QUASI-PERIODIC BREATHERS IN GRANULAR CHAINS WITH HERTZIAN CONTACT POTENTIAL

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ABSTRACT. In this paper, we study the existence and stability of quasi-periodic breathers in granular chains of coupled Duffing oscillators with Hertzian interaction potential which is of a finite smoothness. Using Jackson-Moser-Zehnder analytic approximation technique and KAM iterations, not only do we obtain the existence and linear stability of quasi-periodic breathers, but also we provide concrete estimates on the localization rate etc, depending on the smoothness order and the number of oscillating frequencies.

1. INTRODUCTION AND MAIN RESULT

Breathers or quasi-periodic breathers, i.e., time periodic or quasi-periodic, spatially localized solutions, as important dynamics subjects in describing coherent and localized structures or waves in physical applications, have been extensively studied in Hamiltonian chains or lattices including discrete nonlinear Schrödinger equations, Fermi-Pasta-Ulam chains, Frenkel-Kontorova lattices etc. We refer the readers to [2]-[6], [8]-[10], [14, 17, 19, 31] and references therein for modern development in the subjects.

Comparing with classical lattices mentioned above, a less studied case on breathers and quasi-periodic breathers is the granular chains

$$\ddot{x}_n + V_n'(x_n) = W'(x_{n+1} - x_n) - W'(x_n - x_{n-1}), \quad x_n \in \mathbb{R},\ n \in \mathbb{Z}, \tag{1.1}$$

where $V_n(x)$'s are the on-site potentials and

$$W_n(x) = (W_n^0(x) + o(|x|^{1+\alpha_n}))u_0(-x), \tag{1.2}$$

in which u_0 is the unit step function and $W_n^0(x) = \frac{1}{1+\alpha_n}|x|^{1+\alpha_n}$, $n \in \mathbb{Z}$ for a bounded sequence of positive real numbers $\{\alpha_n : n \in \mathbb{Z}\}$, are the Hertzian interaction potentials that describe the contact of finitely smooth non-conforming surfaces. As observed in [22], due to the high nonlinear character of the Hertzian potential, the existence and natures of breathers

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in granular chains are more subtle than that of classical chains or lattices for which pre-compressions provided by the onsite potential plays a crucial role. When (1.1) is a Newton's cradle chain, i.e, $\alpha_n \equiv \alpha$ for some $\alpha > 0$ and $V_n \equiv V$ is a harmonic potential describing identical spherical beads which are attached to linear pendula, the existence and properties of breathers have been well studied in [1, 20, 21, 24] both numerically and analytically. For the existence of quasi-periodic breathers in Newton's cradle chains, Hertzian interaction of non-identical harmonic oscillators need to be considered in order to generate multi-frequencies. In [16], the authors of the present paper studied this problem and rigorously showed the existence of quasi-periodic breathers in a parametrized family of Newton's cradle chains by treating a finite number of natural frequencies as parameters. This work uses both KAM method and the Jackson-Moser-Zehnder (JMZ) analytic approximation technique in order to encounter finite smoothness of the Hertzian potential (we note that, while [16] deals with a fixed fundamental part $W^0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}$ of Hertzian potentials, its main result remain valid for general Hertzian potentials (1.2)).

In the present work, we will explore the general existence and stability of quasi-periodic breathers for non-parametrized granular chains. While our method will work for general (even finitely smooth) anharmonic onsite potentials, we will choose the Duffing-like potentials

$$V_n(x) = \frac{\beta_n^2 x^2}{2} + \frac{\eta_n x^4}{4}, \qquad n \in \mathbb{Z}, \tag{1.3}$$

where $\{\beta_n > 0\}$, $\{\eta_n \neq 0\}$ are given bounded sequences of real numbers, to present the result and carry out the analysis, simply because both the frequencies of quasi-periodic breathers and the required minimal smoothness order

$$\alpha = \inf_{n\in\mathbb{Z}}\{\alpha_n\}$$

of the Hertzian potentials for the existence will not only depend on the number of frequencies under consideration but also on the non-harmonic part of the onsite potential. More precisely, we will consider the following granular chain

$$\ddot{x}_n + \beta_n^2 x_n + \eta_n x_n^3 = W_n'(x_{n+1} - x_n) - W_n'(x_n - x_{n-1}), \quad x_n \in \mathbb{R},\ n \in \mathbb{Z}, \tag{1.4}$$

where β_n's and η_n's are as in (1.3) and W_n's are the Hertzian interaction potentials given in (1.2) which are only of the class $\mathcal{C}^{1+\alpha_n}$ in general. We note that the granular chain (1.4) forms a Hamiltonian lattice with the Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{2}\dot{x}_n^2 + \frac{\beta_n^2}{2}x_n^2 + \frac{\eta_n}{4}x_n^4 + W_n(x_{n+1} - x_n).$$

Our main result states as follows.

Main Theorem. *Consider* (1.4) *and assume that* $\alpha = \inf_{n\in\mathbb{Z}}\{\alpha_n\} > \alpha_* := 6b + 24$ *for a fixed integer* $b \geq 1$. *Let a lattice set* $\mathcal{I} = \{i_1, \cdots, i_b\} \subset \mathbb{Z}$ *and* *a bounded close region* $\mathcal{O} \subseteq \mathbb{R}_+^b$ *be given. Then there exist real numbers* $0 < \varepsilon_0 = \varepsilon_0(b, \alpha) \ll 1$, $2 < \mathfrak{s}_* = \mathfrak{s}_*(b, \alpha) < \alpha + 1$, *a family of Cantor set* $\mathcal{O}_\varepsilon \subset \mathcal{O}$, $0 < \varepsilon \le \varepsilon_0$, *satisfying* $\mathrm{meas}(\mathcal{O} \setminus \mathcal{O}_\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *and a Whitney smooth map* $\omega : \mathcal{O}_\varepsilon \to \mathbb{R}^b$ *satisfying* $\omega(\xi) = (\beta_{i_1}, \cdots, \beta_{i_b}) + \frac{3\varepsilon^2}{4}(\frac{\eta_{i_1}}{\beta_{i_1}^2}\xi_{i_1}, \cdots, \frac{\eta_{i_b}}{\beta_{i_b}^2}\xi_{i_b}) + o(\varepsilon^2)$ *as* $\varepsilon \to 0$, *such that for each* $\xi = (\xi_{i1}, \cdots, \xi_{i_b}) \in \mathcal{O}_\varepsilon$, (1.4) *admits a linearly stable, quasi-periodic breather* $x(t) = (x_n(t))_{n\in\mathbb{Z}}$ *with the frequency vector* $\omega(\xi)$ *which satisfies the following properties:*

(1) *For each* $n \in \mathbb{Z}$, $x_n(t) = X_n(\omega(\xi)t)$, *where* $X_n \in \mathcal{C}^{\mathfrak{s}}(\mathbb{T}^b, \mathbb{R})$ *for any* $2 \le \mathfrak{s} < \mathfrak{s}_*$;
(2) $\sum_{n\in\mathbb{Z}} |x_n(0)|\langle n\rangle^{1+\alpha} \le \varepsilon$, *where* $\langle n\rangle = \max\{1, |n|\}$;
(3) $x_n(t) \sim \frac{1}{|n|^{1+\alpha}}$ *as* $|n| \gg 1$ *uniformly in* t.

When $b = 1$, the Main Theorem gives the existence and stability of (periodic) breathers. In fact, in this case, as no small divisors are involved in the KAM iterations, $\mathcal{O} \equiv \mathcal{O}_\varepsilon$ and the lower bound of the smoothness order α in the Main Theorem can be any real number greater than 1. However, differing from those shown in [22], the (periodic) breathers obtained from the Main Theorem above are still of small amplitudes.

Quasi-periodic motions in non-analytic Hamiltonian systems have received considerable attentions. For finitely smooth, finite-dimensional Hamiltonian systems, KAM theory has been well developed in [25]-[28], [32, 33] for nearly integrable cases to show the existence of full dimensional, quasi-periodic, invariant tori, and in [13] for partially nearly integrable cases to show the existence of lower dimensional, quasi-periodic, invariant tori (see also [29] for the case of response solutions in finitely smooth, quasi-periodically forced oscillators). For non-analytic Hamiltonian PDEs including various nonlinear Schrödinger equations, KdV equations, and nonlinear wave equations, a Nash-Moser approach to KAM theory is recently developed to show the existence of lower dimensional, quasi-periodic, invariant tori (see [7, 11, 12, 15, 23] and references therein). As remarked in [16], though the Nash-Moser method is an important (and more flexible) technique in dealing with quasi-periodic motions in finitely smooth, infinite dimensional Hamiltonian systems, it does not seem to apply to the problem of quasi-periodic breathers in Hamiltonian lattices to capture sharp localization rates.

The Main Theorem above will be proved using the approach of [16] by conducting a Hamiltonian normal form reduction, applying the Jackson-Moser-Zehnder analytic approximation technique, and performing KAM iterations. However, the non-harmonic nature of the onsite Duffing potential (1.3) leads to a more complicated normal form which requires more delicate KAM iterations comparing to those in [16]. Several key ingredients of the proof is summarized as follows. The normal form reduction will be conducted by one step KAM transformation which not only introduces frequency parameters but also separates the perturbation into a higher order real analytic part and a finitely smooth part. The analytic approximations to the finitely smooth part will be made using the JMZ approximation approach in [16] that refines the finite dimensional JMZ approximation result (e.g. [13, Lemma 2.1]) by giving an explicit error bound of the approximations in terms of the order of smoothness and dimensions of domains of the approximate functions. We note that such a refined approximation lemma is necessary in infinite dimension simply because KAM iterations require the analytic approximations of truncations of perturbations at each KAM step whose domains, though being finite-dimensional, will have increased dimensions along the iterations. The KAM iterations then apply only to the real analytic parts and the real analytic approximation parts of the truncations. During the KAM iterations, the loss of regularity of perturbations is compensated by shrinking the complex domains of the angle variables and the increase of dimensions of the truncated domains are controlled by a finite number as a result of the short range coupling of the granular chains.

We note that due to the consideration of small amplitude quasi-periodic breathers, our proof and result only depend on the minimal smoothness order of the Hertzian potentials W_n's at the origin rather than their particular forms. The Main theorem remain valid under the weaker condition $\alpha_i \neq 0$ for $i \in \mathcal{I}$ is assumed, i.e, α_i for some or all $i \notin \mathcal{I}$ are allowed to be 0. However, if $\alpha_i = 0$ for some $i \in \mathcal{I}$, then the special forms and the "on-off" forcing nature of the Hertzian potentials at these sites will play crucial roles for the possible existence of quasi-periodic breathers oscillating at these sites. We will leave this problem to a future study.

The rest of the paper is organized as follows. Section 2 is a preliminary section in which we introduce terminologies and symbols to be used in the rest of the paper and also give a refined JMZ analytic approximation lemma (Lemma 2.1) with concrete error estimates on the smoothness and dimension. In Section 3, we construct a symplectic transformation and establish analytic approximations of the finite smooth part of the new perturbation to derive a normal form for the Hamiltonian (3.7). In Section 4, for the approximated Hamiltonian, we sketch our KAM scheme by describing abstract KAM steps along with the construction of the symplectic transformations. Section 5 is devoted to the proof of the Main Theorem by providing all necessary estimates and showing the convergence and measure estimates.

2. Preliminary

In this section, we define some notions to be used in the rest of the paper and re-formulate the JMZ analytic approximation lemma in [16] for the present application.

2.1. **Notions.** Let $\mathcal{I} = \{i_1, \cdots, i_b\} \subset \mathbb{Z}$ be given as in the Main Theorem and denote $\mathbb{Z}_1 = \mathbb{Z} \setminus \{j \in \mathbb{Z} : j \in \mathcal{I}\}$.

For simplicity, symbol $|\cdot|$ will be used to denote the absolute value of a complex number and the norm of a vector space supported on both $\mathbb{Z}$ and

$\mathbb{Z}_1$. However, if an integer vector is considered, then $|\cdot|$ is specified as the l^1-norm.

For each non-negative integer k, let d_k be the cardinality of the set

$$\mathcal{B}_k := \{n \in \mathbb{Z}_1 : |n-i| \le k,\ i \in \mathcal{I}\}.$$

We note that $d_k \le 2kb$, $d_k - d_{k-1} \le 2b$, for any $k = 1, 2, \cdots$.

For each complex sequence $w = (w_n)_{n\in\mathbb{Z}_1}$, denote $\bar{w} = (\bar{w}_n)_{n\in\mathbb{Z}_1}$ as its complex conjugate. Then

$$\ell := \{w = (w_n)_{n\in\mathbb{Z}_1} : \|w\|_\ell := \sum_{n\in\mathbb{Z}_1} |w_n|\langle n\rangle^{\alpha+1} < \infty\},$$

where $\langle n\rangle = \max\{1, |n|\}$, $n \in \mathbb{Z}$, is a Banach space with the norm $\|\cdot\|_\ell$. For any matrix $A = (a_{ij})_{m\times n}$, we use the norm

$$\|A\| = \max\{\sup_{1\le j\le n} \sum_{1\le i\le m} |a_{ij}|, \sup_{1\le i\le m} \sum_{1\le j\le n} |a_{ij}|\}.$$

For a given compact set $\mathcal{O}$ in the parameter space $\mathbb{R}^b$, we use the short notions $|\cdot|_{\mathcal{O}}$ or $\|\cdot\|_{\mathcal{O}}$ to denote the $\mathcal{C}^1$-norm of a function on $\mathcal{O}$ in the sense of Whitney. Let $f \in \mathcal{C}^{l,1}(X\times\mathcal{O})$, i.e., f is a $\mathcal{C}^l$ function on a Banach space X depending on a parameter in $\mathcal{O}$ $\mathcal{C}^1$-smoothly, where $l = p+\delta \in \mathbb{R}^+$ with $p \in \mathbb{Z}_+$ and $\delta \in [0, 1)$. We use the norm

$$|f|_{\mathcal{C}^l(X),\mathcal{O}} := \sup_{x\in X} \sum_{0\le|\eta|\le p-1} |\partial_x^\eta f(x,\beta)|_{\mathcal{O}} + \sup_{x,y\in X} \sum_{\substack{|\eta|=p-1\\ 0<|x-y|<1}} \frac{|\partial_x^\eta f(x) - \partial_x^\eta f(y)|_{\mathcal{O}}}{|x-y|^\delta}$$

in $\mathcal{C}^{l,1}(X\times\mathcal{O})$. For short, we simply denote $|f|_{X,\mathcal{O}} = |f|_{\mathcal{C}^0(X),\mathcal{O}}$.

Consider a complex neighborhood

$$D(r,s) = \{(\theta, I, w, \bar{w}) : |\mathrm{Im}\theta| < r, |I| < s^2, \|w\|_\ell < s, \|\bar{w}\|_\ell < s\} \tag{2.5}$$

of the d-torus $T^* =: \mathbb{T}^b\times\{I=0\}\times\{w=0\}\times\{\bar{w}=0\}$ in $\mathbb{C}^b/2\pi\mathbb{Z}^b\times\mathbb{C}^b\times\ell\times\ell$, where $|\mathrm{Im}\theta| = \sum_{j\in\mathcal{I}} |\mathrm{Im}\theta_j|\langle j\rangle^{\alpha+1}$, $|I| = \sum_{j\in\mathcal{I}} |I_j|\langle j\rangle^{\alpha+1}$. For any $\mathcal{C}^1$ function $F = F(\theta, I, w, \bar{w}; \xi)$ on $D(r,s)\times\mathcal{O}$, we denote its associated, parametrized Hamiltonian vector field by

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{w_n}\}_{n\in\mathbb{Z}_1}, \{-\mathrm{i}F_{\bar{w}_n}\}_{n\in\mathbb{Z}_1})$$

and define the following weighted norm

$$\begin{aligned}\|X_F\|_{D(r,s),\mathcal{O}} &\equiv \|F_I\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D(r,s),\mathcal{O}}\\ &+ \frac{1}{s}(\sum_{n\in\mathbb{Z}_1} \|F_{w_n}\|_{D(r,s),\mathcal{O}} + \sum_{n\in\mathbb{Z}_1} \|F_{\bar{w}_n}\|_{D(r,s),\mathcal{O}}),\end{aligned}$$

where $\|\partial^\iota F\|_{D(r,s),\mathcal{O}} := \sup_{D(r,s)} \|\partial^\iota F\|_{\mathcal{O}}$ for $|\iota| = 1$.

2.2. **A refined analytic approximation lemma.** For given $0<\rho,\sigma<1$ and $k\in\mathbb{Z}_+$, we consider the following subsets of $\mathbb{T}^b\times\mathbb{R}^b\times\mathrm{Re}(\ell)\times\mathrm{Re}(\ell)$:

$$D_\rho=\left\{(x,y)=((x_n),(y_n)):\begin{array}{cc} x_n\in\mathbb{T},n\in\mathcal{I}; & \sum\limits_{n\in\mathbb{Z}_1}|x_n|\langle n\rangle^{\alpha+1}\le\rho \\ \sum\limits_{n\in\mathcal{I}}|y_n|\langle n\rangle^{\alpha+1}\le\rho; & \sum\limits_{n\in\mathbb{Z}_1}|y_n|\langle n\rangle^{\alpha+1}\le\rho\end{array}\right\},$$

$$\Omega^k=\left\{(x,y)=((x_n),(y_n)):\begin{array}{c} x_n=y_n=0 \quad \text{if}|n-i|>k, i\in\mathcal{I},n\in\mathbb{Z}_1 \\ \|\varsigma=(x,y)\|:=\|x\|+\|y\|<\infty\end{array}\right\},$$

where $\|x\|:=\sum\limits_{n\in\mathbb{Z}}|x_n|\langle n\rangle^{\alpha+1}$ and $\|y\|:=\sum\limits_{n\in\mathbb{Z}}|y_n|\langle n\rangle^{\alpha+1}$. We also consider the following complex strips of Δ_σ:

$$\Delta_\sigma^k=\left\{(x,y):\begin{array}{l}(\mathrm{Re}x,\mathrm{Re}y)\in D_\sigma,\\ \sum\limits_{j\in\mathcal{I}}|\mathrm{Im}x_j|\langle j\rangle^{\alpha+1}\le\sigma;\ \sum\limits_{j\in\mathcal{B}_k}|\mathrm{Im}x_j|\langle j\rangle^{\alpha+1}\le\sigma\\ \sum\limits_{j\in\mathcal{I}}|\mathrm{Im}y_j|\langle j\rangle^{\alpha+1}\le\sigma;\ \sum\limits_{j\in\mathcal{B}_k}|\mathrm{Im}y_j|\langle j\rangle^{\alpha+1}\le\sigma\\ |\mathrm{Im}x_j|=|\mathrm{Im}y_j|=0, \text{ if } |j-i|\ge k+1, i\in\mathcal{I}, j\in\mathbb{Z}_1\end{array}\right\}.$$

For each $k\in\mathbb{Z}_+$, let ϕ_k denote a radially symmetric, $\mathcal{C}^\infty(\Omega^k)$ cut-off function satisfying

$$\phi_k(\varsigma)=1,\quad\forall\|\varsigma\|\le1,\quad \textit{and}\quad \phi_k(\varsigma)=0,\quad\forall\|\varsigma\|\ge2.$$

and let $K_k=\hat{\phi_k}$ be the Fourier transform of ϕ_k.

The following JMZ analytic approximation lemma will be used in the rest of the paper.

Lemma 2.1. *Let D_ρ, $\mathcal{O}$, and Ω^k, $k\in\mathbb{Z}_+$ be domains defined in the above for some given $\rho>0$. Consider a function $f\in\mathcal{C}^{l,1}(D_\rho\times\mathcal{O})$ for some $l>2$, and, for each $\sigma>0$ and $k\in\mathbb{Z}_+$, denote*

$$f_{\sigma,k}(\psi,\xi):=K_k^\sigma*f^k(\psi,\xi)=\sigma^{-(b+d_k)}\int_{\Omega^k}K_k(\frac{\psi-y}{\sigma})f^k(y,\xi)dy,$$

where $f^k=f|_{\Omega_k\times\mathcal{O}}$. Then there exists a constant $c_1\ge1$, depending only on l and b, such that for each $\sigma>0$ and $k\in\mathbb{Z}_+$, $f_{\sigma,k}(\psi)$ is a real analytic function on Δ_σ^k satisfying

$$\begin{aligned}&\left|f_{\sigma,k}(\psi)-\sum_{|\varrho|\le l-1}\frac{\partial^\varrho f^k(\mathrm{Re}\psi)}{\varrho!}(\mathrm{iIm}\psi)^\varrho\right|_{\mathcal{C}^1(\Delta_\sigma^k)\times\mathcal{O}}\\ \le& c_1|f|_{\mathcal{C}^l(D_\rho),\mathcal{O}}\sigma^{l-1}(b+d_k)^{l+1},\end{aligned}$$

where $\varrho=((\varrho_j)_{j\in\mathcal{B}_k},\{0\})$ with $(\varrho_j)_{j\in\mathcal{B}_k}\in\mathbb{N}^{b+d_k}$ and $\varrho!:=\prod\limits_{j\in\mathcal{B}_k}\varrho_j!$. Moreover, if there exists a constant $c_0>1$, depending only on l and b, such that

$$\left|f^{k+1}-f^k\right|_{\mathcal{C}^l(D_\rho),\mathcal{O}}\le c_0$$

for all $k \in \mathbb{Z}_+$, then for any decreasing sequence $\{\sigma_k\}$ of positive numbers, there exists a constant $c_2 > 1$, depending only on l and b, such that

$$\left|f_{\sigma_{k+1},k+1} - f_{\sigma_k,k}\right|_{\mathcal{C}^1(\Delta^{k+1}_{\sigma_{k+1}}),\mathcal{O}} \le c_2 |f|_{\mathcal{C}^l(D_\rho),\mathcal{O}} \sigma_k^{l-1}(b+d_{k+1})^{l+1}.$$

Proof. This lemma is proved in [16] for the special case that $\mathcal{I} = \{n \in \mathbb{Z} : |n| \le b\}$ for which $\#\mathcal{I} = 2b+1$ and $d_k = 2k$ for all $k \in \mathbb{N}$. The general case considered here follows from the exact same arguments with respective estimates modified accordingly.

□

3. Normal form reduction

For simplicity and without loss of generality, we only prove the Main Theorem for the special case that $\beta_n = \eta_n \equiv 1$, $\alpha_n \equiv \alpha$, and

$$W_n(x) \equiv W(x) =: \frac{1}{1+\alpha}|x|^{1+\alpha}u_0(-x), \tag{3.6}$$

where $u_0(x)$ is the unit step function. In other word, we will consider the simplified Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{2}y_n^2 + \frac{1}{2}x_n^2 + \frac{1}{4}x_n^4 + W(x_{n+1} - x_n), \tag{3.7}$$

where W is as in (3.6) and $y_n = \dot{x}_n$, $n \in \mathbb{Z}$.

In this section, we introduce the standard action-angle-normal variables to the Hamiltonian (3.7) and convert it into a normal form.

3.1. **Partial Birkhoff normal form.** Let $\varepsilon > 0$ be a small parameter and consider the re-scalings $y_n, x_n \to \varepsilon y_n, \varepsilon x_n$, $n \in \mathbb{Z}$, $H \to \varepsilon^{-2}H$ to the Hamiltonian (3.7). Then the re-scaled Hamiltonian reads

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{2}y_n^2 + \frac{1}{2}x_n^2 + \varepsilon^2\frac{1}{4}x_n^4 + \varepsilon^{\alpha-1}W(x_{n+1} - x_n).$$

With the complex coordinates

$$w_n = \frac{1}{\sqrt{2}}(x_n + \mathrm{i}y_n), \qquad \bar{w}_n = \frac{1}{\sqrt{2}}(x_n - \mathrm{i}y_n), \quad n \in \mathbb{Z}, \tag{3.8}$$

the corresponding Hamiltonian equation becomes

$$w_t = \mathrm{i}\frac{\partial H}{\partial \bar{w}},$$

where

$$H = H(w, \bar{w}) = N + P_1 + P_2,$$

with

$$\begin{aligned} N &= \sum_{n\in\mathbb{Z}} w_n\bar{w}_n, \\ P_1 &= \varepsilon^2 \sum_{n\in\mathbb{Z}} \frac{1}{16}w_n^4 + \frac{1}{4}w_n^3\bar{w}_n + \frac{3}{8}w_n^2\bar{w}_n^2 + \frac{1}{4}w_n\bar{w}_n^3 + \frac{1}{16}\bar{w}_n^4, \end{aligned}$$

$$P_2 \;=\; \varepsilon^{\alpha-1}\sum_{n\in\mathbb{Z}} W(w_{n+1}+\bar{w}_{n+1}-w_n-\bar{w}_n).$$

For fixed $\rho>0$, we will restrict P_1, P_2 on the domain

$$\breve{D}_{4\rho} =: \left\{(w,\bar{w}) : \sum_n |w_n|\langle n\rangle^{\alpha+1}\le 4\rho, \sum_n |\bar{w}_n|\langle n\rangle^{\alpha+1}\le 4\rho\right\}.$$

Consider the Hamiltonian function

$$F=\sum_n \frac{-\mathrm{i}\varepsilon^2}{16}w_n^4+\frac{-\mathrm{i}\varepsilon^2}{8}w_n^3\bar{w}_n+\frac{\mathrm{i}\varepsilon^2}{8}w_n\bar{w}_n^3+\frac{\mathrm{i}\varepsilon^2}{16}\bar{w}_n^4.$$

Then, by making ε further small if necessary, the time-1 map $\Gamma=\phi_F^1$ of the flow generated from the Hamiltonian vector field X_F is a symplectic transformation on $\breve{D}_{4\rho}$, under which the transformed Hamiltonian reads

$$\begin{aligned}\tilde{H}=H\circ\phi_F^1 &= H+\{H,F\}+\int_0^1(1-t)\{\{H,F\},F\}\circ\phi_F^t dt\\ &:= \tilde{N}+\tilde{P}_1+\tilde{P}_2\end{aligned}$$

with

$$\begin{aligned}\tilde{N} &= N+\{N,F\}=\sum_{n\in\mathbb{Z}}w_n\bar{w}_n+\frac{3\varepsilon^2}{8}\sum_{n\in\mathbb{Z}}w_n^2\bar{w}_n^2,\\ \tilde{P}_1 &= P_1+\{P_1,F\}+\int_0^1(1-t)\{\{N+P_1,F\},F\}\circ\phi_F^t dt\\ &=: \varepsilon^4\sum_{n\in\mathbb{Z}}\tilde{P}_{1,n}(w_n,\bar{w}_n),\\ P_2 &= P_2+\{P_2,F\}+\int_0^1(1-t)\{\{P_2,F\},F\}\circ\phi_F^t dt\\ &=: \varepsilon^{\alpha-1}\sum_{n\in\mathbb{Z}}\tilde{P}_{2,n}(w_n,\bar{w}_n,w_{n+1},\bar{w}_{n+1}),\end{aligned}$$

where, for each $n\in\mathbb{Z}$, $\tilde{P}_{1,n}$ is real analytic and $\tilde{P}_{2,n}$ is of the class $\mathcal{C}^{\alpha+1}$ on $\breve{D}_{4\rho}$.

Let $\mathcal{I}$ be the lattice set given in the Main Theorem and fix a bounded closed region $\mathcal{O}\subset\mathbb{R}_+^b$ as the parameter set. We convert the variables $w_j,\bar{w}_j$, $j\in\mathcal{I}$ into parametrized action-angle variables $I=(I_j)_{j\in\mathcal{I}}\in\mathbb{R}^b$, $\theta=(\theta_j)_{j\in\mathcal{I}}\in\mathbb{T}^b$ via

$$w_j=\sqrt{\xi_j+I_j}e^{\mathrm{i}\theta_j},\quad \bar{w}_j=\sqrt{\xi_j+I_j}e^{-\mathrm{i}\theta_j},\ j\in\mathcal{I},$$

where $\xi=(\xi_j)_{j\in\mathcal{I}}\in\mathcal{O}$ is the parameter. For simplicity, we still use the same symbol $\breve{D}_{4\rho}$ to denote its converted set

$$\left\{(\theta,I,w,\bar{w}):\theta\in\mathbb{T}^b,|I|\le 4\rho,\|w\|_\ell\le 4\rho,\|\bar{w}\|_\ell\le 4\rho\right\}$$

in term of the action-angle-normal variables $I \in \mathbb{R}^b$, $\theta \in \mathbb{T}^b$, and $w = (w_j)_{j\in\mathbb{Z}_1} \in \ell$. Then the Hamiltonian $\tilde{H}$ is written in the parametrized form

$$H^* = N^* + P_1^* + P_2^*, \tag{3.9}$$

where

$$N^* = e^*(\xi) + \langle \omega, I\rangle + \sum_{n\in\mathbb{Z}_1} w_n \bar{w}_n$$

with

$$\omega = \omega(\xi) = (\omega_j)_{j\in\mathcal{I}} = (1 + \frac{3\varepsilon^2}{4}\xi_j)_{j\in\mathcal{I}},$$

and P_1^*, P_2^* have the form

$$\begin{aligned} P_1^* &= \frac{3}{8}\varepsilon^2|I|^2 + \varepsilon^4 \sum_{n\in\mathbb{Z}_1} P_*^{1,n}(\theta, I, w_n, \bar{w}_n; \xi), \\ P_2^* &= \varepsilon^{\alpha-1} \sum_{n\in\mathbb{Z}_1} P_*^{2,n}(\theta, I, w_n, \bar{w}_n, w_{n+1}, \bar{w}_{n+1}; \xi), \end{aligned}$$

with $P_*^{1,n}$ being real analytic in $(\theta, I, w, \bar{w}) \in \breve{D}_{4\rho}$, $\mathcal{C}^1$ in $\xi \in \mathcal{O}$ and $P_*^{2,n} \in \mathcal{C}^{\alpha+1,1}(\breve{D}_{4\rho} \times \mathcal{O})$ for all $n \in \mathbb{Z}_1$.

3.2. **Analytic approximations.** We use Lemma 2.1 to construct analytic approximations of P_2^* in (3.9) as follows. By (3.8), we can express P_2^* as a function of $(\theta, I, \{x_n\}_{n\in\mathbb{Z}_1}, \{y_n\}_{n\in\mathbb{Z}_1}; \xi)$ in $\breve{D}_{2\rho} \times \mathcal{O}$, which we still denote as P_2^* for simplicity.

For each $\nu = 1, 2, \cdots$, let

$$P_2^{*\nu} \quad = \quad P_2^*|_{\Omega^\nu}.$$

Then, by tracing back the definition of P_2^* through $\varepsilon^{\alpha-1}\sum_n W(x_{n+1} - x_n)$ and noticing the fact that $\#\{\mathcal{B}_{\nu+1} \setminus \mathcal{B}_\nu\} \le 2b$, it is easy to see that there is a constant $c_0 = c_0(\alpha, b) > 0$ such that

$$|P_2^{*\nu+1} - P_2^{*\nu}|_{\mathcal{C}^{\alpha-1}(\breve{D}_{2\rho}),\mathcal{O}} \le \varepsilon^{\alpha-2} \sum_{n\in\mathcal{B}_{\nu+1}\setminus\mathcal{B}_\nu} |\hat{P}_{2,n}^*|_{\mathcal{C}^{\alpha-1}(\breve{D}_{2\rho}),\mathcal{O}} < c_0.$$

For a given monotonically decreasing sequence $\sigma_\nu \downarrow 0$ of positive numbers (to be specified later), we have by Lemma 2.1 that

$$P_{\sigma_\nu} := K_\nu^{2\sigma_\nu} * P_2^{*\nu}$$

is a sequence of real analytic approximations of $P_2^{*\nu}$ for each $\nu = 1, 2, \cdots$ Let

$$z_n = \frac{x_n}{\sqrt{2}} + \mathrm{i}\frac{y_n}{\sqrt{2}}, \quad \bar{z}_n = \frac{x_n}{\sqrt{2}} - \mathrm{i}\frac{y_n}{\sqrt{2}}, \;\; x_n, y_n \in \mathbb{C},$$

$n \in \mathbb{Z}_1$, and,

$$\Delta^{\nu}_{\sigma_\nu} = \left\{ \psi := (\theta, I, z, \bar{z}) : \begin{array}{c} \mathrm{Re}(\theta, I, z, \bar{z}) \in \breve{D}_\rho \\ \sum_{j \in \mathcal{I}} |\mathrm{Im}\theta_j| \langle j \rangle^{\alpha+1} \le \sigma_\nu \\ \sum_{j \in \mathcal{I}} |\mathrm{Im} I_j| \langle j \rangle^{\alpha+1} \le \sigma_\nu \\ \sum_{n \in \mathcal{B}_\nu} |\mathrm{Im} z_n| \langle n \rangle^{\alpha+1} \le \sigma_\nu \\ \sum_{n \in \mathcal{B}_\nu} |\mathrm{Im} \bar{z}_n| \langle n \rangle^{\alpha+1} \le \sigma_\nu \\ |\mathrm{Im} z_n| = |\mathrm{Im} \bar{z}_n| = 0, n \in \mathbb{Z}_1 \setminus \mathcal{B}_\nu \end{array} \right\},$$

$\nu = 1, 2, \cdots$. We note that for each $n \in \mathbb{Z}^n$, $w_n = z_n$ when $x_n, y_n \in \mathbb{R}$, and, for each $\nu = 1, 2, \cdots$, each P_{σ_ν} can be expressed as a real analytic function in $(\theta, I, z, \bar{z}) \in \Delta^{\nu}_{\sigma_\nu}$ which is $\mathcal{C}^1$ in $\xi \in \mathcal{O}$. For simplicity, we still use the same symbol to denote the function P_{σ_ν}, for each ν, under the new variables.

It follows from Lemma 2.1 that there exists a constant $c_2 = c_2(\alpha, b) > 0$ such that

$$\left|P_{\sigma_{\nu+1}} - P_{\sigma_\nu}\right|_{\mathcal{C}^1(\Delta^{\nu+1}_{\sigma_{\nu+1}}), \mathcal{O}} \le c_2 |P_2^*|_{\mathcal{C}^{\alpha+1}(\breve{D}_{4\rho}), \mathcal{O}} \sigma_\nu^{\alpha} (b + d_{\nu+1})^{\alpha+2}$$

for all $\nu = 1, 2, \cdots$.

For each $\nu = 1, 2, \cdots$, let

$$P_{*,\nu}(\theta, I, z, \bar{z}; \xi) = P_1^*(\theta, I, \{z_n\}_{n \in \mathcal{B}_\nu}, \{0\}, \{\bar{z}_n\}_{n \in \mathcal{B}_\nu}, \{0\}; \xi).$$

Then for each $\nu = 1, 2, \cdots$,

$$H^{\nu}_{*} = N^* + P_{*,\nu} + P_{\sigma_\nu}, \tag{3.10}$$

is a function which is real analytic in $(\theta, I, z, \bar{z}) \in \Delta^{\nu}_{\sigma_\nu}$ and $\mathcal{C}^1$ in $\xi \in \mathcal{O}$. Moreover, the sequence $\{H^{\nu}_{*}\}$ converges to H_* under $|\cdot|_{\mathcal{C}^1(\breve{D}_\rho), \mathcal{O}}$-norm as $\nu \to \infty$.

Remark 3.1. *Due to the fact that P_2^* depends on an infinite dimensional vector, it is difficult to directly construct the analytic approximation for it. In this article we approximates the truncation $P_2^{*\nu}$ of P_2^* to construct the sequence $\{P_{\sigma_\nu}\}$. As $\nu \to \infty$, while the dimension of the variable in P_{σ_ν} increases, its analytical radius σ_ν rapidly decreases, ensuring the convergence of P_{σ_ν} to P_2^*.*

4. KAM Scheme

For $\nu = 1, 2, \cdots$, we will use KAM method to construct a sequence of symplectic transformations Φ_ν, on appropriate phase and parameter domains, to transform the approximated Hamiltonians $H^{\nu+1}_{*}$ in (3.10), so that, as $\nu \to \infty$, the limit transform Φ_∞ of Φ_ν will transform the Hamiltonian H_* to the limit Hamiltonian H_∞ which yields the desired quasi-periodic invariant b-tori.

The transformations Φ_ν, $\nu = 1, 2, \cdots$, will be constructed as follows. Let

$$P_1 := P_{*,1} + P_{\sigma_1}, \quad N_1 := N^*.$$

Starting from $\nu = 1$, we will use KAM iterations to obtain parametrized sequences of updated partially integrable Hamiltonians N_ν, of the from

$$N_\nu = e_\nu(\xi) + \langle \omega_\nu(\xi), I\rangle + \langle A_\nu(\xi) Z_\nu, \bar{Z}_\nu\rangle + \sum_{n\in\mathbb{Z}_1\setminus\mathcal{B}_{\nu-1}} z_n\bar{z}_n, \quad Z_\nu = (z_n)_{n\in\mathcal{B}_{\nu-1}},$$

perturbations P_ν, and symplectic transformations ϕ_ν, such that

$$(N_\nu + P_{*,\nu} + P_{\sigma_\nu}) \circ \phi_\nu = N_{\nu+1} + P_{\nu+1}, \tag{4.11}$$

for all $\nu = 1, 2, \cdots$. Having constructed all ϕ_ν's, we obtain

$$\Phi_\nu := \phi_1 \circ \cdots \circ \phi_\nu, \quad \nu = 1, 2, \cdots$$

as desired. Let

$$P'_1 = P_1, \tag{4.12}$$

$$P'_\nu = P_\nu + (P_{*,\nu} - P_{*,\nu-1}) \circ \Phi_{\nu-1} + (P_{\sigma_\nu} - P_{\sigma_{\nu-1}}) \circ \Phi_{\nu-1}, \tag{4.13}$$

$\nu = 2, 3, \cdots$. Then it follows from an induction argument that (4.11) can be re-written as

$$H'_\nu \circ \phi_\nu = H_{\nu+1}, \tag{4.14}$$

where

$$H'_\nu = N_\nu + P'_\nu,$$
$$H_\nu = N_\nu + P_\nu,$$

$\nu = 1, 2, \cdots$. Further induction arguments using (4.14) yields that

$$H_*^{\nu+1} \circ \Phi_\nu = H'_{\nu+1} \tag{4.15}$$

for all $\nu = 1, 2, \cdots$.

Let $\nu \to \infty$ in (4.15). Because the sequence $\{H_*^\nu\}$ converges to H_* as shown in Section 3.2, we achieve the main goal of the iteration that

$$H^* \circ \Phi_\infty = H'_\infty = N_\infty = e_\infty + \langle \omega_\infty, I\rangle + \langle A_\infty Z_\infty, \bar{Z}_\infty\rangle.$$

4.1. **Iteration sequences and Technical Lemmas.** Denote $\otimes$ as the tensor product of matrixes. The following two technical lemmas on linear, algebraic systems will be needed in solving the homological equation and estimating the transformations in our KAM iterations.

Lemma 4.1. *Let A, B be $n \times n$, $m \times m$ real symmetric matrices respectively and C, X be $n \times m$ matrices. Then the matrix equation*

$$AX - XB = C$$

is uniquely solvable if and only if $I_m \otimes A - B \otimes I_n$ is invertible. Moreover, if the equation is uniquely solvable, then its solution X satisfies

$$\|X\| \le \|(I_m \otimes A - B \otimes I_n)^{-1}\| \|C\|.$$

Proof. See e.g. [30]. $\square$

Lemma 4.2. *Let $A(\xi)$, $\xi \in \mathcal{O}$, be a $\mathcal{C}^1$, matrix-valued function which is invertible for each $\xi \in \mathcal{O}$. Then*

$$\|\partial_\xi A^{-1}\| \le \|A^{-1}\|^2 \|\partial_\xi A\|.$$

Proof. See [16, Lemma 4.3]. □

The following lemma will be needed in estimating the new perturbations from our KAM iterations.

Lemma 4.3. *Consider $\mathcal{C}^1$ families $\{G_1, G_2\}_{\xi\in\mathcal{O}}$ of real analytic Hamiltonians on $D(r,s)$ for some $r,s>0$. If there are constants $\varepsilon', \varepsilon'' > 0$ such that*

$$\|X_{G_1}\|_{D(r,s),\mathcal{O}} < \varepsilon', \ \|X_{G_2}\|_{D(r,s),\mathcal{O}} < \varepsilon'',$$

then there is a constant $c_* > 0$ such that

$$\|X_{\{G_1,G_2\}}\|_{D(r-\sigma,\eta s),\mathcal{O}} < c_*\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon''$$

for all $0<\sigma<r, 0<\eta\ll 1$.

Proof. See [18, Lemma 7.3]. □

We now specify the phase spaces and frequency domains for the above Hamiltonian sequences. To do so, we let $0 < q < 1 < \kappa < \frac{4}{3}$ be fixed constants satisfying

$$\frac{\kappa^2}{4-3\kappa} = \frac{\alpha-1}{\alpha_*-1}, \quad q = \frac{4-3\kappa}{2\tau+4}.$$

We choose Diophantine constants $\gamma = \varepsilon^{1-\kappa}$ and $\tau = b+2$, and let $\gamma_* = \varepsilon^2\gamma = \varepsilon^{3-\kappa}$ and $\alpha_* := 3(2\tau+4)+1$. To define the iteration sequences, we set $\varepsilon_1 = \varepsilon^{2+\kappa}$, $r_1 = \frac{\varepsilon}{2}$ and choose $s_1 > 0$, by making ε sufficiently small, such that

$$\frac{\varepsilon_1^q}{2} > s_1 \ge \varepsilon_1^\kappa.$$

For $\nu = 1,2,\cdots$, we recursively define the following sequences:

$$\varepsilon_{\nu+1} = \varepsilon_\nu^\kappa, \ \sigma_\nu = \varepsilon_\nu^q, \ \eta_\nu = \varepsilon_\nu^{\kappa-1}, \ r_\nu = \frac{1}{2}\sigma_\nu, \ s_{\nu+1} = \eta_\nu s_\nu.$$

We can make ε further small if necessary so that $D(r_1,s_1) \subset \breve{D}_{4\rho}$ and

$$s_\nu \ge \varepsilon_\nu^\kappa, \quad s_\nu < \sigma_\nu, \quad 2^{\nu+3}\varepsilon_\nu^{3(\kappa-1)} \ll 1,$$

for all $\nu = 1,2,\cdots$. Having defined the sequences r_ν, s_ν, we specify the phase domains of the Hamiltonians H_ν, H'_ν as $D(r_\nu, s_\nu)$, for $\nu = 1,2,\cdots$, respectively. To specify the frequency domains for these Hamiltonians, we note that, for each $\nu = 1,2,\cdots$, the normal form N_ν can be rewritten as

$$\begin{aligned} N_\nu &= e_\nu + \langle\omega_\nu, I\rangle + \langle A_\nu Z_\nu, \bar{Z}_\nu\rangle + \sum_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}} z_n\bar{z}_n + \sum_{n\in\mathbb{Z}_1\setminus\mathcal{B}_\nu} z_n\bar{z}_n \\ &:= e_\nu + \langle\omega_\nu, I\rangle + \langle \tilde{A}_\nu Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle + \sum_{n\in\mathbb{Z}_1\setminus\mathcal{B}_\nu} z_n\bar{z}_n, \end{aligned} \tag{4.16}$$

where $Z_{\nu+1} = (z_n)_{n\in\mathcal{B}_\nu}$,

$$\tilde{A}_\nu = \begin{pmatrix} A_\nu & 0 \\ 0 & I_{d_\nu - d_{\nu-1}} \end{pmatrix},$$

and $I_{d_\nu - d_{\nu-1}}$ is the $(d_\nu - d_{\nu-1}) \times (d_\nu - d_{\nu-1})$ identity matrix. We note that $e_\nu, \omega_\nu, A_\nu, \tilde{A}_\nu$ are functions of the parameter $\xi \in \mathcal{O}$. The dimension of the tangential frequencies ω_ν, which determine the low-dimensional torus, is always fixed at b. However the normal ones associated with A_ν will have changing dimensions because A_ν has dimension $d_{\nu-1} \times d_{\nu-1}$ which increases as ν does.

Define

$$K_\nu = \frac{\kappa^\nu}{\sigma_\nu},$$

$\nu = 1, 2, \cdots$. Let $\mathcal{O}_1 := \mathcal{O}$, and, for each $\nu = 1, 2, \cdots$, define

$$\mathcal{O}_{\nu+1} = \left\{ \xi \in \mathcal{O}_\nu : \begin{array}{l} |\langle k, \omega_\nu\rangle^{-1}| \le \frac{|K_{\nu+1}|^\tau}{\gamma_*}, \quad k \neq 0; \\ \|(\langle k, \omega_\nu\rangle I \pm \tilde{A}_\nu)^{-1}\| \le \frac{(d_\nu)^2 |K_{\nu+1}|^\tau}{\gamma_*}; \\ \|(\langle k, \omega_\nu\rangle I + I \otimes \tilde{A}_\nu + \tilde{A}_\nu \otimes I)^{-1}\| \le \frac{(d_\nu)^4 |K_{\nu+1}|^\tau}{\gamma_*}; \\ \|(\langle k, \omega_\nu\rangle I + I \otimes \tilde{A}_\nu - \tilde{A}_\nu \otimes I)^{-1}\| \le \frac{(d_\nu)^4 |K_{\nu+1}|^\tau}{\gamma_*}; \\ |k| \le K_{\nu+1} \end{array} \right\}.$$

For each $\nu = 1, 2, \cdots$, $\mathcal{O}_\nu$ will be the frequency domain for both H_ν and H'_ν.

4.2. **Homological equations.** Let $\nu = 1, 2, \cdots$ be given. We would like to find, by solving a homological equation, the transformation $\phi_\nu : D(r_\nu, s_\nu) \times \mathcal{O}_\nu \to D(r_{\nu+1}, s_{\nu+1})$ in (4.14) which is real analytic and symplectic in phase variables and of the class $\mathcal{C}^1$ in the parameter.

Expanding P'_ν into the Taylor-Fourier series, we have

$$\begin{aligned} P'_\nu &= \sum_{k,l,\varrho,\varrho'} \bar{P}_{k,l,\varrho,\varrho'}(\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^l Z_\nu^\varrho \bar{Z}_\nu^{\varrho'} \\ &\quad + \sum_{\substack{\varrho_n + \varrho'_m \ge 1 \\ n,m\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}, k,l,\varrho,\varrho'}} \bar{P}_{k,l,\varrho,\varrho',\varrho_n,\varrho'_m}(\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^l Z_\nu^\varrho \bar{Z}_\nu^{\varrho'} z_n^{\varrho_n} \bar{z}_m^{\varrho'_m}, \end{aligned}$$

where $Z_\nu = (z_n)_{n\in\mathcal{B}_{\nu-1}}$, $k \in \mathbb{Z}^b$, $l \in \mathbb{N}^b$, and indices ϱ, ϱ' run over the set of non-negative integer vectors $(\cdots, \varrho_n, \cdots)_{n\in\mathcal{B}_{\nu-1}}$, $(\cdots, \varrho'_n, \cdots)_{n\in\mathcal{B}_{\nu-1}}$.

Consider the following truncation of P'_ν:

$$\begin{aligned} R_\nu &= \sum_{\substack{|k|\le K_{\nu+1} \\ 2|l|+|\varrho|+|\varrho'|\le 2}} \bar{P}_{k,l,\varrho,\varrho'}(\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^l Z_\nu^\varrho \bar{Z}_\nu^{\varrho'} \\ &\quad + \sum_{\substack{\varrho_n + \varrho'_m + |\varrho| + |\varrho'| \le 2 \\ n,m\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}, |k|\le K_{\nu+1}, l=0}} \bar{P}_{k,l,\varrho,\varrho',\varrho_n,\varrho'_m}(\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^l Z_\nu^\varrho \bar{Z}_\nu^{\varrho'} z_n^{\varrho_n} \bar{z}_m^{\varrho'_m}. \end{aligned}$$

which could be rewriten as

$$\begin{aligned}
R_\nu &= \sum_{\substack{|l|\le 1\\ |k|\le K_{\nu+1}}} \bar P_{kl00}(\xi)e^{\mathrm{i}\langle k,\theta\rangle} I^l + \sum_{|k|\le K_{\nu+1}} (\langle \bar P_k^{10}(\xi), Z_\nu\rangle + \langle \bar P_k^{01}(\xi), \bar Z_\nu\rangle)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+ \sum_{|k|\le K_{\nu+1}} (\langle \bar P_k^{20}(\xi) Z_\nu, Z_\nu\rangle + \langle \bar P_k^{11}(\xi) Z_\nu, \bar Z_\nu\rangle + \langle \bar P_k^{02}(\xi)\bar Z_\nu, \bar Z_\nu\rangle)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+ \sum_{\substack{|k|\le K_{\nu+1}\\ n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}} (\langle \bar P_{kn}^{20}(\xi) z_n, Z_\nu\rangle + \langle \bar P_{kn}^{02}(\xi)\bar z_n, \bar Z_\nu\rangle)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+ \sum_{\substack{|k|\le K_{\nu+1}\\ n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}} (\langle \bar P_{kn}^{11}(\xi) z_n, \bar Z_\nu\rangle + \langle \bar P_{kn}^{11}(\xi)\bar z_n, Z_\nu\rangle)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+ \sum_{\substack{|k|\le K_{\nu+1}\\ m,n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}} (\bar P_{knm}^{20}(\xi) z_n z_m + \bar P_{knm}^{11}(\xi) z_n\bar z_m + \bar P_{knm}^{02}(\xi)\bar z_n\bar z_m)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+ \sum_{\substack{|k|\le K_{\nu+1}\\ n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}} (\bar P_{kn}^{10}(\xi) z_n + \bar P_{kn}^{01}(\xi)\bar z_n)e^{\mathrm{i}\langle k,\theta\rangle}\\
&= R_0 + R_1 + R_2,
\end{aligned}$$

where

$$\begin{aligned}
R_0 &= \sum_{\substack{|l|\le 1\\ |k|\le K_{\nu+1}}} \bar P_{kl00}(\xi)e^{\mathrm{i}\langle k,\theta\rangle} I^l,\\
R_1 &= \sum_{|k|\le K_{\nu+1}} (\langle R_k^{10}(\xi), Z_{\nu+1}\rangle + \langle R_k^{01}(\xi), \bar Z_{\nu+1}\rangle)e^{\mathrm{i}\langle k,\theta\rangle},\\
R_2 &= \sum_{|k|\le K_{\nu+1}} (\langle R_k^{20}(\xi) Z_{\nu+1}, Z_{\nu+1}\rangle + \langle R_k^{11}(\xi) Z_{\nu+1}, \bar Z_{\nu+1}\rangle\\
&\quad + \langle R_k^{02}(\xi)\bar Z_{\nu+1}, \bar Z_{\nu+1}\rangle)e^{\mathrm{i}\langle k,\theta\rangle}
\end{aligned}$$

with

$$\begin{aligned}
R_k^{10} &= \begin{pmatrix} \bar P_k^{10}\\ \bar P_{kn}^{10}\end{pmatrix}_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}, \quad R_k^{01} = \begin{pmatrix} \bar P_k^{01}\\ \bar P_{kn}^{01}\end{pmatrix}_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}},\\
R_k^{20} &= \begin{pmatrix} \bar P_k^{20} & \frac12 \bar P_{kn}^{20}\\ \frac12 (\bar P_{kn}^{20})^{\mathrm{T}} & (\bar P_{knm}^{20})_{m\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}\end{pmatrix}_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}},\\
R_k^{02} &= \begin{pmatrix} \bar P_k^{02} & \frac12 \bar P_{kn}^{02}\\ \frac12 (\bar P_{kn}^{02})^{\mathrm{T}} & (\bar P_{knm}^{02})_{m\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}\end{pmatrix}_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}},\\
R_k^{11} &= \begin{pmatrix} \bar P_k^{11} & \bar P_{kn}^{11}\\ \bar P_{kn}^{11} & (\bar P_{knm}^{11})_{m\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}\end{pmatrix}_{n\in\mathcal{B}_\nu\setminus\mathcal{B}_{\nu-1}}.
\end{aligned}$$

When $|1| = |\varrho'| = 0, |\varrho| = 1$, we define

$$\bar P_k^{10}(\xi) := \bar P_{k,l,\varrho,\varrho'}(\xi)|_{|1|=|\varrho'|=0,|\varrho|=1},$$

similarly when $|1| = |\varrho'| = |\varrho| = \varrho'_m = 0, \varrho'_n = 1$, let

$$\bar{P}^{10}_{kn} := \bar{P}_{k,l,\varrho,\varrho',\varrho_n,\varrho'_m}(\xi)|_{|1|=|\varrho'|=|\varrho|=\varrho'_m=0,\varrho'_n=1}.$$

Other symbols of R_ν are defined analogously.

We will construct the transformation ϕ_ν as the time-1 map $\phi^1_{F_\nu}$ of the Hamiltonian flow $\phi^t_{F_\nu}$ generated from the generating function F_ν that satisfies the following homological equation

$$\{N_\nu, F\} + R - \langle R\rangle = 0. \tag{4.17}$$

with resonant paprt $\langle R\rangle := \sum_{|l|\le 1} \bar{P}_{0l00} I^l - \langle R^{11}_0 Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle$.

For the sake of simplicity, we will omit the subscript ν of R_ν and F_ν in the following text.

Consider generating function F of the form

$$F = F_0 + F_1 + F_2, \tag{4.18}$$

where

$$\begin{aligned}
F_0 &= \sum_{\substack{|l|\le 1\\ |k|\le K_{\nu+1}}} f_{kl00}(\beta) e^{\mathrm{i}\langle k,\theta\rangle} I^l,\\
F_1 &= \sum_{\substack{|k|\le K_{\nu+1}\\ n\in\mathcal{B}_\nu}} (f^{k10}_n z_n + f^{k01}_n \bar{z}_n) e^{\mathrm{i}\langle k,\theta\rangle}\\
&:= \sum_{|k|\le K_{\nu+1}} (\langle F^{10}_k, Z_{\nu+1}\rangle + \langle F^{01}_k, \bar{Z}_{\nu+1}\rangle) e^{\mathrm{i}\langle k,\theta\rangle},\\
F_2 &= \sum_{\substack{|k|\le K_{\nu+1}\\ m,n\in\mathcal{B}_\nu}} (f^{k20}_{nm} z_n z_m + f^{k02}_{nm} \bar{z}_n \bar{z}_m) e^{\mathrm{i}\langle k,\theta\rangle}\\
&\quad + \sum_{\substack{0<|k|\le K_{\nu+1}\\ m,n\in\mathcal{B}_\nu}} f^{k11}_{nm} z_n \bar{z}_m e^{\mathrm{i}\langle k,\theta\rangle}\\
&:= \sum_{|k|\le K_{\nu+1}} (\langle F^{20}_k Z_{\nu+1}, Z_{\nu+1}\rangle + \langle F^{02}_k \bar{Z}_{\nu+1}, \bar{Z}_{\nu+1}\rangle) e^{\mathrm{i}\langle k,\theta\rangle}\\
&\quad + \sum_{0<|k|\le K_{\nu+1}} \langle F^{11}_k Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle}.
\end{aligned}$$

Then we have the following result.

Lemma 4.4. *With F being given in the form* (4.18)*, the homological equation* (4.17) *is equivalent to*

$$(4.19)\quad \begin{array}{l} (\langle k,\omega_\nu\rangle) f_{kl00} = -\mathrm{i}\bar{P}_{kl00},\ \ 0<|k|\le K_{\nu+1},\ |l|\le 1,\\ (\langle k,\omega_\nu\rangle I - \tilde{A}_\nu))F_k^{10} = -\mathrm{i}R_k^{10},\ \ 0<|k|\le K_{\nu+1},\\ (\langle k,\omega_\nu\rangle I + \tilde{A}_\nu)F_k^{01} = -\mathrm{i}R_k^{01},\ \ 0<|k|\le K_{\nu+1},\\ (\langle k,\omega_\nu\rangle I - I\otimes\tilde{A}_\nu - \tilde{A}_\nu\otimes I)F_k^{20} = -\mathrm{i}R_k^{20},\ \ 0<|k|\le K_{\nu+1},\\ (\langle k,\omega\rangle I + I\otimes\tilde{A}_\nu + \tilde{A}_\nu\otimes I)F_k^{02} = -\mathrm{i}R_k^{02},\ \ 0<|k|\le K_{\nu+1},\\ (\langle k,\omega\rangle I + I\otimes\tilde{A}_\nu - \tilde{A}_\nu\otimes I)F_k^{11} = -\mathrm{i}R_k^{11},\ \ 0<|k|\le K_{\nu+1}. \end{array}$$

Consequently, the homological equation (4.17) *is solvable on* $D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$ *to obtain a unique solution F of the form* (4.18)*.*

Proof. By substituting F in (4.18) into (4.17) and comparing coefficients, we see that (4.17) is decomposed into the following equations

$$(4.20)\quad \begin{array}{c} \{N_\nu, F_0\} + R_0 = \sum_{|l|\le 1} \bar{P}_{0l00} I^l,\\ \{N_\nu, F_1\} + R_1 = 0,\\ \{N_\nu, F_2\} + R_2 = \langle R_0^{11} Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle. \end{array}$$

that (4.19) and (4.20) coincide.

It is clear that the respective first equations of (4.19) and (4.20) coincide with each other. To explore the remaining two equations of (4.20), we note that

$$\begin{aligned} \{N_\nu, F_1\} &= \mathrm{i}\sum_{|k|\le K_{\nu+1}} \langle(\langle k,\omega_\nu\rangle I - \tilde{A}_\nu)F_k^{10}, Z_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle}\\ &\quad + \langle(\langle k,\omega_\nu\rangle I + \tilde{A}_\nu)F_k^{01}, \bar{Z}_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned}$$

and

$$\begin{aligned} \{N_\nu, F_2\} &= \mathrm{i}\sum_{|k|\le K_{\nu+1}} \langle(\langle k,\omega_\nu\rangle F_k^{20} - \tilde{A}_\nu F_k^{20} - F_k^{20}\tilde{A}_\nu)Z_{\nu+1}, Z_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle}\\ &\quad + \mathrm{i}\sum_{|k|\le K_{\nu+1}} \langle(\langle k,\omega_\nu\rangle F_k^{02} + \tilde{A}_\nu F_k^{02} + F_k^{02}\tilde{A}_\nu)\bar{Z}_{\nu+1}, \bar{Z}_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle}\\ &\quad + \mathrm{i}\sum_{0<|k|\le K_{\nu+1}} \langle(\langle k,\omega_\nu\rangle F_k^{11} + \tilde{A}_\nu F_k^{11} - F_k^{11}\tilde{A}_\nu)Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle e^{\mathrm{i}\langle k,\theta\rangle}\ , \end{aligned}$$

from which we see clearly that the remaining two equations of (4.20) yield the remaining equations of (4.19).

The solvability of (4.20) on $D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$ simply follows from (4.19) and Lemma 4.1. □

Now let

$$(4.21)\quad \begin{aligned} N_{\nu+1} &= N_\nu + \langle R\rangle,\\ P_{\nu+1} &= \int_0^1 (1-t)\{\{N_{\nu+1}, F\}, F\} \circ \phi_F^t dt + \int_0^1 \{R, F\} \circ \phi_F^t dt \end{aligned}$$

$$+ \quad (P'_\nu - R)\circ\phi_F^1.$$

Then

$$\begin{aligned}
H'_\nu\circ\phi_\nu &= H'_\nu\circ\phi_F^1 = (N_\nu + R)\circ\phi_F^1 + (P'_\nu - R)\circ\phi_F^1\\
&= N_\nu + \{N_\nu, F\} + R + \int_0^1 (1-t)\{\{N_\nu, F\}, F\}\circ\phi_F^t dt\\
&\quad + \int_0^1 \{R, F\}\circ\phi_F^t dt + (P'_\nu - R)\circ\phi_F^1\\
&= N_{\nu+1} + P_{\nu+1} + (\{N_\nu, F\} + R - \sum_{|l|\le 1}\bar{P}_{0l00}I^l - \langle R_0^{11}Z_{\nu+1}, \bar{Z}_{\nu+1}\rangle)\\
&= N_{\nu+1} + P_{\nu+1}.
\end{aligned}$$

We note that, at each νth KAM step, the transformation ϕ_ν is constructed to be such that, among all the normal variables, $(z_n, \bar{z}_n)_{n\in\mathbb{Z}_1\setminus\mathcal{B}_{\nu+1}}$ do not participate in the iteration and the ones participating in the iteration have their dimensions increased from $d_{\nu-1}$ to d_ν.

5. Proof of Main Theorem

In this section, we prove the Main Theorem by showing the convergence of KAM iterations and conducting measure estimates. To do so, we will provide all necessary estimates on the transformations and iterated Hamiltonians.

Throughout the section, we use the symbol "$\preccurlyeq$" to denote "$\le$" up to a constant multiple that is independent of the KAM iterations.

5.1. **Estimates on the transformations.** Let $\nu\in\mathbb{N}$ be given. We make the following regularity assumption:

(A) $\|X_{P'_\nu}\|_{D(r_\nu,s_\nu),\mathcal{O}_\nu}$, $\|X_{P_\nu}\|_{D(r_\nu,s_\nu),\mathcal{O}_\nu} \preccurlyeq \varepsilon_\nu$.

We note that this assumption implies the decay property of the coefficients of P'_ν, i.e.,

$$|\bar{P}_{k,l,\varrho,\varrho'}|_{\mathcal{O}_\nu} \le \varepsilon_\nu e^{-|k|r_\nu}, \quad |\bar{P}_{k,l,\varrho,\varrho',\varrho_n,\varrho'_m}|_{\mathcal{O}_\nu} \le \varepsilon_\nu e^{-|k|r_\nu},$$

for all ν, k and indexes $l, \varrho, \varrho', \varrho_n, \varrho'_m$.

Below, we give some estimates of the transformation $\phi_\nu = \phi_F^1$ at the νth KAM step, where F is the solution of the homological equation (4.17) as stated in Lemma 4.4 and ϕ_F^1 is the time-1 map of the Hamiltonian flow ϕ_F^t generated from the Hamiltonian vector field X_F.

Lemma 5.1. *Denote $D_i = D(r_\nu, \frac{i}{4}s_\nu)$, $i = 1, 2, 3, 4$. If* **(A)** *holds, then*

$$\|X_F\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} \preccurlyeq \gamma_*^{-2}r_\nu^{-(2\tau+1)}(d_\nu)^{10}\varepsilon_\nu. \tag{5.22}$$

Proof. By Lemmas 4.2, 4.4, we have

$$\begin{aligned}
|f_{kl00}|_{\mathcal{O}_{\nu+1}} &\le \gamma_*{}^{-2}(K_{\nu+1})^{2\tau+1}|\bar{P}_{kl00}|_{\mathcal{O}_{\nu+1}}, \quad |k|\ne 0, |l|\le 1,\\
\|F_k^{10}\|_{\mathcal{O}_{\nu+1}} &\le \gamma_*{}^{-2}(K_{\nu+1})^{2\tau+1}(d_\nu)^5\|R_k^{10}\|_{\mathcal{O}_{\nu+1}}, \ 0<|k|\le K_{\nu+1},
\end{aligned}$$

$$\begin{aligned}
\|F_k^{01}\|_{\mathcal{O}_{\nu+1}} &\leq {\gamma_*}^{-2}(K_{\nu+1})^{2\tau+1}(d_\nu)^5\|R_k^{01}\|_{\mathcal{O}_{\nu+1}},\ 0<|k|\leq K_{\nu+1},\\
\|F_k^{20}\|_{\mathcal{O}_{\nu+1}} &\leq {\gamma_*}^{-2}(K_{\nu+1})^{2\tau+1}(d_\nu)^{10}\|R_k^{20}\|_{\mathcal{O}_{\nu+1}},\ 0<|k|\leq K_{\nu+1},\\
\|F_k^{11}\|_{\mathcal{O}_{\nu+1}} &\leq {\gamma_*}^{-2}(K_{\nu+1})^{2\tau+1}(d_\nu)^{10}\|R_k^{11}\|_{\mathcal{O}_{\nu+1}},\quad |k|\neq 0,\\
\|F_k^{02}\|_{\mathcal{O}_{\nu+1}} &\leq {\gamma_*}^{-2}(K_{\nu+1})^{2\tau+1}(d_\nu)^{10}\|R_k^{02}\|_{\mathcal{O}_{\nu+1}}\ 0<|k|\leq K_{\nu+1}.
\end{aligned}$$

Applying **(A)** and the standard Cauchy estimate, we have

$$\|X_R\|_{D_3,\mathcal{O}_{\nu+1}} \preccurlyeq \varepsilon_\nu.$$

It follows that

$$\begin{aligned}
\frac{1}{s_\nu^2}\|F_\theta\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} &\leq \frac{1}{s_\nu^2}\Big(\sum_{|k|\leq K_{\nu+1},|l|\leq 1}|f_{kl00}|\cdot s_\nu^{2|l|}\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\\
&+\sum_{|k|\leq K_{\nu+1}}\|F_k^{10}\|\cdot\|Z_{\nu+1}\|\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\\
&+\sum_{|k|\leq K_{\nu+1}}\|F_k^{01}\|\cdot\|\bar{Z}_{\nu+1}\|\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\\
&+\sum_{|k|\leq K_{\nu+1}}\|F_k^{20}\|\cdot\|Z_{\nu+1}\|^2\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\\
&+\sum_{0<|k|\leq K_{\nu+1}}\|F_k^{11}\|\cdot\|Z_{\nu+1}\|\cdot\|\bar{Z}_{\nu+1}\|\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\\
&+\sum_{|k|\leq K_{\nu+1}}\|F_k^{02}\|\cdot\|\bar{Z}_{\nu+1}\|^2\cdot|k|\cdot e^{|k|r_\nu}\cdot e^{-|k|\frac{r_\nu}{2}}\Big)\\
&\preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\|X_R\|_{D_3,\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\varepsilon_\nu.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|F_I\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} &= \sum_{\substack{|l|=1\\0<|k|\leq K_{\nu+1}}}|f_{kl00}|e^{|k|r_\nu} \preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\varepsilon_\nu,\\
\frac{1}{s_\nu}\|F_{Z_{\nu+1}}\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} &\preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\varepsilon_\nu,\\
\frac{1}{s_\nu}\|F_{\bar{Z}_{\nu+1}}\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} &\preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\varepsilon_\nu.
\end{aligned}$$

(5.22) now easily follows. □

Lemma 5.2. *Assume* **(A)** *and denote* $\eta_\nu = \varepsilon_\nu^{\kappa-1}$ *and* $D_{i\eta_\nu} = D(\frac{i}{4}r_\nu, i\eta s_\nu)$, $i=1,2,3,4$. *Then, for* ε *sufficiently small,* $\forall t\in[-1,1]$,

$$\phi_F^t : D_{2\eta_\nu}\times\mathcal{O}_{\nu+1}\to D_{3\eta_\nu}$$

is a $\mathcal{C}^1$ *family of real analytic, invertible, symplectic transformations such that*

$$\|\phi_F^t - id\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}},\ \|D\phi_F^t - Id\|_{D_{1\eta_\nu},\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2}r_\nu^{-(2\tau+2)}(d_\nu)^{10}\varepsilon_\nu.$$

Proof. We note that F is a polynomial of degree 1 in I and degree 2 in z. By (2.6), (5.22) and Cauchy estimates, we have

$$\|D^m F\|_{D_2,\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2} r_\nu^{-(2\tau+2)} (d_\nu)^{10} \varepsilon_\nu, \ \ m = 0, 1, 2. \tag{5.23}$$

Note that

$$\begin{aligned} \phi_F^t &= id + \int_0^t X_F \circ \phi_F^s \, ds, \\ D\phi_F^t &= Id + \int_0^t (DX_F) D\phi_F^s \, ds = Id + \int_0^t J(D^2F) D\phi_F^s \, ds, \end{aligned}$$

where J denotes the standard symplectic matrix. It follows easily from (5.23) that

$$\phi_F^t : D_{2\eta_\nu} \times \mathcal{O}_{\nu+1} \to D_{3\eta_\nu}$$

and

$$\begin{aligned} \|\phi_F^t - id\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} &\le \|X_F\|_{D(\frac{r_\nu}{2},\frac{3}{4}s_\nu),\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2} r_\nu^{-(2\tau+1)} (d_\nu)^{10} \varepsilon_\nu, \\ \|D\phi_F^t - Id\|_{D_{1\eta_\nu},\mathcal{O}_{\nu+1}} &\le 2\|D^2F\|_{D_2,\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2} r_\nu^{-(2\tau+2)} (d_\nu)^{10} \varepsilon_\nu. \end{aligned}$$

In particular, as ε sufficiently small, ϕ_F^t is a near identity, hence invertible transformation for each $\xi \in \mathcal{O}_{\nu+1}$. □

5.2. **Estimates on iterated Hamiltonians.** We would like to estimate the iterated Hamiltonians

$$\begin{aligned} H_\nu &= N_\nu + P_\nu, \\ H'_\nu &= N_\nu + P'_\nu, \end{aligned}$$

$\nu = 1, 2, \cdots$. According to (4.16), the estimates for N_ν's amount to that for sequences e_ν, ω_ν, A_ν, $\tilde{A}_\nu$, $\nu = 1, 2, \cdots$, for which we have the following result.

Lemma 5.3. *For any $\nu = 1, 2, \cdots$, if* **(A)** *holds, then*

$$|e_{\nu+1} - e_\nu|_{\mathcal{O}_{\nu+1}}, \ |\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_{\nu+1}}, \ \|A_{\nu+1} - \tilde{A}_\nu\|_{\mathcal{O}_{\nu+1}} \preccurlyeq \varepsilon_\nu.$$

Proof. For each $\nu = 1, 2, \cdots$, it follows from (4.21) that

$$\begin{aligned} e_{\nu+1} - e_\nu &= \bar{P}_{0000}, \\ \omega_{\nu+1} - \omega_\nu &= (\bar{P}_{0l00})_{|l|=1}, \\ A_{\nu+1} - \tilde{A}_\nu &= R_0^{11}, \end{aligned}$$

where $\bar{P}_{0000}$, $\{\bar{P}_{0l00}\}_{|l|=1}$, and R_0^{11} are defined in Section 4.2 from the truncation R of P'_ν. The lemma simply follows from **(A)** and Cauchy estimates. □

For estimations on the perturbations P_ν, P'_ν, we have the following result.

Lemma 5.4. *The condition* **(A)** *holds for all $\nu = 1, 2, \cdots$.*

Proof. We proceed with the proof by induction. For $\nu = 1$, we have by Lemma 2.1 that

$$\begin{aligned}|P_{\sigma_1}|_{C^1(\Delta^1_{\sigma_1}),\mathcal{O}} &\leq \left| P_{\sigma_1} - \sum_{|\varrho|\preccurlyeq\alpha-1} \frac{\partial^\varrho P_2^{*,1}(\mathrm{Re}\, X)}{\varrho!}(\mathrm{iIm}X)^\varrho \right|_{C^1(\Delta^1_{\sigma_1}),\mathcal{O}} \\ &\quad + \left| \sum_{|\varrho|\leq\alpha-1} \frac{\partial^\varrho P_2^{*,1}(\mathrm{Re}\, X)}{\varrho!}(\mathrm{iIm}X)^\varrho \right|_{C^1(\Delta^1_{\sigma_1}),\mathcal{O}} \\ &\quad \preccurlyeq \varepsilon^{\alpha-2}\sigma_1^{\alpha-1} + \varepsilon^{\alpha-2} \preccurlyeq \varepsilon^{\alpha-2}.\end{aligned}$$

It follows that

$$\|X_{P_1}\|_{D(r_1,s_1),\mathcal{O}_1} = \|X_{P_1'}\|_{D(r_1,s_1),\mathcal{O}_1} \preccurlyeq \varepsilon^3 + s_1^{-2}\varepsilon^{\alpha-2} \preccurlyeq \varepsilon_1.$$

Suppose that **(A)** holds for some $\nu \geq 1$. We note that

$$P_{\nu+1} = \int_0^1 \{R(t), F\} \circ \phi_F^t dt + (P_\nu' - R) \circ \phi_F^1,$$

and hence,

$$X_{P_{\nu+1}} = \int_0^1 (\phi_F^t)^* X_{\{R(t),F\}} dt + (\phi_F^1)^* X_{(P_\nu'-R)},$$

where $R(t) = (1-t)(N_{\nu+1} - N_\nu) + tR$ with R being the truncation defined in Section 4,2, F is the solution of the homological equation (4.17), and P_ν' is as in (4.12) and (4.13).

By Lemma 5.1, we have

$$\|D\phi_F^t\|_{D_{1\eta_\nu},\mathcal{O}_{\nu+1}} \leq 1 + \|D\phi_F^t - Id\|_{D_{1\eta_\nu},\mathcal{O}_{\nu+1}} \leq 2, \ |t| \leq 1,$$

and by Lemma 4.3, we have

$$\|X_{\{R(t),F\}}\|_{D_{2\eta_\nu},\mathcal{O}_{\nu+1}} \preccurlyeq {\gamma_*}^{-2} r_\nu^{-(2\tau+2)} (d_\nu)^{10} \eta_\nu^{-2} \varepsilon_\nu^2.$$

Making use of the truncation order of R from P_ν', it is not hard to check that

$$\|X_{(P_\nu'-R)}\|_{D_{2\eta_\nu}\mathcal{O}_{\nu+1}} \preccurlyeq \eta_\nu \varepsilon_\nu.$$

We thus have

$$\begin{aligned}\|X_{P_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} &\preccurlyeq \eta_\nu\varepsilon_\nu + \gamma^{-2} r_\nu^{-(2\tau+2)} (d_\nu)^{10} \eta_\nu^{-2} \varepsilon_\nu^2 \\ &\preccurlyeq {\gamma_*}^{-2} (d_\nu)^{10} \varepsilon_\nu^{4-2\kappa-(2\tau+2)q}.\end{aligned}$$

Recall that $q = \frac{4-3\kappa}{2\tau+4}$. Since $4 - 2\kappa - (2\tau+2)q > \kappa$, we have

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} \preccurlyeq \varepsilon_{\nu+1}.$$

To estimate $X_{P_{\nu+1}'}$, we note from (4.13) that

$$P_{\nu+1}' - P_{\nu+1} = (P_{\sigma_{\nu+1}} - P_{\sigma_\nu}) \circ \Phi_\nu + (P_{1,\nu+1}^* - P_{1,\nu}^*) \circ \Phi_\nu,$$

where $\Phi_\nu = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_\nu$.

Since

$$2^{\nu+1}r_\nu^{-2\tau-2}\varepsilon_\nu = 2^{\nu+1}r_\nu^2\varepsilon_\nu^{3(\kappa-1)} \le 2^{\nu+3}\varepsilon_\nu^{3(\kappa-1)} \le 1,$$

we have by Lemma 5.1 that

$$\begin{aligned}
&\|D\Phi_\nu\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\\
&= \|(\partial\phi_1\circ\phi_2\circ\cdots\circ\phi_\nu)(\partial\phi_2\circ\phi_3\circ\cdots\circ\phi_\nu)\cdots(\partial\phi_\nu)\|_{D(r_{\nu+1},s_{\nu+1})}\\
&\preccurlyeq \prod_{j=1}^{\nu}[1+\gamma_*{}^{-2}(r_j)^{-2\tau-2}(2d_{j-1}^j)^2\varepsilon_j] \le \prod_{j=1}^{\nu}(1+\frac{1}{2^{j+1}}) \le 2.
\end{aligned} \tag{5.24}$$

Since $s_{\nu+1}\le\sigma_{\nu+1}$, it follows from Lemma 2.1 that

$$\begin{aligned}
&\|D[(P_{\sigma_{\nu+1}}-P_{\sigma_\nu})\circ\Phi_\nu]+D[(P^*_{1,\nu+1}-P^*_{1,\nu})\circ\Phi_\nu]\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\\
\le\ & [\|D(P_{\sigma_{\nu+1}}-P_{\sigma_\nu})\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}+\|D(P^*_{1,\nu+1}-P^*_{1,\nu})\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}]\\
&\cdot|D\Phi_\nu|_{\mathcal{C}^1(\Delta^{\nu+1}_{\sigma_{\nu+1}}),\mathcal{O}_{\nu+1}} \preccurlyeq |P_{\sigma_{\nu+1}}-P_{\sigma_\nu}|_{\mathcal{C}^1(\Delta^{\nu+1}_{\sigma_{\nu+1}}),\mathcal{O}_{\nu+1}}+s^3_{\nu+1}\\
\preccurlyeq\ & (\sigma_\nu)^{\alpha-1}(b+d_{\nu+1})^{1+\alpha}+s^6_{\nu+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|X_{(P_{\sigma_{\nu+1}}-P_{\sigma_\nu})\circ\Phi_\nu+(P^*_{1,\nu+1}-P^*_{1,\nu})\circ\Phi_\nu}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\\
\preccurlyeq\ & s^{-2}_{\nu+1}\|D[(P_{\sigma_{\nu+1}}-P_{\sigma_\nu})\circ\Phi_\nu]+D[(P^*_{1,\nu+1}-P^*_{1,\nu})\circ\Phi_\nu]\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\\
\preccurlyeq\ & s^{-2}_{\nu+1}(\sigma_\nu)^{\alpha-1}(b+d_{\nu+1})^{1+\alpha}+s^3_{\nu+1}\\
\preccurlyeq\ & (\varepsilon_\nu)^{\alpha q-q}s^{-2}_{\nu+1}(b+d_{\nu+1})^{1+\alpha}+\varepsilon^3_{\nu+1}\\
\preccurlyeq\ & \varepsilon_\nu^{3\kappa^2}s^{-2}_{\nu+1}(b+d_{\nu+1})^{1+\alpha}+\varepsilon^3_{\nu+1}\\
\le\ & \frac{1}{4}\varepsilon^{3\kappa-2\kappa}_{\nu+1}(b+d_{\nu+1})^{1+\alpha}+\frac{1}{4}\varepsilon_{\nu+1}\\
=\ & \frac{\varepsilon_{\nu+1}}{4}\varepsilon^{\kappa-1}_{\nu+1}(b+d_{\nu+1})^{1+\alpha}+\frac{1}{4}\varepsilon_{\nu+1}\preccurlyeq\varepsilon_{\nu+1}.
\end{aligned}$$

It follows that,

$$\|X_{P'_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\preccurlyeq\varepsilon_{\nu+1}.$$

□

5.3. **Convergence.** Let $\mathbb{T}^* = D(0,0)$ and $\tilde{\mathcal{O}} = \bigcap_{\nu=1}^{\infty}\mathcal{O}_\nu$. Then $\mathbb{T}^*$ is a b-torus in $D(r_1,s_1)$. On one hand, by estimates in Section 5.1, we see that the transformations

$$\Phi_\nu=\phi_1\circ\phi_2\circ\cdots\circ\phi_\nu : D(r_{\nu+1},s_{\nu+1})\times\mathcal{O}_{\nu+1}\to D(r_1,s_1),$$

$\nu=1,2,\cdots$, converge in $\mathcal{C}^1(\mathbb{T}^*,D(r_1,s_1))$-norm, $\mathcal{C}^1$ uniformly Whitney s-moothly on $\tilde{\mathcal{O}}$. We denote $\Phi_\infty=\lim_{\nu\to\infty}\Phi_\nu$.

On the other hand, by estimates in Section 5.2., we see that, under the $\mathcal{C}^1(\mathbb{T}^*)$-norm,

$$N_\nu=e_\nu+\langle\omega_\nu,I\rangle+\langle A_\nu Z_\nu,\bar{Z}_\nu\rangle,\ \nu=1,2,\cdots,$$

converge $\mathcal{C}^1$ uniformly Whitney smoothly on $\tilde{\mathcal{O}}$, say to

$$N_\infty =: e_\infty + \langle \omega_\infty, I\rangle + \langle A_\infty Z_\infty, \bar{Z}_\infty\rangle,$$

and P'_ν, $\nu = 1, 2, \cdots$, converge to 0, $\mathcal{C}^1$ uniformly Whitney smoothly on $\tilde{\mathcal{O}}$.

As shown in Section 3.2, $\{H_*^\nu\}$, $\nu = 1, 2, \cdots$, converge to H_* under $\mathcal{C}^1(\breve{D}_\rho \times \mathcal{O})$-norm, as $\nu \to \infty$. Taking the limit $\nu \to \infty$ in (4.15), we have

$$H^* \circ \Phi_\infty = H'_\infty =: N_\infty, \text{ on } \mathbb{T}^* \times \tilde{\mathcal{O}},$$

and consequently,

$$\phi^t_{H^*} \circ \Phi_\infty = \Phi_\infty \circ \phi^t_{N_\infty}, \ \ t \in \mathbb{R}.$$

For each $\xi \in \tilde{\mathcal{O}}$, since $\mathbb{T}^*$ is clearly an invariant, quasi-periodic d-torus of the Hamiltonian flow $\phi^t_{N_\infty}$ with frequency vector $\omega_\infty(\xi)$, we see that $\Phi_\infty(\mathbb{T}^* \times \{\xi\})$ is an invariant, quasi-periodic b-torus of the Hamiltonian flow $\phi^t_{H^*}$ with the same frequency vector. It follows from the constancy of the normal matrix $A_\infty(\xi)$ that the invariant torus $\Phi_\infty(\mathbb{T}^b \times \{\xi\})$ is linearly stable. Moreover, in virtue of the weighted norm used in the range $D(r_1, s_1)$ of $\Phi_\infty(\mathbb{T}_* \times \{\xi\})$, we see that the normal components of the quasi-periodic orbits in $\Phi_\infty(\theta + \omega_\infty(\xi)t, \xi)$ are spatially localized with rates $\frac{1}{|n|^{1+\alpha}}$ as $|n| \gg 1$.

To characterize the smoothness order of the invariant torus $\Phi_\infty(\mathbb{T}^b \times \{\xi\})$, we note by Lemma 5.2 and (5.24) that

$$|\Phi_\nu(\theta, 0, 0, 0; \xi) - \Phi_{\nu-1}(\theta, 0, 0, 0; \xi)| \preccurlyeq \varepsilon_\nu^{1-q(2\tau+4)} = \varepsilon_\nu^{3(\kappa-1)},$$

for all $(\theta, \xi) \in \mathbb{T}^b \times \tilde{\mathcal{O}}$ and $\nu = 1, 2, \cdots$. Let

$$\mathfrak{s}_* = \frac{(\kappa-1)(\alpha-1)}{\kappa^2}.$$

Then $3(\kappa-1) - q\mathfrak{s} = 3(\kappa-1)\frac{\mathfrak{s}_*-\mathfrak{s}}{\mathfrak{s}_*}$. For any $(\theta, \xi) \in \mathbb{T}^b \times \tilde{\mathcal{O}}$ and $2 \le |\beta| = \mathfrak{s} < \mathfrak{s}_*$, we have by Cauchy estimates that

$$\begin{aligned}
&|\partial_\theta^\beta[\Phi_\infty(\theta, \xi) - \theta]| \\
= \ & |\partial_\theta^\beta[(\Phi_1(\theta, 0, 0, 0; \xi) - \theta) + \sum_{\nu=2}^{\infty}(\Phi_\nu(\theta, 0, 0, 0; \xi) - \Phi_{\nu-1}(\theta, 0, 0, 0; \xi))]| \\
\preccurlyeq \ & {\gamma_*}^{-2}\varepsilon_\nu^{3(\kappa-1)} r_\nu^{-\mathfrak{s}} \preccurlyeq \varepsilon_\nu^{3(\kappa-1)-q\mathfrak{s}} \preccurlyeq \varepsilon_\nu^{3(\kappa-1)\frac{\mathfrak{s}_*-\mathfrak{s}}{\mathfrak{s}_*}}.
\end{aligned}$$

It follows that each embedded torus $\Phi_\infty(\mathbb{T}^b \times \{\xi\})$ is $\mathcal{C}^{\mathfrak{s}}$ smooth.

5.4. **Measure Estimates.** We note that

$$\mathcal{O} \setminus \tilde{\mathcal{O}} = \bigcup_{\nu=1}^{\infty} \bigcup_{|k| \le K_\nu} (\mathcal{R}_k^{\nu,1} \bigcup \mathcal{R}_k^{\nu,2} \bigcup \mathcal{R}_k^{\nu,3} \bigcup \mathcal{R}_k^{\nu,4}),$$

where, for each $|k| \le K_\nu$ and $\nu = 1, 2, \cdots$,

$$\mathcal{R}_k^{\nu,1} = \left\{\xi \in \mathcal{O}_\nu : |(A^1_{k,\nu})^{-1}| > \frac{K_\nu^\tau}{\gamma_*}\right\},$$

$$\mathcal{R}_k^{\nu,2} = \left\{\xi \in \mathcal{O}_\nu : \|(A_{k,\nu}^2)^{-1}\| > \frac{(d_\nu)^2 K_\nu^\tau}{\gamma_*}\right\},$$
$$\mathcal{R}_k^{\nu,3} = \left\{\xi \in \mathcal{O}_\nu : \|(A_{k,\nu}^3)^{-1}\| > \frac{(d_\nu)^4 K_\nu^\tau}{\gamma_*}\right\},$$
$$\mathcal{R}_k^{\nu,4} = \left\{\xi \in \mathcal{O}_\nu : \|(A_{k,\nu}^4)^{-1}\| > \frac{(d_\nu)^4 K_\nu^\tau}{\gamma_*}\right\}$$

with

$$A_{k,\nu}^1 = \langle k, \omega_\nu \rangle,$$
$$A_{k,\nu}^2 = \langle k, \omega_\nu(\xi)\rangle I \pm \tilde{A}_\nu,$$
$$A_{k,\nu}^3 = \langle k, \omega_\nu(\xi)\rangle I + I \otimes \tilde{A}_\nu + \tilde{A}_\nu \otimes I,$$
$$A_{k,\nu}^3 = \langle k, \omega_\nu(\xi)\rangle I \pm I \otimes \tilde{A}_\nu - \tilde{A}_\nu \otimes I.$$

Let $m_\nu^i = \mathrm{dim} A_{k,\nu}^i$, $i = 1,2,3,4$. Then it is clear that $m_\nu^1 = 1$, $m_\nu^2 = d_\nu$, and $m_\nu^i = (d_\nu)^2$, $i = 3,4$.

Let $i = 1,2,3,4$, $\nu \in \mathbb{N}$, and $0 < |k| < K_\nu$ be given. Since $A_{k,\nu}^i$ is Hermitian, a simple linear algebra argument yields that if all eigenvalues of $A_{k,\nu}^i$ are greater or equal to $\frac{\gamma_*}{K_\nu^\tau}$ in absolute value, then $\|(A_{k,\nu}^i)^{-1}\| \le (m_\nu^i)^2 \frac{\gamma_*}{K_\nu^\tau}$ (see [17]). It follows that

$$\mathcal{R}_k^{\nu,i} \subset \tilde{\mathcal{R}}_k^{\nu,i} =: \{\xi \in \mathcal{O}_\nu : \exists \text{ an eigenvalue } \lambda \text{ of } A_{k,\nu}^i(\xi) \text{ such that } |\lambda| < \tfrac{\gamma_*}{K_\nu^\tau}\}.$$

Since $A_{k,\nu}^i = A_{k,\nu}^i(\xi)$ is Hermitian for each $\xi \in \mathcal{O}_\nu$ and depends on ξ s-moothly, its eigenvalues are smooth functions on $\mathcal{O}_\nu$. Let $\lambda = \lambda(\xi)$ be such an eigenvalue function. Then there exists a unit eigenvector $\psi = \psi(\xi)$ of $\lambda(\xi)$ for each $\xi \in \mathcal{O}_\nu$ which also depends on ξ smoothly (see e.g. [17]). Since

$$\lambda = \langle A_{k,\nu}^i \psi, \psi \rangle,$$

it follows from the fact that ψ is a unit eigenvector for each ξ that

$$\partial_\xi \lambda = \langle \partial_\xi A_{k,\nu}^i \psi, \psi \rangle.$$

By Lemma 5.3, we have

$$|\partial_\xi \lambda| \succcurlyeq |\langle \partial_\xi(\langle k, \omega_1(\xi)\rangle)\psi, \psi\rangle| - \varepsilon_1 |k| = O(\varepsilon^2 |k|).$$

It follows from the standard measure estimate that

$$\mathrm{meas}(\bigcup_{|k| \le K_\nu} \tilde{\mathcal{R}}_k^{\nu,i}) \preccurlyeq \sum_{\substack{|k| \le K_\nu \\ 1 \le l \le (d_\nu)^2}} \frac{\gamma}{K_\nu^{\tau+1}} \le \frac{(d_\nu)^2 \gamma}{K_\nu^{\tau-b-1}} \le \frac{\varepsilon^{1-\kappa}}{K_\nu^{\tau-b}}.$$

Thus

$$\begin{aligned} \mathrm{meas}(\mathcal{O} \setminus \tilde{\mathcal{O}}) &\ge \mathrm{meas}(\bigcup_{\nu \ge 1} \bigcup_{|k| \le K_\nu} (\tilde{\mathcal{R}}_k^{\nu,1} \bigcup \tilde{\mathcal{R}}_k^{\nu,2} \bigcup \tilde{\mathcal{R}}_k^{\nu,3} \bigcup \tilde{\mathcal{R}}_k^{\nu,4})) \\ &\succcurlyeq \sum_{\nu \ge 1} \frac{\varepsilon^{1-\kappa}}{K_\nu^{\tau-b}} = \sum_{\nu \ge 1} \frac{\varepsilon^{1-\kappa}}{K_\nu^2} = O(\varepsilon^{1-\kappa}). \end{aligned}$$

This completes the measure estimate.

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