Journal of Nonlinear Science (2023) 33:114
https://doi.org/10.1007/s00332-023-09974-2

Response Solutions in Singularly Perturbed, Quasi-Periodically Forced Nonlinear Oscillators

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Received: 4 January 2023 / Accepted: 14 September 2023

Abstract
For a quasi-periodically forced oscillator, response solutions are quasi-periodic ones having the same frequencies as that of the forcing function. Typically being the most stable or robust ones, they form an important class of oscillatory solutions of the oscillator. Since the introduction of the notion in the 1950 s, response solutions have been extensively studied in regularly perturbed, quasi-periodically forced oscillators with large, small, or zero damping coefficients with recent advances being made toward some singularly perturbed and highly or completely degenerate cases. The aim of the present paper is to make a general investigation toward the existence and stability properties of response solutions in singularly perturbed, quasi-periodically forced

Communicated by Paul Newton.

The first author was partially supported by NSFC grants 12001315,11971261,11571201,12071255, and Shandong Provincial Natural Science Foundation grant ZR2020MA015. The Second author was partially supported by NSFC grants 12271204, and the Department project of Science and Technology of Jilin Province (Grant No. 20200201265JC). The third author was partially supported by NSERC discovery grant RGPIN-2020-04451, a faculty development grant from University of Alberta, and a Scholarship from Jilin University.

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Published online: 04 October 2023

oscillators of the normal form

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = \epsilon^{\alpha} A(\epsilon) z + \epsilon^{\alpha+\beta} f(\theta, z, \epsilon), \qquad (\theta, z) \in \mathbb{T}^d \times \mathbb{R}^2, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $\beta > 0$ are constants, $\omega \in \mathbb{R}^d$ is the forcing frequency vector, $0 < \epsilon \ll 1$ is a parameter, and f is of a finite order of smoothness. The normal form includes strongly damped oscillators of the form

$$\ddot{x} + \frac{1}{\epsilon}\dot{x} - g(x) = \epsilon^{\chi_1} f(\omega t), \qquad x \in \mathbb{R}^1$$

and damping-free oscillators with large potentials of the form

$$\ddot{x} - \frac{1}{\epsilon} g(x) = \epsilon^{\chi_2} f(\omega t), \qquad x \in \mathbb{R}^1,$$

where χ_1, χ_2 are constants. With respect to the normal form, we show the existence of Floquet response tori for all or the majority of sufficiently small $\epsilon > 0$ in three typical cases. Not only do our results on response solutions and their stabilities extend some existing ones in both regularly and singularly perturbed cases by allowing finite smoothness of potential and forcing functions, but also they provide new insides to the nature of these solutions, for instance the coexistence of response solutions of both hyperbolic and elliptic types in a given quasi-periodically forced, degenerate nonlinear oscillator.

Keywords Response solutions · Degenerate oscillators · Quasi-periodic forcing · Singular perturbations

Mathematics Subject Classification Primary 34J40 · 34C27; Secondary 34E20

1 Introduction

Consider quasi-periodically forced, perturbed second-order differential equations of the form

$$\ddot{x} + c\dot{x} + \lambda g(x) = \epsilon^{\chi} f(\omega t, x, \dot{x}), \qquad x \in \mathbb{R}^1, \tag{1.1}$$

where $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$ is a d-dimensional forcing frequency vector for some integer $d > 1$, $0 < \epsilon \ll 1$ is a small parameter, χ is a constant, c and λ are either constants or dependent on ϵ, and $f : \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}^1$ and $g : \mathbb{R}^1 \to \mathbb{R}^1$ are smooth functions with $g(0) = 0$. A solution of (1.1) is said to be a *response solution* if it is quasi-periodic with the same frequency vector ω as that of the forcing function f. These solutions form an important class of oscillatory solutions of equations (1.1) especially when they are oscillators. They are typically the most stable or robust oscillatory solutions, represent the simplest harmonic responses to the external forcing, and reveal some synchronizing behaviors of the oscillators (Corsi and Gentile 2012).

Response solutions of (1.1) have been extensively studied when f, g are real analytic and the equation is regularly perturbed, i.e., c, λ, χ are constants with $\chi > 0$. In the case that $(0, 0)$ is a non-degenerate equilibrium of the first-order system corresponding to the unperturbed equation of (1.1), the existence of response solutions of (1.1) for ϵ sufficiently small has been shown in classical works of Stoker (1950) and Moser (1965) for cases $|c| \gg 1$ and $c = 0$, respectively. The work Moser (1965) is in fact among one of the pioneer works in KAM theory. While the existence problem is simple in the former case because of the hyperbolic nature of the equilibrium, it is however highly non-trivial in the later case because of the perturbation of the elliptic equilibrium and the involvement of small divisor problems in KAM iterations, in which Diophantine forcing frequencies, the reversibility of f in t, and a coupling non-resonance condition between ω and eigenvalues of the linearization at $(0, 0)$ - known as a Melnikov condition in the modern literature, need to be assumed, and a small Lebesgue measure set of ϵ in the vicinity of 0 needs to be excluded for the existence of response solutions. The result in Moser (1965) was later shown to also hold in certain cases when c is sufficiently small (Braaksma and Broer 1987; Friedman 1967). More recent studies of response solution of (1.1) focused on cases when $(0, 0)$ is a degenerate equilibrium of the first-order system corresponding to the unperturbed equation of (1.1). In the degenerate cases, response solutions are found, as an important mechanism, by perturbing relative equilibria of certain averaged equation of (1.1). It turns out that response solutions exist for all ϵ sufficiently small if the relative equilibria are hyperbolic and ω is of Brjuno type, while they exist for all ϵ sufficiently small, with the exception of a small Lebesgue measure set, if ω is of Diophantine type and the relative equilibria are elliptic, without reversibility and Melnikov conditions (Corsi and Gentile 2012, 2015, 2017; Gentile 2007; Hu and Liu 2018; Si and Yi 2020, 2022; You 1998). We refer the reader to Broer et al. (2005, 2006, 2013, 1996), Hanßmann (1998, 2004, 2007), Lou and Geng (2017), Wang et al. (2017) for other studies relating to response solutions under regular perturbations.

Comparing with the regular perturbation cases, response solutions are much less known when (1.1) is singularly perturbed, i.e., when c or λ depends on ϵ in a singular way and/or $\chi < 0$. One relatively well-studied singularly perturbed case of (1.1) is the strongly damped oscillator

$$\ddot{x} + \frac{1}{\epsilon}\dot{x} - g(x) = \epsilon^{\chi_1} f(\omega t), \qquad x \in \mathbb{R}^1 \tag{1.2}$$

with the so-called resistor-inductor-varactor circuit system (Matsumoto et al. (1984)) and 'ship roll and capsize' system (Thompson (1997)) as particular physical examples. To be more specific, when f, g are real analytic, $\chi_1 = 0$, and the equation $g(x)+[f] = 0$, where

$$[f] = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta)d\theta$$

is the average of f, admits a root of odd order, it is shown in Calleja et al. (2013), Corsi et al. (2013, 2014), Gentile (2010, 2012), Gentile and Vaia (2021) that response solutions of (1.2) exists for all ϵ sufficiently small. In the case that f, g are of finite order of smoothness, it is shown in Wang and de la Llave (2020) that if $[f] = 0$, $g(0) = 0$, and $g'(0) \neq 0$, then for sufficiently small $\sigma > 0$, (1.2) admits a response solution for each $\epsilon \in (\sigma, 2\sigma)$. Except these results on the strongly damped oscillator (1.2), the existence of response solutions in singularly perturbed cases of (1.1) is widely open, including the case of a damping-free oscillator with large potential

$$\ddot{x} - \frac{1}{\epsilon} g(x) = \epsilon^{\chi_2} f(\omega t), \qquad x \in \mathbb{R}^1 \tag{1.3}$$

which has roots in describing oscillatory standing waves of forced Kdv equations with small dispersion (see e.g. Blyuss (2002)).

The present work aims at making some general studies of response solutions of (1.1) in the singularly perturbed cases. For the sake of generality, we will consider the following finitely smooth, first-order normal form

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = A(\epsilon) z + \epsilon^{\alpha+\beta} f(\theta, z, \epsilon), \qquad (\theta, z) \in \mathbb{T}^d \times \mathbb{R}^2, \end{cases} \tag{1.4}$$

where $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector with Diophantine constants $\gamma > 0$ and $\tau > d - 1$, i.e.,

$$|\langle k, \omega \rangle| > \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\},$$

$\alpha \in \mathbb{R}$ and $\beta > 0$ are constants, $f \in C^{m+\tilde{\mu}}(\mathbb{T}^d \times B_r \times \Pi_{\epsilon_*}, \mathbb{R}^2)$ for some natural number m and real number $0 < \tilde{\mu} < 1$, with $B_r := \{z = (z_1, z_2) \in \mathbb{R}^2 : |z| \leq r\}$ for a fixed $r > 0$ and $\Pi_{\epsilon_*} = (0, \epsilon_*)$ for some $\epsilon_* > 0$ sufficiently small, and A is a 2×2 - matrix-valued function of ϵ with possible singularity at $\epsilon = 0$. We refer to an invariant, quasi-periodic d-torus of (1.4) with the frequency vector ω as a *response torus*. In applying the normal form to the problem of response solutions of (1.1) in the singularly perturbed cases, one first reduces the oscillators, in the vicinity of appropriate relative equilibria, into the normal form whose response tori then correspond to response solutions of the original oscillators.

For the normal form (1.4), we will consider the following canonical cases of the matrices $A(\epsilon)$:

(C1): $A(\epsilon) = \begin{pmatrix} \epsilon^{\alpha_1}\lambda_1(\epsilon) & 0 \\ 0 & \epsilon^{\alpha_2}\lambda_2(\epsilon) \end{pmatrix}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\max\{\alpha_1, \alpha_2\} = \alpha$ and $\lambda_i(\epsilon) \in C^1[0, 1]$ with $\lambda_i(0) \neq 0$, $i = 1, 2$;

(C2): $A(\epsilon) = \epsilon^\alpha \begin{pmatrix} \lambda_2(\epsilon) & \lambda_1(\epsilon) \\ -\lambda_1(\epsilon) & \lambda_2(\epsilon) \end{pmatrix}$, where $\lambda_i(\epsilon) \in C^1[0, 1]$ with $\lambda_i(0) \neq 0$, $i = 1, 2$;

(C3): $A(\epsilon) = \epsilon^\alpha \begin{pmatrix} 0 & \lambda(\epsilon) \\ -\lambda(\epsilon) & 0 \end{pmatrix}$, where $\lambda(\epsilon) \in C^1[0, 1]$ with $\lambda(0) \neq 0$ and $\alpha \neq 0$ is assumed in this case.

We note that the matrices are hyperbolic in both cases **(C1)** and **(C2)**, and are elliptic in the case **(C3)**. Our main result of the paper states as follows.

Main Theorem *Consider* (1.4) *and assume* $m \geq 3d$ *in cases* **(C1)**, **(C2)** *and* $m > 6\tau + 5$ *in the case* **(C3)**. *Denote*

$$\tilde{m} = \begin{cases} m - 2d, & \text{in cases } (\mathbf{C1}),\ (\mathbf{C2}); \\ m - 4[\tau] - 5, & \text{in the case } (\mathbf{C3}). \end{cases}$$

Then, there exists a $\epsilon_* > 0$ *sufficiently small and a* C^1 *family of invertible,* $C^{\tilde{m}}$*-smooth transformations* $\Phi^\epsilon : \mathbb{T}^d \times B_r \to \mathbb{T}^d \times B_{r/2}$, $\epsilon \in \mathcal{E}_{\epsilon_*}$, *where* $\mathcal{E}_{\epsilon_*}$ *equals* $(0, \epsilon_*)$ *in cases* **(C1)**, **(C2)** *and is a Cantor subset of* $(0, \epsilon_*)$ *with Lebesgue measure* $\text{meas}(\mathcal{E}_{\epsilon_*}) \sim \epsilon_*$ *in the case* **(C3)**, *which, for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$, *transforms* (1.4) *into the system*

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = \breve{A} z + \breve{H}(\theta, z, \epsilon), \end{cases} \tag{1.5}$$

where $\breve{A} = A + \mathcal{O}(\epsilon^{\alpha+\beta})$ *is a constant matrix for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$ *and* $\breve{H} = \mathcal{O}(|z|^2)$. *Consequently, for each* $\epsilon \in \mathcal{E}_{\epsilon_*}$, (1.4) *admits a Floquet,* $C^{\tilde{m}}$*-smooth response torus.*

Remark

(1) We remark that the theorem above holds in the real analytic context, i.e., if (1.4) is real analytic in θ, then so are the transformations and response tori stated in the Main Theorem. This can be easily seen from our proof of the theorem.

(2) The theorems will be proved via KAM iterations and analytic approximation technique. Concerning cases **(C1)** and **(C2)** in particular, we do so in order to obtain Floquet response tori which not only characterize their stabilities but also their nearby dynamical behaviors. We remark that the existence of response tori in cases **(C1)** and **(C2)** can be proved simply via the uniform contraction mapping principle by considering the family of maps

$$T_\epsilon x(\theta) = \epsilon^{\alpha+\beta} \int_\infty^0 e^{-As} P f(x(\theta + \omega s)), \theta + \omega s) ds$$
$$+ \epsilon^{\alpha+\beta} \int_{-\infty}^0 e^{-As} (I - P) f(x(\theta + \omega s), \theta + \omega s) ds$$

on an appropriate function space in $C^0(\mathbb{T}^2, \mathbb{R}^2)$, where P denotes the project to the stable eigenspace of A. For cases **(C1)** and **(C2)**, not only will this approach requires no Diophantine condition on ω, but also it will yield response tori of the class C^m (see e.g. Yi (1993a) for a much more general situation).

(3) With the Floquet form (1.5), the stability of the response tori can be analyzed. Using certain stability results of parameterized, normally hyperbolic invariant manifolds (see e.g. Yi (1993b)) after a time re-scaling if necessary, one can conclude that a response torus obtained in the Main Theorem is asymptotically stable (resp. unstable) if both $\lambda_1(0)$ and $\lambda_2(0)$ are negative in the case **(C1)** or $\lambda_2(0)$ is negative

in the case **(C2)** (resp. one of $\lambda_1(0)$ and $\lambda_2(0)$ is positive in the case **(C1)** or $\lambda_2(0)$ is positive in the case **(C2)**). In the case **(C3)**, the stability type of response tori will depend on the nature of eigenvalues $\lambda(\epsilon)$ of $\breve{A}(\epsilon)$ for $\epsilon \in \mathcal{E}_{\epsilon_*}$, i.e., for each $\epsilon \in \mathcal{E}_{\epsilon_*}$, the response torus is asymptotically unstable (resp. stable) if $\mathrm{Re}\lambda(\epsilon) > 0$ (resp. $\mathrm{Re}\lambda(\epsilon) < 0$), and if $\mathrm{Re}\lambda(\epsilon) = 0$, then $\breve{A}(\epsilon)$ is elliptic and only linear stability of the response torus can be concluded.

(4) For finitely smooth Hamiltonian systems, KAM theory has been well-developed using the Jackson-Moser-Zehnder analytic approximation technique (see e.g., Salamon and Zehnder 1989; Salamon 2004; Zehnder 1975, 1976 for nearly integrable cases and Chierchia and Qian (2004) for partially nearly integrable cases). The proof of the Main Theorem above will use the same approximation technique. However, due to the multi-scale nature of (1.5), special cares are needed for both analytic approximations and KAM iterations in the proof.

We will show under certain conditions that the Main Theorem above indeed yields the existence of response solutions for singularly perturbed oscillators (1.2) and (1.3) when f, g are sufficiently smooth and ω is Diophantine. More precisely, with respect to (1.2) with $\chi_1 \in (-1, \infty)$, we will show that if ω is Diophantine and the equation $g(x) + \epsilon^{\chi_1}[f] = 0$ admits a non-degenerate root for each $0 \le \epsilon \ll 1$, then (1.2) has a response solution for each $0 < \epsilon \ll 1$. In the case $\chi_1 = 0$, this result partially extends those of Corsi et al. (2013, 2014), Gentile (2010, 2012), Gentile and Vaia (2021) to the finitely smooth case, and those of Calleja et al. (2013); Wang and de la Llave (2020) to allow a wider range of parameters. With respect to (1.3) with $\chi_2 \in (-1, \infty)$, we will show that if $g(x_0) = 0,\ g'(x_0) > 0$ for some x_0, then (1.3) admits a response solution for each $0 < \epsilon \ll 1$, and if $g(x_0) = 0,\ g'(x_0) < 0$ for some x_0, then there is an almost full Lebegue measure Cantor subset $\mathcal{D}$ of $0 < \epsilon \ll 1$ such that (1.3) admits a response solution for each $\epsilon \in \mathcal{D}$. Besides singularly perturbed problems, the Main Theorem above can also be applied to regularly perturbed, degenerate oscillators of the form

$$\ddot{x} + \lambda x^l = \epsilon f(\omega t), \qquad x \in \mathbb{R}^1,$$

where $\lambda \neq 0$ is a constant, $0 < \epsilon \ll 1$ is a parameter, $l > 1$ is an integer, and $\omega \in \mathbb{R}^d$ is Diophantine. Not only does such an application extend the results of Si and Yi (2020, 2022) to finitely smooth cases, but also it asserts the coexistence of both hyperbolic and elliptic types of response solutions for the same λ.

In applying the Main Theorem to some of these oscillators in the hyperbolic cases, a novelty of our normal form reduction is to solve finitely smooth homological equations using Diophantine conditions instead of hyperbolicity of eigenvalues at relative equilibria, simply because these eigenvalues can depend on the parameter in a singular way.

The rest of this paper is organized as follows. Section 2 is a preliminary section in which we summarize some notions, recall the analytic approximation technique for smooth functions, and establish some technical lemmas. We prove the Main Theorem in Sect. 3 using analytic approximations and KAM iterations. In Sect. 4, we consider applications of our Main Theorem to three type of quasi-periodically forced nonlinear oscillators, i.e., strongly damped oscillators, damping-free oscillators with large potentials, and degenerate harmonic oscillators.

2 Preliminary

In this section, we give some notations, review the classical approach of analytic approximations, and establish some technical lemmas.

2.1 Notations

For simplicity, we use the same symbol $|\cdot|$ to denote both absolute values of a real number and a norm in an Euclidean space. Let U be a region in a Euclidean space $\mathbb{R}^n$. For any bounded function f on U valued in an Euclidean space $\mathbb{R}^m$, we use $|f|_U$ to denote the sup-norm of f on U. For any C^ℓ function f on U with respect to some real number $\ell > 0$, if $0 < \ell < 1$, then we denote

$$|f|_{C^\ell(U)} = \sup_{x,y\in U,\ 0<|x-y|<1} \frac{|f(x) - f(y)|}{|x-y|^\ell} + \sup_{x\in U} |f(x)|,$$

and if $\ell > 1$, then we denote

$$|f|_{C^\ell(U)} = \sum_{|k|\le[\ell]} |\partial^k f|_{C^\mu(U)},$$

where $\mu = \ell - [\ell] < 1$, and for each multi-index $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$, $|k| = k_1 + \cdots + k_n$. If f also depends on a parameter ϵ in an open set $\mathcal{D} \subset (0, 1)$ and is C^1 in ϵ, then we denote

$$|f|_{C^\ell(U),\mathcal{D}} = \sup_{\epsilon\in\mathcal{D}} (|f|_{C^\ell(U)} + \epsilon|\frac{\partial f}{\partial\epsilon}|_{C^\ell(U)}).$$

Let $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ be the standard d-torus. For given $r, s > 0$, we denote

$$\mathbb{T}^d_s := \{\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : |\mathrm{Im}\theta_j| \le s, \quad j = 1, 2, \ldots, d\}$$

as the complex s-strip neighborhood of $\mathbb{T}^d$, denote

$$\mathbf{B}_r := \{z \in \mathbb{C}^n : |z| \le r\}$$

as the complex extension of the r-ball $B_r := \{z \in \mathbb{R}^n : |z| \le r\}$ in an Euclidean space $\mathbb{R}^n$, and denote

$$D(s, r) = \mathbb{T}^d_s \times \mathbf{B}_r.$$

For any analytic function F on $D(s, r)$ valued in some complex vector space $\mathbb{C}^m$, we define

$$\|F\|_{s,r} = \sup_{(\theta,z)\in D(s,r)} |F(\theta, z)|,$$

and, if F also depends on a parameter ϵ in a set $\mathcal{D} \subset (0, 1)$ C^1-Whitney smoothly, then we define

$$\|F\|_{s,r,\mathcal{D}} = \sup_{\epsilon\in\mathcal{D}} \left(\|F\|_{s,r} + \|\epsilon \frac{\partial F}{\partial \epsilon}\|_{s,r} \right),$$

where the derivative with respect to ϵ is taken in the sense of Whitney. For any $n \times m$ matrix-valued function $P = (P_{ij})$ on $D(s, r) \times \mathcal{D}$ which is analytic in $(\theta, z) \in D(s, r)$ and C^1-Whitney smooth in $\epsilon \in \mathcal{D}$, we define

$$\|P\|_{s,r,\mathcal{D}} = \max_{1\le i\le n} \sum_{j=1}^{m} \|P_{ij}\|_{s,r,\mathcal{D}}.$$

If F or P is independent of z, then we define $\|F\|_{r,\mathcal{D}}$ and $\|P\|_{r,\mathcal{D}}$ similarly.

2.2 Analytic Approximations

The following approximation result is commonly referred to as the JMZ (Jackson-Moser-Zehnder) Lemma, whose proof can be found in Chierchia and Qian (2004), Salamon and Zehnder (1989), Salamon (2004), Zehnder (1975), Zehnder (1976).

Lemma 2.1 *Let $\mathcal{K}$ be a radially symmetric, C^∞ function in $\mathbb{R}^n$ which is compactly supported in a ball B_a and satisfies*

$$\partial^\alpha \mathcal{K}(0) = \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0. \end{cases}$$

For each $0 < r \le 1$ and $F \in \mathrm{C}^0(\mathbb{R}^n)$, consider the family of convolutions

$$S_r F(x) = \frac{1}{r^n} \int_{\mathbb{R}^n} K(\frac{x-y}{r}) F(y) dy,$$

where K is the inverse Fourier transform of $\mathcal{K}$. Then, S_r, $0 < r \le 1$, is a family of operators from $\mathrm{C}^0(\mathbb{R}^n)$ into the linear space of entire functions on $\mathbb{C}^n$ satisfying the following properties for any $\ell > 0$:

(1) *There exists a constant $c = c(\ell, n) > 0$ such that for each $0 < r \le 1$, $F \in \mathrm{C}^\ell(\mathbb{R}^n)$, and $\alpha \in \mathbb{N}^n$ with $|\alpha| \le \ell$, we have*

$$\sup_{|\mathrm{Im} x|\le r} |\partial^\alpha (S_r F(x)) - \sum_{|\beta|\le \ell-|\alpha|} \partial^{\alpha+\beta} \frac{F(\mathrm{Re} x)(\mathrm{i}\,\mathrm{Im} x)^\beta}{\beta!}| \le c|F|_{\mathrm{C}^\ell} r^{\ell-|\alpha|}, \quad x \in \mathbb{C}^n,$$

$$\sup_{|\mathrm{Im}x|\le\delta}|\partial^\alpha(S_r(x))-\partial^\alpha(S_\delta(x))|\le c|F|_{\mathrm{C}^\ell}r^{\ell-|\alpha|},\quad x\in\mathbb{C}^n,\ 0\le\delta\le r,$$
$$|S_rF-F|_{\mathrm{C}^p(\mathbb{R}^n)}\le c|F|_{\mathrm{C}^\ell(\mathbb{R}^n)}r^{\ell-p},\quad 0\le p\le\ell,$$
$$|S_rF|_{\mathrm{C}^p(\mathbb{R}^n)}\le c|F|_{\mathrm{C}^\ell(\mathbb{R}^n)}r^{\ell-p},\quad p\le\ell.$$

(2) *If* $F\in\mathrm{C}^\ell(\mathbb{R}^n)$ *is periodic in some variables, then so are the approximating functions* S_rF *in the same variables.*

In performing KAM iterations to a finitely smooth differential system, one needs to use a sequence of real analytic approximations to its vector field, for which estimates on the convergent rate of the approximations are ultimately important. To approximate the ϵ-dependent vector field in (1.5) under the norms defined above, we will use the following convergent rate result which is a reformulation of that in Chierchia and Qian (2004) (see also Li et al. (2023)).

Lemma 2.2 *Let* $F:\mathbb{T}^d\times B_r\times\mathcal{D}\to\mathbb{R}^m$ *be a function which is of the class* C^ℓ *in* $(\theta,z)\in\mathbb{T}^d\times B_r$ *and of the class* C^1 *in* $\epsilon\in\mathcal{D}$*, where* B_r *is the r-ball in* $\mathbb{R}^n$ *for some* $r>0$ *and* $\mathcal{D}\subset(0,1)$ *is an open set, and* $d,n,m\ge1$ *are integers. Then for any given deceasing sequence* $\{r_\nu:\nu=0,1,\cdots\}$ *with* $r_\nu\to0$ *as* $\nu\to\infty$*, there is a sequence* $F^\nu:\mathbb{T}^d_{r_\nu}\times\mathbb{R}^n_{r_\nu}\times\mathcal{D}\to\mathbb{C}^m$*, where* $\mathbb{R}^n_{r_\nu}=\{y\in\mathbb{C}^n:|\mathrm{Im}y|\le r_\nu\}$*,* $\nu=0,1,\cdots$*, which are real analytic for each fixed* ϵ *and of the class* C^1 *in* ϵ*, such that*

$$|F^\nu-F|_{C^p(\mathbb{T}^d\times B_r),\mathcal{D}}\le c|F|_{C^\ell(\mathbb{T}^d\times B_r),\mathcal{D}}r_\nu^\ell,\quad 0\le p\le\ell,$$
$$\|F^{\nu+1}-F^\nu\|_{r_{\nu+1},r_{\nu+1},\mathcal{D}}\le c|F|_{C^\ell(\mathbb{T}^d\times B_r),\mathcal{D}}r_{\nu+1}^\ell,$$

$\nu=0,1,\cdots$*, where c is a constant depending only on* ℓ,d,n,m*.*

Proof Without loss of generality, we let $m=1$. Using a standard C^∞ cut-off function, we can easily obtain an extension $\widetilde{F}$ of F to $\mathbb{T}^d\times\mathbb{R}^n\times\mathcal{D}$ such that $\widetilde{F}$ is C^ℓ in $(\theta,z)\in\mathbb{T}^d\times\mathbb{R}^n$ and C^1 in $\epsilon\in\mathcal{D}$, and

$$|\widetilde{F}|_{C^{|\alpha|}(\mathbb{T}^d\times\mathbb{R}^n),\mathcal{D}}\le C|F|_{C^{|\alpha|}(\mathbb{T}^d\times B_r,\mathcal{D})},\quad\forall\alpha\in\mathbb{Z}^n_+,\ |\alpha|\le\ell,$$

where C is a constant depending only ℓ,d,n.

For a monotone decreasing sequence r_ν, $\nu=0,1,\cdots$, with $r_\nu\to0$, let

$$F^\nu=S_{r_\nu}\widetilde{F},\quad\nu=0,1,\cdots.$$

Using Lemma 2.1 and the fact that

$$\frac{\partial(S_r\widetilde{F})}{\partial\epsilon}=S_r\frac{\partial\widetilde{F}}{\partial\epsilon},\quad r>0,$$

we have

$$\sup_{\epsilon\in\mathcal{D}}\left\{|F^{\nu}-F|_{C^{p}(\mathbb{T}^d\times B_r)}+\epsilon|\frac{\partial(F^{\nu}-F)}{\partial\epsilon}|_{C^{p}(\mathbb{T}^d\times B_r)}\right\}$$
$$\leq c|F|_{C^{\ell}(\mathbb{T}^d\times B_r),\mathcal{D}}r_{\nu}^{\ell},\quad 0\leq p\leq r,$$
$$\sup_{\epsilon\in\mathcal{D}}\left\{\sup_{\mathbb{T}^n_{r_{\nu+1}}\times\mathbb{R}^n_{r_{\nu+1}}}|F^{\nu+1}-F^{\nu}|+\epsilon\sup_{\mathbb{T}^n_{r_{\nu+1}}\times\mathbb{R}^n_{r_{\nu+1}}}|\frac{\partial(F^{\nu+1}-F^{\nu})}{\partial\epsilon}|\right\}$$
$$\leq c|F|_{C^{\ell}(\mathbb{T}^d\times B_r),\mathcal{D}}r_{\nu}^{\ell},$$

$\nu=0,1,\cdots$, where $c>0$ is a constant depending only on ℓ,d,n. □

2.3 A Finitely Smooth Homological Equation

The following lemma will be used for reducing a finitely smooth, quasi-periodically forced nonlinear oscillator into a normal form.

Lemma 2.3 *Consider the homological equation*

$$\partial_{\omega}V(\theta)-\lambda V(\theta)=F(\theta),\qquad \theta\in\mathbb{T}^d,$$

where $d>1$ is an integer, λ is a real constant, $\partial_{\omega}=\omega\cdot\frac{\partial}{\partial\theta}$ for some Diophantine vector $\omega\in\mathbb{R}^d$ with Diophantine constants $\gamma>0$ and $\tau>d-1$, $F\in C^{m+\mu}(\mathbb{T}^d)$ with average $[F]=0$ for some integer $m\geq d+[\tau]+2$ and real number $0<\mu<1$. Then, the equation admits a unique solution $V\in C^{m-d-[\tau]-1}(\mathbb{T}^d)$ with zero-average such that

$$\|V\|_{C^{m-d-[\tau]-1}}\leq C\|F\|_{C^{m+\mu}},$$

where $C=\frac{C_d}{\gamma}\sum_{j=0}^{\infty}\frac{1}{j^{2-(\tau-[\tau])}}$ with $C_d>0$ being a constant depending only on d.

Proof Substituting Fourier expansions $V(\theta)=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}V_ke^{\mathrm{i}\langle k,\theta\rangle}$ and $F(\theta)=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}F_ke^{\mathrm{i}\langle k,\theta\rangle}$ into the homological equation and comparing the coefficients, we have

$$V_k=\frac{F_k}{\mathrm{i}\langle k,\omega\rangle-\lambda},\quad k\in\mathbb{Z}^d\setminus\{0\},$$

implying that V is uniquely solvable from the homological equation if its Fourier series converges. Now,

$$\begin{aligned}\|V\|_{C^{m-d-[\tau]-1}} &\leq \sum_{|k|>0} |V_k||k|^{m-d-[\tau]-1} \leq \sum_{|k|>0} \frac{|F_k|}{|\mathrm{i}\langle k, \omega\rangle - \lambda|}|k|^{m-d-[\tau]-1} \\ &\leq \sum_{|k|>0} \frac{\|F\|_{C^m(\mathbb{T}^d)}|k|^\tau}{\gamma|k|^m}|k|^{m-d-[\tau]-1} \\ &\leq \|F\|_{C^{m+\mu}(\mathbb{T}^d)} \sum_{j=0}^{\infty} \frac{C_d}{\gamma} j^{d-1} j^{-m+\tau} j^{m-d-[\tau]-1} \\ &\leq \frac{C_d}{\gamma}\|F\|_{C^{m+\mu}(\mathbb{T}^d)} \sum_{j=0}^{\infty} \frac{1}{j^{2-(\tau-[\tau])}}.\end{aligned}$$

Hence, the Fourier series of V converges and the lemma is proved. □

Remark 2.1 We note that if $\lambda \neq 0$, then the solution V of homological equation above can be solved and estimated without assuming a Diophantine condition on ω, so that both lower bound of m and the regularity of V can be significantly improved. However, in our applications considered in this paper, λ typically depends on a small parameter and we would like to have a parameter-independent upper bound of an appropriate norm of V in order to obtain a desired norm form (1.4).

3 Proof of Main Theorem

In this section, we prove the Main Theorem by using analytic approximations, KAM iterations, and Whitney extensions. For simplicity, we make the ϵ-dependence of all ϵ-dependent functions implicit and use the symbol '$\preceq$' to denote '$\leq$' up to a positive constant multiple that is independent of KAM iterations.

We fix $0 < \mu_1 < \min\{\beta, \tilde{\mu}\}$ and $0 < \epsilon_* \ll 1$, and denote

$$\Pi_{\epsilon_*} = (0, \epsilon_*),\ \ \delta = \epsilon_*^{\frac{\beta}{2}},\ \ s = \eta^{\frac{2}{m}},\ \ r = s^{1+\mu_1}\ \text{ such that } \eta s^{\frac{m-2}{2}} r = \epsilon_*^{\frac{\beta}{2}}.$$

Then for each $\epsilon \in \Pi_{\epsilon_*}$ (1.4) becomes

$$\begin{cases}\dot{\theta} = \omega, \\ \dot{z} = \epsilon^\alpha A_* z + \epsilon^\alpha \delta\eta s^{\frac{m-2}{2}} r f(\theta, z),\end{cases} \tag{3.1}$$

where $A_* = \epsilon^{-\alpha} A$.

3.1 KAM Iterations

In the case **(C3)**, we fix a $0 < \mu_2 < \frac{m-6\tau+5}{6}$. Then, $m > 6\tau_1 - 1$, where $\tau_1 = \tau+1+\mu_2$. Let $\mu > 0$ be given such that

$$\mu < \begin{cases} \min\left\{\frac{m+\tilde{\mu}}{m+\mu_1} - 1, \frac{m-2d+\mu_1}{m+\mu_1}, \frac{m+1-3d+2\mu_1}{m-1}\right\}, & \text{in cases } (\mathbf{C1}),\ (\mathbf{C2}), \\ \min\left\{\frac{m+\tilde{\mu}}{m+\mu_1} - 1, \frac{m-4\tau_1+\mu_1}{m+\mu_1}, \frac{m+1-6\tau_1+2\mu_1}{m-1}\right\}, & \text{in the case } (\mathbf{C3}). \end{cases}$$

Setting $\eta_0 = \eta$, $s_0 = s$, $r_0 = r$, we consider the sequences

$$\eta_{\nu+1} = \eta_\nu^{1+\mu},\ s_\nu = \eta_\nu^{\frac{2}{m}},\ r_\nu = s_\nu^{1+\mu_1},$$
$$s_\nu^{(i)} = \begin{cases} s_\nu - \frac{(s_\nu - s_{\nu+1})i}{m-2d+2}, & i = 0, 1, \cdots, m-2d+2, \text{ in cases } (\mathbf{C1}),\ (\mathbf{C2}), \\ s_\nu - \frac{(s_\nu - s_{\nu+1})i}{m-4[\tau_1]}, & i = 0, 1, \cdots, m-[4\tau_1], \quad \text{in the case } (\mathbf{C3}), \end{cases}$$

$\nu = 0, 1, \cdots$. In the case **(C3)**, we also let $\gamma_0 = \gamma^2$ and consider the sequence

$$\gamma_\nu = \frac{\gamma_0}{2^\nu}.$$

According to Lemma 2.2, we can rewrite (3.1) as

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = Az + \epsilon^\alpha \delta \sum_{\nu=0}^{\infty} f_\nu(\theta, z), \end{cases}$$

where, as $\epsilon \in \Pi_{\epsilon_*}$ is varying, $f_\nu : \mathbb{T}^d_{s_\nu} \times \mathbb{R}^2_{r_\nu} \to \mathbb{C}^2$ is a smooth family of real analytic functions satisfying

$$\|\sum_{i=0}^{\nu} f_i - \eta s^{\frac{m-2}{2}} r f\|_{C^{m+\tilde{\mu}}(\mathbb{T}^d \times B_r), \Pi_{\epsilon_*}}, \quad \|f_\nu\|_{s_\nu, r_\nu, \Pi_{\epsilon_*}} \preceq \eta s^{\frac{m-2}{2}} r s_\nu^{m+\tilde{\mu}},$$

for all $\nu = 0, 1, \cdots$.

The KAM iterations concern the construction of a sequence of invertible transformations $\phi_0, \phi_1, \cdots$ such that for each $\nu = 0, 1, \cdots$, the inverse of the transformation $\phi^\nu = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_\nu$ transforms

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = \epsilon^\alpha A_* z + \epsilon^\alpha \delta \sum_{i=0}^{\nu} f_i(\theta, z), \end{cases} \tag{0_ν}$$

to a system of the form

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (\epsilon^\alpha A_\nu + \epsilon^\alpha \delta Q_\nu(\theta))z + \epsilon^\alpha \delta F_\nu(\theta) + \epsilon^\alpha \delta H_\nu(\theta, z), \quad (\theta, z) \in D(\frac{s_\nu}{2}, r_\nu),\ \epsilon \in \mathcal{E}_\nu, \end{cases} \tag{1_ν}$$

where $\mathcal{E}_\nu \equiv \Pi_{\epsilon_*}$ in cases **(C1)**, **(C2)** for all $\nu = 0, 1, \cdots$, and

$$\mathcal{E}_\nu = \begin{cases} \Pi_{\epsilon_*}, & \text{if } \nu = 0, \\ \mathcal{E}_\nu^1 \cap \mathcal{E}_\nu^2, & \text{if } \nu > 0; \end{cases} \quad \text{with}$$

$$\mathcal{E}_\nu^1 := \left\{ \epsilon \in \Pi_{\epsilon_*} : |\langle k, \omega \rangle - \epsilon^\alpha \mathrm{Im}\lambda_{\nu-1}^i(\epsilon)| \geq \frac{\epsilon^{\tilde{\alpha}} \gamma_\nu^2}{|k|^{\tau_1}}, \forall k \in \mathbb{Z}^d \setminus \{0\},\ i = 1, 2 \right\},$$

$$\mathcal{E}_\nu^2 := \left\{ \epsilon \in \Pi_{\epsilon_*} : |\langle k, \omega \rangle - \epsilon^\alpha \mathrm{Im}\lambda_\nu^i + \epsilon^\alpha \mathrm{Im}\lambda_\nu^j| \geq \frac{\epsilon^{\tilde{\alpha}} \gamma_\nu^2}{|k|^{\tau_1}}, \forall k \in \mathbb{Z}^d \setminus \{0\},\ i, j = 1, 2 \right\}$$

in the case **(C3)**, in which λ_ν^i, $i = 1, 2$, are eigenvalues of A_ν for each $\nu = 0, 1, \cdots$, and $\tilde{\alpha} = 0$ if $\alpha > 0$ and $\tilde{\alpha} = \alpha$ if $\alpha < 0$. Moreover, for all $\nu = 1, 2, \cdots$,

$$\|A_\nu - A_{\nu-1}\|_{\mathcal{E}_\nu} \preceq s_{\nu-1}^{m_*}, \tag{3.2}$$

$$\|Q_\nu\|_{\frac{s_\nu}{2}, \mathcal{E}_\nu} \preceq \eta_\nu s_\nu^{\frac{m-2}{2}}, \tag{3.3}$$

$$\|F_\nu\|_{\frac{s_\nu}{2}, \mathcal{E}_\nu} \preceq \eta_\nu s_\nu^{\frac{m-2}{2}} r_\nu, \tag{3.4}$$

$$\|H_\nu - H_{\nu-1}\|_{\frac{s_\nu}{2}, r_\nu, \mathcal{E}_\nu} \preceq s_{\nu-1}^{m_*}, \tag{3.5}$$

$$H_\nu(\theta, z) = \mathcal{O}(|z|^2) \text{ uniformly in } \theta,\ \epsilon,\ \text{and } \nu, \tag{3.6}$$

where

$$m_* = \begin{cases} m - 2d + \mu_1, & \text{in cases } (\mathbf{C1}),\ (\mathbf{C2}), \\ m - 4\tau_1 + \frac{\mu_1}{2}, & \text{in the case } (\mathbf{C3}). \end{cases} \tag{3.7}$$

.

3.2 Transformations and Homological Equations

We now construct the transformations ϕ_ν, $\nu = 0, 1, \cdots$, by solving suitable homological equations. We do so by taking advantage of the special nature of A and considering diagonalizations of the coefficients matrices of the homological equations in order to obtain a better lower bound of the smoothness order of ϕ_ν's.

Lemma 3.1 *For given $0 < \epsilon_* \ll 1$, there exists a smooth family of real analytic transformations $\phi_\nu : D(s_\nu/2, r_\nu) \to D(s_{\nu-1}/2, r_{\nu-1})$, $\epsilon \in \mathcal{E}_\nu$, $\nu = 0, 1, \cdots$, such that, for each $\nu = 0, 1, \cdots$, the inverse of $\phi^\nu = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_\nu$ transforms (0_ν) to (1_ν) and ϕ^ν satisfies*

$$\|\phi^\nu - id\|_{\frac{s_\nu}{2}, r_\nu, \mathcal{E}_\nu} \preceq s_{\nu-1}^{m_*}, \tag{3.8}$$

$$\|D\phi^\nu - Id\|_{\frac{s_\nu}{2}, r_\nu, \mathcal{E}_\nu} \preceq s_{\nu-1}^{m_*-1}, \tag{3.9}$$

$$\|D^j(\phi^\nu - \phi^{\nu-1})\|_{\frac{s_\nu}{2}, r_\nu, \mathcal{E}_\nu} \preceq s_\nu^{m_*-j},\ j = 0, 1, \cdots, \tilde{m},\ \nu \geq 1, \tag{3.10}$$

where m_* is as in (3.7).

Proof We use induction argument. For $\nu = 0$, we let $A_0 = A_*$, $Q_0 = 0$, $F_0 = 0$, and $H_0 = f_0$. Then, we can simply choose $\phi_0 = id$. Now suppose that for some given $\nu = 1, 2, \cdots, \phi_i, i = 1, 2, \cdots, \nu - 1$, are already constructed as stated in the lemma, i.e., the inverse of $\phi^{\nu-1} = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{\nu-1}$ transforms $(0_{\nu-1})$ to $(1_{\nu-1})$ and $\phi^{\nu-1}$ satisfies (3.2)-(3.6) with $\nu - 1$ in place of ν. Consider the homological equations

$$\partial_\omega V_\nu(\theta) = \epsilon^\alpha A_{\nu-1} V_\nu(\theta) + \epsilon^\alpha \delta F_{\nu-1}(\theta), \tag{3.11}$$

$$\partial_\omega U_\nu(\theta) = \epsilon^\alpha A_\nu U_\nu(\theta) - \epsilon^\alpha U_\nu(\theta) A_\nu + \epsilon^\alpha \delta G_{\nu-1}(\theta), \tag{3.12}$$

where $G_{\nu-1}(\theta) = Q_{\nu-1}(\theta) - [Q_{\nu-1}(\cdot)] + \frac{\partial H_{\nu-1}(\theta, V_\nu(\theta))}{\partial z} - [\frac{\partial H_{\nu-1}(\cdot, V_\nu(\cdot))}{\partial z}]$ and

$$A_\nu = A_{\nu-1} + \delta[Q_{\nu-1}(\cdot)] + \delta\left[\frac{\partial H_{\nu-1}(\cdot, V_\nu(\cdot))}{\partial z}\right]. \tag{3.13}$$

If the equations are solvable, then the transformation ϕ_ν: $(\theta, Z) \mapsto (\theta, z)$:

$$z = (I + U_\nu(\theta))Z + V_\nu(\theta)$$

transforms $(1)_{\nu-1}$ to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{Z} = (A_\nu + \epsilon^\alpha \delta \tilde{Q}_\nu(\theta))Z + \epsilon^\alpha \delta \tilde{F}_\nu(\theta) + \epsilon^\alpha \delta \tilde{H}_\nu(\theta, Z), \end{cases} \tag{2_ν}$$

where A_ν is as in (3.13) and

$$\begin{aligned} \tilde{F}_\nu(\theta) &= (I + U_\nu(\theta))^{-1} \left(Q_{\nu-1}(\theta) V_\nu(\theta) + H_{\nu-1}(\theta, V_\nu(\theta))\right), \\ \tilde{Q}_\nu(\theta) &= (I + U_\nu(\theta))^{-1}((Q_{\nu-1}(\theta) - [Q_{\nu-1}(\cdot)])U_\nu(\theta) \\ &\quad + \left(\frac{\partial H_{\nu-1}(\theta, V_\nu(\theta))}{\partial z} - \left[\frac{\partial H_{\nu-1}(\cdot, V_\nu(\cdot))}{\partial z}\right]\right) U_\nu(\theta)), \\ \tilde{H}_\nu(\theta, Z) &= (I + U_\nu(\theta))^{-1}\Big(H_{\nu-1}(\theta, (I + U_\nu(\theta))Z + V_\nu(\theta)) - H_{\nu-1}(\theta, V_\nu(\theta)) \\ &\quad - \frac{\partial H_{\nu-1}(\theta, V_\nu(\theta))}{\partial z}(I + U_\nu(\theta))Z\Big). \end{aligned}$$

Let

$$G_\nu(\theta, Z) = (D\phi^\nu)^{-1} \cdot (f_\nu(\theta, z)) \circ \phi^\nu|_{z=\phi^\nu(Z)} = G_\nu^0(\theta) + G_\nu^1(\theta)Z + G_\nu^2(\theta, Z),$$

where $G_\nu^0(\theta) = G_\nu(\theta, 0)$, $G_\nu^1(\theta) = \frac{\partial G_\nu(\theta, 0)}{\partial Z}$ and $G_\nu^2(\theta, Z) = G_\nu(\theta, Z) - G_\nu^0(\theta) + G_\nu^1(\theta)Z = \mathcal{O}(|Z|^2)$. Then it is clear that the inverse of $\phi^\nu = \phi_0 \circ \cdots \circ \phi_{\nu-1} \circ \phi_\nu$: $(\theta, Z) \mapsto (\theta, z)$, transforms (0_ν) to (1_ν) with A_ν being defined in (3.13) and

$$\begin{aligned} Q_\nu(\theta) &= \tilde{Q}_\nu(\theta) + G_\nu^1(\theta), \\ F_\nu(\theta) &= \tilde{F}_\nu(\theta) + G_\nu^0(\theta), \\ H_\nu(\theta, Z) &= \tilde{H}_\nu(\theta, Z) + G_\nu^2(\theta, Z). \end{aligned}$$

We now analyze the solvability of homological equations (3.11), (3.12) and conduct the corresponding estimates. Let $B_{\nu-1}$ be such that $B_{\nu-1}^{-1}A_{\nu-1}B_{\nu-1} = diag\{\lambda_{\nu-1}^1, \lambda_{\nu-1}^2\}$. As $\|A_{\nu-1} - A_0\|_{\mathcal{E}_\nu} \ll 1$ and A_0 is a 2×2 diagonalizable matrix, we can choose $B_{\nu-1}$ such that

$$\|B_{\nu-1}\|_{\mathcal{E}_\nu}\|B_{\nu-1}^{-1}\|_{\mathcal{E}_\nu} \preceq 1.$$

Denote $\hat{V}_\nu = B_{\nu-1}^{-1}V_\nu$. Then, (3.11) becomes

$$\partial_\omega \hat{V}_\nu(\theta) = \epsilon^\alpha \tilde{A}_{\nu-1}\hat{V}_\nu(\theta) + \epsilon^\alpha\delta\hat{F}_{\nu-1}(\theta), \tag{3.14}$$

where $\tilde{A}_{\nu-1} = diag\{\lambda_{\nu-1}^1, \lambda_{\nu-1}^2\}$ and $\hat{F}_{\nu-1}(\theta) = B_{\nu-1}^{-1}F_{\nu-1}(\theta)$. By witting $\hat{V}_\nu(\theta) = (\hat{V}_\nu^1(\theta), \hat{V}_\nu^2(\theta))^\top$ and $\hat{F}_{\nu-1}(\theta) = (\hat{F}_{\nu-1}^1(\theta), \hat{F}_{\nu-1}^2(\theta))^\top$, (3.14) is equivalent to

$$\partial_\omega \hat{V}_\nu^i(\theta) = \epsilon^\alpha\lambda_{\nu-1}^i\hat{V}_\nu^i(\theta) + \epsilon^\alpha\delta\hat{F}_{\nu-1}^i(\theta), \quad i = 1, 2. \tag{3.15}$$

To solve (3.15), we consider Fourier expansions

$$\hat{V}_\nu^i(\theta) = \sum_{k\in\mathbb{Z}^d}\hat{V}_{\nu,k}^i e^{\mathrm{i}\langle k,\theta\rangle}, \quad \hat{F}_{\nu-1}^i(\theta) = \sum_{k\in\mathbb{Z}^d}\hat{F}_{\nu-1,k}^i e^{\mathrm{i}\langle k,\theta\rangle}.$$

Substituting these Fourier series and comparing coefficients in (3.15) yields

$$\hat{V}_{\nu,k}^i = \frac{\epsilon^\alpha\delta\hat{F}_{\nu-1,k}^i}{\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i}, \quad i = 1, 2, \ k \in \mathbb{Z}^d.$$

Hence

$$\begin{aligned}
&\|\hat{V}_\nu^i\|_{\frac{s_{\nu-1}^{(1)}}{2},\mathcal{E}_\nu} \\
&\preceq \delta\sum_{k\in\mathbb{Z}^d}\sup_{\epsilon\in\mathcal{E}_\nu}\left(\frac{\epsilon^\alpha|\hat{F}_{\nu-1,k}^i|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|} + \frac{\epsilon^{\alpha+1}|\frac{\partial\hat{F}_{\nu-1,k}^i}{\partial\epsilon}|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|}\right. \\
&\quad\left. + \frac{\epsilon^{2\alpha+1}|\hat{F}_{\nu-1,k}^i|\cdot|\lambda_{\nu-1}^i|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|^2}\right)e^{\frac{s_{\nu-1}^{(1)}}{2}|k|} \\
&\preceq \delta\|F_{\nu-1}^i\|_{\frac{s_{\nu-1}}{2},\mathcal{E}_\nu}\sum_{k\in\mathbb{Z}^d}\sup_{\epsilon\in\mathcal{E}_\nu}\left(\frac{\epsilon^\alpha}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|} + \frac{\epsilon^{2\alpha+1}\cdot|\lambda_{\nu-1}^i|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|^2}\right)e^{(\frac{s_{\nu-1}^{(1)}-s_{\nu-1}}{2})|k|} \\
&\preceq \delta\eta_{\nu-1}s_{\nu-1}^{\frac{m-2}{2}}r_{\nu-1}\sum_{k\in\mathbb{Z}^d}\sup_{\epsilon\in\mathcal{E}_\nu}\left(\frac{\epsilon^\alpha}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|}. + \frac{\epsilon^{2\alpha+1}\cdot|\lambda_{\nu-1}^i|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|^2}\right)e^{\left(\frac{s_{\nu-1}^{(1)}-s_{\nu-1}}{2}\right)|k|} \\
&= \delta s_{\nu-1}^{m+\mu_1}\sum_{k\in\mathbb{Z}^d}\sup_{\epsilon\in\mathcal{E}_\nu}\left(\frac{\epsilon^\alpha}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|} + \frac{\epsilon^{2\alpha+1}\cdot|\lambda_{\nu-1}^i|}{|\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha\lambda_{\nu-1}^i|^2}\right)e^{(\frac{s_{\nu-1}^{(1)}-s_{\nu-1}}{2})|k|}.
\end{aligned}$$

In cases **(C1)**, **(C2)**, since for each $i = 1, 2$, $\lambda^i_{\nu-1}$ is a small perturbation of λ^i_0 which has non-zero real part, we have by noting $\mathcal{E}_\nu \equiv \Pi_{\epsilon_*}$ that

$$\begin{aligned}\|\hat{V}^i_\nu\|_{\frac{s^{(1)}_{\nu-1}}{2},\mathcal{E}_\nu} &\preceq \delta s^{m+\mu_1}_{\nu-1} \frac{1}{\inf_{\epsilon\in\Pi_{\epsilon_*}} |\mathrm{Re}\lambda^i_{\nu-1}|} \sum_{k\in\mathbb{Z}^d} e^{(\frac{s^{(1)}_{\nu-1}-s_{\nu-1}}{2})|k|} \\ &\preceq \delta s^{m+\mu_1}_{\nu-1} \frac{1}{s^d_{\nu-1}} = \delta s^{m+\mu_1-d}_{\nu-1}.\end{aligned}$$

In the case **(C3)**, we have by definition of $\mathcal{E}_\nu$ that

$$\begin{aligned}\|\hat{V}^i_\nu\|_{\frac{s^{(1)}_{\nu-1}}{2},\mathcal{E}_\nu} &\preceq \delta s^{m+\mu_1}_{\nu-1} \sum_{k\in\mathbb{Z}^d} \sup_{\epsilon\in\mathcal{E}_\nu} \left(\frac{\epsilon^\alpha}{|\langle k,\omega\rangle - \epsilon^\alpha \mathrm{Im}\lambda^i_{\nu-1}|} \right. \\ &\quad \left. + \frac{\epsilon^{2\alpha+1}}{|\langle k,\omega\rangle - \epsilon^\alpha \mathrm{Im}\lambda^i_{\nu-1}|^2} \right) e^{\left(\frac{s^{(1)}_{\nu-1}-s_{\nu-1}}{2}\right)|k|} \\ &\preceq \delta s^{m+\mu_1}_{\nu-1} \left(1 + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \sup_{\epsilon\in\mathcal{E}_\nu} \left(\frac{\epsilon^\alpha |k|^{\tau_1}}{\epsilon^{\tilde{\alpha}}\gamma_{\nu-1}} + \frac{\epsilon^{2\alpha+1}|k|^{2\tau_1}}{\epsilon^{2\tilde{\alpha}}\gamma^2_{\nu-1}} \right) e^{\left(\frac{s^{(1)}_{\nu-1}-s_{\nu-1}}{2}\right)|k|} \right) \\ &\preceq \delta \gamma^{-2}_{\nu-1} s^{m+\mu_1}_{\nu-1} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \sup_{\epsilon\in\mathcal{E}_\nu} |k|^{2\tau_1} e^{\left(\frac{s^{(1)}_{\nu-1}-s_{\nu-1}}{2}\right)|k|} \preceq \delta \gamma^{-2}_{\nu-1} s^{m+\mu_1}_{\nu-1} s^{-2\tau_1}_{\nu-1} \\ &= \delta \gamma^{-2}_{\nu-1} s^{m+\mu_1-2\tau_1}_{\nu-1} \preceq \delta s^{m-2\tau_1+\frac{\mu_1}{2}}_{\nu-1}.\end{aligned}$$

Thus, in all three cases, (3.11) is uniquely solvable to yield a solution V_ν which is smooth in $\epsilon \in \mathcal{E}_\nu$ and analytic in $\theta \in \mathbb{T}^d_{s^{(1)}_{\nu-1}/2}$ such that $\|V_\nu\|_{s^{(1)}_{\nu-1}/2,\mathcal{E}_\nu} \preceq \delta s^{a_1}_{\nu-1}$. It follows that A_ν in (3.13) is well-defined and satisfies (3.2).

Similarly, we let B_ν be the matrix such that $B^{-1}_\nu A_\nu B_\nu = diag\{\lambda^1_{\nu-1}, \lambda^2_{\nu-1}\} \equiv \tilde{A}_\nu$ and

$$\|B_\nu\|_{\mathcal{E}_\nu} \|B^{-1}_\nu\|_{\mathcal{E}_\nu} \preceq 1.$$

Denote $\hat{U}_\nu = B_\nu U_\nu B^{-1}_\nu \equiv (\hat{U}^{i,j}_\nu(\theta))_{1\le i,j\le 2}$. Then (3.12) becomes

$$\partial_\omega \hat{U}_\nu(\theta) = \tilde{A}_\nu \hat{U}_\nu(\theta) - \hat{U}_\nu(\theta)\tilde{A}_{\nu+1} + \epsilon^\alpha \delta \hat{G}_{\nu-1}(\theta), \tag{3.16}$$

where $\hat{G}_{\nu-1}(\theta) = B^{-1}_\nu G_{\nu-1}(\theta) B_\nu \equiv (\hat{G}^{i,j}_\nu(\theta))_{1\le i,j\le 2}$, which, in components, reads

$$\partial_\omega \hat{U}^{i,j}_\nu(\theta) = \begin{cases} (\lambda^i_\nu - \lambda^j_\nu)\hat{U}^{i,j}_\nu(\theta) + \epsilon^\alpha \delta \hat{G}^{i,j}_{\nu-1}(\theta), & i \ne j, \\ \epsilon^\alpha \delta \hat{G}^{i,i}_{\nu-1}(\theta), & i = j, \end{cases} \tag{3.17}$$

$i, j = 1, 2$. Since $[G_{\nu-1}] = 0$, a solution $\hat{U}_\nu$ of (3.16) must satisfy $[\hat{U}_\nu] = 0$. Consider Fourier expansions

$$\hat{U}_\nu^{i,j}(\theta) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{U}_{\nu,k}^{i,j} e^{\mathrm{i}\langle k,\theta\rangle}, \quad \hat{G}_{\nu-1}^{i,j}(\theta) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{G}_{\nu-1,k}^{i,j} e^{\mathrm{i}\langle k,\theta\rangle}.$$

Then, substituting these Fourier series and comparing coefficients in (3.17) yields

$$\hat{U}_{\nu,k}^{i,j} = \begin{cases} \dfrac{\epsilon^\alpha \delta \hat{G}_{\nu-1,k}^{i,j}}{\mathrm{i}\langle k,\omega\rangle - \epsilon^\alpha \lambda_\nu^i - \epsilon^\alpha \lambda_\nu^j}, & i \neq j, \\[2ex] \dfrac{\epsilon^\alpha \delta \hat{G}_{\nu-1,k}^{i,i}}{\mathrm{i}\langle k,\omega\rangle}, & i = j, \end{cases}$$

for all $i, j = 1, 2$ and $k \in \mathbb{Z}^d\setminus\{0\}$. Similar to arguments for V_ν, we see from the above Fourier coefficients that (3.16) is solvable to yields a solution U_ν which is smooth in $\epsilon \in \mathcal{E}_\nu$ and analytic in $\theta \in \mathbb{T}^d_{s_{\nu-1}^{(2)}/2}$ such that $\|U_\nu\|_{s_{\nu-1}^{(2)}/2,\mathcal{E}_\nu} \preceq \delta s_{\nu-1}^{a_1}$.

By making ϵ_* further small if necessary, one can check that

$$r_\nu(1 + \|U_\nu\|_{\frac{s_{\nu-1}^{(2)}}{2},\mathcal{E}_\nu}) + \|V_\nu\|_{\frac{s_{\nu-1}^{(2)}}{2},\mathcal{E}_\nu} \leq r_{\nu-1},$$

which implies that

$$\phi_\nu : D\left(\frac{s_\nu}{2}, r_\nu\right) \to D\left(\frac{s_{\nu-1}}{2}, r_{\nu-1}\right), \quad \epsilon \in \mathcal{E}_\nu.$$

It also follows from the estimates of V_ν, U_ν and Cauchy's estimate that

$$\begin{aligned} &\|\phi_\nu - id\|_{\frac{s_{\nu-1}^{(1)}}{2}, r_{\nu-1},\mathcal{E}_\nu} \preceq s_{\nu-1}^{a_1}, \\ &\|D\phi_\nu - Id\|_{\frac{s_{\nu-1}^{(2)}}{2}, r_{\nu-1},\mathcal{E}_\nu} \preceq s_{\nu-1}^{a_1-1}, \\ &\|D^j\phi_\nu\|_{\frac{s_{\nu-1}^{(j+1)}}{2}, r_{\nu-1},\mathcal{E}_\nu} \preceq s_{\nu-1}^{a_1-j}, \ \ j = 2, \cdots, \tilde{m}, \end{aligned}$$

where

$$a_1 = \begin{cases} m - 2d + \mu_1, & \text{in cases } (\mathbf{C1}), \ (\mathbf{C2}), \\ m - 2\tau_1 + \frac{\mu_1}{2}, & \text{in the case } (\mathbf{C3}). \end{cases}$$

Since $\phi^\nu = \phi^{\nu-1} \circ \phi_\nu$, (3.8)-(3.10) follow easily from (3.18)-(3.20) and the induction hypothesis. Properties (3.3)-(3.6) can be also shown by the definitions of Q_ν, F_ν, H_ν, the estimates of V_ν, U_ν, and the induction hypothesis. □

3.3 Whitney Extensions and Convergence

Consider the sequence $\beta_\nu = \sum_{i=0}^{\nu} \frac{\gamma}{2^\nu}$, $\nu = 0, 1, \cdots$. By Whitney extension theorem (Pöschel 1982; Stein 1970), for each $\nu = 1, 2, \cdots$ and $\epsilon \in \mathcal{E}_\nu$, $\psi^\nu =: \phi^\nu - \phi^{\nu-1}$ can be extended $C^{\tilde{m}}$-smoothly from $D(s_\nu/2, r_\nu)$ to a function on $D(s_\nu/2, \beta_\nu/2)$, still denoted by ψ^ν, satisfying

$$\|D^j \psi^\nu\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \preceq \|D^j \psi^\nu\|_{\frac{s_\nu}{2}, r_\nu, \mathcal{E}_\nu} \preceq s_\nu^{m_*-j}, \;\; j = 0, 1, \cdots, \tilde{m}, \;\; \nu = 1, 2, \cdots.$$

This, for each $\nu = 1, 2, \cdots$, results in extensions of $\phi^\nu = \psi^\nu + \psi^{\nu-1} + \cdots + \psi^1 + id$ from $D(s_\nu/2, r_\nu)$ to $D(s_\nu/2, \beta_\nu/2)$ which we still denote by ϕ^ν. By making ϵ_* further small if necessary, it follows that

$$\|\phi^\nu - id\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \le \sum_{i=1}^{\nu} \|\psi^i\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \preceq \sum_{i=1}^{\nu} s_{i-1}^{m_*} \le \frac{\beta_\nu}{2}, \tag{3.18}$$

$$\|D\phi^\nu - Id\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \le \sum_{i=1}^{\nu} \|D\psi^i\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \preceq \sum_{i=1}^{\nu} s_{i-1}^{m_*-1} \le \beta_\nu, \tag{3.19}$$

$$\|D^j(\phi^\nu - \phi^{\nu-1})\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} = \|D^j \psi^\nu\|_{\frac{s_\nu}{2}, \frac{\beta_\nu}{2}, \mathcal{E}_\nu} \preceq s_\nu^{m_*-j}, \;\; j = 0, 1, \cdots, \tilde{m}, \tag{3.20}$$

for all $\nu = 1, 2, \cdots$. Consequently, for each $\nu = 0, 1, \cdots$, the inverse of ϕ^ν : $D(s_\nu/2, \beta_\nu/2) \times \mathcal{E}_\nu \to D(s_0, \beta_0) \times \mathcal{E}_0$ transforms (0_ν) on $D(s_0, \beta_0) \times \mathcal{E}_0$ to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (\epsilon^\alpha A_\nu + \epsilon^\alpha \delta Q_\nu(\theta))z + \epsilon^\alpha \delta F_\nu(\theta) + \epsilon^\alpha \delta \tilde{H}_\nu(\theta, z), \;\; (\theta, z) \in D(\frac{s_\nu}{2}, \frac{\beta_\nu}{2}), \; \epsilon \in \mathcal{E}_\nu, \end{cases} \tag{1_ν}$$

where, in virtual of the extension of ϕ^ν, $\tilde{H}_\nu$ is a $C^{\tilde{m},1}$ extension of H_ν from $D(\frac{s_\nu}{2}, r_\nu) \times \mathcal{E}_\nu$ to $D(\frac{s_\nu}{2}, \frac{\beta_\nu}{2}) \times \mathcal{E}_\nu$. Denote

$$\mathcal{E}_\infty = \cap_{\nu=0}^{\infty} \mathcal{E}_\nu.$$

Then, (3.18)-(3.20) also imply that ϕ^ν is convergent on $\mathbb{T}^d \times B_{r/2} \times \mathcal{E}_\infty$ in $C^{\tilde{m},1}$-norm in the sense of Whitney to an invertible transformation $\phi^\infty : \mathbb{T}^d \times B_{r/2} \times \mathcal{E}_\infty \to \mathbb{T}^d \times B_r \times \mathcal{E}_0$. By noting that the vector field of (0_ν) converges to that of (4.5) in $C^{\tilde{m},1}$-norm and $\lim_{\nu\to\infty} Q_\nu = 0$, $\lim_{\nu\to\infty} F_\nu = 0$ by (3.3), (3.4) respectively, it is clear that the inverse of ϕ^∞ transforms (4.5), on $\mathbb{T}^d \times B_r \times \mathcal{E}_0$, to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = \epsilon^\alpha A_\infty z + \epsilon^\alpha \delta \tilde{H}_\infty(\theta, z), \;\; (\theta, z) \in \mathbb{T}^d \times B_{r/2}, \; \epsilon \in \mathcal{E}_\infty, \end{cases}$$

where $A_\infty = \lim_{\nu\to\infty} A_\nu$ and $\tilde{H}_\infty = \lim_{\nu\to\infty} \tilde{H}_\nu$. It is clear that $\tilde{H}_\infty(\theta, z) = \mathcal{O}(|z|^2)$.

3.4 Measure Estimate

It remains to estimate the measure of $\mathcal{E}_\infty = \cap_{\nu=0}^{\infty}\mathcal{E}_\nu$ in the case **(C3)**. For each $k \in \mathbb{Z}^d \setminus \{0\}$ and $\nu = 0, 1, \cdots$, consider

$$f_{\nu k}^{i}(\epsilon) = \langle k, \omega\rangle - \epsilon^{\alpha}\mathrm{Im}\lambda_{\nu}^{i}(\epsilon),\ i = 1, 2,$$
$$f_{\nu k}^{ij}(\epsilon) = \langle k, \omega\rangle - \epsilon^{\alpha}\mathrm{Im}\lambda_{\nu+1}^{i} + \epsilon^{\alpha}\mathrm{Im}\lambda_{\nu+1}^{j},\ i, j = 1, 2,$$

and, in the case $\alpha > 0$, also consider

$$R_{\nu k}^{i} := \left\{\epsilon \in \mathcal{E}_\nu : |f_{\nu k}^{i}(\epsilon)| < \frac{\gamma_\nu^2}{|k|^{\tau_1}}\right\},\ i = 1, 2,$$
$$R_{\nu k}^{ij} := \left\{\epsilon \in \mathcal{E}_\nu : |f_{\nu k}^{ij}(\epsilon)| < \frac{\gamma_\nu^2}{|k|^{\tau_1}}\right\},\ i, j = 1, 2.$$

Since

$$\frac{d(\det A_\nu(\epsilon))}{d\epsilon} = \frac{d(\det(A_0 + \sum_{i=1}^{\nu}([Q_i(\theta)] + [\frac{\partial G_i(\theta, V_\nu(\theta))}{\partial z}])))}{d\epsilon},$$

there is a constant $c > 0$ such that

$$|\frac{df_{\nu k}^{i}(\epsilon)}{d\epsilon}| > c\epsilon^{\alpha-1},\ |\frac{df_{\nu k}^{ij}(\epsilon)}{d\epsilon}| > c\epsilon^{\alpha-1}, \qquad \epsilon \in \mathcal{E}_\nu,$$

for all $k \in \mathbb{Z}^d\setminus\{0\}$, $\nu = 0, 1, \cdots$, and $i = 1, 2$.

For given $i = 1, 2$ and $\nu = 0, 1, \cdots$, if $k \in \mathbb{Z}^d\setminus\{0\}$ satisfies $\frac{\gamma}{|k|^\tau} \geq 2c\epsilon_*^\alpha$, then

$$|f_{\nu k}^{i}(\epsilon)| \geq \frac{\gamma}{|k|^\tau} - c\epsilon_*^\alpha \geq \frac{\gamma}{2|k|^\tau} \geq \frac{\gamma_\nu^2}{|k|^{\tau_1}},\ \ \epsilon \in \mathcal{E}_\nu,$$

implying that $R_{\nu,k}^{i} = \emptyset$. Therefore, $\frac{\gamma}{|k|^\tau} < 2c\epsilon_*^\alpha$, and consequently,

$$\mathrm{meas} R_{\nu k}^{i} \leq \frac{\gamma_\nu^2}{c|k|^{\tau_1}\epsilon_*^{\alpha-1}} \preceq \frac{\gamma_\nu^2\epsilon_*^\alpha}{|k|^{1+\mu_2}\epsilon_*^{\alpha-1}} \preceq \frac{\gamma_\nu^2\epsilon_*}{|k|^{1+\mu_2}}.$$

It follows that

$$\mathrm{meas}\left(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ \nu=0,1,\cdots}} R_{\nu k}^{i}\right) \preceq \epsilon_* \sum_{\nu=0}^{\infty}\gamma_\nu^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^{1+\mu_2}} \preceq \gamma^2\epsilon_*,\ \ i = 1, 2.$$

Similarly,

$$\text{meas}\left(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ \nu=0,1,\cdots}} R^{ij}_{\nu k}\right) \preceq \epsilon_* \sum_{\nu=0}^{\infty}\gamma_\nu^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^{1+\mu_2}} \preceq \gamma^2\epsilon_*,\ i=1,2.$$

Thus, there is a constant $c_1 > 0$ such that

$$\begin{aligned}\text{meas}\mathcal{E}^\infty \geq \text{meas}\Pi_{\epsilon_*} - \text{meas}&\left(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\},i=1,2\\ \nu=0,1,\cdots}} R^{i}_{\nu k}\right)\\ -\text{meas}&\left(\bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\},i,j=1,2,\\ \nu=0,1,\cdots}} R^{ij}_{\nu k}\right) \geq \epsilon_*(1-c_1\gamma^2).\end{aligned}$$

In the case $\alpha < 0$, we consider sets

$$\begin{aligned}R^i_{\nu k} &:= \left\{\epsilon\in\Pi_{\epsilon_*} : |f^i_{\nu k}(\epsilon)| < \frac{\epsilon^\alpha\gamma_\nu^2}{|k|^{\tau_1}}\right\},\ i=1,2,\\ R^{ij}_{\nu k} &:= \left\{\epsilon\in\Pi_{\epsilon_*} : |f^{ij}_{\nu k}(\epsilon)| < \frac{\epsilon^\alpha\gamma_\nu^2}{|k|^{\tau_1}}\right\},\quad i,j=1,2,\end{aligned}$$

$k\in\mathbb{Z}^d\setminus\{0\}$, $\nu=0,1,\cdots$. Then similar arguments as the above yield that

$$\text{meas}\mathcal{E}^\infty \geq \epsilon_*(1-c_2\gamma^2)$$

for some constant $c_2 > 0$. In both cases, we see that $\text{meas}\mathcal{E}^\infty \sim \epsilon_*$ as γ sufficiently small.

The Main Theorem is now proved by taking $\Phi^\epsilon = (\phi^\infty)^{-1}$, $\mathcal{E}_{\epsilon_*} = \mathcal{E}^\infty$, $\breve{A} = \epsilon^\alpha A_\infty$, and $\breve{H} = \epsilon^\alpha\delta\tilde{H}_\infty$.

4 Response Solutions in Quasi-Periodically Forced Nonlinear Oscillators

In this section, we consider applications of our Main Theorem to three classes of quasi-periodically forced nonlinear oscillators: (1) strongly damped oscillators; (2) damping-free oscillators with large potentials; (3) degenerate harmonic oscillators.

For a forced nonlinear oscillator with d-dimensional frequencies, we refer the smoothness order of a response solution as that of the response torus of the corresponding reduced system in $\mathbb{T}^d\times\mathbb{R}^2$. The response solution is said to be Floquet if its corresponding response torus is.

4.1 Forced Oscillators with Strong Dampings

Consider the following strongly damped, quasi-periodically forced oscillators

$$\ddot{x} + \frac{1}{\epsilon}\dot{x} - g(x) = \epsilon^{\chi_1} f(\omega t), \qquad x \in \mathbb{R}^1, \tag{4.1}$$

where $f(\omega t)$ is a quasi-periodic forcing function, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector with Diophantine constants $\gamma > 0$ and $\tau > d - 1$, $0 < \epsilon \ll 1$ is a parameter, $\chi_1 \in (-1, \infty)$ is a constant, and $g \in C^{m+\tilde{\mu}}(\mathbb{R}^1)$, $f \in C^{m+\tilde{\mu}}(\mathbb{T}^d)$ for some integer $m > 4d + [\tau]$ and real number $0 < \tilde{\mu} < 1$.

Extending existing results Calleja et al. (2013), Corsi et al. (2013, 2014), Gentile (2010, 2012), Gentile and Vaia (2021), Wang and de la Llave (2020) on strongly damped oscillators, we consider more general cases of (4.1) by allowing not only finite smoothness, a full range of small parameter ϵ, but also the possibility of large forcing when $\chi_1 \in (-1, 0)$. Our result will also provide stability information of response solutions.

Consider the equivalent reduced system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = -\frac{1}{\epsilon} y + g(x) + \epsilon^{\chi_1} f(\theta) \end{cases} \tag{4.2}$$

corresponding to (4.1) and assume the following condition:
(H1) The equation $g(x) + \epsilon^{\chi_1}[f] = 0$ admits a smooth family of solutions $x = c(\epsilon)$ for $0 \leq \epsilon \ll 1$ such that $\frac{dg(c(0))}{dx} \neq 0$.

Remark 4.1 We note that if $\chi \in (-1, 0)$, then for **(H1)** to hold it is necessary that $[f] = 0$.

We have the following result.

Corollary 1 *Assume the condition* **(H1)**. *Then, there exists a* $0 < \epsilon_0 \ll 1$ *such that, for all* $\epsilon \in (0, \epsilon_0)$, (4.1) *admits a* C^1 *family of Floquet,* $C^{m-3d-[\tau]-1}$*-smooth response solutions* $x(t, \epsilon) = c(\epsilon) + X(\omega t, \epsilon)$, C^1*-uniformly close to* $c(0)$ *as* $\epsilon \to 0$, *such that their corresponding response tori of* (4.2) *are asymptotically stable* (*resp. unstable*) *if* $\frac{dg(c(0))}{dx} < 0$ (*resp.* $\frac{dg(c(0))}{dx} > 0$).

Proof For simplicity, we suspend the explicit dependence of $c(\epsilon)$ on ϵ and denote $g' = \frac{dg}{dx}$. We first rewrite (4.2) as

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = g'(c)(x - c) - \frac{1}{\epsilon} y + h(x - c) + \epsilon^{\chi_1} \tilde{f}(\theta), \end{cases} \tag{4.3}$$

where $\tilde{f}(\theta) = f(\theta) - [f]$ and $h(x-c) = g(x) - g(c) - g'(c)(x-c)$. Let $\lambda_1 = \lambda_1(\epsilon) = \frac{-1-\sqrt{4g'(c)\epsilon^2+1}}{2\epsilon}$ and $\lambda_2 = \lambda_2(\epsilon) = \frac{-1+\sqrt{4g'(c)\epsilon^2+1}}{2\epsilon}$. Then, the transformation $\phi_{1\epsilon}$:

$$x = x_1 + y_1 + c,$$
$$y = \lambda_2 x_1 + \lambda_1 y_1$$

transforms (4.3) to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x}_1 = \lambda_1 x_1 - \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}(\epsilon^{\chi_1}\tilde{f}(\theta) + h(x_1+y_1)), \\ \dot{y}_1 = \lambda_2 y_1 + \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}(\epsilon^{\chi_1}\tilde{f}(\theta) + h(x_1+y_1)). \end{cases} \tag{4.4}$$

Consider the homological equations

$$\partial_\omega v_1(\theta) - \lambda_1 v_1(\theta) = -\frac{\epsilon^{1+\chi_1}}{\sqrt{1+4\dot{g}(c)\epsilon^2}}\tilde{f}(\theta),$$
$$\partial_\omega v_2(\theta) - \lambda_2 v_2(\theta) = \frac{\epsilon^{1+\chi_1}}{\sqrt{1+4\dot{g}(c)\epsilon^2}}\tilde{f}(\theta).$$

By Lemma 2.3, these homological equations can be solved to yield solutions $v_i \in C^{m-d-[\tau]-1}(\mathbb{T}^d)$, $i = 1, 2$, which also depend on ϵ C^1-smoothly. Moreover, it follows from Lemma 2.3 and simple estimates on derivatives with respect to ϵ that

$$\|v_i(\theta)\|_{C^{m-d-[\tau]-1},\Pi_\epsilon} \le C\frac{\epsilon^{1+\chi_1}}{\sqrt{1+4g'(c)\epsilon^2}}\|\tilde{f}\|_{C^{m+\mu}(\mathbb{T}^d),\Pi_\epsilon}, \quad i = 1, 2, \ 0 < \epsilon \ll 1,$$

where $C > 0$ is a constant only depending on d and τ. Using v_1, v_2, the transformation $\phi_{2\epsilon}$:

$$\theta = \theta, \ x_1 = x_2 + v_1(\theta), \ y_1 = y_2 + v_2(\theta)$$

transforms the system (4.4) into

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x_2} = \lambda_1 x_2 - \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}h'(v_1(\theta)+v_2(\theta))(x_2+y_2) - \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}h(v_1(\theta) \\ \qquad +v_2(\theta)) - \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}\tilde{h}(x_2+y_2,\theta), \\ \dot{y_2} = \lambda_2 y_2 + \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}h'(v_1(\theta)+v_2(\theta))(x_2+y_2) + \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}h(v_1(\theta) \\ \qquad +v_2(\theta)) - \frac{\epsilon}{\sqrt{1+4g'(c)\epsilon^2}}\tilde{h}(x_2+y_2,\theta), \end{cases}$$

where $h' = \frac{dh}{dx}$ and $\tilde{h}(x_2+y_2,\theta) = h(v_1(\theta)+v_2(\theta)+x_2+y_2) - h'(v_1(\theta)+v_2(\theta))(x_2+y_2)$. Now consider the re-scaling $\phi_{3\epsilon}$: $x_2 \to \epsilon^{\frac{1+\chi_1}{2}}x_2$, $y_2 \to \epsilon^{\frac{1+\chi_1}{2}}y_2$

and let $\phi_\epsilon = \phi_{1\epsilon} \circ \phi_{2\epsilon} \circ \phi_{3\epsilon}$. Then $\{\phi_\epsilon\}_{0<\epsilon\ll 1}$ is a C^1 family of $C^{m-d-[\tau]-1}$-smooth, invertible transformations whose inverses transform (4.2) to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = A(\epsilon)z + \epsilon^{1+\frac{1+\chi_1}{2}} F(\theta, z, \epsilon), \quad (\theta, z) \in \mathbb{T}^d \times \mathbb{R}^2, \end{cases} \tag{4.5}$$

where $A(\epsilon) = diag\{\lambda_1(\epsilon), \lambda_2(\epsilon)\}$. The existence of $C^{m-3d-[\tau]-1}$, Floquet response tori of (4.5) for all $0 < \epsilon \ll 1$ follows from the Main Theorem in the case **(C1)** with $\alpha_1 = -1$ and $\alpha_2 = 1$. By noting that $\lambda_1(\epsilon) = -\frac{2}{\epsilon} + \mathcal{O}(\epsilon)$ and $\lambda_2(\epsilon) = g'(c(0))\epsilon + \mathcal{O}(\epsilon^2)$, the stability of these response tori follow from Remark (3) after the Main Theorem. □

4.2 Forced, Damping-Free Oscillators with Large Potentials

Consider the following quasi-periodic forced, damping-free oscillators with large potentials:

$$\ddot{x} - \frac{1}{\epsilon} g(x) = \epsilon^{\chi_2} f(\omega t), \quad x \in \mathbb{R}^1, \tag{4.6}$$

where $f(\omega t)$ is a quasi-periodic forcing function, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector with Diophantine constants $\gamma > 0$ and $\tau > d - 1$, $0 < \epsilon \ll 1$ is a parameter, $\chi_2 \in (-1, \infty)$ is a constant, and $g \in C^{m+\tilde{\mu}}(\mathbb{R}^1)$, $f \in C^{m+\tilde{\mu}}(\mathbb{T}^d)$ for some integer $m \geq 1$ and real number $0 < \tilde{\mu} < 1$. We note that, like the case of strongly damped oscillators, the forcing coefficients of (4.6) are also allowed to be large when $\chi_2 \in (-1, 0)$.

We assume the following condition:

(H2) The equation $g(x) = 0$ admits a non-degenerate real root c_0.

Consider the reduced system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = \frac{1}{\epsilon} g(x) + \epsilon^{\chi_2} f(\theta) \end{cases} \tag{4.7}$$

corresponding to (4.6) and denote $g' = \frac{dg}{dx}$. Then, we have the following results.

Corollary 2 *The followings hold under the condition* **(H2)**.

(a) *If $g'(c_0) > 0$ and $m \geq 3d$, then there exists a $0 < \epsilon_0 \ll 1$ such that for all $\epsilon \in (0, \epsilon_0)$, (4.6) admits a C^1 family of Floquet, C^{m-2d}-smooth response solutions $x(t, \epsilon) = c_0 + X(\omega t, \epsilon)$ which are C^1 uniformly close to c_0 as $\epsilon \to 0$. Moreover, the response tori of (4.7) corresponding to these response solutions are unstable (in fact, they are of saddle type).*

(b) *If $g'(c_0) < 0$ and $m > 6(\tau + 1)$, then there exists a $0 < \epsilon_0 \ll 1$ and a Cantor set $\mathcal{D} \subset (0, \epsilon_0)$ with almost full Lebesgue measure such that for all $\epsilon \in \mathcal{D}$, (4.6) admits a C^1 family of Floquet, $C^{m-4[\tau]-5}$-smooth response solutions $x(t, \epsilon) =$*

$c_0 + X(\omega t, \epsilon)$ which are C^1 uniformly close to c_0 in the sense of Whitney as $\epsilon \to 0$. Moreover, the response tori of (4.7) corresponding to these response solutions are linearly stable.

Proof We first re-write (4.7) as

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = \frac{1}{\epsilon} g'(c_0)(x - c_0) + \frac{1}{\epsilon} h(x - c_0) + \epsilon^{\chi_2} f(\theta), \end{cases} \tag{4.8}$$

where $h(x - c_0) = g(x) - g(c) - g'(c_0)(x - c_0)$. Then the transformation

$$x = x_1 + c_0, \quad y = \frac{1}{\sqrt{\epsilon}} y_1$$

transforms system (4.8) to

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x_1} = \frac{1}{\sqrt{\epsilon}} y_1, \\ \dot{y_1} = \frac{1}{\sqrt{\epsilon}} g'(c_0) x_1 + \frac{1}{\sqrt{\epsilon}} h(x_1) + \epsilon^{\frac{1}{2} + \chi_2} f(\theta). \end{cases} \tag{4.9}$$

Under the re-scaling $x_1 = \epsilon^{\frac{1+\chi_2}{2}} x_2$, $y_1 = \epsilon^{\frac{1+\chi_2}{2}} y_2$, system (4.9) then becomes

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x_1} = \frac{1}{\sqrt{\epsilon}} y_1, \\ \dot{y_1} = \frac{1}{\sqrt{\epsilon}} g'(c_0) x_1 + \frac{1}{\sqrt{\epsilon}} \epsilon^{\frac{1+\chi_2}{2}} \frac{h(\epsilon^{\frac{1+\chi_2}{2}} x_1)}{\epsilon^{1+\chi_2}} + \frac{1}{\sqrt{\epsilon}} \epsilon^{\frac{1+\chi_2}{2}} f(\theta), \end{cases}$$

which, under a linear, constant transformation in x_1, y_1, becomes

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = A(\epsilon) z + \epsilon^{-\frac{1}{2}} \epsilon^{\frac{1+\chi_2}{2}} F(\theta, z, \epsilon), \quad (\theta, z) \in \mathbb{T}^d \times \mathbb{R}^2, \end{cases} \tag{4.10}$$

where $F(\cdot, \epsilon) \in C^{m+\tilde{\mu}}(\mathbb{T}^d \times \mathbb{R}^2)$ and is of the class C^1 in $0 < \epsilon \ll 1$, and

$$A = A(\epsilon) = \begin{pmatrix} 0 & \frac{1}{\sqrt{\epsilon}} \sqrt{|g'(c_0)|} \\ \frac{\operatorname{sgn} g'(c_0))}{\sqrt{\epsilon}} \sqrt{|g'(c_0)|} & 0 \end{pmatrix}.$$

Applying the Main Theorem to (4.10) with $\alpha = -\frac{1}{2}$, we obtain (a) corresponding to the case **(C1)** and (b) corresponding to the case **(C3)**. We note in the case (b) that, because (4.6) is damping-free, its corresponding limiting matrices $\breve{A}(\epsilon)$ as in (1.5) are of elliptic type. The stability of the resulting response tori now follows from Remark (3) after the Main Theorem. □

4.3 Forced, Degenerate Harmonic Oscillators

Consider the following degenerate harmonic oscillators with small quasi-periodic forcing:

$$\ddot{x} - \lambda x^l = \epsilon f(\omega t), \qquad x \in \mathbb{R}^1, \tag{4.11}$$

where $\lambda \neq 0$ is a constant, $f(\omega t)$ is a quasi-periodic forcing function, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector with Diophantine constants $\gamma > 0$ and $\tau > d-1$, $0 < \epsilon \ll 1$ is a parameter, and $f \in C^{m+\tilde{\mu}}(\mathbb{T}^d)$ for some integer $m > 7[\tau] + d + 8$ and real number $0 < \tilde{\mu} < 1$.

When f is real analytic with $[f] \neq 0$, the existence of response solutions of (4.11) of hyperbolic, elliptic types is shown in Si and Yi (2020, 2022), respectively, for either positive or negative λ values. Now applying the Main Theorem, not only can we treat the finitely smooth case of f, but also we can show the co-existence of both type of response solutions for a given λ when ϵ is sufficiently small.

We assume the following condition that

(H3) l is even and $\lambda[f] < 0$.

Consider the reduced system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = y, \\ \dot{y} = \lambda x^l + \epsilon f(\theta) \end{cases} \tag{4.12}$$

corresponding to (4.11). Then, the following result holds.

Corollary 3 *Assume that the given l, λ, f satisfy the condition* **(H3)**. *Then there exists a $0 < \epsilon_0 \ll 1$ and a Cantor set $\mathcal{D} \subset (0, \epsilon_0)$ of almost full Lebesgue measure such that* (4.11) *admits two C^1 families of response solutions:* (a) $\{X_\epsilon^1(\omega t)\}_{\epsilon \in (0,\epsilon_0)}$, *which are hyperbolic* (*hence unstable*) *and of the class $C^{m-3d-[\tau]-1}$; and* (b) $\{X_\epsilon^2(\omega t)\}_{\epsilon \in \mathcal{D}}$, *which are elliptic* (*hence linearly stable*) *and of the class $C^{m-d-5[\tau]-6}$*.

Proof Under the condition **(H3)**, it is clear that the equation

$$\lambda x^l + \epsilon[f] = 0$$

admits two roots $x_\pm = \pm(\frac{-\epsilon[f]}{\lambda})^{\frac{1}{l}}$. The Jacobian matrices of (4.12) at the relative equilibria $(x_\pm, 0)$ read as

$$A_\pm = \begin{pmatrix} 0 & 1 \\ \pm l(\frac{-\epsilon[f]}{\lambda})^{\frac{l-1}{l}} & 0 \end{pmatrix},$$

among which A_+ is hyperbolic and A_- is elliptic. By translating the relative equilibria to the origin and using a normal form reduction similar to that in the proof of Corollary 1, we can construct a C^1 family of invertible, $C^{m-d-[\tau]-1}$-smooth transformations which, for $\epsilon > 0$ sufficiently small, transform (4.12) to the form

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z} = \tilde{A}_{\pm}(\epsilon)z + \epsilon^{\frac{1}{2}} F_{\pm}(\theta, z, \epsilon), \quad (\theta, z) \in \mathbb{T}^d \times \mathbb{R}^2, \end{cases} \tag{4.13}$$

where

$$\tilde{A}_{\pm}(\epsilon) = \epsilon^{\frac{l-1}{2l}} \begin{pmatrix} 0 & 1 \\ \pm l(\frac{-[f]}{\lambda})^{\frac{l-1}{l}} & 0 \end{pmatrix}.$$

The corollary now follows from an application of the Main Theorem to (4.13) by taking $\alpha = \frac{l-1}{2l}$, with the cases **(C1)** and **(C3)** corresponding to '+' and '-' cases of (4.13), respectively. Again, in the '-' case, because (4.11) is damping-free, its corresponding limiting matrices $\breve{A}(\epsilon)$ as in (1.5) are of elliptic type. Hence, the stability properties of the response tori follow from the Remark (3) under the Main Theorem. □

Remark 4.2 We have not considered applications of the Main Theorem in the case **(C2)**. Such applications should naturally arise when some appropriate damping terms are incorporated in oscillators (4.6) or (4.11).

Acknowledgements We would like to thank the referees for their valuable comments and suggestions.

Author Contributions All authors wrote the main manuscript text, all authors reviewed the manuscript carefully, and all the authors revised the manuscript carefully.

Data Availability No date were used to support this manuscript.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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