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Steady States of Fokker–Planck Equations: III. Degenerate Diffusion

Wen Huang[1,2] · **Min Ji**[3] · **Zhenxin Liu**[4,6] · **Yingfei Yi**[5,6,7]

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Abstract This is the third paper in a series concerning the study of steady states of a Fokker–Planck equation in a general domain in $\mathbb{R}^n$ with L^p_{loc} drift term and $W^{1,p}_{loc}$ diffusion term for any $p > n$. In this paper, we give some existence results of stationary measures of the Fokker–Planck equation under Lyapunov conditions which allow the degeneracy of diffusion.

Keywords Fokker–Planck equation · Degenerate diffusion · Existence · Stationary solution · Stationary measure · Level set method

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✉ Min Ji
jimin@math.ac.cn

Wen Huang
wenh@mail.ustc.edu.cn

Zhenxin Liu
zxliu@dlut.edu.cn; zxliu@jlu.edu.cn

Yingfei Yi
yingfei@ualberta.ca; yi@math.gatech.edu

1 School of Mathematics, Sichuan University, Chengdu 610064, Sichuan, People's Republic of China

2 Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences, Hefei 230026, Anhui, People's Republic of China

3 Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, People's Republic of China

4 School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

5 Department of Mathematical & Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada

6 School of Mathematics, Jilin University, Changchun 130012, People's Republic of China

7 School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

1 Introduction

Consider the *stationary Fokker–Planck equation* on $\mathcal{U}$:

$$\begin{cases} Lu(x) = 0, & x \in \mathcal{U}, \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.1}$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set which can be bounded, unbounded, or the entire space $\mathbb{R}^n$, L is the *Fokker–Planck operator* on $\mathcal{U}$ defined by

$$Lg(x) = \partial_{ij}^2\big(a^{ij}(x)g(x)\big) - \partial_i\big(V^i(x)g(x)\big), \quad g \in C^2(\mathcal{U})$$

with $V = (V^i)$ being the *drift field* and $(a^{ij}) \geq 0$ being the *diffusion matrix*, on $\mathcal{U}$.

In the whole paper, we use short notations $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \ldots, n$ whenever applicable.

We make the following standard hypothesis:

(A) $a^{ij} \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, $V^i \in L^p_{\mathrm{loc}}(\mathcal{U})$ for all $i, j = 1, \ldots, n$, where $p > n$ is fixed.

A continuous function u on $\mathcal{U}$ is said to be a *weak stationary solution* of the Fokker–Planck equation corresponding to L if it is a weak solution of the stationary Fokker–Planck equation (1.1), i.e.,

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}f(x)u(x)\mathrm{d}x = 0, & \text{for all } f \in C_0^\infty(\mathcal{U}), \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.2}$$

where

$$\mathcal{L} = a^{ij}\partial_{ij}^2 + V^i\partial_i$$

is the adjoint Fokker–Planck operator and $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports. More generally, a Borel probability measure μ on $\mathcal{U}$ is said to be a *stationary measure* of the Fokker–Planck equation corresponding to L if it is a measure solution of the stationary Fokker–Planck equation (1.1), i.e.,

$$V^i \in L^1_{loc}(\mathcal{U}, \mu), \quad i = 1, 2, \ldots, n, \text{ and,} \tag{1.3}$$

$$\int_{\mathcal{U}} \mathcal{L}f(x)\mathrm{d}\mu(x) = 0, \quad \text{for all } f \in C_0^\infty(\mathcal{U}). \tag{1.4}$$

A stationary measure μ of the Fokker–Planck equation corresponding to L is called *regular* if it admits a continuous density function u, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$. It is clear that such a density u is a weak stationary solution of the Fokker–Planck equation corresponding to L, and vice versa. Indeed, under the condition (A), if (a^{ij}) is everywhere positive definite in $\mathcal{U}$, then it follows from a regularity theorem due to Bogachev et al. [5] (also recalled in Theorem 2.5 below) that all stationary measures of the Fokker–Planck equation corresponding to L are regular with densities lying in $W^{1,p}_{loc}(\mathcal{U})$. If $a^{ij} \in C^{2,\alpha}_{loc}(\mathcal{U})$, $V^i \in C^{1,\alpha}_{loc}(\mathcal{U})$, $i, j = 1, \ldots, n$, for some $\alpha \in (0, 1)$, then it follows from the standard Schauder theory that the density functions become classical solutions of (1.1).

When (a^{ij}) is everywhere positive definite in $\mathcal{U}$, the existence of stationary measures corresponding to L has been extensively studied. We refer the reader to [1–11,18–21] and references therein for the case $\mathcal{U} = \mathbb{R}^n$. For a general domain $\mathcal{U}$, we have given several new existence, respectively, non-existence results for stationary measures corresponding to L in the previous two parts of the series [15,16], by using Lyapunov-like, respectively anti-Lyapunov-like functions. We note that existence results in part I of the series [15] are still applicable to obtain the existence of a regular stationary measure corresponding to L when (a^{ij}) is degenerate on the boundary of $\mathcal{U}$, though the standard theory of elliptic equations fails to apply even when $\mathcal{U}$ is bounded, due to the degeneracy on the boundary.

When (a^{ij}) is not necessarily everywhere positive definite in $\mathcal{U}$, some conditions are given in part II of the series [16] which ensure the non-existence of a regular stationary measure with positive density corresponding to L. Still, stationary measures which are not necessarily regular may exist under the conditions, as pointed out in Remark 3.2 (1) in [16]. In [7], without assuming (a^{ij}) to be everywhere positive definite in $\mathbb{R}^n$, Bogachev–Röckner showed the existence of a stationary measure corresponding to L in $\mathbb{R}^n$ when (A) holds, V is continuous on $\mathbb{R}^n$, and there exists a non-negative function $U \in C^2(\mathbb{R}^n)$ with

$$\lim_{x\to\infty} U(x) = +\infty \tag{1.5}$$

and

$$\lim_{x\to\infty} \mathcal{L}U(x) = -\infty. \tag{1.6}$$

In this paper, we will study the existence of stationary measures of the Fokker–Planck equation corresponding to L in a general domain $\mathcal{U}$ without assuming (a^{ij}) to be everywhere positive definite in $\mathcal{U}$. Such a generality of the domain does allow a wide range of applications as already remarked in part I of the series [15]. Our result, extending the corresponding one of [7] even in the case of $\mathbb{R}^n$, makes use of a *Lyapunov function* $U \in C^2(\mathcal{U})$ with respect to (1.1) which is a so-called *compact function* in $\mathcal{U}$ (a notion generalizing condition (1.5), see Sect. 2.1 for details) satisfying the "dissipation" property that

$$\limsup_{x\to\partial\mathcal{U}} \mathcal{L}U(x) = \limsup_{x\to\partial\mathcal{U}} \left(a^{ij}(x)\partial^2_{ij}U(x) + V^i(x)\partial_i U(x)\right) = -\gamma \tag{1.7}$$

for some constant $\gamma > 0$. A compact function U satisfying (1.7) with $\gamma = +\infty$ is called a *strong Lyapunov function*. In the above, the notion $\partial\mathcal{U}$ and limit $x \to \partial\mathcal{U}$ are defined according to the topology of the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ defined in [15], which identifies $\mathbb{E}^n$ with the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ so that $\mathbb{R}^n$ is identified with $\mathbb{B}^n$, and $\partial\mathbb{R}^n$, consisting of infinity elements of all rays, is identified with $\partial\mathbb{B}^n$. Therefore, when $\mathcal{U} = \mathbb{R}^n$, the limit $x \to \partial\mathbb{R}^n$ is simply equivalent to $x \to \infty$. Our result also involves the class $\mathcal{B}^*(A)$ which consists of functions U with controlled growth rates of $a^{ij}\partial_i U \partial_j U$ near $\partial\mathcal{U}$ (see Sect. 2 for details).

More precisely, we will show the following result.

Theorem A *Assume* (A) *and that* $V \in C(\mathcal{U}, \mathbb{R}^n)$. *If, with respect to* (1.1)*, there exists either a strong Lyapunov function or a Lyapunov function of the class* $\mathcal{B}^*(A)$*, then the Fokker–Planck equation corresponding to L admits a stationary measure.*

The proof of Theorem A above uses the level set method in particular the integral identity which we derived in [14] (also recalled in Theorem 2.2 below). It also follows the approach in [7] by perturbing the diffusion matrix via a family of non-degenerate ones and then taking weak*-limits of the corresponding stationary measures. More generally, consider a family of adjoint Fokker–Planck operators:

$$\mathcal{L}_{V,A} = a^{ij}\partial^2_{ij} + V^i\partial_i, \quad V = \left(V^i\right) \in \mathcal{V}, \quad A = \left(a^{ij}\right) \in \mathcal{A},$$

where $\mathcal{V}$ is a set of continuous vector-valued functions from $\mathcal{U}$ to $\mathbb{R}^n$ under the compact-open topology and $\mathcal{A}$ is a set of $n \times n$ matrix-valued, everywhere positive definite functions on $\mathcal{U}$ of the class $W^{1,p}_{loc}$. Then one can speak of a *uniform Lyapunov* or a *strong uniform Lyapunov* *function* with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V},\ A \in \mathcal{A}\}$ and the *class* $\mathcal{B}^*(\mathcal{A})$ with respect to the family $\mathcal{A}$ (see Sect. 2.2 for details).

For each $V \in \mathcal{V},\ A \in \mathcal{A}$, let $\mathcal{M}_{V,A}$ denote the set of stationary measures $\mu_{V,A}$ of the Fokker–Planck equations with drift field V and diffusion matrix A. The proof of Theorem A will rely on the following result on the upper semi-continuity of the stationary measure set $\mathcal{M}_{V,A}$ in V, A which is of interest on its own rights.

Theorem B *Assume that there is a uniform Lyapunov function U with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V}, A \in \mathcal{A}\}$ such that either U is a uniform strong Lyapunov function with*

$$\sup_{V\in\mathcal{V},A\in\mathcal{A}} (|V|_{C(\Omega)} + |A|_{C(\Omega)}) < \infty, \quad \forall \Omega \subset\subset \mathcal{U},$$

or U is of the class $\mathcal{B}^*(\mathcal{A})$. Then the following properties hold:

(a) *The set $\mathcal{M} = \bigcup_{V\in\mathcal{V},A\in\mathcal{A}} \mathcal{M}_{V,A}$ is relatively sequentially compact in the space $M(\mathcal{U})$ of Borel probability measures on $\mathcal{U}$;*
(b) *For any $(V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$, any sequence $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ with $(V_k, A_k) \to (V_0, A_0)$, and any sequence of corresponding stationary measures μ_{V_k,A_k}, there is a subsequence (V_{k_i}, A_{k_i}) such that $\mu_{V_{k_i},A_{k_i}}$ converges to a stationary measure $\mu_0 \in \mathcal{M}_{V_0,A_0}$. In particular, $\mathcal{M}_{V_0,A_0} \neq \emptyset$.*

We remark that if the Eq. (1.1) is defined on $\mathcal{U} \times M$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set and M is a smooth, compact manifold without boundary, then one can modify the definitions of (uniform) Lyapunov, strong (uniform) Lyapunov functions in Sect. 2 in an obvious way by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$. Then the proofs in later sections can be modified accordingly so that Theorems A and B still hold with respect to such a generalized domain.

This paper is organized as follows. Section 2 is a preliminary section in which we will recall notions of compact and Lyapunov functions and define new notions of uniform Lyapunov and strong uniform Lyapunov functions. We will also recall the Prokhorov's theorem on tightness and sequential compactness of probability measures, main ingredients of the level set method from [14], and a regularity result for stationary measures from [5]. In Sect. 3, we study the upper-semi-continuity of stationary measures on drift and diffusions in a general domain. Theorem B will be proved under a slightly more general condition. In Sect. 4, we show Theorem A in a general domain and also give a corollary in $\mathbb{R}^n$ involving more explicit conditions.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote absolute value of a number, and norm of a vector or a matrix.

2 Preliminary

2.1 Compact Functions

Recall from [15] that a non-negative continuous function U in $\mathcal{U}$ is a *compact function* if

(i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
(ii) $\lim_{x\to\partial\mathcal{U}} U(x) = \rho_M$,

where $\rho_M = \sup_{x\in\mathcal{U}} U(x)$ is called the *essential upper bound of U*.

When $\mathcal{U}$ is unbounded, the notion $\partial\mathcal{U}$ and the limit $x \to \partial\mathcal{U}$ in (ii) above should be understood under the topology which is defined through a fixed homeomorphism between the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ and the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ in $\mathbb{R}^n$ which identifies $\mathbb{R}^n$ with B^n and $\partial\mathbb{R}^n$ with $\mathbb{S}^{n-1}$, and in particular, identifies each $x_* \in \mathbb{S}^{n-1}$ with the infinity element $x_*^\infty \in \partial\mathbb{R}^n$ of the ray through x_* (see [15] for details). Consequently, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathbb{R}^n$ under this topology simply means $x \to \infty$ in the usual sense, and it is easy to see that an unbounded, non-negative function $U \in C(\mathcal{U})$ is a compact function in $\mathcal{U}$ if and only if

$$\lim_{x \to \partial\mathcal{U}} U(x) = +\infty.$$

We recall from [16] the following definition of class $\mathcal{B}^*(A)$ of compact functions.

Definition 2.1 Let $A = (a^{ij})$ be an everywhere positive semi-definite, $n \times n$ matrix-valued function on $\mathcal{U}$. A compact function $U \in C^1(\mathcal{U})$ with essential upper bound ρ_M is said to be *of the class* $\mathcal{B}^*(A)$ if there exist $\rho_m \in (0, \rho_M)$ and a non-negative, locally bounded function H defined on $[\rho_m, \rho_M)$ such that

$$\nabla U(x) \neq 0, \quad \forall x \in U^{-1}(\rho) \quad \text{for } a.e.\ \rho \in \big[\rho_m, \rho_M\big), \tag{2.1}$$

$$a^{ij}(x)\partial_i U(x)\partial_j U(x) \leq H(\rho), \quad x \in U^{-1}(\rho), \quad \rho \in \big[\rho_m, \rho_M\big), \text{ and} \tag{2.2}$$

$$\int_{\rho_0}^{\rho_M} \frac{1}{H(\rho)} \mathrm{d}\rho = +\infty, \quad \forall \rho_0 \in \big(\rho_m, \rho_M\big). \tag{2.3}$$

Remark 2.1 As remarked in [16], (2.1) says that the set of regular values of the function U is of full Lebesgue measure in $[\rho_m, \rho_M)$. By Sard's theorem (see e.g. [13]), if $U \in C^n(\mathcal{U})$, then the set of regular values of U is of full Lebesgue measure in $[0, \rho_M)$. Consequently, (2.1) is satisfied by any compact function $U \in C^n(\mathcal{U})$.

The following proposition proved in [16] gives some useful sufficient conditions for a C^2 function in $\mathbb{R}^n$ to be of the class $\mathcal{B}^*(A)$.

Proposition 2.1 *Let $U \in C^2(\mathbb{R}^n)$ be a function such that the Hessian matrix D^2U is bounded under the sup-norm and uniformly positive definite in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ for some $r_0 > 0$. Then the following properties hold.*

(a) *There is a constant $c \geq 0$ such that $U + c$ is an unbounded compact function in $\mathbb{R}^n$.*
(b) *$U + c$ is of the class $\mathcal{B}^*(A)$ with respect to any $n \times n$ matrix-valued function A which is bounded under the sup-norm.*

2.2 Lyapunov-Like Functions

We recall the following definition from [15].

Definition 2.2 Let U be a C^2 compact function in $\mathcal{U}$ with essential upper bound ρ_M. For each $\rho \in [0, \rho_M)$, we denote $\Omega_\rho = \{x \in \mathcal{U} : U(x) < \rho\}$ as the ρ-sublevel set of U.

(1) U is called a *Lyapunov function* in $\mathcal{U}$ with respect to (1.1) or $\mathcal{L}$, if there exists $\rho_m \in (0, \rho_M)$, called *essential lower bound of* U, and a constant $\gamma > 0$, called *Lyapunov constant* of U, such that

$$\mathcal{L}U(x) \leq -\gamma, \quad x \in \tilde{\mathcal{U}} =: \mathcal{U} \setminus \bar{\Omega}_{\rho_m}, \tag{2.4}$$

where $\tilde{\mathcal{U}}$ is called the *essential domain of* U.

(2) U is called a *strong Lyapunov function* in $\mathcal{U}$ with respect to (1.1) or $\mathcal{L}$, if

$$\lim_{x\to\partial\mathcal{U}} \mathcal{L}U(x) = -\infty.$$

Now consider a family of adjoint Fokker–Planck operators:

$$\mathcal{L}_{V,A} = a^{ij}\partial^2_{ij} + V^i\partial_i, \quad V = \left(V^i\right) \in \mathcal{V},\ A = \left(a^{ij}\right) \in \mathcal{A},$$

where $\mathcal{V}$ is a set of continuous functions from $\mathcal{U}$ to $\mathbb{R}^n$ under the compact-open topology and $\mathcal{A}$ is a set of $n \times n$ matrix-valued, $\mathcal{U}$-everywhere positive definite functions of the class $W^{1,p}_{loc}$ under the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$. By Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ for any pre-compact set $\Omega \subset \mathcal{U}$, the topology on $\mathcal{A}$ implies the compact-open topology in the space of $n \times n$ matrix-valued, continuous functions on $\mathcal{U}$.

Definition 2.3 Let $\mathcal{V}$, $\mathcal{A}$ be the families defined as above and U be a C^2 compact function on $\mathcal{U}$.

(1) U is said to be a *uniform Lyapunov function* with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V},\ A \in \mathcal{A}\}$ if it is a Lyapunov function in $\mathcal{U}$ with respect to each $\mathcal{L}_{V,A}$, $V \in \mathcal{V}$, $A \in \mathcal{A}$, with uniform essential lower bound ρ_m and Lyapunov constant γ which are independent of $V \in \mathcal{V}$ and $A \in \mathcal{A}$.
(2) U is said to be a *strong uniform Lyapunov function* with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V},\ A \in \mathcal{A}\}$ if $\mathcal{L}_{V,A}U \to -\infty$ uniformly in $V \in \mathcal{V}$ and $A \in \mathcal{A}$ as $x \to \partial\mathcal{U}$.
(3) U is said to be *of the class* $\mathcal{B}^*(\mathcal{A})$ with respect to the family $\mathcal{A}$ if it satisfies (2.1)–(2.3) in Definition 2.1 for each $A \in \mathcal{A}$ with some $\rho_m \in (0, \rho_M)$ and function H independent of $A \in \mathcal{A}$.

2.3 Relative Sequential Compactness and Tightness

For a Borel set $\Omega \subset \mathbb{R}^n$, we denote by $M(\Omega)$ the set of Borel probability measures on Ω equipped with the *weak***-topology*, i.e., $\mu_k \to \mu$ if and only if

$$\int_\Omega f(x)\mathrm{d}\mu_k(x) \to \int_\Omega f(x)\mathrm{d}\mu(x),$$

for every $f \in C_b(\Omega)$—the space of bounded, continuous functions in Ω. It is well-known that $M(\Omega)$ with the weak*-topology is metrizable.

Definition 2.4 A subset $\mathcal{M} \subset M(\Omega)$ is said to be *tight* if for any $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\epsilon) < \epsilon$ for all $\mu \in \mathcal{M}$.

Theorem 2.1 (Prokhorov's Theorem, [12, III-59]) *If a subset* $\mathcal{M} \subset M(\Omega)$ *is tight, then it is relatively sequentially compact in* $M(\Omega)$.

We note that if Ω is compact, then any subset of $M(\Omega)$ is tight.

2.4 Level Set Method

The level set method introduced in [14] contains two main ingredients: an integral identity and a derivative formula, both will play important roles in the measure estimates of stationary measures when the diffusion is degenerate.

Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *generalized Lipschitz domain* if

(i) Ω is a disjoint union of finitely many Lipschitz sub-domains;

(ii) Intersections of boundaries among these Lipschitz sub-domains only occur at finitely many points.

Theorem 2.2 (Integral identity, [14, Theorem 2.1]) *Assume that* (A) *holds in a domain* $\Omega \subset \mathbb{R}^n$ *and let* $u \in W^{1,p}_{loc}(\Omega)$ *be a weak stationary solution of the Fokker–Planck equation corresponding to* L *in* Ω*. Then for any generalized Lipschitz domain* $\Omega' \subset\subset \Omega$ *and any function* $F \in C^2(\bar{\Omega}')$ *with* $F|_{\partial\Omega'}$ *=constant,*

$$\int_{\Omega'} (\mathcal{L}F)u \, \mathrm{d}x = \int_{\partial\Omega'} \left(a^{ij}\partial_i F \nu_j\right) u \, \mathrm{d}s, \tag{2.5}$$

where for a.e. $x \in \partial\Omega'$, $(\nu_j(x))$ *denotes the unit outward normal vector of* $\partial\Omega'$ *at* x.

Theorem 2.3 (Derivative formula, [14, Theorem 2.2]) *For a given compact function* $U \in C^1(\mathcal{U})$ *and a function* $u \in C(\mathcal{U})$*, consider the function*

$$y(\rho) := \int_{\Omega_\rho} u \, \mathrm{d}x, \quad \rho \in (0, \rho_M)$$

and the open set of regular values of U,

$$\mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, \ x \in U^{-1}(\rho)\}, \tag{2.6}$$

where ρ_M *is the essential upper bound of* U *and* Ω_ρ *is the* ρ*-sublevel set of* U *for each* $\rho \in (0, \rho_M)$*. Then* y *is of class* C^1 *on* $\mathcal{I}$ *with derivatives*

$$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \, \mathrm{d}s, \quad \rho \in \mathcal{I}. \tag{2.7}$$

2.5 Existence and Regularity in the Case of Non-degenerate Diffusions

The following existence result is obtained in [15].

Theorem 2.4 ([15, Theorem A]) *Assume that* (A) *holds in* $\mathcal{U}$ *and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$*. If there exists a Lyapunov function in* $\mathcal{U}$ *with respect to the stationary Fokker–Planck Eq.* (1.1)*, then the Fokker–Planck equation corresponding to* L *admits a regular stationary measure in* $\mathcal{U}$ *with positive density function lying in the space* $W^{1,p}_{loc}(\mathcal{U})$*. If, in addition, the Lyapunov function is unbounded, then the stationary measure is unique in* $\mathcal{U}$*.*

The following regularity result for stationary measures of Fokker–Planck equations is proved in [5].

Theorem 2.5 (Bogachev–Krylov–Röckner [5]) *Assume that* (A) holds and (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$*. Then any stationary measure* μ *of the Fokker–Planck equation corresponding to* L *on* $\mathcal{U}$ *admits a positive density function* $u \in W^{1,p}_{loc}(\mathcal{U})$.

3 Upper-Semi-Continuity of Stationary Measures in Drift and Diffusion

3.1 Measure Estimates Via Level Set Method

Applying the level set method, we have the following measure estimates for a stationary measure in terms of the diffusion and a Lyapunov function for (1.1).

Lemma 3.1 *Assume that* (A) *holds and* (1.1) *has a Lyapunov function U in $\mathcal{U}$ with Lyapunov constant γ. Denote Ω_ρ as the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$, where ρ_m, respectively ρ_M, is the essential lower, respectively upper, bound of U. Then the following properties hold for any weak solution $u \in W^{1,p}_{loc}(\mathcal{U})$ of* (1.1)*:*

(i) $\int_{\mathcal{U}} |\mathcal{L}U(x)|u(x)\mathrm{d}x \le 2\int_{\Omega_{\rho_m}} |\mathcal{L}U(x)|u(x)\mathrm{d}x.$

(ii) ([14, Theorem A (b)]) *If U satisfies* (2.1) *and* (2.2) *for some non-negative, locally bounded function H defined on $[\rho_m, \rho_M)$, then*

$$\mu(\mathcal{U}\setminus\Omega_\rho) \le \mathrm{e}^{-\gamma\int_{\rho_m}^{\rho}\frac{1}{H(t)}\mathrm{d}t}, \quad \rho \in [\rho_m, \rho_M),$$

where μ is the stationary measure with density u.

Proof (i) For any $\rho \in (\rho_m, \rho_M)$, we fix a $\rho^* \in (\rho, \rho_M)$. Since Morse functions are dense in $C^2(\mathcal{U})$, there is a sequence of Morse functions $U_k \in C^2(\mathcal{U})$ such that $U_k \to U$ in $C^2(\mathcal{U})$. In particular, $U_k \to U$ in $C^2(\bar{\Omega}_{\rho^*})$. For each k, denote

$$\Omega_\rho^k = \left\{x \in \Omega_{\rho^*} : \text{ either } U_k(x) < \rho \text{ or } x \text{ is a local maximum of } U_k \text{ lying in } U_k^{-1}(\rho)\right\}.$$

By Lemmas 3.1, 3.2 in [14], there is a positive integer $k(\rho)$ such that for each $k \ge k(\rho)$, $\Omega_\rho^k \subset\subset \Omega_{\rho^*}$ and Ω_ρ^k is a generalized Lipschitz domain. An application of Theorem 2.2 with $F = U_k$, $\Omega = \Omega_{\rho^*}$, $\Omega' = \Omega_\rho^k$ for each $k \ge k(\rho)$ yields that

$$\int_{\Omega_\rho^k} (\mathcal{L}U_k)u \,\mathrm{d}x = \int_{\partial\Omega_\rho^k} ua^{ij}\partial_i U_k \nu_j \,\mathrm{d}s,$$

where (ν_j) are the unit outward normal vectors of $\partial\Omega_\rho^k$. Since (a^{ij}) is everywhere positive semi-definite in $\mathcal{U}$ and (ν_j) are well-defined and equal to $\nabla U_k/|\nabla U_k|$ except a finite number of points on $\partial\Omega_\rho^k$, $a^{ij}\partial_i U_k \nu_j \ge 0$ a.e. on $\partial\Omega_\rho^k$. It follows that

$$\int_{\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x \ge 0,$$

i.e.,

$$\begin{aligned}\int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho^k}(\mathcal{L}U_k)u \,\mathrm{d}x &= \int_{\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x - \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x \\ &\ge -\int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x, \end{aligned} \tag{3.1}$$

where for any Borel subset E of Ω_{ρ^*}, χ_E denotes the indicator function of E in Ω_{ρ^*}. Since U is a Lyapunov function and $\rho > \rho_m$, we have

$$\begin{aligned}\int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x &\le \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big|\mathcal{L}U_k - \mathcal{L}U\big|u \,\mathrm{d}x + \int_{U^{-1}(\rho)\cap\Omega_\rho^k} (\mathcal{L}U)u \,\mathrm{d}x \\ &\le \big|U_k - U\big|_{C^2(\Omega_{\rho^*})}\int_{\Omega_{\rho^*}} \big(|A| + |V|\big)u \,\mathrm{d}x - \gamma\mu\big(U^{-1}(\rho)\cap\Omega_\rho^k\big).\end{aligned}$$

It follows from the facts $u \in C(\bar{\Omega}_{\rho^*})$ and $U_k \to U$ in $C^2(\bar{\Omega}_{\rho^*})$ that

$$\limsup_{k\to\infty}\int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big(\mathcal{L}U_k\big)u \,\mathrm{d}x \le 0. \tag{3.2}$$

Since for any $x \in \Omega_{\rho^*} \setminus U^{-1}(\rho)$, $\chi_{\Omega_\rho^k}(x) \to \chi_{\Omega_\rho}(x)$ as $k \to \infty$, (3.1) and (3.2) imply that

$$0 \le \lim_{k\to\infty} \int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho^k}\big(\mathcal{L}U_k\big)u\,\mathrm{d}x = \int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho}\big(\mathcal{L}U\big)u\,\mathrm{d}x = \int_{\Omega_\rho} \big(\mathcal{L}U\big)u\,\mathrm{d}x.$$

By letting $\rho \to \rho_M$ in the above expression, we have

$$\int_{\mathcal{U}} \big(\mathcal{L}U\big)u\,\mathrm{d}x \ge 0,$$

or equivalently,

$$-\int_{\mathcal{U}\setminus\Omega_{\rho m}} \mathcal{L}U\mathrm{d}\mu(x) \le \int_{\Omega_{\rho m}} \mathcal{L}U\mathrm{d}\mu(x) \le \int_{\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x). \tag{3.3}$$

Since $\mathcal{L}U < 0$ on $\mathcal{U} \setminus \bar{\Omega}_{\rho_m}$, (3.3) becomes

$$\int_{\mathcal{U}\setminus\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x) = -\int_{\mathcal{U}\setminus\Omega_{\rho m}} \mathcal{L}U\mathrm{d}\mu(x) \le \int_{\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x).$$

Hence

$$\int_{\mathcal{U}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x) = \int_{\mathcal{U}\setminus\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x) + \int_{\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x) \le 2\int_{\Omega_{\rho m}} \big|\mathcal{L}U\big|\mathrm{d}\mu(x).$$

(ii) The original proof of [14, Theorem A (b)]) involves a complicated partition of $[\rho_m, \rho_M)$ so that the integral identity and the derivative formula are applicable on each partitioning sub-interval. To highlight the key idea of the proof, we give the proof below in the special case where $\nabla U \ne 0$ everywhere in the essential domain $\mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ of U.

In this case, we note that for each $\eta \in (\rho_m, \rho_M)$, Ω_η is a C^2 domain, whose boundary $\partial\Omega_\eta$ coincides with $U^{-1}(\eta)$, and the unit outward normal vector $\nu(x)$ of $\partial\Omega_\eta$ at each x is well-defined and equals $\nabla U(x)/|\nabla U(x)|$.

Let $\eta^* \in (\rho_m, \rho_M)$. For any $\eta \in (\rho_m, \eta^*)$, applications of Theorem 2.2 with $F = U$ on $\Omega' = \Omega_{\eta^*}, \Omega_\eta$, respectively, yield that

$$\int_{\partial\Omega_\eta} ua^{ij}\frac{\partial_i U \partial_j U}{|\nabla U|}\,\mathrm{d}s + \int_{\Omega_{\eta^*}\setminus\Omega_\eta} (a^{ij}\partial_{ij}^2 U + V^i\partial_i U)u\,\mathrm{d}x = \int_{\partial\Omega_{\eta^*}} ua^{ij}\frac{\partial_i U \partial_j U}{|\nabla U|}\,\mathrm{d}s.$$

In the above identity, since the right hand side is non-negative, applications of (2.2) with $H + \epsilon$ in place of H, where $0 < \epsilon \ll 1$, to the first term of the left hand side, and the definition of Lyapunov function to the second term of the left hand side yield that

$$\gamma\int_{\Omega_{\eta^*}\setminus\Omega_\eta} u\,\mathrm{d}x \le (H(\eta)+\epsilon)\int_{\partial\Omega_\eta}\frac{u}{|\nabla U|}\,\mathrm{d}s, \quad \eta \in \big[\rho_m, \eta^*\big). \tag{3.4}$$

Consider the function

$$y(\eta) = \mu\big(\Omega_{\eta^*}\setminus\Omega_\eta\big) = \int_{\Omega_{\eta^*}\setminus\Omega_\eta} u\,\mathrm{d}x, \quad \eta \in \big(\rho_m, \eta^*\big).$$

By Theorem 2.3, $y(\eta)$ is of the class C^1 on (ρ_m, η^*) and

$$y'(\eta) = -\int_{\partial\Omega_\eta}\frac{u}{|\nabla U|}\,\mathrm{d}s, \quad \eta \in \big(\rho_m, \eta^*\big).$$

Hence by (3.4),

$$y'(\eta) + \frac{\gamma}{H(\eta)+\epsilon} y(\eta) \le 0, \quad \eta \in \big(\rho_m, \eta^*\big). \tag{3.5}$$

For any $\rho_0 \in (\rho_m, \eta^*)$, $\rho \in (\rho_0, \eta^*)$, a direct integration of (3.5) in the interval $[\rho_0, \rho]$ yields that

$$\mu\big(\Omega_{\eta^*} \setminus \Omega_\rho\big) \le \mu\big(\Omega_{\eta^*} \setminus \Omega_{\rho_0}\big) e^{-\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)+\epsilon} dt} < e^{-\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)+\epsilon} dt}.$$

The proof is complete by letting $\eta^* \to \rho_M$, $\rho_0 \to \rho_m$, and $\epsilon \to 0$ in the above inequality. □

Remark 3.1 As to be seen in a separate work [17], part (ii) of the above lemma will also be useful in characterizing local concentration of a family of stationary measures associated with a so-called null family of diffusion matrices, as noises tend to zero.

3.2 Upper Semi-Continuity of a Family of Stationary Measures

Consider a family of adjoint Fokker–Planck operators:

$$\mathcal{L}_{V,A} = a^{ij}\partial^2_{ij} + V^i\partial_i, \quad V = \big(V^i\big) \in \mathcal{V}, \quad A = \big(a^{ij}\big) \in \mathcal{A},$$

where $\mathcal{V}$ is a set of continuous functions from $\mathcal{U}$ to $\mathbb{R}^n$ under the compact-open topology and $\mathcal{A}$ is a set of $n \times n$ matrix-valued, $\mathcal{U}$-everywhere positive definite functions of the class $W^{1,p}_{loc}$ under the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$. For each $V \in \bar{\mathcal{V}}$ and $A \in \bar{\mathcal{A}}$, we denote $\mathcal{M}_{V,A}$ the set of stationary measures of the Fokker–Planck equation corresponding to $L = L_{V,A}$, i.e.,

$$\mathcal{M}_{V,A} = \big\{\mu \in M(\mathcal{U}) : \ \mu \text{ satisfies (1.4) with } \mathcal{L} = \mathcal{L}_{V,A}\big\}, \tag{3.6}$$

where $M(\mathcal{U})$ denotes the space of the Borel probability measures on $\mathcal{U}$ equipped with the weak*-topology.

Assume that there exists a uniform Lyapunov function in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V}, A \in \mathcal{A}\}$. Then it follows from Theorem 2.4 that $\mathcal{M}_{V,A}$ is non-empty for each $V \in \mathcal{V}$ and $A \in \mathcal{A}$. The following theorem, from which Theorem B in Sect. 1 follows, says that under some additional conditions on the uniform Lyapunov function on $\mathcal{U}$ the map $(V, A) \in \mathcal{V} \times \mathcal{A} \mapsto \mathcal{M}_{V,A}$ is upper semi-continuous and can be extended to $\bar{\mathcal{V}} \times \bar{\mathcal{A}}$ in an upper semi-continuous way. This amounts to showing the relative sequential compactness of $\mathcal{M} = \bigcup_{V\in\mathcal{V},A\in\mathcal{A}} \mathcal{M}_{\mathcal{V},\mathcal{A}}$ in $M(\mathcal{U})$.

Theorem 3.1 *Assume that there is a uniform Lyapunov function U in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_{V,A} : V \in \mathcal{V}, A \in \mathcal{A}\}$. Let ρ_m and ρ_M be the essential lower and upper bound of U respectively, and Ω_ρ be the ρ-sublevel set of U in $\mathcal{U}$ for each $\rho \in [\rho_m, \rho_M)$. Further assume that*

(i) *Either U is a uniform strong Lyapunov function such that*

$$\sup_{V\in\mathcal{V},A\in\mathcal{A}} \Big(|V|_{C(\bar{\Omega}_{\rho_m})} + |A|_{C(\bar{\Omega}_{\rho_m})}\Big) < \infty; \tag{3.7}$$

or

(ii) *U is of the class $\mathcal{B}^*(\mathcal{A})$.*

Then the following properties hold:

(a) *The set $\mathcal{M} = \bigcup_{V\in\mathcal{V},A\in\mathcal{A}} \mathcal{M}_{V,A}$ is relatively sequentially compact in $M(\mathcal{U})$;*

(b) *For any* $(V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$, *any sequence* $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ *with* $(V_k, A_k) \to (V_0, A_0)$, *and any sequence of corresponding stationary measures* μ_{V_k,A_k}, *there is a subsequence* (V_{k_i}, A_{k_i}) *such that* $\mu_{V_{k_i},A_{k_i}}$ *converges to a stationary measure* $\mu_0 \in \mathcal{M}_{V_0,A_0}$. *In particular,* $\mathcal{M}_{V_0,A_0} \neq \emptyset$.

Proof By Theorem 2.5, every stationary measure $\mu_{V,A}$, for $V \in \mathcal{V}$, $A \in \mathcal{A}$, admits a positive density function in $W^{1,p}_{loc}(\mathcal{U})$ which is necessarily a weak solution of (1.1).

In the case (i), we have

$$\gamma_{V,A}(\rho) =: \inf_{x \in \mathcal{U} \setminus \bar{\Omega}_\rho} |\mathcal{L}_{V,A} U(x)| \to +\infty,$$

as $\rho \to \rho_M$, uniformly with respect to $V \in \mathcal{V}$, $A \in \mathcal{A}$. By (3.7),

$$C =: \sup_{V \in \mathcal{V}, A \in \mathcal{A}} |\mathcal{L}_{V,A} U|_{C(\bar{\Omega}_{\rho_m})} < \infty.$$

It then follows from Lemma 3.1 (i) that

$$\begin{aligned} \mu_{V,A}\big(\mathcal{U} \setminus \bar{\Omega}_\rho\big) &\leq \gamma^{-1}_{V,A}(\rho) \int_{\mathcal{U} \setminus \bar{\Omega}_\rho} \big|\mathcal{L}_{V,A} U(x)\big| \mathrm{d}\mu_{V,A}(x) \\ &\leq \gamma^{-1}_{V,A}(\rho) \int_{\mathcal{U}} \big|\mathcal{L}_{V,A} U(x)\big| \mathrm{d}\mu_{V,A}(x) \leq 2C\gamma^{-1}_{V,A}(\rho) \to 0, \end{aligned} \tag{3.8}$$

as $\rho \to \rho_M$, uniformly with respect to $V \in \mathcal{V}$, $A \in \mathcal{A}$.

In the case (ii), we let γ be a Lyapunov constant of U and H be a function on $[\rho_m, \rho_M)$ satisfying Definition 2.3 (3) with respect to the family $\mathcal{A}$. It follows from Lemma 3.1 (ii) and the condition (2.3) that

$$\mu_{V,A}\big(\mathcal{U} \setminus \bar{\Omega}_\rho\big) \leq \mu_{V,A}\big(\mathcal{U} \setminus \Omega_\rho\big) \leq \mathrm{e}^{-\gamma \int_{\rho_m}^{\rho} \frac{1}{H(t)} \mathrm{d}t} \to 0, \tag{3.9}$$

as $\rho \to \rho_M$, uniformly with respect to $V \in \mathcal{V}$, $A \in \mathcal{A}$.

It now follows from (3.8), (3.9) that $\mathcal{M}$ is tight, hence by Theorem 2.1 it is relatively sequentially compact. This proves (a).

To prove (b), we let $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ be a sequence such that $(V_k, A_k) \to (V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$. By Theorem 2.4, $\mathcal{M}_{V_k,A_k} \neq \emptyset$ for each k. By (a), we may assume without loss of generality that $\mu_k =: \mu_{V_k,A_k}$ converges, under weak*-topology, to a measure $\mu_0 \in M(\mathcal{U})$. Since

$$\int_{\mathcal{U}} \mathcal{L}_{V_k,A_k} f(x) \mathrm{d}\mu_k(x) = 0, \quad f \in C_0^\infty(\mathcal{U}),\ k = 1, 2, \ldots,$$

and $\mathcal{L}_{V_k,A_k} f \to \mathcal{L}_{V_0,A_0} f$ uniformly on the support of f as $k \to \infty$, passing to the limit $k \to \infty$ yields that

$$\int_{\mathcal{U}} \mathcal{L}_{V_0,A_0} f(x) \mathrm{d}\mu_0(x) = 0, \quad f \in C_0^\infty(\mathcal{U}),$$

i.e., $\mu_0 \in \mathcal{M}_{V_0,A_0}$. Hence $\mathcal{M}_{V_0,A_0} \neq \emptyset$. □

Remark 3.2 The upper semi-continuity of the set-valued map $\bar{\mathcal{V}} \times \bar{\mathcal{A}} \to \bar{\mathcal{M}} : (V, A) \mapsto \mathcal{M}_{V,A} \cap \bar{\mathcal{M}}$, shown in the above theorem, resembles the deterministic case for which it is well-known that compact global attractors associated with a continuous family of dissipative systems vary upper semi-continuously.

4 Stationary Measures Corresponding to Degenerate Diffusion

In this section, we will consider the problem of the existence of stationary measures of the Fokker–Planck equation corresponding to L when the diffusion matrix $A = (a^{ij})$ is only positive semi-definite in $\mathcal{U}$. As in [7], we will tackle this problem by perturbing A with a family of positive definite matrixes then pass weak*-limit of the corresponding stationary measures as the perturbations tend to zero.

The following theorem is just Theorem A in Sect. 1.

Theorem 4.1 *Assume that there exists a Lyapunov function U with respect to the operator $\mathcal{L}_{V_0,A_0}$, where $A_0 = (a_0^{ij})$ with $a_0^{ij} \in W^{1,p}_{loc}(\mathcal{U})$, $i, j = 1, \ldots, n$, and $V_0 \in C(\mathcal{U}, \mathbb{R}^n)$. Also assume that*

(i) *Either U is a strong Lyapunov function in $\mathcal{U}$ with respect to $\mathcal{L}_{V_0,A_0}$; or*
(ii) *U is of the class $\mathcal{B}^*(A_0)$.*

Then the Fokker–Planck equation corresponding to $L = L_{V_0,A_0}$ admits a stationary measure in $\mathcal{U}$.

Proof Let

$$H^*\big(\rho\big) = \max_{x \in U^{-1}\left(\rho\right)} a_0^{ij}\big(x\big)\partial_i U\big(x\big)\partial_j U\big(x\big), \quad \rho \in \big[0, \rho_M\big).$$

It is clear that the function H^* is locally bounded.

We will first prove the following claim:

Claim 1 If U is of the class $\mathcal{B}^*(A_0)$, then there is a positive, locally bounded function H_1 on $[0, \rho_M)$ such that $\min_{t \in [0,\rho]}(H_1 - H^*)(t) =: c_\rho > 0$ for any $\rho \in [0, \rho_M)$ and $\int_0^{\rho_M} \frac{1}{H_1(t)} \mathrm{d}t = +\infty$.

Since U is of the class $\mathcal{B}^*(A_0)$, there exist a constant $\rho_m \geq 0$ and a non-negative, locally bounded function H on $[\rho_m, \rho_M)$ such that

$$\begin{aligned} &a_0^{ij}(x)\partial_i U(x)\partial_j U(x) \leq H(\rho), \quad x \in U^{-1}(\rho), \quad \rho \in \big[\rho_m, \rho_M\big), \text{ and} \\ &\int_{\rho_0}^{\rho_M} \frac{1}{H(t)} \mathrm{d}t = +\infty, \quad \forall \rho_0 \in \big(\rho_m, \rho_M\big). \end{aligned} \tag{4.1}$$

We let $\{s_i : i = 0, 1, \ldots\}$ be a strictly increasing sequence in $[\rho_m, \rho_M)$ such that $s_0 = \rho_m$ and $\lim\limits_{i\to\infty} s_i = \rho_M$. For each $i = 1, 2, \ldots$, choose a continuous function $u_i : [s_{i-1}, s_i] \to [\frac{1}{2}, 1]$ such that $u_i(s_{i-1}) = u_i(s_i) = 1$ and $u_i(t) < 1, t \in (s_{i-1}, s_i)$. Since $\lim_{n\to\infty} \int_{s_{i-1}}^{s_i} \frac{1}{H+u_i^n} \mathrm{d}t = \int_{s_{i-1}}^{s_i} \frac{1}{H} \mathrm{d}t$, there exists an integer $n(i) \in \mathbb{N}$ such that $\int_{s_{i-1}}^{s_i} \frac{1}{H+u_i^{n(i)}} \mathrm{d}t \geq \frac{1}{2} \min\{\int_{s_{i-1}}^{s_i} \frac{1}{H} \mathrm{d}t, i\}$.

Let $H_1(t) = H^*(t) + u_i^{n(i)}(t)$, $t \in [s_{i-1}, s_i]$, $i = 1, 2, \cdots$, and $H_1(t) = H^*(t) + 1$, $t \in [0, \rho_m]$. It is clear that H_1 is locally bounded. For a given $\rho \in [0, \rho_M)$, there exists i_0 sufficiently large such that $\rho \leq s_{i_0}$. It is clear that for $t \in [0, \rho]$, $(H_1 - H^*)(t) \geq \min_{i \leq i_0} u_i^{n(i)}(t) \geq \min_{i \leq i_0} 2^{-n(i)}$. Hence $c_\rho > 0$. Moreover, since $H^* \leq H$ and (4.1) holds, we have

$$\begin{aligned}\int_{\rho_m}^{\rho_M} \frac{1}{H_1}\mathrm{d}t &\geq \sum_{i=1}^{\infty}\int_{s_{i-1}}^{s_i} \frac{1}{H+u_i^{n(i)}}\mathrm{d}t \geq \frac{1}{2}\sum_{i=1}^{\infty}\min\left\{\int_{s_{i-1}}^{s_i}\frac{1}{H}\mathrm{d}t, i\right\}\\ &\geq \frac{1}{2}\liminf_{k\to+\infty}\sum_{i=k}^{\infty}\min\left\{\int_{s_{i-1}}^{s_i}\frac{1}{H}\mathrm{d}t, k\right\} \geq \frac{1}{2}\liminf_{k\to+\infty}\min\left\{\sum_{i=k}^{\infty}\int_{s_{i-1}}^{s_i}\frac{1}{H}\mathrm{d}t, k\right\}\\ &= \liminf_{k\to+\infty} k = +\infty.\end{aligned}$$

This proves *Claim 1*.

Now we let V_0 be fixed and perturb $A_0 = (a_0^{ij})$ by a set $\mathcal{A}$ of everywhere positive definite, matrix-valued functions in $\mathcal{U}$, in the following way. Consider the function

$$\epsilon(x) = \frac{\min\left\{\epsilon_0, H_1\big(U(x)\big) - H^*\big(U(x)\big)\right\}}{\big|D^2U(x)\big| + \big|\nabla U(x)\big|^2 + 1}, \quad x \in \mathcal{U},$$

where D^2U denotes the Hessian matrix of U and

$$\epsilon_0 = \begin{cases} 1, & \text{in the case (i);}\\ \frac{\gamma}{2}, & \text{in the case (ii)}\end{cases}$$

with γ being the Lyapunov constant of U in the case (ii). By *Claim 1*, the function $\epsilon(x)$ has a positive minimum on any compact subset of $\mathcal{U}$.

Let $\mathcal{A}$ be the set of all everywhere positive definite, matrix-valued functions $A = (a^{ij})$ in $\mathcal{U}$ of the class $W^{1,p}_{loc}$ such that

$$\max_{1\leq i,j\leq n}\left|a^{ij}(x) - a_0^{ij}(x)\right| \leq \epsilon(x), \quad x \in \mathcal{U}.$$

Then for each $A \in \mathcal{A}$, we clearly have

$$\begin{aligned}&\left|\big(\mathcal{L}_{V_0,A} - \mathcal{L}_{V_0,A_0}\big)U(x)\right| \leq \epsilon_0,\\ &\left|\big(a^{ij}(x) - a_0^{ij}(x)\big)\partial_i U(x)\partial_j U(x)\right| \leq H_1\big(U(x)\big) - H^*\big(U(x)\big).\end{aligned}$$

It follows that, with respect to the family $\{\mathcal{L}_{V_0,A} : A \in \mathcal{A}\}$, U is a uniform strong Lyapunov function in the case (i) and a uniform Lyapunov function of the class $\mathcal{B}^*(\mathcal{A})$ for the function H_1 in the case (ii) by *Claim 1*.

We now prove the next claim:

Claim 2 $\mathcal{A}$ contains A_0 as a limit point.

It suffices to construct a family $A_\epsilon, \epsilon \in (0, 1]$, such that $A_\epsilon \in \mathcal{A}$ and $A_\epsilon \to A_0$ as $\epsilon \to 0$. Using a partition of unity, we show that there exists a sequence $\{f_i : i \in \mathcal{I}\}$, $\mathcal{I} \subset \mathbb{N}$, of non-negative, C^∞ functions on $\mathcal{U}$ such that $\mathrm{supp}(f_i) \subset\subset \mathcal{U}$, $i \in \mathcal{I}$, $\{\mathrm{supp}(f_i) : i \in \mathcal{I}\}$ is a locally finite cover of $\mathcal{U}$, and

$$\sum_{i\in I} f_i(x) = 1, \quad x \in \mathcal{U}.$$

For each $i \in \mathcal{I}$, denote $c_i = \min_{x\in\mathrm{supp}(f_i)}\epsilon(x)$. Since each $\mathrm{supp}(f_i)$ is a compact subset of $\mathcal{U}$, we have $c_i > 0$ for each $i \in \mathcal{I}$. Consider the function $G(x) = \sum_{i\in\mathcal{I}} c_i f_i(x)$, $x \in \mathcal{U}$. Then it is not hard to see that G is a well-defined C^∞ function, and $0 < G(x) \leq \epsilon(x)$, $x \in \mathcal{U}$.

Let $A_\epsilon(x) = A_0 + \epsilon G(x)I$, $\epsilon \in (0, 1]$, $x \in \mathcal{U}$, where I denotes the identity matrix. Then it is clear that the family $\{A_\epsilon\}$ satisfies the desired property. This proves *Claim 2*.

Since $A_0 \in \bar{\mathcal{A}}$, an application of Theorem 3.1 yields the existence of a stationary measure of the Fokker–Planck equation in $\mathcal{U}$ corresponding to L with $V = V_0$, $A = A_0$. □

Remark 4.1 We note that when A is degenerate in $\mathcal{U}$ the stationary measures need not admit density functions. For example, consider $\mathcal{U} = \mathbb{R}^1$, $A \equiv 0$, and $V(x) = x$, $x \in R^1$. It is easy to see by a simple calculation that the corresponding Fokker–Planck equation admits no weak stationary solution at all. However, it is clear that the Dirac measure at the origin is a stationary measure of the corresponding Fokker–Planck equation.

We now consider the special case $\mathcal{U} = \mathbb{R}^n$ in Theorem 4.1. In the following corollary, we note that part (a) generalizes the result of [7] in the case of degenerate diffusion.

Corollary 4.1 *Let* $\mathcal{U} = \mathbb{R}^n$, $A = (a^{ij})$ *with* $a^{ij} \in W^{1,p}_{loc}(\mathbb{R}^n)$, $i, j = 1, \ldots, n$, *and* $V \in C(\mathbb{R}^n, \mathbb{R}^n)$. *Further assume that there is a function* $U \in C^2(\mathbb{R}^n)$ *such that either*

(a) *U is a compact function in $\mathbb{R}^n$ satisfying (1.6); or*
(b) *U satisfies*

$$\limsup_{x\to\infty} \mathcal{L}U(x) = -\gamma \tag{4.2}$$

for some constant $\gamma > 0$, both A and D^2U are bounded under the sup-norm, and there is a constant $r_0 > 0$ such that D^2U is uniformly positive definite in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$.

Then the Fokker–Planck equation corresponding to $L = L_{V,A}$ admits a stationary measure in $\mathbb{R}^n$.

Proof Case (a) follows immediately from Theorem 4.1(i).

In the case (b), Proposition 2.1 implies that there is a constant $c \geq 0$ such that $U + c$ is an unbounded compact function in $\mathbb{R}^n$ which is of the class $\mathcal{B}^*(A)$. Since $U + c$ also satisfies (4.2), it is a Lyapunov function in $\mathbb{R}^n$. Therefore, all conditions in Theorem 4.1(ii) are satisfied. □

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