_0=\varepsilon^\iota$ for some fixed $0<\iota<\frac{1}{3}$, and $\mathcal{O}_0=\mathcal{O}$. We also denote $\omega=(\hat{\omega}^{0,0},\hat{\omega}^{0,1},\cdots,\hat{\omega}^{0,a})^\top$ and $\lambda^{0,j}_p(\omega)=:\lambda^j_p(\omega)$, $j=0,\cdots,\alpha$, $p=1,2,\cdots,2m_\alpha$. For $b=4m^2N$, we let $\gamma^b_0=\varepsilon^{\frac{\iota}{2}}$, $\mu_0=\varepsilon^{\frac{1-3\iota}{2}}$. Since P^0 is at least of the order of $O(\varepsilon^{\frac{1}{2}})$, it is clear that

$$|\partial^l_\omega P^0(x,y,z,\omega,\varepsilon)|_{D(r_0,s_0)\times\mathcal{O}_0}\le\gamma^b_0 s^2_0\mu_0,\ \ |l|\le N.$$

Suppose that we are at the ν-th KAM step for some integer $\nu \geq 0$ with the associated Hamiltonian normal form:

$$\begin{aligned} H(x,y,z,\omega,\varepsilon) &= e_\varepsilon(\omega) + \langle \omega_\varepsilon(\omega), y\rangle \\ &+ \frac{1}{2}\langle M_\varepsilon(\omega)z,\ z\rangle + \varepsilon^{\bar{m}+\frac{1}{2}} P(x,y,z,\omega,\varepsilon),\ \ (x,y,z,\omega) \in D(r,s)\times \mathcal{O} \end{aligned} \tag{2.2}$$

which is real analytic in (x,y,z) and C^N smooth in $\omega,\ \varepsilon$, where $0 < r \leq r_0$, $0 < s \leq s_0$, $\mathcal{O} \subset \mathcal{O}_0$ is a bounded closed region,

$$\omega_\varepsilon(\omega) = (\varepsilon^{\tau_0}\hat{\omega}^{0,0} + O(\varepsilon^{\bar{m}+\frac{1}{2}}),\ \varepsilon^{\tau_1}\hat{\omega}^{0,1} + O(\varepsilon^{\bar{m}+\frac{1}{2}}), \cdots, \varepsilon^{\tau_a}\hat{\omega}^{0,a} + O(\varepsilon^{\bar{m}+\frac{1}{2}}))^\top, \tag{2.3}$$

$M_\varepsilon(\omega)$ is non-singular on $\mathcal{O}$. In the above, the ν-dependency in each term is made implicit for the sake of simplicity. We denote

$$N_\varepsilon := e_\varepsilon(\omega) + \langle \omega_\varepsilon(\omega), y\rangle + \frac{1}{2}\langle M_\varepsilon(\omega)z, z\rangle \tag{2.4}$$

and assume that the perturbation P satisfies

$$|\partial_\omega^l P|_{D(r,s)\times\mathcal{O}} < \gamma_0^b s^2\mu, \quad |l| \leq N$$

for some constant $0 < \mu \leq \mu_0$.

For $+ =: \nu+1$, we will find a symplectic transformation Φ^+, such that, on a smaller phase domain $D(r_+,s_+)$ and a smaller frequency domain $\mathcal{O}_+$, the Hamiltonian (2.2) is transformed to a new Hamiltonian

$$H^+ = H\circ\Phi^+ = e_\varepsilon^+(\omega) + \langle \omega_\varepsilon^+(\omega), y\rangle + \frac{1}{2}\langle z, M_\varepsilon^+(\omega)\ z\rangle + \varepsilon^{\bar{m}+\frac{1}{2}} P^+(x,y,z,\omega,\varepsilon)$$

which satisfies similar properties as (2.3) but with much smaller perturbation P^+.

Through the rest of the paper, all constants $c_i,\ i = 0,1,2,\cdots,7$ are positive constants independent of ε and the iteration process. For simplicity, we also use c and asymptotic symbols "O, o" to denote any intermediate constant and term that are independent of ε and the iteration process.

2.1. **Truncation.** Define

$$\begin{aligned} r_+ &= \frac{r}{2} + \frac{r_0}{4}, \\ s_+ &= \frac{1}{8}\alpha s,\ \ \alpha = \mu^{\frac{1}{3}}, \\ \mu_+ &= \mu^{1+\hat{\imath}}, \qquad \text{for some fixed } \ \hat{\imath} \in (0,\imath), \\ K_+ &= ([\log\frac{1}{\mu}] + 1)^3, \\ D_i &= D(r_+ + \frac{i-1}{8}(r-r_+), \frac{i}{8}\alpha s),\ \ i = 1,\cdots,8, \\ D_+ &= D_1 = D(r_+, s_+), \\ \hat{D}(s) &= D(r_+ + \frac{7}{8}(r-r_+), s), \\ \Gamma(r-r_+) &= \sum_{0<|k|\leq K_+} |k|^{N+(N+1)4m^2\tau} e^{-|k|\frac{r-r_+}{8}}. \end{aligned}$$

We first express P in (2.2) into the following Taylor-Fourier series

$$P = \sum_{k\in\mathbb{Z}^n, \imath\in\mathbb{Z}_+^n, \jmath\in\mathbb{Z}_+^{2m}} p_{k\imath\jmath} y^\imath z^\jmath e^{\sqrt{-1}\langle k,x\rangle}$$

and let R be the truncation as follow

$$R := \sum_{|k|\le K_+} (p_{k00} + \langle p_{k10}, y\rangle + \langle p_{k01}, z\rangle + \langle z, p_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle}. \tag{2.5}$$

Lemma 2.1. *Assume that*

H1) $\displaystyle\int_{K_+}^{\infty} t^{n+3}\mathrm{e}^{-t\frac{r-r_+}{16}}\mathrm{d}t \le \mu.$

Then, there is a constant c_1, such that for any $|l|\le N$, we have

$$|\partial_\omega^l (P-R)|_{D_7\times\mathcal{O}} \le c_1\gamma_0^b s^2\mu^2, \quad |\partial_\omega^l R|_{D_7\times\mathcal{O}} \le c_1\gamma_0^b s^2\mu.$$

Proof. The proof follows from [4, Section 3] or [8, Lemma 3.1]. For the reader's convenience, we outline the key steps below.

Write

$$P = R + I + II,$$

where

$$\begin{aligned} I &= \sum_{|k|>K_+, \imath\in\mathbb{Z}_+^n, \jmath\in\mathbb{Z}_+^{2m}} p_{k\jmath} y^\imath z^\jmath e^{\sqrt{-1}\langle k,x\rangle}, \\ II &= \sum_{|k|\le K_+, 2|\imath|+|\jmath|\ge 3} p_{k\jmath} y^\imath z^\jmath e^{\sqrt{-1}\langle k,x\rangle}. \end{aligned}$$

The standard Cauchy estimate and H1) yield that

$$|\partial_\omega^l I|_{\hat{D}(s)\times\mathcal{O}} \le \sum_{|k|>K_+} |\partial_\omega^l P|_{D(r,s)\times\mathcal{O}} e^{-|k|\frac{r-r_+}{8}} \le \gamma_0^b s^2 \mu \int_{K_+}^{\infty} t^{n+3} e^{-t\frac{r-r_+}{16}} \mathrm{d}t \le \gamma_0^b s^2\mu^2.$$

It follows that

$$|\partial_\omega^l (P-I)|_{\hat{D}(s)\times\mathcal{O}} \le |\partial_\omega^l P|_{D(r,s)\times\mathcal{O}} + |\partial_\omega^l I|_{\hat{D}(s)\times\mathcal{O}} \le 2\gamma_0^b s^2\mu.$$

Using Cauchy estimate again, we have

$$|\partial_\omega^l II|_{D_7\times\mathcal{O}} \le \frac{c}{s^3}\left|\int \partial_{(y,z)}^{(p,q)} |\partial_\omega^l (P-I-R)|_{\hat{D}(s)\times\mathcal{O}} \mathrm{d}y\mathrm{d}z\right|_{D_7\times\mathcal{O}} \le c\gamma_0^b s^2\mu^2,$$

where $\int$ is the obvious anti-derivative of fold $2|p|+|q|=3$. Thus there exits a constant c_1 such that

$$\begin{aligned} &|\partial_\omega^l (P-R)|_{D_7\times\mathcal{O}} \le c_1\gamma_0^b s^2\mu^2, \\ &|\partial_\omega^l R|_{D_7\times\mathcal{O}} \le |\partial_\omega^l (P-R)|_{D_7\times\mathcal{O}} + |\partial_\omega^l P|_{D(r,s)\times\mathcal{O}} \le c_1\gamma_0^b s^2\mu. \end{aligned}$$

□

2.2. **Homological Equation.** We will eliminate the resonant terms in R by constructing a symplectic transformation which is the time-1 map ϕ_F^1 of the Hamiltonian flow generated by a Hamiltonian F of the form

$$F = \sum_{0<|k|\le K_+} (f_{k00} + \langle f_{k10}, y\rangle + \langle f_{k01}, z\rangle + \langle z, f_{k02} z\rangle) \mathrm{e}^{\sqrt{-1}\langle k,x\rangle}, \tag{2.6}$$

where f_{kij}'s are (matrix valued) functions of (ω,ε) and f_{k02} is symmetric for each k. To do so, F needs to satisfy the following homological equation

$$\{N_\varepsilon, F\} + \varepsilon^{\bar{m}+\frac{1}{2}}(R - [R]) = 0, \tag{2.7}$$

where N_ε is defined in (2.4) and

$$[R] := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} R(x,\cdot)\mathrm{d}x \tag{2.8}$$

is the average of the truncation R defined in (2.5). Substituting (2.6) and (2.8) into (2.7) yields

$$\begin{aligned}
& - \sum_{0<|k|\le K_+} \sqrt{-1}\langle k,\omega_\varepsilon\rangle (f_{k00} + \langle f_{k10}, y\rangle + \langle f_{k01}, z\rangle + \langle z, f_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} \\
& + \sum_{0<|k|\le K_+} (\langle M_\varepsilon(\omega) z, J f_{k01}\rangle + 2\langle M_\varepsilon(\omega) z, J f_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} \\
= & - \sum_{0<|k|\le K_+} \varepsilon^{\bar{m}+\frac{1}{2}} (p_{k00} + \langle p_{k10}, y\rangle + \langle p_{k01}, z\rangle + \langle z, p_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle}.
\end{aligned}$$

Comparing the coefficients in the above, we deduce the following linear equations for all $0 < |k| \le K_+$:

$$\tilde{L}_{0k} f_{k00} = \varepsilon^{\frac{1}{2}} p_{k00}, \tag{2.9}$$

$$\tilde{L}_{0k} f_{k10} = \varepsilon^{\frac{1}{2}} p_{k10}, \tag{2.10}$$

$$\tilde{L}_{1k} f_{k01} = \varepsilon^{\frac{1}{2}} p_{k01}, \tag{2.11}$$

$$\tilde{L}_{2k} f_{k02} = \varepsilon^{\frac{1}{2}} p_{k02}, \tag{2.12}$$

where

$$\begin{aligned}
\tilde{L}_{0k} &= \frac{1}{\varepsilon^{\bar{m}}}\sqrt{-1}\langle k,\omega_\varepsilon\rangle, \\
\tilde{L}_{1k} &= \frac{1}{\varepsilon^{\bar{m}}}(\sqrt{-1}\langle k,\omega_\varepsilon\rangle I_{2m} - J M_\varepsilon(\omega)), \\
\tilde{L}_{2k} &= \frac{1}{\varepsilon^{\bar{m}}}\left[\sqrt{-1}\langle k,\omega_\varepsilon\rangle I_{4m^2} - (J M_\varepsilon(\omega))\otimes I_{2m} - I_{2m}\otimes (J M_\varepsilon(\omega))\right]
\end{aligned}$$

with $\otimes$ standing for the tensor product of matrices.

We would like to solve (2.9)-(2.12) on the frequency domain

$$\mathcal{O}_+ = \{\omega\in\mathcal{O}:\ |\tilde{L}_{0k}^0| > \frac{\gamma_0}{|k|^\tau},\ |\det \tilde{L}_{1k}^0| > \frac{\gamma_0^{2m}}{|k|^{2m\tau}},\ |\det \tilde{L}_{2k}^0| > \frac{\gamma_0^{4m^2}}{|k|^{4m^2\tau}},\ \forall\ 0<|k|\le K_+\}, \tag{2.13}$$

where

$$\begin{aligned}
\tilde{L}_{0k}^0 &= \frac{1}{\varepsilon^{\bar{m}}}\sqrt{-1}\langle k,\omega_\varepsilon^0\rangle, \\
\tilde{L}_{1k}^0 &= \frac{1}{\varepsilon^{\bar{m}}}(\sqrt{-1}\langle k,\omega_\varepsilon^0\rangle I_{2m} - J M_\varepsilon^0(\omega)), \\
\tilde{L}_{2k}^0 &= \frac{1}{\varepsilon^{\bar{m}}}\left[\sqrt{-1}\langle k,\omega_\varepsilon^0\rangle I_{4m^2} - (J M_\varepsilon^0(\omega))\otimes I_{2m} - I_{2m}\otimes (J M_\varepsilon^0(\omega))\right].
\end{aligned}$$

Lemma 2.2. *Assume*

H2) $\varepsilon^{\frac{1}{2}} K_+^{4m^2(\tau+1)} = o(\gamma_0)$,
H3) $|\partial_\omega^l (M_\varepsilon - M_\varepsilon^0)|_{\mathcal{O}} = O(\varepsilon^{\bar{m}+\frac{1}{2}})$, $|\partial_\omega^l (\omega_\varepsilon - \omega_\varepsilon^0)|_{\mathcal{O}} = O(\varepsilon^{\bar{m}+\frac{1}{2}})$, $\forall |l| \le N$.

Then for all $0 < |k| \le K_+$, $\tilde{L}_{qk}$, $q = 0,1,2$, *are invertible on* $\mathcal{O}_+$, *and moreover, there is a constant* $c_2 > 0$ *such that for all* $q = 0,1,2$ *and* $0 < |k| \le K_+$,

$$|\partial_\omega^l \tilde{L}_{qk}^{-1}| \le c_2 \frac{|k|^{|l|+(|l|+1)(2m)^q \tau}}{\gamma_0^{(2m)^q(|l|+1)}}, \qquad |l| \le N, \omega \in \ \mathcal{O}_+,$$

Proof. For any $0 < |k| \le K_+$, denote

$$\bar{L} = \frac{1}{\varepsilon^{\bar{m}+\frac{1}{2}}} (\sqrt{-1}\langle k, \omega_\varepsilon I_{2m} - \omega_\varepsilon^0 \rangle + J(M_\varepsilon - M_\varepsilon^0)).$$

It follows from H3) that $\bar{L}$ is bounded from the above by a constant independent of k, ω and ε. Since

$$\tilde{L}_{1k} = \tilde{L}_{1k}^0 + \varepsilon^{\frac{1}{2}} \bar{L},$$

we have by H2) that

$$|\det \tilde{L}_{1k}|_{\mathcal{O}_+} \ge |\det \tilde{L}_{1k}^0|_{\mathcal{O}_+} - O(\varepsilon^{\frac{1}{2}}) K_+^{2m} \ge \frac{\gamma_0^{2m}}{2|k|^{2m\tau}},$$

and consequently,

$$|\tilde{L}_{1k}^{-1}|_{\mathcal{O}_+} \ = \ |(\tilde{L}_{1k}^0 + \varepsilon^{\frac{1}{2}} \bar{L})^{-1}| \le c \frac{|(\tilde{L}_{1k}^0)^{-1}|}{1 - |(\tilde{L}_{1k}^0)^{-1}||\varepsilon^{\frac{1}{2}} \bar{L}|} \le c \frac{|k|^{2m\tau+2m-1}}{\gamma_0^{2m}}. \tag{2.14}$$

Similarly,

$$|\det \tilde{L}_{2k}|_{\mathcal{O}_+} \ge \frac{\gamma_0^{4m^2}}{2|k|^{4m^2\tau}}, \quad |\tilde{L}_{2k}^{-1}|_{\mathcal{O}_+} \le c \frac{|k|^{4m^2\tau+4m^2-1}}{\gamma_0^{4m^2}}. \tag{2.15}$$

The lemma now follows from (2.14), (2.15) and the following inductive equations

$$\partial_\omega^l \tilde{L}_{qk}^{-1} = -\sum_{l'=1}^{l} C_{l'}^l (\partial_\omega^{l-l'} \tilde{L}_{qk}^{-1} \partial_\omega^{l'} \tilde{L}_{qk}) \tilde{L}_{qk}^{-1}, \quad |l| \le N, \, q = 0,1,2.$$

□

It follows from Lemma 2.2 that the equations (2.9)-(2.12) are uniquely solvable on $\mathcal{O}_+$ to yield solutions which satisfy estimates

$$|\partial_\omega^l f_{k00}|_{\mathcal{O}_+} \ \le \ c_3 |k|^{|l|+(|l|+1)\tau} s^2 \mu e^{-|k|^\tau}, \tag{2.16}$$

$$|\partial_\omega^l f_{k10}|_{\mathcal{O}_+} \ \le \ c_3 |k|^{|l|+(|l|+1)\tau} s \mu e^{-|k|^\tau}, \tag{2.17}$$

$$|\partial_\omega^l f_{k01}|_{\mathcal{O}_+} \ \le \ c_3 |k|^{|l|+(|l|+1)2m\tau} s \mu e^{-|k|^\tau}, \tag{2.18}$$

$$|\partial_\omega^l f_{k02}|_{\mathcal{O}_+} \ \le \ c_3 |k|^{|l|+(|l|+1)4m^2\tau} \mu e^{-|k|^\tau}, \tag{2.19}$$

for all $|l| \le N$, $0 < |k| \le K_+$, where $c_3 > 0$ is a constant.

Lemma 2.3. *Assume* H2), H3) *and let* F *be as in* (2.6). *Then there is a constant* $c_4 > 0$ *such that on* $\hat{D}(s) \times \mathcal{O}_+$,

$$|F|,\ |F_x|,\ s^2|F_y|,\ s|F_z| \le c_4 s^2 \mu \Gamma(r - r_+),$$

and

$$|\partial_\omega^l \partial_x^i \partial_{(y,z)}^{(p,q)} F| \le c_4 \mu \Gamma(r - r_+)$$

for all $0 \le |l| + |i| \le N$, $0 \le 2|p| + |q| \le 2$.

Proof. It follows easily from (2.16)-(2.19). □

Lemma 2.4. *Let* ϕ_F^t *be the flow generated by* F *and assume*

H4) $c_4 \mu \Gamma(r - r_+) < \frac{1}{8}(r - r_+)$,
H5) $c_4 s^2 \mu \Gamma(r - r_+) < s_+^2$.

Then the followings hold.

1) *For all* $0 \le t \le 1$,
$$\phi_F^t : D_3 \to D_4$$
are well defined, real analytic and smoothly depend on $\omega \in \mathcal{O}_+$.
2) *Let* $\Phi^+ = \phi_F^1$. *Then for each* $\omega \in \mathcal{O}_+$,
$$\Phi^+ : D_+ \to D(r, s).$$
3) *There is a constant* $c_5 > 0$ *such that*
$$\begin{aligned} |\partial_\omega^l(\phi_F^t - \mathrm{Id})|_{D_+ \times \mathcal{O}_+} &\le c_5 s \mu \Gamma(r - r_+),\ \forall\ t \in [0,1],\ 0 \le |l| \le N. \\ |\partial_\omega^l \partial_x^i \partial_{(y,z)}^j (\Phi^+ - \mathrm{Id})|_{D_+ \times \mathcal{O}_+} &\le c_5 \mu \Gamma(r - r_+),\ \forall\ 0 \le |l| + |i| \le N,\ |j| = 0, 1. \end{aligned}$$

Proof. Write

$$\phi_F^t = \mathrm{Id} + \int_0^t X_F \circ \phi_F^u \mathrm{d}u = (\phi_{F1}^t, \phi_{F2}^t, \phi_{F3}^t)^\top, \tag{2.20}$$

where $X_F = (F_y,\ -F_x,\ JF_z)^\top$ is the vector field generated by F and

$$\begin{aligned} \phi_{F1}^t(x,y,z) &= x + \int_0^t F_y \circ \phi_F^u \mathrm{d}u, \\ \phi_{F2}^t(x,y,z) &= y + \int_0^t -F_x \circ \phi_F^u \mathrm{d}u, \\ \phi_{F2}^t(x,y,z) &= z + \int_0^t JF_z \circ \phi_F^u \mathrm{d}u. \end{aligned}$$

Then for any $(x,y,z) \in D_3$ and $t \in [0,1]$, we have by Lemma 2.3 and H4), H5) that

$$\begin{aligned} |\phi_{F1}^t(x,y,z)| &\le |x| + |F_y|_{\hat{D}(s)\times\mathcal{O}} \le r_+ + \frac{2}{8}(r - r_+) + c_4 \mu \Gamma(r - r_+) < r_+ + \frac{3}{8}(r - r_+), \\ |\phi_{F2}^t(x,y,z)| &\le |y| + |F_x|_{\hat{D}(s)\times\mathcal{O}} \le (3s_+)^2 + c_4 s^2 \mu \Gamma(r - r_+) < (3s_+)^2 + s_+^2 < (4s_+)^2, \\ |\phi_{F3}^t(x,y,z)| &\le |z| + |F_z|_{\hat{D}(s)\times\mathcal{O}} \le 3s_+ + c_4 s \mu \Gamma(r - r_+) < 3s_+ + \frac{s_+^2}{s} < 4s_+, \end{aligned}$$

i.e., $\phi_F^t(x,y,z) \in D_4$ for $t \in [0,1]$. Hence 2) holds.

Similar to that in [4, Section 3] or [8, Lemma 3.3], the proof of 3) simply follows from (2.20) and the estimates in Lemma 2.3. . □

2.3. **New Hamiltonian.** Lemma 2.4 shows that for each $\omega \in \mathcal{O}_+$, $\Phi^+ : D(r_+, s_+) \to D_4 \subset D(r,s)$ is a well defined, real analytic symplectic transformation. Applying this transformation to the Hamiltonian H, we obtain the new Hamiltonian

$$\begin{aligned} H^+ &=: H \circ \Phi^+ = H \circ \phi_F^1 = (N + \varepsilon^{\bar{m}+\frac{1}{2}} R) \circ \phi_F^1 + \varepsilon^{\bar{m}+\frac{1}{2}} (P - R) \circ \phi_F^1 \\ &= e_\varepsilon^+(\omega) + \langle \omega_\varepsilon^+(\omega), y \rangle + \frac{1}{2} \langle M_\varepsilon^+(\omega) z, z \rangle + \varepsilon^{\bar{m}+\frac{1}{2}} P^+ \end{aligned}$$

on $D(r_+, s_+) \times \mathcal{O}_+$, where e_ε^+ is a smooth function on $\mathcal{O}_+$ and

$$\omega_\varepsilon^+(\omega) = \omega_\varepsilon + \varepsilon^{\bar{m}+\frac{1}{2}} p_{010}, \tag{2.21}$$

$$M_\varepsilon^+(\omega) = M_\varepsilon(\omega) + \varepsilon^{\bar{m}+\frac{1}{2}} p_{002}, \tag{2.22}$$

$$P^+ = \int_0^1 \{(1-t)(R - [R]) + R, F\} \mathrm{d}t + (P - R) \circ \phi_F^1 \tag{2.23}$$

with $[R]$ being defined in (2.8). We denote the new unperturbed part as

$$N_\varepsilon^+ =: e_\varepsilon^+(\omega) + \langle \omega_\varepsilon^+(\omega), y \rangle + \frac{1}{2} \langle M_\varepsilon^+(\omega) z, z \rangle.$$

Lemma 2.5. *There exits a constant $c_6 > 0$ such that the followings hold for all $0 \le |l| \le N$:*

$$\begin{aligned} |\partial_\omega^l (\omega_\varepsilon^+ - \omega_\varepsilon)|_{\mathcal{O}_+} &\le c_6 \varepsilon^{\bar{m}+\frac{1}{2}} s\mu, \\ |\partial_\omega^l (M_\varepsilon^+(\omega) - M_\varepsilon(\omega))|_{\mathcal{O}_+} &\le c_6 \varepsilon^{\bar{m}+\frac{1}{2}} \mu. \end{aligned}$$

Proof. It follows from (2.21), (2.22) and estimate of truncation R in Lemma 2.1. □

For the new perturbation P^+, we have the following estimate.

Lemma 2.6. *There exists a constant $c_7 > 0$ such that*

$$|\partial_\omega^l P^+|_{D_+ \times \mathcal{O}_+} \le c_7 \gamma_0^b (s^3 \mu^2 \Gamma(r - r_+) + s^3 \mu^2 + s^2 \mu^2), \quad |l| \le N,$$

and consequently, if

H6) $c_0 \gamma_0^b (s^3 \mu^2 \Gamma(r - r_+) + s^3 \mu^2) \le \gamma_0^b s_+^2 \mu_+$,

where $c_0 = \max\{1, c_1, c_2, \cdots, c_7\}$, then

$$|\partial_\omega^l P^+|_{D_+ \times \mathcal{O}_+} \le \gamma_0^b s_+^2 \mu_+, \qquad |l| \le N.$$

Proof. We note that $[R] = O(\varepsilon^{\bar{m}+\frac{1}{2}})$. The lemma follows easily from the expression of P^+ in (2.23) and estimates of F in Lemma 2.3. □

2.4. **Finite Steps of KAM Iteration.** Recursively applying the definitions of quantities in the above for the νth step with $\nu+1$ in place of "+" for $\nu = 0, 1, \cdots$, we obtain the following iterative sequences:

$$
\begin{aligned}
r_\nu &= r_0\Big(1 - \sum_{i=1}^{\nu} \frac{1}{2^{i+1}}\Big),\\
s_\nu &= \frac{1}{8}\alpha_{\nu-1}s_{\nu-1},\ \alpha_\nu = \mu_\nu^{\frac{1}{3}},\\
\mu_\nu &= \mu_{\nu-1}^{1+\hat{\iota}}, \quad \text{for some fixed } \hat{\iota} \in (0, \iota)\\
H^\nu &= H^{\nu-1} \circ \Phi^\nu = e_\varepsilon^\nu(\omega) + \langle \omega_\varepsilon^\nu(\omega), y\rangle + \frac{1}{2}\langle M_\varepsilon^\nu(\omega)z, z\rangle + \varepsilon^{\bar{m}+\frac{1}{2}} P^\nu(x, y, z, \omega, \varepsilon),\\
&\qquad\qquad (x, y, z, \omega) \in D(r_\nu, s_\nu) \times \mathcal{O}_\nu,\\
K_\nu &= \Big(\Big[\log\Big(\frac{1}{\mu_{\nu-1}}\Big)\Big] + 1\Big)^3,\\
\mathcal{O}_\nu &= \Big\{\omega \in \mathcal{O}_{\nu-1} : \ |\tilde{L}_{0k}^0| > \frac{\gamma_0}{|k|^\tau},\ |\det \tilde{L}_{1k}^0| > \frac{\gamma_0^{2m}}{|k|^{2m\tau}}, |\det \tilde{L}_{2k}^0| > \frac{\gamma_0^{4m^2}}{|k|^{4m^2\tau}},\ \forall\, 0 < |k| \le K_\nu\Big\},
\end{aligned}
$$

for $\nu = 1, 2, \cdots$.

Let

$$
\nu_* = \left[\frac{\log 2(N+6)c_* + \log 8m^2(N+1) - \log(\frac{1-3\iota}{2})}{\log(1+\hat{\iota})}\right] + 1, \tag{2.24}
$$

where $c_* = 4m^2 \sum_{i=1}^{a} \tau_i(n_i - n_{i-1})$ and $[x]$ denotes the maximum integer less than x.

Proposition 2.1. *The iteration scheme can be performed inductively for $\nu = 0, 1, \cdots, \nu_* - 1$ to yield the Hamiltonian*

$$
H^* =: H^{\nu_*} = e_\varepsilon^*(\omega) + \langle \omega_\varepsilon^*(\omega), y\rangle + \frac{1}{2}\langle M_\varepsilon^*(\omega)z, z\rangle + \varepsilon^{\frac{1}{2}} P^*(x, y, z, \omega, \varepsilon) \tag{2.25}
$$

defined on $D(r_*, s_*) \times \mathcal{O}_*$, where $r_* = r_{\nu_*}$, $s_* = s_{\nu_*}$, $e_\varepsilon^* = e_\varepsilon^{\nu_*}$, $\mathcal{O}_* = \mathcal{O}_{\nu_*}$, $\omega_\varepsilon^* = \omega_\varepsilon^{\nu_*}$, $M_\varepsilon^* = M_\varepsilon^{\nu_*}$, $P^* = \varepsilon^{\bar{m}} P^{\nu_*}$, such that

$$
\begin{aligned}
|\partial_\omega^l P^*|_{D(r_*, s_*) \times O^*} &\le \varepsilon^{\bar{m}} \gamma_0^b s_*^2 \mu_{\nu_*} \le s_*^2 \varepsilon^{\bar{m}+\frac{\iota}{2}+16d^2(N+1)(N+6)c_*}\\
&\le \gamma_*^{16d^2(N+1)(N+6)} s_*^2 \mu_*^2, \qquad |l| \le N,
\end{aligned} \tag{2.26}
$$

where $\gamma_* = \varepsilon^{c_*}$, $\mu_* = \varepsilon^{\frac{2\bar{m}+\iota}{4}}$.

Proof. To show the validity of the iteration scheme, we need to check conditions H1)-H6) for all $\nu = 0, 1, \cdots, \nu_*$.

First of all, for any $|l| \le N$, we have

$$
\begin{aligned}
|\partial_\omega^l (M_\varepsilon^\nu - M_\varepsilon^0)|_{\mathcal{O}_\nu} &\le c\varepsilon^{\bar{m}+\frac{1}{2}}(\mu_{\nu-1} + \mu_{\nu-2} + \cdots + \mu_0) = O(\varepsilon^{\bar{m}+\frac{1}{2}}),\\
|\partial_\omega^l (\omega_\varepsilon^\nu - \omega_\varepsilon^0)|_{\mathcal{O}_\nu} &\le c\varepsilon^{\bar{m}+\frac{1}{2}}(\mu_{\nu-1} + \mu_{\nu-2} + \cdots + \mu_0) = O(\varepsilon^{\bar{m}+\frac{1}{2}}),
\end{aligned}
$$

which verifies H3) for all $\nu = 0, 1, 2, \cdots$, as ε sufficiently small. Next, it is easy to deduce that

$$
r_\nu - r_{\nu+1} = \frac{r_0}{2^{\nu+2}}, \quad \mu_\nu = \mu_0^{(1+\hat{\iota})^\nu} = \varepsilon^{(\frac{1-3\iota}{2})(1+\hat{\iota})^\nu} \quad \nu = 1, 2, \cdots, \tag{2.27}
$$

from which it is easy to see that conditions H1), H4) - H6) hold for all $\nu = 1, 2, \cdots$, as ε sufficiently small. However, we note that H2) only holds for a finite number of ν's. In fact, by definitions of

K_ν's, γ_0 and ν_*, we see that H2) holds for all $\nu = 0, 1, \cdots, \nu_* - 1$ as long as $\varepsilon \ll 1$ is sufficiently small.

Now, by (2.24) and (2.27), we have

$$\mu_{\nu_*} = \mu_0^{1+\hat{\iota}} \le \varepsilon^{16m^2(N+1)(N+6)c_*},$$

from which (2.26) follows by applying Lemma 2.6 to the step $\nu_* - 1$. □

2.5. **Measure estimates.** We now estimate the excluded measure of $\mathcal{O}_0$ after ν^*th iteration.

Lemma 2.7. *For all $k \in \mathbb{Z}^n$,*

$$\begin{aligned} \det \tilde{L}_{1k}^0 &= \prod_{j=0}^{\alpha} \prod_{p=2m_{j-1}+1}^{2m_j} \frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0(\omega)\rangle - \varepsilon^{\chi_j} \lambda_p^{0,j}(\omega)), \\ \det \tilde{L}_{2k}^0 &= \prod_{j=0}^{\alpha} \prod_{p,q=2m_{j-1}+1}^{2m_j} \frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0(\omega)\rangle - \varepsilon^{\chi_j} \lambda_p^{0,j}(\omega) - \varepsilon^{\chi_j} \lambda_q^{0,j}(\omega)). \end{aligned}$$

Proof. Recall that the eigenvalues of $JM_\varepsilon^0(\omega)$ are of the form

$$\begin{aligned} &\varepsilon^{\chi_0} \lambda_1^{0,0}(\omega), \cdots, \varepsilon^{\chi_0} \lambda_{2m_0}^{0,0}(\omega), \\ &\varepsilon^{\chi_1} \lambda_{2m_0+1}^{0,1}(\omega), \cdots, \varepsilon^{\chi_1} \lambda_{2m_1}^{0,1}(\omega), \\ &\cdots, \\ &\varepsilon^{\chi_\alpha} \lambda_{2m_{\alpha-1}+1}^{0,\alpha}(\omega), \cdots, \varepsilon^{\chi_\alpha} \lambda_{2m_\alpha}^{0,\alpha}(\omega), \end{aligned}$$

where α, m_j are positive integers, $m_0 < m_1 < \cdots < m_\alpha := m$, $\{\chi_0, \cdots, \chi_\alpha\}$ is a sequence of distinct, non-negative integers.

Let E be the Jordan canonical form of JM_ε^0 and T be the non-singular matrix such that $T^{-1}(JM_\varepsilon^0)T = E$. Then

$$\begin{aligned} \det \tilde{L}_{1k}^0 &= \det \left[\frac{1}{\varepsilon^{\bar{m}}} T^{-1} (\sqrt{-1}\langle k, \omega_\varepsilon^0\rangle I_{2m} - JM_\varepsilon^0) T \right] \\ &= \det \left[\frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0\rangle I_{2m} - E) \right] \\ &= \prod_{j=0}^{\alpha} \prod_{p=2m_{j-1}+1}^{2m_j} \frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0(\omega)\rangle - \varepsilon^{\chi_j} \lambda_p^{0,j}(\omega)). \end{aligned}$$

Since

$$(T^{-1} \otimes T^{-1}) \left[(JM_\varepsilon^0) \otimes I_{2m} + I_{2m} \otimes (JM_\varepsilon^0) \right] (T \otimes T) = E \otimes I_{2m} + I_{2m} \otimes E,$$

it follows directly follows from [8, Lemma 5.1] that

$$\begin{aligned} \det \tilde{L}_{2k}^0 &= \det \left[\frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0\rangle I_{4m^2} - E \otimes I_{2m} - I_{2m} \otimes E) \right] \\ &= \prod_{j=0}^{\alpha} \prod_{p,q=2m_{j-1}+1}^{2m_j} \frac{1}{\varepsilon^{\bar{m}}} (\sqrt{-1}\langle k, \omega_\varepsilon^0(\omega)\rangle - \varepsilon^{\chi_j} \lambda_p^{0,j}(\omega) - \varepsilon^{\chi_j} \lambda_q^{0,j}(\omega)). \end{aligned}$$

□

Proposition 2.2. $|\mathcal{O}_0 \setminus \mathcal{O}_*| \to 0$ *as* $\varepsilon \to 0$.

Proof. We note that

$$\mathcal{O}_* = \{\omega \in \mathcal{O}_0 : \ |\tilde{L}^0_{0k}| > \frac{\gamma_0}{|k|^\tau}, \ |\det \tilde{L}^0_{1k}| > \frac{\gamma_0^{2m}}{|k|^{2m\tau}}, |\det \tilde{L}^0_{2k}| > \frac{\gamma_0^{4m^2}}{|k|^{4m^2\tau}}, \ \forall\ 0 < |k| \le K_{\nu_*}\}.$$

Consider the set $O^* = O^0_1 \bigcap O^0_2 \bigcap O^0_3$, where

$$\begin{aligned} O^0_1 &= \{\omega \in \mathcal{O}_0 : \ |\langle k, \omega\rangle| > \frac{2\gamma_0}{|k|^\tau}, \ \forall\ 0 < |k| \le K_{\nu_*}\}, \\ O^0_2 &= \{\omega \in \mathcal{O}_0 : \ |\lambda^j_p(\omega)| > \frac{2\gamma_0}{|k|^\tau}, \ \forall \text{ lower scale indexes } j, \ 0 < |k| \le K_{\nu_*} \text{ and } p = 2m_{j-1}+1, \cdots, \ 2m_j\}, \\ O^0_3 &= \{\omega \in \mathcal{O}_0 : \ |\sqrt{-1}\langle \hat{k}^i, \ \hat{\omega}^{0,i}\rangle - \lambda^{0,j}_p(\omega)|, \ |\sqrt{-1}\langle \hat{k}^i, \ \hat{\omega}^{0,i}\rangle - \lambda^{0,j}_p(\omega) - \lambda^{0,j}_q(\omega)| > \frac{2\gamma_0}{|k|^\tau}, \\ &\qquad \forall \text{ equal scale index pairs } \{i,j\}, \ p, \ q = 2m_{j-1}+1, \cdots, \ 2m_j, \ \hat{k}^i \in \mathbb{Z}^{n_i - n_{i-1}}, \\ &\qquad i = 0, 1, \cdots, a, \ j = 0, 1, \cdots, \alpha, \ 0 < |k| \le K_{\nu_*}\}. \end{aligned}$$

Let $\omega \in O^*$. For given $k = (\hat{k}^0, \cdots, \hat{k}^i, \cdots, \hat{k}^\alpha)^\top$ with $0 < |k| \le K_{\nu_*}$, let $\hat{k}^i$, for some $i = 0, 1, \cdots, a$, be the first nonzero components of k and $\tau_{i_0} = \min\{\tau_i, \cdots, \tau_a\}$.

Since $\omega \in \ O^0_1$ and H2) holds for $\nu = \nu_*$, we have

$$\begin{aligned} |\tilde{L}_{0k}| &= \frac{1}{\varepsilon^{\bar{m}}} |\varepsilon^{\tau_{i_0}} \langle k^{i_0}, \omega^{0,i_0}_\varepsilon\rangle + O(\varepsilon^{\tau_{i_0}+1}) K_1^{\tau+1}| \ge \frac{1}{\varepsilon^{\bar{m}-\tau_{i_0}}} [|\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle| - |o(\gamma_0)|] \\ &\ge \frac{1}{2\varepsilon^{\bar{m}-\tau_{i_0}}} |\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle| > \frac{\gamma_0}{|k|^\tau}. \end{aligned}$$

In vitro of Lemma 2.7, we consider a typical term $l^{0,j,p}_{1k} := \frac{1}{\varepsilon^{\bar{m}}}(\sqrt{-1}\langle k, \omega^0_\varepsilon\rangle - \varepsilon^{\chi_j} \lambda^{0,j}_p(\omega))$ of $\tilde{L}^0_{1k}$. Comparing τ_{i_0} with χ_j, we have the following three cases:

(1) When $\tau_{i_0} = \chi_j$, since $\omega \in O^0_3$, it follows from H2) for $\nu = \nu_*$ that

$$|l^{0,j,p}_{1k}| \ge \frac{1}{\varepsilon^{\bar{m}-\tau_{i_0}}} \Big[|\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle - \lambda^{0,j}_p(\omega)| - |o(\gamma_0)|\Big] \ge \frac{1}{2\varepsilon^{\bar{m}-\tau_{i_0}}} |\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle - \lambda^{0,j}_p(\omega)| \ge \frac{\gamma_0}{|k|^\tau}.$$

(2) When $\tau_{i_0} > \chi_j$, since $\omega \in O^2_2$, it follows from H2) for $\nu = \nu_*$ that

$$|l^{0,j,p}_{1k}| \ge \frac{1}{\varepsilon^{\bar{m}-\chi_j}} \left[|\lambda^{0,j}_p(\omega)| - |O(\varepsilon) K_+^{\tau+1}|\right] \ge \frac{1}{2\varepsilon^{\bar{m}-\chi_j}} |\lambda^{0,j}_p(\omega)| \ge \frac{\gamma_0}{|k|^\tau}.$$

(3) When $\tau_{i_0} < \chi_j$, since $\omega \in O^0_1$, it follows from H2) for $\nu = \nu_*$ that

$$|l^{0,j,p}_{1k}| \ge \frac{1}{\varepsilon^{\bar{m}-\tau_{i_0}}} \Big[|\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle| - |O(\varepsilon) K_+^{\tau+1}| - |O(\varepsilon)|\Big] \ge \frac{1}{2\varepsilon^{\bar{m}-\tau_{i_0}}} |\langle \hat{k}^{i_0}, \hat{\omega}^{0,i_0}\rangle| \ge \frac{\gamma_0}{|k|^\tau}.$$

Using Lemma 2.7, we then have

$$|\det \tilde{L}^0_{1k}| > \frac{\gamma_0^{2m}}{|k|^{2m\tau}}.$$

Arguing the same way, we also conclude that

$$\det \tilde{L}^0_{2k} > \frac{\gamma_0^{4m^2}}{|k|^{4m^2\tau}}.$$

Above all, we have shown that $O^* \subset \mathcal{O}_*$. As ε sufficiently small, we note that $|\mathcal{O}_0 \setminus O_1^0| = O(\gamma_0)$ and $\mathcal{O}_0 = O_2^0$. For each equal index pair $\{i,j\}$, we note that

$$\{\omega \in \mathcal{O}_0 : \sqrt{-1}\langle \hat{k}^i, \ \hat{\omega}^{0,i}\rangle - \lambda_p^{0,j}(\omega) - \lambda_q^{0,j}(\omega) \neq 0, \ \hat{k}^i \in \mathbb{Z}^{n_i - n_{i-1}}\}$$
$$\subset \{\omega \in \mathcal{O}_0 : \sqrt{-1}\langle \hat{k}^i, \ \hat{\omega}^{0,i}\rangle - \lambda_p^{0,j}(\omega) \neq 0, \quad \hat{k}^i \in \mathbb{Z}^{n_i - n_{i-1}}\}.$$

It then follows from A2) that $|\mathcal{O}_0 \setminus O_3^0| \to 0$ as $\varepsilon \to 0$, and consequently, $|\mathcal{O}_0 \setminus \mathcal{O}_*| \to 0$ as $\varepsilon \to 0$. □

3. Proof of main result

In this section, we will perform infinite steps of standard KAM iterations to the normal form (2.25) to prove the Main Theorem.

3.1. Rescaling.

We consider the following rescalings

$$y \to \gamma_*^{8m^2(N+1)(N+6)} s_* \mu_* y, \quad z \to \gamma_*^{8m^2(N+1)(N+6)} s_* \mu_* z, \quad H^* \to \frac{H^*}{\gamma_*^{8m^2(N+1)(N+6)} s_* \mu_*}$$

to the normal form (2.25). The re-scaled Hamiltonian then reads

$$H_0 =: \frac{H^*}{\gamma_*^{8m^2(N+1)(N+6)} \mu_*} =: e_0(\omega, \varepsilon) + \langle \omega_0(\omega, \varepsilon), y\rangle + \frac{1}{2}\langle z, M_0(\omega, \varepsilon) z\rangle + \varepsilon^{\frac{1}{2}} P_0(x, y, z, \omega, \varepsilon),$$

$(x, y, z) \in D(r_0, s_0)$, $\omega \in O_0 =: \mathcal{O}_*$, where $r_0 =: r_*$, $s_0 =: s_*$, $e_0 = e_\varepsilon^*$, $\omega_0 = \omega_\varepsilon^*$, $M_0 := \gamma_*^{8m^2(N+1)(N+6)} s_* \mu_* M_\varepsilon^*$, and

$$P_0 = \frac{P^*}{\gamma_*^{8m^2(N+1)(N+6)} s_* \mu_*}.$$

It follows from (2.26) that

$$|\partial_\omega^l P_0|_{D(r_0, s_0) \times O_0} \leq \frac{|\partial_\omega^l P^*|}{\gamma_*^{8m^2(N+1)(N+6)} s_* \mu_*} \leq \gamma_*^{8m^2(N+1)(N+6)} s_* \mu_* = \gamma_0^{4m^2 N} s_0 \mu_0, \ |l| \leq N,$$

where $\gamma_0 =: \gamma_*^{2(N+6)}$, $\mu_0 =: \mu_*$.

3.2. **Iteration and convergence.** Consider the following sequences

$$
\begin{aligned}
&r_\nu = r_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
&s_\nu = \frac{1}{8}\alpha_{\nu-1}s_{\nu-1},\\
&\alpha_\nu = \mu_\nu^{\frac{1}{2}},\\
&\mu_\nu = c_0\mu_{\nu-1}^{\frac{6}{5}},\\
&\gamma_\nu = \gamma_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
&K_\nu = ([\log(\frac{1}{\mu_{\nu-1}})]+1)^{3\eta},\\
&L_{1k,\nu-1} = \sqrt{-1}\langle k,\omega_{\nu-1}\rangle I_{2m} - M_{\nu-1}J, \quad 0<|k|\le K_\nu,\\
&L_{2k,\nu-1} = \sqrt{-1}\langle k,\omega_{\nu-1}\rangle I_{4m^2} - (M_{\nu-1}J)\otimes I_{2m} + I_{2m}\otimes(JM_{\nu-1}), \quad 0<|k|\le K_\nu,\\
&O_\nu = \{\omega\in O_{\nu-1}: |\langle k,\omega_{\nu-1}\rangle| > \frac{\gamma_{\nu-1}}{|k|^\tau},\ |\det L_{1k,\nu-1}| > \frac{\gamma_{\nu-1}^{2m}}{|k|^{2m\tau}},\\
&\qquad\qquad |\det L_{2k,\nu-1}| > \frac{\gamma_{\nu-1}^{4m^2}}{|k|^{4m^2\tau}},\ 0<|k|\le K_\nu\},
\end{aligned}
$$

$\nu = 1,2,\cdots$, where $\eta \ge \frac{\log 2}{\log 6 - \log 5}$ is a fixed constant. Then we have the following result.

Proposition 3.1. *Let ε be sufficiently small. Then the followings hold for all $\nu = 1,2,\cdots$.*

1) *There is a sequence of smooth families of symplectic, real analytic, near identity transformations*

$$\Phi_\omega^\nu : D(r_\nu, s_\nu) \to D(r_{\nu-1}, s_{\nu-1}),\ \omega\in O_\nu$$

such that

$$H_\nu \quad = \quad H_{\nu-1}\circ\Phi_\omega^\nu =: e_\nu(\omega,\varepsilon) + \langle\omega_\nu(\omega,\varepsilon), y\rangle + \frac{1}{2}\langle z, M_\nu(\omega,\varepsilon)z\rangle + \varepsilon^{\frac{1}{2}}P_\nu,$$

where

$$
\begin{aligned}
&|\partial_\omega^l\omega_\nu - \partial_\omega^l\omega_0|_{O_\nu} \le \gamma_0^{4m^2N}\mu_0^{\frac{1}{4}},\\
&|\partial_\omega^l M_\nu - \partial_\omega^l M_0|_{O_\nu} \le \gamma_0^{4m^2N}\mu_0^{\frac{1}{4}},\\
&|\partial_\omega^l P_\nu|_{D_\nu\times O_\nu} \le \gamma_\nu^{4m^2N}s_\nu\mu_\nu
\end{aligned}
$$

for all $|l|\le N$.

2) $O_\nu = \{\omega\in O_{\nu-1}: |\langle k,\omega_{\nu-1}\rangle| > \frac{\gamma_{\nu-1}}{|k|^\tau},\ |\det L_{1k,\nu-1}| > \frac{\gamma_{\nu-1}^{2m}}{|k|^{2m\tau}},\ |\det L_{2k,\nu-1}| > \frac{\gamma_{\nu-1}^{4m^2}}{|k|^{4m^2\tau}},$ $K_{\nu-1} < |k| \le K_\nu\}$.

3) *The Whitney extensions of*

$$\Psi^\nu =: \Phi_\omega^1\circ\Phi_\omega^2\circ\cdots\circ\Phi_\omega^\nu$$

converge C^N uniformly to a smooth family of symplectic maps Ψ^∞, on $D(\frac{r_0}{2},\frac{s_0}{2})\times O_\infty$, where

$$O_\infty = \bigcap_{\nu\ge 0} O_\nu,$$

such that

$$H_\nu = H_0\circ\Psi^{\nu-1} \to H_\infty =: H_0\circ\Psi^\infty = e_\infty + \langle\omega_\infty, y\rangle + \frac{1}{2}\langle z, M_\infty z\rangle + \varepsilon^{\frac{1}{2}}P_\infty$$

with $e_\infty = \lim_{\nu\to\infty} e_\nu$, $\omega_\infty = \lim_{\nu\to\infty}\omega_\nu$, $M_\infty = \lim_{\nu\to\infty} M_\nu$, $P_\infty = \lim_{\nu\to\infty} P_\nu$, *and moreover,*

$$\partial^j_{(y,z)} P_\infty|_{D(\frac{r_0}{2},0)\times O_\infty} = 0, \qquad |j| \le 2.$$

Proof. This is a special case of the iteration lemma and convergence theorem contained in [15], [16]. □

3.3. **Measure estimate.** We have the following estimate for the excluding measure of O_0 after infinite steps of KAM iterations.

Proposition 3.2. $|O_0 \setminus O_\infty| = O(\varepsilon^{\frac{12c_*}{N}})$, *where* $c_* = 4m^2\sum_{i=1}^a \tau_i(n_i - n_{i-1})$.

Proof. It follows from the same argument as that in [15, Subsection 4.2]. □

Proof of Main Theorem. Propostion 3.1 shows that, for each $0<\varepsilon \ll 1$ and $\omega \in O_\infty =: \mathcal{O}_\varepsilon$, $\mathbb{T}^n \times \{0\}\times\{0\}$ is an analytic, invariant, Diophantine torus of H_∞ with the frequency ω_∞ of Diophantine type (γ_∞, τ), where $\gamma_\infty = \lim_{\nu\to\infty}\gamma_\nu$. Moreover, these invariant n-tori form a Whitney smooth family.

By Propositions 2.2, 3.2, we have $|\mathcal{O}_0 \setminus \mathcal{O}_\varepsilon| \to 0$ as $\varepsilon \to 0$, which completes the proof. □

4. An Example

In this section, we will adopt an example from [2], concerning a five body problem involving Jupiter and its Galilean satellites, to illustrate the application of our Main Theorem. After certain averaging and reduction procedures, the associated Hamiltonian has the normal form

$$F = \langle \omega_{\mu,e}, y\rangle + \frac{1}{2}\langle M_{\mu,e}(\omega)z, z\rangle + F_{\rm rem}(x,y,z,\omega,\mu,e), \quad (x,y,z)\in \mathbb{T}^2\times\mathbb{R}^2\times\mathbb{R}^{10}, \tag{4.1}$$

where μ is associated with the mass ratio of satellites with respect to Jupiter and e is associated with the eccentricity of some lower dimensional obits so that $0<\mu\ll e\ll 1$, and $M_{\mu,e}$, whose entries are of orders $\sqrt{\mu}$, $\sqrt{\mu}e$, $\sqrt{\mu^3}/e^3$, consists of the Hessian Matrix of some Kepler terms plus some terms splitting from the perturbation so that the eigenvalues of $JM_{\mu,e}$ are non-zero and admit two different scales $\sqrt{\mu e}$ and μ/e. Following [1], the persistence of quasi-periodic, invariant 2-tori is shown for the equal scale case besides that eigenvalues of $JM_{\mu,e}$ are simple and the perturbation F_{rem} is of the order $O(\mu^L)+O(\mu e^L)$ for a sufficiently large L.

We would like to present a general persistence result of quasi-periodic, invariant 2-tori for (4.1) by allowing all possible lower, higher, and equal scales, multiple eigenvalues of $JM_{\mu,e}$, and not necessarily very high order of perturbation F_{rem}. To do so, we rewrite $\mu = e^k$, where $k = \frac{\ln\mu}{\ln e}$. Without loss of generality, we assume that k is a constant. Since $\mu \ll e$, we have $k\gg 1$. It follows that the orders of the eigenvalues of $JM_{\mu,e}$ become $e^{\frac{k+1}{2}}$, e^{k-1} and the orders of the entries of $M_{\mu,e}$ become $e^{\frac{k}{2}}$, $e^{\frac{k}{2}+1}$, $e^{\frac{3k}{2}-3}$. Since $k\gg 1$, these orders are positive and satisfy $\frac{k}{2} < \frac{k+1}{2} < \frac{k}{2}+1 < k-1 < \frac{3k}{2}-3$.

Now let $\varepsilon := e$ and rewrite the Hamiltonian (4.1) as

$$F = \langle \omega_\varepsilon, y\rangle + \frac{1}{2}\langle M_\varepsilon(\omega)z, z\rangle + F_{\rm rem}(x,y,z,\omega,\varepsilon), \quad (x,y,z) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^{10}, \tag{4.2}$$

where $\omega_\varepsilon = (\varepsilon^{\tau_1}\omega_1,\ \varepsilon^{\tau_2}\omega_2)^\top$ for some appropriate constants $\tau_1,\ \tau_2 > 0$, the entries m_{ij}'s of M_ε are of the order of $O(\varepsilon^{\bar{\chi}_{ij}})$'s for positive constants $\bar{\chi}_{ij} := \frac{k}{2},\ \frac{k}{2}+1,\ \frac{3k}{2}-3$, $1 \le i,j \le 10$, and the eigenvalues of JM_ε have the form

$$\varepsilon^{\chi_1}\lambda_1^1(\omega),\ \cdots,\ \varepsilon^{\chi_1}\lambda_{2m_1}^1(\omega),$$
$$\varepsilon^{\chi_2}\lambda_{2m_1+1}^2(\omega),\ \cdots,\ \varepsilon^{\chi_2}\lambda_{2m_2}^2(\omega)$$

with $m_0 := 0 < m_1 < m_2 := 5$, $\chi_1 := \frac{k+1}{2}$, $\chi_2 := k-1$. Depending on the values of τ_1, τ_2 in comparing with that of χ_1, χ_2, all cases of lower, higher, and equal scales are allowed in (4.2).

Let $\omega = (\omega_1,\ \omega_2)^\top$ vary in a bounded closed region $\mathcal{O} \subset \mathbb{R}^2$ and $\bar{m} := \max\{\tau_1,\ \tau_2,\ \chi_1,\ \chi_2,\ \bar{\chi}_{ij}\}$. When $F_{\rm rem} = 0$, we denote the family of unperturbed 2-tori by $T_{\omega,\varepsilon}$. Since $|\chi_2 - \chi_1| > 1$ as $k \gg 1$, an application of the Main Theorem together with its remark 3) yields the following result.

Corollary. *Suppose $F_{rem} = O(\varepsilon^{\bar{m}+1})$ and that the Hamiltonian* (4.2) *satisfies the non-resonant condition* A2)*. Then, as ε sufficiently small, there exist Cantor-like sets $\mathcal{O}_\varepsilon \subset \mathcal{O}$ with $|\mathcal{O}\setminus\mathcal{O}_\varepsilon| \to 0$, as $\varepsilon \to 0$, such that the unperturbed tori $T_{\omega,\varepsilon}$, $\omega \in \mathcal{O}_\varepsilon$, persist and give rise to a family of quasi-periodic, invariant* 2*-tori with slight deformed Diophantine frequencies.*

We remark that the above corollary improves the result in [2] in several ways as follows. Firstly, it applies to all cases of lower, higher and equal scales as already mentioned. Secondly, the least order of ε in F_{rem} is made precise according to the orders of the tangential and normal frequencies, whereas the order of ε in F_{rem} is only assumed to be sufficiently high in [2]. Finally, the corollary provides an additional information that the excluded measure tends to zero as $\varepsilon \to 0$.

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