STOCHASTIC STABILITY OF MEASURES IN GRADIENT SYSTEMS

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ABSTRACT. Stochastic stability of a compact invariant set of a finite dimensional, dissipative system is studied in [20] for general white noise perturbations. In particular, it is shown under some Lyapunov conditions that the global attractor of the systems is always stable under general noise perturbations and any strong local attractor in it can be stabilized by a particular family of noise perturbations. Nevertheless, not much is known about the stochastic stability of an invariant measure in such a system. In this paper, we will study the issue of stochastic stability of invariant measures with respect to a finite dimensional, dissipative gradient system with potential function f. As we will show, a special property of such a system is that it is the set of equilibria which is stable under general noise perturbations and the set S_f of global minimal points of f which is stable under additive noise perturbations. For stochastic stability of invariant measures in such a system, we will characterize two cases of f, one corresponding to the case of finite S_f and the other one corresponding to the case when S_f is of positive Lebesgue measure, such that either some combined Dirac measures or the normalized Lebesgue measure on S_f is stable under additive noise perturbations. However, we will show by constructing an example that such measure stability can fail even in the simplest situation, i.e., in 1-dimension there exists a potential function f such that S_f consists of merely two points but no invariant measure of the corresponding gradient system is stable under additive noise perturbations. Crucial roles played by multiplicative and additive noise perturbations to the measure stability of a gradient system will also be discussed. In particular, the nature of instabilities of the normalized Lebesgue measure on S_f under multiplicative noise perturbations will be exhibited by an example in 2-dimension.

1. INTRODUCTION

Consider a system of ordinary differential equations

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $V = (V^i) \in C(\mathbb{R}^n, \mathbb{R}^n)$. Adding multiplicative including additive white noise $G(x)\dot{W}$ to (1.3), we obtain the following Itô stochastic differential equations

$$\mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \quad x \in \mathbb{R}^n, \tag{1.2}$$

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where W is a standard m-dimensional Brownian motion for some integer $m \geq n$ and $G = (g^{ij})_{n\times m}$ is a matrix-valued function on $\mathbb{R}^n$.

Regarded as a physical model, system (1.1) is often subject to white noise perturbations either from its surrounding environment or from intrinsic uncertainties of a coupling system due to high complexity, large degree of freedom, lack of full knowledge of mechanisms, the need for organizing a large amount of data, etc. Suppose that (1.1) generates a local flow on $\mathbb{R}^n$. Analyzing the impact of noise perturbations on the dynamics of the system is a fundamental issue with respect to both modeling and dynamics. The study of this fundamental issue from a distribution point of view naturally gives rise to the analysis of limit behaviors of stationary measures of the Fokker-Planck equations associated with (1.2) as $G \to 0$ under an appropriate topology.

More precisely, consider noise coefficient matrices lying in the class

$$\tilde{\mathcal{G}} = \{G = (g^{ij}) : \mathrm{Rank}(G) \equiv n,\, g^{ij} \in W^{1,2p}_{loc}(\mathbb{R}^n),\, i = 1, 2, \cdots, n,\, j = 1, 2, \cdots, m\}$$

for some fixed $p > n$. The class $\tilde{\mathcal{G}}$ gives rise to the following class of diffusion matrices:

$$\tilde{\mathcal{A}} = \{A = (a^{ij}) \in W^{1,p}_{loc}(\mathbb{R}^n, GL(n,\mathbb{R})) :\ A = \frac{GG^\top}{2} \text{ for some } G \in \tilde{\mathcal{G}}\}.$$

For each $A = \frac{GG^\top}{2} = (a^{ij}) \in \tilde{\mathcal{A}}$, the stationary process generated by (1.2) is described by its corresponding stationary measures $\{\mu_A\}$ which are measure-valued solutions of the stationary Fokker-Planck equation associated with (1.2) (see Section 2.1 for details).

Let $\tilde{\mathcal{A}}$ be furnished with the uniform topology of $C(\mathbb{R}^n)$ and consider an *admissible null family* $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, i.e., $\mathcal{A}$ is a directed net with $A_\alpha \to 0$, and the Fokker-Planck equation corresponding to each $A_\alpha \in \mathcal{A}$ admits a stationary measure. Stability of dynamics of (1.1) under the noise family $\mathcal{A}$ can be characterized by the behaviors of $\mathcal{A}$-*limit measures*, i.e., sequential limit points of all stationary measures $\{\mu_{A_\alpha}\}$ of the Fokker-Planck equations corresponding to $\mathcal{A} = \{A_\alpha\}$, as $A_\alpha \to 0$, in the space $M(\mathbb{R}^n)$ of Borel probability measures on $\mathbb{R}^n$ endowed with the weak*-topology. We recall from [20] that a compact invariant set Ω of (1.1) is said to be $\mathcal{A}$-*stable* if for any $\epsilon > 0$ and any open neighborhood W of Ω there exists a $\delta > 0$ such that $\mu_{A_\alpha}(\mathbb{R}^n \setminus W) < \epsilon$ whenever $|A_\alpha| < \delta$. An invariant measure μ of (1.1) is said to be $\mathcal{A}$-*stable* if any sequence of $\{\mu_{A_\alpha}\}$ converges to μ in $M(\mathbb{R}^n)$ as $A_\alpha \to 0$, i.e., μ is the only $\mathcal{A}$-limit measure.

$\mathcal{A}$-stability of a compact invariant set of (1.1) has been extensively investigated in [20]. In particular, it is shown in [20] that if (1.1) admits a Lyapunov function whose second derivatives are bounded, then its global attractor is $\mathcal{A}$-stable with respect to any null family $\mathcal{A}$, and moreover, if the global attractor contains a strong local attractor then there is an admissible null family $\mathcal{A}$ such that the local attractor is $\mathcal{A}$-stable. To the contrary, if the global attractor contains a strong local repeller, then there is an admissible null family $\mathcal{A}$ such that the local repeller is *strongly* $\mathcal{A}$-*unstable*, i.e., no $\mathcal{A}$-limit measure can be concentrated on the repeller. Moreover, if the repeller is a so-called strongly repelling equilibrium, then it is strongly $\mathcal{A}$-unstable with respect to any so-called normal null family $\mathcal{A}$.

In contrast to the case of compact invariant sets, not much is known about the $\mathcal{A}$-stability of invariant measures of (1.1). Of course, if an $\mathcal{A}$-stable compact invariant set is uniquely ergodic, then it is clear that the unique invariant measure is also $\mathcal{A}$-stable. In general, stochastic stability of a non-ergodic invariant measure is much harder to be characterized. We mention some well-known studies in this regard for flows on a 2-torus ([22]), flows on a circle ([26]), flows whose ω-limit sets consist of a finite number of fixed points and periodic orbits ([27]), and flows on a compact manifold admitting SRB measures ([23], see also [11, 28] for the case of random perturbations of maps on a compact manifold).

In this paper, we pay particular attention to the stochastic stability of compact invariant sets and invariant measures of a gradient system

$$\dot{x} = -\nabla f(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

where $f \in C^2(\mathbb{R}^n)$.

As to be seen in this paper, not only does the stochastic stability of compact invariant sets of (1.3) has very special natures, but also the stochastic stability of invariant measures of (1.3) can be characterized in various situations. Besides analyzing the impact of noise perturbations on a gradient system, the study of stochastic stability of invariant measures in a gradient system is closely related to the problem of ergodicity when taking the thermodynamic limit in a huge particle system ([1]) and the problem of noise stabilizing a multi-stable gradient system ([10]).

It is well-known that if the gradient system (1.3) admits a weak Lyapunov function then it always generates a positive semiflow in $\mathbb{R}^n$. Another special property of a gradient system is that when it is dissipative, its global attractor is typically simple by consisting of equilibria together with connecting orbits among them. However, it follows from general results of [20] that noise perturbations can remove all the connecting orbits among the equilibria in the global attractor of a dissipative gradient system. More precisely, if (1.3) admits a C^2 Lyapunov function whose second derivative is bounded, then a) the set E of critical points of f is $\mathcal{A}$-stable with respect to any null family $\mathcal{A}$; b) any finite set $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) of isolated local minimal (maximal) points of f is $\mathcal{A}$-stable (resp. strongly $\mathcal{A}$-unstable) with respect to a particular null family $\mathcal{A}$; c) any simple local maximal point of f is strongly $\mathcal{A}$-unstable with respect to any normal null family $\mathcal{A}$ (see Theorem 2.1 for details). When either the set E or $\mathcal{J}_0$ is a singleton, the $\mathcal{A}$-stability of the corresponding Dirac measure obviously follows from that of the E or $\mathcal{J}_0$ with respect to all or a particular null family $\mathcal{A}$.

However, stochastic stability of (1.3) has very special natures under the additive white noise perturbation $\sqrt{2\epsilon}\dot{W}$ to (1.3), where $\epsilon > 0$ is a small parameter. Under the condition that

H) there are positive constants R and β such that $f(x) \geq \beta \log |x|$ for all $|x| \geq R$,

the Fokker-Planck equation corresponding to

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}W, \quad x \in \mathbb{R}^n \tag{1.4}$$

for each $\epsilon > 0$ admits a stationary measure μ_ϵ, called *Gibbs measure*, with density

$$u^\epsilon(x) = k_\epsilon \mathrm{e}^{-\frac{f(x)}{\epsilon}}, \qquad x \in \mathbb{R}^n,$$

called *Gibbs state*, where

$$k_\epsilon = \frac{1}{\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{f(x)}{\epsilon}} \mathrm{d}x}.$$

Limit behaviors of (continuum) Gibbs measures have been explicitly investigated in many situations (see, e.g., [3, 21]). These limit behaviors lead to various $\mathcal{A}_0$-stability results of (1.3), where $\mathcal{A}_0$ denote the family of diffusion matrices $\{\epsilon I\}$ corresponding to the additive noise perturbations. More precisely, it follows from the limit characterizations of Gibbs measures in [21] that if the condition **H)** holds, then the set S_f of all global minimal points of f is $\mathcal{A}_0$-stable (see Theorem 2.2 a)). This result actually reveals a special stochastic stability feature of (1.3) under additive noise perturbations because it implies that any compact invariant set of (1.3) consisting of relative minimal points of f as well as any invariant measure on the set will not be $\mathcal{A}_0$-stable though by Theorem 2.1 they can be $\mathcal{A}$-stable with respect to some null family $\mathcal{A}$ of multiplicative noise perturbations.

If S_f is a singleton, then the stability of S_f under additive noise perturbation implies that of its Dirac measure. In the case that S_f is not a singleton, $\mathcal{A}_0$-stability of a broader class of non-ergodic invariant measures on S_f can also be studied using existing convergence results of Gibbs measures ([3, 21]). It turns out that such measure stability depends on the nature of absolute minimal points of f. For instance, there are $\mathcal{A}_0$-stable invariant measures in the following two special cases:

C1) $f \in C^2(\mathbb{R}^n)$, $S_f = \{x_1, x_2, \cdots, x_m\}$ for some $m \in \mathbb{N}$, and near each x_i, $i = 1, 2, \cdots, m$, f has the form

$$f(x) = P_i(x - x_i) + o(|x - x_i|^{2k_i}), \qquad x \in B_i := B_\delta(x_i), \tag{1.5}$$

where P_i is a homogeneous polynomial of order $2k_i$ for some natural number k_i and $B_\delta(x_i)$ is the open δ-ball centered at x_i for some $\delta > 0$.

C2) S_f admits positive Lebesgue measure.

More precisely, in the case **C1)**, the $\mathcal{A}_0$-stable measure has the form $\sum_{i=1}^m c_i \delta_{x_i}$, where, for each $i = 1, \cdots, m$, δ_{x_i} denotes the Dirac measure at x_i and c_i depends on P_i, and in the case **C2)**, the $\mathcal{A}_0$-stable measure is precisely the normalized Lebesgue measure μ_f on S_f (see Theorem 2.2 b), c)). We note that (1.5) holds automatically when $n = 1$. In general, when S_f consists of a finite number of points and each point is an isolated minimum point of f, it is only know that the leading term of the Taylor expansion near each of these points must be of even order.

Stochastic stability of an invariant measure is a much restricted property than that of a compact invariant set. One of the main goals of this paper is to show, even in the simplest situation, that if an $\mathcal{A}$-stable compact invariant set of a finite dimensional flow is non-uniquely ergodic, then it can occur that none of its invariant measures is $\mathcal{A}$-stable. We will demonstrate such an instability phenomenon by giving a 1-dimensional example under additive noise perturbation such that the condition in **C1)** fails in the simplest case $m = 2$ (Note that if $m = 1$ then the Dirac measure on S_f is always $\mathcal{A}_0$-stable). Main features of this example are summarized as follows.

Theorem 1. (Measure instability under additive noise) *In the case $n = 1$ there exists a potential function $f \in C^2(\mathbb{R})$ for which the following properties hold under additive noise perturbation $\sqrt{2\epsilon}\dot{W}$.*

a) *S_f consists of two points x_1, x_2 but neither is an isolated critical point of f.*
b) *The corresponding Fokker-Planck equation admits no stationary measures other than the Gibbs measures $\{\mu_\epsilon\}$.*
c) *There are two sequences $\epsilon_k^1, \epsilon_k^2 \to 0$ such that $\mu_{\epsilon_k^i} \to \lambda_i \delta_{x_1} + (1-\lambda_i)\delta_{x_2}$ as $k \to \infty$, for $i = 1, 2$ respectively, where $\lambda_1, \lambda_2 \in (0,1)$ are two distinct constants. Consequently, no invariant measure on S_f is $\mathcal{A}_0$-stable.*

Theorem 1 is also a counter example when the condition in **C2)** fails. When the condition in **C2)** holds, we note that the stability of the normalized Lebesgue measure on S_f can be however destroyed by a multiplicative noise perturbation, indicating an impact of multiplicative noise perturbations on the instability of non-singular measures. We will give an example in 2-dimension to demonstrate such an instability phenomenon. Main features of this example are summarized as follows.

Theorem 2. (Measure instability under multiplicative noise) *In the case $n = 2$ there exists a potential function $f \in C^2(\mathbb{R}^2)$ satisfying the condition in* **C2)** *and infinitely many null families $\mathcal{A}$ satisfying the following properties:*
a) *S_f is $\mathcal{A}$-stable.*
b) *Any $\mathcal{A}$-limit measure differs from the normalized Lebesgue measure μ_f on S_f, i.e., μ_f is $\mathcal{A}$-unstable.*

This paper is organized as follows. In Section 2, we review some basic results on stationary measures and stochastic stability, and give some results on the stochastic stability of compact invariant sets of (1.3) under both multiplicative and additive noise perturbations. In Section 3, we construct a 1-dimensional example having the features described in Theorem 1 whose Gibbs measures are not convergent as $\epsilon \to 0$. In Section 4, we construct an example of 2-dimensional gradient system having the features described in Theorem 2.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote the absolute value of a number, the norm of a vector or a matrix, and the Lebesgue measure of a set.

2. Stability under additive v.s. multiplicative noises

In this section, we will discuss stochastic stability of invariant sets and invariant measures of (1.3) under both additive and multiplicative noise perturbations. Through the rest of the section, for simplicity, we will use short notions $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \cdots, n$ whenever applicable.

2.1. **Lyapunov functions and stationary measures.** Adding multiplicative white noise $G(x)\dot{W}$ to (1.3), where W is a standard m-dimensional Brownian motion for some integer $m \geq n$ and $G = (g^{ij})_{n\times m}$ is a matrix-valued function on $\mathbb{R}^n$, we obtain the following Itô stochastic differential equations

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t + G(x)\mathrm{d}W, \quad x \in \mathbb{R}^n. \tag{2.1}$$

Let $A = \frac{GG^\top}{2} = (a^{ij})$ and consider the adjoint Fokker-Planck operator

$$\mathcal{L}_A = a^{ij}\partial_{ij}^2 - \partial_i f \partial_i$$

corresponding to A. A *stationary measure corresponding to* $\mathcal{L}_A$ is a stationary measure of the Fokker-Planck equation associated with (2.1), i.e., a Borel probability measure μ satisfying

$$\int_{\mathbb{R}^n} \mathcal{L}_A F(x)\mathrm{d}\mu(x) = 0, \qquad \forall F \in C_0^\infty(\mathbb{R}^n). \tag{2.2}$$

A *weak stationary solution corresponding to* $\mathcal{L}_A$ is a weak stationary solution of the Fokker-Planck equation associated with (2.1), i.e., a continuous function u on $\mathbb{R}^n$ satisfying

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}_A F(x)u(x)\mathrm{d}x = 0, \qquad \forall F \in C_0^\infty(\mathcal{U}), \\ u(x) \geq 0, \quad \int_{\mathbb{R}^n} u(x)\mathrm{d}x = 1. \end{cases} \tag{2.3}$$

Corresponding to $\mathcal{L}_A$, if u is a weak stationary solution, then it is clear that μ with $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$ is a stationary measure. Conversely, by the regularity theorem in [4], if $A \in \tilde{\mathcal{A}}$ then a stationary measure μ corresponding to $\mathcal{L}_A$ must be regular in the sense that $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$ for a density function $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ which is a weak stationary solution corresponding to $\mathcal{L}_A$. We note that if $a^{ij} \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$, $i, j = 1, \cdots, n$, and $f \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$, for some $\alpha \in (0, 1)$, then it follows from the standard Schauder theory that a weak stationary solution corresponding to $\mathcal{L}_A$ becomes a classical solution of the stationary Fokker-Planck equation

$$\begin{cases} \partial^2_{ij}(a^{ij}u) + \partial_i(\partial_i f u) = 0, \\ u(x) \geq 0, \quad \int_{\mathbb{R}^n} u(x)\mathrm{d}x = 1 \end{cases} \tag{2.4}$$

associated with (2.1).

We recall from [16, 17] that a non-negative continuous function U on $\mathbb{R}^n$ is a *compact function* if $U(x) < \rho_M$, $x \in \mathbb{R}^n$, and $\lim_{x\to\infty} U(x) = \rho_M$, where $\rho_M = \sup_{x\in\mathbb{R}^n} U(x) \leq \infty$. It is easy to see that a non-negative function U is an unbounded compact function if and only if $\lim_{x\to\infty} U(x) = +\infty$. A C^2 compact function U on $\mathbb{R}^n$ is called a *Lyapunov function with respect to* (2.1) *or* $\mathcal{L}_A$ if there is a constant $\rho_m < \rho_M$ and a constant $\gamma > 0$, called *Lyapunov constant of* U, such that

$$\mathcal{L}_A U(x) \leq -\gamma, \qquad \text{whenever } U(x) > \rho_m. \tag{2.5}$$

Let $\mathcal{A} = \{A_\alpha\}$ be a null family. Recall from [19] that a compact function $U \in C^2(\mathbb{R}^n)$ is a *uniform Lyapunov function with respect to the family* $\{\mathcal{L}_{A_\alpha}\}$ if (2.5) holds for constants ρ_m, γ that are independent of $A_\alpha \in \mathcal{A}$.

The concept of Lyapunov function above generalizes the usual one for a deterministic system. Recall that a C^1 compact function U on $\mathbb{R}^n$ is called a *Lyapunov function of* (1.3) if there exist constants $0 \leq \rho_m < \rho_M, \gamma > 0$ such that

$$-\nabla f(x) \cdot \nabla U(x) \leq -\gamma, \qquad \text{whenever } U(x) > \rho_m. \tag{2.6}$$

If (2.6) holds with $\gamma = 0$ and some $\rho_m < \rho_M$, then U is called a *weak Lyapunov function* of (1.3). A weak Lyapunov function U is called an *entire weak Lyapunov function* of (1.3) if $\rho_m = 0$ in the above. In fact, the notions of compact, Lyapunov, weak Lyapounov, and entire weak Lyapunov functions can be introduced on any connected open subset of $\mathbb{R}^n$ instead of the entire $\mathbb{R}^n$ (see [17] for details). In particular, for a connected open set $\mathcal{U} \subset \mathbb{R}^n$, a compact function $U \in C^1(\mathcal{U}, \mathbb{R})$ is called an *entire weak Lyapunov function of* (1.3) *in* $\mathcal{U}$ if

$$-\nabla f(x) \cdot \nabla U(x) \leq 0, \qquad x \in \mathcal{U}.$$

Remark 2.1. Let φ^t denote the local flow generated from (1.3). According to the general theory of dissipative systems (see e.g., [18, Proposition 6.3]), if (1.3) admits a weak Lyapunov function, then φ^t becomes a semiflow on $\mathbb{R}^n$, and moreover, if (1.3) admits a Lyapunov function, then φ^t must be dissipative and hence admit a global attractor. Let E denote the set of equilibria of (1.3). It is well-known that the global attractor of φ^t, if exists, equals the unstable set of E, which, in typical situations, consists of equilibria of (1.3) along with their connecting orbits. The global attractor of φ^t is actually invariant with respect to φ^t, i.e., φ^t can be extended to a flow on it.

We note that for any null family $\mathcal{A} = \{A_\alpha\}$ a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ is necessarily a Lyapunov function of (1.3). Conversely, if (1.3) admits a C^2 Lyapunov function whose second derivative is bounded, then with respect to any null family $\mathcal{A} = \{A_\alpha\}$, this Lyapunov function is clearly a uniform Lyapunov function with respect to the family $\{\mathcal{L}_{A_\alpha}\}$ as $|A_\alpha| \ll 1$. In the case $\mathcal{A} = \{\epsilon I\}$, where $\epsilon > 0$ is a small parameter, it is easy to see that if U is a Lyapunov function of (1.3), then it is a uniform Lyapunov function with respect to the family $\{\mathcal{L}_{\epsilon I}\}$ if either ΔU is bounded on $\mathbb{R}^n$ or U is sub-harmonic and a Lyapunov function with respect to $\mathcal{L}_I$.

For a given null family, the following result summaries important roles played by a uniform Lyapunov function to the corresponding stationary measures.

Lemma 2.1. *Consider a null family $\mathcal{A} = \{A_\alpha\}$ and assume that there is a uniform Lyapunov function with respect to the family $\{\mathcal{L}_{A_\alpha}\}$. Then the following holds.*

a) (Admissibility) *There is a stationary measure corresponding to each $\mathcal{L}_{A_\alpha}$ which is regular with density function lying in the space $W^{1,p}_{loc}(\mathbb{R}^n)$. Consequently, $\mathcal{A}$ is admissible.*
b) (Tightness) *As $A_\alpha \to 0$, the family $\{\mu_{A_\alpha}\}$ of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ is relatively sequentially compact in $M(\mathbb{R}^n)$.*
c) (Concentration) *Any limit measure of $\{\mu_{A_\alpha}\}$, as $A_\alpha \to 0$, is an invariant measure of φ^t supported on the global attractor of φ^t.*

In the above lemma, a) follows from general existence results contained in [7, 17], and b), c) follow from [20, Theorem B]. We refer the reader to [2, 6, 7, 17] and references therein for more information on the study of the existence of stationary measures of a Fokker-Planck equation in general.

2.2. **Stochastic stability.** The following equivalence result on $\mathcal{A}$-stability is proved in [20, Proposition 3.4].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a compact invariant set of φ^t and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family. Then Ω is $\mathcal{A}$-stable if and only if as $A_\alpha \to 0$, the set $\{\mu_{A_\alpha}\}$ of stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ is relatively sequentially compact in $M(\mathbb{R}^n)$ and all $\mathcal{A}$-limit measures are supported on Ω.*

For an admissible null family $\mathcal{A}$, denote

$$\mathcal{J}_{\mathcal{A}} = \overline{\cup\{\text{supp}(\mu) : \mu \,\text{is an}\, \mathcal{A}\text{-limit measure}\}}.$$

The following result follows from [20, Proposition 3.3 b)].

Lemma 2.3. (Stochastic LaSalle invariance principle) *Let $\mathcal{A}$ be an admissible null family and suppose that $\mathcal{J}_{\mathcal{A}}$ is contained in a compact set $\mathcal{J}$ and* (1.3) *admits an entire weak Lyapunov function U_0 in a neighborhood of $\mathcal{J}$. Then any $\mathcal{A}$-limit measure is supported on the set $\{x \in \mathcal{J} : \nabla f(x) \cdot \nabla U_0(x) = 0\}$.*

We first give the following result concerning the stability of compact invariant set of (1.3) under general noise perturbations.

Theorem 2.1. (Set stability under general noise) *Consider* (1.3) *and assume that it admits a C^2 Lyapunov function whose second derivative is bounded. Then the following holds.*

a) *The set E of critical points of f is $\mathcal{A}$-stable with respect to any null family $\mathcal{A}$;*
b) *For any finite set $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) of isolated local minimal (maximal) points of f, there is a null family $\mathcal{A}$ such that $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) is $\mathcal{A}$-stable (resp. strongly $\mathcal{A}$-unstable).*
c) *Any simple local maximal point of f is strongly $\mathcal{A}$-unstable with respect to any normal null family $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{A_\alpha\}$ be a given null family. Since (1.3) admits a C^2 Lyapunov function U whose second derivative is bounded, U becomes a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ as $|A_\alpha| \ll 1$. By Lemma 2.1 a), $\mathcal{A}$ is admissible.

Let $\{\mu_{A_\alpha}\}$ denote the set of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$. It follows from Lemma 2.1 b) that $\{\mu_{A_\alpha}\}$ is relatively sequentially compact in $M(\mathbb{R}^n)$ as $A_\alpha \to 0$, and from Lemma 2.1 c) that the set $\mathcal{J}_{\mathcal{A}}$ is contained in the global attractor $\mathcal{J}$ of φ^t. It is easy to see that there is a neighborhood of $\mathcal{J}$ and a constant $c > 0$ such that $U_0 =: f + c$ is an entire weak Lyapunov function of (1.3) in that neighborhood. It follows from Lemma 2.3 that any $\mathcal{A}$-limit measure is actually supported on the set

$$\{x \in \mathcal{J} : \nabla f(x) \cdot \nabla U_0(x) = 0\} = \{x \in \mathcal{J} : |\nabla f(x)|^2 = 0\}$$

which is precisely the set E. By Lemma 2.2, E is $\mathcal{A}$-stable. This proves a).

b) follows from [20, Theorem C a), b)] because the set $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) is a strong local attractor (resp. repeller) of φ^t.

c) follows from [20, Theorem C c)] because each non-degenerate local maximal point of f is a strongly repelling equilibrium of φ^t in the sense of [20, Definition 6.4] when taking $U = f$ in that definition. □

In the above theorem, a normal null family is a restricted null family defined in [20]. We now consider additive noise perturbation $\sqrt{2\epsilon}\dot{W}$ to (1.3).

Lemma 2.4. *Assume the condition* **H)**. *For each $\epsilon > 0$, denote the Gibbs measure by μ_ϵ. Then the following holds.*

a) (Tightness) *There exists an* $\epsilon_0 > 0$*, depending on* n *and* β*, such that* $\{\mu_\epsilon\}_{0<\epsilon\le\epsilon_0}$ *is relatively sequentially compact in* $M(\mathbb{R}^n)$*.*

b) (Concentration) *As* $\epsilon \to 0$*, any limit measure of* $\{\mu_\epsilon\}$ *is supported on the set* S_f *of global minimal points of* f*.*

In the above, a) follows from [21, Proposition 2.3] and b) follows from [21, Corollary 2.1].

We note that, under the condition **H)**, f itself is a weak Lyapunov function of (1.3). Hence φ^t becomes a semiflow on $\mathbb{R}^n$.

Theorem 2.2. (Stabilities under additive noise) *Assume the condition* **H)***. Then the following holds.*

a) S_f *is* $\mathcal{A}_0$*-stable.*

b) *In the case* **C1)***, the measure* $\delta_* =: \sum_{i=1}^m c_i\delta_{x_i}$ *is* $\mathcal{A}_0$*-stable, where, for each* $i = 1, \cdots, m$,

$$c_i = \begin{cases} 0, & \text{if } k_i < k, \\ \dfrac{\int_{\mathbb{R}^n} \mathrm{e}^{-P_i(y)}\mathrm{d}y}{\sum\limits_{j:k_j=k} \int_{\mathbb{R}^n} \mathrm{e}^{-P_j(y)}\mathrm{d}y}, & \text{if } k_i = k, \end{cases}$$

with $k = \max\{k_i : i = 1, 2, \cdots, m\}$*.*

c) *In the case* **C2)***, the normalized Lebesgue measure* μ_f *on* S_f *is* $\mathcal{A}_0$*-stable.*

Proof. For each $\epsilon > 0$, an application of [8, Example 5.4], with $A = \epsilon I$ and $b \equiv \beta_{\mu,A}$ there, shows the uniqueness of stationary measures corresponding to $\mathcal{L}_{\epsilon I}$. Hence the Gibbs measure μ_ϵ is actually the unique stationary measure corresponding to $\mathcal{L}_{\epsilon I}$. Let μ be a limit point of $\{\mu_\epsilon\}$ as $\epsilon \to 0$. By Lemma 2.4 b), $\mathrm{supp}(\mu) \subset S_f$. It follows from Lemma 2.2 and Lemma 2.4 a) that S_f is $\mathcal{A}_0$-stable. This proves a).

In the case **C1)**, it follows from [3, Theorem 5.2] that $\lim_{\epsilon\to 0}\mu_\epsilon = \delta_*$, and in the case **C2)**, it follows from [21, Proposition 2.2] that $\lim_{\epsilon\to 0}\mu_\epsilon = \mu_f$. This proves b) and c). ☐

Remark 2.2. 1) The proof of [8, Example 5.4] actually implies that the Gibbs measure μ_ϵ, for a fixed $\epsilon > 0$, is an invariant measure of the semigroup generated by the closure of the operator $\mathcal{L}_{\epsilon I}$ on $L^1(\mathbb{R}^n, \mu_\epsilon)$ (see also [5] for some interesting discussions in this regard). In fact, it can be further shown that if (1.4) generates a diffusion process then μ_ϵ is an invariant measure of that process.

2) We would like to make some comments about the measure δ_* in Theorem 2.2. By the expression of c_i in the above, we see that among the global minimal points of f, $\{\mu_\epsilon\}$ are eventually concentrated only on the most degenerate ones, i.e., those with the greatest order $2k$ in the dominated terms of their Taylor expansions. If there are more than one most degenerate global minimal points, then $\{\mu_\epsilon\}$ are eventually concentrated on all these points with certain weights c_i determined by the coefficients of P_i. For instance, consider the case $n = 1$. Then $P_i(x) = A_i x^{2k_i}$ for some $A_i > 0$. Hence the Taylor expansion of f in $B_\delta(x_i)$ reads

$$f(x) = A_i(x - x_i)^{2k_i} + o(|x - x_i|^{2k_i}),$$

from which it is not hard to see that

$$c_i = \begin{cases} 0, & \text{if } k_i < k, \\ \dfrac{\sqrt[2k]{\frac{1}{A_i}}}{\sum\limits_{j:k_j=k} \sqrt[2k]{\frac{1}{A_j}}}, & \text{if } k_i = k. \end{cases}$$

Hence it is clear that $\{\mu_\epsilon\}$ are eventually concentrated with larger probability on those global minimal points having smaller coefficients A_i.

3. An example of non-convergence

In this section, we construct an example of potential function f so that the set of stationary measures $\{\mu_\epsilon\}$ corresponding to $\{\mathcal{L}_{\epsilon I}\}$ admits more than one limit measures as $\epsilon \to 0$. In virtue of Theorem 2.2 b), c), such a potential function thus satisfies neither condition in **C1)** nor condition in **C2)**. Main features of this example have already been summarized in Theorem 1 of Section 1.

The construction of the example will be divided into several steps as follows.

Step 1. A simple computation shows that

$$\lim_{x\to+\infty} \sqrt{2}(\sum_{j=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{x^{2j}}} \cdot \frac{1}{2^j}) = \sqrt{2}(\sum_{j=1}^{+\infty} \frac{1}{2^j} + \frac{1}{\mathrm{e}})$$
$$= \sqrt{2}(1 + \frac{1}{\mathrm{e}}) < 2 = \sum_{j=0}^{+\infty} \frac{1}{2^j} = \lim_{x\to+\infty} \sum_{j=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{x^{2j+1}}} \cdot \frac{1}{2^j}.$$

Thus there exists a constant $M_* > 0$ such that for any $a \geq M_*$, if $A := \sum_{j=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j}$ and $B := \sum_{j=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j}$, then $\sqrt{2}A < B$, i.e., $\frac{A}{A+B} < \frac{B}{2A+B}$.

Fix $a \geq \max\{8, M_*\}$. We first consider a measurable function $f(x)$, $x \in \mathbb{R}$, defined in the following way:

For $x \in E_1 =: (-\frac{3}{2}, -\frac{1}{2})$, f is a step function attaining minimal value at $x = -1$ and the graph of f is symmetric with respect to the line $x = -1$. More precisely, for all $i \in \mathbb{N}$,

$$f(-1+t) = \begin{cases} \frac{1}{a^{2i}}, & t \in (\frac{1}{2} - \sum_{j=1}^{i} \frac{1}{2^{j+1}}, \frac{1}{2} - \sum_{j=1}^{i-1} \frac{1}{2^{j+1}}] \cup [-\frac{1}{2} + \sum_{j=1}^{i-1} \frac{1}{2^{j+1}}, -\frac{1}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}}), \\ 0, & t = 0. \end{cases}$$

For $x \in E_2 =: (\frac{1}{2}, \frac{3}{2})$, f has a similar shape, i.e., for all $i \in \mathbb{N}$,

$$f(1+t) = \begin{cases} \frac{1}{a^{2i+1}}, & t \in (\frac{1}{2} - \sum_{j=1}^{i} \frac{1}{2^{j+1}}, \frac{1}{2} - \sum_{j=1}^{i-1} \frac{1}{2^{j+1}}] \cup [-\frac{1}{2} + \sum_{j=1}^{i-1} \frac{1}{2^{j+1}}, -\frac{1}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}}), \\ 0, & t = 0. \end{cases}$$

On the set $E =: \mathbb{R} \setminus (E_1 \cup E_2)$, f is C^∞ smooth and smoothly connecting at the points $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Moreover, $f(x) \geq \frac{1}{a^3}$ on E, and, $f(x) = x^2$ when $|x| \gg 1$.

For each $i \in \mathbb{N}$, we let $\varepsilon_i = \frac{1}{a^{2i}}$, $\delta_i = \frac{1}{a^{2i-1}}$, and denote by μ_i, ν_i the Gibbs measures associated with $u_{\varepsilon_i} =: \frac{\mathrm{e}^{-\frac{f}{\varepsilon_i}}}{\int_{\mathbb{R}} \mathrm{e}^{-\frac{f}{\varepsilon_i}}\,\mathrm{d}x}$, $u_{\delta_i} =: \frac{\mathrm{e}^{-\frac{f}{\delta_i}}}{\int_{\mathbb{R}} \mathrm{e}^{-\frac{f}{\delta_i}}\,\mathrm{d}x}$, respectively; i.e., $\mathrm{d}\mu_i(x) = u_{\varepsilon_i}(x)\mathrm{d}x$, $\mathrm{d}\nu_i(x) = u_{\delta_i}(x)\mathrm{d}x$. Then, as $i \to +\infty$, we have

$$\begin{aligned}
\mu_i(E_1) &= \frac{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}\varepsilon_i}} \cdot \frac{1}{2^j}}{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}\varepsilon_i}} \cdot \frac{1}{2^j} + \sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}\varepsilon_i}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\varepsilon_i}}\,\mathrm{d}x} \\
&= \frac{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i)}}} \cdot \frac{1}{2^j}}{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i)}}} \cdot \frac{1}{2^j} + \sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i)+1}}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\varepsilon_i}}\,\mathrm{d}x} \\
&= \frac{\frac{1}{2^i}\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j}}{\frac{1}{2^i}\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j} + \frac{1}{2^i}\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\varepsilon_i}}\,\mathrm{d}x} \\
&= \frac{\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j}}{\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j} + \sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j} + 2^i\int_E \mathrm{e}^{-\frac{f(x)}{\varepsilon_i}}\,\mathrm{d}x} \\
&\longrightarrow \frac{A}{A+B},
\end{aligned}$$

and

$$\begin{aligned}
\nu_i(E_1) &= \frac{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}\delta_i}} \cdot \frac{1}{2^j}}{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}\delta_i}} \cdot \frac{1}{2^j} + \sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}\delta_i}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\delta_i}}\,\mathrm{d}x} \\
&= \frac{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i)+1}}} \cdot \frac{1}{2^j}}{\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i)+1}}} \cdot \frac{1}{2^j} + \sum_{j=1}^{\infty} \mathrm{e}^{-\frac{1}{a^{2(j-i+1)}}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\delta_i}}\,\mathrm{d}x} \\
&= \frac{\frac{1}{2^i}\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j}}{\frac{1}{2^i}\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j} + \frac{1}{2^{i-1}}\sum_{j=2-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j} + \int_E \mathrm{e}^{-\frac{f(x)}{\delta_i}}\,\mathrm{d}x} \\
&= \frac{\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j}}{\sum_{j=1-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j+1}}} \cdot \frac{1}{2^j} + 2\sum_{j=2-i}^{\infty} \mathrm{e}^{-\frac{1}{a^{2j}}} \cdot \frac{1}{2^j} + 2^i\int_E \mathrm{e}^{-\frac{f(x)}{\delta_i}}\,\mathrm{d}x} \\
&\longrightarrow \frac{B}{2A+B}.
\end{aligned}$$

Step 2. Fix two C^∞ functions $g_1, g_2 : \mathbb{R} \to [0,1]$ satisfying

$$g_1(x) = \begin{cases} 0, & \text{for } x \le 0, \\ 1, & \text{for } x \ge 1, \end{cases} \qquad g_2(x) = \begin{cases} 1, & x \le 0, \\ 0, & x \ge 1, \end{cases}$$

and $\sup_{x\in\mathbb{R}} \max\{|g_1'(x)|, |g_2'(x)|, |g_1''(x)|, |g_2''(x)|\} \le M$ for some constant $M > 0$. For given $L, R, z \in \mathbb{R}, d > 0$ with $L \ne R$, consider the function $g_{L,R,d,z} : \mathbb{R} \to [\min\{L,R\}, \max\{L,R\}]$:

$$g_{L,R,d,z}(x) = \begin{cases} (R-L)g_1(\frac{x-z}{d}) + L, & \text{when } L < R; \\ (L-R)g_2(\frac{x-z}{d}) + R, & \text{when } L > R. \end{cases}$$

Then it is clear that

$$g_{L,R,d,z}(x) = \begin{cases} L, & x \le z, \\ R, & x \ge z+d, \end{cases}$$

$|g'_{L,R,d,z}(x)| \le \frac{|R-L|}{d} M$, and $|g''_{L,R,d,z}(x)| \le \frac{|R-L|}{d^2} M$.

For a fixed small constant $\eta > 0$, we now use the function $g_{L,R,d,z}$ to construct a function $f_\eta \in C^2(\mathbb{R})$ satisfying the following properties:

P1) $f_\eta \equiv f$ on E and $f_\eta \ge f$ on E_i, $i = 1, 2$;

P2) f_η attains global minima at $\{\pm 1\}$ which are however not isolated critical points of f_η;

P3) For each constant $c \ge 0$,

$$\int_{E_i} \mathrm{e}^{-cf_\eta} \mathrm{d}x \ge (1-\eta) \int_{E_i} \mathrm{e}^{-cf} \mathrm{d}x, \qquad i = 1, 2.$$

The construction is a smoothing process. At each point of discontinuity $z = -1 - \frac{1}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}} \in (-\frac{3}{2}, -1)$, we let $L = \frac{1}{a^{2i}}$, $R = \frac{1}{a^{2(i+1)}}$, and $d = \eta \cdot \frac{1}{2^{i+2}}$; at each point of discontinuity $z = -1 + \frac{1}{2} - \sum_{j=1}^{i} \frac{1}{2^{j+1}} \in (-1, -\frac{1}{2})$, we let $L = \frac{1}{a^{2(i+1)}}$, $R = \frac{1}{a^{2i}}$, and $d = \eta \cdot \frac{1}{2^{i+2}}$; at each point of discontinuity $z = 1 - \frac{1}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}} \in (\frac{1}{2}, 1)$, we let $L = \frac{1}{a^{2i+1}}$, $R = \frac{1}{a^{2(i+1)+1}}$, and $d = \eta \cdot \frac{1}{2^{i+2}}$; and at each point of discontinuity $z = 1 + \frac{1}{2} - \sum_{j=1}^{i} \frac{1}{2^{j+1}} \in (1, \frac{3}{2})$, we let $L = \frac{1}{a^{2(i+1)+1}}$, $R = \frac{1}{a^{2i+1}}$, and $d = \eta \cdot \frac{1}{2^{i+2}}$. We let $f_\eta(x) = g_{L,R,d,z}(x)$ for x lying in each interval of the form $[z, z+d] \subset (-\frac{3}{2}, -1) \cup (\frac{1}{2}, 1)$, $f_\eta(x) = g_{L,R,d,z-d}(x)$ for x lying in each interval of the form $[z-d, z] \subset (-1, -\frac{1}{2}) \cup (1, \frac{3}{2})$, and $f_\eta(x) = f(x)$ for x lying in the complement of these intervals.

The function f_η is C^∞ everywhere just with possible exception at the points $x = -1$ and $x = 1$. Note that when $x \in [-\frac{3}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}}, -\frac{3}{2} + \sum_{j=1}^{i} \frac{1}{2^{j+1}} + \eta \cdot \frac{1}{2^{i+2}}]$, we have $\frac{1}{2^{i+2}} \le |x - (-1)| \le \frac{1}{2^{i+1}}$. Using the fact $a \ge 8$, it follows that, when $|x - (-1)| \ll 1$,

$$|f'_\eta(x)| \le \frac{|L-R|}{d} M \le \frac{\frac{1}{a^{2i}}}{\eta \cdot \frac{1}{2^{i+2}}} M = \frac{M 2^{4(i+2)}}{\eta a^{2i}} \cdot \frac{1}{2^{3(i+2)}} \le \frac{M}{\eta} |x - (-1)|^3,$$

$$|f_\eta''(x)| \le \frac{|L-R|}{d^2} M \le \frac{\frac{1}{a^{2i}}}{\eta^2 \cdot \frac{1}{2^{2(i+2)}}} M = \frac{16M}{\eta^2} \left(\frac{2}{a}\right)^{2i} \frac{2^{2(i+2)}}{2^{2(i+2)}} \le \frac{M}{\eta^2} |x - (-1)|^2.$$

Therefore $f'_\eta(-1-0) = 0$ and $f''_\eta(-1-0) = 0$. Similarly, $f'_\eta(-1+0) = 0$ and $f''_\eta(-1+0) = 0$. Thus both $f'_\eta(-1)$ and $f_\eta''(-1)$ exist and equal 0. By the definition of f_η, we have

$$|f_\eta(x) - f_\eta(-1)| = |f_\eta(x)| \le \frac{1}{a^{2i}} \le |x - (-1)|^4$$

when $|x - (-1)| \ll 1$. Hence f'_η is continuous at -1. Since

$$|f'_\eta(x) - f'_\eta(-1)| = |f'_\eta(x)| \le \frac{M}{\eta} |x - (-1)|^3$$

when $|x - (-1)| \ll 1$, f_η'' is continuous at -1. The same arguments show that $f'_\eta(1), f_\eta''(1)$ exist, both equal 0, and f'_η, f_η'' are continuous at 1.

It is clear that f_η so constructed satisfies all properties **P1)-P3)** above.

Step 3. Consider the noise perturbed gradient system

$$\dot{x} = -f'_\eta(x) + \sqrt{2\epsilon}\, \dot{W}, \qquad x \in \mathbb{R}^1. \tag{3.1}$$

We note that with f_η being a weak Lyapunov function the unperturbed gradient system corresponding to (3.1) generates a semiflow on $\mathbb{R}^1$, which we denote by φ^t.

For each $\epsilon > 0$, an application of [8, Example 5.4] shows that the Gibbs state

$$u_\epsilon^\eta(x) = \frac{\mathrm{e}^{-\frac{f_\eta(x)}{\epsilon}}}{\int_{\mathbb{R}} \mathrm{e}^{-\frac{f_\eta(x)}{\epsilon}} \mathrm{d}x}$$

is the unique stationary solution of the Fokker-Planck equation associated with (3.1). This also follows from [8, Example 5.1] because $U(x) =: f_\eta(x) = x^2$ as $|x| \gg 1$, is an unbounded, uniform Lyapunov function with respect to the family $\{\mathcal{L}_{\epsilon I}\}$ of the adjoint Fokker-Planck operators corresponding to (3.1) (see Remark 2.1). By either Lemma 2.1 b) or Lemma 2.4 a), the set of limit measures of the corresponding Gibbs measures as $\epsilon \to 0$ is non-empty.

For each $i \in \mathbb{N}$, we let $\varepsilon_i = \frac{1}{a^{2i}}$, $\delta_i = \frac{1}{a^{2i-1}}$ be the sequences used in Step 1 and let μ_i^η, ν_i^η be the stationary measures associated with the Gibbs states $u_{\varepsilon_i}^\eta$, $u_{\delta_i}^\eta$, respectively. Choose any limit measure μ of $\{\mu_i^\eta\}$ and any limit measure ν of $\{\nu_i^\eta\}$, as $i \to \infty$. We have by Theorem 2.2 a) that both μ and ν are invariant measures of φ^t supported on $S_{f_\eta} = \{-1, 1\}$.

Using the properties **P1)-P3)** of f_η, it is easy to see that, as $i \to \infty$,

$$\begin{aligned}
\mu_i^\eta(E_1) &= \frac{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x + \int_{E_2} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x + \int_E \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x} \\
&\le \frac{\int_{E_1} \mathrm{e}^{-f \cdot a^{2i}} \mathrm{d}x}{(1-\eta)\int_{E_1} \mathrm{e}^{-f \cdot a^{2i}} \mathrm{d}x + (1-\eta)\int_{E_2} \mathrm{e}^{-f \cdot a^{2i}} \mathrm{d}x + \int_E \mathrm{e}^{-f \cdot a^{2i}} \mathrm{d}x} \\
&\longrightarrow \frac{1}{(1-\eta)} \cdot \frac{A}{A+B},
\end{aligned}$$

$$\begin{aligned}
\nu_i^\eta(E_1) &= \frac{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i-1}} \mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i-1}} \mathrm{d}x + \int_{E_2} \mathrm{e}^{-f_\eta \cdot a^{2i-1}} \mathrm{d}x + \int_E \mathrm{e}^{-f_\eta \cdot a^{2i-1}} \mathrm{d}x} \\
&\ge \frac{(1-\eta)\int_{E_1} \mathrm{e}^{-f \cdot a^{2i-1}} \mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f \cdot a^{2i-1}} \mathrm{d}x + \int_{E_2} \mathrm{e}^{-f \cdot a^{2i-1}} \mathrm{d}x + \int_E \mathrm{e}^{-f \cdot a^{2i-1}} \mathrm{d}x} \\
&\longrightarrow (1-\eta) \cdot \frac{B}{2A+B}.
\end{aligned}$$

It follow that

$$\mu(E_1) \le \frac{1}{(1-\eta)} \cdot \frac{A}{A+B} < (1-\eta) \cdot \frac{B}{2A+B} \le \nu(E_1) \tag{3.2}$$

when η is sufficiently small. Hence $\mu \ne \nu$.

This shows that the family $\{\mu_\epsilon^\eta\}$ of all stationary measures corresponding to $\{\mathcal{L}_{\epsilon I}\}$ admits at least two limit measures as $\epsilon \to 0$. Consequently, no invariant measure of φ^t can be $\{\epsilon I\}$-stable.

We now show that μ, ν have the desired forms. Since both μ and ν are supported on $S_{f_\eta} = \{-1, 1\}$ and $\mu \ne \nu$, $\mu = \lambda_1 \delta_{-1} + (1-\lambda_1)\delta_1$ and $\nu = \lambda_2 \delta_{-1} + (1-\lambda_2)\delta_1$ for some constants $\lambda_1 \ne \lambda_2$. We note that, as $i \to \infty$,

$$\mu_i^\eta(E_1) = \frac{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x + \int_{E_2} \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x + \int_E \mathrm{e}^{-f_\eta \cdot a^{2i}} \mathrm{d}x}$$

$$\geq \frac{(1-\eta)\cdot\int_{E_1} \mathrm{e}^{-f\cdot a^{2i}}\mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f\cdot a^{2i}}\mathrm{d}x + \int_{E_2} \mathrm{e}^{-f\cdot a^{2i}}\mathrm{d}x + \int_{E} \mathrm{e}^{-f\cdot a^{2i}}\mathrm{d}x}$$
$$\longrightarrow (1-\eta)\cdot\frac{A}{A+B},$$

and

$$\nu_i^\eta(E_1) = \frac{\int_{E_1} \mathrm{e}^{-f_\eta\cdot a^{2i-1}}\mathrm{d}x}{\int_{E_1} \mathrm{e}^{-f_\eta\cdot a^{2i-1}}\mathrm{d}x + \int_{E_2} \mathrm{e}^{-f_\eta\cdot a^{2i-1}}\mathrm{d}x + \int_{E} \mathrm{e}^{-f_\eta\cdot a^{2i-1}}\mathrm{d}x}$$
$$\leq \frac{\int_{E_1} \mathrm{e}^{-f\cdot a^{2i-1}}\mathrm{d}x}{(1-\eta)\cdot\int_{E_1} \mathrm{e}^{-f\cdot a^{2i-1}}\mathrm{d}x + (1-\eta)\cdot\int_{E_2} \mathrm{e}^{-f\cdot a^{2i-1}}\mathrm{d}x + \int_{E} \mathrm{e}^{-f_\eta\cdot a^{2i-1}}\mathrm{d}x}$$
$$\longrightarrow \frac{1}{1-\eta}\cdot\frac{B}{2A+B}.$$

It follows that

$$\mu(E_1) \geq (1-\eta)\cdot\frac{A}{A+B} \qquad \text{and} \qquad \nu(E_1) \leq \frac{1}{1-\eta}\cdot\frac{B}{2A+B},$$

which, together with (3.2), yields that $\lambda_1, \lambda_2 \in (0,1)$ when η is small. This completes the proof.

4. De-stabilization of normalized Lebesgue measure via multiplicative noises

For stochastic stability of singular measures, we have seen from Theorem 2.2 that those concentrated on relative minimal points of f are never stable under additive noise perturbations, though they can be stable under some multiplicative noise perturbations according to Theorem 2.1 b).

The situation with absolutely continuous measures is more complicated. We see from Theorem 2.2 c) that if the condition in **C2)** holds, then the normalized Lebesgue measure on S_f is stable under additive noise perturbations. Below we give an example of a gradient system in 2-dimension satisfying the condition in **C2)** for which there are infinitely many null families $\{\mathcal{A}\}$ of multiplicative noise perturbations such that with respect to each family $\mathcal{A}$ the set S_f remains $\mathcal{A}$-stable but any $\mathcal{A}$-limit measure differs from the normalized Lebesgue measure on S_f, i.e., the normalized Lebesgue measure is $\mathcal{A}$-unstable. In fact, this example contains more information than that stated in Theorem 2.

We consider (1.3) with $n=2$ for a potential function $f \in C^2(\mathbb{R}^2)$ such that

$$f(x) = \begin{cases} 0, & |x| \leq 1; \\ |x|^2, & |x| \geq 2, \end{cases}$$

and $f(x) > 0$, $\nabla f(x) \neq 0$, $1 < |x| < 2$. It is clear that $E = S_f$ which is precisely the closed unit disk in $\mathbb{R}^2$ centered at the origin. We note that f itself is an unbounded C^2 Lyapunov function of (1.3) whose second derivatives are bounded in $\mathbb{R}^2$. Hence for any null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, f is a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ and it follows from Theorem 2.1 a) that S_f is $\mathcal{A}$-stable.

Now consider the noise perturbed system

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t + \sqrt{2\epsilon a(x)}\, I\ \mathrm{d}W,$$

where $a \in C^2(\mathbb{R}^2)$ is a bounded positive function. We then obtain the stationary Fokker-Planck equation

$$\begin{cases} \epsilon\Delta(au) + \nabla\cdot(u\nabla f) = 0, \\ u(x) \geq 0, \quad \int_{\mathbb{R}^n} u(x)\mathrm{d}x = 1. \end{cases} \tag{4.1}$$

Since f is an unbounded uniform Lyapunov function with respect to the family $\{\mathcal{L}_{\epsilon aI}\}$, Lemma 2.1, together with [8, Example 5.1], implies that, for each a and $\epsilon > 0$, (4.1) admits a unique classical, positive solution u^a_ϵ which gives rise to a stationary measure μ^a_ϵ corresponding to $\mathcal{L}_{\epsilon aI}$ with density u^a_ϵ. Moreover, by Lemma 2.1 b), the set of limit measures of $\{\mu^a_\epsilon\}$ as $\epsilon \to 0$ is non-empty.

Consider the function

$$f_\epsilon(\rho) := \frac{1}{\rho}\int_{\partial B_\rho} au^a_\epsilon \mathrm{d}s,$$

where B_ρ is the ball with radius ρ centered at the origin for each $\rho \in (0,1]$. Using the radial coordinate $r = |x|$ and letting $\omega = \dfrac{x}{r}$, we have $u^a_\epsilon(x) = u^a_\epsilon(r\omega)$. Moreover,

$$\begin{aligned}\int_{\partial B_\rho} \frac{\partial(au^a_\epsilon)}{\partial\vec{n}}\mathrm{d}s &= \int_{\partial B_\rho} \frac{\partial(au^a_\epsilon)}{\partial r}(\rho\omega)\mathrm{d}s = \rho\int_{|\omega|=1} \frac{\partial(au^a_\epsilon)}{\partial r}(\rho\omega)\mathrm{d}\omega \\ &= \rho\frac{\partial}{\partial\rho}\int_{|\omega|=1}(au^a_\epsilon)(\rho\omega)\mathrm{d}\omega = \rho\frac{\partial}{\partial\rho}\left(\rho^{-1}\int_{\partial B_\rho}(au^a_\epsilon)(x)\mathrm{d}s\right),\end{aligned}$$

i.e.,

$$f'_\epsilon(\rho) = \frac{1}{\rho}\int_{\partial B_\rho}\frac{\partial(au^a_\epsilon)}{\partial\vec{n}}\mathrm{d}s.$$

But the Divergence Theorem together with (4.1) imply that

$$\int_{\partial B_\rho}\frac{\partial(au^a_\epsilon)}{\partial\vec{n}}\mathrm{d}s = \int_{\partial B_\rho}\nabla(au^a_\epsilon)\cdot\vec{n}\mathrm{d}s = \int_{B_\rho}\Delta(au^a_\epsilon)\mathrm{d}x = -\frac{1}{\epsilon}\int_{B_\rho}\nabla\cdot(u^a_\epsilon\nabla f)\mathrm{d}x = 0.$$

Hence

$$f'_\epsilon(\rho) = 0, \qquad \text{for all } \rho \in (0,1].$$

It follows that there is a positive constant C, depending on a and ϵ, such that

$$\int_{\partial B_\rho} au^a_\epsilon\mathrm{d}s = C\rho, \qquad \text{for all } \rho \in (0,1],$$

from which we deduce that

$$\int_{B_r} au^a_\epsilon\mathrm{d}x = \int_0^r\int_{\partial B_\rho} au^a_\epsilon\mathrm{d}s\mathrm{d}\rho = \int_0^r C\rho\mathrm{d}\rho = \frac{1}{2}Cr^2,$$

for any $0 < r \leq 1$. Thus, for any $0 < r < R \leq 1$, we have

$$\frac{\int_{B_r} a\mathrm{d}\mu^a_\epsilon}{\int_{B_R\setminus B_r} a\mathrm{d}\mu^a_\epsilon} = \frac{\int_{B_r} au^a_\epsilon\mathrm{d}x}{\int_{B_R\setminus B_r} au^a_\epsilon\mathrm{d}x} = \frac{r^2}{R^2 - r^2}.$$

Let μ^a be any limit measure of $\{\mu^a_\epsilon\}$ as $\epsilon \to 0$. Then

$$\frac{\int_{B_r} a\mathrm{d}\mu^a}{\int_{B_R\setminus B_r} a\mathrm{d}\mu^a} = \frac{r^2}{R^2 - r^2} \tag{4.2}$$

for any $0 < r < R \leq 1$. The following properties follow immediately from (4.2):

i) Each μ^a is supported on $\bar{B}_1$ but not on any proper compact subset of it.

ii) For any two fixed number $0 < r < R \leq 1$ and any two functions a_1, a_2 with $a_1(x) < a_2(x)$, $x \in B_r$; $a_1(x) = a_2(x)$, $x \in \partial B_r$; and $a_1(x) > a_2(x)$, $x \in B_R \setminus \bar{B}_r$, any limit measure μ^{a_1} corresponding to a_1 is different from any limit measure μ^{a_2} corresponding to a_2.

In particular, by varying a, we obtain infinitely many limit measures differing from the normalized Lebesgue measure on $\bar{B}_1 = S_f$.

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