RESPONSE SOLUTIONS IN DEGENERATE OSCILLATORS UNDER DEGENERATE PERTURBATIONS

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Abstract. For a quasi-periodically forced differential equation, response solutions are quasi-periodic ones whose frequency vector coincides with that of the forcing function and they are known to play a fundamental role in the harmonic and synchronizing behaviors of quasi-periodically forced oscillators. These solutions are well-understood in quasi-periodically perturbed nonlinear oscillators either in the presence of large damping or in the non-degenerate cases with small or free damping.

In this paper, we consider the existence of response solutions in quasi-periodically perturbed, second order differential equations, including nonlinear oscillators, of the form

$$\ddot{x} - \lambda x^l = \epsilon f(\omega t, x, \dot{x}), \qquad x \in \mathbb{R},$$

where λ is a constant, $0 < \epsilon \ll 1$ is a small parameter, $l > 1$ is an integer, $\omega \in \mathbb{R}^d$ is a frequency vector, and $f : \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}^1$ is real analytic and non-degenerate in x up to a given order $p \geq 0$, i.e., $[f(\cdot, 0, 0)] = [\frac{\partial f(\cdot,0,0)}{\partial x}] = [\frac{\partial^2 f(\cdot,0,0)}{\partial x^2}] = \cdots = [\frac{\partial^{p-1} f(\cdot,0,0)}{\partial x^{p-1}}] = 0$ and $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] \neq 0$, where $[\ ]$ denotes the average value of a continuous function on $\mathbb{T}^d$. In the case that $\lambda = 0$ and f is independent of $\dot{x}$, the existence of response solutions was first shown by Gentile in [12] when $p = 1$. This result was later generalized by Corsi and Gentile in [8]-[10] to some cases of $p > 1$. In the case $\lambda \neq 0$, the existence of response solutions is studied by the authors in [27] when $p = 0$.

The present paper is devoted to the study of response solutions of the above quasi-periodically differential equations for the case $\lambda \neq 0$ by allowing $p > 0$. Under the conditions that $0 \leq p < l/2$ and $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]>0$ when $l - p$ is even, we obtain a general result which particularly implies the following: (1) If either l is odd and $\lambda<0$ or l is even and $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]>0$, then as ϵ sufficiently small response solutions exist for each ω satisfying a Brjuno-like non-resonant condition; (2) If either l is odd and $\lambda>0$ or l is even and $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]<0$, then there exists an $\epsilon_* > 0$ sufficiently small and a Cantor set $\mathcal{E} \in (0, \epsilon_*)$ with almost full Lebesgue measure such that response solutions exist for each $\epsilon \in \mathcal{E}$ and ω satisfying a Diophantine condition.

Similar results are also obtained in the case $\lambda = \pm\epsilon$ which particularly concern the existence of large amplitude response solutions.

1. Introduction

Consider a quasi-periodically forced, perturbative, second order nonlinear differential equation

$$\ddot{x} + c\dot{x} + a^2 x + \lambda x^l = \epsilon f(\omega t, x, \dot{x}), \tag{1.1}$$

where c, a, λ are constants, $0 < \epsilon \ll 1$ is a small parameter, $l > 1$ is an integer, $f : \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}$, for some $d \geq 1$, is a real analytic function, and $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$ is the forcing frequency vector. A solution of (1.1) is said to be a *response solution* if it is quasi-periodic with the same frequency vector ω as that of the forcing function. These solutions play a fundamental role in

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understanding the harmonic responses and oscillatory properties of the forced nonlinear equation (1.1) as well as their related synchronizing behaviors ([8]). Response solutions are well expected when c in (1.1) is sufficiently large in absolute value comparing to the size of λ and f, as shown in the classical work of Stoker ([28]). When c in (1.1) is in the intermediate size, i.e., neither too large nor too small, response solutions are not generally expected. Instead, (1.1) tends to have weakly quasi-periodic or almost automorphic type of oscillatory solutions responding to the forcing frequency vector ([31]).

The study for the existence of response solutions of (1.1) becomes much more complicated when c in (1.1) is sufficiently small or equal to 0 because of the unavoidability of the small divisor problem. In the case $c = 0$, $a \neq 0$, (1.1) becomes a small perturbation of a harmonic or a nonlinear oscillator. As an early introduction of the KAM method, the first breakthrough in this case was made by Moser ([24]), who showed the existence of response solutions of (1.1) when ω is Diophantine and f satisfying the reversible condition that $f(-\omega t, x, -\dot{x}, \epsilon) \equiv f(\omega t, x, \dot{x}, \epsilon)$. This result was later extended in [1, 11] to the case when $c \neq 0$ but sufficiently small, by using KAM method and normal form theory. In general, response solutions do not need to be of Diophantine type. For the case $c = 0$, it is recently shown in [23, 29] for the case $d = 2$ and in [6, 30]) for the general case that response solutions of (1.1) with Liouvillean type of forcing frequencies also exist. Using Lyapunov-Schmidt reduction, the existence of response solutions of Brjuno type is investigated in [13, 14] for some general strongly dissipative, quasi-periodically forced systems, including some cases of (1.1).

The most challenging case of (1.1) with respect to the study of the existence of response solutions is when $c = a = 0$, i.e.,

$$\ddot{x} + \lambda x^l = \epsilon f(\omega t, x, \dot{x}). \tag{1.2}$$

We note that the unperturbed part of (1.2) is a degenerate nonlinear oscillator when $\lambda > 0$. When f is independent of $\dot{x}$, (1.2), in the extended phase space $\mathbb{T}^d \times \mathbb{R}^{d+2}$ with standard symplectic structure, becomes a Hamiltonian system with the following normally degenerate Hamiltonian

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I \rangle + \frac{y^2}{2} + \lambda \frac{x^{l+1}}{l+1} + \epsilon P(\theta, x, \epsilon), \qquad (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2}, \tag{1.3}$$

where $P(\theta, x, \epsilon) = -\int_0^x f(\theta, u)du$, whose quasi-periodic invariant d-tori with frequency vector ω are *response tori* corresponding to response solutions of (1.2). Such normal degeneracy brings in extra difficulties into the KAM method when studying quasi-periodic invariant tori even not necessary response ones, as already demonstrated in [17, 22, 32] for general normally degenerate Hamiltonian systems. However, as part of quasi-periodic bifurcation theory, the existence of quasi-periodic invariant tori, including response tori, is studied in [2]-[5], [18]-[20] with respect to the universal unfolding of certain quasi-periodically forced, normally degenerate Hamiltonian systems of form (1.3) in which certain perturbative, one parameter families can be embedded. When l is odd, $\lambda < 0$, and ω is Diophantine, the existence of an invariant, quasi-periodic d-torus with frequency vector close to ω is shown in [32] for Hamiltonian of the form (1.3) but with a more general I-dependent perturbation P. This result is later generalized in [21] for non-Hamiltonian but time-reversible cases. We note that the method of [32] does apply to (1.3) to yield response solutions when l is odd, $\lambda < 0$, and ω is Diophantine.

Recently, we have made a general study in [27] on the existence of response tori even for systems which are more general than (1.3). More precisely, we considered in [27] the following family of normally degenerate, including completely degenerate, Hamiltonian systems with Hamiltonians:

$$H(\theta, I, x, y, \epsilon) = \langle \omega, I \rangle + \frac{y^m}{m} + \lambda \frac{x^n}{n} + \epsilon P(\theta, x, y, \epsilon), \qquad (\theta, I, x, y) \in \mathbb{T}^d \times \mathbb{R}^{d+2}, \tag{1.4}$$

where $\lambda \neq 0$, m, n are positive integers greater or equal to 2, P is real analytic in θ, x, y and C^1-Whitney smooth in ϵ. Under the condition that the θ-averages of the linear coefficients of $P(\theta, x, y, 0)$ with respect to x, y are non-zero, the following results are shown in [27]: (i) When $\lambda < 0$, there exist a sufficiently small ϵ_* such that, for each $\epsilon \in (0, \epsilon_*)$, the Hamiltonian system associated to (1.4) admits quasi-periodic response tori whose frequency vector ω satisfies the following Brjuno-like non-resonant condition, weaker than the Brjuno's, that

$$|\langle k,\omega\rangle| > \frac{\gamma}{\Omega(|k|)}, \qquad \forall\ k\in\mathbb{Z}^d\backslash\{0\}, \tag{1.5}$$

where $\gamma > 0$ is a fixed constant and $\Omega(t) : [1,\infty) \to [1,\infty)$ is a continuous function with $\sup_{t\geq 1}\Omega(t)e^{-\eta t} < \infty$ for some constant $\eta > 0$. (ii) When $\lambda > 0$, there exist a sufficiently small ϵ_* and a Cantor set $\mathcal{E}\subset(0,\epsilon_*)$ with almost full Lebesgue measure such that, for each $\epsilon\in\mathcal{E}$, the Hamiltonian system associated to (1.4) admits quasi-periodic response tori whose frequency vector ω satisfies the Diopantine condition that

$$|\langle k,\omega\rangle| > \gamma|k|^{-\tau}, \quad \forall k\in\mathbb{Z}^d\setminus\{0\}, \tag{1.6}$$

where $\gamma > 0,\ \tau > d-1$ are fixed constants.

We note that when applying the above results to (1.2) in the case either $\lambda > 0$ and l is odd or $\lambda\neq 0$ and l is even, the condition on P becomes $[f(\cdot,0,0)]\neq 0$. For the first order, quasi-periodically forced equation

$$\dot{x} = x^l + h(\omega t,x) + \epsilon f(\omega t,x), \qquad x\in\mathbb{R}, \tag{1.7}$$

where $l\geq 2$ is an integer, $0<\epsilon\ll 1$ is a parameter, $h=\mathcal{O}(|x|^{l+1})$, and f, h: $\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ are real analytic, it is shown by Cheng, de la Llave, and Wang in [7], as a generalization to [26], that response solutions of (1.7) exist if $\omega\in\mathbb{R}^d$ is Diophantine, $[f(\cdot,0)]=0$ and $[\frac{\partial f(\cdot,0)}{\partial x}]\neq 0$. However, as it is also shown in [7], the same method when applying to the case of degenerate nonlinear oscillator in (1.2) still requires that $[f(\cdot,0,0)]\neq 0$. In the above and also below, for any continuous function f on $\mathbb{T}^d$, $[f]$ denotes its average, i.e., $[f]=\int_{\mathbb{T}^d} f(\theta)d\theta$.

The existence of response solutions of (1.2) with zero average perturbation was first investigated by Gentile in [12]. For the case that $\lambda=0$ in (1.2), i.e.,

$$\ddot{x} = \epsilon f(\omega t,x,\dot{x}), \tag{1.8}$$

it is proved in [12] that if f is independent of $\dot{x}$ and $[f(\cdot,x)]$ admits a non-degenerate zero, then response solutions with Brjuno frequencies exist for sufficiently small ϵ. This result was later generalized by Corsi and Gentile, in [8] for the case that f is independent of $\dot{x}$ and in [9] for the general case, to allow the existence of a degenerate zero of $[f(\cdot,x,0)]$ of any odd order. Similar results were also obtained by the same authors in [10] when x is higher dimensional, f is independent of $\dot{x}$ and reversible in time, and $[f(\cdot,x)]$ admits a finitely degenerate zero.

In this paper, we consider the existence of response solutions of (1.2) in the case $\lambda\neq 0$ by allowing any finite order of degeneracy of the perturbation f in x. More precisely, we consider (1.2) with $[f(\cdot,x,0)]\not\equiv 0$ and let $p\geq 0$ be the leading order of non-degeneracy of $[f(\cdot,x,0)]$ at $x=0$, i.e., p is such that

$$[f(\cdot,0,0)] = [\frac{\partial f(\cdot,0,0)}{\partial x}] = [\frac{\partial^2 f(\cdot,0,0)}{\partial x^2}] = \cdots = [\frac{\partial^{p-1} f(\cdot,0,0)}{\partial x^{p-1}}] = 0, \qquad [\frac{\partial^{p} f(\cdot,0,0)}{\partial x^{p}}]\neq 0.$$

We pay particular attentions to the following two cases:

I. $\lambda\neq 0$ is given;
II. $\lambda=\pm\epsilon$.

We assume that

(H) $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]>0$ when $l-p$ is even

and denote

$$\hat{\lambda} =: -(l-p)\lambda(\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{l-1}{l-p}}.$$

We will show the following result.

Main Theorem. *Consider cases I, II for* (1.2) *and assume* **(H)**. *Further assume the following conditions:*

(H1) $0 \le 2p < l$ *in case I;*

(H2) $p \neq l$ *and* $f(\cdot, x, 0)$ *is a p-th order polynomial of* x *in case II.*

Then the following holds:

(1) *If* $\hat{\lambda} > 0$, *then for any* ω *satisfying the Brjuno-like non-resonant condition* (1.5) *with fixed* $\gamma, \eta > 0$, *there exists an* $\epsilon_* > 0$ *sufficiently small, depending on* λ *in case I, such that, as* ϵ *varies in* $(0, \epsilon_*)$, (1.2) *admits a* C^1 *smooth family of response solutions* $x_\epsilon(t) = (\frac{\epsilon}{\lambda}[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{1}{l-p}} + o(\epsilon^{\frac{1}{|l-p|}})$ *with the frequency vector* ω;

(2) *If* $\hat{\lambda} < 0$, *then for any* ω *satisfying the Diophantine condition* (1.6) *with fixed* $\gamma > 0$ *and* $\tau > d-1$, *there exists an* $\epsilon_* > 0$ *sufficiently small, depending on* λ *in case I, and a Cantor set* $\mathcal{E} \subset (0, \epsilon_*)$ *with almost full Lebesgue measure such that, as* ϵ *varies in* $\mathcal{E}$, (1.2) *admits a* C^1*-Whitney smooth family of response solutions* $x_\epsilon(t) = (\frac{\epsilon}{\lambda}[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{1}{l-p}} + o(\epsilon^{\frac{1}{|l-p|}})$ *with the frequency vector* ω.

By examining the sign of $\hat{\lambda}$ v.s. that of λ and $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]$, we easily obtain the following results for l being odd or even.

Corollary A. *Let* l *be odd and assume conditions of the* Main Theorem. *Then* $\hat{\lambda} > 0$ *iff* $\lambda < 0$ *in case I and* $(l-p)\lambda < 0$ *in case II. Consequently, the conclusion of* Main Theorem (1) (*resp.* (2)) *holds when* $\lambda < 0$ *in case I and* $(l-p)\lambda < 0$ *in case II* (*resp.* $\lambda > 0$ *in case I and* $(l-p)\lambda > 0$ *in case II*).

Corollary B. *Let* l *be even and assume conditions of the* Main Theorem. *Then* $\hat{\lambda} > 0$ *iff* $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] < 0$ *in case I and* $(l-p)[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] < 0$ *in case II. Consequently, the conclusion of* Main Theorem (1) (*resp.* (2)) *holds when* $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] < 0$ *in case I and* $(l-p)[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] < 0$ *in case II* (*resp.* $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] > 0$ *in case I and* $(l-p)[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] > 0$ *in case II*).

Remark. 1) In the above results, amplitudes of response solutions are in the scale of $\epsilon^{\frac{1}{l-p}}$ in case I and in the scale of $(|[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]|)^{\frac{1}{l-p}}$ in case II. We thus obtain large amplitude response solutions in case II for any ϵ sufficiently small.

2) The condition **(H)** assumed in the Main Theorem above is necessary in the following sense: If the condition **(H)** fails, i.e., $l-p$ is even but $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]<0$, then (1.2) admits no response solution of amplitude in the scale which is greater than or equal to $\epsilon^{\frac{1}{|l-p|}}$ for any ϵ sufficiently small independent of λ (see Section 4 for detail).

3) The existence of response solutions showed in the Main Theorem above follows from a special mechanism of perturbing relative equilibria that exist due to the nonlinearity and the p-th order non-degeneracy of $[f(\cdot, x, 0)]$. Such a mechanism is different than the ones considered in existing works on response solutions of quasi-periodically forced, second order nonlinear differential equations. Thus, our results, though overlap with some of the existing ones, do not generalize them completely. For instance, for fix $\lambda < 0$, the existence of response solutions of (1.2) follows from the approach of [32] when l is odd without any restriction on the perturbation. Our existence result in this case requires additional conditions **(H)** and (a) but with a weaker non-resonance condition on the forcing frequency vector. We also obtain results for the cases l is even and $\lambda > 0$ which do not follow from the approach of [32]. When $\lambda \neq 0$, $0 < p < l$ is odd, and $[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}] < 0$, the re-scaling $x \to \epsilon^{\frac{1}{l}} x$ to (1.8) yields

$$\ddot{x} = \epsilon^{\frac{l-1}{l}} F(\omega t, x, \dot{x}, \epsilon), \tag{1.9}$$

where $F(\omega t, x, \dot{x}, \epsilon) = \lambda x^l + f(\omega t, \epsilon^{\frac{1}{l}} x, \epsilon^{\frac{1}{l}} \dot{x})$. It follows that $x = 0$ is a zero of $[F(\cdot, x, 0, 0)]$ of odd order p, and consequently, by [9, Theorem ???] (see also [8, Theorem 2.1] for the $\dot{x}$-independent case), (1.9) hence (1.8) admits a response solution. This result does not follow from our Main

Theorem. Nevertheless, our results, though have additional conditions, do allow p to be even as well.

4) The condition **(H1)**, respectively **(H2)**, imposed in the Main Theorem above is a technical condition ensuing that the implicit equation (2.9), respectively (2.8), is perturbative hence solvable in case I, respectively case II. The existence of response solutions in either case without these conditions or with weaker conditions is definitely an interesting question worthy for a further investigation.

Our Main Theorem will be proved by conducting normal form reduction and using KAM iterations, for which mathematical difficulties caused by the degeneracies of both unperturbed oscillator and the perturbation need to be encountered. Our approach is divided into three steps. The first step is to eliminate the zero-averaging terms of the perturbation and re-scale the x-variable by solving a degenerate implicit function equation. The second step is to transform the system to the vicinity of the relative equilibrium then improve the order of perturbation by eliminating the zero-th and the first order resonant terms. With the normal form whose order of perturbation doubles that of the linear part, KAM iterations are performed in the third step.

The rest of this paper is organized as follows. In Section 2, we reduce the system to a normal form by taking the normal degeneracy into considerations. In Section 3, we perform KAM iterations to the normal form to obtain results which are more general than the Main Theorem. In Section 4, we present a non-existence result for response solutions when the condition **(H)** fails.

Throughout of the paper, we will use the symbol $\preceq$ to denote $\leq$ up to a positive constant multiple. Unless specified otherwise, $|\cdot|$ will be used to denote the absolute value of numbers, the norm of vectors, and the norm of matrices. When ϵ lies in a bounded set, all derivatives with respect to ϵ are taken in the sense of Whitney.

2. Normal form reduction

In this section, we reduce (1.2) into a suitable normal form in order to apply KAM iterations. In fact, we will work with the following first order system equivalent to (1.2):

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = -\lambda x^l + \epsilon f(\theta, x, y). \end{cases} \tag{2.1}$$

It is clear that response solutions of (1.2) correspond to response tori of (2.1).

For given $r, s, \epsilon_* > 0$, we denote

$$D(s,r) = \mathbb{T}^d_s \times \mathbf{B}_r,$$

where

$$\begin{aligned} \mathbb{T}^d_s &:= \{\theta = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : \ |\mathrm{Im}\theta_j| \leq s, \quad j = 1, 2, \dots, d\}, \\ \mathbf{B}_r &:= \{z = (z_1, z_2) \in \mathbb{C}^2 : \ |z| \leq r\}. \end{aligned}$$

We note that $\mathbb{T}^d_s$ is the strip neighborhood of size s of the d-torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ in $\mathbb{C}^d$. Let $B \subset (0, \epsilon_*)$ be a given subset. For any function $F: \ D(s,r) \times B \to \mathbb{C}$,

$$F(\theta, z, \epsilon) = \sum_{k \in \mathbb{Z}^d} F_k(z, \epsilon) e^{\mathrm{i}\langle k, \theta \rangle}$$

which is analytic in $(\theta, z) \in D(s,r)$ and C^1-Whitney smooth in $\epsilon \in B$, we define its $\|\cdot\|_{s,r,B}$-norm by

$$\|F(\theta, z, \epsilon)\|_{s,r,B} = \sum_{k \in \mathbb{Z}^d} \|F_k(z, \epsilon)\|_{r,B} e^{s|k|},$$

where $\|F_k\|_{r,B} = \sup_{\epsilon \in B, z \in \mathbf{B}_r} (|f_k(z,\epsilon)| + \epsilon|\frac{\partial f_k(z,\epsilon)}{\partial \epsilon}|)$ and $|k| = \sum_{j=1}^d |k_j|$ for $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$. The space of all such functions with finite $\|\cdot\|_{s,r,B}$-norm is denoted by $C^{\omega,1}(D(s,r) \times B, \mathbb{C})$ which is easily seen to be a Banach algebra under the norm $\|\cdot\|_{r,s,B}$. Taking $r=0$ in the above, we can similarly define the $\|\cdot\|_{s,B}$-norm of any function $F:\mathbb{T}_s^d\times B\to\mathbb{C}$,

$$F(\theta,\epsilon)=\sum_{k\in\mathbb{Z}^d}F_k(\epsilon)e^{\mathrm{i}\langle k,\theta\rangle}$$

which is analytic in $\theta\in\mathbb{T}_s^d$ and C^1-Whitney smooth in $\epsilon\in B$. The Banach algebra of all such functions under the $\|\cdot\|_{s,B}$-norm is denoted by $C^{\omega,1}(\mathbb{T}_s^d\times B,\mathbb{C})$. For any matrix-valued function $P(\theta,z,\epsilon)=(P_{ij}(\theta,z,\epsilon))_{n\times m}$ with $P_{ij}(\theta,z,\epsilon)\in C^{\omega,1}(D\ (s,r)\times B,\mathbb{C})$, $1\le i\le n,\ 1\le j\le m$, we still use the same symbol to denote its $\|\cdot\|_{r,s,B}$-norm which is defined by

$$\|P\|_{r,s,B}=\max_{1\le i\le n}\sum_{j=1}^{m}\|P_{ij}\|_{r,s,B}.$$

The $\|\cdot\|_{s,B}$ norm of a matrix-valued function $P(\theta,\epsilon)=(P_{ij}(\theta,\epsilon))_{n\times m}$ with $P_{ij}(\theta,\epsilon)\in C^{\omega,1}(\mathbb{T}_s^d\times B)$, $1\le i\le n,\ 1\le j\le m$, is defined similarly by taking $r=0$ in the above. We use $C^{\omega,1}(\mathbb{T}_s^d\times B,\mathbb{C}^n)$ (resp. $C^{\omega,1}(\mathbb{T}_s^d\times B,gl(n,\mathbb{C}))$) to denote the Banach algebra of all n-vector-valued (resp. $n\times n$ matrix-valued) functions, under the $\|\cdot\|_{r,s,B}$ norm. For any bounded function F on a domain Ω of an Euclidean space, we use the short notion $|F|_\Omega$ to denote the sup-norm of F on Ω.

Through the rest of the paper, we fix $s,r,\epsilon_*>0$ so that in (1.2), f is real analytic in $D(s,r)$ and $\epsilon\in(0,\epsilon_*)$. Denote

$$\tilde\lambda:=-(l-p)\lambda(\lambda^{-1}[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{l-1}{l-p}}.$$

It is clear that $\hat\lambda$ and $\tilde\lambda$ always have the same sign.

We will show the following result in this section.

Proposition 2.1. *Consider system* (2.1) *in* $D(s,r)\times(0,\epsilon_*)$ *with* ω *satisfying* (1.6)*, respectively* (1.5)*, in the case* $\tilde\lambda<0$*, respectively* $\tilde\lambda>0$*, with respect to fixed* $\gamma,\eta>0$ *and* $\tau>d-1$*. Assume* **(H)**, **(H1)** *in case I, and* **(H2)** *in case II. If* ϵ_* *is sufficiently small, then there exists a family of real analytic transformations*

$$\phi^\epsilon:D(\tilde s,\tilde r)\to D(s,r),\quad \epsilon\in\mathcal{D},$$

where $\tilde s=s-\sigma$, $\tilde r=r-\sigma$, $0<\sigma\le\min\{r/4,s/4\}$ *is a fixed constant,* $\mathcal{D}=\mathcal{D}_{\epsilon_*}$ *is a Cantor set with Lebesgue measure* meas $(\mathcal{D})=\mathcal{O}(\epsilon_*)$ *in the case* $\tilde\lambda<0$*, and* $\mathcal{D}=(0,\epsilon_*)$ *in the case* $\tilde\lambda>0$*, such that the following hold.*

(a) ϕ^ϵ, $\epsilon\in\mathcal{D}$, *is a* C^1*-Whitney smooth family in the case* $\tilde\lambda<0$ *and a* C^1*-smooth family in the case* $\tilde\lambda>0$.

(b) *For each* $\epsilon\in\mathcal{D}$, ϕ^ϵ *transforms* (2.1) *into the system*

$$\begin{cases}\dot\theta=\omega,\\ \dot z_1=(\delta\breve A(\epsilon)+\breve Q(\theta,\epsilon))z_1+\delta\breve G(\theta,z_1,\epsilon)+\breve E(\theta,\epsilon),\end{cases}\tag{2.2}$$

where $\delta=\epsilon^{\frac{l-1}{2|l-p|}}$, $\breve A(\epsilon)=A(\epsilon)+\mathcal{O}(\delta)$ *with*

$$A(\epsilon)=\begin{pmatrix}0&1\\ \tilde\lambda&0\end{pmatrix}+\mathcal{O}(\epsilon^{\frac{p-1}{|l-p|}}),\tag{2.3}$$

$\breve G(\theta,z_1,\epsilon)=\mathcal{O}(|z_1|^2)\in C^{\omega,1}(D(\tilde s,\tilde r)\times\mathcal{D},\mathbb{C}^2)$*, and* $\breve Q(\theta,\epsilon)\in C^{\omega,1}(\mathbb{T}_{\tilde s}^d\times\mathcal{D},gl(2,\mathbb{C}))$, $\breve E(\theta,\epsilon)\in C^{\omega,1}(\mathbb{T}_{\tilde s}^d\times\mathcal{D},\mathbb{C}^2)$ *satisfy*

$$\|\breve Q(\theta,\epsilon)\|_{\tilde s,\mathcal{D}}\preceq\delta^2,\quad \|\breve E(\theta,\epsilon)\|_{\tilde s,\mathcal{D}}\preceq\delta^2.$$

We note that, for ϵ sufficiently small, $\breve A(\epsilon)$ is of elliptic type if $\tilde\lambda<0$ and of hyperbolic type if $\tilde\lambda>0$.

Below, we prove the proposition in two steps. The first step is to average the perturbation up to order p in x and eliminate the averaged terms by transforming the system to the vicinity of the relative equilibrium. The second step is to remove both the zero-th and the first order resonant terms in the perturbation in order to make the new perturbation sufficiently small for carrying out KAM iterations.

2.1. **Averaging and relative equilibrium.** We first state the following technical lemma.

Lemma 2.1. *Consider the equation*

$$\partial_\omega V(\theta) = F(\theta),$$

where $\partial_\omega = \omega \cdot \frac{\partial}{\partial\theta}$ *for some* $\omega \in \mathbb{R}^d$ *and* $F \in C^\omega(\mathbb{T}_s^d, \mathbb{C})$ *with* $[F] = 0$. *If* ω *satisfies the non-resonant condition* (1.5) *with respect to fixed* $\gamma > 0$, $0 < \eta < s$, *then the equation admits a unique solution* $V \in C^\omega(\mathbb{T}_{s-\eta}^d, \mathbb{C})$ *with zero-average such that*

$$\|V\|_{s-\eta} \leq C\|F\|_s,$$

where $C = \frac{\sup_{t>0} \Omega(t)e^{-\eta t}}{\gamma}$.

Proof. The lemma simply follows from substituting the Fourier expansions of F and V into the equation and compariing coefficients. □

The following is our first reduction result.

Lemma 2.2. *Consider the system* (2.1) *in* $D(s, r)\times(0, \epsilon_*)$ *with* ω *satisfying either the non-resonant condition* (1.5) *for fixed* $\gamma > 0$ *and* $0 < \eta < s$ *or the Diophantine condition* (1.6) *for fixed* $\gamma > 0$ *and* $\tau > d-1$. *Assume* **(H)**, **(H1)** *in case I, and* **(H2)** *in case II. If* ϵ_* *is sufficiently small, then there exists a* C^1 *smooth family of real analytic transformations*

$$\phi_0 = \phi_0(\epsilon) : D(s - \sigma/4, r - \sigma/4) \to D(s, r), \qquad \epsilon \in (0, \epsilon_*),$$

where $0 < \sigma \leq \min\{r/4, s/4\}$ *is a fixed constant, which, for each* $\epsilon \in (0, \epsilon_*)$, *transforms the system* (2.1) *into*

$$\begin{cases} \dot\theta = \omega, \\ \dot z = (\epsilon^{\frac{l-1}{2|l-p|}} A(\epsilon) + Q(\theta, \epsilon))z + \epsilon^{\frac{l-1}{2|l-p|}} G(\theta, z, \epsilon) + E(\theta, \epsilon), \end{cases} \tag{2.4}$$

where $z = (x, y)^\top$, $[E(\cdot, \epsilon)] = 0$, $[Q(\cdot, \epsilon))] = 0$, $G(\theta, z, \epsilon) = \mathcal{O}(|z|^2) \in C^{\omega,1}(D(s - \frac{\sigma}{4}, r - \frac{\sigma}{4}) \times (0, \epsilon), \mathbb{C}^2)$, $A(\epsilon)$ *is as in* (2.3), *and* $E(\theta, \epsilon) \in C^{\omega,1}(\mathbb{T}^d_{s-\frac{\sigma}{4}} \times (0, \epsilon), \mathbb{C}^2)$, $Q(\theta, \epsilon) \in C^{\omega,1}(\mathbb{T}^d_{s-\frac{\sigma}{4}} \times (0, \epsilon), gl(2, \mathbb{C}))$ *with*

$$\|Q\|_{s-\frac{\sigma}{4},(0,\epsilon)} \preceq \epsilon^{\frac{l-1}{2|l-p|}}, \quad \|E\|_{s-\frac{\sigma}{4},(0,\epsilon)} \preceq \epsilon^{\frac{l-1}{2|l-p|}}.$$

Proof. We only treat the case when ω satisfies the Brjuno-like condition (1.5) because the Diophantine condition (1.6) is stronger. Rewrite $f(\theta, x, y)$ in (2.1) as

$$f(\theta, x, y) = f_0(\theta, x) + f_1(\theta, x, y)y,$$

where

$$f_0(\theta, x) = \sum_{i=0}^{p} f_{i0}(\theta)x^i + f_2(\theta, x), \ \ f_1(\theta, x, y)y = f(\theta, x, y) - f(\theta, x, 0),$$

with $f_{i0}(\theta) = \frac{\partial^i f(\theta,0,0)}{\partial x^i}$, $i = 1, \cdots, p$, and

$$f_2(\theta, x) = \begin{cases} \mathcal{O}(|x|^{p+1}), & \text{in case I,} \\ 0, & \text{in case II.} \end{cases}$$

For fixed $0 < \sigma \leq \min\{r/4, s/4\}$, suppose without loss of generality that $\eta < \sigma/8$. By Lemma 2.1, the equation

$$\partial_\omega v_i(\theta) = f_{i0}(\theta) - [f_{i0}(\theta)] \tag{2.5}$$

for each $i = 0, \cdots, p$, is uniquely solvable in $C^{\omega,1}(\mathbb{T}^d_{s-\frac{1}{8}\sigma}, \mathbb{C})$. Consider the transformation $\tilde{\phi}_1 : D(s-\sigma/8, r-\sigma/8) \to D(s, r)$ defined by

$$x = x_1, \quad y = y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x^i,$$

where, for each $i = 0, \cdots, p$, $v_i(\theta) \in C^{\omega,1}(\mathbb{T}^d_{s-\frac{1}{8}\sigma}, \mathbb{C})$ is the solution of (2.5). Since $[f_{i0}] = 0$, $i = 0, \cdots, p-1$, system (2.1) under this transformation becomes

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x}_1 = y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i, \\ \dot{y}_1 = -\lambda x_1^l + \epsilon [f_{p0}] x_1^p + \epsilon f_2(\theta, x_1) + \epsilon f_1(\theta, x_1, y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i)(y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i) \\ \quad -\epsilon \sum_{i=1}^{p} v_i(\theta) x_1^{i-1} (y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i). \end{cases} \tag{2.6}$$

Since $[v_{i0}] = 0$, $i = 0, \cdots, p$, equilibria of the averaged system of (2.6) satisfy the degenerate implicit equation

$$\begin{cases} y_1 = 0, \\ -\lambda x_1^l + \epsilon [f_{p0}] x_1^p + \epsilon [f_2(\cdot, x_1)] + \epsilon [f_1(\cdot, x_1, y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i)] y_1 + \epsilon^2 [f_3(\cdot, x_1, y_1)] = 0, \end{cases} \tag{2.7}$$

i.e.,

$$-\lambda x_1^l + \epsilon [f_{p0}] x_1^p + \epsilon [f_2(\cdot, x_1)] + \epsilon^2 [f_3(\cdot, x_1, 0)] = 0, \tag{2.8}$$

where

$$f_3(\theta, x_1, y_1) = f_1(\theta, x_1, y_1 + \epsilon \sum_{i=0}^{p} v_i(\theta) x_1^i) \sum_{i=0}^{p} v_i(\theta) x_1^i + \sum_{i=1}^{p} v_i(\theta) x_1^{i-1} \sum_{i=0}^{p} v_i(\theta) x_1^i.$$

In case II, since $\lambda = \pm\epsilon$ and $f_2 \equiv 0$, the implicit function theorem can be applied to the degenerate implicit equation (2.8) immediately to yield a smooth family of solutions $x(\epsilon) = (\pm[f_{p0}])^{\frac{1}{l-p}} + \mathcal{O}(\epsilon)$. To solve the degenerate implicit equation (2.8) in case I, we consider the re-scaling $x_1 \to \epsilon^{\frac{1}{l-p}} x_1$ so that the equilibrium equation (2.8) becomes

$$H_1(x_1, \epsilon) =: -\lambda x_1^l + [f_{p0}] x_1^p + \mathcal{O}(\epsilon^{\frac{1}{l-p}}) + \epsilon^{\frac{l-2p}{l-p}} [f_3(\cdot, \epsilon^{\frac{1}{l-p}} x_1, 0)] = 0 \tag{2.9}$$

which is perturbative because $p < l/2$ in this case. Since **(H)** holds, $x_* =: (\frac{1}{\lambda}[f_{p0}])^{\frac{1}{l-p}}$ is well-defined and satisfies

$$H_1(x_*, 0) = 0, \quad \frac{\partial H_1(x_*, 0)}{\partial x_1} \neq 0.$$

It follows from the implicit function theorem that the equation (2.9) admits a smooth family of solutions $\tilde{x}(\epsilon) = x_* + \mathcal{O}(\epsilon^{\frac{1}{l-p}})$. By tracing back the re-scaling in case II, we have obtained, in both cases, a smooth family of solutions $(x(\epsilon), y(\epsilon))$ of (2.7), as ϵ sufficiently small, where $y(\epsilon) \equiv 0$ and

$$x(\epsilon) = (\frac{\epsilon}{\lambda}[f_{p0}])^{\frac{1}{l-p}} + \mathcal{O}(\epsilon^{\frac{2}{l-p}}).$$

To eliminate the averaged term in the perturbation of the system (2.6), we consider the transformation $\tilde{\phi}_2 : D(s-\sigma/4, r-\sigma/4) \to D(s-\sigma/8, r-\sigma/8)$,

$$x_1 \to \epsilon^{\frac{1}{|l-p|}} x_1 + x(\epsilon), \quad y_1 \to \epsilon^{\frac{l+1}{2|l-p|}} y_1$$

in both cases. We note that $|l-p|$ is used in the above because p is allowed to be bigger than l in case II. Then system (2.6) under this transformation becomes

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x}_1 = \epsilon^{\frac{l-1}{2|l-p|}} y_1 + F_1(\theta, x_1, y_1, \epsilon), \\ \dot{y}_1 = \epsilon^{\frac{l-1}{2|l-p|}} (-l\lambda \tilde{x}^{l-1}(\epsilon) + p[f_{p0}] \tilde{x}^{p-1}(\epsilon)) x_1 - \lambda \sum_{k=2}^{l} \binom{l}{k} \epsilon^{\frac{l-1}{2|l-p|}} \tilde{x}^{l-k}(\epsilon) x_1^k \\ \quad + [f_{p0}] \sum_{k=2}^{p} \binom{p}{k} \epsilon^{\frac{l-1}{2|l-p|}} \tilde{x}^{p-k}(\epsilon) x_1^k + F_2(\theta, x_1, y_1, \epsilon), \end{cases}$$

where

$$\begin{aligned}
F_1(\theta,x_1,y_1,\epsilon) &= \epsilon^{\frac{l-p-1}{|l-p|}}\sum_{i=0}^{p} v_i(\theta)(\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon))^i,\\
F_2(\theta,x_1,y_1,\epsilon) &= \epsilon^{\frac{l-2p-1}{2|l-p|}} f_2(\theta,\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon))\\
&+ \epsilon f_1(\theta,\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon),\epsilon^{\frac{l+1}{2|l-p|}}y_1+\epsilon\sum_{i=0}^{p} v_i(\theta)(\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon))^i)y_1\\
&+ \epsilon^{\frac{3l-4p-1}{2|l-p|}} f_3(\theta,\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon),\epsilon^{\frac{l+1}{2|l-p|}}y_1)+\epsilon^{\frac{l-1}{2|l-p|}}(-\lambda\tilde{x}^l(\epsilon)+[f_{p0}]\tilde{x}^l(\epsilon))\\
&+ \epsilon\sum_{i=1}^{p} v_i(\theta)(\epsilon^{\frac{1}{|l-p|}}x_1+\epsilon^{\frac{1}{|l-p|}}\tilde{x}(\epsilon))^{i-1}y_1.
\end{aligned}$$

Now, with the composed transformations $\phi_0 = \tilde{\phi}_1\circ\tilde{\phi}_2$, the system (2.1) becomes (2.4) with $G(\theta,z,\epsilon)=\mathcal{O}(|z|^2)\in C^{\omega,1}(D(s-\frac{\sigma}{4},r-\frac{\sigma}{4})\times(0,\epsilon),\mathbb{C}^2)$, $E(\theta,\epsilon)=(E_1(\theta,\epsilon),E_2(\theta,\epsilon))^\top\in C^{\omega,1}(\mathbb{T}^d_{s-\frac{\sigma}{4}}\times(0,\epsilon),\mathbb{C}^2)$, $E_1(\theta,\epsilon)=F_1(\theta,0,0,\epsilon)$, $E_2(\theta,\epsilon)=F_2(\theta,0,0,\epsilon)$, $Q(\theta,\epsilon)=(Q_{ij}(\theta,\epsilon))_{1\le i,j\le 2}-([Q_{ij}(\theta,\epsilon)])_{1\le i,j\le 2}\in C^{\omega,1}(\mathbb{T}^d_{s-\frac{\sigma}{4}}\times(0,\epsilon),gl(2,\mathbb{C}))$,

$$\begin{aligned}
Q_{11}(\theta,\epsilon)&=\tfrac{\partial F_1(\theta,0,0,\epsilon)}{\partial x_1}, & Q_{12}(\theta,\epsilon)&=\tfrac{\partial F_1(\theta,0,0,\epsilon)}{\partial y_1},\\
Q_{21}(\theta,\epsilon)&=\tfrac{\partial F_2(\theta,0,0,\epsilon)}{\partial x_1}, & Q_{22}(\theta,\epsilon)&=\tfrac{\partial F_2(\theta,0,0,\epsilon)}{\partial y_1},
\end{aligned}$$

and

$$A(\epsilon)=\begin{pmatrix}0 & 1\\ -l\lambda\tilde{x}^{l-1}(\epsilon)+p[f_{p0}]\tilde{x}^{p-1}(\epsilon) & 0\end{pmatrix}+\epsilon^{-\frac{l-1}{2|l-p|}}([Q_{ij}(\theta,\epsilon)])_{1\le i,j\le 2}. \tag{2.10}$$

By (2.9), $[E(\theta,\epsilon)]=0$. It is also easy to see that

$$\|Q\|_{s-\frac{\sigma}{4},(0,\epsilon)}\preceq\epsilon^{\frac{l+1}{2|l-p|}},\quad \|E\|_{s-\frac{\sigma}{4},(0,\epsilon)}\preceq\epsilon^{\frac{l-1}{2|l-p|}}. \tag{2.11}$$

By (2.10), (2.11), and the definition of $\tilde{x}(\epsilon)$, it is clear that $A(\epsilon)$ has the desired form. □

2.2. **Improving the order of perturbation.** As stated in the following result, the order of perturbation in (2.4) can be raised to double that of the linear part.

Lemma 2.3. *Let $0<\sigma\le\min\{r/4,s/4\}$ be fixed as in* Lemma 2.2 *and consider system* (2.4) *in $D(s-\frac{\sigma}{4},r-\frac{\sigma}{4})\times(0,\epsilon_*)$ with ω satisfying* (1.6)*, respectively* (1.5)*, in the case $\tilde{\lambda}<0$, respectively $\tilde{\lambda}>0$, satisfies* (2.11)*, with respect to fixed $\gamma>0$, $0<\eta<s$, and $\tau>d-1$. Then there exists a family of real analytic transformations*

$$\tilde{\phi}^\epsilon: D(\tilde{s},\tilde{r})\to D(s-\frac{\sigma}{4},r-\frac{\sigma}{4}),\quad \epsilon\in\mathcal{D},$$

where $\tilde{s}=s-\sigma$, $\tilde{r}=r-\sigma$, $\mathcal{D}=\mathcal{D}_{\epsilon_*}$ is a Cantor set with meas$(\mathcal{D})=\mathcal{O}(\epsilon_*)$ *in the case $\tilde{\lambda}<0$, and $\mathcal{D}=(0,\epsilon_*)$ in the case $\tilde{\lambda}>0$, such that the following hold.*

(a) *$\tilde{\phi}^\epsilon$, $\epsilon\in\mathcal{D}$, is a C^1-Whitney smooth family in the case $\tilde{\lambda}<0$ and a C^1-smooth family in the case $\tilde{\lambda}>0$, having the form*

$$z=(Id+u(\theta,\epsilon))z_1+v(\theta,\epsilon),\ \ \theta=\theta,$$

where $u(\theta,\epsilon)\in C^{\omega,1}(\mathbb{T}^d_{\tilde{s}}\times\mathcal{D},gl(2,\mathbb{C}))$, $v(\theta,\epsilon)\in C^{\omega,1}(\mathbb{T}^d_{\tilde{s}}\times\mathcal{D},\mathbb{C}^2)$ satisfying the estimates

$$\|u\|_{\tilde{s},\mathcal{D}},\ \|v\|_{\tilde{s},\mathcal{D}}\preceq\delta.$$

(b) *For each $\epsilon\in\mathcal{D}$, $\tilde{\phi}^\epsilon$ transforms* (2.4) *into* (2.2).

Proof. In the proof, we suspend the explicit dependence of all terms on ϵ for the sake of simplicity. We intend to find a transformation $\tilde{\phi} : (\theta, z) \mapsto (\theta, z_1)$ of the form

$$z = z_1 + u(\theta)z_1 + v(\theta),\ \ \theta = \theta, \tag{2.12}$$

which transforms the system (2.4) into the system (2.2) with

$$\begin{aligned}
\breve{A} &= A + [\frac{\partial G(\theta, v(\theta))}{\partial z}],\\
\breve{E}(\theta) &= (I + u(\theta))^{-1} \left(Q(\theta)v(\theta) + \delta G(\theta, v(\theta))\right),\\
\breve{Q}(\theta) &= (I + u(\theta))^{-1} \left(Q(\theta)u(\theta) + \delta(\frac{\partial G(\theta, v(\theta))}{\partial z} - [\frac{\partial G(\theta, v(\theta))}{\partial z}])u(\theta)\right),\\
\breve{G}(\theta, z_1) &= (I + u(\theta))^{-1}\Big(G(\theta, (I + u(\theta))z_1 + v(\theta)) - G(\theta, v(\theta))\\
&\quad - \frac{\partial G(\theta, v(\theta))}{\partial y}(I + u(\theta))z_1\Big).
\end{aligned} \tag{2.13}$$

To achieve this, it is easy to see that functions u and v in the transformation (2.12) need to satisfy the following homological equations

$$\partial_\omega v(\theta) = \delta A v(\theta) + E(\theta), \tag{2.14}$$

$$\partial_\omega u(\theta) = \delta \breve{A} u(\theta) - \delta u(\theta)\breve{A} + H(\theta), \tag{2.15}$$

where $H(\theta) = Q(\theta) + \delta(\frac{\partial G(\theta,v(\theta))}{\partial z} - [\frac{\partial G(\theta,v(\theta))}{\partial z}])$. Consider Fourier expansions

$$v(\theta) = \sum_{k\in\mathbb{Z}^d} v_k e^{\mathrm{i}\langle k,\theta\rangle},\ E(\theta) = \sum_{k\in\mathbb{Z}^d} E_k e^{\mathrm{i}\langle k,\theta\rangle},\ u(\theta) = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}\langle k,\theta\rangle},\ H(\theta) = \sum_{k\in\mathbb{Z}^d} H_k e^{\mathrm{i}\langle k,\theta\rangle}.$$

Case 1. $\tilde{\lambda} < 0$ (the elliptic case): Substituting the expansions of v and E into the equation (2.14) yields

$$\mathrm{i}\langle k, \omega\rangle v_k = \delta A v_k + E_k, \quad k \in \mathbb{Z}^d. \tag{2.16}$$

Since $E_0 = 0$, we can choose $v_0 = 0$. Consider the set

$$\mathcal{D}_1 := \left\{\epsilon \in (0, \epsilon_*) :\ |\langle k, \omega\rangle^2 - \det \delta A(\epsilon)| \geq \frac{\gamma^3}{|k|^{5\tau}},\ \forall k \in \mathbb{Z}^d \setminus \{0\}\right\},$$

where $\delta = \epsilon^{\frac{l-1}{2|l-p|}}$. Then for each $\epsilon \in \mathcal{D}_1$, the equation (2.16) is uniquely solvable for each $k \neq 0$ to yield the solution

$$v_k = (\mathrm{i}\langle k, \omega\rangle I_2 - \delta A)^{-1} E_k = \frac{\mathrm{adj}(\mathrm{i}\langle k, \omega\rangle I_2 - \delta A)}{\det(\mathrm{i}\langle k, \omega\rangle I_2 - \delta A)} E_k,$$

where "adj" stands for the adjugate of a matrix, satisfying

$$|v_k|_{\mathcal{D}_1} \leq \left|\frac{\mathrm{adj}(\mathrm{i}\langle k, \omega\rangle I_2 - \delta A)}{\det(\mathrm{i}\langle k, \omega\rangle I_2 - \delta A)}\right|_{\mathcal{D}_1} |E_k|_{\mathcal{D}_1} \leq \frac{|k|^{5\tau+1}}{\gamma^3} |E_k|_{\mathcal{D}_1}.$$

It follows that, for each $\epsilon \in \mathcal{D}_1$, the equation (2.14) admits a unique solution $v(\theta)$ satisfying

$$\begin{aligned}
\|v\|_{\tilde{s},\mathcal{D}_1} &\leq \sum_{k\neq 0} |v_k|_{\mathcal{D}_1} e^{|k|\tilde{s}} \leq \sum_{k\neq 0} \gamma^{-3}|k|^{5\tau+1}|E_k|_{\mathcal{D}_1} e^{|k|\tilde{s}}\\
&= \sum_{k\neq 0} \gamma^{-3}|k|^{5\tau+1}e^{-|k|\frac{3\sigma}{4}}|E_k|_{\mathcal{D}_1} e^{|k|(s-\frac{\sigma}{4})}\\
&\preceq \sum_{k\neq 0} |E_k|_{\mathcal{D}_1} e^{|k|(s-\frac{\sigma}{4})} \preceq \|E(\theta)\|_{s-\frac{\sigma}{4},\mathcal{D}_1} \preceq \delta.
\end{aligned} \tag{2.17}$$

Since

$$v_{-k}=\frac{\mathrm{adj}(-\mathrm{i}\langle k,\omega\rangle I_2-\delta A)}{\det(-\mathrm{i}\langle k,\omega\rangle I_2-\delta A)}E_{-k}=\overline{\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_2-\delta A)}{\det(\mathrm{i}\langle k,\omega\rangle I_2-\delta A)}E_k}=\bar v_k,\quad k\neq 0, \tag{2.18}$$

$v(\theta)$ is a real analytic function.

We note that the equation (2.15) is equivalent to

$$\partial_\omega u(\theta)=\delta\tilde A u(\theta)-\delta u(\theta)\tilde A+H(\theta), \tag{2.19}$$

where $\tilde A=\breve A-\mathrm{tr}(\breve A)I_2$ with $\breve A$ being defined in (2.13) using the solution $v(\theta)$ of (2.14) solved above. We regard a 2×2 matrix $X=(X_{ij})_{1\le i,j\le 2}$ as a four-dimensional vector

$$(X)_v=(X_{11},X_{12},X_{21},X_{22})^\top. \tag{2.20}$$

Then the linear operator $L_A: gl(2,\mathbb{C})\to gl(2,\mathbb{C})$: $L_AX=XA-AX$, induces a linear operator $\mathcal{L}_A:(X)_v\to\mathcal{L}_A(X)_v$ on $\mathbb{C}^4$, where, for simplicity, we have used the same symbol to denote the 4×4 matrix corresponding to the operator $\mathcal{L}_A$. With the vector representation, the linear equation (2.19) becomes

$$\partial_\omega(u)_v+\mathcal{L}_{\delta\tilde A}(u)_v=(H)_v. \tag{2.21}$$

The characteristic polynomial of $\mathcal{L}_{\delta\tilde A}$ is simply $\det(\mathcal{L}_{\delta\tilde A}-\lambda I_4)=\lambda^2(\lambda^2+4\det\delta\tilde A)$. By substituting the Fourier expansions of u and H into (2.21), we find that $(u_0)_v=0$ and

$$(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})(u_k)_v=(H_k)_v,\qquad k\neq 0. \tag{2.22}$$

Consider the set

$$\mathcal{D}_2:=\left\{\epsilon\in(0,\epsilon_*):\ |\langle k,\omega\rangle^2-4\det\delta\tilde A(\epsilon)|\ge\frac{\gamma^3}{|k|^{5\tau}},\ \forall k\in\mathbb{Z}^d\setminus\{0\}\right\},$$

where $\delta=\epsilon^{\frac{l-1}{2|l-p|}}$. For each $\epsilon\in\mathcal{D}_2$, it is clear that (2.22) is uniquely solvable for each $k\neq0$ to yield the solution

$$(u_k)_v=(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})^{-1}(H_k)_v=\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})}{\det(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})}(H_k)_v$$

satisfying

$$\begin{aligned}|(u_k)_v|_{\mathcal{D}_2}&\le\left|\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})}{\det(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})}\right|_{\mathcal{D}_2}|(H_k)_v|_{\mathcal{D}_2}\\&\le\left|\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4+\mathcal{L}_{\delta\tilde A})}{\langle k,\omega\rangle^2(\langle k,\omega\rangle^2-4\det\delta\tilde A)}\right|_{\mathcal{D}_2}|(H_k)_v|_{\mathcal{D}_2}\preceq\frac{|k|^{15\tau}}{\gamma^3}|(H_k)_v|_{\mathcal{D}_2}.\end{aligned}$$

Thus, for each $\epsilon\in\mathcal{D}_2$, the equation (2.21) admits a unique solution $u(\theta)$ satisfying $\|(u)_v\|_{\tilde s,\mathcal{D}_2}\preceq\|(H)_v\|_{s,\mathcal{D}_2}\preceq\delta$, which implies that $\{u(\theta)\}_{\epsilon\in\mathcal{D}_2}$ is a C^1-Whitney smooth family of solutions of (2.15) satisfying

$$\|u\|_{\tilde s,\mathcal{D}_2}\preceq\delta. \tag{2.23}$$

Using identities similar to (2.18), we see, similar to the case of v, that u is real analytic.

Let $\mathcal{D}=\mathcal{D}_1\cap\mathcal{D}_2$. For the case $\tilde\lambda<0$, we have now obtained desired real analytic transformations $\tilde\phi$ defined by (2.12) which form a C^1-Whitney smooth family as $\epsilon\in\mathcal{D}$ varying.

Case 2. $\tilde\lambda>0$ (the hyperbolic case): In this case, A has two distinct real eigenvalues, say λ_1, λ_2. Since $E_0=0$, we can choose $v_0=0$. For each $k\neq0$, we have by substituting the Fourier expansions of v,E into (2.16) that

$$v_k=(\mathrm{i}\langle k,\omega\rangle I_2-\delta A)^{-1}E_k.$$

Let P be a non-singular real matrix such that $A = P\mathrm{diag}\{\lambda_1,\lambda_2\}P^{-1}$. Then

$$(\mathrm{i}\langle k,\omega\rangle I_2 - \delta A)^{-1} = P^{-1}\begin{pmatrix} \frac{1}{\mathrm{i}\langle k,\omega\rangle-\delta\lambda_1} & 0 \\ 0 & \frac{1}{\mathrm{i}\langle k,\omega\rangle-\delta\lambda_2}\end{pmatrix}P, \qquad k\neq 0.$$

As ϵ_* sufficiently small, we have by the non-resonant condition (1.5) that $|(\mathrm{i}\langle k,\omega\rangle - \lambda_i)^{-1}| < |(\mathrm{i}\langle k,\omega\rangle)^{-1}| \preceq \Omega(|k|)$ for all $k\neq 0$ and $\epsilon\in\mathcal{D} =: (0,\epsilon_*)$. This implies that, for each $\epsilon\in\mathcal{D}$, the equation (2.14) admits a unique real analytic solution $v(\theta)$ satisfying

$$\begin{aligned}\|v\|_{\tilde{s},\mathcal{D}} &= \sum_{k\neq 0}|v_k|_{\mathcal{D}}e^{\tilde{s}|k|} < \sum_{k\neq 0}|\Omega(|k|)||E_k|_{\mathcal{D}}e^{-\frac{3}{4\sigma}|k|}e^{(s-\frac{1}{4\sigma})|k|}\\ &\leq \sup_{|t|>0} e^{-\eta t}\Omega(t)\|E\|_{s,\mathcal{D}} \preceq \delta.\end{aligned}$$

Similarly, $\breve{A}$ in (2.15) has two distinct real eigenvalues, say $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. Let P be a non-singular matrix such that $u = P\tilde{u}P^{-1}$. Then (2.15) becomes

$$\partial_\omega \tilde{u}(\theta) = \delta\tilde{A}\tilde{u}(\theta) - \delta\tilde{u}(\theta)\tilde{A} + \tilde{H}(\theta),$$

where $\tilde{A} = P^{-1}\breve{A}P$ and $\tilde{H}(\theta) = P^{-1}\breve{H}(\theta)P$. Writing $\tilde{u}(\theta) = (\tilde{u}_{ij}(\theta))_{1\leq i,j\leq 2}$, $\tilde{H}(\theta) = (\tilde{H}_{ij}(\theta))_{1\leq i,j\leq 2}$, we have

$$\partial_\omega \tilde{u}_{ij}(\theta) = \delta(\lambda_i - \lambda_j)\tilde{u}_{ij}(\theta) + \tilde{H}_{ij}(\theta), \quad 1\leq i,j\leq 2. \tag{2.24}$$

Similar to the case of $v(\theta)$, for each $\epsilon\in\mathcal{D} = (0,\epsilon_*)$ with ϵ_* sufficiently small, (2.24) is uniquely solvable for each $1\leq i,j\leq 2$ whose real analytic solution u_{ij} satisfyies the estimate

$$\|\tilde{u}_{ij}\|_{\tilde{s},\mathcal{D}} \leq \sup_{|t|>0} e^{-\eta t}\Omega(t)\|\tilde{H}(\theta)\|_{s-\frac{\sigma}{4},\mathcal{D}} \preceq \epsilon^{\frac{1}{2}}.$$

This implies that the equation (2.15) is uniquely solvable for each $\epsilon\in\mathcal{D}$ to yield real analytic solution $u(\theta)$ satisfying

$$\|u\|_{\tilde{s},\mathcal{D}} \preceq \delta.$$

Thus, for the case $\tilde{\lambda} > 0$, we have obtained desired real analytic transformations $\tilde{\phi}$ defined by (2.12) which form a C^1 smooth family as $\epsilon\in\mathcal{D}$ varying.

By (2.13), (2.17) and (2.23), we have

$$\|\breve{Q}\|_{\tilde{s},\mathcal{D}} \preceq \delta^2,\ \|\breve{E}\|_{\tilde{s},\mathcal{D}} \preceq \delta^2,\ \breve{G} = \mathcal{O}(|z_1|^2).$$

It remains to estimate the Lebesgue measure of $\mathcal{D} = \mathcal{D}_1\cap\mathcal{D}_2$ in the case $\tilde{\lambda} < 0$. For each $k\neq 0$, we let $f_k^1(\epsilon) = \langle k,\omega\rangle^2 - \det\delta A(\epsilon)$ and $f_k^2(\epsilon) = \langle k,\omega\rangle^2 - 4\det\delta\tilde{A}(\epsilon)$ and consider sets

$$R_k^i := \{\epsilon\in(0,\epsilon_*) : |f_k^i(\epsilon)| < \gamma^3/|k|^{5\tau}\}, \qquad i = 1,2.$$

Then the Diophantine condition (1.6) implies that there is a constant $C > 0$ such that $|f_k^i(\epsilon)| \geq \frac{\gamma^2}{|k|^{2\tau}} - C\epsilon^{\frac{l-1}{|l-p|}}$ and $|\frac{df_k^i(\epsilon)}{d\epsilon}| > C\epsilon^{\frac{l-1}{|l-p|}-1}$, $k\neq 0$, $i = 1,2$. If $\frac{\gamma^2}{|k|^{2\tau}} \geq 2C\epsilon^{\frac{l-1}{|l-p|}}$ for some $k\neq 0$, then

$$|f_k^i(\epsilon)| \geq \frac{\gamma^2}{|k|^{2\tau}} - C\epsilon^{\frac{l-1}{|l-p|}} \geq \frac{\gamma^2}{2|k|^{2\tau}} \geq \frac{\gamma^3}{|k|^{5\tau}},$$

implying that meas $R_k^i = 0$, $i = 1,2$, where 'meas' denotes the Lebesgue measure of a set. If $\frac{\gamma^2}{|k|^{2\tau}} < 2C\epsilon^{\frac{l-1}{|l-p|}}$ for some $k\neq 0$, then

$$\text{meas } R_k^i \preceq \frac{\gamma^3}{|k|^{5\tau}\epsilon^{\frac{l-1}{|l-p|}-1}} \preceq \frac{\epsilon^{\frac{2(l-1)}{|l-p|}}}{|k|^{\tau}\epsilon^{\frac{l-1}{|l-p|}-1}} \preceq \frac{1}{|k|^\tau}\epsilon^{1+\frac{l-1}{|l-p|}}, \qquad i = 1,2.$$

It follows that

$$\text{meas } (\bigcup_{k\in\mathbb{Z}^d\setminus\{0\}} R_k^i) \preceq \epsilon^{1+\frac{l-1}{|l-p|}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{1}{|k|^\tau} \preceq \epsilon^{1+\frac{l-1}{|l-p|}}$$

for $i = 1, 2$. Therefore,

$$\text{meas } \mathcal{D} \geq \epsilon_* - \text{meas}\Big(\bigcup_{\substack{k \in \mathbb{Z}^d \setminus \{0\}, \\ 1 \leq i \leq 2}} R_k^i \Big) \geq \epsilon_* \big(1 - C \epsilon_*^{\frac{l-1}{|l-p|}}\big)$$

for some constant $C > 0$. □

Proof of Proposition 2.1: It is an immediate consequence of Lemmas 2.2, 2.3, by letting $\phi^\epsilon = \phi_0 \circ \tilde{\phi}^\epsilon$. □

3. KAM Iterations

In this section, we use KAM iterations to prove the following theorems, one for the hyperbolic case and the other one for the elliptic case, from which the Main Theorem and Corollaries A, B will follow.

Theorem 3.1. *Consider the system* (2.1) *in the domain* $D(s, r) \times (0, \epsilon_*)$ *for fixed* $r, s, \epsilon_* > 0$. *Assume* $\hat{\lambda} > 0$, *the condition* **(H)**, *the condition* **(H1)** *in case I, the condition* **(H2)** *in case II, and that* ω *satisfies the Brjuno-like non-resonant condition* (1.5) *with respect to fixed* $\gamma > 0$ *and* $0 < \eta < s$. *Then for* ϵ_* *sufficiently small, there exists a* C^1 *smooth family of real analytic transformations*

$$\Phi_\epsilon : D(\frac{s}{2}, \frac{r}{2}) \to D(s, r), \quad \epsilon \in (0, \epsilon_*)$$

such that each Φ_ϵ *transforms the system* (2.1) *into the form*

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = A_*^\epsilon(\theta) z + G_*^\epsilon(\theta, z), \end{cases}$$

where $z = (x, y)^\top$, $G_*^\epsilon(\theta, z) = \mathcal{O}(|z|^2)$ *and each* $A_*^\epsilon(\theta)$ *is a matrix-valued, real analytic function. Consequently, for each* $\epsilon \in (0, \epsilon_*)$, *the system* (2.1) *admits a real analytic, quasi-periodic response d-torus with the frequency vector* ω.

Theorem 3.2. *Consider the system* (2.1) *in the domain* $D(s, r) \times (0, \epsilon_*)$ *for fixed* $r, s, \epsilon_* > 0$. *Assume* $\hat{\lambda} > 0$, *the condition* **(H)**, *the condition* **(H1)** *in case I, the condition* **(H2)** *in case II, and that* ω *satisfies the Diophantine condition* (1.6) *with respect to fixed* $\gamma > 0$ *and* $\tau > d - 1$. *Then for* ϵ_* *sufficiently small, there exist a Cantor set* $\mathcal{E}_{\epsilon_*} \subset (0, \epsilon_*)$ *with* $\lim_{\epsilon_* \to 0} \frac{Meas(\mathcal{E}_{\epsilon_*})}{\epsilon_*} = 1$ *and a* C^1*-Whitney smooth family of real analytic transformations*

$$\Psi_\epsilon : D(\frac{s}{2}, \frac{r}{2}) \to D(s, r), \qquad \epsilon \in \mathcal{E}_{\epsilon_*}$$

such that each Ψ_ϵ *transforms the system* (2.1) *into the form*

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = A_*^\epsilon z + G_*^\epsilon(\theta, z), \end{cases}$$

where $z = (x, y)^\top$, $G_*^\epsilon(\theta, z) = \mathcal{O}(|z|^2)$ *and each* A_*^ϵ *is a constant matrix. Consequently, for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$, *the system* (2.1) *admits a real analytic, Diophantine response torus with the frequency vector* ω.

Through the rest of the section, for the sake of simplicity, we suspend the explicit dependence of all terms on ϵ. To begin with the KAM iterations, we set $A_0 = \check{A}, Q_0 = \check{Q}, G_0 = \check{G}, E_0 = \check{E}, r_0 = \tilde{r} = r - \sigma, s_0 = \tilde{s} = s - \sigma$ in (2.2), i.e., we start with the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (\delta A_0 + Q_0(\theta)) z + E_0(\theta) + \delta G_0(\theta, z), \end{cases}$$

where $z = (x, y)^\top$, $(\theta, z) \in D(s_0, r_0)$. We recall from Lemma 2.3 that

$$A_0 = A + \mathcal{O}(\delta), \; \|Q_0\|_{s_0, \mathcal{D}} \leq C_0 \delta^2, \; \|E_0\|_{s_0, \mathcal{D}} \leq C_0 \delta^2, \; G_0 = \mathcal{O}(\|z\|^2),$$

where $C_0 > 0$ is a suitable constant. Fix a $0 < \sigma_0 \le \min\{\frac{r_0}{4}, \frac{s_0}{4}\}$, $\delta_0 = C_0\delta$ and consider the iterative sequences

$$\begin{cases} \delta_{\nu+1} = \delta_\nu^{\frac{6}{5}}, \quad \sigma_{\nu+1} = \sigma_\nu/2, \\ s_{\nu+1} = s_\nu - \sigma_\nu, \quad r_{\nu+1} = r_\nu - \sigma_\nu, \end{cases} \tag{3.1}$$

$\nu = 0, 1, \cdots$. It is clear that $\lim_{\nu\to\infty} s_\nu =: s_* \ge s_0/2$, $\lim_{\nu\to\infty} r_\nu =: r_* \ge r_0/2$, and $\lim_{\nu\to\infty} \epsilon_\nu = 0$.

3.1. **Transformations for the case $\tilde{\lambda} > 0$.** We implement a KAM scheme to iterate the following sequence of C^1-smooth family of real analytic systems

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (\delta A_0 + Q_\nu(\theta))z + E_\nu(\theta) + \delta G_\nu(\theta, z), \end{cases} \tag{1_ν}$$

where $z = (x, y)^\top$, $(\theta, z) \in D(s_\nu, r_\nu)$, $\epsilon \in (0, \epsilon_*)$, $\nu = 0, 1, \cdots$, whose terms on the right hand side satisfy the following properties:

$$\|Q_\nu - Q_{\nu-1}\|_{s_\nu,(0,\epsilon_*)} \le \delta_\nu^{\frac{1}{2}}, \tag{3.2}$$

$$G_\nu = \mathcal{O}(|(x, y)|^2),$$

$$\|G_\nu - G_{\nu-1}\|_{s_\nu,r_\nu,(0,\epsilon_*)} \le \delta_\nu^{\frac{1}{2}}, \tag{3.3}$$

$$\|E_\nu\|_{s_\nu,r_\nu,(0,\epsilon_*)} \le \delta\delta_\nu, \tag{3.4}$$

for all $\nu = 1, 2, \cdots$.

Lemma 3.1. *Consider the case $\tilde{\lambda} > 0$ and assume conditions of* Theorem 3.1. *Then, as ϵ_* sufficiently small, for each $\nu = 0, 1, \cdots$, there exists a C^1-smooth family of real analytic transformation $\Phi_\nu : D(s_{\nu+1}, r_{\nu+1}) \to D(s_\nu, r_\nu)$, $\epsilon \in (0, \epsilon_*)$, satisfying*

$$\begin{aligned} &\|\Phi_\nu - id\|_{s_\nu,r_\nu,(0,\epsilon_*)} < \delta_\nu^{\frac{1}{2}}, \\ &\|D\Phi_\nu - Id\|_{s_\nu,r_\nu,(0,\epsilon_*)} < \delta_\nu^{\frac{1}{2}}, \end{aligned} \tag{3.5}$$

which transforms (1_ν) to $(1_{\nu+1})$.

Proof. Consider the homological equation

$$\partial_\omega V(\theta) = (\delta A_0 + Q_\nu(\theta))V(\theta) + E_\nu(\theta),$$

which is equivalent to

$$(T + W_\nu)V = E_\nu, \tag{3.6}$$

where T, W_ν: $C^{\omega,1}(\mathbb{T}^d_{s_\nu} \times (0, \epsilon_*), \mathbb{C}^2) \to C^{\omega,1}(\mathbb{T}^d_{s_\nu} \times (0, \epsilon_*), \mathbb{C}^2)$ are operators defined by

$$T(V(\theta)) := \partial_\omega V(\theta) - \delta A_0 V(\theta) = \sum_{k\in\mathbb{Z}^d} (\mathrm{i}\langle k, \omega\rangle I_2 - \delta A_0)V_k e^{\mathrm{i}\langle k,\theta\rangle}, \qquad V = \sum_{k\in\mathbb{Z}^d} V_k e^{\mathrm{i}\langle k,\theta\rangle},$$

$$W_\nu(V(\theta)) = -Q_\nu(\theta)V(\theta).$$

We note that A_0 is hyperbolic in this case with real distinct eigenvalues λ_i, $i = 1, 2$. Thus, there is a non-singular matrix P such that $A_0 = P\mathrm{diag}\{\lambda_1, \lambda_2\}P^{-1}$ and a constant $c > 0$ such that $|\mathrm{i}\langle k, \omega\rangle - \delta\lambda_i| \ge c\delta$, $i = 1, 2$, $k \in \mathbb{Z}^d$. It follows that T^{-1} : $C^{\omega,1}$ $(\mathbb{T}^d_{s_\nu} \times (0, \epsilon_*), \mathbb{C}^2) \to C^{\omega,1}(\mathbb{T}^d_{s_\nu} \times (0, \epsilon_*), \mathbb{C}^2)$:

$$\begin{aligned} T^{-1}(V(\theta)) &= \sum_{k\in\mathbb{Z}^d} (\mathrm{i}\langle k, \omega\rangle I_2 - \delta A_0)^{-1} V_k e^{\mathrm{i}\langle k,\theta\rangle} \\ &= \sum_{k\in\mathbb{Z}^d} P^{-1} \begin{pmatrix} (\mathrm{i}\langle k, \omega\rangle - \delta\lambda_1)^{-1} & 0 \\ 0 & (\mathrm{i}\langle k, \omega\rangle - \delta\lambda_2)^{-1} \end{pmatrix} PV_k e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned}$$

exists, and

$$\begin{aligned}\|T^{-1}\| &= \sup_{V(\theta)\in C^{\omega,1}(\mathbb{T}^d_{s_\nu}\times(0,\epsilon_*),\mathbb{C}^2)\setminus\{0\}} \frac{\|T^{-1}(V(\theta))\|_{s_\nu(0,\epsilon_*)}}{\|V(\theta)\|_{s_\nu(0,\epsilon_*)}}\\ &\preceq \sup_{V(\theta)\in C^{\omega,1}(\mathbb{T}^d_{s_\nu}\times(0,\epsilon_*),\mathbb{C}^2)\setminus\{0\}} \frac{\delta^{-1}\|V(\theta)\|_{s_\nu(0,\epsilon_*)}}{\|V(\theta)\|_{s_\nu(0,\epsilon_*)}} \preceq \delta^{-1}.\end{aligned}$$

Since $\|Q_\nu\|_{s_\nu,(0,\epsilon_*)} \le \delta\sum_{l=0}^{\nu-1}\|\frac{G_\nu(\theta,V_l)}{\partial z}\|_{s_\nu,(0,\epsilon_*)} \preceq \delta^{\frac{3}{2}}$, we have

$$\|W_\nu\| \le \|Q_\nu(\theta)\|_{s_\nu(0,\epsilon_*)} \preceq \delta^{\frac{3}{2}}.$$

Thus, $\|T^{-1}W_\nu\| \le \|T^{-1}\|\|W_\nu\| \preceq \delta$, which implies that the operator $T+W_\nu$ is invertible and

$$\|(T+W_\nu)^{-1}\| \le \|(1+T^{-1}W_\nu)\|\|T^{-1}\| \preceq \delta^{-1}.$$

It follows that the equation (3.6) is solvable to yield a unique solution $V_\nu(\theta) = (T+W_\nu)^{-1}(E_\nu(\theta))$ satisfying

$$\|V_\nu\|_{s_\nu,(0,\epsilon_*)} \preceq \delta^{-1}\|E_\nu\|_{s_\nu,(0,\epsilon_*)} \preceq \delta^{-1}\delta\delta_\nu \preceq \delta_\nu. \tag{3.7}$$

Let $\Phi_\nu : (\theta, z) \mapsto (\theta, z^+)$ be the transformation such that

$$z^+ = z - V_\nu(\theta).$$

Then it is clear that, as ϵ varies, Φ_ν is a C^1 smooth family of real analytic transformations, which, for each $\epsilon\in(0,\epsilon_*)$, transforms (1_ν) to $(1_{\nu+1})$ with

$$\begin{array}{l} Q_{\nu+1}(\theta) = Q_\nu(\theta) + \delta\frac{G_\nu(\theta,V_\nu(\theta))}{\partial z},\\ E_{\nu+1}(\theta) = \delta G_\nu(\theta, V_\nu(\theta)),\\ G_{\nu+1}(\theta,z^+) = G_\nu(\theta,V_\nu(\theta)+z^+) - G_\nu(\theta,V_\nu(\theta)) - \frac{G_\nu(\theta,V_\nu(\theta))}{\partial z}z^+. \end{array} \tag{3.8}$$

By making ϵ_* sufficiently small if necessary, we have $r_{\nu+1} + \|V_\nu(\theta)\|_{s_\nu,(0,\epsilon_*)} \le r_\nu$, which implies that

$$\Phi_\nu(D(s_{\nu+1},r_{\nu+1})) \subset D(s_\nu,r_\nu), \qquad \epsilon\in(0,\epsilon_*),$$

and moreover, (3.5) holds.

Since $G_{\nu+1} = \mathcal{O}(|z|^2)$, we have by (3.7), (3.8), and Cauchy estimate that

$$\begin{aligned}\|E_{\nu+1}\|_{s_{\nu+1},(0,\epsilon_*)} &\le \|\delta G_\nu(\cdot,V_\nu(\cdot))\|_{s_{\nu+1},(0,\epsilon_*)} \le \delta\delta_\nu^2 \le \delta\delta_{\nu+1},\\ \|Q_{\nu+1}-Q_\nu\|_{s_{\nu+1},(0,\epsilon_*)} &\le \|\delta\frac{G_\nu(\cdot,V_\nu(\cdot))}{\partial z}\|_{s_{\nu+1},(0,\epsilon_*)} \le \delta_\nu^{\frac{1}{2}},\\ \|G_{\nu+1}-G_\nu\|_{s_{\nu+1},r_{\nu+1},(0,\epsilon_*)} &\preceq \|G_\nu(\cdot,V_\nu(\cdot))\|_{s_{\nu+1},(0,\epsilon_*)} + \|\frac{G_\nu(\cdot,V_\nu(\cdot))}{\partial z}\|_{s_{\nu+1},(0,\epsilon_*)}\\ &+ \|V_\nu\|_{s_\nu,(0,\epsilon_*)} \le \delta_\nu^{\frac{1}{2}},\end{aligned}$$

i.e., (3.2)-(3.4) hold with $\nu+1$ in place of ν. □

3.2. **Transformations for the case $\tilde\lambda < 0$.** Setting $\gamma_0 = \gamma$, we consider a new sequence $\{\gamma_\nu\}$ defined by $\gamma_{\nu+1} = \gamma_\nu/2$, $\nu = 0,1,\cdots$, in addition to sequences in (3.1). For each $\nu = 0,1,\cdots$, denote $\mathcal{E}_\nu := \mathcal{E}_\nu^1 \cap \mathcal{E}_\nu^2$, where

$$\begin{aligned}\mathcal{E}_\nu^1 &:= \{\epsilon\in\mathcal{D} : |\langle k,\omega\rangle^2 - \det\delta A_\nu(\epsilon)| \ge \frac{\gamma_\nu^3}{|k|^{5\tau}}, \forall k\in\mathbb{Z}^d\setminus\{0\}\},\\ \mathcal{E}_\nu^2 &:= \{\epsilon\in\mathcal{D} : |\langle k,\omega\rangle^2 - 4\det\delta\tilde A_{\nu+1}(\epsilon)| \ge \frac{\gamma_\nu^3}{|k|^{5\tau}}, \forall k\in\mathbb{Z}^d\setminus\{0\}\}\end{aligned}$$

with $\tilde A_{\nu+1} = A_{\nu+1} - \frac{1}{2}\mathrm{tr}(A_{\nu+1})I_2$. We would like to implement a KAM scheme to iterate the following sequence of C^1-Whitney smooth family of real analytic systems

$$\left\{\begin{array}{l}\dot\theta = \omega,\\ \dot z = (\delta A_\nu + Q_\nu(\theta))z + E_\nu(\theta) + \delta G_\nu(\theta,z),\end{array}\right. \tag{2_ν}$$

where $z = (x, y)^\top$, $(\theta, z) \in D(s_\nu, r_\nu)$, $\epsilon \in \mathcal{E}_\nu$, $\nu = 0, 1, \cdots$, whose terms on the right hand side satisfy the following properties:

$$|A_\nu - A_{\nu-1}|_{\mathcal{E}_\nu} \leq \delta_\nu^{\frac{1}{2}}, \tag{3.9}$$

$$\|Q_\nu\|_{s_\nu, \mathcal{E}_\nu} \leq \delta\delta_\nu, \tag{3.10}$$

$$G_\nu = \mathcal{O}(|(x, y)|^2),$$

$$\|G_\nu - G_{\nu-1}\|_{s_\nu, r_\nu, \mathcal{E}_\nu} \leq \delta_\nu^{\frac{1}{2}}, \tag{3.11}$$

$$\|E_\nu\|_{s_\nu, \mathcal{E}_\nu} \leq \delta\delta_\nu, \tag{3.12}$$

$\nu = 1, 2, \cdots$.

Lemma 3.2. *Consider the case* $\tilde{\lambda} < 0$ *and assume conditions of* Theorem 3.2. *Then, as* ϵ_* *sufficiently small, for each* $\nu = 0, 1, \cdots$, *there exists a* C^1*-Whitney smooth family of real analytic transformation* $\Phi_\nu : D(s_{\nu+1}, r_{\nu+1}) \to D(s_\nu, r_\nu)$, $\epsilon \in \mathcal{E}_\nu$, *satisfying*

$$\begin{aligned} \|\Phi_\nu - id\|_{s_{\nu+1}, r_{\nu+1}, \mathcal{E}_{\nu+1}} &\leq \delta_\nu^{\frac{1}{2}}, \\ \|D\Phi_\nu - Id\|_{s_{\nu+1}, r_{\nu+1}, \mathcal{E}_{\nu+1}} &\leq \delta_\nu^{\frac{1}{2}}, \end{aligned} \tag{3.13}$$

which transforms (2_ν) *to* $(2_{\nu+1})$.

Proof. Consider the homological equations

$$\partial_\omega V(\theta) = \delta A_\nu V(\theta) + E_\nu(\theta), \tag{3.14}$$

$$\partial_\omega U(\theta) = \delta A_{\nu+1} U(\theta) - \delta U(\theta) A_{\nu+1} + H_\nu(\theta), \tag{3.15}$$

where $H_\nu(\theta) = Q_\nu(\theta) - [Q_\nu(\cdot)] + \delta\frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z} - \delta[\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial z}]$. Substituting Fourier expansions

$$V(\theta) = \sum_{k \in \mathbb{Z}^d} V_k e^{\mathrm{i}\langle k, \theta \rangle}, \quad E_\nu(\theta) = \sum_{k \in \mathbb{Z}^d} E_{\nu,k} e^{\mathrm{i}\langle k, \theta \rangle}$$

into (3.14), we have

$$\mathrm{i}\langle k, \omega \rangle V_k = \delta A_\nu V_k + E_{\nu,k}, \quad k \in \mathbb{Z}^d,$$

which admit solutions

$$V_k = (\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)^{-1} E_{\nu,k} = \frac{\mathrm{adj}(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)}{\det(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)} E_{\nu,k}, \qquad k \in \mathbb{Z}^d.$$

Clearly, if $k = 0$, then

$$|V_0|_{\mathcal{E}_\nu} \leq \left|\frac{\mathrm{adj}(\delta A_\nu)}{\det(\delta A_\nu)}\right|_{\mathcal{E}_\nu} |E_{\nu,0}|_{\mathcal{E}_\nu} \preceq \delta^{-1} |E_{\nu,0}|_{\mathcal{E}_\nu},$$

and if $k \neq 0$, then

$$|V_k|_{\mathcal{E}_\nu} \leq \left|\frac{\mathrm{adj}(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)}{\det(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)}\right|_{\mathcal{E}_\nu} |E_{\nu,k}|_{\mathcal{E}_\nu} \leq \frac{|k|^{5\tau+1}}{\gamma_\nu^3} |E_{\nu,k}|_{\mathcal{E}_\nu}.$$

It follows that the equation (3.14) is solvable to yield a unique solution $V_\nu(\theta)$ satisfying

$$\begin{aligned} \|V_\nu\|_{s_{\nu+1}, \mathcal{E}_\nu} &\leq \sum_{k \in \mathbb{Z}^d} |V_k|_{\mathcal{E}_\nu} e^{|k| s_{\nu+1}} \preceq \delta^{-1} |E_{\nu,0}|_{\mathcal{E}_\nu} + \sum_{k \neq 0} \gamma_\nu^{-3} |k|^{5\tau+1} |E_{\nu,k}|_{\mathcal{E}_\nu} e^{|k| s_{\nu+1}} \\ &\preceq \delta^{-1} \sum_{k \neq 0} \gamma_\nu^{-3} |k|^{5\tau+1} e^{-|k|\sigma_\nu} |E_{\nu,k}|_{\mathcal{E}_\nu} e^{|k| s_\nu} \\ &\preceq \delta^{-1} \frac{1}{\gamma_\nu^3 \sigma_\nu} \|E_\nu(\theta)\|_{s_\nu, \mathcal{E}_\nu} \preceq \frac{1}{\gamma_\nu^3 \sigma_\nu} \delta_\nu. \end{aligned} \tag{3.16}$$

Since

$$V_{\nu,-k} = \frac{\mathrm{adj}(-\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)}{\det(-\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)} E_{\nu,-k} = \overline{\frac{\mathrm{adj}(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)}{\det(\mathrm{i}\langle k, \omega \rangle I_2 - \delta A_\nu)} E_{\nu,k}} = \overline{V}_{\nu,k}, \qquad k \neq 0, \tag{3.17}$$

$V_\nu(\theta)$ is a real analytic function.

We note that the equation (3.15) is equivalent to

$$\partial_\omega U(\theta) \quad = \quad \delta\tilde{A}_{\nu+1}U(\theta) - \delta U(\theta)\tilde{A}_{\nu+1} + H_\nu(\theta),$$

which is further equivalent to the vector form

$$\partial_\omega (U)_v + \mathcal{L}_{\delta\tilde{A}_{\nu+1}}(U)_v = (H_\nu)_v, \tag{3.18}$$

where the vector representation $(\cdot)_v$ of a matrix and the operator $\mathcal{L}_{\delta\tilde{A}_{\nu+1}}$ are defined in (2.20) and (2.21), respectively. We note that the characteristic polynomial of $\mathcal{L}_{\delta\tilde{A}_{\nu+1}}$ is simply $\det(\mathcal{L}_{\delta\tilde{A}_{\nu+1}} - \lambda I_4) = \lambda^2(\lambda^2 + 4\det\delta\tilde{A}_{\nu+1})$. Substituting the Fourier expansions

$$U(\theta) = \sum_{k\in\mathbb{Z}^d} U_k e^{\mathrm{i}\langle k,\theta\rangle}, \quad H_\nu(\theta) = \sum_{k\in\mathbb{Z}^d} H_{\nu,k} e^{\mathrm{i}\langle k,\theta\rangle}$$

into (3.18) yields

$$(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})(U_k)_v = (H_{\nu,k})_v,$$

i.e.,

$$(U_k)_v = (\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})^{-1}(H_{\nu,k})_v = \frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})}{\det(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})}(H_{\nu,k})_v, \qquad k\in\mathbb{Z}^d.$$

Since $\|(U_0)\|_{s_{\nu+1},\mathcal{E}_\nu} \preceq \delta^{-1}\frac{1}{\gamma_\nu^3\sigma_\nu}\|(H_{\nu,0})\|_{s_\nu,\mathcal{E}_\nu}$ and

$$\begin{aligned} |(U_k)_v|_{\mathcal{E}_\nu} \quad &\leq \quad \left|\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})}{\det(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})}\right| |(H_{\nu,k})_v|_{\mathcal{E}_\nu} \\ &\leq \quad \left|\frac{\mathrm{adj}(\mathrm{i}\langle k,\omega\rangle I_4 + \mathcal{L}_{\delta\tilde{A}_{\nu+1}})}{\langle k,\omega\rangle^2(\langle k,\omega\rangle^2 - 4\det\delta\tilde{A}_{\nu+1})}\right|_{\mathcal{E}_\nu} |(H_{\nu,k})_v|_{\mathcal{E}_\nu} \preceq \frac{|k|^{15\tau}}{\gamma_\nu^3}|(H_{\nu,k})_v|_{\mathcal{E}_\nu}, \qquad k\neq 0, \end{aligned}$$

we obtain a unique solution $U_\nu(\theta)$ of (3.15) satisfying

$$\|U_\nu\|_{s_{\nu+1},\mathcal{E}_\nu} \preceq \frac{1}{\gamma_\nu^6\sigma_\nu^2}\delta_\nu. \tag{3.19}$$

Using an identity similar to (3.17), we also see that U_ν is a real analytic function.

Let $\Phi_\nu : (\theta, z) \mapsto (\theta, z^+)$ be the transformation such that

$$z = z^+ + U_\nu(\theta)z^+ + V_\nu(\theta).$$

Then it is easy to see that Φ_ν is a C^1-Whitney smooth family of real analytic transformations, which, for each $\epsilon\in\mathcal{E}_\nu$, transforms (2_ν) into $(2_{\nu+1})$ with

$$\begin{aligned} A_{\nu+1} \quad &= \quad A_\nu + \delta^{-1}[Q_\nu(\cdot)] + [\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial z}], \\ E_{\nu+1}(\theta) \quad &= \quad (I+U_\nu(\theta))^{-1}\left(Q_\nu(\theta)V_\nu(\theta) + \delta G_\nu(\theta, V_\nu(\theta))\right), \\ Q_{\nu+1}(\theta) \quad &= \quad (I+U_\nu(\theta))^{-1}((Q_\nu(\theta) - [Q_\nu(\cdot)])U_\nu(\theta) \\ &+ \quad \delta(\frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z} - [\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial z}])U_\nu(\theta)), \\ G_{\nu+1}(\theta, z^+) \quad &= \quad (I+U_\nu(\theta))^{-1}\Big(G_\nu(\theta, (I+U_\nu(\theta))z^+ + V_\nu(\theta)) - G_\nu(\theta, V_\nu(\theta)) \\ &- \quad \frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial y}(I+U_\nu(\theta))z^+\Big). \end{aligned} \tag{3.20}$$

By making ϵ_* sufficiently small if necessary, we have

$$r_\nu - \|V_\nu + (I+U_\nu)z^+\|_{s_{\nu+1},r_{\nu+1}} > \sigma_\nu - \delta_\nu^{\frac{1}{2}} > 0,$$

which implies that

$$\Phi_\nu(D(s_{\nu+1}, r_{\nu+1})) \subset D(s_\nu, r_\nu), \qquad \epsilon\in\mathcal{E}_\nu,$$

and moreover, (3.13) holds.

Since $G_{\nu+1} = \mathcal{O}(|z|^2)$, we have by (3.16), (3.19), (3.20), and Cauchy estimate that

$$\begin{aligned}
|A_{\nu+1} - A_\nu|_{\mathcal{E}_{\nu+1}} &\leq \delta_{\nu+1}^{\frac{1}{2}},\\
\|E_{\nu+1}\|_{s_{\nu+1},\mathcal{E}_\nu} &\leq \|(I+U_\nu)^{-1}\|_{s_{\nu+1},\mathcal{E}_\nu}\|(Q_\nu V_\nu + \delta G_\nu(\cdot, V_\nu(\cdot))\|_{s_{\nu+1},\mathcal{E}_\nu}\\
&\leq 2(\delta\delta_\nu\delta_\nu^{\frac{1}{2}} + \delta\delta_\nu\delta_\nu^{\frac{1}{2}}) \leq \delta\delta_{\nu+1},\\
\|Q_{\nu+1}\|_{s_{\nu+1},\mathcal{E}_\nu} &\leq \|(I+U_\nu)^{-1}\|_{s_{\nu+1},\mathcal{E}_\nu}(\|(Q_\nu - [Q_\nu])U_\nu\|_{s_{\nu+1},\mathcal{E}_\nu}\\
&+ \|\delta(\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial z} - [\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial z}])U_\nu\|_{s_{\nu+1},\mathcal{E}_\nu})\\
&\leq 2(\delta\delta_\nu\delta_\nu^{\frac{1}{2}} + \delta\delta_\nu\delta_\nu^{\frac{1}{2}}) \leq \delta\delta_{\nu+1},\\
\|G_{\nu+1} - G_\nu\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_\nu} &\leq \|(G_\nu(\cdot,(I+U_\nu(\cdot))z + V_\nu) - G_\nu\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_\nu}\\
&+ \|G_\nu(\cdot, V_\nu(\cdot))\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_\nu} + \|\frac{\partial G_\nu(\cdot, V_\nu(\cdot))}{\partial y}\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_\nu}\\
&+ \|U_\nu(\theta)\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_\nu} \leq \delta_\nu^{\frac{1}{2}},
\end{aligned}$$

i.e., (3.9)-(3.12) hold with $\nu+1$ in place of ν. □

3.3. **Proof of Main Results.** For simplicity, we use the same notions $\mathcal{E}_\nu$, $\nu = 1, 2, \cdots$, to denote the parameter sets during the KAM iterations for both cases of $\tilde{\lambda} > 0$ and $\tilde{\lambda} < 0$, i.e., we let $\mathcal{E}_\nu \equiv (0, \epsilon_*)$, $\nu = 1, 2, \cdots$, when $\tilde{\lambda} > 0$. For each $\nu = 0, 1, \cdots$, let

$$\tilde{\Phi}^\nu = \phi_0 \circ \phi^\epsilon \circ \Phi^\nu,$$

where ϕ_0 and ϕ^ϵ are transformations defined in Section 2 and

$$\Phi^\nu := \Phi_0 \circ \Phi_1 \circ \cdots \Phi_\nu \,:\, D(s_{\nu+1}, r_{\nu+1}) \times \mathcal{E}_{\nu+1} \to D(s_0, r_0) \times \mathcal{E}_0$$

with Φ_j's being given by Lemmas 3.1, 3.2. Since, by (3.5) and (3.13),

$$\|D\tilde{\Phi}^{\nu-1}\|_{s_\nu, r_\nu, \mathcal{E}_\nu} \leq \prod_{\nu\geq 0}(1 + \epsilon_\nu^{\frac{2n+3}{4n}}) \preceq 1,$$

we have

$$\begin{aligned}
\|\tilde{\Phi}^\nu - \tilde{\Phi}^{\nu-1}\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_{\nu+1}} &= \|\tilde{\Phi}^{\nu-1}(\Phi_\nu) - \tilde{\Phi}^{\nu-1}\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_{\nu+1}}\\
&\leq \|D\tilde{\Phi}^{\nu-1}\|_{s_\nu,r_\nu,\mathcal{E}_\nu} \cdot \|\Phi_\nu - id\|_{s_{\nu+1},r_{\nu+1},\mathcal{E}_{\nu+1}}\\
&\leq \delta_\nu^{\frac{1}{2}},
\end{aligned}$$

$\nu = 1, 2, \cdots$. It follows that, as $\nu \to \infty$, the sequence $\{\tilde{\Phi}^\nu\}$ converges uniformly on

$$\bigcap_{\nu\geq 0} D(s_{\nu+1}, r_{\nu+1}) \times \mathcal{E}_{\nu+1} = D(s_*, r_*) \times \mathcal{E}_\infty,$$

where $\mathcal{E}_\infty = \cap_{\nu=0}^\infty \mathcal{E}_\nu$, say, to a parametrized family of invertible, real analytic transformations $\tilde{\Phi}^*_\epsilon$, $\epsilon \in \mathcal{E}_\infty$, which depends on ϵ C^1 smoothly when $\tilde{\lambda} > 0$ and C^1-Whitney smoothly when $\tilde{\lambda} < 0$. By (3.2)-(3.4) and (3.9)-(3.12), we also see the convergence of all coefficients of (1_ν) and (2_ν) on the same domain, as $\nu \to \infty$, to the respective forms stated in Theorems 3.1, 3.2.

We are now ready to prove our main results, i.e., Theorems 3.1, 3.2, the Main Theorem, and Corollaries A, B.

Proof of Theorems 3.1, 3.2: As $\tilde{\lambda}$ and $\hat{\lambda}$ have the same sign, all results in this and the previous section hold with $\hat{\lambda}$ in place of $\tilde{\lambda}$. With the convergence results above, the proof of Theorem 3.1 is already completed. To finish the proof of Theorem 3.2, we note that, by considering sets

$$R^i_{\nu k} := \{\epsilon \in \mathcal{D} : |f^i_{\nu k}(\epsilon)| < \gamma_\nu^3/|k|^{5\tau}\}, \qquad i = 1, 2,$$

where

$$\begin{aligned} f^1_{\nu k}(\epsilon) &= \langle k,\omega\rangle^2 - \det \delta A_\nu(\epsilon),\\ f^2_{\nu k}(\epsilon) &= \langle k,\omega\rangle^2 - 4\det \delta \tilde{A}_{\nu+1}(\epsilon), \end{aligned}$$

$k\neq 0$, $\nu = 1,2,\cdots$, measure estimates similar to that in the proof of Lemma 2.3 can be conducted to conclude that the Lebesgue measure of the Cantor set $\mathcal{E}_{\epsilon_*} =: \mathcal{E}_\infty$ is of the order of ϵ_* as ϵ_* sufficiently small. This completes the proof of Theorem 3.2. □

Proof of Main Theorem and Corollaries A, B: The Main Theorem (1) and (2) are direct consequences of Theorem 3.1 and Theorem 3.2, respectively.

Corollaries A, B follow from the Main Theorem by examining the signs of $\hat\lambda$. Indeed, when l is odd, $(\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{l-1}{l-p}} > 0$ and hence $\hat\lambda$ has the opposite sign as that of λ, and when l is even, $(\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}])^{\frac{l-1}{l-p}}$ and $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]$ have the same sign and hence $\hat\lambda$ has the same sign as that of $-\lambda(\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]) = -\lambda^2[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]$. □

4. Non-existence of response solutions

In the case that the condition **(H)** fails, i.e., $l-p$ is even but $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]<0$, we have the following result which implies that the system (1.2) admits no response solution of amplitude in the scale which is greater or equal to $(\epsilon/\lambda)^{\frac{1}{|l-p|}}$ for any ϵ sufficiently small, independent of λ.

Proposition 4.1. *Let $l-p$ be even and assume that $\lambda[\frac{\partial^p f(\cdot,0,0)}{\partial x^p}]<0$. Then for any $x_*>0$, there exists an $\epsilon(x_*)>0$ such that the system* (2.1) *admits no invariant torus, in particular no response torus, in the region $|x| > \epsilon^{\frac{1}{|l-p|}} x_*$ for any $\epsilon\in(0,\epsilon(x_*))$.*

Proof. Since $[f_{i0}] = 0$, $i = 0,\cdots,p-1$, we see that the transformation

$$x = x_1, \quad y = y_1 + \epsilon\sum_{i=0}^{p} v_i(\theta)x^i,$$

where $v_i(\theta)$ is the solutions of equation (2.5), transforms the system (2.1) to

$$\begin{cases} \dot\theta = \omega,\\ \dot x_1 = y_1 + \epsilon\sum_{i=0}^p v_i(\theta)x_1^i,\\ \dot y_1 = -\lambda x_1^l + \epsilon[f_{p0}]x_1^p + \epsilon f_2(\theta,x) + \epsilon f_1(\theta,x_1,y_1+\epsilon\sum_{i=0}^p v_i(\theta)x_1^i)(y_1+\epsilon\sum_{i=0}^p v_i(\theta)x_1^i)\\ \quad -\epsilon\sum_{i=1}^p v_i(\theta)x_1^{i-1}(y_1+\epsilon\sum_{i=0}^p v_i(\theta)x_1^i), \end{cases}$$

which, under the re-scaling $x_1 = \epsilon^{\frac{1}{|l-p|}}x_2$, $y_1 = \epsilon^{\frac{l+1}{2|l-p|}}y_2$, becomes

$$\begin{cases} \dot\theta = \omega,\\ \dot x_2 = \epsilon^{\frac{l-1}{2|l-p|}}y_2 + \epsilon^{\frac{|l-p|-1}{|l-p|}}\sum_{i=0}^p v_i(\theta)(\epsilon^{\frac{1}{|l-p|}}x_2)^i,\\ \dot y_2 = \epsilon^{\frac{l-1}{2|l-p|}}(-\lambda x_2^l + [f_{p0}]x_2^p + \mathcal{O}(\epsilon^\alpha)) \end{cases} \tag{4.1}$$

for a suitable constant $\alpha>0$. For any $x_*>0$, the conditions that $l-p$ is even and $\lambda[f_{p0}]<0$ imply that

$$|-\lambda x_2^l + [f_{p0}]x_2^p| > |x_*^p[f_{p0}]|,$$

for all $|x_2|>x_*$. Thus, there exist an $\epsilon_* = \epsilon(x_*)$ such that, for any $\epsilon\in(0,\epsilon_*)$, we have

$$|\dot y_2| > \epsilon^{\frac{l-1}{2|l-p|}}(|\lambda x_2^l + [f_{p0}]x_2^p| - |\mathcal{O}(\epsilon^\alpha)|) > \epsilon^{\frac{l-1}{2|l-p|}}\frac{|x_*^p[f_{p0}]|}{2} > 0,$$

as $|x_2|>x_*$. This implies that the system (4.1) admits no invariant torus in the region $|x_2|>x_*$. By tracing back the re-scaling, we see that the system (2.1) admits no invariant torus in the region $|x| > \epsilon^{\frac{1}{|l-p|}}x_*$ for any $\epsilon\in(0,\epsilon_*)$. □

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