INTERIOR REGULARITY FOR REGIONAL FRACTIONAL LAPLACIAN

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Abstract. In this paper, we study interior regularity properties for the regional fractional Laplacian operator. We obtain the integer order differentiability of the regional fractional Laplacian, which solves a conjecture of Guan and Ma [9]. We further extend the integer order differentiability to the fractional order of the regional fractional Laplacian. Schauder estimates for the regional fractional Laplacian are also provided.

1. Introduction

This paper is devoted to the study of interior regularity for the regional fractional Laplacian operators. Given real numbers $0 < s < 2$, $\epsilon > 0$, and an open set $\Omega \subset \mathbb{R}^n$, denote

$$\Delta_{\Omega,\epsilon}^{\frac{s}{2}} u(x) = \mathcal{A}(n,-s) \int_{\Omega\cap B_\epsilon^c(x)} \frac{u(y)-u(x)}{|x-y|^{n+s}} dy, \tag{1.1}$$

where $\mathcal{A}(n,-s) = \frac{|s|2^{s-1}\Gamma(\frac{n+s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\frac{s}{2})}$, $B_\epsilon(x)$ is the open ϵ-ball in $\mathbb{R}^n$ centered at x, and $u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$, i.e., $\int_\Omega \frac{|u(x)|}{(1+|x|)^{n+s}} dx < \infty$. The *regional s-fractional Laplacian* $\Delta_\Omega^{\frac{s}{2}}$ *on* Ω is defined as

$$\Delta_\Omega^{\frac{s}{2}} u(x) = \lim_{\epsilon\to 0} \Delta_{\Omega,\epsilon}^{\frac{s}{2}} u(x), \qquad u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}}), \tag{1.2}$$

provided that the limit exists. The regional s-fractional Laplacian can be also defined on the closure $\bar{\Omega}$ of Ω by talking $\bar{\Omega}$ in place of Ω in the above. We note that, if $x \in \Omega$, then $\Delta_{\bar{\Omega}}^{\frac{s}{2}} u(x) = \Delta_\Omega^{\frac{s}{2}} u(x)$.

When Ω is a bounded Lipschitz open set, the regional s-fractional Laplacian $\Delta_{\bar{\Omega}}^{\frac{s}{2}}$ is in fact the Feller generator of the so-called reflected symmetric s-stable process $(X_t)_{t\geq 0}$ on $\bar{\Omega}$, i.e., a Hunt process associated with the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\bar{\Omega}, dx)$:

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}(n,-s)\int_{\bar{\Omega}}\int_{\bar{\Omega}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+s}} dxdy,$$

$$\mathcal{F} = \Big\{u \in L^2(\bar{\Omega}), \int_{\bar{\Omega}}\int_{\bar{\Omega}} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} dxdy < \infty\Big\}.$$

It is first shown in [1] that if $0 < s \leq 1$, then the censored s-stable process in Ω is essentially the reflected s-stable process $(X_t)_{t\geq 0}$, and if $1 < s < 2$, then the censored s-stable process in Ω is identified as a proper subprocess of $(X_t)_{t\geq 0}$ killed upon leaving Ω. Later, it is

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shown in [3] that $(X_t)_{t\geq 0}$ can be refined to be a Feller process starting from each point of $\bar{\Omega}$ which admits a Hölder continuous transition density function. In [9], not only does the Feller generator of $(X_t)_{t\geq 0}$ on $\bar{\Omega}$ shown to be the regional s-fractional Laplacian, but also a semi-martingale decomposition of $(X_t)_{t\geq 0}$ is obtained by studying the differentiability of the regional fractional Laplacian and its integration by parts property. For other studies on regional fractional Laplacians, we refer the reader to [7] for a more general integration by parts formula of the regional fractional and fractional-like Laplacian, and to [8] for some boundary Harnack inequalities for the regional fractional Laplacian on $C^{1,\beta-1}(\Omega)$, $s<\beta\leq 2$.

If $\Omega=\mathbb{R}^n$, the regional fractional Laplacian $\Delta^{\frac{s}{2}}_{\mathbb{R}^n}$ becomes the usual fractional Laplacian $-(-\Delta)^{\frac{s}{2}}$ defined via Fourier transform: $\mathcal{F}((-\Delta)^{\frac{s}{2}}u)(\xi)=|\xi|^s\mathcal{F}(u)(\xi)$ (see [4]). If we furtherly let s tend to 2, then the fractional Laplacian $-(-\Delta)^{\frac{s}{2}}$ becomes the classical Laplacian Δ, and it is clear that $u\in C^{\alpha}$ for some integer $\alpha>2$ implies that $\Delta u\in C^{\alpha-2}$. In the case that $u\in C^{\alpha}$ for some $\alpha>s$ with $\alpha-s$ not being an integer, one also has $-(-\Delta)^{\frac{s}{2}}u\in C^{\alpha-s}$ ([11, Proposition 2.7]). A natural problem then is whether the regional fractional Laplacian shares similar regularity properties as that of the classical and fractional Laplacian. This problem is first investigated in [9] in which the following results are proved.

Theorem. *Let Ω be an open set in $\mathbb{R}^n$ and $u\in L^1(\Omega,\frac{dx}{(1+|x|)^{n+s}})$ for some $0<s<2$. Then the following holds.*

a) ([9, Proposition 8.3]) *If $u\in C^{1,\alpha}(\Omega)$ for some $\alpha>s$ when $0<s<1$ or $u\in C^{2,\alpha}(\Omega)$ for some $\alpha>s-1$ when $1\leq s<2$, then $\Delta^{\frac{s}{2}}_{\Omega}u\in C^{1,0}(\Omega)$.*

b) ([9, Theorem 8.1]) *In the case $n=1$, if r is a non-negative integer such that $u\in C^{r,\alpha}(\Omega)$ for some some $\alpha>s$ when $0<s<1$ or $u\in C^{r+1,\alpha}(\Omega)$ for some $\alpha>s-1$ when $1\leq s<2$, then $\Delta^{\frac{s}{2}}_{\Omega}u\in C^{r,0}(\Omega)$.*

In the above and also through the rest of the paper, for each non-negative integer r and $0\leq\alpha\leq 1$, we denote by $C^{r,\alpha}(\Omega)$ ($C^{r,\alpha}(\bar{\Omega})$) the subspace of $C^{r,0}(\Omega)$ ($C^{r,0}(\bar{\Omega})$) consisting functions whose rth partial derivatives are locally (uniformly) α-Hölder continuous in Ω.

It is conjectured in [9] that part b) of the above theorem should hold for higher dimensions as well.

In this paper, we give an affirmative answer to this conjecture. More precisely, we will prove the following result.

Theorem A. *Let $\Omega\subset\mathbb{R}^n$ be an open set and $u\in L^1(\Omega,\frac{1}{(1+|x|)^{n+s}})$ for some $0<s<2$. If r is a non-negative integer such that $u\in C^{r,\alpha}(\Omega)$ for some $1\geq\alpha>s$ or $u\in C^{r+1,\alpha}(\Omega)$ for some α with $2\geq 1+\alpha>s\geq\alpha$, then $\Delta^{\frac{s}{2}}_{\Omega}u\in C^{r,0}(\Omega)$.*

Unlike the fractional Laplacian, the differential operator and the regional fractional Laplacian are not exchangeable in order. To overcome this difficulty in the proof of Theorem A, we derive a class of integral identities (see Lemma 2.1) and use them to conclude that all possible singular terms of $D^r(\Delta^{\frac{s}{2}}_{\Omega,\epsilon}u)$ as $\epsilon\to 0^+$ are in fact non-singular.

Making further estimates, we are able to extend the integer order differentiability result in Theorem A to a fractional order. Throughout the paper, for each $j=(j_1,j_2\cdots j_n)\in\mathbb{N}^n$, we denote $|j|=j_1+j_2+\cdots+j_n$ and $\partial^j u=\frac{\partial^{|j|}u}{(\partial x_1)^{j_1}(\partial x_2)^{j_2}\cdots(\partial x_n)^{j_n}}$. For any $u\in C^{r,\alpha}(\Omega)$, where r

is a non-negative integer and $0 \le \alpha \le 1$, define

$$[u]_{r,\alpha;\Omega} = \begin{cases} \sup_{x\in\Omega,|j|=r} |\partial^j u(x)|, & \text{if}\, \alpha = 0; \\ \sup_{x,y\in\Omega,x\neq y,|j|=r} \frac{|\partial^j u(x)-\partial^j u(y)|}{|x-y|^\alpha}, & \text{if}\, \alpha > 0. \end{cases}$$

Then we have the follows result analogous to [11, Proposition 2.7] in the case of regional fractional Laplacian.

Theorem B. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}})$ for some $0 < s < 2$. Then the following holds.*

(i) *If $u \in C^{r,\alpha}(\Omega)$ for some positive number α with $s < \alpha \le 1$ and a non-negative integer r, then $\Delta_\Omega^{\frac{s}{2}} u(x) \in C^{r,\alpha-s}(\Omega)$, and moreover,*

$$[\Delta_\Omega^{\frac{s}{2}} u]_{r,\alpha-s;\Omega} \le C[u]_{r,\alpha;\Omega}.$$

(ii) *If $u \in C^{r,\alpha}(\Omega)$ for some positive number α with $\alpha < s < 1+\alpha \le 2$ and a positive integer r, then $\Delta_\Omega^{\frac{s}{2}} u(x) \in C^{r-1,1+\alpha-s}(\Omega)$, and moreover,*

$$[\Delta_\Omega^{\frac{s}{2}} u]_{r-1,1+\alpha-s;\Omega} \le C[u]_{r,\alpha;\Omega}.$$

Schauder estimate is well-known for the classical Laplacian Δ (see [6]) as well as for the fractional Laplacian (see [2, 5, 10, 11]). We refer the reader to [2, 5] for interior and boundary regularity theory for more general fractional operators. Using Schauder estimates for the fractional Laplacian, we are able to show a similar Schauder estimate holds for the regional fractional Laplacian. For each $u \in C^{r,\alpha}(\bar{\Omega})$, denote

$$\|u\|_{C^{r,\alpha}(\bar{\Omega})} = \begin{cases} \sum_{j=0}^{r} [u]_{j,0,\Omega}, & \text{if}\, \alpha = 0; \\ \|u\|_{C^{r,0}(\bar{\Omega})} + [u]_{r,\alpha;\Omega}, & \text{if}\, \alpha > 0. \end{cases}$$

For simplicity, through the rest of the paper, we use the notation $C^\alpha(\Omega)$ $(C^\alpha(\bar{\Omega}))$, where $\alpha > 0$, to denote the space $C^{r,\alpha'}(\Omega)$ $(C^{r,\alpha'}(\bar{\Omega}))$, where r is the largest integer smaller than α and $\alpha' = \alpha - r$. We note that if α is an integer r, then $C^\alpha(\Omega) = C^{\alpha-1,1}(\Omega) \neq C^{\alpha,0}(\Omega)$ $(C^\alpha(\bar{\Omega}) = C^{\alpha-1,1}(\bar{\Omega}) \neq C^{\alpha,0}(\bar{\Omega}))$.

For a given open set Ω in $\mathbb{R}^n$ with $\partial\Omega \neq \emptyset$, let

$$d_x = dist(x, \Omega) \text{ and } \Omega_\delta = \{x \in \Omega; d_x < \delta\}.$$

We denote $C_c^\infty(\Omega)$ as the space of C^∞ functions with compact support in Ω, $\mathcal{S}$ as the Schwartz space of rapidly decreasing C^∞ function in $\mathbb{R}^n$, and $\Lambda_*(\bar{\Omega})$ as the Zygmund space of all bounded functions on $\bar{\Omega}$ such that

$$[u]_{\Lambda_*(\bar{\Omega})} := \sup_{x,x+h,x-h\in\bar{\Omega}} \frac{|u(x+h)+u(x-h)-2u(x)|}{|h|} < \infty.$$

We equip the space $\Lambda_*(\bar{\Omega})$ with the norm $\|u\|_{\Lambda_*(\bar{\Omega})} := \|u\|_{L^\infty(\bar{\Omega})} + [u]_{\Lambda_*(\bar{\Omega})}$. It is easy to see that the regional fractional Laplacian $\Delta_\Omega^{\frac{s}{2}}$ is well-defined for functions $u \in C^2(\Omega) \cap L^\infty(\Omega)$. We can then extend the definition of the regional fractional Laplacian to the space $L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$. For any $u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$, we define

$$< \Delta_\Omega^{\frac{s}{2}} u, \varphi >_\Omega := \int_\Omega u(y) \Delta_\Omega^{\frac{s}{2}} \varphi(y) dy, \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{1.3}$$

In Theorems C and D below, the definition of $\Delta_{\Omega}^{\frac{s}{2}}$ is understood in the sense of (1.3).

Theorem C. *Let $0 < \alpha \le 1$ and $0 < s < 2$. If, for some $w \in C^{\alpha}(\bar{\Omega})$, $u \in L^{\infty}(\Omega)$ solves the equation $\Delta_{\Omega}^{\frac{s}{2}} u = w$ in Ω, then*

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_{\delta})} \le C(\|u\|_{L^{\infty}(\Omega)} + \|w\|_{C^{\alpha}(\bar{\Omega})}),$$

where $\delta > 0$ is sufficiently small and C is a constant depending only on n, s, δ and α.

Remark 1.1. *The notions of $C^{\alpha+s}(\bar{\Omega}_{\delta})$ and $C^{\alpha}(\bar{\Omega})$ we defined earlier have unified different cases for $\alpha, \alpha+s$ being or not being natural numbers in the statement of Theorem C. We note that $C^1(\bar{\Omega}_{\delta}) = C^{0,1}(\bar{\Omega}_{\delta}) \neq C^{1,0}(\bar{\Omega}_{\delta})$, $C^1(\bar{\Omega}) = C^{0,1}(\bar{\Omega}) \neq C^{1,0}(\bar{\Omega})$, and $C^2(\bar{\Omega}_{\delta}) = C^{1,1}(\bar{\Omega}_{\delta}) \neq C^{2,0}(\bar{\Omega}_{\delta})$.*

Theorem D. *Let $0 < s < 2$. Suppose that, for some $w \in L^{\infty}(\Omega)$, $u \in L^{\infty}(\Omega)$ solves the equation $\Delta_{\Omega}^{\frac{s}{2}} u = w$ in Ω. Then, for any sufficiently small $\delta > 0$, there exists a constant $C > 0$ depending only on n, s, and δ such that the following holds:*

(i) *If $s \neq 1$, then*

$$\|u\|_{C^{s}(\bar{\Omega}_{\delta})} \le C(\|u\|_{L^{\infty}(\Omega)} + \|w\|_{L^{\infty}(\Omega)}).$$

(ii) *If $s = 1$, then*

$$\|u\|_{\Lambda_{*}(\bar{\Omega}_{\delta})} \le C(\|u\|_{L^{\infty}(\Omega)} + \|w\|_{L^{\infty}(\Omega)}).$$

The rest of sections are devoted to the proof of these results. In Section 2, we will give a class of integral identities and use them to obtain the integer order differentiability of the regional fractional Laplacian $\Delta_{\Omega}^{\frac{s}{2}}$. Both Theorem A and B will be proved in this section. In Section 3, Schauder estimate for the regional fractional Laplacian will be given and Theorem C and D will be proved.

2. Differentiability

In this section, we will study the differentiability of the regional fractional Laplacian in an open set Ω and prove Theorems A, B. The proof will be based on some integral identities in $\mathbb{R}^n$.

2.1. Integral identities.

Given $n' \in \mathbb{N}$, $z = (z_1, z_2, \cdots, z_{n'}) \in \mathbb{R}^{n'}$ and $k = (k_1, k_2, \cdots, k_{n'}) \in \mathbb{N}^{n'}$, we denote by z^k the monomial $\prod_{i=1}^{n'} z_i^{k_i}$. Also, for each $j = 1, 2, \cdots, n$, we let e_j to denote the jth standard coordinate vector in $\mathbb{R}^n$.

Lemma 2.1. *Consider an annulus domain $R_{\delta\epsilon}(0) := \{z \in \mathbb{R}^n : \epsilon < |z| < \delta\}$, where $0 < \epsilon < \delta$. Then for any $i, j, m \in \mathbb{N}$ and $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$, we have*

$$\frac{1}{k_i + 1} \int_{R_{\delta\epsilon}(0)} \frac{z^{k+2e_i}}{|z|^m} dz = \frac{1}{k_j + 1} \int_{R_{\delta\epsilon}(0)} \frac{z^{k+2e_j}}{|z|^m} dz.$$

Proof. The lemma clearly holds if $i = j$. Let $i \neq j$, and without loss of generality, we assume that $i = n-1$ and $j = n$. Denote $\tilde{z} = (z_1, z_2, \cdots, z_{n-2}) \in \mathbb{R}^{n-2}$, $\tilde{k} = (k_1, k_2, \cdots, k_{n-2})$, and consider polar coordinate (r, θ) in the (z_{n-1}, z_n)-plane. Then under the new coordinate

$R_{\delta\epsilon}(0)$ becomes $\tilde{R}_{\delta\epsilon}(0)\times(0,2\pi)=\{(\tilde{z},r)\in\mathbb{R}^{n-2}\times\mathbb{R}_+ : \epsilon^2<|\tilde{z}|^2+r^2<\delta^2\}\times(0,2\pi)$, and integration by parts yields that

$$\begin{aligned}
&\int_{R_{\delta\epsilon}(0)}\frac{z^k z_{n-1}^2}{|z|^m}dz\\
=&\int_{\tilde{R}_{\delta\epsilon}(0)}\frac{\tilde{z}^{\tilde{k}}}{(|\tilde{z}|^2+r^2)^{\frac{m}{2}}}r^{k_{n-1}+k_n+3}d\tilde{z}dr\int_0^{2\pi}cos^{k_{n-1}+2}\theta sin^{k_n}\theta d\theta\\
=&(k_{n-1}+1)\int_{\tilde{R}_{\delta\epsilon}(0)}\frac{\tilde{z}^{\tilde{k}}}{(|\tilde{z}|^2+r^2)^{\frac{m}{2}}}r^{k_{n-1}+k_n+3}d\tilde{z}dr\int_0^{2\pi}cos^{k_{n-1}}\theta sin^{k_n+2}\theta d\theta\\
&-k_n\int_{\tilde{R}_{\delta\epsilon}(0)}\frac{\tilde{z}^{\tilde{k}}}{(|\tilde{z}|^2+r^2)^{\frac{m}{2}}}r^{k_{n-1}+k_n+3}d\tilde{z}dr\int_0^{2\pi}cos^{k_{n-1}+2}\theta sin^{k_n}\theta d\theta\\
=&(k_{n-1}+1)\int_{R_{\delta\epsilon}(0)}\frac{z^k z_n^2}{|z|^m}dz-k_n\int_{R_{\delta\epsilon}(0)}\frac{z^k z_{n-1}^2}{|z|^m}dz,
\end{aligned}$$

from which the lemma follows. □

Lemma 2.2. ([9, Lemma 8.2]) *Let Ω be an open subset of $\mathbb{R}^n$ and $u\in L^1(\Omega)$. Suppose that u is continuous in an open neighborhood U of $x_0=(x_1,\cdots,x_n)\in\Omega$ and $dist(x_0,\partial U)>\epsilon>0$. Then the function $f(x)=\int_{\Omega\cap B_\epsilon^c(x)}u(y)dy$ is differentiable at x_0 and*

$$\frac{\partial f}{\partial x_i}(x_0)=\int_{\partial B_\epsilon(x_0)}u(y)\frac{x_i-y_i}{|x-y|}m(dy),$$

where $m(dy)$ is the $n-1$ dimensional surface Lebesgue measure.

2.2. **Proof of Theorems A.** We first prepare some technical lemmas concerning derivatives of $\Delta_{\Omega,\epsilon}^{\frac{s}{2}}u(x)$.

Lemma 2.3. *Let Ω be an open set in $\mathbb{R}^n$ and $0<s<2$. Suppose that $u\in L^1(\Omega,\frac{1}{(1+|x|)^{n+s}})\cap C^{1,0}(\Omega)$. Then for any $x\in\Omega$, $\delta<d_x=dist(x,\partial\Omega)$, and $0<\epsilon<\delta$, we have*

$$\begin{aligned}
&\frac{2}{\mathcal{A}(n,-s)}\frac{\partial}{\partial x_i}\Delta_{\Omega,\epsilon}^{\frac{s}{2}}u(x)\\
=&-\int_{R_{\delta\epsilon}(x)}\frac{\sum_{j=1}^n(\frac{\partial u}{\partial x_j}(y)-\frac{\partial u}{\partial x_j}(x))(y_j-x_j)(x_i-y_i)}{|x-y|^{n+s+2}}dy\\
&-(n-1)\int_{R_{\delta\epsilon}(x)}\frac{[u(y)-u(x)+\sum_{j=1}^n\frac{\partial u}{\partial x_j}(x)(x_j-y_j)](x_i-y_i)}{|x-y|^{n+s+2}}dy\\
&-\int_{B_\delta^c(x)\cap\Omega}\frac{\frac{\partial u}{\partial x_i}(x)}{|x-y|^{n+s}}dy+\int_{\partial B_\delta(x)\cap\Omega}\frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+s+1}}m(dy)\\
&-(n+s)\int_{B_\delta^c(x)\cap\Omega}\frac{(u(y)-u(x))(x_i-y_i)}{|y-x|^{n+s+2}}dy,
\end{aligned}\tag{2.1}$$

where $R_{\delta\epsilon}(x):=B_\delta(x)\cap B_\epsilon^c(x)$.

Proof. The proof follows from that of [9, Proposition 8.3]. □

Lemma 2.4. *Let Ω be an open set in $\mathbb{R}^n$ and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r+1,0}(\Omega)$ for some positive integer r. For any $x \in \Omega$, $\epsilon > 0$, $\delta < d_x = dist(x, \partial\Omega)$, and $l = (l_1, l_2, \cdots l_n)$, $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$, if $k_i = 0$ for some $1 \le i \le n$, then*

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy \\
= &\ -(n+2p-m-1)\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}}(x-y)^{k+e_i} dy \\
&-\int_{R_{\delta\epsilon}(x)} \Big[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j'+j} u(x)}{|x-y|^{n+s+2p+2}}(y-x)^{j'}\Big](x-y)^{k+e_i} dy,
\end{aligned}$$

where $A_j = \frac{|j|!}{j_1! j_2! \cdots j_n!}$ for each $j = (j_1, j_2, \cdots, j_n) \in \mathbb{N}$ and $m, p \in \mathbb{N}$ are such that $|l| + m = r$ and $m + |k| = 2p$.

Proof. Since $k_i = 0$, we have by Lemma 2.2 that

$$\begin{aligned}
I : &= \frac{\partial}{\partial x_i}\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= -\frac{1}{m!}\sum_{|j|=m} A_j \partial^{l+j+e_i} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p}} dy, \\
I_2 &:= -(n+s+2p)\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}}(x-y)^{k+e_i} dy, \\
I_3 &:= \int_{\partial B_\epsilon(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+1}}(x-y)^{k+e_i} m(dy) \\
&\quad - \int_{\partial B_\delta(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+1}}(x-y)^{k+e_i} m(dy).
\end{aligned}$$

Using the identity $m + |k| = 2p$, integration by parts yields

$$\begin{aligned}
I_2 &= -(n+s+2p)\int_\epsilon^\delta r^{-n-s-2p-1} dr \int_{\partial B_1(x)} r^{n+2p-m-1}(x-y)^{k+e_i} \\
&\quad \Big[\partial^l u(x + r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x) r^{|j|}\Big] m(dy) \\
&= \int_\epsilon^\delta r^{n+2p-m-1}\Big\{\int_{\partial B_1(x)} \big[\partial^l u(x + r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x) r^{|j|}\big] \\
&\quad (x-y)^{k+e_i} m(dy)\Big\} dr^{-(n+s+2p)} \\
&= -I_3 - (n+2p-m-1)\int_\epsilon^\delta \int_{\partial B_1(x)} r^{-m-2-s}(x-y)^{k+e_i}
\end{aligned}$$

$$\Big[\partial^l u(x+r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j} u(x) r^{|j|}\Big] m(dy)dr$$

$$-\int_\epsilon^\delta \int_{\partial B_1(x)} r^{-m-1-s}(x-y)^{k+e_i}$$

$$\sum_{|j'|=1} \Big[\partial^{l+j'} u(x+r(y-x))(y-x)^{j'} - \sum_{|j|=0}^{|j|=m-1} \frac{A_j}{|j|!}(y-x)^{j'+j} \partial^{l+j'+j} u(x) r^{|j|}\Big] m(dy)dr$$

$$= -I_3 - (n+2p-m-1)\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j}u(x)}{|x-y|^{n+s+2p+2}}(x-y)^{k+e_i}dy$$

$$-\int_{R_{\delta\epsilon}(x)} \Big[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m-1} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j'+j}u(x)}{|x-y|^{n+s+2p+2}}(y-x)^{j'}\Big](x-y)^{k+e_i}dy.$$

Therefore,

$$I_2 + I_3 = I_{23}^{(1)} + I_{23}^{(2)} + I_{23}^{(3)},$$

where

$$I_{23}^{(1)} = -(n+2p-m-1)\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j}u(x)}{|x-y|^{n+s+2p+2}}(x-y)^{k+e_i}dy,$$

$$I_{23}^{(2)} = -\frac{n+2p}{(m+1)!}\sum_{|j|=m+1} A_j \partial^{l+j}u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy,$$

$$I_{23}^{(3)} = -\int_{R_{\delta\epsilon}(x)} \Big[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!}(y-x)^j \partial^{l+j'+j}u(x)}{|x-y|^{n+s+2p+2}}(y-x)^{j'}\Big](x-y)^{k+e_i}dy.$$

Observe that

$$I_1 + I_{23}^{(2)}$$

$$= -\sum_{t\in\mathbb{N}, t\le \frac{m}{2}} \frac{C_m^{2t}}{m!} \sum_{|j|=m, j_i=2t} A_{j-j_ie_i}\partial^{l+j+e_i}u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1}(x-y)^{2j'}dy$$

$$-\frac{n+2p}{(m+1)!}\sum_{t\in\mathbb{N}, t\le \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_ie_i}\partial^{l+j}u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy$$

$$= -\sum_{t\in\mathbb{N}, t\le \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_ie_i}(2t+1)\partial^{l+j+e_i}u(x)$$

$$\int_{R_{\delta\epsilon(x)}} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1}(x-y)^{2j'}dy$$

$$-\frac{n+2p}{(m+1)!}\sum_{t\in\mathbb{N}, t\le \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_ie_i}\partial^{l+j}u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy,$$

where $j=(j_1, j_2, \cdots, j_n)$. An application of Lemma 2.1 yields

$$I_1 + I_{23}^{(2)}$$

$$= \sum_{t\in\mathbb{N}, t\le \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_ie_i}\Big[n+2p-(|k|+n+m)\Big]\partial^{l+j+e_i}u(x)$$
$$\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}}dy = 0.$$

Thus, $I = I_{23}^{(1)} + I_{23}^{(3)}$ and the lemma is proved. □

Lemma 2.5. *Let Ω be an open set in $\mathbb{R}^n$ and $0<s<2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r+1,0}(\Omega)$ for some positive integer r. For any $x\in\Omega$, $\epsilon>0$, $\delta<d_x = dist(x,\partial\Omega)$, and $l=(l_1,l_2,\cdots l_n)$, $k=(k_1,k_2,\cdots,k_n)\in\mathbb{N}^n$, if $k_i\neq 0$ for some $1\le i\le n$, then*

$$\frac{\partial}{\partial x_i}\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy$$
$$= k_i\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^{k-e_i} dy$$
$$-(n+2p-m-1)\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p+2}}(x-y)^{k+e_i} dy$$
$$-\int_{R_{\delta\epsilon}(x)}\Big[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j'+j}u(x)}{|x-y|^{n+s+2p+2}}(y-x)^{j'}\Big](x-y)^{k+e_i} dy,$$

where $m,p\in\mathbb{N}$ are such that $|l|+m=r$ and $m+|k|=2p$.

Proof. Since $k_i\neq 0$, we have by Lemma 2.2 that

$$I = I_1 + \bar{I}_1 + I_2 + I_3,$$

where I and I_i, $i=1,2,3$, are as in the proof of Lemma 2.4 and

$$\bar{I}_1 = k_i\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^{k-e_i} dy.$$

By the proof of Lemma 2.4, we have

$$I_2 + I_3 = I_{23}^{(1)} + I_{23}^{(2)} + I_{23}^{(3)}.$$

where $I_{23}^{(i)}$, $i=1,2,3$, are defined in the proof of Lemma 2.4. Write

$$\bar{I}_1 = \bar{I}_1^{(1)} + \bar{I}_1^{(2)},$$

where

$$\bar{I}_1^{(1)} = k_i\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^{k-e_i} dy,$$
$$\bar{I}_1^{(2)} = \frac{k_i}{(m+1)!}\sum_{|j|=m+1} A_j\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p}}dy.$$

Note that

$$I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)}$$

$$
\begin{aligned}
= \quad & -\frac{1}{m!}\sum_{|j|=m} A_j \partial^{l+j+e_i}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^k}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy \\
& +\frac{k_i}{(m+1)!}\sum_{|j|=m+1} A_j \partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy \\
& -\frac{n+2p}{(m+1)!}\sum_{|j|=m+1} A_j \partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy.
\end{aligned}
$$

First let k_i be an even number. Then

$$
\begin{aligned}
& I_1+\bar{I_1}^{(2)}+I_{23}^{(2)} \\
= \quad & -\sum_{t\in\mathbb{N},t\le\frac{m}{2}}\frac{C_m^{2t}}{m!}\sum_{|j|=m,j_i=2t} A_{j-j_ie_i}\partial^{l+j+e_i}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^k}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy \\
& +\frac{k_i}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m}{2}} C_{m+1}^{2t+1}\sum_{|j|=m+1,j_i=2t+1} A_{j-j_ie_i}\partial^{l+j}u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy \\
& -\frac{n+2p}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m}{2}} C_{m+1}^{2t+1}\sum_{|j|=m+1,j_i=2t+1} A_{j-j_ie_i}\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy \\
= \quad & -\sum_{t\in\mathbb{N},t\le\frac{m}{2}}\frac{C_{m+1}^{2t+1}}{(m+1)!}\sum_{|j|=m,j_i=2t} A_{j-j_ie_i}(2t+k_i+1)\partial^{l+j+e_i}u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^k}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy \\
& +\sum_{t\in\mathbb{N},t\le\frac{m}{2}}\frac{C_{m+1}^{2t+1}}{(m+1)!}\sum_{|j|=m,j_i=2t} A_{j-j_ie_i}(n+2p)\partial^{l+j+e_i}u(x)\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}}dy.
\end{aligned}
$$

It follows from Lemma 2.1 that

$$
\begin{aligned}
& I_1+\bar{I_1}^{(2)}+I_{23}^{(2)} \\
= \quad & \sum_{t\in\mathbb{N},t\le\frac{m}{2}}\frac{C_{m+1}^{2t+1}}{(m+1)!}\sum_{|j|=m,j_i=2t} A_{j-j_ie_i}\Big[(n+2p)-(|k|+n+m)\Big]\partial^{l+j+e_i}u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}}dy=0.
\end{aligned}
$$

Now let k_i be an odd number. Then

$$
\begin{aligned}
& I_1+\bar{I_1}^{(2)}+I_{23}^{(2)} \\
= \quad & -\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}}\frac{C_m^{2t+1}}{m!}\sum_{|j|=m,j_i=2t+1} A_{j-j_ie_i}\partial^{l+j+e_i}u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^k}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k_i}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m+1}{2}} C_{m+1}^{2t}\sum_{|j|=m+1,j_i=2t} A_{j-j_ie_i}\partial^{l+j}u(x)\\
&\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy\\
&-\frac{n+2p}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m+1}{2}} C_{m+1}^{2t}\sum_{|j|=m+1,j_i=2t} A_{j-j_ie_i}\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy\\
=\;&\frac{k_i}{(m+1)!}\sum_{|j|=m+1,j_i=0} A_j\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy\\
&-\frac{(n+2p)}{(m+1)!}\sum_{|j|=m+1,j_i=0} A_j\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy\\
&-\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}}\frac{C_m^{2t+1}}{m!}\sum_{|j|=m,j_i=2t+1} A_{j-j_ie_i}\partial^{l+j+e_i}u(x)\\
&\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy\\
&+\frac{k_i}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}} C_{m+1}^{2t+2}\sum_{|j|=m+1,j_i=2t+2} A_{j-j_ie_i}\partial^{l+j}u(x)\\
&\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy\\
&-\frac{n+2p}{(m+1)!}\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}} C_{m+1}^{2t+2}\sum_{|j|=m+1,j_i=2t+2} A_{j-j_ie_i}\partial^{l+j}u(x)\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy.
\end{aligned}
$$

It again follows from Lemma 2.1 that

$$
\begin{aligned}
&I_1+\bar{I}_1^{(2)}+I_{23}^{(2)}\\
=\;&\frac{1}{(m+1)!}\sum_{|j|=m+1,j_i=0} A_j\Big[(m+n+|k|-(n+2p))\Big]\partial^{l+j}u(x)\\
&\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}}dy\\
&-\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}}\frac{C_{m+1}^{2t+2}}{(m+1)!}\sum_{|j|=m,j_i=2t+1} A_{j-j_ie_i}(2t+2+k_i)\partial^{l+j+e_i}u(x)\\
&\quad\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k}}{|x-y|^{n+s+2p+2}}\sum_{|j'|=1}(x-y)^{2j'}dy\\
&+\sum_{t\in\mathbb{N},t\le\frac{m-1}{2}}\frac{C_{m+1}^{2t+2}}{(m+1)!}\sum_{|j|=m,j_i=2t+1} A_{j-j_ie_i}(n+2p)\partial^{l+j+e_i}u(x)\\
&\quad\int_{R_{\delta\epsilon}(x)}\frac{(y-x)^j(x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}}dy
\end{aligned}
$$

$$= \sum_{t\in\mathbb{N}, t\le\frac{m-1}{2}} \frac{C_{m+1}^{2t+2}}{(m+1)!} \sum_{|j|=m, j_i=2t+1} A_{j-j_ie_i}\Big[(n+2p)-(|k|+n+m)\Big]\partial^{l+j+e_i}u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}}dy = 0.$$

Thus, for any $k_i \neq 0$, $I = \bar{I_1}^{(1)} + I_{23}^{(1)} + I_{23}^{(3)}$ and the lemma is proved. ☐

Lemma 2.6. *Let Ω be an open set in $\mathbb{R}^n$ and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r,0}(\Omega)$ for some positive integer r. Then for any $x \in \Omega$, $\epsilon > 0$, $\hat{r} = (r_1, r_2, \cdots, r_n) \in \mathbb{N}^n$ with $|\hat{r}| = r$, $\delta < d_x = dist(x, \partial\Omega)$, all ϵ-dependent terms of $\partial^{\hat{r}}\Delta_{\Omega,\epsilon}^{\frac{s}{2}}u(x)$ have the form*

$$I_{l,k,m,p}^{\epsilon,\delta}(x) = \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy, \tag{2.2}$$

where $l = (l_1, l_2, \cdots l_n)$, $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n, m, p \in \mathbb{N}$ are such that $|l| + m = r$ and $m + |k| = 2p$.

Proof. We will prove the lemma by induction. In the case of $r = 1$, we observe that the only ϵ-dependent terms on the right hand side of (2.1) are its first two terms. They clearly have the form (2.2) with the first term corresponding to $|l| = 1$, $m = 0$, $|k| = 2$ and $p = 1$, and the second term corresponding to $|l| = 0$, $m = 1$, $|k| = 1$ and $p = 1$.

Now suppose that (2.2) is satisfied when $r = q$, where q is a fixed positive integer. We want to show that it is also satisfied when $r = q + 1$, i.e., for any $l, k \in \mathbb{N}$ with $|l| + m = q$ and $m + |k| = 2p$, all ϵ-dependent terms of

$$I = \frac{\partial}{\partial x_i}\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy$$

have the form

$$\int_{R_{\delta\epsilon}(x)} \frac{\partial^{l'} u(y) - \sum_{|j|=0}^{|j|=m'}\frac{A_j}{|j|!}(y-x)^j\partial^{l'+j}u(x)}{|x-y|^{n+s+2p'}}(x-y)^{k'} dy, \tag{2.3}$$

where $|l'| + m' = q + 1$ and $m' + |k'| = 2p'$.

If $k_i = 0$, then, by Lemma 2.4, we have $I = I_{23}^{(1)} + I_{23}^{(3)}$, where $I_{23}^{(1)}$, $I_{23}^{(3)}$ are as in the proof of Lemma 2.4 which clearly have the form (2.3) with $|l'| + m' = |l| + m + 1 = q + 1$ and $m' + |k'| = m + |k| + 2 = 2(p + 1)$.

If $k_i \neq 0$, then, by Lemma 2.5, we have $I = \bar{I_1}^{(1)} + I_{23}^{(1)} + I_{23}^{(3)}$, where $\bar{I_1}^{(1)}$ is as in the proof of Lemma 2.5 which is clearly of the form (2.3) with $l' + m' = |l| + m + 1 = q + 1$ and $m' + |k'| = m + k = 2p$. ☐

Proof of Theorem A. For any $\epsilon > 0$, $x \in \Omega$, $\hat{r} = (r_1, r_2, \cdots, r_n) \in \mathbb{N}^n$ with $|\hat{r}| = r$, and $\delta < d_x = dist(x, \partial\Omega)$, we have by Lemma 2.6 that all ϵ-dependent terms of $\partial^{\hat{r}}\Delta_{\Omega,\epsilon}^{\frac{s}{2}}u(x)$ have the form (2.2).

In the case $1 \geq \alpha > s$, we note that any integral of the form (2.2) is bounded above in absolute value by a constant times $\int_\epsilon^\delta \rho^{\alpha-s-1}d\rho$ which is convergent as $\epsilon \to 0$. It follows that

$\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega,\epsilon}u$ converges uniformly on any compact subset of Ω as $\epsilon \to 0$. Thus, $\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega}u \in C(\Omega)$, i.e., $\Delta^{\frac{s}{2}}_{\Omega}u \in C^{r,0}(\Omega)$.

In the case $2 \geq 1+\alpha > s \geq \alpha$, we again consider an integral $I^{\epsilon,\delta}_{l,k,m,p}(x)$ of the form (2.2) for some $l,k \in \mathbb{N}^n$, $m,p \in \mathbb{N}$ satisfying $|l|+m = r$ and $m+|k| = 2p$. Since $m+1+|k| = 2p+1$ is an odd number, we have, for any $j \in \mathbb{N}^n$ with $|j| = m+1$, that

$$\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j(x-y)^k}{|x-y|^{n+s+2p}}dy = 0.$$

Hence $I^{\epsilon,\delta}_{l,k,m,p}(x)$ can be re-written as

$$\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy$$

which is bounded above in absolute value by a constant times $\int_\epsilon^\delta \rho^{\alpha-s}d\rho$ that is convergent as $\epsilon \to 0$. It follows again that $\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega,\epsilon}u$ converges uniformly on any compact subset of Ω as $\epsilon \to 0$. Thus, $\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega}u(x) \in C(\Omega)$, i.e., $\Delta^{\frac{s}{2}}_{\Omega}u \in C^{r,0}(\Omega)$. □

2.3. **Proof of Theorem B.** Let $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r,\alpha}(\Omega)$ for some positive integer r and some real numbers $0 < s < 2$, $0 < \alpha \leq 1$. For each $\epsilon > 0$ sufficiently small, $x \in \Omega$, and any $\hat r \in \mathbb{N}^n$ with $|\hat r| \leq r$, we write

$$\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega,\epsilon}u(x) = I_\epsilon(x) + I_*(x),$$

where $I_\epsilon(x)$ denotes the ϵ-dependent term of $\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega,\epsilon}u(x)$ and $I_*(x)$ denotes the remaining term.

Lemma 2.7. *Let u, s, r, α be as in the above. If either $1 \geq \alpha > s$ or $2 \geq 1+\alpha > s \geq \alpha$, then*

$$\partial^{\hat r}\Delta^{\frac{s}{2}}_{\Omega}u(x) = I_0(x) + I_*(x), \quad \text{if } |\hat r| \leq r,$$

where $I_0(x) = \lim_{\epsilon\to 0} I_\epsilon(x)$ which consists of terms of the form

$$I^{\delta}_{l,k,m,p}(x) = \int_{B_\delta(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m}\frac{A_j}{|j|!}(y-x)^j\partial^{l+j}u(x)}{|x-y|^{n+s+2p}}(x-y)^k dy, \tag{2.4}$$

for any $\delta < d_x = dist(x, \partial\Omega)$ and some $l = (l_1, l_2, \cdots l_n)$, $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n, m, p \in \mathbb{N}$ with $|l| + m = |\hat r|$ and $m + |k| = 2p$.

Proof. It follows immediately from Lemma 2.6 and the proof of Theorem A. □

Lemma 2.8. *Let u, s, r, α be as in the above. Then the following holds.*

a) *If $1 \geq \alpha > s$ and $|\hat r| = r$, then there exists a constant $C > 0$ such that*

$$|I_*(x) - I_*(y)| \leq C[u]_{r,\alpha.\Omega}|x-y|^{\alpha-s}, \;\; x, y \in \Omega, \; |x-y| \ll 1.$$

b) *If $2 \geq 1+\alpha > s \geq \alpha$ and $|\hat r| = r-1$, then there exists a constant $C > 0$ such that*

$$|I_*(x) - I_*(y)| \leq C[u]_{r,\alpha;\Omega}|x-y|^{1+\alpha-s}, \;\; x, y \in \Omega, \; |x-y| \ll 1.$$

Proof. The function I_* can be derived simply by taking higher order derivatives of the right hand side of (2.1) and identifying all ϵ-independent terms of the derivatives. As these terms involves only regular integrals, the lemma follows from straightforward estimates. □

Proof of Theorem B. Let $x, y \in \Omega$ and take $\delta < d_{x,y} = min\{d_x, d_y\}$. For given $l = (l_1, l_2, \cdots l_n),\ k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n, m, p \in \mathbb{N}$, consider

$$J = I^{\delta}_{l,k,m,p}(x) - I^{\delta}_{l,k,m,p}(y),$$

where $I^{\delta}_{l,k,m,p}$ is as in (2.4). It is clear that

$$J = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &:= \int_{B_\eta(0)} \Big[\frac{\partial^l u(x+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \\ &\quad - \frac{\partial^l u(y+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}}\Big] z^k dz, \\ J_2 &:= \int_{R_{\delta\eta}(0)} \Big[\frac{\partial^l u(x+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \\ &\quad - \frac{\partial^l u(y+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}}\Big] z^k dz, \end{aligned}$$

and $\eta = |x - y| < \delta$.

(i) In this case, we let $|l| + m = r$ and $m + |k| = 2p$ in J_1, J_2. On one hand, since $|l| + m = r$ and $u \in C^{r,\alpha}(\Omega)$, there exists a constant $C_1 > 0$ such that

$$|\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} \big(\partial^{l+j} u(x) - \partial^{l+j} u(y)\big) z^j| \le C_1 [u]_{r,\alpha;\Omega} |z|^{m+\alpha},\ z \in B_\eta(0).$$

Using the fact $m + |k| = 2p$, it follows that

$$\begin{aligned} |J_1| &\le |\int_{B_\eta(0)} \frac{C_1 [u]_{r,\alpha;\Omega} |z|^{2p+\alpha}}{|z|^{n+s+2p}} dz| \\ &\le C_2 [u]_{r,\alpha;\Omega} \eta^{\alpha-s} = C_2 [u]_{r,\alpha;\Omega} |x-y|^{\alpha-s} \end{aligned}$$

for some constant $C_2 > 0$. On the other hand, we also have

$$\begin{aligned} &|\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} \big(\partial^{l+j} u(x) - \partial^{l+j} u(y)\big) z^j| \\ \le\ & C_3 [u]_{r,\alpha;\Omega} [\sum_{i=0}^{m} |x-y|^{m+\alpha-i} |z|^i + \sum_{i=1}^{m} |z|^{m+\alpha-i} |x-y|^i],\ z \in B^c_\eta(0), \end{aligned}$$

where $C_3 > 0$ is a constant. It follows that

$$|J_2| \le |\int_{B^c_\eta(0)} \frac{C_3 [u]_{r,\alpha;\Omega} (\sum_{i=0}^{m} |x-y|^{m+\alpha-i} |z|^i + \sum_{i=1}^{m} |z|^{m+\alpha-i} |x-y|^i)}{|z|^{n+s+m}} dz|$$

$$\leq \quad C[u]_{r,\alpha;\Omega}(\sum_{i=0}^{m}\eta^{i-m-s}|x-y|^{m+\alpha-i}+\sum_{i=1}^{m}\eta^{\alpha-s-i}|x-y|^{i})\leq C_4[u]_{r,\alpha;\Omega}|x-y|^{\alpha-s}$$

for some constant $C_4>0$. Hence

$$|J|\leq |J_1|+|J_2|\leq (C_2+C_4)[u]_{r,\alpha.\Omega}|x-y|^{\alpha-s}. \tag{2.5}$$

Let I_0 be as in Lemma 2.7. Then Lemma 2.7 together with (2.5) imply that

$$|I_0(x)-I_0(y)|\leq C_5[u]_{r,\alpha.\Omega}|x-y|^{\alpha-s}$$

for some constant $C_5>0$. With this estimate, the proof is now complete by Lemma 2.7 and Lemma 2.8 a).

(ii) In this case, we let $|l|+m=r-1$ and $m+|k|=2p$ in J_1, J_2. Since $m+1+|k|=2p+1$ is an odd number, we have, for any $j\in\mathbb{N}^n$ with $|j|=m+1$, any $w\in\Omega$, and any $\rho<d_w$ that

$$\int_{B_\rho(w)}\frac{z^{j+k}}{|z|^{n+s+2p}}dy=0.$$

It follows that

$$\begin{aligned} J_1 &= \int_{B_\eta(0)}\Big[\frac{\partial^l u(x+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}z^j\partial^{l+j}u(x)}{|z|^{n+s+2p}} \\ &\quad -\frac{\partial^l u(y+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}z^j\partial^{l+j}u(y)}{|z|^{n+s+2p}}\Big]z^k dz, \end{aligned}$$

$$\begin{aligned} J_2 &= \int_{R_{\delta\eta}(0)}\Big[\frac{\partial^l u(x+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}z^j\partial^{l+j}u(x)}{|z|^{n+s+2p}} \\ &\quad -\frac{\partial^l u(y+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}z^j\partial^{l+j}u(y)}{|z|^{n+s+2p}}\Big]z^k dz. \end{aligned}$$

The rest of the proof is similar to that of (i). We only note that using facts $|l|+m=r-1$ and $u\in C^{r,\alpha}(\Omega)$, the estimate of J_1 follows from the inequality

$$|\partial^l u(x+z)-\partial^l u(y+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}\big(\partial^{l+j}u(x)-\partial^{l+j}u(y)\big)z^j|\leq C_5[u]_{r,\alpha;\Omega}|z|^{m+1+\alpha},$$

$z\in B_\eta(0)$, where $C_5>0$ is a constant, while, using facts $m+|k|=2p$ and $s<1+\alpha$, the estimate of J_2 follows from the inequality

$$\begin{aligned} &|\partial^l u(x+z)-\partial^l u(y+z)-\sum_{|j|=0}^{|j|=m+1}\frac{A_j}{|j|!}\big(\partial^{l+j}u(x)-\partial^{l+j}u(y)\big)z^j| \\ \leq\ & C_6[u]_{r,\alpha;\Omega}(\sum_{i=0}^{m+1}|x-y|^{m+1+\alpha-i}|z|^i+\sum_{i=1}^{m+1}|z|^{m+1+\alpha-i}|x-y|^i),\ z\in B^c_\eta(0), \end{aligned}$$

where $C_6>0$ is a constant. □

3. Schauder estimates for the regional fractional Laplacian

In this section, we will show the Schauder estimates for the regional fractional Laplacian using those for the fractional Laplacian. Recall that the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ is well-defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$, and we can then extend its definition to the space $L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+s}})$ by

$$< (-\Delta)^{\frac{s}{2}} u, \varphi >_{\mathbb{R}^n} = \int_{\mathbb{R}^n} u(y)(-\Delta)^{\frac{s}{2}}\varphi(y)dy, \quad \forall \varphi \in \mathcal{S}, \tag{3.1}$$

for any $u \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+s}})$.

3.1. **Schauder estimates for the fractional Laplacian.** In the following Lemma 3.1 and 3.2, the definition of $(-\Delta)^{\frac{s}{2}}$ is understood in the sense of (3.1). We refer the reader to [11] for a more general definition of the fractional Laplacian.

Lemma 3.1. *Let $0 < \alpha \le 1$ and $0 < s < 2$. If, for some $w \in C^\alpha(\bar{\Omega})$, $u \in L^\infty(\mathbb{R}^n)$ solves the equation $(-\Delta)^{\frac{s}{2}} u = w$ in Ω, then for any $\delta > 0$ sufficiently small there exists a constant $C > 0$ depending only on n, s, δ and α such that*

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_\delta)} \le C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{C^\alpha(\bar{\Omega})}).$$

Proof. The proof follows from that of [11, Proposition 2.8]. □

Lemma 3.2. *Let $0 < s < 2$. Suppose that, for some $w \in L^\infty(\Omega)$, $u \in L^\infty(\mathbb{R}^n)$ solves the equation $(-\Delta)^{\frac{s}{2}} u = w$ in Ω. Then, for any sufficiently small $\delta > 0$, there exists a constant $C > 0$ depending only on n, s and δ such that the following holds:*

(i) *If $s \ne 1$, then*

$$\|u\|_{C^s(\bar{\Omega}_\delta)} \le C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\Omega)}).$$

(ii) *If $s = 1$, then*

$$\|u\|_{\Lambda_*(\bar{\Omega}_\delta)} \le C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\Omega)}).$$

Proof. We first use the argument in the proof of [11, Proposition 2.8]. By covering and rescaling arguments, we only need to consider the case $\Omega_\delta = B_{\frac{1}{2}}(0)$ and $\Omega = B_1(0)$. Let $\eta \in C_c^\infty(\mathbb{R})$ be such that range$(\eta) \subset [0,1]$, supp$(\eta) \subset B_1(0)$, and $\eta(x) = 1$ for any $x \in B_{\frac{3}{4}}(0)$. Denote

$$u_0(x) := \mathcal{A}(n,s) \int_{\mathbb{R}^n} \frac{\eta(y)w(y)}{|x-y|^{n-s}} dy = (-\Delta)^{-\frac{s}{2}} \eta w(x).$$

Then $(-\Delta)^{\frac{s}{2}} u_0 = w = (-\Delta)^{\frac{s}{2}} u$ in $B_{\frac{3}{4}}(0)$. It follows that $u - u_0 \in C^2(\bar{B}_{\frac{1}{2}}(0))$ and

$$\|u - u_0\|_{C^2(\bar{B}_{\frac{1}{2}}(0))} \le C\|u - u_0\|_{L^\infty(\mathbb{R}^n)} \le C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(B_1(0))}),$$

where $C > 0$ is a constant depending only on n. We note that $C^2(\bar{B}_{\frac{1}{2}}(0)) = C^{1,1}(\bar{B}_{\frac{1}{2}}(0)) \ne C^{2,0}(\bar{B}_{\frac{1}{2}}(0))$. The lemma now follows from [2, Proposition 5.2]. □

3.2. **Proof of Theorems C and D.** Let $0 < s < 2$, $w \in L^\infty(\Omega)$, and $u \in L^\infty(\Omega)$ solves the equation $\Delta_\Omega^{\frac{s}{2}} u = w$ in Ω. Also let $\bar{u} \in L^\infty(\mathbb{R}^n)$ be such that $\bar{u} \equiv u$ in Ω and $\bar{u} \equiv 0$ outside of Ω. Then for any $\varphi \in \mathcal{S}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(-\Delta_{\Omega,\epsilon}^{\frac{s}{2}} \bar{u}(x) + \mathcal{A}(n,-s)\bar{u}(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy \Big) \varphi(x) dx \\
= \;& \mathcal{A}(n,-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x) - \bar{u}(y)}{|x-y|^{n+s}} dy \varphi(x) dx \\
= \;& \mathcal{A}(n,-s) \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x)\varphi(x)}{|x-y|^{n+s}} dy dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(y)} \frac{\bar{u}(y)\varphi(x)}{|x-y|^{n+s}} dx dy \Big) \\
= \;& \mathcal{A}(n,-s) \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x)\varphi(x)}{|x-y|^{n+s}} dy dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x)\varphi(y)}{|x-y|^{n+s}} dy dx \Big) \\
= \;& \mathcal{A}(n,-s) \int_{\mathbb{R}^n} \bar{u}(x) \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} dy dx.
\end{aligned}$$

In particular, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
&- \int_\Omega u(x) \Delta_{\Omega,\epsilon}^{\frac{s}{2}} \varphi(x) dx + \mathcal{A}(n,-s) \int_{\mathrm{supp}(\varphi)} u(x)\varphi(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy dx \\
= \;& \int_{\mathbb{R}^n} \Big(-\Delta_{\Omega,\epsilon}^{\frac{s}{2}} \bar{u}(x) + \mathcal{A}(n,-s)\bar{u}(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy \Big) \varphi(x) dx \\
= \;& \mathcal{A}(n,-s) \int_{\mathbb{R}^n} \bar{u}(x) \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} dy dx.
\end{aligned}$$

Letting $\epsilon \to 0$ in the above, we easily obtain that, for any $x \in \Omega$,

$$(-\Delta)^{\frac{s}{2}} \bar{u}(x) = -w(x) + \mathcal{A}(n,-s) u(x) \int_{\Omega^c} \frac{1}{|x-y|^{n+s}} dy. \tag{3.2}$$

Proof of Theorems C, D. Let $\delta > 0$ be sufficiently small. We have by Lemma 3.2 and (3.2) that there exists a constant C depending on n, s, and δ such that

$$\begin{aligned}
\|u\|_{C^s(\bar{\Omega}_\delta)} &\le C(\|\bar{u}\|_{L^\infty(\mathbb{R}^n)} + \| -w + \mathcal{A}(n,-s)u \int_{\Omega^c} \frac{1}{|\cdot - y|^{n+s}} dy \|_{L^\infty(\Omega_{\frac{\delta}{2}})}) \\
&\le C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)})
\end{aligned}$$

when $s \neq 1$. Similarly,

$$\|u\|_{\Lambda_*(\bar{\Omega}_\delta)} \le C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}).$$

when $s = 1$. This proves Theorem D.

Using bootstrap arguments, Lemma 3.1 and (3.2) for both cases of $\alpha, \alpha + s$ being or not being natural numbers, we have, similarly to the above, that

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_\delta)} \le C(\|u\|_{L^\infty(\Omega)} + \|w\|_{C^\alpha(\bar{\Omega})}),$$

where $C > 0$ is a constant depending on n, s, δ and α. This proves Theorem C. □

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