COMPLETELY DEGENERATE RESPONSE TORI IN HAMILTONIAN SYSTEMS

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Abstract. We consider the existence of response tori for the completely degenerate Hamiltonian system with the following Hamiltonian

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I\rangle + \lambda \frac{x^n}{n} + \frac{y^m}{m} + \epsilon P(\theta, x, y, \epsilon),\ (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2},$$

where $\lambda = \pm 1$ and $n > 2$, $n \geq m \geq 2$ are integers. With P satisfying certain non-degenerate conditions, we obtain the following results: (1) For $\lambda = -1$ and ϵ sufficiently small, response tori exist for each ω satisfying a weak non-resonant condition; (2) For $\lambda = 1$ and ϵ_* sufficiently small, there exists a Cantor set $\mathcal{E} \in (0, \epsilon_*)$ with almost full Lebesgue measure such that response tori exist for each $\epsilon \in \mathcal{E}$ if ω satisfies a Diophantine condition. Non-existence of response tori are also discussed when P fails to satisfy the non-degenerate condition. Our results are directly applicable to the existence problem of quasi-periodic response solutions of degenerate harmonic oscillators.

1. Introduction

Many vibration problems in physics, mechanics, and engineering applications are modeled by the following quasi-periodically forced harmonic oscillators

$$\ddot{x} + c\dot{x} + a^2 x + \lambda x^l = f(\omega t, x, \dot{x}) \tag{1.1}$$

where c, a, λ are parameters, $l > 1$ is an integer, and f is quasi-periodic in t with frequency vector $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$. J. Stoker in [22] raised the question of the existence of a *response solution*, i.e., whether (1.1) has a quasi-periodic solution with the same frequency vector as that of the forcing function f. He gave an affirmative answer to this question for the case of large damping, i.e., when $|c|$ is large comparing to the size of f, but left the problem open for the case of small damping in which small divisor problems arise. As an early introduction of the KAM method, the first breakthrough to this open problem was made by J. Moser ([20]) who showed the existence of Diophantine response solutions for the case $c = 0$ when f is perturbative and satisfies a reversible condition. Later, M. Friedman ([9]) and B. Braaksma and H. Broer ([1]) treated the case of small non-zero $|c|$ using KAM and normal form methods and showed the existence of Diophantine response solutions. Some more recent studies show that response solutions do not need to be of Diophantine type. For instance, G. Gentile ([10, 11]) investigated, using Lyapunov-Schmidt reduction, the existence of response solutions for some general strongly dissipative, quasi-periodically forced systems whose forcing frequencies satisfy the Brjuno condition that

$$|\langle k, \omega\rangle| > \frac{\gamma}{\Delta(|k|)},\ \forall k \in \mathbb{Z}^d/\{0\}, \tag{1.2}$$

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where $\Delta : [1,\infty) \to [1,\infty)$ is an approximation function which is continuous, increasing, unbounded such that $\Delta(1) = 1$ and

$$\int_1^\infty \frac{\ln \Delta(t)}{t^2} dt < \infty.$$

Results in [10, 11] are recently improved in [12] without restrictions on ω and also extended in [13] to higher dimensions. In 2017, J. Wang, J. You and Q. Zhou ([23]) and Z. Lou and J. Geng ([19]) showed the existence of response solutions for the case $c = 0$, $a \neq 0$ in (1.1) whose frequencies are of Liouvillean type.

The most challenging case for studying the existence of response solutions is when $c = a = 0$ in (1.1), i.e., (1.1) becomes

$$\ddot{x} + \lambda x^l = \epsilon f(\omega t, x, \epsilon), \qquad \lambda \neq 0, \tag{1.3}$$

whose Hamiltonian simply reads

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I \rangle + \frac{y^2}{2} + \lambda \frac{x^{l+1}}{l+1} + \epsilon P(\theta, x, \epsilon),\ (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2}, \tag{1.4}$$

where $P(\theta, x, \epsilon) = -\int_0^x f(\theta, u, \epsilon)du$. In this case, essential difficulty arises due the degeneracy of the normal part of (1.4). Such degeneracy creates obstacles in controlling the drift of relative equilibria for a direct application of the standard KAM method. This degenerate problem was first investigated by J. You ([24]) for the Hamiltonian

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I \rangle + \frac{y^2}{2} - x^{2n} + \epsilon P(\theta, I, x, y, \epsilon),\ (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2}. \tag{1.5}$$

Without any conditions on the perturbation except its smoothness, he showed the existence of an invariant, quasi-periodic d-torus of the system corresponding to (1.5) with frequency close to ω if ω satisfies a Diophantine condition.

Later, Y. Han, Y. Li and Y. Yi ([14]) considered general normally degenerate Hamiltonians of the form

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I \rangle + \langle z, A(\omega) z \rangle + \epsilon P(\theta, I, z, \omega, \epsilon),\ (\theta, I, z) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{2m}, \tag{1.6}$$

where the normal matrix $A(\omega)$ can admit zero eigenvalues. Under a singularity-removal condition on the perturbation P and a Melnikov non-resonant condition between the tangential and normal frequencies, existence of invariant, quasi-periodic d-tori with frequency ω is shown for system with the Hamiltonian (1.6). More recently, S. Hu and B. Liu ([18]) considered the following completely degenerate Hamiltonian

$$H = \langle \omega, y \rangle + \frac{u^{2p}}{2p} + \frac{v^{2q}}{2q} + y_1^m u + y_2^l v + \epsilon P(x, y, u, v),\ (x, y, u, v) \in \mathbb{T}^d \times \mathbb{R}^{d+2}, \tag{1.7}$$

where y_1, y_2 are the first and second component of y, p, $q > 1$ and m, l are positive integers. The existence of invariant, quasi-periodic d-tori with frequency ω is shown for system corresponding to (1.7) with arbitrary perturbation. We remark results of [14, 18] are not applicable to (1.3) directly to obtain response solutions. As part of quasi-periodic bifurcation theory, H. W. Broer, H. Hanßmann, and collaborators ([2]-[5], [15]-[17]) showed the existence of quasi-periodic, including responsive, solutions, in the universal unfolding of various quasi-periodically forced, normally degenerate Hamiltonian systems in which certain perturbative, one parameter families can be embedded. In particular, the following real analytic Hamiltonian is considered in [2]:

$$H = \langle \omega, I \rangle + x^n + y^2 + \sum_{i=1}^{n-2} \lambda_i x^i + p(\theta, I, x, y, \lambda, \omega), \qquad (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2},$$

where $n > 2$ and $p = \mathcal{O}(\epsilon)$. It is shown that if ω is Diophantine then there exists a near identity symplectic transformation which transforms the Hamiltonian into the form

$$H_\infty = \langle \omega, I\rangle + x^n + y^2 + \sum_{i=1}^{n-2} \tilde{\lambda}_i(\epsilon)x^i + p_\infty(\theta, I, x, y, \lambda, \omega), \qquad (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2},$$

where $\tilde{\lambda}_i = \lambda_i + \mathcal{O}(\epsilon)$, $i = 1, \cdots, n-2$, and $\frac{\partial^{|l|+i+j+|h|} p_\infty}{\partial \tilde{I}^l \partial \tilde{x}^i \partial \tilde{y}^j \partial \tilde{\lambda}^h}(\theta, 0, 0, 0, 0, \omega) = 0$ for all $\theta \in \mathbb{T}$ and all i, j, l, $h = (h_1, \cdots, h_{l-2})$ satisfing

$$2n|l| + ni + 2j + (2n-2)h_1 + (2n-4)h_2 + \cdots + 4h_{n-2} \leq 2l.$$

It follows that responsive quasi-periodic invariant tori exist for small ϵ if $\tilde{\lambda}_1(\epsilon) \equiv 0$. Because that $\tilde{\lambda}_1(\epsilon)$ is defined through KAM iterations, such a condition does not seem to be explicitly applicable to (1.3) to yield the existence of responsive quasi-periodic solutions.

The present paper aims at making a closer look at the problem of response solutions for the quasi-periodically forced, degenerate harmonic oscillators (1.3). Firstly, we note in the normally hyperbolic case $\lambda < 0$ that if l is even and $f(\omega t, x, \epsilon) > 0$ (1.3), then no response solution can exist for any $\epsilon > 0$ simply because $\ddot{x} > 0$. Given the existence of response solutions when $\lambda < 0$ and l is odd following the result of [24], a natural question is to find proper conditions on the perturbation to ensure *the existence of response solutions when* $\lambda < 0$ *and* l *is even*. Secondly, we note in the normally elliptic case $\lambda > 0$ that successful KAM iterations to the Hamiltonian (1.4) require a proper control of the measure of parameters, in addition to the control of the drift of relative equilibria. This raises the second question that *whether and under what conditions system* (1.3) *admit response solutions in the case* $\lambda > 0$. Finally, the existence of non-Diophantine response solutions in the non-degenerate case raises the third question that *whether system* (1.3) *can also admit non-Diophantine response solutions.*

We will answer these questions with respect to the following family of normally degenerate, including completely degenerate, Hamiltonian functions:

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I\rangle + \lambda\frac{x^n}{n} + \frac{y^m}{m} + \epsilon P(\theta, x, y, \epsilon),\ (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2}, \tag{1.8}$$

where m, n are positive integers greater or equal to 2. We note that the perturbation P in the above does not depend on I, and, if $\lambda \neq 0$, then it can be normalized to ± 1. For each ϵ, the Hamiltonian $H(\theta, I, x, y, \epsilon)$ is associated with the standard symplectic structure $d\theta \wedge dI \wedge dx \wedge dy$. We assume that P is real analytic in θ, x, y and Whitney smooth in ϵ. We want to mention that the existence of responsive invariant tori of (1.8) was studied by Corsi and Gentile in [6]-[8] for the case of complete degeneracy in x which corresponds to $m = 2$ and either $n = \infty$ or $\lambda = 0$. In particular, to our knowledge, [8] is the only known result concerning lower-dimensional invariant tori with codimension greater than 1 in the degenerate case.

Under certain non-degenerate conditions on the perturbation (see condition **(H)** in section 2), we will prove the following results:

(1) When $\lambda = -1$, there exist a sufficiently small ϵ_* such that, for each $\epsilon \in (0, \epsilon_*)$, the Hamiltonian system associated with (1.8) admits quasi-periodic response tori with frequency ω satisfying a non-resonant condition weaker than the Brjuno condition (1.2).
(2) When $\lambda = 1$, there exist a sufficiently small ϵ_* and a Cantor set $\mathcal{E} \in (0, \epsilon_*)$ with almost full Lebesgue measure such that, for each $\epsilon \in \mathcal{E}$, Diophantine quasi-periodic response tori exist in the Hamiltonian system associated with (1.8).

The mathematical challenge in proving these results is to overcome the degeneracy. To reduce the original Hamiltonian to a normal form for applying KAM iterations, our strategy is to transform the Hamiltonian to the vicinity of the relative equilibrium then to improve the order of perturbation by eliminating the first and second order resonant terms. However, the order of perturbation in the normal form can only reach the double of that of the linear part, which is not sufficient for the validity of the standard KAM iterations. We thus implement a modified KAM scheme by dividing the perturbation into two parts: one at a double order of the linear part and the other one with higher order. It turns out that the iteration of only the higher order part yields the desired results.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and state our main results, which, when applying to the oscillators (1.3), give sufficient conditions for the existence of response solutions. In Section 3, we reduce the Hamiltonian to a normal form by taking the normal degeneracy into consideration. The proof of our main results will be carried out in Sections 4. In Section 5, we give a non-existence result for response solutions when l is even in (1.3).

Throughout of the paper, we will use c, C to denote intermediate positive constants. Unless specified otherwise, $|\cdot|$ will be used to denote the absolute value of numbers, the norm of vectors, and the norm of matrices. All derivatives with respect to ϵ are taken in the sense of Whitney.

2. Notations and main results

For each $r, s > 0$, we denote

$$D(s,r) = \mathbb{T}_s^d \times \mathbf{B}_r,$$

where

$$\mathbf{B}_r := \{z = (z_1, z_2) \in \mathbb{C}^2 : \ |z| \le r\}$$

is the ball of radius r in $\mathbb{C}^2$ and

$$\mathbb{T}_s^d := \{\theta = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : \ |\mathrm{Im}\theta_j| \le s, \quad j = 1, 2, \dots, d\}$$

is the strip domain of size s of the d-torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ in $\mathbb{C}^d$. For given $\delta > 0$, denote $\Pi_\delta = (0, \delta)$ and let $B \subset \Pi_\delta$. If a function $f : \ D(s,r) \times B \to \mathbb{C}$,

$$f(\theta, z, \epsilon) = \sum_{k \in \mathbb{Z}^d} f_k(z, \epsilon) e^{\mathrm{i}\langle k, \theta\rangle},$$

is analytic in θ, z and Whitney smooth in $\epsilon \in B$, then we define the norm $\|\cdot\|_{s,r,B}$ by

$$\|f(\theta, z, \epsilon)\|_{s,r,B} = \sum_{k \in \mathbb{Z}^d} \|f_k(z, \epsilon)\|_{r,B} e^{s|k|},$$

where $\|f_k\|_{r,B} = \sup_{\epsilon \in B, z \in \mathbf{B}_r} (|f_k(z,\epsilon)| + \epsilon|\frac{\partial f_k(z,\epsilon)}{\partial \epsilon}|)$ and $|k| = \sum_{j=1}^d |k_j|$ for $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$. The space of all such functions is denoted by

$$C^\omega(D(s,r) \times B) = \{f(\theta, z, \epsilon) : \|f(\theta, z, \epsilon)\|_{s,r,B} < +\infty\}$$

which is easily seen to be a Banach algebra under the norm $\|\cdot\|_{r,s,B}$. Taking $r = 0$ in the above, we can define the $\|\cdot\|_{s,B}$ norm for any function $f : \mathbb{T}_s^d \to \mathbb{C}$,

$$f(\theta, \epsilon) = \sum_{k \in \mathbb{Z}^d} f_k(\epsilon) e^{\mathrm{i}\langle k, \theta\rangle},$$

which is analytic in θ and Whitney smooth in $\epsilon \in B$. The Banach algebra of all such functions under the $\|\cdot\|_{s,B}$ norm is denoted by

$$C^\omega(\mathbb{T}_s^d \times B) = \{f(\theta, \epsilon) : \|f(\theta, \epsilon)\|_{s,B} < +\infty\}.$$

For a matrix-valued function $P(\theta, z, \epsilon) = (P_{ij}(\theta, z, \epsilon))_{n \times m}$, where $P_{ij}(\theta, z, \epsilon) \in C^\omega(D\ (s,r) \times B)$, we define its $|\cdot|_{r,s,B}$ norm by

$$|P|_{r,s,B} = \max_{1 \le i \le n} \sum_{j=1}^m \|P_{ij}\|_{r,s,B}.$$

The $|\cdot|_{s,B}$ norm of a matrix-valued function $P(\theta, \epsilon) = (P_{ij}(\theta, \epsilon))_{n \times m}$, where $P_{ij}(\theta, \epsilon) \in C^\omega(\mathbb{T}_s^d \times B)$, is defined by taking $r = 0$ in the above. For any function $f(\theta, z)$ in $D(r,s)$, we denote its average with respect to θ by

$$[f(\cdot, z)] = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta, z) d\theta.$$

Without loss of generality, we suppose that $n \geq m \geq 2$ in (1.8). We note that the non-degenerate case $n = m = 2$ has already been well investigated in [21]. Thus we only consider the case when $n > 2$. Moreover, we assume the following condition:

(H) For any $m \geq 2$,

$$\begin{cases} [\frac{\partial P(\cdot,0,0,0)}{\partial x}] \neq 0, & \text{if } n \text{ is even}, \\ [\frac{\partial P(\cdot,0,0,0)}{\partial x}] > 0 \text{ (resp. } < 0), & \text{if } n \text{ is odd and } \lambda = -1 \text{ (resp. } \lambda = 1), \end{cases}$$

and, when $m > 2$,

$$\begin{cases} [\frac{\partial P(\cdot,0,0,0)}{\partial y}] \neq 0, & \text{if } m \text{ is even}, \\ [\frac{\partial P(\cdot,0,0,0)}{\partial y}] < 0, & \text{if } m \text{ is odd}. \end{cases}$$

Our main result for the degenerate hyperbolic case is stated as follows.

Theorem 1. *Consider the Hamiltonian* (1.8) *with* $\lambda = -1$, $n > 2$, $n \geq m \geq 2$ *on the domain* $D(s,r) \times \Pi_{\epsilon_0}$ *for fixed* $r, s, \epsilon_0 > 0$ *sufficiently small. Assume* **(H)** *and that* ω *satisfies the following Brjuno-like non-resonant condition:*

$$|\langle k, \omega \rangle| > \frac{\gamma}{\Omega(|k|)}, \qquad \forall k \in \mathbb{Z}^d/\{0\}, \tag{2.1}$$

where $\Omega(t)$ *is a function such that* $\lim_{t\to\infty} \frac{\ln \Omega(t)}{t} = 0$. *Then there exists an* $\epsilon_* \leq \epsilon_0$ *and a Whitney smooth family of real analytic symplectic transformations*

$$\Phi_\epsilon : D(\frac{s}{2}, \frac{r}{2}) \to D(s,r), \quad \epsilon \in (0, \epsilon_*)$$

such that each Φ_ϵ *transforms the Hamiltonian* (1.8) *into the form*

$$H_* = H \circ \Phi_\epsilon(\theta, I, x, y) = \langle \omega, I \rangle + \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, M(\theta) \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle + G(x,y) + p(\theta),$$

where $G(x,y) = \mathcal{O}(\|(x,y)\|^3)$ *and* $M(\theta)$ *is a matrix-valued function. Consequently, for each* $\epsilon \in (0, \epsilon_*)$, *the Hamiltonian system corresponding to* (1.8) *admits a real analytic, quasi-periodic response* d*-torus with the frequency vector* ω.

For the degenerate elliptic case, we have the following result.

Theorem 2. *Consider the Hamiltonian* (1.8) *with* $\lambda = 1$, $n > 2$, $n \geq m \geq 2$ *on the domain* $D(s,r) \times \Pi_{\epsilon_0}$ *for fixed* $r, s, \epsilon_0 > 0$ *sufficiently small. Assume* **(H)** *and that* ω *satisfies the following Diophantine condition:*

$$|\langle k, \omega \rangle| > \gamma |k|^{-\tau}, \qquad \forall k \in \mathbb{Z}^d/\{0\}, \tag{2.2}$$

where $\tau > d - 1$ *is fixed. Then there exist a Cantor set* $\mathcal{E}_{\epsilon_*} \subset (0, \epsilon_0)$ *with* $\lim_{\epsilon_* \to 0} \frac{Meas(\mathcal{E}_{\epsilon_*})}{\epsilon_*} = 1$ *and a Whitney smooth family of real analytic symplectic transformations*

$$\Psi_\epsilon : D(\frac{s}{2}, \frac{r}{2}) \to D(s,r), \qquad \epsilon \in \mathcal{E}_{\epsilon_*}$$

such that each Ψ_ϵ *transforms the Hamiltonian* (1.8) *into the form*

$$H_* = H \circ \Psi_\epsilon(\theta, I, x, y) = \langle \omega, I \rangle + \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, M \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle + G(x,y) + p(\theta),$$

where $G(x,y) = \mathcal{O}(\|(x,y)\|^3)$ *and* M *is a constant matrix. Consequently, for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$, *the Hamiltonian system corresponding to* (1.8) *admits a real analytic, Diophantine response torus with the frequency vector* ω.

Remark 2.1. (1) It is clear that responsive tori obtained in Theorem 2 are Floquet but the ones obtained in Theorem 1 are not in general.

(2) The non-resonant condition (2.1) is weaker than the Brjuno non-resonant condition (1.2). Indeed, if $\Omega(t)$ is an approximation function in the Brjuno non-resonant condition, then for each $x > 1$, we have

$$\int_x^{2x} \frac{\ln \Omega(t)}{t^2} dt > \int_x^{2x} \frac{\ln \Omega(x)}{t^2} dt > \frac{\ln \Omega(x)}{2x} > 0,$$

which implies that $\lim_{x\to\infty} \frac{\ln \Omega(x)}{x} = 0$ since $\lim_{x\to\infty} \int_x^{2x} \frac{\ln\Omega(t)}{t^2} dt = 0$. If we choose $\Omega(t) = e^{\frac{t}{\ln t}}$, then it is easily seen that $\lim_{t\to\infty} \frac{\ln \Omega(t)}{t} = \lim_{t\to\infty} \frac{1}{\ln t} = 0$ but $\int_1^{\infty} \frac{\ln \Omega(t)}{t^2} dt$ is divergent. The non-resonance condition (2.1) is made weaker because it is only needed in the step of normal reduction, instead of in the KAM iterations. However, if the assumption **(H)** is made weaker, then small divisor problems will be encountered in the KAM iterations for which a stronger non-resonance condition like the Brjuno's is unavoidable.

(3) Condition **(H)** is a strong condition which essentially says that the perturbation is dominated by a non-degenerate linear term in x, and, when $m > 2$, it is also dominated by a non-degenerate linear term in y. This is a reasonable condition to assume when either m or n is even, given the non-existence example contained in Section 5. However, in the case that both m and n are odd, we suspect that this condition may not be necessary to yield the same results. We will investigate this issue in a future work.

As the existence of real analytic, quasi-periodic response solutions of (1.3) is equivalent to the existence of real-analytic, quasi-periodic, invariant d-tori of (1.4), Theorems 1, 2 can be applied directly to yield the following results.

Corollary 1. *Consider* (1.3) *with* $\lambda = \pm 1$ *and* $f : \mathbb{T}^d \times \mathbb{R}^1 \to \mathbb{R}^1$ *being real analytic.*

(1) *In the case* $\lambda = -1$*, assume that* (i) *either* $[f(\cdot, 0, 0)] \neq 0$ *when* l *is odd or* $[f(\cdot, 0, 0)] > 0$ *when* l *is even; and* (ii) ω *satisfies the non-resonant condition* (2.1)*. Then, for each* ϵ *sufficiently small,* (1.3) *admits a real analytic, quasi-periodic response solution with the frequency vector* ω*.*

(2) *In the case* $\lambda = 1$*, assume that* (i) *either* $[f(\cdot, 0, 0)] \neq 0$ *when* l *is odd or* $[f(\cdot, 0, 0)] < 0$ *when* l *is even; and* (ii) ω *is Diophantine. Then, for sufficiently small* ϵ_**, there exists a Cantor set* $\mathcal{E}_{\epsilon_*} \in (0, \epsilon_*)$ *with* $\lim_{\epsilon_*\to 0} \frac{Meas(\mathcal{E}_{\epsilon_*})}{\epsilon_*} = 1$ *such that* (1.3) *admits a real analytic, quasi-periodic response solution with frequency vector* ω *for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$*.*

3. Normal form reduction

In this section, we reduce the Hamiltonian (1.8) into a suitable normal form in order to apply KAM iterations. The reduction will be carried over in the two steps. The first is to average linear terms in the perturbation and eliminate the averaged terms by transforming the Hamiltonian to the vicinity of the relative equilibrium. The second is to remove first and second order resonant terms in the perturbation in order to make it further small.

3.1. Averaging and relative equilibrium.

The purpose of this step is to reduce (1.8) to a Hamiltonian whose perturbation has zero average linear term. We first consider the following implicit equation

$$\begin{cases} y^{m-1} + \epsilon[\frac{\partial P(\cdot,x,y,\epsilon)}{\partial y}] = 0, \\ \lambda x^{n-1} + \epsilon[\frac{\partial P(\cdot,x,y,\epsilon)}{\partial x}] = 0, \end{cases} \tag{3.1}$$

which, with the re-scaling $y \to \epsilon^{\frac{1}{m-1}} y$, $x \to \epsilon^{\frac{1}{n-1}} x$, becomes

$$\begin{cases} H_1(x,y,\epsilon) =: y^{m-1} + [\frac{\partial P(\cdot,0,0,0)}{\partial y}] + \mathcal{O}(\epsilon^{\frac{1}{n-1}}) = 0, \\ H_2(x,y,\epsilon) =: \lambda x^{n-1} + [\frac{\partial P(\cdot,0,0,0)}{\partial x}] + \mathcal{O}(\epsilon^{\frac{1}{n-1}}) = 0. \end{cases} \tag{3.2}$$

Denote $x_* = (-\frac{1}{\lambda}[\frac{\partial P(\cdot,0,0,0)}{\partial x}])^{\frac{1}{n-1}}$, $y_* = (-[\frac{\partial P(\cdot,0,0,0)}{\partial y}])^{\frac{1}{m-1}}$. Since, by condition **(H)**,

$$H_1(x_*, y_*, 0) = 0, \quad H_2(x_*, y_*, 0) = 0,$$
$$\left|\det \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix} \Big|_{(x_*,y_*,0)}\right| = |x_*|^{n-2}|y_*|^{m-2} \neq 0,$$

the Implicit Function Theorem implies that the equation (3.2) has a solution $(\tilde{x}(\epsilon), \tilde{y}(\epsilon)) = (x_* + \mathcal{O}(\epsilon^{\frac{1}{n-1}}), y_* + \mathcal{O}(\epsilon^{\frac{1}{n-1}}))$. Thus the equation (3.1) has a solution $(x(\epsilon), y(\epsilon)) = (\epsilon^{\frac{1}{n-1}}\tilde{x}(\epsilon), \epsilon^{\frac{1}{m-1}}\tilde{y}(\epsilon))$.

We now eliminate the average term in the perturbation. Let $\rho = \epsilon^{\frac{1}{(n-1)(m-1)}}$, $\breve{x}(\rho) = \tilde{x}(\rho^{(m-1)(n-1)})$, $\breve{y}(\rho) = \tilde{y}(\rho^{(m-1)(n-1)})$, and consider the symplectic transformation:

$$\phi_0 : x \to \rho^{\frac{n-m}{2(n-2)}}x + \rho^{m-1}\breve{x}(\rho),\ y \to \rho^{\frac{(n-m)(n-1)}{2(n-2)}}y + \rho^{n-1}\breve{y}(\rho),\ \theta \to \theta,\ I \to \rho^{\frac{(n-m)(n+1)}{n}}I$$

and re-scaling $\breve{H} \to \rho^{-\frac{(n-m)n}{2(n-2)}}H \circ \phi_0$. Then the new Hamiltonian reads

$$\begin{aligned} H &= \langle\omega, I\rangle + (n-1)\lambda\rho^{\frac{2mn-3m-3n+4}{2}}\breve{x}^{n-2}(\rho)x^2 + (m-1)\rho^{\frac{2mn-3m-3n+4}{2}}\breve{y}^{m-2}(\rho)y^2 \\ &+ \rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)n}{2(n-2)}}\breve{P}(\theta, \rho^{\frac{n-m}{2(n-2)}}x + \rho^{m-1}\breve{x}(\rho), \rho^{\frac{(n-m)(n-1)}{2(n-2)}}y + \rho^{n-1}\breve{y}(\rho), \rho) \\ &+ \lambda\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}\breve{x}^{n-1}(\rho)x + \rho^{\frac{2(n-2)(n-1)(m-1)-n+m}{2(n-2)}}\breve{y}^{m-1}(\rho)y + h(x,y,\rho), \end{aligned}$$

where

$$\begin{aligned} h(x,y,\rho) &= \lambda\frac{1}{n}\sum_{k=3}^{n}\binom{n}{k}\rho^{(m-1)(n-k)+\frac{(n-m)k}{2(n-2)}}\breve{x}^{n-k}(\rho)x^k \\ &+ \frac{1}{m}\sum_{k=3}^{m}\binom{m}{k}\rho^{(n-1)(m-k)+\frac{(n-1)(n-m)k}{2(n-2)}}\breve{y}^{m-k}(\rho)y^k, \\ \breve{P}(\theta, I, x, y, \rho) &= P(\theta, I, x, y, \rho^{(m-1)(n-1)}). \end{aligned}$$

Let $\delta_1 = \sqrt{n-1}\breve{x}^{\frac{n-2}{2}}(\rho)$, $\delta_2 = \sqrt{m-1}\breve{y}^{\frac{m-2}{2}}(\rho)$, and $\delta = \delta_1\delta_2$. Then the symplectic re-scaling

$$I \to \frac{1}{\delta_1\delta_2}I,\ x \to \frac{1}{\delta_1}x,\ y \to \frac{1}{\delta_2}y,\ \theta = \theta,\ H \to \delta H(\theta, \delta^{-1}I, \delta_1^{-1}x, \delta_2^{-1}y)$$

further reduces the Hamiltonian to

$$H(\theta, I, x, y, \rho) = N(I, x, y) + G(x, y, \rho) + E(\theta, x, y, \rho) + p(\theta), \tag{3.3}$$

where $p(\theta) = \delta\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)n}{2(n-2)}}\breve{P}(\theta, \rho^{m-1}\breve{x}(\rho), \rho^{n-1}\breve{y}(\rho), \rho)$, $G(x,y,\rho) = \delta h(\delta_1^{-1}x,\ \delta_2^{-1}y, \rho)$,

$$N = \langle\omega, I\rangle + \langle\begin{pmatrix} x \\ y \end{pmatrix}, M\begin{pmatrix} x \\ y \end{pmatrix}\rangle$$

with $M = \delta\rho^p\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $p = \frac{2mn-3m-3n+4}{2}$, and

$$\begin{aligned} &E(\theta, x, y, \rho) \\ &= \delta\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)n}{2(n-2)}}\breve{P}(\theta, \delta_1^{-1}\rho^{\frac{n-m}{2(n-2)}}x + \rho^{m-1}\breve{x}(\rho), \delta_2^{-1}\rho^{\frac{(n-m)(n-1)}{2(n-2)}}y \\ &+\rho^{n-1}\breve{y}(\rho), \rho) + \delta_2\lambda\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}\breve{x}^{n-1}(\rho)x \\ &+\delta_1\rho^{\frac{2(n-2)(n-1)(m-1)-n+m}{2(n-2)}}\breve{y}^{m-1}(\rho)y \\ &-\delta\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)n}{2(n-2)}}\breve{P}(\theta, \rho^{m-1}\breve{x}(\rho), \rho^{2n+1}\breve{y}(\rho), \rho). \end{aligned}$$

One can easily check that $G(\theta, x, y, \rho) = \mathcal{O}(\|(x,y)\|^3)$ and there exist $s, r > 0$ such that

$$\|E\|_{s,r,\Pi_\rho} \leq C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}.$$

Write $E(\theta,x,y,\rho)=\acute{E}(\theta,x,y,\rho)+\grave{E}(\theta,x,y,\rho)$, where

$$\begin{aligned}\acute{E}(\theta,x,y,\rho) &= \acute{E}_1(\theta,x,y,\rho)+\acute{E}_2(\theta,x,y,\rho)\\ &= \sum_{i+j=1}E_{i,j}(\theta,\rho)x^iy^j+\sum_{i+j=2}E_{i,j}(\theta,\rho)x^iy^j\\ &= \sum_{i+j=1,k\in\mathbb{Z}^d}E_{i,j,k}(\rho)e^{\mathrm{i}\langle k,\theta\rangle}x^iy^j+\sum_{i+j=2,k\in\mathbb{Z}^d}E_{i,j,k}(\rho)e^{\mathrm{i}\langle k,\theta\rangle}x^iy^j,\\ \grave{E}(\theta,x,y,\rho) &= \sum_{i+j\geq 3}E_{i,j}(\theta,\rho)x^iy^j=\sum_{i+j\geq 3,k\in\mathbb{Z}^d}E_{i,j,k}(\rho)e^{\mathrm{i}\langle k,\theta\rangle}x^iy^j.\end{aligned}$$

Since $(\rho^{m-1}\breve{x}(\rho),\rho^{n-1}\breve{y}(\rho))$ solve the equation (3.1), we have

$$\begin{aligned}E_{1,0,0}(\rho) &= \delta_2\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}[\frac{\partial\breve{P}(\cdot,\rho^{m-1}\breve{x}(\rho),\rho^{n-1}\breve{y}(\rho),\rho)}{\partial x}]\\ &+ \delta_2\lambda\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}\breve{x}^{n-1}(\rho)=0,\\ E_{0,1,0}(\rho) &= \delta_1\rho^{\frac{2(n-2)(n-1)(m-1)-n+m}{2(n-2)}}[\frac{\partial\breve{P}(\cdot,\rho^{m-1}\breve{x}(\rho),\rho^{n-1}\breve{y}(\rho),\rho)}{\partial y}]\\ &+ \delta_1\rho^{\frac{2(n-2)(n-1)(m-1)-n+m}{2(n-2)}}\breve{y}^{m-1}(\rho)=0.\end{aligned}\tag{3.4}$$

Thus the Hamiltonian (3.3) reads

$$H(\theta,I,x,y,\rho)=N(I,x,y)+h(\theta,x,y,\rho)+\acute{E}(\theta,x,y,\rho)+p(\theta),\tag{3.5}$$

where

$$\begin{aligned}&h(\theta,x,y,\rho)=\grave{E}(\theta,x,y,\rho)+G(x,y,\rho)=\mathcal{O}(\|(x,y)\|^3),\\ &\|\acute{E}(\theta,x,y,\rho)\|_{s,r,\Pi_\rho}\leq C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}.\end{aligned}\tag{3.6}$$

3.2. **Improving the order of perturbation.** The purpose of this step is to improve the order of perturbation to the one which beyond doubles that of the linear part.

In the case $\lambda=1$, the Hamiltonian (3.5) is of elliptic type. We let ω be a Diophantine frequency vector satisfying (2.2) with respect to fixed γ,τ. For given ϵ_* sufficiently small, we consider the set $\mathcal{D}_{\epsilon_*}=\mathcal{D}_1\cap\mathcal{D}_2$, where

$$\begin{aligned}\mathcal{D}_1 &= \{\epsilon\in(0,\epsilon_*):\langle k,\omega\rangle^2-4\delta^2\rho^{2p}>\frac{\gamma^3}{|k|^{5\tau}},\ \forall k\in\mathbb{Z}^d\setminus\{0\}\},\\ \mathcal{D}_2 &= \{\epsilon\in(0,\epsilon_*):\langle k,\omega\rangle^2-16\delta^2\rho^{2p}>\frac{\gamma^3}{|k|^{5\tau}},\ \forall k\in\mathbb{Z}^d\setminus\{0\}\}.\end{aligned}$$

where $\rho=\epsilon^{\frac{1}{(m-1)(n-1)}}$.

In the case $\lambda=-1$, the Hamiltonian (3.5) is of hyperbolic type. We let ω be a frequency vector satisfying the weak non-resonant condition (2.1).

We have the following result.

Lemma 3.1. *Consider the Hamiltonian* (3.5) *in* $D(s,r)\times\Pi_{\rho_*}$ *with* ω *satisfying* (2.2)*, respectively* (2.1)*, in the case* $\lambda=1$*, respectively* $\lambda=-1$*, where* $\rho_*=\epsilon_*^{\frac{1}{(m-1)(n-1)}}$ *satisfies* (3.6)*. Then there exists a family of real analytic symplectic transformations*

$$\phi^\epsilon:D(s-\sigma,r-\sigma)\to D(s,r),\quad \epsilon\in\mathcal{D},$$

where $\sigma\leq\min\{r/4,s/4\}$ *is a fixed constant,* $\mathcal{D}=\mathcal{D}_{\epsilon_*}$ *is a Cantor set with* $\mathrm{meas}(\mathcal{D})\sim\epsilon_*$ *in the case* $\lambda=1$*, and* $\mathcal{D}=(0,\epsilon_*)$ *in the case* $\lambda=-1$*, such that the following hold.*

(a) ϕ^ϵ, $\epsilon\in\mathcal{D}$, *is a Whitney smooth family in the case* $\lambda=1$ *and a smooth family in the case* $\lambda=-1$*;*

(b) *For each* $\epsilon \in \mathcal{D}$, ϕ^ϵ *transforms* (3.5) *into*

$$\tilde{H} = \tilde{N}(I, x, y, \rho) + \tilde{h}(\theta, x, y, \rho) + \tilde{E}(\theta, x, y, \rho) + \tilde{p}(\theta), \tag{3.7}$$

where $\rho = \epsilon^{\frac{1}{(m-1)(n-1)}}$,

$$\tilde{N} = \langle \omega, I\rangle + \langle \begin{pmatrix} x \\ y \end{pmatrix}, \tilde{M}\begin{pmatrix} x \\ y \end{pmatrix}\rangle$$

with $\tilde{M} = M + O(\rho^{p+\frac{1}{2}})$, $\tilde{h}(\theta, x, y, \rho) = \mathcal{O}(\|(x,y)\|^3)$, *and* $\tilde{E}(\theta, x, y, \rho) = \sum_{0<i+j\le 2} \tilde{E}_{ij}(\theta)x^i y^j$ *satisfying*

$$\|\tilde{E}\|_{s-\sigma, r-\sigma, \mathcal{D}} \le C\rho^{2p+1}.$$

Proof. The first step of the reduction is to find a real analytic function $F_1 = F^1_{10}(\theta)x + F^1_{01}(\theta)y$ such that the time-1 map $\phi^1_{F_1}$ of the Hamiltonian flow $\phi^t_{F_1}$ eliminates the first order resonant terms in H, i.e., the term $\acute{E}_1 = E_{10}(\theta)x + E_{01}(\theta)y$ in $\acute{E}$. Since

$$\begin{aligned} H \circ \phi^1_{F_1} &= N \circ \phi^1_{F_1} + h \circ \phi^1_{F_1} + \acute{E} \circ \phi^1_{F_1} + p \circ \phi^1_{F_1} \\ &= N + \{N, F_1\} + \int_0^1 (1-t)\{\{N, F_1\}, F_1\} \circ \phi^t_{F_1} dt \\ &+ \acute{E} + \int_0^1 \{\acute{E}, F_1\} \circ \phi^t_{F_1} dt + h \circ \phi^1_F + p \circ \phi^1_{F_1}, \end{aligned}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket, this elimination amounts to the solvability of the homological equation

$$\{N, F_1\} = \acute{E}_1. \tag{3.8}$$

Denote $\mathbb{F}_1(\theta) = (F^1_{10}(\theta), F^1_{01}(\theta))^\top$ and $\mathbb{E}_1(\theta) = (\tilde{E}_{10}(\theta), \tilde{E}_{01}(\theta))^\top$. Then (3.8) is equivalent to

$$\omega\frac{\partial \mathbb{F}_1(\theta)}{\partial\theta} - 2MJ\mathbb{F}_1(\theta) = \mathbb{E}_1(\theta) \tag{3.9}$$

where J is the standard symplectic matrix. Using Fourier expansions $\mathbb{E}_1(\theta) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}_{1k}e^{\mathrm{i}\langle k,\theta\rangle}$ and $\mathbb{F}_1(\theta) = \sum_{k\in\mathbb{Z}^d} \mathbb{F}_{1k}e^{\mathrm{i}\langle k,\theta\rangle}$, we re-write (3.9) as

$$(\mathrm{i}\langle k, \omega\rangle Id - 2MJ)\mathbb{F}_{1k} = \mathbb{E}_{1k}, \qquad k \in \mathbb{Z}^d. \tag{3.10}$$

By (3.4), we let $\mathbb{F}_{10} = 0$. For each $|k| > 0$, it is clear that

$$\mathbb{F}_{1k} = \frac{\mathrm{adj}(\mathrm{i}\langle k, \omega\rangle Id - 2MJ)}{\det(\mathrm{i}\langle k, \omega\rangle Id - 2MJ)}\mathbb{E}_{1k}$$

formally solves the equation (3.10), where "adj" stands for the adjoint of a matrix.

We claim that these formal solutions are true solutions when $\epsilon \in \mathcal{D}$ to yield the desired real analytic Hamiltonian function F_1. In the case $\lambda = 1$, let $\epsilon \in \mathcal{D}_1$. Then $|\mathbb{F}_{1k}| < C\frac{|k|^{10\tau+3}}{\gamma^6}|\mathbb{E}_{1k}|$, $|k| > 0$ and $\epsilon \in \mathcal{D}_1$. It follows from (3.6) that

$$|\mathbb{F}_1|_{s-\frac{\sigma}{4}, \mathcal{D}_1} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}, \qquad |k| > 0.$$

In the case $\lambda = -1$, let $\epsilon \in \Pi_{\epsilon_*} = (0, \epsilon_*)$. For each $|k| > 0$, since $2MJ = 2\delta\rho^p\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathrm{i}\langle k, \omega\rangle Id - 2MJ$ has eigenvalues $\mathrm{i}\langle k, \omega\rangle \pm 2\delta\rho^p$, which, by (2.1), satisfies the estimate $|(\mathrm{i}\langle k, \omega\rangle \pm 2\delta\rho^p)^{-1}| < |(\mathrm{i}\langle k, \omega\rangle)^{-1}| < C\Omega(|k|)$. Since $\sup_{|t|>0} e^{-\frac{\sigma}{4}t}\Omega(t) < \infty$, we have by (3.6) that

$$\begin{aligned} |\mathbb{F}_1|_{s-\frac{\sigma}{4}, \Pi_{\epsilon_*}} &= \sum_{|k|\ge 0} |\mathbb{F}_{1k}|_{\Pi_{\epsilon_*}} e^{(s-\frac{\sigma}{4})|k|} < \sum_{|k|\ge 0} |\Omega(|k|)||\mathbb{E}_{1k}|_{\Pi_{\epsilon_*}} e^{(s-\frac{\sigma}{4})|k|} \\ &\le \sup_{|t|>0} e^{-\frac{\sigma}{4}t}\Omega(t)|\mathbb{E}_1(\theta)|_{s-\frac{\sigma}{4}, \Pi_{\epsilon_*}} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}, \qquad |k| > 0. \end{aligned}$$

Thus, in both cases, for each $\epsilon \in \mathcal{D}$, (3.8) is solvable to yield a real analytic function F_1 defined on $D(s-\frac{\sigma}{4}, r)$. Moreover, F_1 varies in $\mathcal{D}$ Whitney smoothly when $\lambda = 1$ and smoothly when $\lambda = -1$, and

$$\|F_1\|_{s-\frac{\sigma}{4},r,\mathcal{D}} \leq C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}.$$

By Cauchy's estimates, we have

$$\begin{aligned} |DF_1|_{s-\frac{\sigma}{2},r-\frac{\sigma}{2},\mathcal{D}} &\leq C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}, \\ |D^2F_1|_{s-\frac{\sigma}{2},r-\frac{\sigma}{2},\mathcal{D}} &\leq C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}. \end{aligned} \tag{3.11}$$

A standard argument using integral equations yields that

$$\begin{aligned} |\phi^t_{F_1} - id|_{s-\frac{\sigma}{2},r-\frac{\sigma}{2},\mathcal{D}} &< C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}, \\ |D\phi^t_{F_1} - Id|_{s-\frac{\sigma}{2},r-\frac{\sigma}{2},\mathcal{D}} &< C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}, \end{aligned} \tag{3.12}$$

for all $t \in [0,1]$. Consequently, $\phi^1_{F_1}(D(s-\frac{\sigma}{2}, r-\frac{\sigma}{2}) \times \mathcal{D}) \subset D(s,r) \times \mathcal{D}$, and

$$\begin{aligned} H_1 =: H \circ \phi^1_{F_1} &= N + \int_0^1 (1-t)\{\{N, F_1\}, F_1\} \circ \phi^t_{F_1} dt \\ &+ \acute{E}_2 + \int_0^1 \{\acute{E}, F_1\} \circ \phi^t_{F_1} dt + h \circ \phi^1_{F_1} + p \circ \phi^1_{F_1} \\ &= N + \int_0^1 (1-t)\{\{N, F_1\}, F_1\} \circ \phi^t_{F_1} dt \\ &+ \acute{E}_2 + \int_0^1 \{\acute{E}, F_1\} \circ \phi^t_{F_1} dt + h \circ (\mathbb{F}_1 J) + \langle \frac{\partial h}{\partial z} \circ (\mathbb{F}_1 J), z\rangle \\ &+ \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J) z, z\rangle + \breve{h} + p, \end{aligned}$$

where $\breve{h} = h - h \circ (\mathbb{F}_1 J) - \langle \frac{\partial h}{\partial z} \circ (\mathbb{F}_1 J), z\rangle - \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J)z, z\rangle$ and $z = (x,y)^\top$.

The second step of the reduction is to find a real analytic Hamiltonian of the form $F_2 = F^2_{20}(\theta)x^2 + F^2_{11}(\theta)xy + F^2_{02}(\theta)y^2$ so that the time-1 map $\phi^1_{F_2}$ of the Hamiltonian flow ϕ^t_{F2} removes the quadratic term $\acute{E}_2 + \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J)z, z\rangle$ in the non-integrable part of H_1 and add its average to the integrable part. This amounts to the solvability of the homological equation

$$\{N, F_2\} = \acute{E}_2 + \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J)z, z\rangle - [\acute{E}_2 + \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J)z, z\rangle]. \tag{3.13}$$

Write $\acute{E}_2 + \langle \frac{\partial^2 h}{\partial z^2} \circ (\mathbb{F}_1 J)z, z\rangle \circ \phi^1_{F_2} = R_{20}(\theta)x^2 + R_{11}(\theta)xy + R_{02}(\theta)y^2$ and denote

$$\mathbb{F}_2(\theta) := \begin{pmatrix} F^2_{20}(\theta) & \frac{F^2_{11}(\theta)}{2} \\ \frac{F^2_{11}(\theta)}{2} & F^2_{02}(\theta) \end{pmatrix}, \ \mathbb{R}(\theta) := \begin{pmatrix} R_{20}(\theta) & \frac{R_{11}(\theta)}{2} \\ \frac{R_{11}(\theta)}{2} & R_{02}(\theta) \end{pmatrix}.$$

Then (3.13) is equivalent to

$$\omega \frac{\partial \mathbb{F}_2(\theta)}{\partial \theta} - 4MJ\mathbb{F}_2(\theta) = \mathbb{R}(\theta),$$

which, in terms of Fourier expansions $\mathbb{F}_2(\theta) = \sum_{k \in \mathbb{Z}^d} \mathbb{F}_{2k} e^{\mathrm{i}\langle k,\theta\rangle}$ and $\mathbb{R}(\theta) = \sum_{k\in\mathbb{Z}^d} \mathbb{R}_k e^{\mathrm{i}\langle k,\theta\rangle}$, is further equivalent to

$$(\mathrm{i}\langle k, \omega\rangle Id - 4MJ)\mathbb{F}_{2k} = \mathbb{R}_k, \quad k \in \mathbb{Z}^d. \tag{3.14}$$

Using similar arguments as that for the first step above, (3.14) is solvable when $\lambda = 1$, $\epsilon \in \mathcal{D}_2$ and $\lambda = -1$, $\epsilon \in \Pi_{\epsilon_*} = (0, \epsilon_*)$ with solutions

$$\begin{aligned} \mathbb{F}_{20} &= 0, \\ \mathbb{F}_{2k} &= \frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle Id - 4MJ)}{\det(\mathrm{i}\langle k,\omega\rangle Id - 4MJ)} \mathbb{E}_{1k}, \quad |k| > 0 \end{aligned}$$

satisfying

$$|\mathbb{F}_{2k}| < C\frac{|k|^{5\tau+2}}{\gamma^3}|\mathbb{R}_k|, \quad k \in \mathbb{Z}^d, \epsilon \in \mathcal{D}_2.$$

Moreover, F_2 varies in $\mathcal{D}$ Whitney smoothly when $\lambda = 1$ and smoothly when $\lambda = -1$. It follows from Cauchy's estimates that

$$\begin{aligned} &|\mathbb{F}_2|_{s-\frac{3\sigma}{4},\mathcal{D}} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}},\\ &\|F_2\|_{s-\frac{3\sigma}{4},r-\frac{\sigma}{2},\mathcal{D}} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}},\\ &|DF_2|_{s-\sigma,r-\sigma,\mathcal{D}} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}},\\ &|D^2F_2|_{s-\sigma,r-\sigma,\mathcal{D}} \le C\rho^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}}. \end{aligned} \tag{3.15}$$

Consequently,

$$\begin{aligned} |\phi^t_{F_2} - id|_{s-\sigma,r-\sigma,\mathcal{D}} &< C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}},\\ |D\phi^t_{F_2} - Id|_{s-\sigma,r-\sigma,\mathcal{D}} &< C\rho_*^{\frac{2(n-2)(n-1)(m-1)-(n-m)(n-1)}{2(n-2)}} \end{aligned} \tag{3.16}$$

for all $t \in [0,1]$, and $\phi^1_{F_2}(D(s-\sigma, r-\sigma)\times\mathcal{D}) \subset D(s-\frac{\sigma}{2}, r-\frac{\sigma}{2})\times\mathcal{D}$.

Let $\phi^\epsilon = \phi^1_{F_1}\circ\phi^1_{F_2}$. Then

$$\begin{aligned} \tilde{H} &=: H\circ\phi^\epsilon = H_1\circ\phi^1_{F_2} = \tilde{N} + \int_0^1 (1-t)\{\{N,F_2\},F_2\}\circ\phi^t_{F_2}dt\\ &+ \int_0^1 (1-t)\{\{N,F_1\},F_1\}\circ\phi^t_{F_1}\circ\phi^1_{F_2}dt\\ &+ \int_0^1 \{\acute{E}_2 + \langle\frac{\partial^2 h}{\partial z^2}\circ(\mathbb{F}_1 J)z, z\rangle, F_2\}\circ\phi^t_{F_2}dt + \int_0^1\{\acute{E}, F_1\}\circ\phi^t_{F_1}\circ\phi^1_{F_2}dt\\ &+ (h\circ(\mathbb{F}_1 J) + \langle\frac{\partial h}{\partial z}\circ(\mathbb{F}_1 J), z\rangle + \breve{h})\circ\phi^1_{F_2} + p\circ\phi^1_{F_2}\\ &= \tilde{N} + \tilde{h} + \tilde{E} + \tilde{p}, \end{aligned}$$

where

$$\begin{aligned} \tilde{N} &= N + [\acute{E}_2 + \langle\frac{\partial^2 h}{\partial z^2}\circ(\mathbb{F}_1 J)z, z\rangle] = \langle\omega, I\rangle + \langle\begin{pmatrix}x\\y\end{pmatrix}, (M+[\mathbb{R}])\begin{pmatrix}x\\y\end{pmatrix}\rangle,\\ \tilde{h} &= \breve{h}\circ\phi^1_{F_2},\\ \tilde{E} &= \breve{\acute{E}}(\theta,x,y) - \breve{\acute{E}}(\theta,0,0),\\ \tilde{p} &= p(\theta) + \breve{\acute{E}}(\theta,0,0) \end{aligned}$$

with

$$\begin{aligned} \breve{\acute{E}}(\theta,x,y) &= \int_0^1 (1-t)\{\{N,F_2\},F_2\}\circ\phi^t_{F_2}dt + \int_0^1 (1-t)\{\{N,F_1\},F_1\}\circ\phi^t_{F_1}\circ\phi^1_{F_2}dt\\ &+ \int_0^1 \{\acute{E}_2 + \langle\frac{\partial^2 h}{\partial z^2}\circ(\mathbb{F}_1 J)z, z\rangle, F_2\}\circ\phi^t_{F_2}dt + \int_0^1\{\acute{E}, F_1\}\circ\phi^t_{F_1}\circ\phi^1_{F_2}dt\\ &+ (h\circ\mathbb{F}_1 + \langle\frac{\partial h}{\partial z}\circ(\mathbb{F}_1 J), z\rangle)\circ\phi^1_{F_2}. \end{aligned}$$

By (3.11), (3.12), (3.15), (3.17) and Cauchy's estimates, we have

$$\|\tilde{E}\|_{s-\sigma,r-\sigma,\mathcal{D}} \le C\rho^{\frac{4(n-2)(n-1)(m-1)-2(n-m)(n-1)}{2(n-2)}} \le C\rho^{2p+1}.$$

We now estimate the Lebesgue measure of $\mathcal{D} = \mathcal{D}_1\cap\mathcal{D}_2$ in the case $\lambda = 1$. For each $|k| > 0$, let $f^1_k(\epsilon) = \langle k,\omega\rangle^2 - 4\delta^2\rho^{2p}$ and $f^2_k(\epsilon) = \langle k,\omega\rangle^2 - 16\delta^2\rho^{2p}$, and denote $R^i_k := \{\epsilon \in (0,\epsilon_*)$:

$|f_k^i(\epsilon)| < \gamma^3/|k|^{5\tau}\}$, $i = 1, 2$. For each k, i, one can check easily that $|f_k^i(\epsilon)| \geq \frac{\gamma^2}{|k|^{2\tau}} - C\rho^{2p}$ and $|\frac{df_k^i(\epsilon)}{d\epsilon}| > c\epsilon^{\frac{2p}{(m-1)(n-1)}-1}$. If $\frac{\gamma^2}{|k|^{2\tau}} \geq 2C\rho^{2p}$, then

$$|f_k^i(\epsilon)| \geq \frac{\gamma^2}{|k|^{2\tau}} - C\rho^{2p} \geq \frac{\gamma^2}{2|k|^{2\tau}} \geq \frac{\gamma^3}{|k|^{5\tau}},$$

implying that $\text{meas}(R_k^i) = 0$. If $\frac{\gamma^2}{|k|^{2\tau}} < 2C\rho^{2p}$, then

$$\text{meas}(R_k^i) \leq \frac{\gamma^3}{c|k|^{5\tau}\epsilon^{\frac{2p}{(m-1)(n-1)}-1}} \leq \frac{C\epsilon^{\frac{4p}{(m-1)(n-1)}}}{|k|^{\tau}\epsilon^{\frac{2p}{(m-1)(n-1)}-1}} \leq \frac{C}{|k|^{\tau}}\epsilon^{1+\frac{2p}{(m-1)(n-1)}}.$$

It follows that

$$\text{meas}(\bigcup_{k\in\mathbb{Z}^d\setminus\{0\}} R_k^i) \leq C\epsilon^{1+\frac{2p}{(m-1)(n-1)}} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^{\tau}} \leq C\epsilon^{1+\frac{2p}{(m-1)(n-1)}},$$

$i = 1, 2$, and therefore,

$$\epsilon_* \geq \text{meas}(\mathcal{D}) \geq \epsilon_* - \text{meas}(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\},\\ 1\leq i\leq 2}} R_k^i) \geq \epsilon_*(1 - C\epsilon_*^{\frac{2p}{(m-1)(n-1)}}).$$

□

4. Proof of main results

In this section, we use KAM iterations to prove Theorems 1, 2. Setting $H_0 = \tilde{H}, N_0 = \tilde{N}, M_0 = \tilde{M}, h_0 = \tilde{h}, E_0 = \tilde{E}, p_0 = \tilde{p}, r_0 = r - \sigma, s_0 = s - \sigma$ in (3.7), we obtain the following Hamiltonian

$$H_0 = \langle\omega, I\rangle + \langle\begin{pmatrix}x\\y\end{pmatrix}, M_0\begin{pmatrix}x\\y\end{pmatrix}\rangle + h_0(\theta, x, y) + E_0(\theta, x, y) + p_0(\theta), \tag{4.1}$$

$I \in \mathbb{R}^d, (\theta, (x, y)) \in D(s_0, r_0)$. For simplicity, we have made the ρ- (or ϵ-) dependency implicit in the above. We recall from Lemma 3.1 that

$$\begin{aligned} M_0 &= M + O(\rho^{p+\frac{1}{2}}), \\ h_0(\theta, x, y, \rho) &= \mathcal{O}(\|(x, y)\|^3), \\ E_0(\theta, x, y, \rho) &= \sum_{0<i+j\leq 2} E_{ij}^0(\theta)x^i y^j, \ \|E_0\|_{s_0,r_0} \leq \rho_0^{2p+1}, \end{aligned}$$

where $\rho_0 = C^{\frac{1}{2p+1}}\rho$. Fix a $\sigma_0 \leq \min\{\frac{r_0}{4}, \frac{s_0}{4}\}$ and consider the sequences

$$\begin{cases} \rho_{\nu+1} = \rho_\nu^{\frac{4p+3}{4p+1}}, & \sigma_{\nu+1} = \sigma_\nu/2, \\ s_{\nu+1} = s_\nu - \sigma_\nu, & r_{\nu+1} = r_\nu - \sigma_\nu, \end{cases} \tag{4.2}$$

$\nu = 0, 1, \cdots$. Then it is clear that $\lim_{\nu\to\infty} s_\nu =: s_* \geq s_0/2$, $\lim_{\nu\to\infty} r_\nu =: r_* \geq r_0/2$, and $\lim_{\nu\to\infty} \rho_\nu = 0$.

4.1. **KAM iteration for the case $\lambda = -1$.** We would like to use a KAM scheme to iterate the following sequence of smooth families of real analytic Hamiltonians:

$$H_\nu = \langle\omega, I\rangle + \langle\begin{pmatrix}x\\y\end{pmatrix}, M_\nu(\theta)\begin{pmatrix}x\\y\end{pmatrix}\rangle + h_\nu(\theta, x, y) + E_\nu(\theta, x, y) + p_\nu(\theta),$$

$I \in \mathbb{R}^d, (\theta,(x,y)) \in D(s_\nu, r_\nu),\ \epsilon \in \Pi_{\epsilon_*},\ \nu = 0, 1, \cdots$, satisfying

$$M_\nu = M + O(\rho^{p+\frac{1}{2}}), \tag{4.3}$$

$$|M_\nu - M_{\nu-1}|_{s_\nu,\Pi_{\epsilon_*}} \le \rho_\nu^{p+\frac{1}{2}}, \tag{4.4}$$

$$h_\nu = \mathcal{O}(\|(x,y)\|^3), \tag{4.5}$$

$$\|h_\nu - h_{\nu-1}\|_{s_\nu,r_\nu,\Pi_{\epsilon_*}} \le \rho_\nu^{p+\frac{1}{2}}, \tag{4.6}$$

$$\|E_\nu\|_{s_\nu,r_\nu,\Pi_{\epsilon_*}} \le \rho_\nu^{2p+1}, \tag{4.7}$$

$$\|p_\nu - p_{\nu-1}\|_{s_\nu,r_\nu,\Pi_{\epsilon_*}} \le \rho_\nu^{p+\frac{1}{2}}, \tag{4.8}$$

$\nu = 1, 2, \cdots$.

Lemma 4.1. *Consider $\lambda = -1$. Then for each $\nu = 0, 1, \cdots$, there exists a smooth family of real analytic symplectic transformation $\Phi_\nu : D(s_{\nu+1}, r_{\nu+1}) \to D(s_\nu, r_\nu)$, $\epsilon \in \Pi_{\epsilon_*}$, satisfying*

$$|\Phi_\nu - id|_{s_\nu,r_\nu,\Pi_{\epsilon_*}} < \rho_\nu^{p+\frac{1}{2}},$$
$$|D\Phi_\nu - Id|_{s_\nu,r_\nu,\Pi_{\epsilon_*}} < \rho_\nu^{p+\frac{1}{2}}$$

such that $H_{\nu+1} = H_\nu \circ \Phi_\nu$.

Proof. Write $E_\nu = \sum_{1\le|i+j|\le 2} E_{\nu,ij}(\theta)x^iy^j$ in two parts: $E_\nu = E_\nu^1 + E_\nu^2$ where $E_\nu^1 = \sum_{|i+j|=1} E_{\nu,ij}(\theta)x^iy^j$ and $E_\nu^2 = \sum_{|i+j|=2} E_{\nu,ij}(\theta)x^iy^j$. We will use the time-1-map of the flow $\phi_{F_\nu}^t$ of a undetermined Hamiltonian function $F_\nu := F_{\nu,10}(\theta)x + F_{\nu,01}(\theta)y$ as the desired symplectic transformation. Denote $\mathbb{F}_\nu(\theta) = (F_{\nu,10}(\theta), F_{\nu,01}(\theta))^\top$ and

$$\mathbb{E}_\nu^2(\theta) = \begin{pmatrix} E_{\nu,20}^2(\theta) & \frac{E_{\nu,11}^2(\theta)}{2} \\ \frac{E_{\nu,11}^2(\theta)}{2} & E_{\nu,02}^2(\theta) \end{pmatrix}.$$

Applying the transformation $\phi_{F_\nu}^1$ to Hamilton H_ν, we have

$$H_{\nu+1} = N_{\nu+1} + h_{\nu+1} + E_{\nu+1} + p_{\nu+1},$$

where $N_{\nu+1} = \langle\omega, I\rangle + \langle\binom{x}{y}, M_{\nu+1}(\theta)\binom{x}{y}\rangle$ and

$$M_{\nu+1} = M_\nu + \mathbb{E}_\nu^2 + \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu J), \tag{4.9}$$

$$E_{\nu+1} = \int_0^1 \{E_\nu^2, F_\nu\} \circ \phi_{F_\nu}^t dt + \langle \frac{\partial h_\nu}{\partial z} \circ (\mathbb{F}_\nu J), z_+\rangle, \tag{4.10}$$

$$p_{\nu+1} = p_\nu + h_\nu \circ (\mathbb{F}_\nu J) + \int_0^1 \{E_\nu^1, F_\nu\} \circ \phi_{F_\nu}^t dt + \int_0^1 (1-t)\{\{N_\nu, F_\nu\}, F_\nu\} \circ \phi_{F_\nu}^t dt, \tag{4.11}$$

$$h_{\nu+1} = h_\nu \circ \phi_{F_\nu}^1 - h_\nu \circ (\mathbb{F}_\nu(\theta)J) - \langle \frac{\partial h_\nu}{\partial z} \circ (\mathbb{F}_\nu J), z_+\rangle - \langle \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu J) z_+, z_+\rangle, \tag{4.12}$$

proved that the homological equation

$$\{N_\nu, F_\nu\} = E_\nu^1 \tag{4.13}$$

is solvable.

Denote $\mathbb{E}_\nu^1(\theta) = (E_{\nu 10}^1(\theta), E_{\nu 01}^1(\theta))^\top$. Then (4.13) is equivalent to

$$\omega \frac{\partial \mathbb{F}_\nu(\theta)}{\partial \theta} - (2M_0 J + 2Q_\nu(\theta))\mathbb{F}_\nu(\theta) = \mathbb{E}_\nu^1(\theta), \tag{4.14}$$

where $Q_\nu(\theta) = \sum_{l=1}^\nu \mathbb{E}_l^2(\theta) + \sum_{l=1}^\nu \frac{\partial^2 h_l}{\partial z^2} \circ (\mathbb{F}_l(\theta)J)$. To solve equation (4.14), we let $\mathbb{F}_\nu(\theta) = \sum_{k\in\mathbb{Z}^d} \mathbb{F}_{\nu k} e^{\mathrm{i}\langle k,\theta\rangle}$ and $\mathbb{E}_\nu^1(\theta) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}_{\nu k}^1 e^{\mathrm{i}\langle k,\theta\rangle}$. Defined operator $T : C^\omega(\mathbb{T}_{s_\nu} \times \Pi_{\rho_0}, \mathbb{R}^2) \to$

$C^\omega(\mathbb{T}_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)$ by

$$T(V(\theta)) := \omega\frac{\partial V(\theta)}{\partial\theta} - 2M_0JV(\theta) = \sum_{k\in\mathbb{Z}^d}(\mathrm{i}\langle k,\omega\rangle E - 2M_0J)V_k e^{\mathrm{i}\langle k,\theta\rangle},$$

for each $V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)$ and operator $W_\nu : C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\to C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)$ by

$$W_\nu(V(\theta)) = -2Q_\nu(\theta)V(\theta)$$

for each $V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)$. The operator T is easily seen to be invertible with inverse $T^{-1} : C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\to C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)$,

$$\begin{aligned} T^{-1}(V(\theta)) &= \sum_{k\in\mathbb{Z}^d}(\mathrm{i}\langle k,\omega\rangle Id - 2M_0J)^{-1}V_k e^{\mathrm{i}\langle k,\theta\rangle} \\ &= \sum_{k\in\mathbb{Z}^d}P^{-1}\begin{pmatrix}(\mathrm{i}\langle k,\omega\rangle-2\lambda_1)^{-1} & 0\\ 0 & (\mathrm{i}\langle k,\omega\rangle-2\lambda_2)^{-1}\end{pmatrix}PV_k e^{\mathrm{i}\langle k,\theta\rangle},\end{aligned}$$

where λ_i, $i=1,2$, are eigenvalues of M_0. By hyperbolicity, we have $|\mathrm{i}\langle k,\omega\rangle-\lambda_i|\ge c\rho_0^p$, $i=1,2$. We also note that $|Q_\nu|_{s_\nu,\Pi_{\rho_0}} \le \sum_{l=1}^{\nu}|\mathbb{E}^2_l|_{s_\nu,\Pi_{\rho_0}} + \sum_{l=1}^{\nu}|\frac{\partial^2 h_l}{\partial z^2}\circ(\mathbb{F}_lJ)|_{s_\nu,\Pi_{\rho_0}} \le C\rho_0^{p+\frac12}$. It follows that

$$\begin{aligned}\|T^{-1}\| &= \sup_{V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\setminus\{0\}}\frac{|T^{-1}(V(\theta))|_{s_\nu\Pi_{\rho_0}}}{|V(\theta)|_{s_\nu\Pi_{\rho_0}}}\\ &\le \sup_{V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\setminus\{0\}}\frac{c\rho_0^{-p}|V(\theta)|_{s_\nu\Pi_{\rho_0}}}{|V(\theta)|_{s_\nu\Pi_{\rho_0}}} = C\rho_0^{-p}\end{aligned}$$

and

$$\begin{aligned}\|W_\nu\| &= \sup_{V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\setminus\{0\}}\frac{|W_\nu(V(\theta))|_{s_\nu\Pi_{\rho_0}}}{|V(\theta)|_{s_\nu\Pi_{\rho_0}}}\\ &\le \sup_{V(\theta)\in C^\omega(\mathbb{T}^d_{s_\nu}\times\Pi_{\rho_0},\mathbb{R}^2)\setminus\{0\}}\frac{|2Q_\nu(\theta)|_{s_\nu\Pi_{\rho_0}}|V(\theta)|_{s_\nu\Pi_{\rho_0}}}{|V(\theta)|_{s_\nu\Pi_{\rho_0}}} \le C\rho_0^{p+\frac12}.\end{aligned}$$

Thus $\|T^{-1}W_\nu\|\le\|T^{-1}\|\|W_\nu\|\le C\rho_0^{\frac12}$, which implies that operator $T+W_\nu$ is also invertible and

$$\|(T+W_\nu)^{-1}\|\le\|(1+T^{-1}W_\nu)\|\|T^{-1}\|\le C\rho_0^{-p}.$$

Hence equation (4.14) is solvable to yield solution $\mathbb{F}_\nu(\theta)=(T+W_\nu)^{-1}(\mathbb{E}^1_\nu(\theta))$ satisfying

$$|\mathbb{F}_\nu|_{s_\nu\Pi_{\rho_0}}\le C\rho_0^{-p}|\mathbb{E}^1_\nu(\theta)|_{s_\nu\Pi_{\rho_0}}\le C\rho_0^{-p}\rho_\nu^{2p+1}\le\rho_\nu^{p+1}. \tag{4.15}$$

By (4.9), one can check that $M_{\nu+1}=M+O(\rho^{p+\frac12})$ and

$$|M_{\nu+1}-M_\nu|_{s_{\nu+1}\Pi_{\rho_0}}\le|\mathbb{E}^2_\nu|_{s_{\nu+1}\Pi_{\rho_0}}\le\rho_\nu^{p+\frac12},$$

which means that (4.3) and (4.4) hold with $\nu+1$ in place of ν.

By (4.15), (4.12) and Cauchy's estimate, we have $h_{\nu+1}=\mathcal{O}(\|(x,y)\|^3)$ and

$$\begin{aligned}\|h_{\nu+1}-h_\nu\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} &\le \|h_\nu\circ\phi^1_{F_\nu}-h_\nu\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}}+\|h_\nu\circ(\mathbb{F}_\nu(\theta)J)\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}}\\ &+ \|\langle\frac{\partial h_\nu}{\partial z}\circ(\mathbb{F}_\nu J),z_+\rangle-\langle\frac{\partial^2 h_\nu}{\partial z^2}\circ(\mathbb{F}_\nu J)z_+,z_+\rangle\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}}\\ &\le \rho_\nu^{p+\frac12},\end{aligned}$$

which means that (4.5) and (4.6) hold with $\nu+1$ in place of ν.

By (4.10), (4.11), (4.15) and Cauchy's estimate, we also have

$$\begin{aligned}
\|E_{\nu+1}\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} &\leq \|\int_0^1 \{E_\nu^3, F_\nu\}\circ\phi_{F_\nu}^t dt\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}}\\
&+ \|\langle\frac{\partial h_\nu}{\partial z}\circ(\mathbb{F}_\nu(\theta)J), z_+\rangle\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} \leq \rho_{\nu+1}^{2p+1},\\
\|p_{\nu+1}-p_\nu\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} &= \|h_\nu\circ(\mathbb{F}_\nu(\theta)J) + \int_0^1 \{E_\nu^1, F_\nu\}\circ\phi_{F_\nu}^t dt\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}}\\
&+ \|\int_0^1 (1-t)\{\{N+E_\nu^2, F_\nu\}, F_\nu\}\circ\phi_{F_\nu}^t dt\|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} \leq \rho_{\nu+1}^{2p+1},
\end{aligned}$$

which means that (4.7) and (4.8) hold with $\nu+1$ in place of ν. By integral equations

$$\begin{aligned}
\phi_{F_\nu}^t &= id + \textstyle\int_0^t X_{F_\nu}(\phi_{F_\nu}^s)ds,\\
D\phi_{F_\nu}^t &= Id + \textstyle\int_0^t J(D^2F_\nu)D\phi_{F_\nu}^s ds,
\end{aligned}$$

one can check by Cauchy's estimate that

$$\begin{aligned}
|\phi_{F_\nu}^t - id|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} &\leq \rho_\nu^{p+\frac{1}{2}},\\
|D\phi_{F_\nu}^t - Id|_{s_{\nu+1}r_{\nu+1}\Pi_{\rho_0}} &\leq \rho_\nu^{p+\frac{1}{2}},
\end{aligned}$$

which implies

$$\Phi_\nu(D(s_{\nu+1}, r_{\nu+1})) \subset D(s_\nu, r_\nu),\ \epsilon\in\Pi_{\rho_0}.$$

This completes the proof of the lemma. □

4.2. **KAM iteration for the case $\lambda = 1$.** Setting $\gamma_0 = \gamma$, we consider a new sequence $\gamma_{\nu+1} = \gamma_\nu/2$ in addition to sequences in (4.2). Denote $\mathcal{D}_\nu := \mathcal{D}_\nu^1 \cap \mathcal{D}_\nu^2$, where

$$\begin{aligned}
\mathcal{D}_\nu^1 &:= \{\epsilon\in\mathcal{D} : \langle k,\omega\rangle^2 - 4\det M_\nu > \frac{\gamma_\nu^3}{|k|^{5\tau}}, \rho = \epsilon^{\frac{1}{(m-1)(n-1)}}, \forall k\in\mathbb{Z}^d\setminus\{0\}\},\\
\mathcal{D}_\nu^2 &:= \{\epsilon\in\mathcal{D} : \langle k,\omega\rangle^2 - 16\det M_\nu > \frac{\gamma_\nu^3}{|k|^{5\tau}}, \rho = \epsilon^{\frac{1}{(m-1)(n-1)}}, \forall k\in\mathbb{Z}^d\setminus\{0\}\}.
\end{aligned}$$

We would like to use a KAM scheme to iterate the following sequence of smooth families of real analytic Hamiltonians:

$$H_\nu = \langle\omega, I\rangle + \langle\begin{pmatrix}x\\y\end{pmatrix}, M_\nu\begin{pmatrix}x\\y\end{pmatrix}\rangle + h_\nu(\theta,x,y) + E_\nu(\theta,x,y) + p_\nu(\theta),$$

$I\in\mathbb{R}^d, (\theta,(x,y))\in D(s_\nu, r_\nu),\ \epsilon\in\mathcal{D}_\nu,\ \nu = 0,1,\cdots,$ satisfying

$$M_\nu = M + O(\rho^{p+\frac{1}{2}}), \tag{4.16}$$

$$|M_\nu - M_{\nu-1}|_{\mathcal{D}_\nu} \leq \rho_\nu^{p+\frac{1}{2}}, \tag{4.17}$$

$$h_\nu = \mathcal{O}(\|(x,y)\|^3), \tag{4.18}$$

$$\|h_\nu - h_{\nu-1}\|_{s_\nu, r_\nu, \mathcal{D}_\nu} \leq \rho_\nu^{p+\frac{1}{2}}, \tag{4.19}$$

$$\|E_\nu\|_{s_\nu, r_\nu, \mathcal{D}_\nu} \leq \rho_\nu^{2p+1}, \tag{4.20}$$

$$\|p_\nu - p_{\nu-1}\|_{s_\nu, r_\nu, \mathcal{D}_\nu} \leq \rho_\nu^{p+\frac{1}{2}}, \tag{4.21}$$

$\nu = 1, 2, \cdots$.

Lemma 4.2. *Consider $\lambda = 1$. Then for each $\nu = 0, 1, \cdots$, there exists a smooth family of real analytic symplectic transformation $\Phi_\nu : D(s_{\nu+1}, r_{\nu+1}) \to D(s_\nu, r_\nu)$, $\epsilon\in\mathcal{D}_\nu$, satisfying*

$$\begin{aligned}
|\Phi_\nu - id|_{s_\nu, r_\nu, \mathcal{D}_\nu} &< \rho_\nu^{p+\frac{1}{2}},\\
|D\Phi_\nu - Id|_{s_\nu, r_\nu, \mathcal{D}_\nu} &< \rho_\nu^{p+\frac{1}{2}}
\end{aligned}$$

such that $H_{\nu+1} = H_\nu\circ\Phi_\nu$.

Proof. Write $E_\nu = \sum_{1\le|i+j|\le 2} E_{\nu,ij}(\theta)x^iy^j$ in two parts: $E_\nu = E_\nu^1 + E_\nu^2$, where $E_\nu^1 = \sum_{|i+j|=1} E_{\nu,ij}(\theta)x^iy^j$ and $E_\nu^2 = \sum_{|i+j|=2} E_{\nu,ij}(\theta)x^iy^j$. We would like to use the time-1 maps of flows $\phi^t_{F_\nu^1}$, $\phi^t_{F_\nu^2}$ as desired symplectic transformations, where F_ν^1, F_ν^2 are Hamiltonian functions of the forms $F_\nu^1 := F_{\nu,10}(\theta)x + F_{\nu,01}(\theta)y$, $F_\nu^2 := F_{\nu,20}(\theta)x^2 + F_{\nu,11}(\theta)xy + F_{\nu,02}(\theta)y^2$. Denote $\mathbb{F}_\nu^1(\theta) = (F_{\nu,10}(\theta), F_{\nu,01}(\theta))^\top$,

$$\mathbb{F}_\nu^2(\theta) = \begin{pmatrix} F_{\nu 20}(\theta) & \frac{F_{\nu 11}(\theta)}{2} \\ \frac{F_{\nu 11}(\theta)}{2} & F_{\nu 02}(\theta) \end{pmatrix}.$$

Applying the transformation $\phi^1_{F_\nu^1} \circ \phi^1_{F_\nu^2}$ to Hamilton H_ν, we have

$$H_{\nu+1} = N_{\nu+1} + h_{\nu+1} + E_{\nu+1} + p_{\nu+1}$$

where $N_{\nu+1} = \langle \omega, I\rangle + \langle \binom{x}{y}, M_{\nu+1}\binom{x}{y}\rangle$ with with $M_{\nu+1} = M_\nu + [\mathbb{E}_\nu^2 + \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J)]$ and

$$\begin{aligned} E_{\nu+1} &= \int_0^1 (1-t)\{\{N_\nu, F_\nu^2\}, F_\nu^2\} \circ \phi^t_{F_\nu^2} dt + \int_0^1 \{E_\nu, F_\nu^1\} \circ \phi^t_{F_\nu^1} \circ \phi^1_{F_\nu^2} dt \\ &+ \langle \frac{\partial h_\nu}{\partial z} \circ (\mathbb{F}_\nu^1(\theta)J), e^{\mathbb{F}_\nu^2(\theta)J} z_+\rangle + \int_0^1 \{\grave{E}_\nu + \langle \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J) z_+, z_+\rangle, F_\nu^2\} \circ \phi^t_{F_\nu^2} dt, \quad (4.22) \\ p_{\nu+1} &= p_\nu + \int_0^1 (1-t)\{\{N_\nu, F_\nu^1\}, F_\nu^1\} \circ \phi^t_{F_\nu^1} \circ \phi^1_{F_\nu^2} dt + h_\nu \circ (\mathbb{F}_\nu^1(\theta)J), \quad (4.23) \\ h_{\nu+1} &= h_\nu \circ \phi^1_{F_\nu^1} \circ \phi^1_{F_\nu^2} - h_\nu \circ (\mathbb{F}_\nu^1(\theta)J) - \langle \frac{\partial h_\nu}{\partial z} \circ (\mathbb{F}_\nu^1(\theta)J), e^{\mathbb{F}_\nu^2(\theta)Jt} z_+\rangle \\ &- \langle \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1(\theta)J) e^{\mathbb{F}_\nu^2(\theta)Jt} z_+, e^{\mathbb{F}_\nu^2(\theta)Jt} z_+\rangle, \quad (4.24) \end{aligned}$$

provided that the following homologic equations

$$\{N_\nu, F_\nu^1\} = E_\nu^1 \quad (4.25)$$
$$\{N_\nu, F_\nu^2\} = R_\nu \quad (4.26)$$

can be solved, where $R_\nu = E_\nu^2 + \langle \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J) z_+, z_+\rangle - [E_\nu^2 + \langle \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J) z_+, z_+\rangle]$.

First, we note that (4.25) is equivalent to

$$\omega \frac{\partial \mathbb{F}_\nu^1(\theta)}{\partial \theta} - 2M_\nu J \mathbb{F}_\nu^1(\theta) = \mathbb{E}_\nu^1(\theta), \quad (4.27)$$

where $\mathbb{E}_\nu^1 = (E_{\nu,10}(\theta), E_{\nu,01}(\theta))\top$. Using Fourier expansions $\mathbb{E}_\nu^1 = \sum_{k\in\mathbb{Z}^d} \mathbb{E}^1_{\nu k} e^{\mathrm{i}\langle k,\theta\rangle}$ and $\mathbb{F}_\nu^1 = \sum_{k\in\mathbb{Z}^d} \mathbb{F}^1_{\nu k} e^{\mathrm{i}\langle k,\theta\rangle}$, a similar argument as that for equation (3.10) in Lemma 3.1 shows that (4.27) is solvable when $\epsilon \in \mathcal{D}_\nu^1$ with solution

$$\mathbb{F}^1_{\nu k} = \frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle Id - 2M_\nu J)}{\det(\mathrm{i}\langle k,\omega\rangle Id - 2M_\nu J)} \mathbb{E}^1_{\nu k},\ |k| \ge 0.$$

Since $|\mathbb{F}^1_{\nu k}|_{\mathcal{D}^1} < C\frac{|k|^{10\tau+3}}{\rho_0^p \gamma_\nu^6} |\mathbb{E}^1_{\nu k}|_{\mathcal{D}_\nu^1}$, we have

$$|\mathbb{F}_\nu^1|_{s_\nu - \frac{\sigma_\nu}{4}, \mathcal{D}_\nu^1} \le \sum_{k\in\mathbb{Z}^d/\{0\}} C\frac{|k|^{10\tau+3}}{\rho_0^p \gamma_\nu^6} |\mathbb{E}^1_{\nu k}|_{\mathcal{D}_\nu^1} e^{|k|(s_\nu - \frac{\sigma_\nu}{4})} \le \frac{C}{\rho_0^p \gamma_\nu^6 \sigma_\nu} |\mathbb{E}_\nu^1|_{s_\nu, \mathcal{D}_\nu^1} \le \frac{C\rho_\nu^{p+1}}{\gamma_\nu^6 \sigma_\nu}. \quad (4.28)$$

Next, we note that (4.26) is equivalent to

$$\omega \frac{\partial \mathbb{F}_\nu^2(\theta)}{\partial \theta} - 4M_\nu J \mathbb{F}_\nu^2(\theta) = \mathbb{R}_\nu(\theta), \quad (4.29)$$

where $\mathbb{R}_\nu = \mathbb{E}_\nu^2 + \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J) - [\mathbb{E}_\nu^2 + \frac{\partial^2 h_\nu}{\partial z^2} \circ (\mathbb{F}_\nu^1 J)]$ with

$$\mathbb{E}_\nu^2(\theta) = \begin{pmatrix} E_{\nu 20}(\theta) & \frac{E_{\nu 11}(\theta)}{2} \\ \frac{E_{\nu 11}(\theta)}{2} & E_{\nu 02}(\theta) \end{pmatrix}.$$

Using Fourier expansions $\mathbb{R}_\nu(\theta)=\sum_{k\in\mathbb{Z}^d}\mathbb{R}_{\nu k}e^{\mathrm{i}\langle k,\theta\rangle}$ and $\mathbb{F}^2_\nu(\theta)=\sum_{k\in\mathbb{Z}^d}\mathbb{F}^2_{\nu k}e^{\mathrm{i}\langle k,\theta\rangle}$, a similar argument as that for the equation (3.13) in Lemma 3.1 shows that (4.29) is solvable when $\epsilon\in\mathcal{D}^2_\nu$ with solution

$$\begin{aligned}\mathbb{F}^2_{\nu 0} &= 0,\\ \mathbb{F}^2_{\nu k} &= \frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle Id-4M_\nu J)}{\det(\mathrm{i}\langle k,\omega\rangle Id-4M\nu J)}\mathbb{R}_{\nu k}.\end{aligned}$$

Since $|\mathbb{F}_{2k}|_{\mathcal{D}^2_\nu}<C\frac{|k|^{5\tau+2}}{\gamma^3}|\mathbb{R}_k|_{\mathcal{D}^2_\nu}$, we have

$$|\mathbb{F}^2_\nu|_{s_\nu-\frac{3\sigma_\nu}{4},\mathcal{D}^2_\nu}\le\sum_{k\in\mathbb{Z}^d/\{0\}}C\frac{|k|^{10\tau+3}}{\gamma^6_\nu}|\mathbb{R}_{\nu k}|_{\mathcal{D}^2_\nu}e^{|k|(s_\nu-\frac{3\sigma_\nu}{4})}\le\frac{C}{\gamma^6_\nu\sigma_\nu}|\mathbb{R}_\nu|_{s_\nu,\mathcal{D}^2_\nu}\le\frac{C\rho^{p+1}_\nu}{\gamma^{12}_\nu\sigma^2_\nu}. \tag{4.30}$$

Cauchy's estimates further yield that

$$\begin{aligned}&\|F^1_\nu\|_{s_\nu-\frac{\sigma_\nu}{4},r_\nu,\mathcal{D}_\nu}\le\tfrac{C}{\rho^p_0\gamma^6_\nu\sigma_\nu}|\mathbb{E}^1_\nu|_{s_\nu,\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^6_\nu\sigma_\nu}, &&\|F^2_\nu\|_{s_\nu-\frac{3\sigma_\nu}{4},r_\nu-\frac{\sigma_\nu}{2},\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^{12}_\nu\sigma^2_\nu},\\ &|DF^1_\nu|_{s_\nu-\frac{\sigma_\nu}{2},r_\nu-\frac{\sigma_\nu}{2},\mathcal{D}_\nu}\le\tfrac{C}{\rho^p_0\gamma^6_\nu\sigma^3_\nu}|\mathbb{E}^1_\nu|_{s_\nu,\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^6_\nu\sigma^3_\nu}, &&|DF^2_\nu|_{s_\nu-\sigma_\nu,r_\nu-\sigma_\nu,\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^{12}_\nu\sigma^4_\nu},\\ &|D^2F^1_\nu|_{s_\nu-\frac{\sigma_\nu}{2},r-\frac{\sigma_\nu}{2},\mathcal{D}_\nu}\le\tfrac{C}{\rho^p_0\gamma^6_\nu\sigma^6_\nu}|\mathbb{E}^1_\nu|_{s_\nu,\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^6_\nu\sigma^6_\nu} &&|D^2F^2_\nu|_{s_\nu-\sigma_\nu,r_\nu-\sigma_\nu,\mathcal{D}_\nu}\le\tfrac{C\rho^{p+1}_\nu}{\gamma^{12}_\nu\sigma^6_\nu}.\end{aligned} \tag{4.31}$$

Since $M_\nu=M_0+\sum_{i=1}^\nu\mathbb{R}_i$, we have $M_\nu=M+O(\rho^{p+\frac12})$, and by (4.31), we also have

$$\begin{aligned}|M_{\nu+1}-M_\nu|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &\le |\mathbb{E}^2_\nu|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}+|\frac{\partial^2h_\nu}{\partial z^2}\circ(\mathbb{F}^1_\nu J)|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &\le \rho^{p+\frac12}_{\nu+1},\end{aligned}$$

which means that (4.16) and (4.17) hold when $\nu+1$ in place of ν. By (4.24), (4.28), (4.30) and (4.31), we have $h_{\nu+1}=\mathcal{O}(\|(x,y)\|^3)$ and

$$\begin{aligned}\|h_{\nu+1}-h_\nu\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &\le \|h_\nu\circ\phi^1_{F^1_\nu}\circ\phi^1_{F^2_\nu}-h_\nu\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}-\|h_\nu\circ(\mathbb{F}^1_\nu(\theta)J)\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &+ \|\langle\frac{\partial h_\nu}{\partial z}\circ(\mathbb{F}^1_\nu(\theta)J),e^{\mathbb{F}^2_\nu(\theta)Jt}z_+\rangle\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &+ \|\langle\frac{\partial^2h_\nu}{\partial z^2}\circ(\mathbb{F}^1_\nu(\theta)J)e^{\mathbb{F}^2_\nu(\theta)Jt}z_+,e^{\mathbb{F}^2_\nu(\theta)Jt}z_+\rangle\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &\le \rho^{p+\frac12}_{\nu+1},\end{aligned}$$

which means that (4.18) and (4.19) hold when $\nu+1$ in place of ν. By (4.22), (4.23), (4.28), (4.30), (4.31) and Cauchy's estimate, we have

$$\begin{aligned}\|E_{\nu+1}\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &\le \|\int_0^1(1-t)\{\{N_\nu,F^2_\nu\},F^2_\nu\}\circ\phi^t_{F^2_\nu}dt\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &+ \|\langle\frac{\partial h_\nu}{\partial z}\circ(\mathbb{F}^1_\nu(\theta)J),e^{\mathbb{F}^2_\nu(\theta)J}z_+\rangle+\int_0^1\{E_\nu,F^1_\nu\}\circ\phi^t_{F^1_\nu}\circ\phi^1_{F^2_\nu}dt\\ &+ \int_0^1\{\grave{E}_\nu+\langle\frac{\partial^2h_\nu}{\partial z^2}\circ(\mathbb{F}^1_\nu J)z_+,z_+\rangle,F^2_\nu\}\circ\phi^t_{F^2_\nu}dt\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &\le \frac{C\rho^{2p+2}_\nu}{\gamma^{12}_\nu\sigma^6_\nu}+\frac{C\rho^{3p+1}_\nu}{\gamma^6_\nu\sigma^4_\nu}+\frac{C\rho^{2p+2}_\nu}{\gamma^{12}_\nu\sigma^2_\nu}+\frac{C\rho^{2p+2}_\nu}{\gamma^{18}_\nu\sigma^5_\nu}\le\rho^{2p+1}_{\nu+1},\\ \|p_{\nu+1}-p_\nu\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &\le \|\int_0^1(1-t)\{\{N_\nu,F^1_\nu\},F^1_\nu\}\circ\phi^t_{F^1_\nu}\circ\phi^1_{F^2_\nu}dt\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\\ &+ \|h_\nu\circ(\mathbb{F}^1_\nu(\theta)J)\|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu}\le\frac{C\rho^{2p+2}_\nu}{\gamma^6_\nu\sigma^6_\nu}+\frac{C\rho^{3p+3}_\nu}{\gamma^{18}_\nu\sigma^3_\nu}\le\rho^{2p+1}_{\nu+1},\end{aligned}$$

which means that (4.20) and (4.21) hold when $\nu+1$ in place of ν.

By integral equations

$$\begin{aligned}\phi^t_{F^1_\nu} &= id + \int_0^t X_{F^1_\nu}(\phi^s_{F^1_\nu})ds,\\ D\phi^t_{F^1_\nu} &= Id + \int_0^t J(D^2F^1_\nu)D\phi^s_{F^1_\nu}ds,\end{aligned}$$

we have

$$\begin{aligned}|\phi^t_{F^1_\nu} - id|_{s_\nu-\frac{\sigma_\nu}{2},r_\nu-\frac{\sigma_\nu}{2},\mathcal{D}_\nu} &< \frac{C}{\rho_0^p\gamma_\nu^6\sigma_\nu^3}|\mathbb{E}^1_\nu|_{s_\nu,\mathcal{D}_\nu} \le \frac{C\rho_\nu^{p+1}}{\gamma_\nu^6\sigma_\nu^3},\\ |D\phi^t_{F^1_\nu} - Id|_{s_\nu-\frac{\sigma_\nu}{2},r_\nu-\frac{\sigma_\nu}{2},\mathcal{D}_\nu} &< \frac{C}{\rho_0^p\gamma_\nu^6\sigma_\nu^6}|\mathbb{E}^1_\nu|_{s_\nu,\mathcal{D}_\nu} \le \frac{C\rho_\nu^{p+1}}{\gamma_\nu^6\sigma_\nu^6}.\end{aligned} \tag{4.32}$$

Thus, $\phi_{F^1_\nu}(D(s_\nu - \frac{\sigma_\nu}{2}, r_\nu - \frac{\sigma_\nu}{2})) \subset D(s_\nu, r_\nu)$, $\epsilon \in \mathcal{D}_\nu$. Similarly,

$$\begin{aligned}|\phi^t_{F^2_\nu} - id|_{s_\nu-\sigma_\nu,r_\nu-\sigma_\nu,\mathcal{D}_\nu} &< \frac{C\rho_\nu^{p+1}}{\gamma_\nu^{12}\sigma_\nu^4},\\ |D\phi^t_{F^2_\nu} - Id|_{s_\nu-\sigma_\nu,r_\nu-\sigma_\nu,\mathcal{D}_\nu} &< \frac{C\rho_\nu^{p+1}}{\gamma_\nu^{12}\sigma_\nu^6},\end{aligned} \tag{4.33}$$

i.e., $\phi^1_{F^2_\nu}(D(s_\nu - \sigma_\nu, r_\nu - \sigma_\nu)) \subset D(s_\nu - \frac{\sigma_\nu}{2}, r_\nu - \frac{\sigma_\nu}{2})$, $\epsilon \in \mathcal{D}_\nu$. Now, let $\Phi_\nu = \phi^1_{F^1_\nu} \circ \phi^1_{F^2_\nu}$. Then $\Phi_\nu : D(s_{\nu+1}, r_{\nu+1}) \to D(s_{\nu+1}, r_{\nu+1})$, $\epsilon \in \mathcal{D}_\nu$, and by (4.32) and (4.33), we have

$$\begin{aligned}|\Phi_\nu - id|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &< \frac{C\rho_\nu^{p+1}}{\gamma_\nu^6\sigma_\nu^3} \le \rho_{\nu+1}^{p+\frac12},\\ |D\Phi_\nu - Id|_{s_{\nu+1},r_{\nu+1},\mathcal{D}_\nu} &< \frac{C\rho_\nu^{p+1}}{\gamma_\nu^{12}\sigma_\nu^6} \le \rho_{\nu+1}^{p+\frac12}.\end{aligned}$$

This completes the proof of the lemma. □

4.3. **Convergence.** For each $\nu = 0, 1, \cdots$, let

$$\tilde\Phi^\nu = \phi_0 \circ \phi^\epsilon \circ \Phi^\nu,$$

where

$$\Phi^\nu := \Phi_0 \circ \Phi_1 \circ \cdots \Phi_\nu \,:\, D(s_{\nu+1}, r_{\nu+1}) \to D(s_0, r_0)$$

and ϕ_0, ϕ^ϵ are the transformations defined in Section 3. If ϕ^t and $\phi^t_{\nu+1}$ denote the flows of (1.8) and $H_{\nu+1}$ respectively, then

$$\phi^t \circ \tilde\Phi^\nu = \tilde\Phi^\nu \circ \phi^t_{\nu+1}.$$

By Lemmas 4.1, 4.2 and standard arguments using Whitney extension, we see the convergence of $\tilde\Phi_\nu, \phi_\nu$, as $\nu \to \infty$, say, to $\tilde\Phi_\infty, \phi_\infty$, respectively. It follows that

$$\phi^t \circ \tilde\Phi_\infty = \tilde\Phi_\infty \circ \phi^t_\infty$$

on $D(\frac{s}{2}, \frac{r}{2}) \times \mathcal{D}_\infty$, where $\mathcal{D}_\infty = \Pi_{\epsilon_*}$ when $\lambda = -1$ and $\mathcal{D}_\infty = \cap_{\nu=0}^\infty \mathcal{D}_\nu$ when $\lambda = 1$. Since ϕ^t_∞ is the flow of

$$H_\infty = \langle \omega, I\rangle + \langle z, M_\infty(\theta)z\rangle + h_\infty + p_\infty(\theta),$$

where $M_\infty = \lim_{\nu\to\infty} M_\nu$, $h_\infty = \lim_{\nu\to\infty} h_\nu = \mathcal{O}(\|(x,y)\|^3)$, and $p_\infty = \lim_{\nu\to\infty} p_\nu$, we have

$$\phi^t(\Phi_\infty(\mathbb{T}^d, 0)) = \Phi_\infty\phi^t_\infty(\mathbb{T}^d, 0) = \Phi_\infty(\mathbb{T}^d, 0),$$

which means that the embedding torus $\Phi_\infty(\mathbb{T}^d, 0)$ is invariant under ϕ^t.

4.4. **Measure estimate for the case $\lambda = 1$.** We now estimate the measure of the Cantor set $\mathcal{D}_\infty =: \mathcal{E}_{\epsilon_*} = \cap_{\nu=0}^\infty \mathcal{D}_\nu$ for the elliptic case $\lambda = 1$. We note in this case that the normal matrixes $M_\nu =: M_\nu(\rho)$, $\nu = 0, 1, \cdots$, are independent of θ. For each $|k| > 0$ and $\nu = 0, 1, \cdots$, we let

$$\begin{aligned}f^1_{\nu k}(\epsilon) &= \langle k, \omega\rangle^2 - 4\det M_\nu(\rho),\\ f^2_{\nu k}(\epsilon) &= \langle k, \omega\rangle^2 - 16\det M_\nu(\rho),\end{aligned}$$

where $\rho = \epsilon^{\frac{1}{(m-1)(n-1)}}$. Since

$$\frac{d(\det M_\nu(\rho))}{d\rho} = \frac{d(\det(M_0 + \sum_{i=1}^\nu \mathbb{R}_i))}{d\rho}$$

and $|\frac{d(\det M_0(\rho))}{d\rho}| > c\rho^{2p-1}$, we have that

$$|\frac{d(\det M_\nu(\rho))}{d\rho}| > C\rho^{2p-1}, \qquad \rho \in \mathcal{D}_\nu,$$

$\nu = 0, 1, \cdots$. It follows that

$$|\frac{df^i_{\nu k}(\epsilon)}{d\rho}| > C\rho^{2p-1}, \qquad \rho \in \mathcal{D}_\nu, \tag{4.34}$$

for all $|k| > 0$, $\nu = 0, 1, \cdots$, and $i = 1, 2$. Denote

$$R^i_{\nu k} := \{\epsilon \in \mathcal{D} : |f^i_{\nu k}(\epsilon)| < \gamma^3_\nu / |k|^{5\tau}\},$$

$|k| > 0$, $\nu = 0, 1, \cdots$, and $i = 1, 2$. If $\frac{\gamma^2_\nu}{|k|^{2\tau}} \geq 2C\rho^{2p}$, then

$$|f^i_{\nu k}(\epsilon)| \geq \frac{\gamma^2}{|k|^{2\tau}} - C\rho^{2p} \geq \frac{\gamma^2}{2|k|^{2\tau}} \geq \frac{\gamma^3}{|k|^{5\tau}},$$

implying that meas$R^i_{\nu,k} = 0$. If $\frac{\gamma^2_\nu}{|k|^{2\tau}} < 2C\rho^{2p}$, then we have by (4.34) that

$$\text{meas} R^i_{\nu k} \;\leq\; \frac{\gamma^3_\nu}{c|k|^{5\tau}\epsilon^{\frac{2p}{(m-1)(n-1)}-1}} \leq \frac{C\gamma_\nu \epsilon^{\frac{4p}{(m-1)(n-1)}}}{|k|^\tau \epsilon^{\frac{2p}{(m-1)(n-1)}-1}} \leq \frac{C\gamma_\nu}{|k|^\tau}\epsilon^{1+\frac{2p}{(m-1)(n-1)}}.$$

It follows that

$$\text{meas}(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ \nu\in\mathbb{Z}_+\setminus\{0\}}} R^i_{\nu k}) \;\leq\; C\epsilon^{1+\frac{2p}{(m-1)(n-1)}} \sum_{\nu=0}^{\infty}\gamma_\nu \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^\tau} \leq C\epsilon^{1+\frac{2p}{(m-1)(n-1)}}$$

for $i = 1, 2$. Hence

$$\text{meas}\mathcal{E}_{\epsilon_*} \;\geq\; \text{meas}\mathcal{D} - \text{meas}(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\},\nu\in\mathbb{Z}_+\setminus\{0\},\\ 1\leq i\leq 2}} R^i_{\nu k}) \geq \epsilon_*(1 - C\epsilon_*^{\frac{2p}{(m-1)(n-1)}}),$$

implying that $\mathcal{E}_{\epsilon_*} \sim \epsilon_*$ as ϵ_* sufficiently small.

□

5. NON-EXISTENCE

In this section, we show that quasi-periodic response solutions need not exist when l is even in system (1.3). We note that the Hamiltonian system corresponding to (1.3) in extended phase space simply reads

$$\begin{cases} \dot\theta = \omega, \\ \dot x = y, \\ \dot y = -\lambda x^{2n} + \epsilon f(\theta, x, \epsilon). \end{cases} \tag{5.1}$$

Let

$$c_0 = \inf_{\theta\in\mathbb{T}^d} f(\theta, 0, 0).$$

We have the following result.

Proposition 5.1. *Assume that $c_0 > 0$ when $\lambda = -1$ and $c_0 < 0$ when $\lambda = 1$. Then there exist a $\epsilon(c_0) > 0$ such that system (5.1) admits no response solution for any $\epsilon \in (0, \epsilon(c_0))$.*

Proof. System (5.1) can be written as

$$\begin{cases} \dot\theta = \omega, \\ \dot x = y, \\ \dot y = -\lambda x^{2n} + \epsilon f(\theta, 0, 0) + \epsilon(f(\theta, x, 0) - f(\theta, 0, 0)) + \mathcal{O}(\epsilon^2), \end{cases} \tag{5.2}$$

which, under the transformation $x = \epsilon^{\frac{1}{2n}}\tilde{x}$, $y = \epsilon\tilde{y}$, becomes

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\tilde{x}} = \epsilon^{\frac{2n-1}{2n}} y, \\ \dot{\tilde{y}} = (-\lambda\tilde{x}^{2n} + f(\theta,0,0)) + \mathcal{O}(\epsilon^{\frac{1}{2n}}). \end{cases} \tag{5.3}$$

If $\lambda = -1$, then one can find a $\epsilon(c_0) > 0$ such that $\dot{\tilde{y}} = (-\lambda\tilde{x}^{2n} + f(\theta,0,0)) + \mathcal{O}(\epsilon^{\frac{1}{2n}}) \geq \frac{c_0}{2}\epsilon^{\frac{2n-1}{2n}}$ for each $\epsilon \in (0, \epsilon(c_0))$, and, if $\lambda = 1$, then one can find a $\epsilon(c_0)$ such that $\dot{\tilde{y}} = (-\lambda\tilde{x}^{2n} + f(\theta,0,0)) + \mathcal{O}(\epsilon^{\frac{1}{2n}}) \leq -\frac{c_0}{2}\epsilon^{\frac{2n-1}{2n}}$ for each $\epsilon \in (0, \epsilon(c_0))$. In either case, the $\tilde{y}$ - components of all solutions of (5.3) are unbounded, hence there are no quasi-periodic solutions for system (5.1). □

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