REDUCIBILITY OF QUASI-PERIODIC LINEAR KDV EQUATION

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Dedicated to the memory of Professor Russell A. Johnson

Abstract. In this paper, we consider the following one-dimensional, quasi-periodically forced, linear KdV equations

$$u_t + (1 + a_1(\omega t))u_{xxx} + a_2(\omega t, x)u_{xx} + a_3(\omega t, x)u_x + a_4(\omega t, x)u = 0$$

under the periodic boundary condition $u(t, x+2\pi) = u(t, x)$, where ω's are frequency vectors lying in a bounded closed region $\Pi_* \subset \mathbb{R}^b$ for some $b > 1$, $a_1 : \mathbb{T}^b \to \mathbb{R}$, $a_i : \mathbb{T}^b \times \mathbb{T} \to \mathbb{R}$, $i = 2, 3, 4$, are real analytic, bounded from the above by a small parameter $\epsilon_* > 0$ under a suitable norm, and a_1, a_3 are even, a_2, a_4 are odd. Under the real analyticity assumption of the coefficients, we re-visit a result of Baldi-Berti-Montalto [4] by showing that there exists a Cantor set $\Pi_{\epsilon_*} \subset \Pi_*$ with $|\Pi_* \setminus \Pi_{\epsilon_*}| = O(\epsilon_*^{\frac{1}{100}})$ such that for each $\omega \in \Pi_{\epsilon_*}$, the corresponding equation is smoothly reducible to a constant-coefficient one. Our main result removes a condition originally assumed in [4] and thus can yield general existence and linear stability results for quasi-periodic solutions of a reversible, quasi-periodically forced, nonlinear KdV equation with much less restrictions on the nonlinearity.

The proof of our reducibility result makes use of some special structures of the equations and is based on a refined Kuksin's estimate for solutions of homological equations with variable coefficients.

1. Introduction and Statement of the Main Result

Consider a quasi-periodic linear system

$$X' = A(\omega t)X, \quad X \in \mathcal{H}, \tag{1.1}$$

where $' = \frac{d}{dt}$, $\mathcal{H}$ is a Hilbert space, A is an operator-valued function from $\mathbb{T}^b$ into the space of symmetric operators on $\mathcal{H}$ for some integer $b > 1$, and $\omega = (\omega_1, \omega_2, \cdots, \omega_b)$ is a non-resonant frequency vector. The reducibility problem for (1.1) amounts to finding a quasi-periodic linear transformation with the same basic frequency vector ω so that the transformed linear equation becomes autonomous.

Motivated by the study of spectra, linear stability, the existence of quasi-periodic Bloch waves etc, the reducibility problem has been extensively investigated in finite dimension for linear, quasi-periodic Schrödinger-like equations and their discrete counterparts using either KAM or renormalization techniques for both perturbative (i.e., small linear perturbations of

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a linear autonomous system) and non-perturbative cases. We refer the reader to [2, 3, 7, 8, 9, 11, 12, 17, 20] and references therein for history and some exciting recent developments in quasi-periodic reducibility and almost reducibility in finite dimension.

In contrast to the finite dimensional case, the reducibility problem in infinite dimension has not been widely studied. Even in the perturbative case, one would encounter some mathematical challenges due to the possible unboundedness of perturbations. More precisely, in typical infinite dimensional problems, $\mathcal{H}$ is often a Sobolev space H^s, $s \geq 0$, consisting of functions whose components $z = (z_j)_{j\in\mathbb{Z}}$ with respect to an orthonormal basis satisfies

$$\|z\|_{a,s}^2 = \sum_{j\in\mathbb{Z}} |z_j|^2 e^{2a|j|}|j|^{2s} < \infty$$

for some fixed number $a \geq 0$. Suppose that the operator-valued function $A(\phi)$, $\phi \in \mathbb{T}^b$, in (1.1) is perturbative, i.e.,

$$A(\phi) = A + B(\phi), \quad \phi \in \mathbb{T}^b,$$

where A is an operator and $B(\phi)$ is an operator-valued function on H^s. Then in many infinite dimensional situations images of $A, B(\phi)$ are expected to be different, say $A : H^s \to H^{s-d}$ and $B(\phi) : H^s \to H^{s-\delta}$, $\phi \in \mathbb{T}^b$, in which d and δ are respectively referred to as the orders of A and B. The perturbation $B(\phi)$ is called *bounded* if $\delta \leq 0$ and called *unbounded* if $\delta > 0$. The notation of bounded and unbounded perturbations is actually adopted from that of nonlinear Hamiltonian PDEs

$$w' = Aw + F(w), \quad w \in H^s \tag{1.2}$$

in which the order of the nonlinear operator F and its boundedness or unboundedness are defined in exactly the same way as in the above.

There have been fruitful recent studies concerning either the reducibility of a quasi-periodic linear system (1.1) or the existence of quasi-periodic solutions of a nonlinear Hamiltonian PDE (1.2) that involves unbounded perturbations of order $0 \leq \delta \leq d-1$. Kuksin [14] established a KAM theorem dealing with an unbounded vector field perturbation of order $0 < \delta < d-1$. He proved the existence of KAM tori for the following perturbed Korteweg-de Vries (KdV) equation

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + \epsilon\frac{\partial}{\partial x} f_u'(u,x), \\ u(t,x) = u(t,x+2\pi), \quad \int_0^{2\pi} u dx = 0 \end{cases}$$

which corresponds to the case $\delta = 1$, $d = 3$, where $0 < \epsilon \ll 1$. Using KAM techniques, Bambusi-Graffi [5] successfully showed the reducibility of the abstract, quasi-periodic, perturbative linear Schrödinger equation

$$\mathbf{i}\psi_t = (A + \epsilon P(\omega t))\psi,$$

where A is a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$, $0 < \epsilon \ll 1$, and P is an operator-valued function from $\mathbb{T}^b$ into the space of symmetric operators on $\mathcal{H}$ which is of order $0 \leq \delta < d-1$. By giving new estimates of solutions of the homological equations, Liu-Yuan [15] improved a result contained in [13] by including the case $0 < \delta = d-1$.

As a consequence, they showed the pure-point spectrum of the quasi-periodically perturbed quantum Duffing oscillator

$$\mathbf{i}\psi_t = -\psi_{xx} + (x^4 + \varrho x^2 + \epsilon x V(\omega t))\psi,$$

where $\varrho \in \mathbb{R}$ is a constant, $0 < \epsilon \ll 1$, and V is an analytic function on $\mathbb{T}^b$. In a separate work [16], Liu-Yuan also proved the existence of quasi-periodic solutions for the following derivative-dependent nonlinear Schrödinger equation

$$\mathbf{i}u_t + u_{xx} + \mathbf{i}(f(|u|^2)u)_x = 0$$

subject to the Dirichlet boundary condition, which corresponds to the case $\delta = 1$, $d = 2$. A similar result was shown by Geng-Wu [10] for the equation

$$\mathbf{i}u_t - u_{xx} - \mathbf{i}(|u|^4 u)_x = 0$$

with the periodic boundary condition.

The more challenging case of $d - 1 \leq \delta \leq d$ is recently treated by Baldi-Berti-Montalto [4] who showed the existence and linear stability of quasi-periodic solutions in a class of quasi-periodically forced, quasi-linear or fully nonlinear KdV equations of the form

$$u_t + u_{xxx} + \epsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{1.3}$$

under the periodic boundary condition. The equations correspond to the case $d = 3$ and $\delta = 2$ or 3 in which $\omega = (\omega_1, \cdots, \omega_b)$ is a b-dimensional, non-resonant frequency vector for some $b > 1$ lying in a bounded closed region Π_* in $\mathbb{R}^b$. In particular, to obtain the existence and linear stability of quasi-periodic solutions they showed a semi-reducibility result, corresponding to a nearly full measure set of ω's in Π_*, for the following quasi-periodic, linear KdV equation

$$\begin{cases} u_t + (1 + a_1(\omega t, x))u_{xxx} + a_2(\omega t, x)u_{xx} + a_3(\omega t, x)u_x + a_4(\omega t, x)u = 0, \\ u(t, x) = u(t, x + 2\pi), \end{cases} \tag{1.4}$$

where $\omega \in \Pi_*$ and $a_i(\phi, x)$, $i = 1, 2, 3, 4$, are C^q functions on $\mathbb{T}^b \times \mathbb{T} = \mathbb{R}^b/(2\pi\mathbb{Z})^b \times \mathbb{R}/2\pi\mathbb{Z}$ for some sufficiently large natural number q which are small in some Sobolev norm related to q and satisfy

$$\int_{\mathbb{T}} \frac{a_2(\phi, x)}{1 + a_1(\phi, x)} dx = 0. \tag{1.5}$$

As remarked in [4], this condition puts some restrictions on the choice of nonlinearity f in (1.3). In fact, this is a technical condition assumed in [4] in order to eliminate certain second order derivative terms after a spatial transformation (see [4, Section 3.3]).

In this paper, we re-visit the linear, quasi-periodic KdV equation (1.4) by assuming the real analyticity of a_i's and the spatial independency of a_1, i.e., we consider the following real analytic, quasi-periodic, linear KdV equation

$$\begin{cases} u_t + (1 + a_1(\omega t))u_{xxx} + a_2(\omega t, x)u_{xx} + a_3(\omega t, x)u_x + a_4(\omega t, x)u = 0, \\ u(t, x) = u(t, x + 2\pi). \end{cases} \tag{1.6}$$

Under the real analytic assumptions on coefficients, we show a reducibility result for the equation (1.6) with the removal of the condition (1.5). Such a result will allow a broader class of nonlinear perturbations in (1.3) (see Remark 1.1 for an example). We note that (1.6) is equivalent to the operator form

$$u_t = (A + B(\omega t))u,$$

where

$$A = -\partial_{xxx}, \quad B(\phi) = -[a_1(\phi)\partial_{xxx} + a_2(\phi,\cdot)\partial_{xx} + a_3(\phi,\cdot)\partial_x + a_4(\phi,\cdot)].$$

In the case that $a_1(\omega t) \neq 0$, we clearly have $d = \delta = 3$. As to be seen in Section 3 of this paper, the problem of the existence of quasi-periodic solutions of (1.6) reduces to the solvability of a regularized linear KdV equation (3.11) which corresponds to the case $d = 3$, $\delta = 2$.

Besides our mathematical motivation of treating the case $d-1 \leq \delta \leq d$ of unbounded perturbations in infinite dimensional KAM theory, the consideration of linear, quasi-periodic KdV equations (1.6) is of significant physical interests in its own right. As it is well-known, the nonlinear KdV equation, exhibiting rich quasi-periodic dynamics, is a leading-order approximation of the free-surface shallow water wave equation and the governing equation in the continuum limit of the Fermi-Pasta-Ulam system. Linear, quasi-periodic KdV equations (1.6) naturally arise as linearizations of the nonlinear KdV equation about quasi-periodic solutions or approximate quasi-periodic solutions, whose reducibility then implies the existence and linear stability of true quasi-periodic solutions (see [4]). Recent interests in various areas of physical and engineering sciences including inhomogeneous fluids, anharmonic lattices, superconductors, plasmas and optical-fibre communications also lie in generalized KdV equations with variable coefficients in which the consideration of spatial-temporal varying coefficients through the vertical structure of the density and the background flow often leads to more realistic models (see e.g. [1, 18, 19]). A linear, quasi-periodic KdV equation (1.6) then arises in these generalized KdV equations as a linearization about a trivial or a quasi-periodic solution.

Like in [4], we will consider (1.6) under the following reversibility condition:

(H) $a_1(\phi), a_3(\phi,x)$ are even, $a_2(\phi,x), a_4(\phi,x)$ are odd functions, i.e.,

$$a_1(-\phi) = a_1(\phi), \quad a_3(-\phi,-x) = a_3(\phi,x),$$

$$a_2(-\phi,-x) = -a_2(\phi,x), \quad a_4(-\phi,-x) = -a_4(\phi,x)$$

for all $\phi \in \mathbb{T}^b$, $x \in \mathbb{T}$.

For given $r > 0, \tau > b$, we let

$$\mathbb{T}^b_r = \{\phi = (\phi_1, \cdots, \phi_b) \in \mathbb{C}^b/(2\pi\mathbb{Z})^b : |\Im\phi_i| \leq r,\ i = 1, \cdots, b\}$$

and denote

$$|g|_r = \sum_{l\in\mathbb{Z}^b} |g_l| e^{|l|r}, \quad |g|_{r,\tau} = \sum_{l\in\mathbb{Z}^b} |g_l|\langle l\rangle^\tau e^{|l|r}, \quad \|g\|_{r,\tau} = \sum_{l\in\mathbb{Z}^b\setminus\{0\}} |g_l||l|^\tau e^{|l|r}$$

for a sufficiently smooth function g on $\mathbb{T}^b_r$ with Fourier coefficients $\{g_l\}$. For any integer vector $k = (k_1, k_2, \cdots, k_b)$ in a lattice $\mathbb{Z}^b$, we denote $\langle k\rangle = \max\{1, |k|\}$, where $|k| = |k_1| + |k_2| + \cdots + |k_b|$.

Differing from [4], we will consider coefficients which are real analytic in $\mathbb{T}^b_{r_*} \times \mathbb{T}$ for some $r_* > 0$, and look for non-trivial quasi-periodic solutions $u(\phi, x)$ in the Sobolev space

$$\mathcal{H}^s := \mathcal{H}^s(\mathbb{T}) = \{h = \sum_{j\in\mathbb{Z}} \hat{h}_j e^{\mathbf{i}jx} \in L^2(\mathbb{T}) : |h|_s =: (\sum_{j\in\mathbb{Z}} |\hat{h}_j|^2 \langle j\rangle^{2s})^{\frac{1}{2}} < \infty\},$$

where $s > 7 + \Delta, 1 < \Delta < 2$.

We note that with such a choice of s, $\mathcal{H}^s(\mathbb{T}) \hookrightarrow C^{s-1,\frac{1}{2}}(\mathbb{T}) \hookrightarrow C^{3,\frac{1}{2}}(\mathbb{T})$.

For each coefficient $a =: a_i(\phi, x) = \sum_{j\in\mathbb{Z}} \hat{a}_j(\phi) e^{\mathbf{i}jx}$, $i = 2, 3, 4$, we denote

$$|a|^{\mathcal{L}}_{r_*,s,\Pi_*} =: (\sum_{j\in\mathbb{Z}} (|\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*})^2 \langle j\rangle^{2s})^{\frac{1}{2}},$$

where $|\cdot|^{\mathcal{L}}_{\Pi_*}$ stands for a Lipschitz norm with respect to $\omega \in \Pi_*$ to be defined in Section 2.

Our main result in this paper states as follows.

Main Theorem. *Consider* (1.6) *under the condition* **(H)**. *Assume that for some given constants $0 < r_*, \epsilon_* < 1$, $a_1(\phi)$ is real analytic in $\mathbb{T}^b_{r_*}$, $a_i(\phi, x)$, $i = 2, 3, 4$, are real analytic in $\mathbb{T}^b_{r_*} \times \mathbb{T}$, and $|a_1|^{\mathcal{L}}_{r_*,\tau+1,\Pi_*}, |a_i|^{\mathcal{L}}_{r_*,s,\Pi_*} < \epsilon_*$, $i = 2, 3, 4$. If ϵ_* is sufficiently small, then there exists a Cantor set $\Pi_\infty \subset \Pi_*$, with $|\Pi_* \setminus \Pi_\infty| = O(\epsilon_*^{\frac{1}{100}})$, over which* (1.6) *is reducible. More precisely, there exists a family $\{U_\omega(\phi),\ \phi \in \mathbb{T}^b, \omega \in \Pi_\infty\}$ of linear, invertible, uniformly bounded operators on $\mathcal{H}^{s'}$, with $4 < s' < s - 3 - \Delta, 1 < \Delta < \min\{2, s-7\}$, which is real, C^∞-smooth and reversibility-preserving in ϕ and Lipschitz continuous in ω, such that, for any $\omega \in \Pi_\infty$, the transformation $w(t, x) = U_\omega(\omega t)u(t, x)$ reduces* (1.6) *to the following constant-coefficient partial differential equation*

$$w_t + (1 + e_1(\omega))w_{xxx} + \mathbf{i}e_2(\omega)w_{xx} + e_3(\omega)w_x + \mathbf{i}e_4(\omega)w = 0, \tag{1.7}$$

where $e_i = O(\epsilon_*) \in \mathbb{R}$, $i = 1, 2, 3, 4$, and they only depend on ω Lipschitz continuously.

We refer the reader to Section 2 for the notation of $|\cdot|^{\mathcal{L}}_{r_*,\tau+1,\Pi_*}$ and reversibility-preserving transformation stated in the theorem.

Remark 1.1. Combining our main result above with the Nash-Moser scheme and arguments contained in [4], one is able to obtain a general result on the existence and linear stability of small amplitude, ω-quasi-periodic solutions for nonlinear KdV equations of the form

$$\begin{cases} u_t + (1 + \epsilon \int_0^{2\pi} u_x^2 dx)u_{xxx} + \epsilon f(\omega t, x, u, u_x, u_{xx}) = 0, \\ u(t, x + 2\pi) = u(t, x), \end{cases}$$

where ϵ is a small parameter, $\omega \in \Pi_*$, and $f(\phi, x, z_0, z_1, z_2)$ is a real analytic function satisfying $f(\phi, x, z_0, z_1, z_2) \equiv -f(-\phi, -x, z_0, -z_1, z_2)$. More precisely, for each $0 < \epsilon \ll 1$, one can show that there exists an asymptotically full Lebesgue measure subset Π_ϵ of Π_* such that for each $\omega \in \Pi_\epsilon$ the corresponding nonlinear KdV equation admits a small amplitude, linearly stable, ω-quasi-periodic solution.

Our reduction approach differs from that of Baldi-Berti-Montalto [4] in the following way. Consider the linear operator

$$L_0 := \omega \cdot \partial_\phi + (1 + a_1(\phi))\partial_{xxx} + a_2(\phi, x)\partial_{xx} + a_3(\phi, x)\partial_x + a_4(\phi, x),\ (\phi, x) \in \mathbb{T}^b \times \mathbb{T}$$

associated with (1.6). The approach in [4] is to first semi-conjugate L_0 (in which a_1 depends on both ϕ and x), via a diffeomorphism Φ of a suitable Sobolev space of functions on $\mathbb{T}^b \times \mathbb{T}$ and a quasi-periodic multiplier ρ, to the following third order differential operator

$$L_1 = \omega \cdot \partial_\phi + c_1 \partial_{xxx} + c_2 \partial_x + \mathcal{R}_0,$$

where $c_1 \approx 1,\ c_2 \ll 1, c_1, c_2 \in \mathbb{R}$ are constants depending only on ω and $\mathcal{R}_0$ is of 0th order which is a bounded operator. More precisely, it is shown in [4] that $\rho\Phi^{-1}L_0\Phi = L_1$ through changes of variables induced by diffeomorphisms of $\mathbb{T}$ and $\mathbb{T}^b$, multiplication operator and pseudo-differential operators. A Nash-Moser smoothing regularization is then performed and a quadratic reducibility KAM scheme is adopted in [4] to decrease the size of the perturbation $\mathcal{R}_0$ at each step. Finally, the conjugacy of L_1 to a diagonal operator is completed after the verification of the second Melnikov conditions using the Töplitz-Lipschitz and quasi-Töplitz properties. However, we note that the semi-conjugacy of operators above actually implies the reducibility of the original quasi-periodic, linear KdV equation (1.6) because ρ^{-1} times a diagonal operator leads to a diagonal system of lattice equations in term of Fourier coefficients which can be simply reduced to a constant-coefficient system over a Diophantine frequency set by integrating individual scalar systems (see Section 5 of the present paper).

In our approach, the first step is to only reduce the coefficient $a_1(\phi)$ of the dominant perturbation term ∂_{xxx} to a constant, without eliminating terms like $a_2(\phi, x)\partial_{xx}$ at all. We then use the KAM techniques analogous to that in Liu-Yuan [15] to transform the resulting reversible system to a diagonal one under a lattice setting. After that, two refined Kuksin's lemmas are proved to ensure the validity of the KAM iteration.

This paper is organized as follows. Section 2 is a preliminary section in which we recall the notation of reversibility and introduce various weighted norms. In Section 3, we regularize the linear operator and obtain infinitely many lattice equations. In Section 4, we describe the KAM scheme and solve the small-denominator equations involving variable coefficients and special structures. Our main result is proved in Section 5.

2. Preliminary

Throughout the rest of the paper, we use "$\cdot$" in front of a function to represent a constant multiple of that function. This will be particularly convenient when the constant is independent of the iteration process.

For convenience, we consider the following skew-product system

$$\begin{cases} u_t + (1 + a_1(\phi))u_{xxx} + a_2(\phi, x)u_{xx} + a_3(\phi, x)u_x + a_4(\phi, x)u = 0, \\ \phi_t = \omega, \end{cases} \tag{2.1}$$

where $\phi \in \mathbb{T}^b$, $x \in \mathbb{T}$, and $\omega \in \Pi_*$.

2.1. **Reversibility.** Let r_*, s be given as in the Main Theorem. Consider a family of operators $R = \{R(\phi)\} : \mathcal{H}^s(\mathbb{T}) \to \mathcal{H}^s(\mathbb{T})$, $\phi \in \mathbb{T}^b_{r_*}$. For each $\phi \in \mathbb{T}^b_{r_*}$, Fourier expansions with respect to the Fourier basis $\{e^{\mathbf{i}jx} :\ j \in \mathbb{Z},\, x \in \mathbb{T}\}$ yields the following matrix representation:

$$R(\phi) \sim (R_{i,j}(\phi))_{i,j\in\mathbb{Z}}$$

which defines a family of operators on

$$\ell_s^2(\mathbb{Z}) = \{\chi = (\chi_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) : \|\chi\|_{\ell_s^2(\mathbb{Z})} =: (\sum_{j\in\mathbb{Z}} |\chi_j|^2 \langle j\rangle^{2s})^{\frac{1}{2}} < \infty\}.$$

We note that $\mathcal{H}^s(\mathbb{T})$ and $\ell_s^2(\mathbb{Z})$ are isometric. The operator family $R(\phi)$ or $(R_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, $\phi \in \mathbb{T}^b_{r_*}$, is said to be *reversible* if $R_{i,j}(\phi) = -R_{-i,-j}(-\phi)$, $\forall i,j \in \mathbb{Z}$, $\phi \in \mathbb{T}^b_{r_*}$; *reversibility-preserving* if $R_{i,j}(\phi) = R_{-i,-j}(-\phi)$, $\forall i,j \in \mathbb{Z}$, $\phi \in \mathbb{T}^b_{r_*}$; and *real* if $R_{i,j}(\phi) = \overline{R_{-i,-j}(\phi)}$, $\forall i,j \in \mathbb{Z}$, $\phi \in \mathbb{T}^b_{r_*}$.

Remark 2.1. *Consider the skew-product differential system*

$$\begin{cases} u' = R(\phi)u, & u \in \mathcal{H}^s(\mathbb{T}), \\ \phi' = \omega, & \phi \in \mathbb{T}^b_{r_*}, \end{cases}$$

where $R = \{R(\phi)\}$, $\phi \in \mathbb{T}^b_{r_*}$, is a reversible family of operators on $\mathcal{H}^s(\mathbb{T})$. It is easy to see from the above definition and Fourier expansion that the skew-product system is time-reversible with respect to the involution $G : \mathcal{H}^s(\mathbb{T}) \times \mathbb{T}^b_{r_*} \to \mathcal{H}^s(\mathbb{T}) \times \mathbb{T}^b_{r_*}$: $(u,\phi) \mapsto (Su, -\phi)$, where $S : \mathcal{H}^s(\mathbb{T}) \to \mathcal{H}^s(\mathbb{T})$: $Su(x) = u(-x)$, i.e., G maps a solution to a solution with time reversed.

Lemma 2.1. *Composition of a reversible and a reversibility-preserving operator is reversible.*

Proof. It follows immediately from above definitions. □

Lemma 2.2. *Consider an operator family $R = \{R(\phi) \sim (R_{i,j}(\phi))_{i,j\in\mathbb{Z}},\ \phi \in \mathbb{T}^b_{r_*}\}$ on $\mathcal{H}^s(\mathbb{T})$ or $\mathcal{R} = \{R(\phi) = (R_{i,j}(\phi))_{i,j\in\mathbb{Z}},\ \phi \in \mathbb{T}^b_{r_*}\}$ on $\ell^2_s(\mathbb{Z})$. For each i,j, also consider the Fourier expansion:*

$$R_{i,j}(\phi) = \sum_{k\in\mathbb{Z}^b} R^k_{i,j} e^{\mathbf{i}k\cdot\phi}, \qquad \phi \in \mathbb{T}^b_{r_*}.$$

Then the following holds.

1. *R or $\mathcal{R}$ is reversible iff $R^k_{i,j} = -R^{-k}_{-i,-j}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$.*
2. *R or $\mathcal{R}$ is reversibility-preserving iff $R^k_{i,j} = R^{-k}_{-i,-j}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$.*
3. *R or $\mathcal{R}$ is real iff $R^k_{i,j} = \overline{R^{-k}_{-i,-j}}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$.*
4. *If R or $\mathcal{R}$ is real and reversible, then $\overline{R(\phi)} = -R(-\phi)$ and $R_{j,j}(0) = -R_{-j,-j}(0) = \overline{R_{-j,-j}(0)}$, $\forall j \in \mathbb{Z}$. In particular, $R_{j,j}(0) \in \mathbf{i}\mathbb{R}$, $\forall j \in \mathbb{Z}$.*
5. *If R or $\mathcal{R}$ is reversible such that $\overline{R(-\phi)} = -R(\phi)$, then R or $\mathcal{R}$ is real.*

Proof. Properties (1),(2),(3) follow immediately from definitions of reversible, reversibility-preserving and real operators. Property (4) follows easily from properties (1) and (3).

To show property (5), we note that the condition $\overline{R(-\phi)} = -R(\phi)$ implies that $\overline{R^k_{i,j}} = -R^k_{i,j}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$. By reversibility and property (1) above, we also have $R^{-k}_{-i,-j} = -R^k_{i,j}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$. It follows that $R^k_{i,j} = \overline{R^{-k}_{-i,-j}}$, $\forall i,j \in \mathbb{Z}, k \in \mathbb{Z}^b$. Hence by property (3), R is real. □

Remark 2.2. (1) *As in* [4], *one can consider the following spaces of complex-valued functions on* $\mathbb{T}^b_{r_*} \times \mathbb{T}$*:*

$$\begin{aligned} Z &= \{u(\phi, x) = \overline{u(\phi, x)}\}, \\ X &= \{u(\phi, x) = u(-\phi, -x)\}, \\ Y &= \{u(\phi, x) = -u(-\phi, -x)\} \end{aligned}$$

and view $R = R(\phi) \sim (R_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, $\phi \in \mathbb{T}^b_{r_*}$ *as an operator acting on these spaces via*

$$(Ru)(\phi, x) = R(\phi)u(\phi, x) = \sum_{i,j\in\mathbb{Z}} R_{i,j}(\phi)\hat{u}_j(\phi)e^{\mathbf{i}ix},$$

where

$$u(\phi, x) = \sum_{j\in\mathbb{Z}} \hat{u}_j(\phi)e^{\mathbf{i}jx}.$$

Using arguments of [4, Lemma 2.6, Lemma 4.4], *one concludes that* R *is reversible if* $R : X \to Y$*, reversibility-preserving if* $R : X \to X,\ Y \to Y$*, and real if* $R : Z \to Z$*. In fact, these are how reversibility, reversibility-preservation, and real properties of an operator are defined in* [4], *while definitions adopted in this paper are stated in* [4] *as equivalent properties.*

(2) *We note that if* R *is reversibility-preserving and* $\overline{R(-\phi)} = R(\phi)$*, then it is easy to see that* $R : X \to X$ *or* $Y \to Y$ *is real, and consequently,* e^R *is real.*

2.2. **Weighted norms and estimates.** Let $\Lambda = diag\{\cdots, \langle j\rangle^d, \cdots\}_{j\in\mathbb{Z}}$ and $s > 0$ be given. For $\delta \geq 0$, we denote by $\mathcal{B}^\delta$ the Banach space of all bounded linear operators T on $\ell^2_s(\mathbb{Z})$ with the norm

$$\|T\|^{\mathcal{B}^\delta}_s := \sup_{\|\chi\|_{\ell^2_s(\mathbb{Z})}=1} \|\Lambda^{-\frac{\delta}{d}} T\chi\|_{\ell^2_s(\mathbb{Z})}. \tag{2.2}$$

We denote by $\mathcal{B}_s$ the Banach space of all bounded linear operators T on $\ell^2_s(\mathbb{Z})$, with the norm

$$\|T\|_s := \sup_{\|\chi\|_{\ell^2_s(\mathbb{Z})}=1} \|T\chi\|_{\ell^2_s(\mathbb{Z})}. \tag{2.3}$$

We also denote by $\mathcal{B}_0$ the Banach space of all bounded linear operators T on $\ell^2(\mathbb{Z})$, with the norm

$$\|T\|_0 := \sup_{\|\chi\|_{\ell^2(\mathbb{Z})}=1} \|T\chi\|_{\ell^2(\mathbb{Z})}. \tag{2.4}$$

For $s \geq 0$, let $\mathcal{G}$ be the Banach space of all the bounded linear operators T on $\ell^2_s(\mathbb{Z})$ such that $\Lambda^{-\frac{\delta}{d}}T$ and $\Lambda^{-\frac{\delta}{d}}T\Lambda^{\frac{\delta}{d}}$ can extend to bounded linear operators on $\ell^2_s(\mathbb{Z})$. The norm in $\mathcal{G}$ is defined as

$$\|T\|^{\mathcal{G}}_s = \max\{\|T\|^{\mathcal{B}^\delta}_s, \|\Lambda^{-\frac{\delta}{d}}T\Lambda^{\frac{\delta}{d}}\|_s\}, \qquad \forall T \in \mathcal{G}. \tag{2.5}$$

For simplicity, we will denote $\|T\|^{\mathcal{B}^\delta}_s$, $\|T\|^{\mathcal{G}}_s$, $s > 0$, by $\|T\|^\delta$, $\|T\|^{\mathcal{G}}$, respectively.

For any $r \geq 0$ and any analytic function f from $\mathbb{T}^b_r$ to a Banach space $\mathcal{B}$, we define the norm

$$\|f\|^{\mathcal{B}}_r = \sup_{\theta\in\mathbb{T}^b_r} \|f(\theta)\|_{\mathcal{B}}, \tag{2.6}$$

where $\|\cdot\|_{\mathcal{B}}$ is the norm in $\mathcal{B}$. If f has an additional dependence on $\omega\in\Pi$, where $\Pi\subset\mathbb{R}^b$ is a compact set, then we define the norm

$$\|f\|_{r,\Pi}^{\mathcal{B},\mathcal{L}}=\|f\|_{r,\Pi}^{\mathcal{B}}+\|f\|_{r,\Pi}^{\mathcal{B},lip}, \tag{2.7}$$

where

$$\|f\|_{r,\Pi}^{\mathcal{B}}=\sup_{\omega\in\Pi}\|f\|_r^{\mathcal{B}},\quad \|f\|_{r,\Pi}^{\mathcal{B},lip}:=\sup_{\theta\in\mathbb{T}_r^b}\sup_{\omega,\omega'\in\Pi}\frac{\|f(\theta,\omega)-f(\theta,\omega')\|_{\mathcal{B}}}{|\omega-\omega'|}.$$

For simplicity, when $\mathcal{B}=\mathbb{C},\mathbb{C}^m,\ell^2(\mathbb{Z}),\ell^2_s(\mathbb{Z}),\mathcal{G}$, we will often suspend the dependence of these norms on $\mathcal{B}$ and denote $\|f\|_{r,\Pi}^{\mathcal{B}},\|f\|_{r,\Pi}^{\mathcal{B},\mathcal{L}}$ simply by $\|f\|_{r,\Pi},\|f\|_{r,\Pi}^{\mathcal{L}}$, respectively. If $\mathcal{B}=\mathcal{B}^\delta$, we will then denote $\|f\|_{r,\Pi}^{\mathcal{B}^\delta},\|f\|_{r,\Pi}^{\mathcal{B}^\delta,\mathcal{L}}$ simply by $\|f\|_{r,\Pi}^{\delta},\|f\|_{r,\Pi}^{\delta,\mathcal{L}}$, respectively.

For given $r>0,\ s,\tau\ge 0$ and a real analytic function $u(\phi,x)$, $(\phi,x)\in\mathbb{T}_r^b\times\mathbb{T}$, we expand it into the Fourier series

$$u(\phi,x)=\sum_{j\in\mathbb{Z}}\hat{u}_j(\phi)e^{\mathbf{i}jx}=\sum_{(l,j)\in\mathbb{Z}^{b+1}}u_{l,j}e^{\mathbf{i}(l\cdot\phi+jx)},\quad (\phi,x)\in\mathbb{T}_r^b\times\mathbb{T},$$

where

$$\hat{u}_j(\phi)=\frac{1}{2\pi}\int_{\mathbb{T}}u(\phi,x)e^{-\mathbf{i}jx}dx,\quad u_{l,j}=\frac{1}{(2\pi)^{b+1}}\int_{\mathbb{T}^{b+1}}u(\phi,x)e^{-\mathbf{i}(l\cdot\phi+jx)}d\phi dx \tag{2.8}$$

are Fourier coefficients. We consider the following weighted norms:

$$\begin{aligned}
|\hat{u}_j|_r &= \sum_{l\in\mathbb{Z}^b}|u_{l,j}|e^{|l|r},\\
|\hat{u}_j|_{r,\tau} &= \sum_{l\in\mathbb{Z}^b}|u_{l,j}|\langle l\rangle^\tau e^{|l|r},\\
\|\hat{u}_j\|_{r,\tau} &= \sum_{l\in\mathbb{Z}^b\setminus\{0\}}|u_{l,j}||l|^\tau e^{|l|r},\\
|u|_r &= \sum_{j\in\mathbb{Z}}|\hat{u}_j|_r,\\
|u|_{r,\tau} &= \sum_{j\in\mathbb{Z}}|\hat{u}_j|_{r,\tau},\\
\|u\|_{r,\tau} &= \sum_{j\in\mathbb{Z}}\|\hat{u}_j\|_{r,\tau},\\
\|u\|_{r,\tau,s} &= \sum_{(l,j)\in\mathbb{Z}^{b+1}}|u_{l,j}|\langle l\rangle^\tau e^{|l|r}\langle j\rangle^s.
\end{aligned}$$

It is clear that

$$\begin{aligned}
\|u\|_{r,\tau,s} &= \sum_{j\in\mathbb{Z}}|\hat{u}_j|_{r,\tau}\langle j\rangle^s,\\
|u|_{r,0} &= |u|_r,\quad \|u\|_{r,\tau,0}=|u|_{r,\tau}
\end{aligned}$$

and if u is independent of x, then

$$\|u\|_{r,\tau,s}=|u|_{r,\tau}.$$

If u ranges in a Banach space $\mathcal{B}$, then we define $|u|_{\mathcal{J}}^{\mathcal{B}},\|u\|_{r,\tau}^{\mathcal{B}}$ as in (2.6), for $\mathcal{J}=\{r\},\{r,\tau\}$ similarly to the above by replacing the sup-norm with the $\mathcal{B}$-norm of the coefficients. In case that u also depends on $\omega \in \Pi$, then we define $|u|^{\mathcal{B},lip}_{\mathcal{J},\Pi}$, $\|u\|^{\mathcal{B},lip}_{r,\tau,\Pi}$, $|u|^{\mathcal{B},\mathcal{L}}_{\mathcal{J},\Pi}$, $\|u\|^{\mathcal{B},\mathcal{L}}_{r,\tau,\Pi}$ as in (2.7) by making use of $|u|^{\mathcal{B}}_{\mathcal{J},\Pi}$, $\|u\|^{\mathcal{B}}_{r,\tau,\Pi}$ and their Lipschitz norms in respective places.

For mappings $\chi = (\chi_j)_{j\in\mathbb{Z}} : \mathbb{T}^b_r \to \ell^2_s(\mathbb{Z})$ and $T : \mathbb{T}^b_r \to \mathcal{B}^\delta$, we define $\|\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})}$, $\|T\|^{\mathcal{B}^\delta}_{s,\mathcal{J}}$, where $\mathcal{J} = \{r\}, \{r,\tau\}$, as

$$\|\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})} = (\sum_{j\in\mathbb{Z}} |\chi_j|^2_{\mathcal{J}} \langle j\rangle^{2s})^{\frac{1}{2}},$$

$$\|T\|^{\mathcal{B}^\delta}_{s,\mathcal{J}} = \sup_{\|\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})}=1} \|\Lambda^{-\frac{\delta}{d}} T\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})}.$$

For simplicity, we will denote $\|T\|^{\mathcal{B}^\delta}_{s,\mathcal{J}}$, $s > 0$, by $\|T\|^{\delta}_{\mathcal{J}}$.

For a linear operator $T(\phi) = (T_{i,j}(\phi))_{i,j\in\mathbb{Z}}$ on $\ell^2_s(\mathbb{Z})$, $s \geq 0$, we define

$$\|T\|_{s,\mathcal{J}} := \sup_{\|\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})}=1} \|T\chi\|_{\mathcal{J},\ell^2_s(\mathbb{Z})}.$$

Obviously,

$$\|T\|^{\delta}_{s} = \|\Lambda^{-\frac{\delta}{d}} T\|_{s}, \quad \|T\|^{\delta}_{s,\mathcal{J}} = \|\Lambda^{-\frac{\delta}{d}} T\|_{s,\mathcal{J}}.$$

We also define the s-decay norm as

$$|T|^2_s := \sum_{k\in\mathbb{Z}} \Big(\sup_{i-j=k} |T_{i,j}|\Big)^2 \langle k\rangle^{2s},$$

$$|T|^2_{s,\mathcal{J}} := \sum_{k\in\mathbb{Z}} \Big(\sup_{i-j=k} |T_{i,j}|_{\mathcal{J}}\Big)^2 \langle k\rangle^{2s}.$$

For simplicity, we will denote $|T|_s, |T|_{s,\mathcal{J}}$, $s > 0$, by $|T|, |T|_{\mathcal{J}}$, respectively.

Let χ be the Fourier coefficients vector of $u(\phi, x) = \sum_{j\in\mathbb{Z}} \hat{u}_j(\phi) e^{\mathbf{i}jx} \in \mathcal{H}^s$, we also have $Tu \in \mathcal{H}^s$ with

$$\|T\|_s := \sup_{\|\chi\|_{\ell^2_s(\mathbb{Z})}=1} \|T\chi\|_{\ell^2_s(\mathbb{Z})} = \sup_{|u|_s=1} |Tu|_s,$$

where $|u|^2_s = \sum_{j\in\mathbb{Z}} |\hat{u}_j|^2 \langle j\rangle^{2s}$, $|u|^2_{s,\mathcal{J}} = \sum_{j\in\mathbb{Z}} |\hat{u}_j|^2_{\mathcal{J}} \langle j\rangle^{2s}$.

Lemma 2.3. *For linear operators $A(\phi) = (A_{i,j}(\phi))_{i,j\in\mathbb{Z}}, B(\phi) = (B_{i,j}(\phi))_{i,j\in\mathbb{Z}}$ defined on $\ell^2_s(\mathbb{Z}), s \geq 0$, we have*

$$|AB|_s \leq C(s)|A|_s|B|_s, \quad |AB|_{s,\mathcal{J}} \leq C(s)|A|_{s,\mathcal{J}}|B|_{s,\mathcal{J}}.$$

Proof. We only need to verify the second inequality. Similar to [4, Lemma 2.2], we have

$$\begin{aligned}
|AB|^2_{s,\mathcal{J}} &= \sum_{k\in\mathbb{Z}} \sup_{i-j=k} |\sum_{m\in\mathbb{Z}} A_{i,m}B_{m,j}|^2_{\mathcal{J}} \langle k\rangle^{2s} \\
&\lessdot \sum_{k\in\mathbb{Z},k_1+k_2=k} \Big(\sup_{i-m=k_1} |A_{i,m}|_{\mathcal{J}} \sum_{m\in\mathbb{Z},m-j=k_2} \sup_{m-j=k_2} |B_{m,j}|_{\mathcal{J}}\Big)^2 \langle k_1+k_2\rangle^{2s} \\
&\lessdot \sum_{k\in\mathbb{Z},k_1+k_2=k} \Big(\sup_{i-m=k_1} |A_{i,m}|_{\mathcal{J}} \sum_{m\in\mathbb{Z},m-j=k_2} \sup_{m-j=k_2} |B_{m,j}|_{\mathcal{J}}\Big)^2 \langle k_1\rangle^{2s}\langle k_2\rangle^{2s} \\
&\lessdot \sum_{k_1\in\mathbb{Z}} \Big(\sup_{i-m=k_1} |A_{i,m}|_{\mathcal{J}}\Big)^2 \langle k_1\rangle^{2s} \cdot \sum_{k_2\in\mathbb{Z}} \Big(\sup_{m-j=k_2} |B_{m,j}|_{\mathcal{J}}\Big)^2 \langle k_2\rangle^{2s} \\
&\lessdot |A|^2_{s,\mathcal{J}} \cdot |B|^2_{s,\mathcal{J}}.
\end{aligned}$$

The lemma follows. ∎

Lemma 2.4. *For a linear operator $T(\phi) = (T_{i,j}(\phi))_{i,j\in\mathbb{Z}}$ defined on $\ell^2_s(\mathbb{Z})$, $s \geq 0$, $|T|_s$, respectively $|T|_{s,\mathcal{J}}$, is equivalent to $\|T\|_s$, respectively $\|T\|_{s,\mathcal{J}}$.*

Proof. We only need to verify the equivalence between $|T|_{s,\mathcal{J}}$ and $\|T\|_{s,\mathcal{J}}$.

On one hand, we can regard $u \in \mathcal{H}^s$ as an multiplication operator $h \to uh$ represented by the Töplitz matrix $U_{i,j} = \hat{u}_{i-j}$. Hence $|U|_{s,\mathcal{J}} = |u|_{s,\mathcal{J}}$. By viewing the collection of Fourier coefficients of u as an infinite-dimensional matrix U, we have

$$
\begin{aligned}
|T|^2_{s,\mathcal{J}} &= \sum_{k\in\mathbb{Z}}\Big(\sup_{i-j=k}|T_{i,j}|_{\mathcal{J}}\Big)^2\langle k\rangle^{2s} = \sum_{k\in\mathbb{Z}}\Big(\sup_{i\in\mathbb{Z}}|T_{i,i-k}|_{\mathcal{J}}\Big)^2\langle k\rangle^{2s} \\
&=\cdot \sup_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|T_{i,i-k}|^2_{\mathcal{J}}\langle k\rangle^{2s} = \cdot\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|T_{i,j}|^2_{\mathcal{J}}\langle i-j\rangle^{2s} \\
&=\cdot \sup_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}|T_{i,j}|^2_{\mathcal{J}}\langle i-j\rangle^{2s} =: \cdot\sum_{i\in\mathbb{Z}}|T_{i,j_*}|^2_{\mathcal{J}}\langle i-j_*\rangle^{2s}.
\end{aligned}
$$

It follows from Lemma 2.3 that

$$
\begin{aligned}
\|T\|^2_{s,\mathcal{J}} &= \sup_{|u|_{s,\mathcal{J}}=1}|Tu|^2_{s,\mathcal{J}} = \sup_{|U|_{s,\mathcal{J}}=1}|(\cdots,\sum_{j\in\mathbb{Z}}T_{i,j}U_{j,0},\cdots)|^2_{s,\mathcal{J}} \\
&= \sup_{|U|_{s,\mathcal{J}}=1}\sum_{i\in\mathbb{Z}}|(TU)_{i,0}|^2_{\mathcal{J}}\langle i\rangle^{2s} \leq \sup_{|U|_{s,\mathcal{J}}=1}|TU|^2_{s,\mathcal{J}} \lessdot |T|^2_{s,\mathcal{J}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|T\|^2_{s,\mathcal{J}} &= \sup_{|u|_{s,\mathcal{J}}=1}\sum_{i\in\mathbb{Z}}|\sum_{j\in\mathbb{Z}}T_{i,j}\hat{u}_j|^2_{\mathcal{J}}\langle i\rangle^{2s} \\
&\geq \langle j_*\rangle^{-2s}\sum_{i\in\mathbb{Z}}|T_{i,j_*}|^2_{\mathcal{J}}\langle i\rangle^{2s} \gtrdot \sum_{i\in\mathbb{Z}}|T_{i,j_*}|^2_{\mathcal{J}}\langle i-j_*\rangle^{2s} = \cdot|T|^2_{s,\mathcal{J}}.
\end{aligned}
$$

The lemma follows. □

Lemma 2.5. *The following hold for any analytic functions u, v in $\mathbb{T}^b_r \times \mathbb{T}$, $r > 0$.*

(1) *For any $\tau, s \geq 0$, $0 < r' \leq r, 0 \leq \tau' \leq \tau, 0 \leq s' \leq s$,*

$$
\max\{|u|_r, \|u\|_{r,\tau}\} \leq |u|_{r,\tau} \leq \|u\|_{r,\tau,s}, \quad |u|_{r,\tau} = |\hat{u}_0| + \|u\|_{r,\tau},
$$
$$
|u|_{r'} \leq |u|_r,\ |u|_{r',\tau'} \leq |u|_{r,\tau},\ \|u\|_{r',\tau',s'} \leq \|u\|_{r,\tau,s},
$$
$$
|uv|_r \leq |u|_r|v|_r, \quad |uv|_{r,\tau} \leq 2^\tau[|u|_{r,\tau}|v|_r + |u|_r|v|_{r,\tau}],
$$
$$
\|uv\|_{r,\tau} \leq 2^\tau[||u||_{r,\tau}|v|_r + |u|_r||v||_{r,\tau}],
$$

where

$$
\hat{u}_0 = \frac{1}{(2\pi)^b}\int_{\mathbb{T}^b}u(\phi,x)d\phi := [u(\cdot,x)], \quad |\hat{u}_0| = \sum_{j\in\mathbb{Z}}|u_{0,j}|
$$

for $u_{0,j}$ being defined in (2.8).

(2) *For any $\tau \geq 0, s \geq 1$,*

$$
|u_x|_r \leq \|u\|_{r,0,1}, \quad |u_\phi|_r \leq \|u\|_{r,1,0},
$$
$$
\|u_x\|_{r,\tau,s-1} \leq \|u\|_{r,\tau,s}, \quad \|u_\phi\|_{r,\tau-1,s} \leq \|u\|_{r,\tau,s}.
$$

(3) *For any $\tau, s \geq 0$,*

$$
\|uv\|_{r,\tau,s} \leq 2^{\tau+s+1}[||u||_{r,\tau,s}|v|_{r,\tau} + |u|_{r,\tau}||v||_{r,\tau,s}].
$$

Proof. The lemma follows from straightforward estimates using definitions of various norms defined above. □

Lemma 2.6. *Let $F(\phi) = (F_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, $R(\phi) = (R_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, be two families of linear operators on $\ell^2_s(\mathbb{Z}), s \geq 0$, with analytic entries $\{F_{i,j} : \mathbb{T}^b_r \to \mathbb{C}\}$, $\{R_{i,j} : \mathbb{T}^b_{r'} \to \mathbb{C}\}$ respectively, where $0 < r' < r$. Assume that $R_{i,i} = 0$ and*

$$|R_{i,j}(\phi)|_{r',\tau+1} \leq \frac{1}{|i-j|}|F_{i,j}(\phi)|_{r-\sigma,\tau+1}, \quad i \neq j,\ i,j \in \mathbb{Z},$$

for all $0 < \sigma < r - r'$. Then

$$\|R\|^{\delta}_{\tilde{s},r',\tau+1} \lessdot \frac{1}{\sigma^b}\|F\|^{\delta}_{\tilde{s},r,\tau+1}$$

for all $0 < \sigma < r - r', \tilde{s} \geq 0$.

Proof. We note from [6, Page 22] that

$$\sum_{k\in\mathbb{Z}^b} e^{-2|k|\sigma}|k|^a \leq (\frac{a}{e})^a \frac{1}{\sigma^{a+b}}(1+e)^b, \quad \sigma > 0,\ a \geq 0. \tag{2.9}$$

Thus, for fixed $i \in \mathbb{Z}, \phi \in \mathbb{T}^b_{r-\sigma}$, the above inequality implies that

$$\begin{aligned}
|R_{i,j}(\phi)|_{r',\tau+1} &\leq \frac{1}{|i-j|}|F_{i,j}(\phi)|_{r-\sigma,\tau+1} \\
&\leq \frac{1}{|i-j|}\sum_{l\in\mathbb{Z}^b}|(\hat{F_{i,j}})_l|e^{|l|r}\langle l\rangle^{\tau+1}\sum_{l\in\mathbb{Z}^b}e^{-|l|\sigma} \\
&\leq \frac{(2+2e)^b}{|i-j|\sigma^b}|F_{i,j}(\phi)|_{r,\tau+1}. \qquad (2.10)
\end{aligned}$$

Therefore, for $\tilde{s} \geq 0$, by (2.10), Lemma 2.4 and the proof of it, we have

$$\begin{aligned}
(\|R(\phi)\|^{\delta}_{\tilde{s},r',\tau+1})^2 &\lessdot \sup_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\langle i\rangle^{-2\delta}|R_{i,j}(\phi)|^2_{r',\tau+1}\langle i-j\rangle^{2\tilde{s}} \\
&\lessdot \frac{1}{\sigma^{2b}}\sup_{j\in\mathbb{Z}}\Big(\sum_{i\in\mathbb{Z}}\frac{1}{|i-j|^2}\Big)\sup_{i,j\in\mathbb{Z}}\langle i\rangle^{-2\delta}|F_{i,j}(\phi)|^2_{r,\tau+1}\langle i-j\rangle^{2\tilde{s}} \\
&\lessdot \frac{1}{\sigma^{2b}}(\|F(\phi)\|^{\delta}_{\tilde{s},r,\tau+1})^2.
\end{aligned}$$

The lemma follows. □

Lemma 2.7. *Let $U(\phi) = (U_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, $V(\phi) = (V_{i,j}(\phi))_{i,j\in\mathbb{Z}}$, $\phi \in \mathbb{T}^b_r$, be two families of linear operators on $\ell^2_s(\mathbb{Z})$ for some $s \geq 0$, such that $U \in \mathcal{G}$ and $V \in \mathcal{B}^{\delta}$. Then for any $r > 0,\ \tau \geq 0$,*

$$\|UV\|^{\delta}_{s,\mathcal{J}} \lessdot \|U\|^{\mathcal{G}}_{s,\mathcal{J}}\|V\|^{\delta}_{s,\mathcal{J}},$$

where $\mathcal{J} = \{r, \tau+1\}$. In particular, if $U(\phi), V(\phi)$ are bounded for each $\phi \in \mathbb{T}^b_r$, then so is $UV(\phi)$ for each $\phi \in \mathbb{T}^b_r$.

Proof. In the case $s = 0$, it is easy to see from (2.4) that $\mathcal{B}_0$ is a Banach algebra. Hence

$$||UV||^{\delta}_{0,\mathcal{J}} = ||\Lambda^{-\frac{\delta}{d}}UV||_{0,\mathcal{J}} \leq ||\Lambda^{-\frac{\delta}{d}}U\Lambda^{\frac{\delta}{d}}||_{0,\mathcal{J}}||\Lambda^{-\frac{\delta}{d}}V||_{0,\mathcal{J}} \leq ||U||^{\mathcal{G}}_{0,\mathcal{J}}||V||^{\delta}_{0,\mathcal{J}}.$$

In the case $s>0$, from (2.2), (2.3), Lemmas 2.3 and 2.4, we also have

$$||UV||^{\delta}_{s,\mathcal{J}} = ||\Lambda^{-\frac{\delta}{d}}UV||_{s,\mathcal{J}} \lessdot ||\Lambda^{-\frac{\delta}{d}}U\Lambda^{\frac{\delta}{d}}||_{s,\mathcal{J}}||\Lambda^{-\frac{\delta}{d}}V||_{s,\mathcal{J}} \lessdot ||U||^{\mathcal{G}}_{s,\mathcal{J}}||V||^{\delta}_{s,\mathcal{J}}.$$

The lemma follows. □

3. Regularization of the linearized operator

In this section, we regularize the equation (2.1) to obtain a simpler form in order to perform desired KAM iterations. Let r_*, ϵ_*, s be given as in the Main Theorem and fix a $\tau > b$. Define constants

$$\epsilon_0 = c_*\epsilon_*, \quad r_0 = \epsilon_0^{\frac{1}{\varsigma}}, \quad \alpha_0 = \epsilon_0^{\frac{1}{100}}, \quad \sigma_0 = \frac{r_0}{10}, \tag{3.1}$$

where $\varsigma = 50(4\tau + 5b + 3)$ and $c_* > 1$ is a constant to be specified at the end of the section. We take $d = 3$ and $\delta = 2$ in the weighted operator norms $\|\cdot\|^{\delta}, \|\cdot\|^{\mathcal{G}}$ defined in Section 2.2.

3.1. Eliminating the ϕ-variable in the coefficient of ∂_{xxx}.

To take advantage of the real analyticity of coefficients of (2.1), we extend the ϕ variable to the complex domain $\mathbb{T}^b_{r_*}$ and rewrite it as

$$\begin{cases} u_t + \mathcal{L}_0(\phi)u = 0, \quad u \in \mathcal{H}^s, \\ \phi_t = \omega, \quad \phi \in \mathbb{T}^b_{r_*}, \end{cases} \tag{3.2}$$

where for each $\phi \in \mathbb{T}^b_{r_*}$, $\mathcal{L}_0(\phi) : \mathcal{H}^s \to \mathcal{H}^{s-3}$:

$$\mathcal{L}_0(\phi) = (1 + a_1(\phi))\partial_{xxx} + a_2(\phi,\cdot)\partial_{xx} + a_3(\phi,\cdot)\partial_x + a_4(\phi,\cdot).$$

It is easy to see that the condition **(H)** implies that $\mathcal{L}_0 = \{\mathcal{L}_0(\phi) : \ \phi \in \mathbb{T}^b_{r_*}\}$ is a reversible family.

For each $i = 2,3,4$, we expand $a_i(\phi,x)$ into Fourier series $a =: a_i(\phi,x) = \sum_{j\in\mathbb{Z}} \hat{a}_j(\phi)e^{\mathbf{i}jx}$. Since $|a|^{\mathcal{L}}_{r_*,s,\Pi_*} < \epsilon_*$, we have

$$|\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*} < \langle j\rangle^{-s}\epsilon_*, \qquad \forall j \in \mathbb{Z}.$$

It follows that there exists a constant $1 < \Delta < \min\{2, s\}$ such that

$$\begin{aligned}
(|a|^{\mathcal{L}}_{r_*,0,s-\Delta,\Pi_*})^2 &= \Big(\sum_{j\in\mathbb{Z},|j|<1} |\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*}\langle j\rangle^{s-\Delta} + \sum_{j\in\mathbb{Z},|j|\geq 1} |\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*}\langle j\rangle^{s-\Delta}\Big)^2 \\
&\leq 2\Big[\Big(\sum_{j\in\mathbb{Z},|j|<1} |\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*}\langle j\rangle^{s-\Delta}\Big)^2 + \Big(\sum_{j\in\mathbb{Z},|j|\geq 1} |\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*}\langle j\rangle^{s-\Delta}\Big)^2\Big] \\
&\lessdot \sum_{j\in\mathbb{Z},|j|<1} (|\hat{a}_j|^{\mathcal{L}}_{r_*,\Pi_*})^2\langle j\rangle^{2(s-\Delta)} + \Big(\sum_{j\in\mathbb{Z},|j|\geq 1} \langle j\rangle^{-\Delta}\Big)^2\epsilon_*^2 \\
&\lessdot (|a|^{\mathcal{L}}_{r_*,s,\Pi_*})^2 + \cdot\epsilon_*^2 \lessdot \epsilon_*^2.
\end{aligned}$$

Let $\tilde{r} = \frac{r_*}{2}$. Then it is easy to see that there exists a constant $c(r_*,\tau,b) > 0$ such that

$$\|a_i\|^{\mathcal{L}}_{\tilde{r},\tau+1,s-\Delta,\Pi_*} \leq c(r_*,\tau,b)\|a_i\|^{\mathcal{L}}_{r_*,0,s-\Delta,\Pi_*} \lessdot \epsilon_*, \qquad i = 2,3,4. \tag{3.3}$$

Consider the Diophantine set

$$\Pi_0 = \{\omega \in \Pi_* : |k\cdot\omega| \geq \frac{\alpha_0}{|k|^\tau}, \ k \in \mathbb{Z}^b \setminus \{0\}\}. \tag{3.4}$$

For each $\omega \in \Pi_0$, it is clear that

$$\alpha(\phi) = \frac{1}{1+[a_1]}(\omega \cdot \partial_\phi)^{-1}(a_1(\phi) - [a_1]), \quad \phi \in \mathbb{T}^b_{\tilde{r}} \tag{3.5}$$

is well-defined, real analytic, and odd, where

$$(\omega \cdot \partial_\phi)^{-1} e^{\mathbf{i} l \cdot \phi} := \frac{e^{\mathbf{i} l \cdot \phi}}{\mathbf{i}\omega \cdot l}, \quad l \in \mathbb{Z}^b \setminus \{0\}, \quad (\omega \cdot \partial_\phi)^{-1} 1 = 0$$

and

$$[a_1] = \frac{1}{(2\pi)^b} \int_{\mathbb{T}^b} a_1(\phi) d\phi.$$

In fact,

$$\alpha(\phi) = \frac{1}{1+[a_1]} \sum_{l \in \mathbb{Z}^b \setminus \{0\}} \frac{(\hat{a_1})_l e^{\mathbf{i} l \cdot \phi}}{\mathbf{i}\omega \cdot l},$$

where $(\hat{a_1})_l$'s are Fourier coefficients of a_1. By Lemma 2.5, it is easy to see that $|[a_1]| \lessdot \epsilon_*$ and hence

$$\|a_1(\phi) - [a_1]\|^{\mathcal{L}}_{\tilde{r},\tau+1,\Pi_*} = |a_1(\phi) - [a_1]|^{\mathcal{L}}_{\tilde{r},\tau+1,\Pi_*} \le |a_1(\phi)|^{\mathcal{L}}_{\tilde{r},\tau+1,\Pi_*} + |[a_1]| \lessdot \epsilon_*.$$

This, together with Lemma 2.5, implies that

$$\begin{aligned} |\alpha(\phi)|_{\tilde{r},1} &= \|\alpha(\phi)\|_{\tilde{r},1} \le (1 + \cdot\epsilon_*) \sum_{l \in \mathbb{Z}^b \setminus \{0\}} \alpha_0^{-1} |l|^{\tau+1} |(\hat{a_1})_l| e^{|l|\tilde{r}} \\ &\le (1 + \cdot\epsilon_*)\alpha_0^{-1} \|a_1(\phi) - [a_1]\|_{\tilde{r},\tau+1} \lessdot \alpha_0^{-1}\epsilon_*. \end{aligned} \tag{3.6}$$

Since $r_0 \ll \tilde{r}$ as $\epsilon_* \ll 1$, we can use the Fourier series (3.5) and arguments in (3.6) to deduce that

$$|\omega \cdot \alpha_\phi(\phi)|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} = \|\omega \cdot \alpha_\phi(\phi)\|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} = \frac{1}{|1+[a_1]|} \|a_1(\phi) - [a_1]\|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} \lessdot \epsilon_*. \tag{3.7}$$

Denote

$$\rho(\theta) =: \rho(\theta, \omega) = 1 + \omega \cdot \alpha_\phi(\mathcal{T}^{-1}(\theta)). \tag{3.8}$$

By [4], for each $\omega \in \Pi_0$, the invertible torus transformation

$$\theta = \mathcal{T}(\phi) =: \phi + \omega\alpha(\phi), \quad \phi \in \mathbb{T}^b_{\tilde{r}} \tag{3.9}$$

and the change of time variable

$$\tilde{t} = t + \alpha(\omega t) \tag{3.10}$$

will reduce the coefficient $a_1(\phi)$ in (3.2) to the constant $e_1 =: [a_1]$ with $|e_1| \lessdot \epsilon_*$. More precisely, the transformed equation under (3.9), (3.10) reads

$$\begin{cases} u_{\tilde{t}} + \mathcal{L}_1(\theta)u = 0, \quad u \in \mathcal{H}^s, \\ \theta_{\tilde{t}} = \omega, \quad \theta \in \mathbb{T}^b_{r_0}, \end{cases} \tag{3.11}$$

where

$$\mathcal{L}_1(\theta) = (1 + e_1)\partial_{xxx} + b_2(\theta, \cdot)\partial_{xx} + b_3(\theta, \cdot)\partial_x + b_4(\theta, \cdot),$$

with

$$b_i(\theta, x) = \frac{1}{\rho(\theta)} a_i(\mathcal{T}^{-1}(\theta), x), \qquad i = 2, 3, 4.$$

In fact, due to (3.2) and (3.10), we have

$$\begin{aligned} u_{\tilde{t}} &= u_t \cdot \frac{\partial t}{\partial \tilde{t}} = -\mathcal{L}_0(\phi)u \cdot (1+\omega\cdot\alpha_\phi(\phi))^{-1} \\ &= -\big[(1+a_1(\phi))u_{xxx} + a_2(\phi,x)u_{xx} + a_3(\phi,x)u_x + a_4(\phi,x)u\big]\cdot(1+\omega\cdot\alpha_\phi(\phi))^{-1}. \end{aligned}$$

Hence by (3.5), we can deduce that

$$\frac{1+a_1(\phi)}{1+\omega\cdot\alpha_\phi(\phi)} = \frac{1+a_1(\phi)}{1+\frac{a_1(\phi)-[a_1]}{1+[a_1]}} = 1+[a_1] := 1+e_1.$$

Combining (3.8) and (3.9), we immediately obtain the form of $b_i(\theta,x), i=2,3,4$, i.e., (3.11) is obtained.

Since α is odd and α_ϕ is even, we have that $\mathcal{T}(-\phi)=-\mathcal{T}(\phi)$ and $\rho(-\theta)=\rho(\mathcal{T}(-\phi))=\rho(\theta)$. It follows that $b_3(\theta,x)$ is even and $b_2(\theta,x), b_4(\theta,x)$ are odd. Consequently, $\{\mathcal{L}_1(\theta)\}$ remains a reversible family of operators.

Lemma 3.1. $|b_i|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} \le ||b_i||^{\mathcal{L}}_{r_0,\tau+1,s-\Delta,\Pi_0} \lessdot \epsilon_*, \quad i=2,3,4.$

Proof. The lemma follows easily from (3.3), (3.7) and Lemma 2.5. □

3.2. **Lattice setting.** To prepare for the desired reduction, we would like to further convert (3.11) into an infinite dimensional lattice system. Consider Fourier expansions

$$u(\tilde{t},x) = \sum_{j\in\mathbb{Z}} \hat{u}_j(\tilde{t})e^{\mathbf{i}jx}, \quad b_i(\theta,x) = \sum_{j\in\mathbb{Z}} \hat{b}^i_j(\theta)e^{\mathbf{i}jx}, \quad i=2,3,4, \tag{3.12}$$

where

$$\hat{b}^i_j(\theta) = \frac{1}{2\pi}\int_{\mathbb{T}} b_i(\theta,x)e^{-\mathbf{i}jx}dx, \quad j\in\mathbb{Z},\ i=2,3,4.$$

Denote

$$\cdot = \frac{d}{d\tilde{t}}.$$

Substituting (3.12) into (3.11) and equating the coefficients of each mode $e^{\mathbf{i}jx}$, $j\in\mathbb{Z}$ yields

$$\begin{cases} \dot{\hat{u}}_j = \mathbf{i}(1+e_1)j^3\hat{u}_j + \sum_{j'\in\mathbb{Z}}\Delta^{(j')}_{j-j'}(\theta)\hat{u}_{j'}, \quad j\in\mathbb{Z}, \\ \dot{\theta} = \omega, \end{cases} \tag{3.13}$$

where

$$\Delta^{(j')}_{j-j'}(\theta) = j'^2\hat{b}^2_{j-j'}(\theta) - \mathbf{i}j'\hat{b}^3_{j-j'}(\theta) - \hat{b}^4_{j-j'}(\theta), \quad j,j'\in\mathbb{Z}.$$

Let $\chi=\chi(\tilde{t}) = (\cdots,\hat{u}_j(\tilde{t}),\cdots)^\top$. For each $\tilde{t}$, since $u(\tilde{t},\cdot)\in\mathcal{H}^s$, we have $\chi(\tilde{t})\in\ell^2_s$. Denote

$$\begin{aligned} A_0(\theta) &= diag\Big(\cdots,\lambda^0_j+\mu^0_j(\theta),\cdots\Big)_{j\in\mathbb{Z}}, \\ P^0(\theta) &= \Big((P^0)_{i,j}(\theta)\Big)_{i,j\in\mathbb{Z}}, \end{aligned}$$

where for each $i,j\in\mathbb{Z}$,

$$\begin{aligned} \lambda^0_j &= \mathbf{i}(1+e_1)j^3, \\ \mu^0_j(\theta) &\equiv 0, \\ (P^0)_{i,j}(\theta) &= \Delta^{(j)}_{i-j}(\theta). \end{aligned}$$

Then equations in (3.13) form the following infinite dimensional, skew-product lattice system

$$\begin{cases} \dot{\chi} = (A_0(\theta) + P^0(\theta))\chi, & \chi \in \ell^2_s(\mathbb{Z}), \\ \dot{\theta} = \omega, & \theta \in \mathbb{T}^b_{r_0}. \end{cases} \tag{3.14}$$

In the above, we have written $A_0 = A_0(\theta)$ for the sake of generality though it is actually independent of θ in (3.14). However, as we apply KAM iterations to reduce (3.14), θ-dependency will occur in the new diagonal part after each KAM iteration.

Lemma 3.2. *A_0, P^0 are real, analytic, and reversible, and moreover*

$$\|P^0\|^{\delta,\mathcal{L}}_{s',r_0,\tau+1,\Pi_0} \le \epsilon_0,$$

where $4 < s' < s - 3 - \Delta, 1 < \Delta < \min\{2, s - 7\}$.

Proof. By Lemma 2.2 (1), (3), it is clear that A_0 is real and reversible.

Since $b_3(\theta, x)$ is even, $b_2(\theta, x), b_4(\theta, x)$ are odd, and $b_2(\theta, x), b_3(\theta, x), b_4(\theta, x)$ are real-valued, we have

$$\overline{\hat{b}^2_{-j}(\theta)} = \hat{b}^2_j(\theta) = -\hat{b}^2_{-j}(-\theta),\ \overline{\hat{b}^3_{-j}(\theta)} = \hat{b}^3_j(\theta) = \hat{b}^3_{-j}(-\theta),\ \overline{\hat{b}^4_{-j}(\theta)} = \hat{b}^4_j(\theta) = -\hat{b}^4_{-j}(-\theta),\ j \in \mathbb{Z}.$$

It now follows from the definition of P^0 and Lemma 2.2 (1), (3) that P^0 is real, analytic, and reversible.

Since by Lemma 3.1, $||b_k||^{\mathcal{L}}_{r_0,\tau+1,s-\Delta,\Pi_0} \lessdot \epsilon_*$, $k = 2, 3, 4$, we have

$$|\hat{b}^k_j|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} \langle j \rangle^{s-\Delta} \le ||b_k||^{\mathcal{L}}_{r_0,\tau+1,s-\Delta,\Pi_0} \lessdot \epsilon_*, \quad k = 2, 3, 4,\ j \in \mathbb{Z}. \tag{3.15}$$

Consequently,

$$\begin{aligned} |(P^0)_{i,j}|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} &\lessdot \langle j \rangle^2 \max_{k=2,3,4} |\hat{b}^k_{i-j}|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} \\ &\lessdot \langle j \rangle^2 \langle i - j \rangle^{-(s-\Delta)} \max_{k=2,3,4} ||b_k||^{\mathcal{L}}_{r_0,\tau+1,s-\Delta,\Pi_0} \\ &\lessdot \langle j \rangle^2 \langle i - j \rangle^{-(s-\Delta)} \epsilon_*. \end{aligned} \tag{3.16}$$

Now, we replace $1 < \Delta < \min\{2, s\}$ with $1 < \Delta < \min\{2, s - 7\}$ and choose $4 < s' < s - 3 - \Delta$, by using Lemma 2.4, we can get

$$\begin{aligned} (\|P^0\|^{\delta}_{s',r_0,\tau+1})^2 &\lessdot \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \langle i \rangle^{-2\delta} |(P^0)_{i,j}|^2_{r_0,\tau+1} \langle i - j \rangle^{2s'} \\ &\lessdot \sup_{j\in\mathbb{Z}} \langle j \rangle^4 \sum_{i\in\mathbb{Z}} \langle i \rangle^{-2\delta} \langle i - j \rangle^{2\left(s'-(s-\Delta)\right)} \epsilon^2_* \\ &\lessdot \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \langle i - j \rangle^{-2} \Big(\frac{\langle j \rangle}{\langle i \rangle \langle i - j \rangle} \Big)^4 \epsilon^2_* \\ &\lessdot \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \langle i - j \rangle^{-2} \Big(\frac{1}{\langle i \rangle} + \frac{1}{\langle i - j \rangle} \Big)^4 \epsilon^2_* \lessdot \epsilon^2_*. \end{aligned}$$

Thus,

$$\|P^0\|^{\delta,\mathcal{L}}_{s',r_0,\tau+1,\Pi_0} \le \epsilon_0,$$

where $\epsilon_0 = c_* \epsilon_*$ for an appropriate constant $c_* > 1$.

□

Remark 3.1. *For each $j \in \mathbb{Z}$, since $\lambda_j^0 = \mathbf{i}(1+e_1)j^3, \mu_j^0(\theta) = 0$, we clearly have $[\mu_j^0] = 0$, and for each $j \in \mathbb{Z}$,*

$$\|\lambda_j^0\|_{\Pi_0}^{lip} \lessdot \langle j \rangle^{\delta} \epsilon_*;$$
$$\|\mu_j^0\|_{r_0,\tau+1,\Pi_0}^{\mathcal{L}} \lessdot \langle j \rangle^{\delta} \epsilon_*.$$

4. KAM scheme

In the following sections, the weighted operator norms $\|\cdot\|^{\delta}, \|\cdot\|^{\mathcal{G}}$ are in the sense of $\|\cdot\|_{s'}^{\delta}, \|\cdot\|_{s'}^{\mathcal{G}}$, see s' in Lemma 3.2. Our goal is to use KAM iteration to conjugate the quasi-periodic system (3.14) to a diagonal system. Then a straightforward integration of the diagonal system will further reduce it to a constant-coefficient system.

The aim of this section is to describe one KAM step in this iteration process. Suppose for some $\nu = 0, 1, \cdots$, we have arrived in the following ∞-dimensional, skew-product, lattice system on $\ell_s^2(\mathbb{Z}) \times \mathbb{T}_r^b$ for some $r > 0$ at the νth step:

$$\begin{cases} \dot{\chi} = (A(\theta) + P(\theta))\chi, \qquad \chi \in \ell_s^2(\mathbb{Z}), \\ \dot{\theta} = \omega, \qquad \theta \in \mathbb{T}_r^b, \end{cases} \tag{4.1}$$

where $\omega = (\omega_1, \cdots, \omega_b) \in \Pi$, Π is a bounded closed region in Π_*, and $A(\theta) = A(\theta, \omega)$, $P(\theta) = P(\theta, \omega)$ satisfy the following conditions:

(H1) $A(\theta) = diag(\cdots, \lambda_j(\omega) + \mu_j(\theta, \omega), \cdots)_{j \in \mathbb{Z}}$ is real, analytic, and reversible in θ, Lipschitz-continuous in ω such that $[\mu_j(\cdot, \omega)] = 0,\ \forall j \in \mathbb{Z},\ \omega \in \Pi$ and

$$\|\lambda_j - \lambda_0\|_{\Pi},\ \|\lambda_j\|_{\Pi}^{lip},\ \|\mu_j\|_{r,\tau+1,\Pi}^{\mathcal{L}} \lessdot \langle j \rangle^{\delta} \epsilon_*.$$

(H2) $P(\theta)$ is real, analytic, reversible in θ, Lipschitz-continuous in ω, and there exists $\epsilon > 0$ such that

$$\|P\|_{r,\tau+1,\Pi}^{\delta,\mathcal{L}} \leq \epsilon.$$

Remark 4.1. (1) *Since A, P are real and reversible, it follows from* Lemma 2.2 (4) *that*

$$\overline{A(-\theta)} = -A(\theta), \qquad \overline{P(-\theta)} = -P(\theta). \tag{4.2}$$

Moreover, it also follows from Lemma 2.2 (4) *and* (4.56) *that*

$$\lambda_j \in \mathbf{i}\mathbb{R}, \qquad \forall j \in \mathbb{Z}.$$

(2) *Consider the involution $S : \ell_s^2(\mathbb{Z}) \to \ell_s^2(\mathbb{Z})$: $\chi \mapsto \overline{\chi}$. Since by* (4.2),

$$\begin{aligned} S \circ (A(-\theta) + P(-\theta))\chi(\tilde{t}) &= \overline{(A(-\theta) + P(-\theta))\chi(\tilde{t})} \\ &= -(A(\theta) + P(\theta))\overline{\chi(\tilde{t})} = -(A(\theta) + P(\theta))S\chi(\tilde{t}), \end{aligned}$$

we see that (4.1) *is time-reversible with respect to the involution $G : \ell_s^2(\mathbb{Z}) \times \mathbb{T}_r^b \to \ell_s^2(\mathbb{Z}) \times \mathbb{T}_r^b$: $(\chi, \theta) \mapsto (S\chi, -\theta)$, and consequently it generates an ∞-dimensional, G-time-reversible dynamical system in the usual sense, i.e., G maps a solution to a solution with time reversed.*

(3) *It follows from $\|P\|_{\mathcal{J}}^{\delta} = \|\Lambda^{-\frac{\delta}{d}} P\|_{\mathcal{J}}$ and Lemma 2.4 that*

$$\|P\|_{\mathcal{J}}^{\delta} = \cdot|\Lambda^{-\frac{\delta}{d}} P|_{\mathcal{J}},$$

which implies that

$$|P_{i,j}|_{\mathcal{J}} \lessdot \frac{\langle i\rangle^{\delta}}{\langle i-j\rangle^{s}}\|P\|_{\mathcal{J}}^{\delta}, \quad i,j\in\mathbb{Z},$$

and consequently, by **(H2)**, *we have*

$$|P_{j,j}|^{\mathcal{L}}_{r,\tau+1,\Pi} \lessdot \langle j\rangle^{\delta}\epsilon, \quad j\in\mathbb{Z}. \tag{4.3}$$

We would like to find a reversibility-preserving transformation under which (4.1) is transformed into a new system which satisfies properties similar to **(H1)** and **(H2)** but with much smaller perturbation.

Let $B=(B_{i,j}(\theta))_{i,j\in\mathbb{Z}}$ be a solution of the equation

$$[A,B]-\omega\cdot D_{\theta}B+(P-diag(P))-R=0, \tag{4.4}$$

where $[A,B]=AB-BA$. Denote $P=(P_{i,j}(\theta))_{i,j\in\mathbb{Z}}$ and

$$\sigma=\frac{r}{10}, \qquad K=\frac{|\ln\ \epsilon|}{\sigma}.$$

Let $R=(R_{i,j})_{i,j\in\mathbb{Z}}$ be the truncation matrix whose entries are defined as

$$R_{i,j}=\begin{cases} 0, & |i^3-j^3|<\frac{4K}{1+e_1},\\ (1-\Gamma_K)\Big((\mu_i(\theta)-\mu_j(\theta))B_{i,j}+P_{i,j}\Big), & |i^3-j^3|\geq\frac{4K}{1+e_1},\end{cases}$$

where

$$(\Gamma_K f)(\theta):=\sum_{|k|\leq K}\hat{f}_k e^{\mathbf{i}k\cdot\theta}, \quad \forall f=\sum_{k\in\mathbb{Z}^b}\hat{f}_k e^{\mathbf{i}k\cdot\theta} \tag{4.5}$$

is the truncation operator.

Define

$$A_+ = A+diag(P), \tag{4.6}$$

$$\begin{aligned} P^+ &= R+(e^{-B}Ae^{B}-A-[A,B])\\ &+ (e^{-B}Pe^{B}-P)+(\omega\cdot D_{\theta}B-e^{-B}\omega\cdot D_{\theta}e^{B}). \end{aligned} \tag{4.7}$$

Then the transformation $\chi=e^{B(\theta)}\zeta$ transforms (4.1) into

$$\begin{aligned}
\dot{\zeta} &= \frac{d\zeta(\tilde{t})}{d\tilde{t}}=e^{-B(\theta)}\Big((A(\theta)+P(\theta))e^{B(\theta)}\zeta-\frac{d}{d\tilde{t}}(e^{B(\theta)})\zeta\Big)\\
&= (e^{-B}Ae^{B}+e^{-B}Pe^{B}-e^{-B}\frac{d}{d\tilde{t}}(e^{B}))\zeta\\
&= \Big((A+diag(P))+([A,B]-\frac{dB}{d\tilde{t}}+P-diag(P)-R)\\
&+ R+(e^{-B}Ae^{B}-A-[A,B])+(e^{-B}Pe^{B}-P)+\big(\frac{dB}{d\tilde{t}}-e^{-B}\frac{d}{d\tilde{t}}(e^{B})\big)\Big)\zeta\\
&= (A_+(\theta)+P^+(\theta))\zeta(\tilde{t}).
\end{aligned}$$

4.1. **Solvability of the homological equation.** We observe that (4.4) can be split into the following cases:

Case 1: $i=j$: $B_{i,i}=0$;

Case 2: $0<|i^3-j^3|<\frac{4K}{1+e_1}$:

$$-\omega\cdot\partial_\theta B_{i,j}+(\lambda_i-\lambda_j)B_{i,j}+(\mu_i(\theta)-\mu_j(\theta))B_{i,j}+P_{i,j}=0; \tag{4.8}$$

Case 3: $|i^3-j^3|\geq\frac{4K}{1+e_1}$:

$$-\omega\cdot\partial_\theta B_{i,j}+(\lambda_i-\lambda_j)B_{i,j}+\Gamma_K\Big((\mu_i(\theta)-\mu_j(\theta))B_{i,j}\Big)+\Gamma_K P_{i,j}=0,\quad \Gamma_K B_{i,j}=B_{i,j}. \tag{4.9}$$

The solvability of these equations will rely on the following two refined Kuksin's lemmas for reversible linear KdV equations which are modifications of Theorem 1.4 and Lemma 2.6 in [15]. In these lemmas as well as in the equation (4.4), we suspend the parameter ω in all coefficients for the sake of simplicity. Such ω-dependency will be added back when we actually solve the equation (4.4).

Let f be an analytic function on $\mathbb{T}^b_r$ and denote by $\hat{f}_k$ the kth-Fourier coefficient of f for each $k\in\mathbb{Z}^b$. For any $y\in\mathbb{R}^b$ with $|y|\leq r$, define $\hat{f}_k(y)=\hat{f}_k e^{-k\cdot y}$, $k\in\mathbb{Z}^b$, $\tilde{f}_{kl}(y)=\hat{f}_{k-l}e^{-(k-l)\cdot y}$, $k,l\in\mathbb{Z}^b$. We call

$$\hat{f}(y)=\begin{pmatrix}\vdots\\ \hat{f}_k(y)\\ \vdots\end{pmatrix}_{k\in\mathbb{Z}^b}$$

the *Fourier coefficients vector* of f on $\mathbb{T}^b+y$ and

$$\tilde{f}(y)=(\tilde{f}_{kl}(y))_{k,l\in\mathbb{Z}^b}$$

the *Töplitz matrix* of f on $\mathbb{T}^b+y$.

Lemma 4.1. *Consider the first-order partial differential equation*

$$-\omega\cdot\partial_\theta u+\lambda u+\mu(\theta)u=p(\theta),\quad \theta\in\mathbb{T}^b_r, \tag{4.10}$$

where $\lambda\in\mathbf{i}\mathbb{R}$ is a constant, and μ,p are analytic functions on $\mathbb{T}^b_r$ such that $\overline{\mu(\theta)}=-\mu(-\theta)$, $\mu(\mathbb{T}^b)\subset\mathbf{i}\mathbb{R}$, $\overline{p(\theta)}=-p(-\theta)$,

$$\|\mu\|_{r,\tau+1}\leq C\gamma \tag{4.11}$$

for some constant $C>0$, and $\omega\in\Pi$ is such that

$$|k\cdot\omega|\geq\frac{\alpha_0}{|k|^\tau},\quad k\in\mathbb{Z}^b\setminus\{0\}, \tag{4.12}$$

$$|k\cdot\omega+\mathbf{i}(\lambda+\hat{\mu}_0)|\geq\frac{\alpha\gamma}{1+|k|^\tau},\quad k\in\mathbb{Z}^b \tag{4.13}$$

for some constants $\alpha_0,\alpha,\gamma>0$, where $\hat{\mu}_0=\frac{1}{(2\pi)^b}\int_{\mathbb{T}^b}\mu(\theta)d\theta$.

Then (4.10) admits a unique solution $u(\theta)$ in a narrower domain $\mathbb{T}^b_{r-\sigma}$ for some $0<\sigma<\min\{1,r\}$ which satisfies

$$|u|_{r-\sigma}\leq\frac{c(\tau,b)}{\alpha\gamma\sigma^{\tau+b}}e^{2C\gamma r/\alpha_0}|p|_r \tag{4.14}$$

for some constant $c(\tau,b)>0$. *Moreover,* $\overline{u(\theta)}=u(-\theta)$ *and*

$$|u|_{r-\sigma,\tau+1}\le \frac{c(\tau,b)}{\alpha\gamma\sigma^{2\tau+2b+1}}e^{2C\gamma r/\alpha_0}|p|_{r,\tau+1}. \tag{4.15}$$

Proof. Let

$$U(\theta)=\sum_{k\in\mathbb{Z}^b\setminus\{0\}}\frac{\hat{\mu}_k}{\mathbf{i}k\cdot\omega}e^{\mathbf{i}k\cdot\theta}.$$

By (4.11) and (4.12), $U(\theta)$ is a well-defined analytic function in $\mathbb{T}^b_r$ and the transformation $u=e^Uv$ transforms (4.10) into

$$-\omega\cdot\partial_\theta v+(\lambda+\hat{\mu}_0)v=g(\theta),$$

where $g=e^{-U}p$. Using Fourier expansions, we see that the Fourier coefficients $\{\hat{v}_k\}$ of v satisfy

$$(-\mathbf{i}k\cdot\omega+\lambda+\hat{\mu}_0)\hat{v}_k=\hat{g}_k,\quad k\in\mathbb{Z}^b, \tag{4.16}$$

where $\{\hat{g}_k\}$ are Fourier coefficients of g. This yields a unique solution u of (4.10). Since, by conditions on λ,μ,p, $\overline{u(-\theta)}$ is also a solution, it follows from the uniqueness that $\overline{u(\theta)}=u(-\theta)$.

For any given $y\in\mathbb{R}^b$ with $|y|\le r$, let $\hat{\mu}(y),\hat{v}(y),\hat{p}(y),\hat{g}(y)$ be the Fourier coefficients vectors of u,v,p and g on $\mathbb{T}^b+y$, respectively, and let $\tilde{U}$ be the Töplitz matrix of U on $\mathbb{T}^b+y$. Then

$$\hat{g}(y)=\widehat{e^{-U}p(y)}=\widetilde{e^{-U}}(y)\hat{p}(y)=e^{-\tilde{U}(y)}\hat{p}(y).$$

Since $\mu(\theta)$ satisfies $\bar{\hat{\mu}}_k=-\hat{\mu}_{-k}$, we deduce that $\tilde{U}(0)$ is an anti-selfadjoint, bounded linear operator on $\ell^2(\mathbb{Z}^b)$. Hence $e^{\tilde{U}(0)}$ is a unitary operator on $\ell^2(\mathbb{Z}^b)$. It is also easy to see that $\tilde{U}(y)$ commutes with $\tilde{U}(0)$. As a result,

$$\|\hat{g}(y)\|_{\ell^2(\mathbb{Z}^b)}\le e^{\|\tilde{U}(y)-\tilde{U}(0)\|_{\ell^2(\mathbb{Z}^b)\to\ell^2(\mathbb{Z}^b)}}\|\hat{p}(y)\|_{\ell^2(\mathbb{Z}^b)}.$$

Since

$$(\tilde{U}(y)-\tilde{U}(0))_{kl}=\begin{cases}\dfrac{\hat{\mu}_{k-l}}{\mathbf{i}(k-l)\cdot\omega}(e^{-(k-l)\cdot y}-1), & k\ne l,\\ 0, & k=l,\end{cases}$$

we have by (4.11) and (4.12) that

$$\begin{aligned}\|\tilde{U}(y)-\tilde{U}(0)\|_{\ell^2(\mathbb{Z}^b)\to\ell^2(\mathbb{Z}^b)} &\le \sum_{k\in\mathbb{Z}^b}|(\tilde{U}(y)-\tilde{U}(0))_{k,0}|\\ &\le \sum_{k\in\mathbb{Z}^b\setminus\{0\}}\frac{|\hat{\mu}_k|}{|k\cdot\omega|}e^{|k|r}\cdot|k|r\le\frac{r}{\alpha_0}\|\mu\|_{r,\tau+1}\le\frac{r}{\alpha_0}C\gamma.\end{aligned}$$

Therefore,

$$\|\hat{g}(y)\|_{\ell^2(\mathbb{Z}^b)}\le e^{C\gamma r/\alpha_0}\|\hat{p}(y)\|_{\ell^2(\mathbb{Z}^b)}.$$

Taking $y=r'\kappa$, where κ is a b-vector whose entries are ±1 and $r'=r-\frac{\sigma}{3}$, we have

$$\|\hat{g}(r'\kappa)\|^2_{\ell^2(\mathbb{Z}^b)}\le e^{2C\gamma r/\alpha_0}\|\hat{p}(r'\kappa)\|^2_{\ell^2(\mathbb{Z}^b)}\le e^{2C\gamma r/\alpha_0}\sum_{k\in\mathbb{Z}^b}|\hat{p}_k|^2e^{2|k|r'}.$$

It follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^b}|\hat{g}_k|^2e^{2|k|r'} &\le \sum_{\kappa,k}|\hat{g}_ke^{-k\cdot r'\kappa}|^2=\sum_{\kappa}\|\hat{g}(r'\kappa)\|^2_{\ell^2(\mathbb{Z}^b)}\\ &\le 2^be^{2C\gamma r/\alpha_0}\sum_{k\in\mathbb{Z}^b}|\hat{p}_k|^2e^{2|k|r'}.\end{aligned}\tag{4.17}$$

Let $r''=r-\frac{2\sigma}{3}=r'-\frac{\sigma}{3}$. We have by (4.13), (4.16), and the fact $\sigma\le 1$ that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^b}|\hat{v}_k|^2e^{2|k|r''} &= \sum_{k\in\mathbb{Z}^b}\frac{|\hat{g}_k|^2}{|-\mathbf{i}k\cdot\omega+\lambda+\hat{\mu}_0|^2}e^{2|k|r''}\\ &\le \frac{1}{\alpha^2\gamma^2}\Big[\sup_{k\in\mathbb{Z}^b}(1+|k|^\tau)^2e^{-2|k|\sigma/3}\Big]\sum_{k\in\mathbb{Z}^b}|\hat{g}_k|^2e^{2|k|r'}\\ &\le \frac{1}{\alpha^2\gamma^2\sigma^{2\tau}}\Big[\sup_{t\ge0}(1+t^\tau)e^{-t\sigma/3}\sigma^\tau\Big]^2\sum_{k\in\mathbb{Z}^b}|\hat{g}_k|^2e^{2|k|r'}\\ &\le \frac{[1+(\frac{3\tau}{e})^\tau]^2}{\alpha^2\gamma^2\sigma^{2\tau}}\sum_{k\in\mathbb{Z}^b}|\hat{g}_k|^2e^{2|k|r'}.\end{aligned}\tag{4.18}$$

Hence by (4.17) and (4.18),

$$\begin{aligned}(\sum_{k\in\mathbb{Z}^b}|\hat{u}_k|^2e^{2|k|r''})^{\frac12} &\le 2^{\frac b2}e^{C\gamma r/\alpha_0}(\sum_{k\in\mathbb{Z}^b}|\hat{v}_k|^2e^{2|k|r''})^{\frac12}\\ &\le 2^b\frac{e^{2C\gamma r/\alpha_0}[1+(\frac{3\tau}{e})^\tau]}{\alpha\gamma\sigma^\tau}(\sum_{k\in\mathbb{Z}^b}|\hat{p}_k|^2e^{2|k|r'})^{\frac12}.\end{aligned}$$

(4.14) now follows from the following inequalities contained in [15, Page 1166]:

$$|u|_{r-\sigma}\le(1+e)^{\frac b2}(\frac{\sigma}{3})^{-\frac b2}(\sum_{k\in\mathbb{Z}^b}|\hat{u}_k|^2e^{2|k|r''})^{\frac12},$$

$$(\sum_{k\in\mathbb{Z}^b}|\hat{p}_k|^2e^{2|k|r'})^{\frac12}\le(1+e)^{\frac b2}(\frac{\sigma}{3})^{-\frac b2}|p|_r.$$

Using (2.9), we have

$$\begin{aligned}|u|_{r-\sigma,\tau+1} &\le |u|_{r-\frac{\sigma}{2}}\sum_{k\in\mathbb{Z}^b}e^{-|k|\frac{\sigma}{2}}\langle k\rangle^{\tau+1}\\ &= |u|_{r-\frac{\sigma}{2}}(1+\sum_{k\in\mathbb{Z}^b}e^{-|k|\frac{\sigma}{2}}|k|^{\tau+1})\\ &\le |u|_{r-\frac{\sigma}{2}}\Big(1+(\frac{\tau+1}{e})^{\tau+1}(\frac{4}{\sigma})^{\tau+b+1}(1+e)^b\Big)\\ &\le \frac{c(\tau,b)}{\alpha\gamma\sigma^{2\tau+2b+1}}e^{2C\gamma r/\alpha_0}|p|_{r,\tau+1},\end{aligned}$$

i.e., (4.15) holds. □

Lemma 4.2. *Consider the first-order partial differential equation*

$$-\omega\cdot\partial_\theta u+\lambda u+\Gamma_K(\mu(\theta)u)=\Gamma_Kp(\theta),\quad \theta\in\mathbb{T}^b_r,\tag{4.19}$$

where Γ_K *is the truncation operator defined in* (4.5) *with* $0 < K \le \frac{|\lambda|}{2}$, $\omega \in \Pi$, $\lambda \in \mathbf{i}\mathbb{R}$, *and* μ, p *are analytic functions on* $\mathbb{T}^b_r$ *such that* $\overline{\mu(\theta)} = -\mu(-\theta)$, $\overline{p(\theta)} = -p(-\theta)$, *and*

$$|\mu|_r = \sum_{k\in\mathbb{Z}^b} |\hat{\mu}_k| e^{|k|r} \le \frac{|\lambda|}{4\iota}$$

for some $\iota \ge 1$. *Then* (4.19) *admits a unique solution* $u(\theta)$ *satisfying* $u = \Gamma_K u$, *and there is a constant* $c(b) > 0$ *such that*

$$|u|_{r-\sigma} \le \frac{c(b)}{|\lambda|\sigma^b}|p|_r, \tag{4.20}$$

$$|(1-\Gamma_K)(\mu u)|_{r-\sigma} \le \frac{c(b)}{\iota\sigma^b} e^{-9K\sigma/10}|p|_r \tag{4.21}$$

for any $0 < \sigma < r$. *Moreover,* $\overline{u(\theta)} = u(-\theta)$ *and*

$$|u|_{r-\sigma,\tau+1} \le \frac{c(\tau,b)}{|\lambda|\sigma^{\tau+2b+1}}|p|_{r,\tau+1}, \tag{4.22}$$

$$|(1-\Gamma_K)(\mu u)|_{r-\sigma,\tau+1} \le \frac{c(\tau,b)}{\iota\sigma^{\tau+2b+1}} e^{-9K\sigma/20}|p|_{r,\tau+1} \tag{4.23}$$

for some constant $c(\tau, b) > 0$.

Proof. It is clear that a solution u of (4.19) satisfying $u = \Gamma_K u$ must have the form

$$u(\theta) = \sum_{|k|\le K} \hat{u}_k e^{\mathbf{i}k\cdot\theta}.$$

Substituting this trail solution into (4.19) yields

$$(\Lambda + \tilde{\mu})\hat{u} = \hat{p}, \tag{4.24}$$

where $\Lambda = diag(-\mathbf{i}k\cdot\omega + \lambda : |k| \le K)$, $\tilde{\mu} = (\hat{\mu}_{k-l})_{|k|,|l|\le K}$, $\hat{\mu} = (\hat{\mu}_k)_{|k|\le K}$, $\hat{p} = (\hat{p}_k)_{|k|\le K}$. Since for $|k| \le K \le \frac{|\lambda|}{2}$,

$$|-\mathbf{i}k\cdot\omega + \lambda| \ge \frac{|\lambda|}{2},$$

we see that Λ is invertible, hence (4.24) admits a unique solution which leads to a unique solution u of (4.19) satisfying $u = \Gamma_K u$. Since it is easy to see that $\overline{u(-\theta)}$ is also a solution, the uniqueness implies that $\overline{u(\theta)} = u(-\theta)$.

Denote $\Omega = diag(e^{|k|r'} : |k| \le K)$, where $r' = r - \frac{\sigma}{10}$. Then

$$\Omega\tilde{\mu}\Omega^{-1} = (\hat{\mu}_{k-l}e^{(|k|-|l|)r'})_{|k|,|l|\le K},$$

and hence

$$\|\Omega\tilde{\mu}\Omega^{-1}\|_{\ell^2(\mathbb{Z}^b)\to\ell^2(\mathbb{Z}^b)} \le \max_{|l|\le K}\sum_{|k|\le K}|\hat{\mu}_{k-l}|e^{|k-l|r'} \le \sum_{k\in\mathbb{Z}^b}|\hat{\mu}_k|e^{|k|r'} \le \frac{|\lambda|}{4\iota}.$$

On one hand, since $\iota \ge 1$, we have

$$\begin{aligned}
\|\Omega\hat{p}\|_{\ell^1(\mathbb{Z}^b)} &= \|\Omega(\Lambda+\tilde{\mu})\hat{u}\|_{\ell^1(\mathbb{Z}^b)} \ge \|\Omega\Lambda\hat{u}\|_{\ell^1(\mathbb{Z}^b)} - \|\Omega\tilde{\mu}\hat{u}\|_{\ell^1(\mathbb{Z}^b)} \\
&\ge \|\Lambda\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)} - \|\Omega\tilde{\mu}\Omega^{-1}\|_{\ell^1(\mathbb{Z}^b)\to\ell^1(\mathbb{Z}^b)}\|\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)} \\
&\ge \frac{|\lambda|}{2}\|\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)} - \frac{|\lambda|}{4\iota}\|\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)} \ge \frac{|\lambda|}{4}\|\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)}.
\end{aligned} \tag{4.25}$$

On the other hand, it follows from (2.9) that

$$\|\Omega\hat{p}\|_{\ell^1(\mathbb{Z}^b)} = \sum_{|k|\le K} |\hat{p}_k| e^{|k|r'} \le (\sum_{|k|\le K} e^{-|k|\sigma/10})|p|_r \le \frac{(20e+20)^b}{\sigma^b}|p|_r. \tag{4.26}$$

Combining (4.25) and (4.26) yields that

$$\begin{aligned} |u|_{r-\sigma} \le |u|_{r'} &\le \sum_{|k|\le K} |\hat{u}_k| e^{|k|r'} = \|\Omega\hat{u}\|_{\ell^1(\mathbb{Z}^b)} \le \frac{4}{|\lambda|}\|\Omega\hat{p}\|_{\ell^1(\mathbb{Z}^b)} \le \frac{4(20e+20)^b}{|\lambda|\sigma^b}|p|_r, \\ |(1-\Gamma_K)(\mu u)|_{r-\sigma} &\le \sum_{|k|>K} |\sum_{|l|\le K} \hat{\mu}_{k-l}\hat{u}_l| e^{|k|(r-\sigma)} < e^{-9K\sigma/10} \sum_{|k|>K} |\sum_{|l|\le K} \hat{\mu}_{k-l}\hat{u}_l| e^{|k|r'} \\ &\le e^{-9K\sigma/10} (\sum_{k\in\mathbb{Z}^b} |\hat{\mu}_k| e^{|k|r'})(\sum_{|k|\le K} |\hat{u}_k| e^{|k|r'}) \le \frac{(20e+20)^b}{\iota\sigma^b} e^{-9K\sigma/10}|p|_r, \end{aligned}$$

i.e., (4.20) and (4.21) hold. The proof of (4.22) and (4.23) follows from a similar argument as that for (4.15). □

Let

$$\Pi_+ = \{\omega \in \Pi : |k\cdot\omega + \mathbf{i}\lambda_{ij}| > \frac{\alpha|i^3 - j^3|}{1+|k|^\tau},\ 0 < |k| \le K,\ i,j \in \mathbb{Z}, i \ne j\}, \tag{4.27}$$

where

$$\lambda_{ij} = \lambda_i(\omega) - \lambda_j(\omega),\ \forall i,j \in \mathbb{Z}.$$

Proposition 4.1. *Under conditions* **(H1), (H2)** *the equation* (4.4) *admits a unique solution* $B : \mathbb{T}_r^b \times \Pi_+ \to \mathcal{G}$ *such that* $B(\theta,\omega)$ *is analytic, anti-selfadjoint, reversibility-preserving, real in* θ*, Lipschitz continuous in* ω*, and satisfies the following properties*

$$\overline{B(\theta,\omega)} = B(-\theta,\omega),\ (\theta,\omega) \in \mathbb{T}_r^b \times \Pi_+, \tag{4.28}$$

$$\|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{\epsilon^{\frac{9}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}. \tag{4.29}$$

Proof. Let $i,j \in \mathbb{Z}$ be given. Since $B_{i,i} = 0$, we only need to consider the case $i \ne j$.

In view of equations (4.8) and (4.9), we note that with $\lambda =: \lambda_i(\omega) - \lambda_j(\omega)$, $\mu =: \mu_i(\theta,\omega) - \mu_j(\theta,\omega)$, and $p =: -P_{i,j}(\theta,\omega)$, $u =: B_{i,j}$ satisfies (4.10) or (4.19) in the case $0 < |i^3 - j^3| < \frac{4K}{1+e_1}$ or $|i^3 - j^3| \ge \frac{4K}{1+e_1}$, respectively.

By Remark 4.1 (1), we also see that $\lambda \in \mathbf{i}\mathbb{R}$. Under the conditions **(H1), (H2)** and noting that $\hat{\mu}_0 = 0$, it is clear that all conditions of Lemmas 4.1, 4.2 are satisfied for $\omega \in \Pi_+$. It follows that $B_{i,j}$ exists, is unique, and $\overline{B_{i,j}(\theta,\omega)} = B_{i,j}(-\theta,\omega)$. Since it is easy to see that $B_{-i,-j}(-\theta,\omega)$ is also a solution, it follows from the uniqueness of solution that $B_{i,j}(\theta,\omega) = B_{-i,-j}(-\theta,\omega)$. Now by considering all $i,j \in \mathbb{Z}$ and applying Lemma 2.2 (2), we obtain a unique solution $B(\theta,\omega)$ of (4.4) on $\mathbb{T}_r^b$ for each $\omega \in \Pi_+$ which is reversibility-preserving and satisfies (4.28). It follows from Remark 2.2 (2) that $B(\theta,\omega)$ is real.

To show that $B(\theta,\omega)$ is Lipschitz-continuous in ω and satisfies the estimate in (4.29), we again consider fixed $i,j \in \mathbb{Z}$, $i \ne j$. We note from **(H1)** and (5.7), (5.9) that there exists a constant $c^* > 1$ such that

$$|\lambda_{ij} - \mathbf{i}(1+e_1)(i^3 - j^3)| \le (\langle i\rangle^\delta + \langle j\rangle^\delta)c^*\epsilon_*, \tag{4.30}$$
$$\|\mu_{ij}\|_{r,\tau+1} = |\mu_{ij}|_{r,\tau+1} \le (\langle i\rangle^\delta + \langle j\rangle^\delta)c^*\epsilon_*.$$

Denote

$$\gamma_{ij} = \langle i-j\rangle(\langle i\rangle^\delta + \langle j\rangle^\delta)c^*.$$

Then

$$\|\mu_{ij}\|_{r,\tau+1} = |\mu_{ij}|_{r,\tau+1} \le \epsilon_* \frac{\gamma_{ij}}{\langle i-j\rangle}. \tag{4.31}$$

It follows from Lemma 4.1 that

$$|B_{i,j}|_{r-2\sigma,\tau+1} \lessdot \frac{\langle i-j\rangle \big(e^{2\epsilon_*\gamma_{ij}(r-\sigma)/\alpha_0}\big)^{\frac{1}{\langle i-j\rangle}}}{\alpha\gamma_{ij}\sigma^{2\tau+2b+1}} |P_{i,j}|_{r-\sigma,\tau+1}. \tag{4.32}$$

By examining separate cases of $i=-j$, $i\neq -j$ (for $i=0, j\neq 0$; $i\neq 0, j=0$; and $i\neq 0, j\neq 0$) and using the facts that $c^*>1, \delta=2, e_1\approx 0$, it is easy to see that

$$2|i-j| \le \langle i-j\rangle(\langle i\rangle^2 + \langle j\rangle^2) = \frac{\gamma_{ij}}{c^*} \le 2|i^3-j^3| \le \frac{4|\lambda_{ij}|}{1+e_1}. \tag{4.33}$$

In the case $0 < |i^3 - j^3| < \frac{4K}{1+e_1}$, for fixed $i, j\in\mathbb{Z}$, we can deduce from (4.30) that $|\lambda_{ij}| \le \frac{3(1+e_1)}{2}|i^3-j^3| < 6K$. Using the facts that $\epsilon_* \ll \alpha_0, c^*>1, e_1\approx 0$, we have by (4.33) that

$$e^{2\epsilon_*\gamma_{ij}(r-\sigma)/\alpha_0} = e^{18\epsilon_*\gamma_{ij}\sigma/\alpha_0} \le e^{\frac{72c^*\epsilon_*|\lambda_{ij}|\sigma}{(1+e_1)\alpha_0}} < e^{\frac{432c^*\epsilon_*|\ln\epsilon|}{(1+e_1)\alpha_0}} < \epsilon^{-\frac{1}{20}}.$$

For sufficiently small $\epsilon_*>0$, it is easy to see that

$$\langle i-j\rangle\big(e^{2\epsilon_*\gamma_{ij}(r-\sigma)/\alpha_0}\big)^{\frac{1}{\langle i-j\rangle}} \le e^{2\epsilon_*\gamma_{ij}(r-\sigma)/\alpha_0} < \epsilon^{-\frac{1}{20}}.$$

Thus (4.32) can be further estimated as

$$|B_{i,j}|_{r-2\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\gamma_{ij}\sigma^{2\tau+2b+1}}|P_{i,j}|_{r-\sigma,\tau+1}. \tag{4.34}$$

In the case $|i^3-j^3| \ge \frac{4K}{1+e_1}$, we have by (4.33) that $K \le \frac{1+e_1}{4}|i^3-j^3| \le \frac{|\lambda_{ij}|}{2}$.

It follows from (4.31) that

$$\sum_{k\in\mathbb{Z}^b} |(\hat{\mu}_{ij})_k| e^{|k|r} \le |\mu_{ij}|_{r,\tau+1} \le \epsilon_*\frac{\gamma_{ij}}{\langle i-j\rangle} < \frac{|\lambda_{ij}|}{4\iota_*}, \tag{4.35}$$

where $\iota_* := \frac{(1+e_1)\langle i-j\rangle}{16c^*\epsilon_*}$. Hence by Lemma 4.2,

$$|B_{i,j}|_{r-2\sigma,\tau+1} \lessdot \frac{1}{|\lambda_{ij}|\sigma^{\tau+2b+1}}|P_{i,j}|_{r-\sigma,\tau+1} \lessdot \frac{1}{\gamma_{ij}\sigma^{\tau+2b+1}}|P_{i,j}|_{r-\sigma,\tau+1}, \tag{4.36}$$

$$|(1-\Gamma)(\mu_{ij}B_{i,j})|_{r-2\sigma,\tau+1} \lessdot \frac{\epsilon_* e^{-9K\sigma/20}}{\sigma^{\tau+2b+1}\langle i-j\rangle}|P_{i,j}|_{r-\sigma,\tau+1} \lessdot \frac{\epsilon_*\epsilon^{\frac{9}{20}}}{\sigma^{\tau+2b+1}\langle i-j\rangle}|P_{i,j}|_{r-\sigma,\tau+1}. \tag{4.37}$$

Combining (4.34) with (4.36) and applying Lemma 2.6, we have

$$\|B\|^\delta_{r-2\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+3b+1}}\|P\|^\delta_{r,\tau+1}. \tag{4.38}$$

Combining (4.34) with (4.36) again and using (4.33) as well as the estimate in the proof of Lemma 2.7, we have for each $i \neq j$ that

$$\begin{aligned} |(B\Lambda^{\frac{\delta}{d}})_{i,j}|_{r-2\sigma,\tau+1} &= |B_{i,j}|_{r-2\sigma,\tau+1}\langle j\rangle^{\delta} \\ &\lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+2b+1}}\frac{1}{\gamma_{ij}}|P_{i,j}|_{r-\sigma,\tau+1}\langle j\rangle^{\delta} \\ &\lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+2b+1}}\frac{1}{\langle i-j\rangle(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})}|P_{i,j}|_{r-\sigma,\tau+1}\langle j\rangle^{\delta} \\ &\lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+2b+1}}\frac{1}{|i-j|}|P_{i,j}|_{r-\sigma,\tau+1}. \end{aligned} \tag{4.39}$$

Hence by Lemma 2.6, (4.39) implies that

$$\|B\Lambda^{\frac{\delta}{d}}\|^{\delta}_{r-2\sigma,\tau+1} = \|\Lambda^{-\frac{\delta}{d}}B\Lambda^{\frac{\delta}{d}}\|_{r-2\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+3b+1}}\|P\|^{\delta}_{r,\tau+1}. \tag{4.40}$$

Combining (4.38), (4.40) and using (2.5), we immediately have

$$\|B\|^{\mathcal{G}}_{r-2\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+3b+1}}\|P\|^{\delta}_{r,\tau+1}. \tag{4.41}$$

Denote $\Delta B = B(\theta,\omega) - B(\theta,\omega')$ for fixed θ and $\Delta B_{i,j} = B_{i,j}(\omega) - B_{i,j}(\omega')$ for fixed i, j. Applying (4.8) and (4.9), we have

$$\begin{aligned} &-\partial_\omega \Delta B_{i,j} + \lambda_{ij}(\omega)\Delta B_{i,j} + \mu_{ij}(\omega)\Delta B_{i,j} \\ &\qquad = \partial_{\Delta\omega}B_{i,j}(\omega') - (\Delta\lambda_{ij}+\Delta\mu_{ij})B_{i,j}(\omega') - \Delta P_{i,j} := Q_{i,j} \end{aligned} \tag{4.42}$$

if $0 < |i^3 - j^3| < \frac{4K}{1+e_1}$, and

$$\begin{aligned} &-\partial_\omega \Delta B_{i,j} + \lambda_{ij}(\omega)\Delta B_{i,j} + \Gamma(\mu_{ij}(\omega)\Delta B_{i,j}) \\ &\qquad = \partial_{\Delta\omega}B_{i,j}(\omega') - \Gamma\Big((\Delta\lambda_{ij}+\Delta\mu_{ij})B_{i,j}(\omega') + \Delta P_{i,j}\Big) := T_{i,j} \end{aligned} \tag{4.43}$$

if $|i^3 - j^3| \geq \frac{4K}{1+e_1}$.

In the case $0 < |i^3 - j^3| < \frac{4K}{1+e_1}$, we apply (4.42) and use the same reasoning as (4.34) to conclude that

$$|\Delta B_{i,j}|_{r-4\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\gamma_{ij}\sigma^{2\tau+2b+1}}|Q_{i,j}|_{r-3\sigma,\tau+1}. \tag{4.44}$$

Since (4.33) implies that

$$\frac{1}{\gamma_{ij}} + \frac{\sigma(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})O(\epsilon_*)}{\gamma_{ij}} \leq \frac{1}{2c^*|i-j|} + \frac{\sigma(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})O(\epsilon_*)}{c^*\langle i-j\rangle(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})} < \frac{1}{2} + O(\epsilon_*), \tag{4.45}$$

we have

$$\begin{aligned} |Q_{i,j}|_{r-3\sigma,\tau+1} &\leq |\Delta\omega|\cdot\Big|\partial_\theta B_{i,j}(\omega')\Big|_{r-3\sigma,\tau+1} \\ &+ (|\lambda_{ij}|^{lip}_{r-3\sigma,\Pi_+} + |\mu_{ij}|^{lip}_{r-3\sigma,\tau+1,\Pi_+})|\Delta\omega|\cdot|B_{i,j}|_{r-3\sigma,\tau+1} + |\Delta P_{i,j}|_{r-3\sigma,\tau+1} \\ &\lessdot \frac{1}{\sigma}|\Delta\omega|\Big(1+\sigma(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})O(\epsilon_*)\Big)|B_{i,j}|_{r-2\sigma,\tau+1} + |\Delta P_{i,j}|_{r-2\sigma,\tau+1} \\ &\lessdot \frac{1}{\sigma}|\Delta\omega|\Big(1+\sigma(\langle i\rangle^{\delta}+\langle j\rangle^{\delta})O(\epsilon_*)\Big)\frac{\epsilon^{-\frac{1}{20}}}{\alpha\gamma_{ij}\sigma^{2\tau+2b+1}}|P_{i,j}|_{r-\sigma,\tau+1} + |\Delta P_{i,j}|_{r-2\sigma,\tau+1} \end{aligned}$$

$$\lessdot \frac{\epsilon^{-\frac{1}{20}}}{\alpha\sigma^{2\tau+2b+2}}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1}). \tag{4.46}$$

It then follows from (4.44) and (4.46) that

$$|\Delta B_{i,j}|_{r-4\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\gamma_{ij}\sigma^{4\tau+4b+3}}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1}). \tag{4.47}$$

In the case $|i^3-j^3| \geq \frac{4K}{1+e_1}$, we apply (4.43) and use the same reasoning as (4.36) to conclude that

$$|\Delta B_{i,j}|_{r-4\sigma,\tau+1} \lessdot \frac{1}{\gamma_{ij}\sigma^{\tau+2b+1}}|T_{i,j}|_{r-3\sigma,\tau+1}. \tag{4.48}$$

Applying (4.36) and (4.45), we have

$$\begin{aligned}
|T_{i,j}|_{r-3\sigma,\tau+1} &\lessdot \frac{1}{\sigma}|\Delta\omega|\cdot|B_{i,j}|_{r-2\sigma,\tau+1}\\
&+ (\langle i\rangle^{\delta}+\langle j\rangle^{\delta})O(\epsilon_*)|\Delta\omega|\cdot|B_{i,j}|_{r-2\sigma,\tau+1}+|\Delta P_{i,j}|_{r-2\sigma,\tau+1}\\
&\lessdot \frac{1}{\sigma}(\gamma_{ij}|\Delta\omega|\cdot|B_{i,j}|_{r-2\sigma,\tau+1}+|\Delta P_{i,j}|_{r-2\sigma,\tau+1})\\
&\lessdot \frac{1}{\sigma^{\tau+2b+2}}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1}). \qquad (4.49)
\end{aligned}$$

It then follows from (4.48) and (4.49) that

$$|\Delta B_{i,j}|_{r-4\sigma,\tau+1} \lessdot \frac{1}{\gamma_{ij}\sigma^{2\tau+4b+3}}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1}). \tag{4.50}$$

By Lemma 4.2, (4.35) and (4.49), we also have

$$\begin{aligned}
|(1-\Gamma)(\mu_{ij}\Delta B_{i,j})|_{r-4\sigma,\tau+1} &\lessdot \frac{e^{-9K\sigma/20}}{\frac{(1+e_1)\langle i-j\rangle}{16c^*\epsilon_*}\sigma^{2\tau+4b+3}}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1})\\
&\lessdot \frac{\epsilon_*\epsilon^{\frac{9}{20}}}{\sigma^{2\tau+4b+3}\langle i-j\rangle}(|\Delta\omega|\cdot|P_{i,j}|_{r-\sigma,\tau+1}+|\Delta P_{i,j}|_{r-\sigma,\tau+1}). \qquad (4.51)
\end{aligned}$$

In view of (4.47) and (4.50) and applying Lemma 2.6, we have

$$\|\Delta B\|^{\delta}_{r-4\sigma,\tau+1} \lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}(|\Delta\omega|\cdot\|P\|^{\delta}_{r,\tau+1}+\|\Delta P\|^{\delta}_{r,\tau+1}). \tag{4.52}$$

It now follows from (4.38) and (4.52) that

$$\begin{aligned}
\|B\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} &\lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}(2\|P\|^{\delta}_{r,\tau+1}+\|P\|^{\delta,lip}_{r,\tau+1,\Pi_+})\\
&\lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}\|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+}. \qquad (4.53)
\end{aligned}$$

Proceeding in the same way as (4.40), we also have

$$\|\Lambda^{-\frac{\delta}{d}}B\Lambda^{\frac{\delta}{d}}\|^{lip}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}\|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+}. \tag{4.54}$$

It now follows from (4.53) and (4.54) that

$$\|B\|^{\mathcal{G},lip}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}\|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+}. \tag{4.55}$$

Combining (4.55) and (4.41), we finally have

$$\|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{\epsilon^{-\frac{1}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}\|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+} \lessdot \frac{\epsilon^{\frac{9}{10}}}{\alpha^2\sigma^{4\tau+5b+3}}.$$

□

4.2. **Reversibility of the transformed system.** For simplicity, we will not specify the ω-dependency in B, A, P, A_+, P^+.

Lemma 4.3. *$e^{B(\theta)}$ is reversibility-preserving and real.*

Proof. By Proposition 4.1, B is reversibility-preserving. It follows from Lemma 2.2 (2) that

$$B_{i,j}(\theta) = B_{-i,-j}(-\theta), \ \forall i, j \in \mathbb{Z}.$$

It follows from Lemma 2.2 (2) again that B^n are reversibility-preserving for all $n \in \mathbb{Z}^+$, and consequently, e^B is reversibility-preserving.

Since by Remark 2.2 (2) B is real, we immediately conclude from the Taylor expansion of $e^{B(\theta)}$ that it is also real. □

Lemma 4.4. *Both $A_+(\theta)$ and $P^+(\theta)$ are reversible.*

Proof. We first note from **(H1), (H2)** that both $A(\theta)$ and $P(\theta)$ are reversible. It immediately follows that $A_+(\theta)$ is reversible.

To show the reversibility of $P^+(\theta)$, we first observe via repeated applications of Lemma 2.2 (1) that $R_{-i,-j}(-\theta) = -R_{i,j}(\theta)$ for all $i, j \in \mathbb{Z}$, and consequently, R is reversible. By Lemma 4.3, we also see that $\omega \cdot D_\theta B(\theta)$ and $\omega \cdot D_\theta e^{B(\theta)}$ are reversible. Hence by Lemma 2.1, each term in the expression of P^+, either as a reversible operator or a composition of a reversible operator with a reversibility-preserving operator, is reversible. Thus P^+ is reversible. □

Lemma 4.5. $\overline{A_+(-\theta)} = -A_+(\theta)$, $\overline{P^+(-\theta)} = -P^+(\theta)$.

Proof. Since $B(\theta)$ is anti-selfadjoint, both $e^{B(\theta)}$ and $e^{-B(\theta)}$ are unitary operators. It also follows from the reversibility-preserving property of $B(\theta)$ that $\overline{R(-\theta)} = -R(\theta)$, $\overline{\omega \cdot D_\theta B(-\theta)} = -\omega \cdot D_\theta B(\theta)$, and $\overline{\omega \cdot D_\theta e^{B(-\theta)}} = -\omega \cdot D_\theta e^{B(\theta)}$.

The lemma now follows easily from definitions of A_+, P^+ in (4.6), (4.7). □

Lemma 4.6. *$A_+(\theta)$ and $P^+(\theta)$ are real.*

Proof. It immediately follows from Lemmas 4.4, 4.5 and Lemma 2.2 (5). □

4.3. **Estimate of the transformed system.** We note that

$$A_+ = diag\Big(\cdots, \lambda_j^+(\omega) + \mu_j^+(\theta,\omega), \cdots\Big)_{j\in\mathbb{Z}},$$

where for each $j \in \mathbb{Z}$,

$$\lambda_j^+ = \lambda_j(\omega) + [P_{j,j}], \tag{4.56}$$

$$\mu_j^+ = \mu_j(\theta,\omega) + P_{j,j}(\theta,\omega) - [P_{j,j}]. \tag{4.57}$$

Lemma 4.7. *For each $j \in \mathbb{Z}$ and all $\omega \in \Pi_+$, we have $\lambda_j^+ \in \mathbf{i}\mathbb{R}$, $[\mu_j^+(\cdot,\omega)] = 0$, and*

$$\|\lambda_j^+ - \lambda_j^0\|_{\Pi_+} \lessdot \langle j\rangle^\delta \epsilon_*,$$
$$\|\lambda_j^+\|_{\Pi_+}^{lip} \lessdot \langle j\rangle^\delta \epsilon_*,$$
$$\|\mu_j^+\|_{r_+,\tau+1,\Pi_+}^{\mathcal{L}} \lessdot \langle j\rangle^\delta \epsilon_*.$$

Proof. For each $j \in \mathbb{Z}$ and $\omega \in \Pi_+$, we clearly have $[\mu_j^+(\cdot,\omega)] = 0$. Since $P(\cdot,\omega)$ satisfies **(H2)**, it follows from Lemma 2.2 (4) that $[P_{j,j}] \in \mathbf{i}\mathbb{R}$. Hence $\lambda_j^+ \in \mathbf{i}\mathbb{R}$. For simplicity, we will denote $(P^\nu)_{j,j}(\nu \geq 0)$ by $P^\nu_{j,j}$. Recall from (3.16) and (4.3) that

$$|P^0_{j,j}|^{\mathcal{L}}_{r_0,\tau+1,\Pi} \lessdot \langle j\rangle^2 \epsilon_*, \quad |P^\nu_{j,j}|^{\mathcal{L}}_{r,\tau+1,\Pi} \lessdot \langle j\rangle^\delta \epsilon_\nu \lll \cdot \langle j\rangle^\delta \epsilon_*, \quad \nu \geq 1.$$

The lemma now follows from **(H1)** and definitions of λ_j^+, μ_j^+. □

Next, we estimate the matrix R.

Lemma 4.8. *Assume*
(C1) $\epsilon^{\frac{1}{20}} < \sigma^{2\tau+5b+3}$,
(C2) $\frac{|\ln \epsilon|}{1+e_1} < \epsilon^{-\frac{1}{10}}$.
Then $\|R\|^{\delta,\mathcal{L}}_{r-3\sigma,\tau+1,\Pi_+} < \epsilon^{\frac{7}{5}}$.

Proof. Write $R = R^1 + R^2 + R^3$, where R^1 and R^2 have entries

$$(R^1)_{i,j} = \begin{cases} 0, & |i^3 - j^3| < \frac{4K}{1+e_1}, \\ (1-\Gamma)(\mu_{ij} B_{i,j}), & |i^3 - j^3| \geq \frac{4K}{1+e_1}, \end{cases}$$
$$(R^2)_{i,j} = \begin{cases} -(1-\Gamma)P_{i,j}, & 0 < |i^3 - j^3| < \frac{4K}{1+e_1}, \\ 0, & |i^3 - j^3| \geq \frac{4K}{1+e_1} \quad \text{or} \quad i = j, \end{cases}$$

and $R^3 = (1-\Gamma)(P - diag(P))$. We recall that $\mu_{ij} = \mu_i(\theta) - \mu_j(\theta)$ in the above.

For any function v on Π_+ and fixed $\omega, \omega' \in \Pi_+$, we denote $\Delta v = v(\omega) - v(\omega')$. Then for any given i, j, we clearly have

$$\Delta(\mu_{ij} B_{i,j}) = \mu_{ij}(\omega')\Delta B_{i,j} + (\Delta\mu_{ij})B_{i,j}(\omega).$$

By (4.51), to estimate $\Delta(\mu_{ij}B_{i,j})$, we only need to estimate $(1-\Gamma)\Big((\Delta\mu_{ij})B_{i,j}(\omega)\Big)$. Since

$$\sum_{|k|>K} e^{-2|k|\sigma} \lessdot \frac{e^{-K\sigma}}{\sigma^b}, \tag{4.58}$$

(4.36) and Lemma 2.5 yield that

$$\begin{aligned}
&\Big|(1-\Gamma)\Big((\Delta\mu_{ij})B_{i,j}(\omega)\Big)\Big|_{r-4\sigma,\tau+1} \lessdot \sum_{|k|>K} e^{-2|k|\sigma} |(\Delta\mu_{ij})B_{i,j}(\omega)|_{r-2\sigma,\tau+1} \\
\lessdot\ & \frac{e^{-K\sigma}}{\sigma^b} |\Delta\mu_{ij}|_{r-2\sigma,\tau+1} \cdot |B_{i,j}|_{r-2\sigma,\tau+1} \lessdot \frac{\varepsilon}{\sigma^b} |\Delta\mu_{ij}|_{r-2\sigma,\tau+1} \cdot |B_{i,j}|_{r-2\sigma,\tau+1} \\
\lessdot\ & \frac{\varepsilon}{\gamma_{ij}\sigma^{\tau+3b+1}} |\Delta\mu_{ij}|_{r-2\sigma,\tau+1} \cdot |P_{i,j}|_{r-\sigma,\tau+1} \\
\lessdot\ & \frac{\varepsilon}{\gamma_{ij}\sigma^{\tau+3b+1}} |\mu_{ij}|^{lip}_{r-2\sigma,\tau+1} |\Delta\omega| \cdot |P_{i,j}|_{r-\sigma,\tau+1}
\end{aligned}$$

$$\lessdot \frac{\epsilon_* \varepsilon}{\sigma^{\tau+3b+1}} |\Delta\omega| \cdot \frac{|P_{i,j}|_{r-\sigma,\tau+1}}{|i-j|}. \tag{4.59}$$

Combining (4.37), (4.51), (4.59), **(C1)** and applying Lemma 2.6, we have

$$\|R^1\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{\epsilon_* \varepsilon^{\frac{9}{20}}}{\sigma^{2\tau+5b+3}} \|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+} \lessdot \frac{\epsilon_* \varepsilon^{\frac{29}{20}}}{\sigma^{2\tau+5b+3}} \lessdot \epsilon_* \varepsilon^{\frac{7}{5}}. \tag{4.60}$$

To estimate R^2, we consider $\eta(K) := \max\{|i-j| : 0 < |i^3 - j^3| < \frac{4K}{1+e_1}\}$. Obviously, $\eta(K) < \frac{4K}{1+e_1}$. On one hand, due to (4.58), we have

$$\begin{aligned} |(R^2)_{i,j}|_{r-3\sigma,\tau+1} &= |(1-\Gamma)P_{i,j}|_{r-3\sigma,\tau+1} = |\sum_{|k|>K} (\hat{P}_{i,j})_k e^{\mathbf{i}k\cdot\theta}|_{r-3\sigma,\tau+1} \\ &\le \sum_{|k|>K} |(\hat{P}_{i,j})_k| e^{|k|(r-3\sigma)} \langle k \rangle^{\tau+1} \le |P_{i,j}|_{r-\sigma,\tau+1} \sum_{|k|>K} e^{-2|k|\sigma} \\ &\lessdot \frac{e^{-K\sigma}}{\sigma^b} |P_{i,j}|_{r-\sigma,\tau+1} \lessdot \frac{\varepsilon \eta(K)}{\sigma^b} \frac{|P_{i,j}|_{r-\sigma,\tau+1}}{|i-j|}. \end{aligned} \tag{4.61}$$

Since by **(C1)**,

$$\varepsilon^{\frac{1}{10}} \ll \sigma^{2b+1},$$

we have by Lemma 2.6 and **(C2)** that

$$\|R^2\|^{\delta}_{r-3\sigma,\tau+1} \lessdot \frac{\varepsilon K}{\sigma^{2b}(1+e_1)} \|P\|^{\delta}_{r,\tau+1} \lessdot \frac{\varepsilon \varepsilon^{-\frac{1}{10}}}{\sigma^{2b+1}} \|P\|^{\delta}_{r,\tau+1} < \varepsilon^{\frac{4}{5}} \|P\|^{\delta}_{r,\tau+1}. \tag{4.62}$$

Using estimates similar to (4.61), we also have

$$|\Delta(R^2)_{i,j}|_{r-3\sigma,\tau+1} = |(1-\Gamma)\Delta P_{i,j}|_{r-3\sigma,\tau+1} \lessdot \frac{e^{-K\sigma}}{\sigma^b} |\Delta P_{i,j}|_{r-\sigma,\tau+1} \lessdot \frac{\varepsilon \eta(K)}{\sigma^b} \frac{|\Delta P_{i,j}|_{r-\sigma,\tau+1}}{|i-j|}.$$

Hence,

$$\|\Delta R^2\|^{\delta}_{r-3\sigma,\tau+1} < \varepsilon^{\frac{4}{5}} \|\Delta P\|^{\delta}_{r,\tau+1}. \tag{4.63}$$

It follows from (4.62) and (4.63) that

$$\|R^2\|^{\delta,\mathcal{L}}_{r-3\sigma,\tau+1,\Pi_+} < \varepsilon^{\frac{4}{5}} \|P\|^{\delta,\mathcal{L}}_{r,\tau+1,\Pi_+} < \varepsilon^{\frac{9}{5}}. \tag{4.64}$$

To estimate R^3, we note that

$$\begin{aligned} |(R^3)_{i,j}|_{r-3\sigma,\tau+1} &= |(1-\Gamma)(P_{i,j} - (diag(P))_{i,j})|_{r-3\sigma,\tau+1} \\ &= |(1-\Gamma)P_{i,j}|_{r-3\sigma,\tau+1} = |(R^2)_{i,j}|_{r-3\sigma,\tau+1}, \end{aligned}$$

for $0 < |i^3 - j^3| < \frac{4K}{1+e_1}$.

Hence the estimate of R^3 in this case follows from similar arguments as that of R^2. In the case $|i^3 - j^3| \ge \frac{4K}{1+e_1}$, we have

$$|(R^3)_{i,j}|_{r-3\sigma,\tau+1} \le |P_{i,j}|_{r,\tau+1} \sum_{|k|>K} e^{-3|k|\sigma} \lessdot \frac{e^{-\frac{3}{2}K\sigma}}{\sigma^b} |P_{i,j}|_{r,\tau+1} = \cdot \frac{\epsilon^{\frac{3}{2}}}{\sigma^b} |P_{i,j}|_{r,\tau+1}.$$

Consequently,

$$||R^3||^{\delta}_{r-3\sigma,\tau+1} \lessdot \frac{\epsilon^{\frac{3}{2}}}{\sigma^b} ||P||^{\delta}_{r,\tau+1} \ll \epsilon^{\frac{4}{5}} ||P||^{\delta}_{r,\tau+1}.$$

Finally, estimates in ω similar to those for R^2 yield that

$$\|R^3\|^{\delta,\mathcal{L}}_{r-3\sigma,\tau+1,\Pi_+} < \varepsilon^{\frac{9}{5}}. \tag{4.65}$$

The proof is complete by combining estimates (4.60), (4.64) and (4.65). □

Lemma 4.9. *Under the condition* **(C1)***,*

$$\|e^{-B}Ae^{B} - A - [A,B]\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{1}{\sigma}\varepsilon^{\frac{12}{7}} < \varepsilon^{\frac{23}{14}}.$$

Proof. Using the identity

$$e^{-B}Ae^{B} - A - [A,B] = \int_0^1\int_0^s e^{-s_1B}[[A,B],B]e^{s_1B}ds_1ds,$$

where $[A,B] = AB - BA$, we only need to estimate $[A,B]$ and $[[A,B],B]$. From (4.4), (5.2) and Lemma 2.7, we deduce that

$$\begin{aligned}
\|[A,B]\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} &\le \|\frac{dB}{d\tilde{t}}\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} + 2\|P\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} + \|R\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+}\\
&\lessdot \frac{b}{\sigma}\|B\|^{\mathcal{G},\mathcal{L}}_{r-3\sigma,\tau+1,\Pi_+} + 2\|P\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} + \|R\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+}\\
&\lessdot \frac{b}{\sigma}\varepsilon^{\frac{6}{7}} + 2\varepsilon + \varepsilon^{\frac{7}{5}} \lessdot \frac{b}{\sigma}\varepsilon^{\frac{6}{7}},\\
\|[[A,B],B]\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} &\lessdot \|[A,B]\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+}\cdot\|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{b}{\sigma}\varepsilon^{\frac{12}{7}}.
\end{aligned}$$

By **(C1)** and the fact that $b > 1$, we have

$$\epsilon < \sigma^{20(2\tau+5b+3)} \ll \sigma^{14}. \tag{4.66}$$

The proof follows. □

Lemma 4.10. $\|e^{-B}Pe^{B} - P\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \varepsilon^{\frac{13}{7}}$.

Proof. For simplicity, in the proof below we omit the suffix $\{r-4\sigma,\tau+1,\Pi_+\}$ in the norm $\|\cdot\|_{r-4\sigma,\tau+1,\Pi_+}$. Consider the function $P(t) = e^{-tB}Pe^{tB}$. Then

$$e^{-B}Pe^{B} - P = P(1) - P(0), \tag{4.67}$$

$$P'(t) = -[B,P(t)], \quad P(0) = P. \tag{4.68}$$

Hence by Lemma 2.7, we have

$$\|P'(t)\|^{\delta} \lessdot \|B\|^{\mathcal{G}}\|P(t)\|^{\delta}$$

which yields that

$$\|P(t)\|^{\delta} \le e^{\cdot\|B\|^{\mathcal{G}}t}\|P\|^{\delta}.$$

By (4.68), we also have

$$P(t) - P = -\int_0^t [B,P(s)]ds.$$

Applying (4.67), Lemma 2.7 and using the fact that $0 \le r \le 1$ and $\|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} < \varepsilon^{\frac{6}{7}}$ as in (5.2), we have

$$\begin{aligned} \|e^{-B}Pe^{B} - P\|^{\delta} &= \| - \int_0^1 [B, P(s)]ds\|^{\delta} \le \|[B, P(s)]\|^{\delta} \\ &\lessdot \|B\|^{\mathcal{G}}\|P(s)\|^{\delta} \lessdot \|B\|^{\mathcal{G}} e^{\cdot\|B\|^{\mathcal{G}}s}\|P\|^{\delta} \\ &\lessdot \|B\|^{\mathcal{G}}\|P\|^{\delta}. \end{aligned} \tag{4.69}$$

For any two points $\omega, \omega' \in \Pi$, denote

$$\Delta P(t) = P(\omega)(t) - P(\omega')(t) = e^{-tB(\omega)}P(\omega)e^{tB(\omega)} - e^{-tB(\omega')}P(\omega')e^{tB(\omega')}.$$

Then

$$(\Delta P)^{\cdot} = -[\Delta B, P(\omega)] - [B(\omega'), \Delta P]. \tag{4.70}$$

By Lemma 2.7, we have

$$\|(\Delta P(t))^{\cdot}\|^{\delta} \lessdot \|\Delta B\|^{\mathcal{G}} \cdot \|P(t)\|^{\delta} + \cdot\|B\|^{\mathcal{G}} \cdot \|\Delta P(t)\|^{\delta}.$$

It follows from the Gronwall's inequality that

$$\begin{aligned} \|\Delta P(t)\|^{\delta} &\lessdot \|\Delta B\|^{\mathcal{G}}\|P(t)\|^{\delta} \cdot e^{\cdot\|B\|^{\mathcal{G}}t} \\ &\lessdot \|\Delta B\|^{\mathcal{G}}\|P\|^{\delta} \cdot e^{\cdot\|B\|^{\mathcal{G}}t} \lessdot \|\Delta B\|^{\mathcal{G}} \cdot \|P\|^{\delta}. \end{aligned}$$

Note that (4.70) also implies that

$$\Delta P(t) - \Delta P(0) = -\int_0^t ([\Delta B, P(\omega)(s)] + [B(\omega'), \Delta P(s)])ds.$$

We thus have

$$\begin{aligned} \Delta(e^{-B}Pe^{B} - P) &= \Delta(P(1) - P(0)) = \Delta P(1) - \Delta P(0) \\ &= -\int_0^1 ([\Delta B, P(\omega)(s)] + [B(\omega'), \Delta P(s)])ds. \end{aligned}$$

By Lemma 2.7, it follows that

$$\|e^{-B}Pe^{B} - P\|^{\delta,lip} \lessdot \|B\|^{\mathcal{G},lip} \cdot \|P\|^{\delta} + \cdot\|B\|^{\mathcal{G}} \cdot \|P\|^{\delta,lip}. \tag{4.71}$$

Combining (4.69) with (4.71), we obtain

$$\|e^{-B}Pe^{B} - P\|^{\delta,\mathcal{L}} \lessdot \|B\|^{\mathcal{G},\mathcal{L}} \cdot \|P\|^{\delta,\mathcal{L}} \lessdot \varepsilon^{\frac{13}{7}}.$$

The proof follows. □

Lemma 4.11. $\|\omega \cdot D_\theta B - e^{-B}\omega \cdot D_\theta e^{B}\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lessdot \frac{1}{\sigma}\varepsilon^{\frac{12}{7}} < \epsilon^{\frac{23}{14}}$.

Proof. Denote $\partial_\omega = \omega \cdot D_\theta$. Then

$$\frac{d}{d\lambda}(\partial_\omega(e^{\lambda B})) = \partial_\omega(e^{\lambda B})B + e^{\lambda B}\partial_\omega B.$$

Consider the function $Q(\lambda) = e^{-\lambda B}\partial_\omega(e^{\lambda B})$, $0 \le \lambda \le 1$. It is clear that $Q(\lambda)$ satisfies

$$\frac{d}{d\lambda}Q(\lambda) = [Q(\lambda), B] + \partial_\omega B, \quad Q(0) = \partial_\omega(1) = 0$$

which yields that

$$Q(\lambda) = Q(\lambda) - Q(0) = \int_0^\lambda [Q(\lambda'), B] d\lambda' + (\partial_\omega B)\lambda. \tag{4.72}$$

Hence by Lemma 2.7, we have

$$\begin{aligned} \|Q(\lambda)\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} &\lessdot \|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \int_0^\lambda \|Q(\lambda')\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} d\lambda' \\ &+ \|\partial_\omega B\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \lambda. \end{aligned}$$

Applying the Gronwall's inequality and using the facts that $\|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} < \varepsilon^{\frac{6}{7}}$ as in (5.2) and $0 \le \lambda \le 1$, we have

$$\|Q(\lambda)\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \le \|\partial_\omega B\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \lambda \cdot e^{\cdot \|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \lambda} \le 2 \|\partial_\omega B\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \lambda.$$

Since, by (4.72),

$$\partial_\omega B - e^{-B} \partial_\omega (e^B) = \partial_\omega B - Q(1) = -\int_0^1 [Q(\lambda'), B] d\lambda',$$

we have by (4.66) and Lemma 2.7 that

$$\begin{aligned} \|\partial_\omega B - e^{-B} \partial_\omega (e^B)\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} &\lessdot \|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \int_0^1 \|Q(\lambda')\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} d\lambda' \\ &\lessdot \|B\|^{\mathcal{G},\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \|\partial_\omega B\|^{\delta,\mathcal{L}}_{r-4\sigma,\tau+1,\Pi_+} \cdot \lambda' \\ &\lessdot \frac{b}{\sigma} (\|B\|^{\mathcal{G},\mathcal{L}}_{r-3\sigma,\tau+1,\Pi_+})^2 \lessdot \frac{1}{\sigma} \varepsilon^{\frac{12}{7}} < \epsilon^{\frac{23}{14}}. \end{aligned}$$

The proof follows. □

Proposition 4.2. *Assume* **(C1)**, **(C2)** *and let* $\varepsilon_+ = \varepsilon^{\frac{4}{3}}, r_+ = \frac{r}{2}$. *If*

(C3) $r_+ < r - 4\sigma$,

then

$$\|P^+\|^{\delta,\mathcal{L}}_{r_+,\tau+1,\Pi_+} < \varepsilon_+.$$

Proof. The proof follows from Lemmas 4.8, 4.9, 4.10 and 4.11. □

5. Reduction Theorem

For each $\nu = 0, 1, \cdots$, we label all index-free quantities in Section 4 by ν and all "+"-labeled quantities by $\nu + 1$. This yields the following sequences:

$$\begin{aligned} \epsilon_\nu &= \epsilon_0^{(\frac{4}{3})^\nu}, \quad r_\nu = \frac{r_0}{2^\nu}, \quad \sigma_\nu = \frac{r_\nu}{10}, \quad K_\nu = \frac{|\ln \epsilon_\nu|}{\sigma_\nu}, \quad \alpha_\nu = \frac{\alpha_0}{2^\nu}, \\ \dot{\chi} &= (A_\nu(\omega,\theta) + P^\nu(\omega,\theta))\chi, \ \dot{\theta} = \omega, \ (\chi,\theta) \in \ell^2_s(\mathbb{Z}) \times \mathbb{T}^b_{r_\nu}, \ \omega \in \Pi_\nu, \\ A_\nu &= diag(\cdots, \lambda^\nu_j(\omega) + \mu^\nu_j(\theta,\omega), \cdots)_{j\in\mathbb{Z}}, \\ \Pi_{\nu+1} &= \{\omega \in \Pi_\nu : |k\cdot\omega + \mathbf{i}\lambda^\nu_{ij}| > \frac{\alpha_\nu |i^3 - j^3|}{1+|k|^\tau}, \ 0 < |k| \le K_\nu, \ i,j \in \mathbb{Z}, i \ne j\}, \end{aligned} \tag{5.1}$$

where $\lambda^\nu_{ij} = \lambda^\nu_i - \lambda^\nu_j$, $i, j \in \mathbb{Z}$, $\nu = 0, 1, \cdots$, $\epsilon_0, r_0, \sigma_0, \alpha_0$ are as in (3.1), and Π_0 is as in (3.4).

5.1. Iterative Lemma.

Lemma 5.1. *The following holds for all* $\nu = 0, 1, \cdots$.

(a) A_ν, P^ν *satisfy conditions* **(H1)**,**(H2)**.

(b) *There exists* $B^\nu : \mathbb{T}^b_{r_\nu - 4\sigma_\nu} \times \Pi_{\nu+1} \mapsto \mathcal{G}$, *which is anti-selfadjoint, real, reversibility-preserving for fixed* $\omega \in \Pi_{\nu+1}$, *analytic in* $\theta \in \mathbb{T}^b_{r_\nu - 4\sigma_\nu}$, *Lipschitz-continuous in* $\Pi_{\nu+1}$, *satisfying*

$$\|B^\nu\|^{\mathcal{G},\mathcal{L}}_{r_\nu - 4\sigma_\nu, \tau+1, \Pi_{\nu+1}} < \varepsilon_\nu^{\frac{6}{7}} \tag{5.2}$$

such that the unitary transformation $e^{-B^\nu(\theta,\omega)}$ *transforms* $(5.1)_\nu$ *to* $(5.1)_{\nu+1}$.

Proof. It is easy to see by the choice of r_ν, σ_ν that the condition **(C3)** holds for all $\nu = 0, 1, \cdots$, also by the choice of $\epsilon_0, r_0, \sigma_0$ and the fact $\varsigma = 50(4\tau + 5b + 3) > 50(2\tau + 5b + 3)$ that conditions **(C1)**, **(C2)** hold for $\nu = 0$, if ϵ_* is sufficiently small. Since

$$\epsilon_0^{\frac{1}{20}(\frac{4}{3})^\nu - \frac{1}{50}} < \frac{1}{(10 \cdot 2^\nu)^{2\tau+5b+3}}, \quad \nu \geq 0$$

and $|\ln \epsilon_\nu| - (1 + e_1)\epsilon_\nu^{-\frac{1}{10}}$ decreases in ν, if ϵ_* is sufficiently small, then conditions **(C1)**, **(C2)** also hold for all $\nu = 1, \cdots$. It follows that the KAM step described in the previous section is valid for all $\nu = 0, 1, \cdots$. Hence (a) holds.

The existence of the operator B^ν for each ν follows from Proposition 4.1. Moreover, by (4.29), we have

$$\|B^\nu\|^{\mathcal{G},\mathcal{L}}_{r_\nu - 4\sigma_\nu, \tau+1, \Pi_{\nu+1}} \lessdot \frac{\epsilon_\nu^{\frac{9}{10}}}{\alpha_\nu^2 \sigma_\nu^{4\tau+5b+3}}.$$

We note that $\sigma_\nu = \frac{\epsilon_0^{\frac{1}{\varsigma}}}{10 \cdot 2^\nu}$ for all $\nu \geq 0$. Since ϵ_0 is sufficiently small and $\varsigma = 50(4\tau + 5b + 3)$, we have

$$\epsilon_0^{\frac{3}{70}(\frac{4}{3})^\nu - \frac{1}{25}} < \frac{1}{4^\nu} \frac{1}{(10 \cdot 2^\nu)^{4\tau+5b+3}}, \quad \nu \geq 0.$$

It follows that

$$\frac{\epsilon_\nu^{\frac{9}{10}}}{\alpha_\nu^2 \sigma_\nu^{4\tau+5b+3}} \ll \epsilon_\nu^{\frac{6}{7}}, \quad \nu \geq 0.$$

Hence (5.2) holds and (b) follows. □

5.2. Measure estimates.

Let $\Pi_\infty = \bigcap_{\nu \geq 0} \Pi_\nu$.

Proposition 5.1. $|\Pi_* \setminus \Pi_\infty| = O(\alpha_0) = O(\epsilon_*^{\frac{1}{100}})$.

Proof. We note from (4.27) that

$$\Pi_* \setminus \Pi_\infty = (\Pi_* \setminus \Pi_0) \bigcup \Big(\bigcup_{\nu=1}^{\infty} \bigcup_{i,j \in \mathbb{Z}, i \neq j, |k| \leq K_\nu} \mathcal{R}^\nu_{kij} \Big) \subset (\Pi_* \setminus \Pi_0) \bigcup \Big(\bigcup_{\nu=1}^{\infty} \bigcup_{i,j \in \mathbb{Z}, i \neq j, k \in \mathbb{Z}^b} \mathcal{R}^\nu_{kij} \Big),$$

where

$$\mathcal{R}^\nu_{kij} = \Big\{ \omega \in \Pi_\nu : |k \cdot \omega + \mathbf{i}\lambda^\nu_{ij}| \leq \frac{\alpha_\nu |i^3 - j^3|}{1 + |k|^\tau} \Big\}, \ i, j \in \mathbb{Z}, \ i \neq j, \ k \in \mathbb{Z}^b.$$

For each $i, j \in \mathbb{Z}$ with $i \neq j$, we have by (4.33) that $|\lambda_{ij}^\nu| \geq \frac{1+e_1}{2}|i^3 - j^3|$. It follows that if $|k| \leq \frac{1+e_1}{4}|i^3 - j^3|$, then

$$|k \cdot \omega + \mathbf{i}\lambda_{ij}^\nu| \geq \frac{1+e_1}{4}|i^3 - j^3| \gg \frac{\alpha_\nu |i^3 - j^3|}{1 + |k|^\tau}.$$

Thus,

$$\bigcup_{i \neq j, k \in \mathbb{Z}^b} \mathcal{R}_{kij}^\nu \subset \Big\{ \omega \in \Pi_\nu : |k \cdot \omega + \mathbf{i}\lambda_{ij}^\nu| < \frac{4\alpha_\nu |k|}{(1+e_1)(1+|k|^\tau)} \Big\},\ i, j \in \mathbb{Z},\ i \neq j,\ k \in \mathbb{Z}^b.$$

Since $\tau > b$, we have

$$|\bigcup_{i \neq j, k \in \mathbb{Z}^b} \mathcal{R}_{kij}^\nu| \lessdot \sum_{k \in \mathbb{Z}^b} \frac{\alpha_\nu}{1 + |k|^\tau} \lessdot \alpha_\nu = \cdot \frac{\alpha_0}{2^\nu}$$

and consequently,

$$|\bigcup_{\nu=1}^{\infty} \bigcup_{i \neq j, k \in \mathbb{Z}^b} \mathcal{R}_{kij}^\nu| = O(\alpha_0).$$

It now follows from the standard measure estimate that

$$|\Pi_* \setminus \Pi_0| = O(\alpha_0),$$

i.e., the proposition holds. □

5.3. **Convergence.** Let

$$U^\nu = U^\nu(\theta, \omega) := e^{B^0(\theta,\omega)} e^{B^1(\theta,\omega)} \cdots e^{B^\nu(\theta,\omega)}, \quad \theta \in \mathbb{T}_{r_{\nu+1}}^b,\ \omega \in \Pi_{\nu+1},\ \nu = 0, 1, \cdots.$$

By Lemma 5.1, for each ν, $U^\nu(\theta, \omega) : \ell_s^2(\mathbb{Z}) \to \ell_s^2(\mathbb{Z})$ is unitary, uniformly bounded, real and reversibility-preserving for fixed ω, and depends on θ analytically and on ω Lipschitz-continuously.

Proposition 5.2. *There exist $U^\infty(\theta, \omega) : \ell_s^2(\mathbb{Z}) \to \ell_s^2(\mathbb{Z}), (\theta, \omega) \in \mathbb{T}^b \times \Pi_\infty$, functions $\lambda_j^\infty(\omega)$, μ_j^∞, $j \in \mathbb{Z}$, which depend on θ C^∞-smoothly and on ω Lipschitz-continuously, such that the following hold:*

(1) *$U^\infty(\theta, \omega)$ is unitary, uniformly bounded, real and reversibility-preserving for fixed ω, and satisfies $\|U^\infty - Id\|_{0,\tau+1,\Pi_\infty}^{\mathcal{G},\mathcal{L}} \lessdot \epsilon_*^{\frac{6}{7}}$;*

(2) *For all $j \in \mathbb{Z}$, $\lambda_j^\infty \in \mathbf{i}\mathbb{R}$, $[\mu_j^\infty(\cdot, \omega)] = 0$, $\omega \in \Pi_\infty$, and*

$$\lambda_j^\infty = \mathbf{i}(1+e_1)j^3 + j^2[\hat{b}_0^2] - \mathbf{i}j[\hat{b}_0^3] - [\hat{b}_0^4] + \langle j \rangle^\delta O(\epsilon_*^{\frac{4}{3}}); \quad \mu_j^\infty = \langle j \rangle^\delta O(\epsilon_*).$$

(3) *As $\nu \to 0$,*

$$\|U^\nu - U^\infty\|_{0,\tau+1,\Pi_\infty}^{\mathcal{G},\mathcal{L}},\ \|\lambda_j^\nu - \lambda_j^\infty\|_{\Pi_\infty}^{lip},\ \|\mu_j^\nu - \mu_j^\infty\|_{0,\tau+1,\Pi_\infty}^{\mathcal{L}} \to 0 \text{ uniformly in } j \in \mathbb{Z};$$

(4) *The transformation $\chi = U^\infty(\theta, \omega)\zeta$ transforms (3.14) into*

$$\dot{\zeta}(\tilde{t}) = A_\infty(\theta, \omega)\zeta(\tilde{t}), \quad \dot{\theta} = \omega, \tag{5.3}$$

where

$$A_\infty(\theta, \omega) = diag(\cdots, , \lambda_j^\infty(\omega) + \mu_j^\infty(\theta, \omega), \cdots)_{j \in \mathbb{Z}}.$$

Proof. We note that for any $M, N : \ell^2_s(\mathbb{Z}) \to \ell^2_s(\mathbb{Z})$,

$$e^M - e^N = e^N\Big(\int_0^1 e^{\lambda(M-N)} d\lambda\Big)(M-N). \tag{5.4}$$

It follows from (5.2) that

$$\begin{aligned} &\|U^\nu - Id\|^{\mathcal{G},\mathcal{L}}_{r_\nu-4\sigma_\nu,\tau+1,\Pi_{\nu+1}} \\ \le\ & \int_0^1 \exp(\lambda\sum_{\nu\ge0}\|B^\nu(\theta)\|^{\mathcal{G},\mathcal{L}}_{r_\nu-4\sigma_\nu,\tau+1,\Pi_{\nu+1}})d\lambda\sum_{\nu\ge0}\|B^\nu(\theta)\|^{\mathcal{G},\mathcal{L}}_{r_\nu-4\sigma_\nu,\tau+1,\Pi_{\nu+1}} \\ <\ & \exp(\sum_{\nu\ge0}\epsilon_0^{\frac{6}{7}(\frac{4}{3})^\nu})\sum_{\nu\ge0}\epsilon_0^{\frac{6}{7}(\frac{4}{3})^\nu} \lessdot \epsilon_*^{\frac{6}{7}}. \end{aligned} \tag{5.5}$$

By (5.4), we also have

$$U^{\nu+1} - U^\nu = U^\nu\Big(\int_0^1 e^{\lambda B^{\nu+1}} d\lambda\Big)B^{\nu+1}.$$

It follows from (5.2) and (5.5) that

$$\begin{aligned} &\|U^{\nu+1} - U^\nu\|^{\mathcal{G},\mathcal{L}}_{r_{\nu+1}-4\sigma_{\nu+1},\tau+1,\Pi_{\nu+2}} \\ \le\ & \|U^\nu\|^{\mathcal{G},\mathcal{L}}_{r_\nu-4\sigma_\nu,\tau+1,\Pi_{\nu+1}}\Big(\int_0^1 e^{\lambda\|B^{\nu+1}\|^{\mathcal{G},\mathcal{L}}_{r_{\nu+1}-4\sigma_{\nu+1},\tau+1,\Pi_{\nu+2}}} d\lambda\Big)\|B^{\nu+1}\|^{\mathcal{G},\mathcal{L}}_{r_{\nu+1}-4\sigma_{\nu+1},\tau+1,\Pi_{\nu+2}} \\ \lessdot\ & (1 + \|U^\nu - Id\|^{\mathcal{G},\mathcal{L}}_{r_\nu-4\sigma_\nu,\tau+1,\Pi_{\nu+1}})\|B^{\nu+1}\|^{\mathcal{G},\mathcal{L}}_{r_{\nu+1}-4\sigma_{\nu+1},\tau+1,\Pi_{\nu+2}} \\ \lessdot\ & (1 + \cdot\epsilon_*^{\frac{6}{7}})\epsilon_{\nu+1}^{\frac{6}{7}} \lessdot \epsilon_*^{\frac{6}{7}(\frac{4}{3})^{\nu+1}}. \end{aligned}$$

Thus as $\nu \to \infty$, $\{U^\nu\}$ converges uniformly on $\mathbb{T}^b \times \Pi_\infty$. It follows that the limit U^∞ satisfies all desired properties. In particular, its C^∞-smoothness in θ follows from arguments in [15, Pgs. 1159, 1163].

For each ν, we have by Lemma 5.1 that P^ν satisfies **(H2)**, i.e.,

$$\|P^\nu\|^{\delta,\mathcal{L}}_{r_\nu,\tau+1,\Pi_\nu} \le \epsilon_\nu = \epsilon_0^{(\frac{4}{3})^\nu}.$$

Hence, $P^\nu \to 0$ and $\sum_{k=0}^\nu P^k$ converges as $\nu \to \infty$ uniformly on $\mathbb{T}^b \times \Pi_\infty$. Since from (4.56) and (4.57), for each $j \in \mathbb{Z}$,

$$\lambda_j^\nu = \lambda_j^0 + \sum_{k=0}^{\nu}[P^k_{j,j}], \tag{5.6}$$

$$\mu_j^\nu = \sum_{k=0}^{\nu}\big(P^k_{j,j} - [P^k_{j,j}]\big) \tag{5.7}$$

for $\nu \ge 1$ and

$$\lambda_j^0 = \mathbf{i}(1+e_1)j^3, \tag{5.8}$$

$$\mu_j^0 = 0, \tag{5.9}$$

we see that λ_j^ν, μ_j^ν, converge, as $\nu \to \infty$, to some $\lambda_j^\infty, \mu_j^\infty$ respectively, uniformly in $\theta \in \mathbb{T}^b$, $\omega \in \Pi_\infty$, $j \in \mathbb{Z}$. The limits $\lambda_j^\infty, \mu_j^\infty$ are easily seen to satisfy the desired properties.

The proof is complete by applying Lemma 5.1 (b) and passing limit $\nu \to \infty$ in $(5.1)_\nu$. □

5.4. **Reducibility.** Let

$$\tilde{\rho}(\phi, \omega) = 1 + \omega \cdot \alpha_\phi(\phi).$$

By tracing back the transformations (3.9), (3.10), the limit system (5.3) can be re-written as

$$\begin{cases} \chi' = \Lambda(\phi, \omega)\chi, & \chi \in \ell_s^2(\mathbb{Z}), \\ \phi' = \omega, & \phi \in \mathbb{T}^b, \end{cases} \tag{5.10}$$

where $' = \frac{d}{dt}$.

$$\Lambda(\phi, \omega) = \tilde{\rho}(\phi, \omega) A_\infty(\mathcal{T}(\phi), \omega) = diag(\cdots, \eta_j(\phi, \omega), \cdots)_{j\in\mathbb{Z}},$$

$$\eta_j(\phi, \omega) = \tilde{\rho}(\phi, \omega)(\lambda_j^\infty(\omega) + \mu_j^\infty(\mathcal{T}(\phi), \omega)), \quad j \in \mathbb{Z}.$$

For each $j \in \mathbb{Z}$ and $\omega \in \Pi_\infty$, expanding η_j into the Fourier series in ϕ yields that

$$\eta_j(\phi, \omega) = \eta_j^0(\omega) + \sum_{k\in\mathbb{Z}^b\setminus\{0\}} \eta_j^{(k)}(\omega) e^{\mathbf{i}k\cdot\phi}.$$

Let

$$m_j(\phi, \omega) = \sum_{k\in\mathbb{Z}^b\setminus\{0\}} \frac{\eta_j^{(k)}(\omega)}{\mathbf{i}k \cdot \omega}(e^{\mathbf{i}k\cdot\phi} - 1)$$

and

$$\mathcal{M}(\phi, \omega) = diag(\cdots, e^{m_j(\phi,\omega)}, \cdots)_{j\in\mathbb{Z}}.$$

Then $\mathcal{M}(\phi, \omega) : \ell_s^2(\mathbb{Z}) \to \ell_s^2(\mathbb{Z})$, $(\phi, \omega) \in \mathbb{T}^b \times \Pi_\infty$, depends on ϕ C^∞-smoothly and on ω Lipschitz-continuously, and it is clearly invertible and uniformly bounded.

Proposition 5.3. *For each $\omega \in \Pi_\infty$, $\mathcal{M}(\phi, \omega)$ is reversibility-preserving and real. Moreover, the transformation $\chi = \mathcal{M}(\phi, \omega)z$ transforms* (5.10) *into*

$$\begin{cases} z' = \Lambda_*(\omega)z, & z \in \ell_s^2(\mathbb{Z}), \\ \phi' = \omega, & \phi \in \mathbb{T}^b, \end{cases} \tag{5.11}$$

where

$$\Lambda_*(\omega) = diag(\cdots, \eta_j^0(\omega), \cdots)_{j\in\mathbb{Z}}$$

with

$$\eta_j^0(\omega) = \mathbf{i}(1 + e_1)j^3 + \mathbf{i}e_2 j^2 - \mathbf{i}e_3 j - \mathbf{i}e_4, \qquad j \in \mathbb{Z} \tag{5.12}$$

for some $e_i = e_i(\omega) = O(\epsilon_*) \in \mathbb{R}$, $i = 1, 2, 3, 4$, *which are Lipschitz continuous in* $\omega \in \Pi_\infty$.

Proof. A straightforward verification shows that $\mathcal{M}(\phi, \omega)$ transforms (5.10) into (5.11).

For fixed $\omega \in \Pi_\infty$, since $\mathcal{T}$ is odd, ρ is real and even, and $A_\infty(\theta, \omega)$ is reversible and real, we see that $\Lambda(\phi, \omega)$ is reversible and real. Hence by Lemma 2.2 (1), (3), $\eta_j^{(k)} = -\eta_{-j}^{(-k)}$, $\eta_j^{(k)} = \overline{\eta_{-j}^{(-k)}}$, $j \in \mathbb{Z}, k \in \mathbb{Z}^b$. It follows that

$$\frac{\eta_j^{(k)}}{\mathbf{i}k \cdot \omega} = \frac{\eta_{-j}^{(-k)}}{\mathbf{i}(-k) \cdot \omega}, \quad \frac{\eta_j^{(k)}}{\mathbf{i}k \cdot \omega} = \overline{\left(\frac{\eta_{-j}^{(-k)}}{\mathbf{i}(-k) \cdot \omega}\right)}, \qquad j \in \mathbb{Z}, \;\; k \in \mathbb{Z}^b \setminus \{0\}.$$

By Lemma 2.2 (2), (3), $\mathcal{M}(\phi, \omega)$ is reversibility-preserving and real.

For each $j \in \mathbb{Z}$, since by Lemma 4.7, $[P^k_{j,j}] \in \mathbf{i}\mathbb{R}, k = 0, 1, \cdots$, and $|P^k_{j,j}|^{\mathcal{L}}_{r,\tau+1,\Pi} \lessdot \langle j\rangle^\delta \epsilon_k, k = 0, 1, \cdots$, we have that $[P^k_{j,j}] = \mathbf{i}\langle j\rangle^\delta \tilde{\epsilon}_k$ for some $\tilde{\epsilon}_k \in \mathbb{R}, \tilde{\epsilon}_k \lessdot \epsilon_*^{(\frac{4}{3})^k}, k = 0, 1, \cdots$.

It follows from (5.6), (5.8) and Proposition 5.2 (2) that

$$\lambda^\infty_j(\omega) = \mathbf{i}(1 + e_1)j^3 + j^2[\hat{b}^2_0] - \mathbf{i}j[\hat{b}^3_0] - [\hat{b}^4_0] + \langle j\rangle^2 \mathbf{i}(\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_\infty). \tag{5.13}$$

Using facts that $[\tilde{\rho}(\cdot, \omega)] = 1$, $[\mu^\infty_j] = 0$, $[(\omega \cdot \alpha_\phi(\phi)) \cdot \mu^\infty_j] = 0$, the desired asymptotic orders of $\eta^0_j(\omega)$, $j \in \mathbb{Z}$ in (5.12), follow from (5.13) by taking

$$e_1 = [a_1], \ \ e_2 = -\mathbf{i}[\hat{b}^2_0], \ \ e_3 = [\hat{b}^3_0], \ \ e_4 = -\mathbf{i}[\hat{b}^4_0] - (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_\infty), \quad j \in \mathbb{Z}, |j| \le 1;$$

$$e_1 = [a_1], \ \ e_2 = -\mathbf{i}[\hat{b}^2_0] + (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_\infty), \ \ e_3 = [\hat{b}^3_0], \ \ e_4 = -\mathbf{i}[\hat{b}^4_0], \quad j \in \mathbb{Z}, \ |j| > 1.$$

Since by (3.15), $|\hat{b}^k_0|^{\mathcal{L}}_{r_0,\tau+1,\Pi_0} \lessdot \epsilon_*, k = 2, 3, 4$, and ϵ_* can be chosen sufficiently small, the proof is complete. □

5.5. **Proof of Main Theorem.** Denote $\mathcal{I} : \mathcal{H}^s(\mathbb{T}) \to \ell^2_s(\mathbb{Z})$: $u(x) = \sum_{j\in\mathbb{Z}} \hat{u}_j e^{\mathbf{i}jx} \mapsto (\hat{u}_j)_{j\in\mathbb{Z}}$ as the isometry, and let

$$U_\omega(\phi) = \mathcal{I}^{-1} \circ \mathcal{M}(\phi, \omega) \circ \mathcal{T}^{-1}(\phi) \circ U^\infty \circ \mathcal{I} \circ \mathcal{T}(\phi), \quad \phi \in \mathbb{T}^b, \ \ \omega \in \Pi_\infty.$$

It is clear that $w(t, x) = U_\omega(\phi)u(t, x) \in \mathcal{H}^s(\mathbb{T})$, $\omega \in \Pi_\infty$, is the family of transformations satisfying all desired properties stated in the Main Theorem. In particular, asymptotic orders of coefficients e_i's of (1.7) follow from that of e_i's in Proposition 5.3. □

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