Concentration and limit behaviors of stationary measures

Wen Huang [a], Min Ji [b], Zhenxin Liu [c], Yingfei Yi [d,e,*]

[a] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China
[b] Academy of Mathematics and System Sciences and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, and University of Chinese Academy of Sciences, Beijing 100080, PR China
[c] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China
[d] Department of Mathematical & Statistical Sci, University of Alberta, Edmonton Alberta, Canada T6G 2G1
[e] School of Mathematics, Jilin University, Changchun 130012, PR China

HIGHLIGHTS

- Noise vanishing limits of stationary measures of Fokker–Planck equations are studied.
- Invariance of the limit measures to the ODE system is shown.
- Local concentrations of stationary measures in the vicinity of a local attractor or repeller are characterized.
- Noise stabilization of a local attractor and de-stabilization of a local repeller are shown with respect to particular noises.
- Noise de-stabilization of a repelling equilibrium is shown with respect to general noises.

ARTICLE INFO

ABSTRACT

In this paper, we study limit behaviors of stationary measures of the Fokker–Planck equations associated with a system of ordinary differential equations perturbed by a class of multiplicative noise including additive white noise case. As the noises are vanishing, various results on the invariance and concentration of the limit measures are obtained. In particular, we show that if the noise perturbed systems admit a uniform Lyapunov function, then the stationary measures form a relatively sequentially compact set whose weak*-limits are invariant measures of the unperturbed system concentrated on its global attractor. In the case that the global attractor contains a strong local attractor, we further show that there exists a family of admissible multiplicative noises with respect to which all limit measures are actually concentrated on the local attractor; and on the contrary, in the presence of a strong local repeller in the global attractor, there exists a family of admissible multiplicative noises with respect to which no limit measure can be concentrated on the local repeller. Moreover, we show that if there is a strongly repelling equilibrium in the global attractor, then limit measures with respect to typical families of multiplicative noises are always concentrated away from the equilibrium. As applications of these results, an example of stochastic Hopf bifurcation and an example with non-decomposable ω-limit sets are provided.

Our study is closely related to the problem of noise stability of compact invariant sets and invariant measures of the unperturbed system.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

Regarded as a physical model, a dynamical system generated from ordinary differential equations is often subject to noise perturbations either from its surrounding environment or from intrinsic uncertainties associated with the system. Analyzing the impact of noise perturbations on the dynamics of the system then becomes a fundamental issue with respect to both modeling and dynamics.

There have been many studies toward this dynamics issue using either a trajectory-based or a distribution-based approach. The trajectory-based approach is often adopted under the framework of random dynamical systems, i.e., skew-product flows with ergodic measure-preserving base flows. By assuming vanishing noise at a reference equilibrium, noise perturbations of essential dynamics of a dynamical system are studied under the random dynamical system framework with respect to problems such as noise perturbations of invariant manifolds [1–4], normal forms

[5–8], and stochastic bifurcations (see [5] and references therein). For a system of ordinary differential equations subject to white noise perturbations vanishing at a reference equilibrium, we refer the reader to [9] for some study of stochastic stability of the equilibrium (see also [10] for similar studies in infinite dimension).

With respect to general noise perturbations, the distribution-based approach is useful and seemly necessary to adopt under both frameworks of random dynamical systems and Itô stochastic differential equations. Due to its essential differences from deterministic dynamical systems, much less is known in this direction comparing with cases using the trajectory-based approach. For some import pioneer works on noise perturbations of dynamical systems on a compact manifold from the viewpoint of distributions, we refer the reader to [11,12] for stochastic stability of flows on a 2-torus or a periodic cycle, to [13] for stochastic stability of essentially a finite number of equilibria and periodic cycles by introducing large deviation theory, to [14–17] for stochastic stability of SRB measures, and to [18] for some global stochastic stability characterizations.

In this paper, we adopt the distribution-based approach to study the impact of white noises on basic dynamics of a system of ordinary differential equations in an Euclidean space. More precisely, we consider a system of ordinary differential equations

$$\dot{x} = V(x), \qquad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{U}$ is a connected open set which can be bounded, unbounded, or the entire $\mathbb{R}^n$, and $V = (V^i) \in C(\mathcal{U}, \mathbb{R}^n)$. We assume throughout the paper that (1.1) generates a local flow φ^t on $\mathcal{U}$. The generality of domain $\mathcal{U}$ does allow a wide range of applications because many physical models (e.g., those concerning populations and concentrations) are not necessarily defined in the entire $\mathbb{R}^n$. Adding general multiplicative (i.e., spatially non-homogeneous) including additive (i.e., spatially homogeneous) white noise perturbations, we obtain the following Itô stochastic differential equations (SDE in short)

$$dx = V(x)dt + G(x)dW, \qquad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{1.2}$$

where W is a standard m-dimensional Brownian motion for some integer $m \geq n$, and $G = (g^{ij})_{n \times m}$ is a matrix-valued function on $\mathcal{U}$, called *noise coefficient matrix*. For generality, we assume that $g^{ij} \in W^{1,p}_{loc}(\mathcal{U}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$, for some fixed constant $p > n$.

The stochastic differential equations (1.2) arise naturally as a non-isolated physical system subject to noise perturbations from its surrounding environments, in which the impact of noises on dynamics is often physically measured in term of distributions. They can also arise naturally from the study of a large scale deterministic but seemingly stochastic system, for instance a so-called mesoscopic system which is partially structured but contains intrinsic uncertainties in a fast time scale due to high complexity, large degree of freedom, lack of full knowledge of mechanisms, the need for organizing a large amount of data, etc. Under some exponential mixing assumptions on the fast dynamics, such a mesoscopic system can have a stochastic reduction of the form (1.2) over any finite time interval in which V represents the structured field and G encompasses all dynamical uncertainties (see e.g., [19,20]). Very recently, such stochastic reduction theory is systematically extended to the case of partially hyperbolic fast–slow systems (see e.g., [21–23]). It has been argued for a mesoscopic system that in the case of sufficiently high uncertainty, a trajectory-based approach using either deterministic or random dynamics modeling would not provide much information to its dynamical description. Instead, a distribution-based approach using stochastic differential equations like (1.2) is necessary to adopt in order to synthesize the typical patterns of dynamics (see [24] and references therein).

An important distribution-based approach for studying diffusion process generated by (1.2) is to use its associated *Fokker–Planck equation* (also called *Kolmogorov forward equation*)

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} = L_A u(x, t), & x \in \mathcal{U}, \ t > 0, \\ u(x, t) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x, t)dx = 1, \end{cases} \tag{1.3}$$

where $A = (a^{ij}) = \frac{GG^\top}{2}$, called the *diffusion matrix*, and L_A is the Fokker–Planck operator defined as

$$L_A g(x) = \partial^2_{ij}(a^{ij}(x)g(x)) - \partial_i(V^i(x)g(x)), \qquad g \in C^2(\mathcal{U}).$$

We note that $a^{ij} \in W^{1,p}_{loc}(\mathcal{U}), i, j = 1, 2, \ldots, n$. It is well-known that if the stochastic differential equation (1.2) generates a (local) diffusion process in $\mathcal{U}$ (e.g., when both V and G are locally Lipschitz in $\mathcal{U}$), then its transition probability density function, if it exists, is actually a (local) fundamental solution of the Fokker–Planck equation (1.3).

In the above and also through the rest of the paper, we use short notions $\partial_i = \frac{\partial}{\partial x_i}, \partial^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \ldots, n$ whenever applicable.

Long time behaviors of solutions of the Fokker–Planck equation (1.3) is governed by the stationary Fokker–Planck equation

$$\begin{cases} L_A u = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u) = 0, \\ u(x) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x)dx = 1, \end{cases} \tag{1.4}$$

which, in the weak form, becomes

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}_A f(x)u(x)dx = 0, & \text{for all } f \in C^\infty_0(\mathcal{U}), \\ u(x) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x)dx = 1, \end{cases} \tag{1.5}$$

where $C^\infty_0(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports and

$$\mathcal{L}_A = a^{ij}\partial^2_{ij} + V^i \partial_i$$

is the adjoint Fokker–Planck operator corresponding to A. Solutions of (1.5) are called *weak stationary solutions of* (1.3) or *stationary solutions corresponding to* $\mathcal{L}_A$. More generally, one considers a measure-valued stationary solution μ_A of the Fokker–Planck equation (1.3), called a *stationary measure of the Fokker–Planck equation* (1.3) or a *stationary measure corresponding to* $\mathcal{L}_A$, which is a Borel probability measure satisfying

$$\int_{\mathcal{U}} \mathcal{L}_A f(x)d\mu_A(x) = 0, \qquad \text{for all } f \in C^\infty_0(\mathcal{U}). \tag{1.6}$$

If a stationary measure μ_A is regular, i.e., $d\mu_A(x) = u_A(x)dx$ for some density function $u_A \in C(\mathcal{U})$, then it is clear that u_A is necessarily a weak stationary solution of (1.3), i.e, it satisfies (1.5). Conversely, according to the regularity theorem in [25], if (a^{ij}) is everywhere positive definite in $\mathcal{U}$, then any stationary measure corresponding to $\mathcal{L}_A$ must be regular with positive density function lying in $W^{1,p}_{loc}(\mathcal{U})$. In the case that (1.2) generates a diffusion process on $\mathcal{U}$, it is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the Fokker–Planck equation (1.3), but the converse need not be true. However, under some mild conditions a stationary measure of the Fokker–Planck equation (1.3) is always a sub-invariant measure of some generalized diffusion process (see [26–28] for discussions in this regard in particular with respect to the uniqueness of stationary measures and their invariance). In this sense, stationary measures of (1.3) may be regarded as generalizations of invariant measures of a classical diffusion process.

The existence of stationary measures of Fokker–Planck equations in $\mathbb{R}^n$ has been extensively investigated (see e.g., [9,25,26,29–31,39] and references therein). In our recent work [32,33], such existence is investigated for a general domain under certain relaxed Lyapunov conditions. In addition, results concerning non-existence of stationary measures of Fokker–Planck equations in a general domain are also obtained in our work [34] under some anti-Lyapunov conditions, which, together with the existence results in [32,33], lead to both sufficient and necessary conditions for the existence of stationary measures of Fokker–Planck equations.

While the non-existence of stationary measures of Fokker–Planck equations associated with a family of noise perturbations reflects a strong stochastic instability of the unperturbed deterministic system with respect to these noises, stochastic stability and instability of the deterministic system at a dynamics level can often occur with respect to noise families for which stationary measures do exist for the corresponding Fokker–Planck equations. In order to study the impact of noises on such stochastic stability of the deterministic system when it generates a local flow, a fundamental problem is to classify basic dynamics subjects like compact invariant sets and invariant measures of the local flow that can "survive" from a given family of noise perturbations. This is in fact our main motivation for the present study.

To be more precise, for a fixed drift field $V \in C(\mathcal{U}, \mathbb{R}^n)$, we consider noise coefficients matrices lying in the class

$$\tilde{\mathcal{G}} = \{G = (g^{ij}) : \text{Rank}(G) \equiv n, \ g^{ij} \in W_{loc}^{1,p}(\mathcal{U}),$$
$$i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m\}$$

for some fixed $p > n$. The class $\tilde{\mathcal{G}}$ gives rise to the following class of diffusion matrices:

$$\tilde{\mathcal{A}} = \{A = (a^{ij}) \in W_{loc}^{1,p}(\mathcal{U}, GL(n, \mathbb{R})) : A = \frac{GG^\top}{2}$$
$$\text{for some } G \in \tilde{\mathcal{G}}\}. \tag{1.7}$$

To consider small noise perturbations, we will pay particular attention to the so-called *null family* (resp. *bounded null family*) $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, i.e., a directed net of $\tilde{\mathcal{A}}$ which converges to 0—the zero matrix, under the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$ (resp. L^∞-convergence on $\mathcal{U}$). Then with respect to the system (1.1) and a given null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, our study amounts to the characterization of behaviors of $\mathcal{A}$-*limit measures*, i.e., sequential limit points, as $A_\alpha \to 0$, of stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$ under the weak*-topology. Our particular attention will be paid to issues such as invariance and concentration of $\mathcal{A}$-limit measures, stochastic stability of compact invariant sets under uniform Lyapunov conditions, and the role played by the multiplicative noises $\mathcal{A}$ to the stabilization of a strong local attractor or de-stabilization of a strong local repeller.

Our main results of the paper are as follows.

Theorem A. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family. Then the following holds.*

(a) *(Invariance of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and φ^t is a flow on $\mathcal{U}$, then any $\mathcal{A}$-limit measure must be an invariant measure of φ^t.*

(b) *(Stochastic LaSalle invariance principle) If (1.1) admits an entire weak Lyapunov (resp. anti-Lyapunov) function U_0, then any $\mathcal{A}$-limit measure with compact support is concentrated on the set $S_0 = \{x \in \mathcal{U} : V(x) \cdot \nabla U_0(x) = 0\}$.*

(c) *(Local concentration of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and $\mathcal{E}_0$ is either a strong local attractor or a strong local repeller of φ^t, then there is a neighborhood $\mathcal{W}_0$ of $\mathcal{E}_0$ in $\mathcal{U}$ such that $\mu(\mathcal{W}_0 \setminus \mathcal{E}_0) = 0$ for any $\mathcal{A}$-limit measure μ.*

Thus, when both V and A'_αs are smooth in $\mathcal{U}$, it follows from part (a) of Theorem A that the stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$, by having smooth density functions, can be regarded as smoothers of their limit invariant measures.

We note that parts (b), (c) of Theorem A do not require φ^t be a flow on $\mathcal{U}$, in which case there is no guarantee that an $\mathcal{A}$-limit measure is an invariant measure of φ^t. However, in the case that φ^t is only a semiflow on $\mathcal{U}$, the invariance of an $\mathcal{A}$-limit measure can be also shown under certain conditions (see Theorem 3.1(c) and Theorem B).

Given a null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, if there exists a uniform Lyapunov function in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_{A_\alpha}\}$ (see Section 2), then there must exist a stationary measure μ_α corresponding to each $\mathcal{L}_{A_\alpha}$ (Proposition 2.1) and φ^t is necessarily a dissipative semiflow on $\mathcal{U}$ (Proposition 2.4) whose global attractor necessarily admits a flow extension (Proposition A.1).

Theorem B. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family. Assume that there is a uniform Lyapunov function in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_{A_\alpha}\}$ and denote by $\{\mu_\alpha\}$ the set of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$. Then the following holds.*

(a) *(Tightness of stationary measures) $\{\mu_\alpha\}$ is $\mathcal{A}$-sequentially null compact in $M(\mathcal{U})$—the space of Borel probability measures on $\mathcal{U}$, i.e., for any sequence $\{A_\ell\}_{\ell=1}^\infty \subset \mathcal{A}$ with $A_\ell \to 0$, $\{\mu_\ell\}_{\ell=1}^\infty$ is relatively compact in $M(\mathcal{U})$.*

(b) *(Global concentration of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$, then any $\mathcal{A}$-limit measure is an invariant measure of φ^t concentrated on the global attractor of φ^t.*

Concentration of $\mathcal{A}$-limit measures on a compact invariant set of φ^t is a primary feature of stochastic stability of the invariant set with respect to the noise family $\mathcal{A}$—the so-called $\mathcal{A}$-*stability* which we will define in Section 2. For instance, Theorem B actually implies that the global attractor of φ^t is $\mathcal{A}$-stable if $\mathcal{A}$ is a so-called invariant null family. Such $\mathcal{A}$-stability is also closely related to the so-called $\mathcal{A}$-stability of an invariant measure of φ^t. We refer the reader to Sections 2, 3 for more discussions in these regards.

Multiplicative noises actually play important roles in stabilizing a local attractor or de-stabilizing a local repeller of φ^t. To analyze such roles played by multiplicative noises, we assume that (1.1) admits a Lyapunov function in $\mathcal{U}$ whose second derivatives are bounded. This Lyapunov function then becomes a uniform Lyapunov function with respect to a bounded null family in $\tilde{\mathcal{A}}$ (Proposition 2.5) for which Theorem B is applicable.

Theorem C. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a Lyapunov function in $\mathcal{U}$ whose second derivatives are bounded.[1] Then the following holds.*

(a) *(Noise stabilization of local attractors) If the global attractor of φ^t contains a strong local attractor, then there exists a normal null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ such that stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(\mathcal{U})$, and any $\mathcal{A}$-limit measure is an invariant measure of φ^t concentrated on the local attractor.*

(b) *(Noise de-stabilization of local repellers) If the global attractor of φ^t contains a strong local repeller, then there exists a normal null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ such that stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(\mathcal{U})$, but all $\mathcal{A}$-limit measures are concentrated away from the local repeller.*

(c) *(Noise instability of repelling equilibria) If the global attractor of φ^t contains a strongly repelling equilibrium, then with respect*

---

[1] This condition can be weakened in some sense, see Remark 4.1 for details.

to any normal null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ on $\mathcal{U}$, stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(\mathcal{U})$, but all $\mathcal{A}$-limit measures are concentrated away from the equilibrium.

In the above theorem, a normal null family is a more restricted bounded null family defined in Section 2. For definitions of a strong local attractor or repeller and strongly repelling equilibrium of (1.1), we refer the reader to the Appendix.

The proof of Theorem C actually contains explicit conditions on the noise family $\mathcal{A}$ under which (a) or (b) above holds. For instance, in the case of (a), these conditions actually quantify how strong the noises should be away from the local attractor in order to achieve the desired stabilization. We refer the reader to Section 4 for details.

We note that Theorem C can actually be re-stated as such that the strong local attractor in (a) is $\mathcal{A}$-stable with respect to some invariant, bounded null family $\mathcal{A}$, the strong local repeller in (b) is strongly $\mathcal{A}$-unstable with respect to some invariant, bounded null family $\mathcal{A}$, and the repelling equilibrium in (c) is strongly $\mathcal{A}$-unstable with respect to any invariant, normal null family $\mathcal{A}$.

This paper is organized as follows. Section 2 is a preliminary section in which we mainly review some fundamental properties of stationary measures of Fokker–Planck equations associated with (1.2) and introduce various notions of stochastic stability for compact invariant sets and invariant measures of φ^t. In Section 3, we study behaviors and concentration of limit measures. Possible local concentration of a limit measure on a strong local attractor or repeller will be studied and invariance of limit measures will be discussed. Theorems A, B will be proved in this section. In Section 4, we prove Theorem C by characterizing noise families that stabilize a strong local attractor or de-stabilize a strong local repeller. In Section 5, we demonstrate applications of our main results by considering an example of stochastic Hopf bifurcation in which the existence of a stochastically stable cycle is observed and an example with non-decomposable ω-limit sets in which the complexity of stochastic stability for invariant measures is witnessed. In the Appendix, we summarize some basic notions and dynamical properties for a system of dissipative ordinary differential equations.

We remark that if the system (1.1) is defined on $\mathcal{U} \times M$, where M is a smooth, compact manifold without boundary, then one can modify the definitions of uniform Lyapunov and anti-Lyapunov functions by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$ so that all results in Sections 3, 4 hold with respect to such a generalized domain.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote the absolute value of a number, the norm of a vector, a matrix, and the Lebesgue measure of a set. For any connected open set $\Omega \subset \mathcal{U}$ and any integer $0 \leq k < \infty$ or $k = \infty$, we denote by $C_0^k(\Omega)$ the set of functions in $C^k(\Omega)$ with compact supports in Ω and by $C_b^k(\Omega)$ the set of functions with bounded derivatives up to order k. When $k = 0$, we denote $C_0^0(\Omega), C_b^0(\Omega)$ simply by $C_0(\Omega), C_b(\Omega)$ respectively.

2. Preliminary

In this section, we will review some fundamental properties of stationary measures of the Fokker–Planck equation associated with (1.2) from [32,35] and define various notions of stochastic stability for compact invariant sets and invariant measures. We will also recall some characterizations of invariant measures and a Harnack inequality to be used in later sections.

2.1. Lyapunov function and stationary measures

Let $A = (a^{ij})$ be a given everywhere positive definite, $n \times n$ matrix-valued function on $\mathcal{U}$ such that $a^{ij} \in W_{loc}^{1,p}(\mathcal{U})$, $i, j =$ 1, 2, ..., n. The matrix-valued function corresponds to an adjoint Fokker–Planck operator:

$$\mathcal{L}_A = a^{ij} \partial_{ij}^2 + V^i \partial_i$$

which defines a weak form of stationary Fokker–Planck equation (1.5).

Let Ω be a connected open subset of $\mathcal{U}$. We recall from [32,35] that a non-negative function $U \in C(\Omega)$ is a *compact function* if (i) $U(x) < \rho_M$, $x \in \Omega$; and (ii) $\lim_{x \to \partial\Omega} U(x) = \rho_M$, where $\rho_M = \sup_{x \in \Omega} U(x)$ is called the *essential upper bound* of U. For each $\rho \in [0, \rho_M)$, we denote $\Omega_\rho = \{x \in \Omega : U(x) < \rho\}$ as the ρ-sublevel set of U and $U^{-1}(\rho) = \{x \in \Omega : U(x) = \rho\}$ as the ρ-level set of U. In the above, the notion $\partial\Omega$ and limit $x \to \partial\Omega$ are defined through a unified topology which identifies the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ with the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ in $\mathbb{R}^n$ so that $\partial\mathbb{R}^n$, consisting of infinity elements of all rays, is identified with $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ (see [32,35] for details). Therefore, when $\Omega = \mathbb{R}^n$, the limit $x \to \partial\mathbb{R}^n$ is simply equivalent to $x \to \infty$.

We also recall from [32,34,35] the following notions of Lyapunov-like and anti-Lyapunov-like functions.

Definition 2.1. A C^2 compact function U is called a *Lyapunov function* (resp. *anti-Lyapunov function*) in Ω with respect to $\mathcal{L}_A$, if there is a $\rho_m \in (0, \rho_M)$, called *essential lower bound* of U, and a constant $\gamma > 0$, called *Lyapunov constant* (resp. *anti-Lyapunov constant*) of U, such that

$$\mathcal{L}_A U(x) \leq -\gamma, \qquad (\text{resp. } \geq \gamma), \quad x \in \tilde{\Omega} = \Omega \setminus \bar{\Omega}_{\rho_m}, \qquad (2.1)$$

where $\tilde{\Omega}$ is called *essential domain* of U. If $\gamma = 0$ in the above, then U is referred to as a *weak Lyapunov function* (resp. *weak anti-Lyapunov function*) in Ω with respect to $\mathcal{L}_A$.

Proposition 2.1 ([32, Theorem A]). *If there is a Lyapunov function in $\mathcal{U}$ with respect to $\mathcal{L}_A$, then a stationary measure μ_A corresponding to $\mathcal{L}_A$ exists and is regular in the sense that $d\mu_A(x) = u_A(x)dx$ for some weak stationary solution $u_A \in W_{loc}^{1,p}(\mathcal{U})$ corresponding to $\mathcal{L}_A$. Moreover, if the Lyapunov function is unbounded, then the stationary measure is unique.*

Below, we recall from [32,35] some measure estimates which are derived based on the level set method introduced in these works. In fact, these estimates only require the weaker regularity condition on the drift field V that $V^i \in L_{loc}^p(\mathcal{U})$, $i = 1, \ldots, n$.

Proposition 2.2. *Assume that there is a Lyapunov function U in Ω with respect to $\mathcal{L}_A$, with Lyapunov constant γ, essential lower bound ρ_m and upper bound ρ_M. Let Ω_ρ denote the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$. Then the following holds for any stationary measure μ corresponding to $\mathcal{L}_A$:*

(a) *([32, Lemma 4.1]) For any $\rho_0 \in (\rho_m, \rho_M)$, there exists a constant $C_{\rho_m, \rho_0} > 0$ depending only on ρ_m, ρ_0 such that*

$$\mu(\mathcal{U} \setminus \Omega_{\rho_0}) \leq \gamma^{-1} C_{\rho_m, \rho_0} |A|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})}$$
$$\times |\nabla U|^2_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}).$$

(b) *([35, Theorem A (b)]) If, in addition,*

$$\nabla U(x) \neq 0, \qquad \forall x \in U^{-1}(\rho) \text{ for a.e. } \rho \in [\rho_m, \rho_M), \quad (2.2)$$

$$a^{ij}(x) \partial_i U(x) \partial_j U(x) \leq H(\rho), \qquad x \in \partial\Omega_\rho,$$
$$\rho \in [\rho_m, \rho_M) \qquad (2.3)$$

for some non-negative measurable function H defined on $[\rho_m, \rho_M)$, then

$$\mu(\mathcal{U} \setminus \Omega_\rho) \leq e^{-\gamma \int_{\rho_m}^\rho \frac{1}{H(t)} dt}, \qquad \rho \in [\rho_m, \rho_M).$$

Proposition 2.3 (*[35, Theorem B (a)]*)**.** *Assume that there is an anti-Lyapunov function U in Ω with respect to $\mathcal{L}_A$, with anti-Lyapunov constant γ, essential lower bound ρ_m and upper bound ρ_M. Let Ω_ρ denote the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M]$. If U satisfies* (2.2), (2.3) *with respect to a non-negative measurable function H defined on $[\rho_m, \rho_M]$, then the following holds for any stationary measure μ corresponding to $\mathcal{L}_A$:*

$$\mu(\Omega_\rho \setminus \Omega^*_{\rho_m}) \geq \mu(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)} dt}, \qquad \rho \in (\rho_0, \rho_M),$$

where $\Omega^*_{\rho_m} = \Omega_{\rho_m} \cup U^{-1}(\rho_m) = \{x \in \mathcal{U} : U(x) \leq \rho_m\}$.

2.2. Null family

Consider the class $\tilde{\mathcal{A}}$ of diffusion matrices defined in (1.7).

Definition 2.2. (1) A *null family* (resp. *bounded null family*) $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ is a directed net of $\tilde{\mathcal{A}}$ which converges to 0—the zero matrix, under the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$ (resp. L^∞-convergence on $\mathcal{U}$).

(2) A bounded null family $\{A_\alpha = (a^{ij}_\alpha)\} \subset \tilde{\mathcal{A}}$ is said to be a *normal null family* if for any pre-compact open subset Ω of $\mathcal{U}$,

$$\sup_\alpha \frac{\Lambda_\alpha(\Omega)}{\lambda_\alpha(\Omega)} < \infty, \tag{2.4}$$

where, for each α,

$$\Lambda_\alpha(\Omega) = \sup_{x \in \Omega} \Lambda_\alpha(x), \qquad \lambda_\alpha(\Omega) = \inf_{x \in \Omega} \lambda_\alpha(x)$$

with $\Lambda_\alpha(x) = \sqrt{\sum_{i,j} |a^{ij}_\alpha(x)|^2}$ and $\lambda_\alpha(x) = \inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\xi^\top A_\alpha(x) \xi}{|\xi|^2}$ for each $x \in \Omega$.

We note that in the above $\lambda_\alpha(x)$ is just the smallest eigenvalue of $A_\alpha(x)$ and $\Lambda_\alpha(x)$ is strictly bigger than the largest eigenvalue of $A_\alpha(x)$.

On $\tilde{\mathcal{A}}$, the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$ is metrizable. Through the rest of the paper, we denote an equivalent metric generating this topology by d and denote the L^∞ norm on the subspace of bounded elements of $\tilde{\mathcal{A}}$ by $|\cdot|$ for short.

Remark 2.1. By Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ for any pre-compact open set $\Omega \subset \mathcal{U}$ with C^1 boundary $\partial \Omega$, if $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ is a null family, then $A_\alpha \to 0$ uniformly on any compact subsets of $\mathcal{U}$. This is because $\mathcal{U}$ can be approximated from inside by pre-compact open subsets with smooth boundary.

Definition 2.3. Let $\mathcal{A} \subset \tilde{\mathcal{A}}$ be a null family.

(1) $\mathcal{A}$ is said to be *invariant* if for each $A = \frac{GG^\top}{2} \in \mathcal{A}$, $\mathcal{U}$ is invariant with respect to the diffusion process corresponding to G or A, i.e., with probability one, any solution of (1.2) starting in $\mathcal{U}$ remains in $\mathcal{U}$ for all positive time of existence.

(2) $\mathcal{A}$ is said to be *admissible* if for each $A = \frac{GG^\top}{2} \in \mathcal{A}$, there exists a stationary measure corresponding to $\mathcal{L}_A$.

Remark 2.2. Let $\mathcal{A}$ be a null family.

(1) With our assumptions on V and $G \in \tilde{\mathcal{G}}$, local existence of solutions of (1.2) is guaranteed. The invariance of a null family $\mathcal{A}$ only requires that such solutions corresponding to each $A = \frac{GG^\top}{2} \in \mathcal{A}$ do not escape $\mathcal{U}$ when time evolves, and local uniqueness of them is not even required.

(2) When $\mathcal{U} = \mathbb{R}^n$, it is obvious that any null family $\mathcal{A}$ is automatically invariant. When $\mathcal{U}$ admits a boundary in $\mathbb{R}^n$, the invariance property of a null family $\mathcal{A}$ naturally leads to suitable boundary conditions (on V or $A \in \mathcal{A}$ or both) of reflection, stopping,

or general Wentzell types (see [36–38]) for (1.2) or (1.3). For instance, if (1.1) admits a Lyapunov function in $\mathcal{U}$ and each $A \in \mathcal{A}$ satisfies $A(x) \to 0$ as $x \to \partial \mathcal{U}$, then it is not hard to see that $\mathcal{A}$ is invariant.

(3) For a null family $\mathcal{A}$ to be both invariant and admissible, certain uniform Lyapunov-like conditions (see below) are usually needed. When $\mathcal{U}$ is bounded, it is true by the theory of elliptic equations that if each $A \in \mathcal{A}$ is positive definite on $\bar{\mathcal{U}}$ then there exists a stationary measure in $\mathcal{U}$ corresponding to each $\mathcal{L}_A$ [32,39]. But such a null family will fail to be invariant. To ensure the admissibility of a null family $\mathcal{A}$ in the case of degeneracy of $A \in \mathcal{A}$ on $\partial \mathcal{U}$, uniform Lyapunov-like conditions are still necessary (see [32]).

Definition 2.4. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family and $\Omega \subset \mathcal{U}$ be a connected open set. A C^2 compact function U is a *uniform Lyapunov function* (resp. *uniform anti-Lyapunov function*) in Ω with respect to $\mathcal{A}$ or $\{\mathcal{L}_{A_\alpha}\} =: \{\mathcal{L}_A\}_{A \in \mathcal{A}}$ if it is a Lyapunov function (resp. anti-Lyapunov function) in Ω with respect to each $\mathcal{L}_A$, $A \in \mathcal{A}$, and the essential lower bound ρ_m and Lyapunov (resp. anti-Lyapunov) constant γ, of U, are independent of $A \in \mathcal{A}$.

Remark 2.3. (1) It follows immediately from Proposition 2.1 that if there is a uniform Lyapunov function in $\mathcal{U}$ with respect to a null family $\mathcal{A}$, then $\mathcal{A}$ is admissible.

(2) Consider $A_\epsilon = \epsilon A$, $0 < \epsilon \leq 1$, for a fixed $A = (a^{ij}) \in \tilde{\mathcal{A}}$. Then it is clear that $\{A_\epsilon\}$ is a null family, and in fact a normal null family if A is uniformly positive definite on $\mathcal{U}$. We note that a uniform Lyapunov function can be easily obtained with respect to such a null family. Assume that U is a Lyapunov function in $\Omega \subset \mathcal{U}$ with respect to the operator $\mathcal{L}_A$ with Lyapunov constant γ. If U is quasi-convex (i.e., the Hessian matrix $D^2 U$ is positive semi-definite) near $\partial \Omega$, then

$$\mathcal{L}_{A_\epsilon} = \epsilon a^{ij}(x) \partial^2_{ij} U(x) + V^i(x) \partial_i U(x)$$
$$\leq a^{ij}(x) \partial^2_{ij} U(x) + V^i(x) \partial_i U(x) \leq -\gamma$$

for all $x \in \Omega$ sufficiently close to $\partial \Omega$, i.e., U becomes a uniform Lyapunov function in Ω with respect to the family $\{\mathcal{L}_{A_\epsilon}\}$.

A uniform Lyapunov (resp. anti-Lyapunov) function with respect to a null family is actually a Lyapunov (resp. anti-Lyapunov) function of the system (1.1).

Proposition 2.4. *Suppose that U is a uniform Lyapunov (resp. anti-Lyapunov) function in a connected open set $\Omega \subset \mathcal{U}$ with respect to a null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$. Then U must be a Lyapunov (resp. anti-Lyapunov) function of (1.1) in Ω. Consequently, (1.1) generates a positive (resp. negative) semiflow which is dissipative (resp. anti-dissipative) in Ω.*

Proof. We only prove the case when U is a uniform Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$.

Denote ρ_m, respectively ρ_M, as the essential lower, respectively upper, bound of U, γ as a uniform Lyapunov constant of U, and Ω_ρ as the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M]$. Then

$$\mathcal{L}_{A_\alpha} U(x) = a^{ij}_\alpha(x) \partial^2_{ij} U(x) + V^i(x) \partial_i U(x) \leq -\gamma, \qquad x \in \Omega \setminus \bar{\Omega}_{\rho_m}.$$

By taking limit $A_\alpha \to 0$ in the above, we have

$$V(x) \cdot \nabla U(x) \leq -\gamma, \qquad x \in \Omega \setminus \bar{\Omega}_{\rho_m}, \tag{2.5}$$

i.e., U is a Lyapunov function of (1.1) in Ω. It follows from Proposition A.2 that (1.1) generates a positive semiflow which is dissipative in Ω. $\square$

Conversely, with respect to a bounded null family, a uniform Lyapunov function can be naturally obtained from a Lyapunov function of (1.1).

Proposition 2.5. *Let $\Omega \subset \mathcal{U}$ be a connected open set and suppose that* (1.1) *admits a C^2 Lyapunov (resp. anti-Lyapunov) function U whose second derivatives are bounded in Ω. Then U is a uniform Lyapunov (resp. anti-Lyapunov) function in Ω with respect to any bounded null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ with $|A_\alpha| \ll 1$.*

Proof. We only consider the case that U is a Lyapunov function of φ^t in Ω, i.e.,

$$V(x) \cdot \nabla U(x) \leq -\gamma, \qquad x \in \tilde{\Omega},$$

where $\tilde{\Omega}$, γ are essential domain, Lyapunov constant, of U, respectively. Let $\mathcal{A} = \{A_\alpha\} = \{(a_\alpha^{ij})\} \subset \tilde{\mathcal{A}}$ be a bounded null family. Then with $|A_\alpha| \ll 1$,

$$\mathcal{L}_{A_\alpha} U(x) = a_\alpha^{ij} \partial_{ij}^2 U(x) + V(x) \cdot \nabla U(x) \leq -\frac{\gamma}{2}, \qquad x \in \tilde{\Omega},$$

i.e., U is a uniform Lyapunov function in Ω with respect to $\mathcal{A} = \{A_\alpha\}$ when $|A_\alpha| \ll 1$. $\square$

2.3. Limit measures and stochastic stability

For a Borel set $\Omega \subset \mathbb{R}^n$, we denote by $M(\Omega)$ the set of Borel probability measures on Ω furnished with the *weak *-topology*, i.e., $\mu_k \to \mu$ if and only if

$$\int_\Omega f(x) \mathrm{d}\mu_k(x) \to \int_\Omega f(x) \mathrm{d}\mu(x),$$

for every $f \in C_b(\Omega)$. It is well-known that $M(\Omega)$ with the weak*-topology is metrizable.

The following result is well-known (see e.g., [40, Chapter II, Theorem 6.1]).

Proposition 2.6. *Let $\{\mu_\ell\}_{\ell=1}^\infty$ be a sequence in $M(\Omega)$ and $\mu \in M(\Omega)$. Then the following statements are equivalent.*

(1) $\lim_{\ell \to \infty} \mu_\ell = \mu$ *under the weak*-topology.*
(2) $\limsup_{\ell \to \infty} \mu_\ell(C) \leq \mu(C)$ *for any closed subset C of Ω.*
(3) $\liminf_{\ell \to \infty} \mu_\ell(W) \geq \mu(W)$ *for any open subset W of Ω.*
(4) $\lim_{\ell \to \infty} \mu_\ell(B) = \mu(B)$ *for any Borel subset B of Ω whose boundary has zero μ-measure.*

By Prokhorov's Theorem (see e.g. [41, Theorem 5.1]), a subset $\mathcal{M} \subset M(\Omega)$ is relatively sequentially compact in $\mathcal{M}(\Omega)$ if it is *tight*, i.e., for any $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\epsilon) < \epsilon$ for all $\mu \in \mathcal{M}$. We note that if Ω is compact, then any subset of $M(\Omega)$ is tight.

Definition 2.5. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a Borel set. Consider the set $\{\mu_\alpha\}$ of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$. For each μ_α, we denote $\mu_\alpha^\Omega = \frac{\mu_\alpha|_\Omega}{\mu_\alpha(\Omega)}$ when $\mu_\alpha(\Omega) > 0$, called a *normalized stationary measure in Ω*.

(1) An *$\mathcal{A}$-limit measure μ^Ω in Ω* is a sequential limit point, as $A_\alpha \to 0$, of the set $\{\mu_\alpha^\Omega\}$ in $M(\Omega)$. The set of all $\mathcal{A}$-limit measures in Ω is denoted by $\mathcal{M}_{\mathcal{A}}^\Omega$. If $\Omega = \mathcal{U}$, then we simply denote $\mathcal{M}_{\mathcal{A}}^{\mathcal{U}}$ by $\mathcal{M}_{\mathcal{A}}$ and call each measure $\mu \in \mathcal{M}_{\mathcal{A}}$ an *$\mathcal{A}$-limit measure*.
(2) The set $\{\mu_\alpha^\Omega\}$ is said to be *$\mathcal{A}$-sequentially null compact in $M(\Omega)$* if any sequence $\{\mu_l^\Omega\}$ in the set with $A_l \to 0$ is relatively compact in $M(\Omega)$.

Remark 2.4. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family on $\mathcal{U}$ and $\Omega \subset \mathcal{U}$ be a connected open set.

(1) By the aforementioned regularity theorem in [25], $\mu_\alpha(\Omega) > 0$ in this situation.

(2) We note that when $\Omega = \mathcal{U}$, the normalized stationary measure $\mu_\alpha^{\mathcal{U}}$ coincides with the original stationary measure μ_α corresponding to $\mathcal{L}_{A_\alpha}$, for each α.

(3) If the set $\{\mu_\alpha^\Omega\}$ is $\mathcal{A}$-sequentially null compact in $M(\Omega)$, then $\mathcal{M}_{\mathcal{A}}^\Omega$ is clearly non-empty. In general, $\mathcal{M}_{\mathcal{A}}^\Omega$ can be an empty set. However, when Ω is a pre-compact open subset of $\mathcal{U}$ (i.e., its closure in $\mathbb{R}^n$ is a compact subset of $\mathcal{U}$), it follows from the regularity theorem again that each stationary measure corresponding to $\mathcal{L}_{A_\alpha}$ is regular, hence each μ_α^Ω is also a probability measure on $\bar{\Omega}$. This enables us to consider the set $\mathcal{M}_{\mathcal{A}}^{\bar{\Omega}}$ of all $\mathcal{A}$-limit measures of $\{\mu_\alpha^\Omega\}$ in $M(\bar{\Omega})$, which is always non-empty because $\{\mu_\alpha^\Omega\}$ is tight, hence relatively sequentially compact in $M(\bar{\Omega})$.

(4) If a sub-domain $\Omega \subset \mathcal{U}$ is considered in (1.6) instead of $\mathcal{U}$, then one can speak of a stationary measure corresponding to $\mathcal{L}_A$ in Ω. A normalized stationary measure μ_α^Ω in Ω is necessarily a stationary measure corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω. When Ω is a pre-compact open subset of $\mathcal{U}$, stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω other than the normalized ones $\{\mu_\alpha^\Omega\}$ also exist. In fact, depending on the boundary conditions imposed, there are infinitely many such stationary measures in Ω (see [32,39] for details). Though most of our results in this paper hold for all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω, we only work with the normalized ones $\{\mu_\alpha^\Omega\}$ because they are noise or diffusion relevant.

We now define the following notion of stochastic stability with respect to a given admissible null family of noise perturbations.

Definition 2.6. Let $\Omega \subset \mathcal{U}$ be a connected open set and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an invariant and admissible null family. Let $\{\mu_\alpha\}$ be the set of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ and $\{\mu_\alpha^\Omega\}$ be the set of normalized stationary measures on Ω.

(1) A compact invariant set $\mathcal{J} \subset \Omega$ of φ^t is said to be *relatively $\mathcal{A}$-stable in Ω* if for any $\epsilon > 0$ and any open neighborhood W of $\mathcal{J}$ in $\mathcal{U}$ there exists a $\delta > 0$ such that $\mu_\alpha^\Omega(\Omega \setminus W) < \epsilon$ whenever $d(A_\alpha, 0) < \delta$. An invariant measure μ of φ^t in Ω is said to be *relatively $\mathcal{A}$-stable in Ω* if $\{\mu_\alpha^\Omega\}$ converges to μ in $M(\Omega)$ as $A_\alpha \to 0$.

(2) A compact invariant set or invariant measure is said to be *relatively $\mathcal{A}$-unstable in Ω* if it is not relatively $\mathcal{A}$-stable in Ω.

(3) A relatively $\mathcal{A}$-stable (resp. relatively $\mathcal{A}$-unstable) compact invariant set or invariant measure in $\mathcal{U}$ is simply said to be *$\mathcal{A}$-stable* (resp. *$\mathcal{A}$-unstable*). A compact invariant set $\mathcal{J}$ is said to be *strongly $\mathcal{A}$-unstable* if $\mathrm{supp}(\mu) \cap \mathcal{J} = \emptyset$ for all $\mu \in \mathcal{M}_{\mathcal{A}}$.

Remark 2.5. (1) Clearly the admissibility of a null family $\mathcal{A}$ is necessary in defining (relative) $\mathcal{A}$-stability, otherwise the (relative) $\mathcal{A}$-stability is void.

(2) The consideration of stationary measures instead of invariant measures allows a general characterization of stochastic stability because they often exist regardless whether (1.2) generates a diffusion process or (1.3) generates a generalized diffusion process, not talking about the fact that even when a diffusion or a generalized diffusion process can be defined its invariant measures need not exist. Such a consideration is natural in the sense that compact invariant sets, respectively invariant measures, of (1.1) are indeed stationary with respect to its induced flow on compact sets, respectively on the space of Borel probability measures.

(3) The invariance of $\mathcal{A}$ is also necessary in defining (relative) $\mathcal{A}$-stability because for such a stability one would like to only restrict the consideration to "noise relevant" stationary measures. This is particularly so when $\mathcal{U}$ is bounded. In this case, stationary measures corresponding to $\mathcal{L}_A$ for any $A \in \tilde{\mathcal{A}}$ always exist, but depending on the boundary conditions imposed, many of which need not be associated with true stationary processes if $\mathcal{A}$ fails to be invariant (see also Remark 2.2(2) and Remark 2.4(4)).

(4) Relative $\mathcal{A}$-stability of a compact invariant set or an invariant measure of φ^t in Ω only says that the normalized stationary measures $\{\mu_\alpha^\Omega\}$ rather than the restricted stationary measures $\{\mu_\alpha|_\Omega\}$ are "attracted" to the set or the measure as $A_\alpha \to 0$, i.e., it can happen that eventually all or part of the original stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$ still "escape" from Ω and concentrate elsewhere. To ensure the eventual concentration of all restricted stationary measures $\{\mu_\alpha|_\Omega\}$ in Ω, an additional condition $\operatorname{supp}(\mu) \cap \Omega \neq \emptyset$, $\mu \in \mathcal{M}_\mathcal{A}$ needs to be applied.

2.4. Harnack inequality

Consider the differential operator

$$Lu := \partial_i(a^{ij}(x)\partial_j u + b^i(x)u) + c^i(x)\partial_i u + d(x)u, \qquad (2.6)$$

where $a^{ij}, b^i, c^i, d, i, j = 1, \ldots, n$, are measurable and bounded functions on a connected open set $\Omega \subset \mathbb{R}^n$. Assume that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad x \in \Omega, \ \xi \in \mathbb{R}^n \qquad (2.7)$$

for some constant $\lambda > 0$, and there exist constants $\Lambda > 0$, $\nu \geq 0$ such that

$$\begin{aligned}
&\sum_{i,j} \left|a^{ij}(x)\right|^2 \leq \Lambda^2, \\
&\lambda^{-2}\sum_i (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d(x)| \leq \nu^2
\end{aligned} \qquad (2.8)$$

for all $x \in \Omega$.

The following Harnack inequality is well-known (see Theorem 8.20 of [42]).

Proposition 2.7 (*Harnack Inequality*). *Assume that the operator L satisfies* (2.7) *and* (2.8)*. Let $u \in W^{1,2}(\Omega)$ be any non-negative solution of*

$$Lu(x) = 0, \qquad x \in \Omega.$$

Then for any ball $B_{4R}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u,$$

where the constant C can be estimated by

$$C \leq C_0^{(\Lambda/\lambda + \nu R)}$$

for some constant $C_0 = C_0(n)$ depending only on n.

2.5. Characterization of invariant measures

The following characterizations of invariant and semi-invariant measures of φ^t in a subset of $\mathbb{R}^n$ (see Section 6 for their definitions) via the weak form of the Liouville's equation should be well-known. As we cannot find an appropriate reference, we include the proof for the sake of completeness.

Proposition 2.8. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and let $\Omega \subset \mathcal{U}$ be an open invariant set of φ^t. Then $\mu \in M(\Omega)$ is an invariant measure of φ^t on Ω if and only if*

$$\int_\Omega V(x) \cdot \nabla h(x)\,\mathrm{d}\mu(x) = 0, \qquad h \in C_0^1(\Omega). \qquad (2.9)$$

Proof. It follows from Riesz Representation Theorem that $\nu_1 = \nu_2 \in M(\Omega)$ if and only if

$$\int_\Omega h(x)\,\mathrm{d}\nu_1(x) = \int_\Omega h(x)\,\mathrm{d}\nu_2(x), \qquad h \in C_b(\Omega). \qquad (2.10)$$

As any $\nu \in M(\Omega)$ is a Borel regular measure, (2.10) is equivalent to

$$\int_\Omega h(x)\,\mathrm{d}\nu_1(x) = \int_\Omega h(x)\,\mathrm{d}\nu_2(x), \qquad h \in C_0(\Omega). \qquad (2.11)$$

For a given $t \in \mathbb{R}$, define $\mu^t \in M(\Omega)$: $\mu^t(B) =: \mu(\varphi^{-t}(B))$ for any Borel set $B \subset \Omega$. Then the invariance of μ is equivalent to $\mu = \mu^t$ for any $t \in \mathbb{R}$, which, by (2.10) and (2.11), is equivalent to

$$\int_\Omega h(x)\,\mathrm{d}\mu(x) = \int_\Omega h(x)\,\mathrm{d}\mu^t(x) = \int_\Omega h(\varphi^t(x))\,\mathrm{d}\mu(x),$$
$$t \in \mathbb{R}, \qquad (2.12)$$

for all $h \in C_b(\Omega)$ or all $h \in C_0(\Omega)$.

For any $h \in C_0^1(\Omega)$, consider the function

$$f_h(t) := \int_\Omega h(\varphi^t(x))\,\mathrm{d}\mu(x), \qquad t \in \mathbb{R}.$$

Then, for any $t, s \in \mathbb{R}$, we have by the flow property that

$$\begin{aligned}
f_h(t+s) &= \int_\Omega h(\varphi^{(t+s)}(x))\,\mathrm{d}\mu(x) = \int_\Omega h \circ \varphi^t(\varphi^s(x))\,\mathrm{d}\mu(x) \\
&= f_{h \circ \varphi^t}(s).
\end{aligned}$$

It follows that

$$f_h'(t) = f_{h \circ \varphi^t}'(0), \qquad h \in C_0^1(\Omega), \ t \in \mathbb{R}. \qquad (2.13)$$

If μ is an invariant measure of φ^t in Ω, then for any $h \in C_0^1(\Omega)$, (2.12) implies that $f_h(t) = f_h(0)$ for all $t \in \mathbb{R}$. Taking derivatives yields that $f_h'(t) \equiv 0$. In particular, $f_h'(0) = 0$, i.e., (2.9) holds.

Conversely, suppose that (2.9) holds. For any $h \in C_0^1(\Omega)$ and any $t \in \mathbb{R}$, using the fact that $\varphi^t : \Omega \to \Omega$ is a C^1 diffeomorphism, it is easy to see that $h \circ \varphi^t \in C_0^1(\Omega)$. Applications of (2.13) and (2.9) with $h \circ \varphi^t$ in place h yield that

$$f_h'(t) = f_{h \circ \varphi^t}'(0) = \int_\Omega V(x) \cdot \nabla(h \circ \varphi^t)(x)\,\mathrm{d}\mu(x) = 0,$$

i.e.,

$$\int_\Omega h(x)\,\mathrm{d}\mu(x) = \int_\Omega h(\varphi^t(x))\,\mathrm{d}\mu(x), \qquad h \in C_0^1(\Omega), \ t \in \mathbb{R}.$$

Since $C_0^1(\Omega)$ is dense in $C_0(\Omega)$, (2.12) holds. Hence μ is an invariant measure of φ^t in Ω. $\square$

Proposition 2.9. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and let $\Omega \subset \mathcal{U}$ be an open, pre-compact, positively (resp. negatively) invariant set of φ^t. Then $\mu \in M(\Omega)$ is a positively (resp. negatively) invariant measure of φ^t on Ω if and only if*

$$\int_\Omega V(x) \cdot \nabla h(x)\,\mathrm{d}\mu(x) = 0, \qquad h \in C_b^1(\Omega).$$

Proof. The proof follows from the same argument as that of Proposition 2.8 with $C_b(\Omega), C_b^1(\Omega), \mathbb{R}_+$ (resp. $\mathbb{R}_-$) in places of $C_0(\Omega)$, $C_0^1(\Omega), \mathbb{R}$ respectively. $\square$

3. General properties of limit measures

In this section, we consider some general limit properties of stationary measures of a family of Fokker–Planck equations as diffusion coefficients tend to zero. Among these properties, the concentration of limit measures will play an important role in characterizing stochastic stability of compact invariant sets of the corresponding unperturbed deterministic system at both local and global levels.

For a fixed vector field $V \in C(\mathcal{U}, \mathbb{R}^n)$ in (1.1), we consider the class $\tilde{\mathcal{A}}$ of diffusion matrices defined in (1.7). When $V \in C^1(\mathcal{U}, \mathbb{R}^n)$,

(1.1) generates a C^1 local flow on $\mathcal{U}$ which will be denoted by φ^t through this and the next section.

This section (as well as next section) uses some basic dynamics notions of a local flow such as dissipation and anti-dissipation, (strong) attractors and repellers, (entire, weak) Lyapunov and anti-Lyapunov functions, and invariant measures. The definition of these notions and some of their fundamental properties are reviewed in the Appendix.

3.1. Invariance of limit measures

Part (a) of Theorem A follows from part (b) of the following result when taking $\Omega = \mathcal{U}$.

Theorem 3.1. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. Then the following holds for any $\mu \in \mathcal{M}_\mathcal{A}^\Omega$.*

(a) *$\int_\Omega V(x) \cdot \nabla h(x)\, d\mu(x) = 0$ for all $h \in C_0^1(\Omega)$.*

(b) *Suppose that $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that Ω is an invariant set of φ^t. Then μ is an invariant measure of φ^t on Ω.*

(c) *Suppose that $V \in C^1(\mathcal{U}, \mathbb{R}^n)$, and that there exists an open, pre-compact, positively (resp. negatively) invariant set of φ^t in Ω containing $\mathrm{supp}(\mu)$. Then μ is an invariant measure of φ^t on Ω.*

Proof. Denote $\{\mu_\alpha^\Omega\}$ as the set of all normalized stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω. Let $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ and $\{\mu_l =: \mu_l^\Omega\} \subset \{\mu_\alpha^\Omega\}$ be sequences such that $A_l \to 0$ and $\mu_l \to \mu$ in $M(\Omega)$, as $l \to \infty$.

We note by the definition of stationary measures that

$$\int_\Omega (a_l^{ij}\partial_{ij}^2 h(x) + V^i\partial_i h(x))\, d\mu_l(x) = 0, \qquad h \in C_0^\infty(\Omega), \tag{3.1}$$

for all l. Using the uniform convergence of (a_l^{ij}) to 0 on any compact subsets of $\mathcal{U}$ and the weak*-convergence of μ_l to μ, we have by simply taking limit $l \to \infty$ in (3.1) that

$$\int_\Omega V(x) \cdot \nabla h(x)\, d\mu(x) = 0, \qquad h \in C_0^\infty(\Omega).$$

It follows that (a) holds because $C_0^\infty(\Omega)$ is dense in $C_0^1(\Omega)$.

When Ω is an invariant set of φ^t, the invariance of μ follows from Proposition 2.8. This proves (b).

To prove (c), we let $\tilde{\Omega}$ be an open, pre-compact, positively (resp. negatively) invariant set of φ^t in Ω containing $\mathrm{supp}(\mu)$. For any $\tilde{h} \in C_b^1(\tilde{\Omega})$, we let $h \in C_0^1(\Omega)$ be an extension of $\tilde{h}$. Then by (a), we have

$$\int_{\tilde{\Omega}} V(x) \cdot \nabla \tilde{h}(x)\, d\mu(x) = \int_\Omega V(x) \cdot \nabla h(x)\, d\mu(x) = 0.$$

It follows from Proposition 2.9 that μ is a positively (resp. negatively) invariant measure of φ^t in $\tilde{\Omega}$. It follows from Proposition A.5 that μ is an invariant measure of φ^t in $\tilde{\Omega}$ supported on the ω-limit set $\omega(\tilde{\Omega})$ (resp. α-limit set $\alpha(\tilde{\Omega})$) of $\tilde{\Omega}$. It follows that μ is an invariant measure of φ^t on Ω because $\mu(\Omega \setminus \tilde{\Omega}) = 0$. $\quad\square$

3.2. Stochastic counterpart of the LaSalle invariance principle

For a deterministic dynamical system, the classical LaSalle invariance principle (see Proposition A.3) is an important tool in locating ω-limit sets using a Lyapunov-like function. For the stochastic system (1.2), we show below that a similar principle can be obtained for locating the support of a limit measure.

For a Borel set $\Omega \subset \mathcal{U}$ and an admissible null family $A = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, we denote

$$\mathcal{J}_\mathcal{A}^\Omega := \overline{\bigcup\{\mathrm{supp}(\mu) : \mu \in \mathcal{M}_\mathcal{A}^\Omega\}}. \tag{3.2}$$

In the case $\Omega = \mathcal{U}$, we simply denote $\mathcal{J}_\mathcal{A}^\mathcal{U}$ by $\mathcal{J}_\mathcal{A}$.

Proposition 3.1. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. Then the following holds.*

(a) *If (1.1) admits an entire weak Lyapunov (resp. anti-Lyapunov) function U in Ω, then any $\mu \in \mathcal{M}_\mathcal{A}^\Omega$ with compact support satisfies*

$$\mathrm{supp}(\mu) \subset S = \{x \in \Omega : V(x) \cdot \nabla U(x) = 0\}.$$

(b) *If the set $\mathcal{J}_\mathcal{A}^\Omega$ is contained in a compact set $\mathcal{J} \subset \Omega$ and there is an entire weak Lyapunov (resp. anti-Lyapunov) function U_0 of (1.1) defined in a neighborhood of $\mathcal{J}$, then any $\mu \in \mathcal{M}_\mathcal{A}^\Omega$ satisfies*

$$\mathrm{supp}(\mu) \subset S_0 = \{x \in \mathcal{J} : V(x) \cdot \nabla U_0(x) = 0\}.$$

Proof. To prove (a), we let $\rho_0 > 0$ be such that the ρ_0-sublevel set Ω_{ρ_0} of U contains the support of μ. Applying Theorem 3.1(a) to a function $h \in C_0^1(\Omega)$ satisfying $h \equiv U$ on Ω_{ρ_0} yields that

$$0 = \int_\Omega V(x) \cdot \nabla h(x)\, d\mu(x) = \int_{\Omega_{\rho_0}} V(x) \cdot \nabla U(x)\, d\mu(x)$$

$$= \int_{\Omega_{\rho_0} \cap (\Omega \setminus S)} V(x) \cdot \nabla U(x)\, d\mu(x).$$

Since $V(x) \cdot \nabla U(x)$ is of constant sign on $\Omega_{\rho_0} \cap (\Omega \setminus S)$, $\mu(\Omega_{\rho_0} \cap (\Omega \setminus S)) = 0$, i.e., $\mathrm{supp}(\mu) \subset \Omega_{\rho_0} \cap S \subset S$.

To prove (b), we note by (a) that $\mathrm{supp}(\mu) \subset S_0 =: \{x \in \Omega_0 : V(x) \cdot \nabla U_0(x) = 0\}$, where Ω_0 is a neighborhood of $\mathcal{J}$ in which U_0 is defined. By the definition of $\mathcal{J}_\mathcal{A}^\Omega$ in (3.2), we also have $\mathrm{supp}(\mu) \subset \mathcal{J}$. It follows that $\mathrm{supp}(\mu) \subset \mathcal{J} \cap S_0$, from which (b) follows. $\quad\square$

We note that part (a) of Proposition 3.1 with $\Omega = \mathcal{U}$ is precisely the part (b) of Theorem A.

3.3. Some general properties on $\mathcal{A}$-stability

We first give an equivalent condition for a compact invariant set of φ^t to be relatively $\mathcal{A}$-stable or $\mathcal{A}$-stable.

Proposition 3.2. *Let $\Omega \subset \mathcal{U}$ be a connected open set, $\mathcal{J} \subset \Omega$ (resp. $\mathcal{J} \subset \mathcal{U}$) be a compact invariant set of φ^t, and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an invariant and admissible null family. Then $\mathcal{J}$ is relatively $\mathcal{A}$-stable in Ω (resp. $\mathcal{A}$-stable) if and only if the set $\{\mu_\alpha^\Omega\}$ of all normalized (resp. the set $\{\mu_\alpha\}$ of all) stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω (resp. in $\mathcal{U}$) is $\mathcal{A}$-sequentially null compact in $M(\Omega)$ (resp. in $M(\mathcal{U})$) and $\mathcal{J}_\mathcal{A}^\Omega \subset \mathcal{J}$ (resp. $\mathcal{J}_\mathcal{A} \subset \mathcal{J}$), i.e., $\mathrm{supp}(\mu) \subset \mathcal{J}$ for all $\mu \in \mathcal{M}_\mathcal{A}^\Omega$ (resp. $\mu \in \mathcal{M}_\mathcal{A}$).*

Proof. To show the sufficiency, we suppose for contradiction that $\mathcal{J}$ is not relatively $\mathcal{A}$-stable in Ω. Then there exists an $\epsilon_0 > 0$, an open neighborhood W of $\mathcal{J}$ in Ω and a sequence $\{A_l\} \subset \mathcal{A}$ such that $A_l \to 0$ but $\mu_{A_l}^\Omega(\Omega \setminus W) \geq \epsilon_0$ for all l. Since $\{\mu_{A_l}^\Omega\}$ is relatively sequentially compact, we may assume without loss of generality that $\mu_{A_l}^\Omega$ converges, say, to some $\mu \in \mathcal{M}_\mathcal{A}^\Omega$. Since $\Omega \setminus W$ is a closed subset of Ω under the restricted topology, Proposition 2.6 implies that $\mu(\Omega \setminus W) \geq \limsup_{l\to\infty} \mu_{A_l}^\Omega(\Omega \setminus W) \geq \epsilon_0$. Thus, $\mathrm{supp}(\mu) \not\subset \mathcal{J}$, a contradiction to the fact that $\mathcal{J}_\mathcal{A}^\Omega \subset \mathcal{J}$.

To show the necessity, we let $\{A_l\} \subset \mathcal{A}$ be a sequence such that $A_l \to 0$. Let $\epsilon > 0$ be given. We also take any compact neighborhood W of $\mathcal{J}$ in Ω. Since $\mathcal{J}$ is relatively $\mathcal{A}$-stable, there exists a positive integer N such that

$$\mu_{A_\ell}^\Omega(\Omega \setminus W) < \epsilon, \qquad \ell \geq N, \tag{3.3}$$

i.e., $\mu_{A_\ell}^\Omega(W) \geq 1 - \epsilon$ for all $\ell \geq N$. Since each $\mu_{A_\ell}^\Omega$ is a Borel regular measure, there is a compact subset B of Ω such that $\mu_{A_\ell}^\Omega(B) \geq 1 - \epsilon$

for all $1 \le \ell \le N$. Let $K = B \cup W$. Then K is a compact subset of Ω and $\mu_{A_\ell}^\Omega(K) \ge 1 - \epsilon$ for all ℓ. This shows that $\{\mu_{A_\ell}^\Omega\}_{\ell=1}^\infty$ is tight, hence relatively compact in $M(\Omega)$.

Let $\mu \in \mathcal{M}_\mathcal{A}^\Omega$. Then there exists a sequence $\{A_l\} \subset \mathcal{A}$ with $A_l \to 0$ such that $\lim_{\ell\to\infty}\mu_{A_\ell}^\Omega = \mu$ under the weak*-topology. Again, using the relative $\mathcal{A}$-stability of $\mathcal{J}$ in Ω, we see that for any $\epsilon > 0$ and any compact neighborhood W of $\mathcal{J}$ in Ω, (3.3) holds with the present ϵ, W, and $\{\mu_{A_\ell}^\Omega\}$. Since $\Omega \setminus W$ is an open subset of Ω, Proposition 2.6 implies that

$$\mu(\Omega \setminus W) \le \liminf_{\ell\to\infty}\mu_{A_\ell}^\Omega(\Omega \setminus W) \le \epsilon.$$

Since ϵ and W are arbitrary, we have $\mu(\Omega \setminus \mathcal{J}) = 0$ by the continuity of probability measures, i.e., $\mathrm{supp}(\mu) \subset \mathcal{J}$. $\square$

Corollary 3.1. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an invariant and admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. If $\mathcal{J}_\mathcal{A}^\Omega$ (resp. $\mathcal{J}_\mathcal{A}$) is a relatively $\mathcal{A}$-stable set in Ω (resp. $\mathcal{A}$-stable set), then it is the smallest relatively $\mathcal{A}$-stable set in Ω (resp. $\mathcal{A}$-stable set), i.e., it contains no proper relatively $\mathcal{A}$-stable subset in Ω (resp. $\mathcal{A}$-stable set).*

Proof. It follows from Proposition 3.2 immediately. $\square$

The following result establishes some connections between (relatively) $\mathcal{A}$-stable sets and measures.

Proposition 3.3. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an invariant and admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. Then the following holds.*

(a) *Let $\mu \in M(\Omega)$ (resp. $\mu \in M(\mathcal{U})$) be an invariant measure of φ^t with compact support. If μ is relatively $\mathcal{A}$-stable in Ω (resp. $\mathcal{A}$-stable), then so is $\mathrm{supp}(\mu)$ as a compact invariant set.*

(b) *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. Let $\mathcal{J} \subset \Omega$ (resp. $\mathcal{J} \subset \mathcal{U}$) be a compact invariant set of φ^t which is uniquely ergodic and contained in an open, pre-compact, positively or negatively invariant subset of φ^t in Ω (resp. in $\mathcal{U}$). If $\mathcal{J}$ is relatively $\mathcal{A}$-stable in Ω (resp. $\mathcal{A}$-stable), then so is the unique ergodic measure on $\mathcal{J}$.*

Proof. Denote $\{\mu_\alpha^\Omega\}$ as the set of all normalized stationary measures in Ω corresponding to the family $\{\mathcal{L}_{A_\alpha}\}$.

(a) We note that $\mathrm{supp}(\mu)$ is a compact invariant set of φ^t in Ω. Since μ is relatively $\mathcal{A}$-stable in Ω, $\{\mu_\alpha^\Omega\}$ converges to μ in $M(\Omega)$ as $A_\alpha \to 0$. Let W be an open neighborhood of $\mathrm{supp}(\mu)$ in Ω. Then it follows from Proposition 2.6 and the fact $\mu(\Omega \setminus \mathrm{supp}(\mu)) = 0$ that

$$\limsup_{A_\alpha\to 0}\mu_\alpha^\Omega(\Omega \setminus W) \le \mu(\Omega \setminus W) = 0.$$

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu_\alpha^\Omega(\Omega \setminus W) < \epsilon$ whenever $d(A_\alpha, 0) < \delta$, i.e., $\mathrm{supp}(\mu)$ is relatively $\mathcal{A}$-stable in Ω.

(b) Let ν be the unique invariant measure of φ^t on $\mathcal{J}$. Suppose for contradiction that ν is not relatively $\mathcal{A}$-stable in Ω. Then there are sequences $\{A_l = (a_l^{ij})\}_{l=1}^\infty \subset \mathcal{A}$ and $\{\mu_l^\Omega\} \subset \{\mu_\alpha^\Omega\}$ such that $A_l \to 0$ but μ_l^Ω is not convergent to ν in $M(\Omega)$, as $l \to \infty$. Since $\mathcal{J}$ is relatively $\mathcal{A}$-stable in Ω, Proposition 3.2 implies that $\{\mu_{A_\ell}^\Omega\}$ is relatively sequentially compact. Without loss of generality, we assume $\lim_{\ell\to\infty}\mu_{A_\ell}^\Omega = \nu^* \ne \nu$. By Theorem 3.1(c) and Proposition 3.2, ν^* is an invariant measure of φ^t on $\mathcal{J}$. But since $\mathcal{J}$ is uniquely ergodic, we must have $\nu^* = \nu$, a contradiction. $\square$

We now consider the case when there exists a uniform Lyapunov function corresponding to $\{\mathcal{L}_{A_\alpha}\}$ in Ω. In this case, we recall from Proposition 2.4 that φ^t must be dissipative in Ω, in particular, Ω is a positively invariant set and contains a maximal attractor of φ^t (Proposition A.1).

Lemma 3.1. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. Suppose that there is a uniform Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$ with essential lower bound ρ_m and upper bound ρ_M. Then the following holds.*

(a) *The family $\{\mu_\alpha^\Omega\}$ of all normalized stationary measures in Ω corresponding to $\{\mathcal{L}_{A_\alpha}\}$ is $\mathcal{A}$-sequentially null compact in $M(\Omega)$;*

(b) *$\mathrm{supp}(\mu) \subset \bar{\Omega}_{\rho_m}$ for any $\mu \in \mathcal{M}_\mathcal{A}^\Omega$, where Ω_{ρ_m} denotes the ρ_m-sublevel set of U.*

Proof. Denote γ as the uniform Lyapunov constant of U and Ω_ρ as the ρ-sublevel set of U in Ω for each $\rho \in [\rho_m, \rho_M)$.

To prove (a), we let $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ be a sequence with $A_l \to 0$ as $l \to \infty$, and denote $\mu_l := \mu_\alpha^\Omega$ for all l. For a fixed $\rho_0 \in (\rho_m, \rho_M)$, we have by Proposition 2.2(a) that

$$\mu_l(\Omega \setminus \bar{\Omega}_{\rho_0}) \le \mu_l(\Omega \setminus \Omega_{\rho_0}) \le \gamma^{-1}C_{\rho_m,\rho_0}|A_l|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})}$$
$$\times |\nabla U|^2_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})}\mu_l(\Omega_{\rho_0} \setminus \Omega_{\rho_m}), \quad (3.4)$$

for all $l = 1, 2, \ldots$, where C_{ρ_m,ρ_0} is a constant depending only on Ω, ρ_0, ρ_m. It follows that for any $\epsilon > 0$, there is a natural number l_0 such that

$$\mu_l(\Omega \setminus \bar{\Omega}_{\rho_0}) < \epsilon, \qquad l = l_0 + 1, \ldots.$$

For each $l = 1, \ldots, l_0$, clearly we can find a compact set $S_l \subset \Omega$ such that $\mu_l(\Omega\setminus S_l) < \epsilon$. Let $K_\epsilon := \bar{\Omega}_{\rho_0}\cup(\bigcup_{l=1}^{l_0}S_l)$. Then $\mu_l(\Omega\setminus K_\epsilon) < \epsilon$ for all l. This shows that the sequence $\{\mu_l\}$ is tight, hence relatively sequentially compact in $M(\Omega)$.

To show (b), we note that $\Omega \setminus \bar{\Omega}_{\rho_0}$ is open. Applying Proposition 2.6, we have by taking limit $l \to \infty$ in (3.4) that

$$0 = \liminf_{l\to\infty}\mu_l(\Omega \setminus \bar{\Omega}_{\rho_0}) \ge \mu(\Omega \setminus \bar{\Omega}_{\rho_0}). \quad (3.5)$$

This implies $\mathrm{supp}(\mu) \subset \bar{\Omega}_{\rho_m}$ by the continuity of probability measures and the arbitrariness of $\rho_0 \in (\rho_m, \rho_M)$. $\square$

Note that Theorem B(a) is just Lemma 3.1(a) when taking $\Omega = \mathcal{U}$.

Theorem 3.2. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. If there is a uniform Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$, then the following holds:*

(a) *$\mathcal{M}_\mathcal{A}^\Omega \ne \emptyset$ and each $\mu \in \mathcal{M}_\mathcal{A}^\Omega$ is an invariant measure of φ^t in Ω supported on $\mathcal{J}$—the maximal attractor of φ^t in Ω.*

(b) *If $\mathcal{A}$ is invariant, then $\mathcal{J}_\mathcal{A}^\Omega$ and $\mathcal{J}$ are the smallest and largest relatively $\mathcal{A}$-stable sets in Ω, respectively.*

Proof. (a) Denote by $\{\mu_\alpha^\Omega\}$ the set of all normalized stationary measures in Ω corresponding to $\{\mathcal{L}_{A_\alpha}\}$. By Lemma 3.1(a), $\mathcal{M}_\mathcal{A}^\Omega \ne \emptyset$. Let $\mu \in \mathcal{M}_\mathcal{A}^\Omega$. Then there is a sequence $\{\mu_l := \mu_{A_\ell}^\Omega\}$ such that $\mu = \lim_{l\to\infty}\mu_l$, where $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ is a sequence such that $A_l \to 0$ as $l \to \infty$. Denote U as a uniform Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$, ρ_m as an essential lower bound of U, ρ_M as the essential upper bound of U, and Ω_ρ as the ρ-sublevel set of U for each $\rho > 0$. By Lemma 3.1(b), we have $\mathrm{supp}(\mu) \subseteq \bar{\Omega}_{\rho_m}$.

For fixed $\rho \in (\rho_m, \rho_M)$, by Proposition 2.4, Ω_ρ is an open, pre-compact, positively invariant set of φ^t in Ω containing $\mathrm{supp}(\mu)$, so it follows from Theorem 3.1 c) and Proposition A.5 that μ is an invariant measure of φ^t supported on the maximal attractor $\mathcal{J} = \omega(\Omega_{\rho_m})$ of φ^t in Ω_{ρ_m}, which is also the maximal attractor of φ^t in Ω.

(b) We note by the invariance of any $\mu \in \mathcal{M}_\mathcal{A}^\Omega$ that $\mathcal{J}_\mathcal{A}^\Omega$ is a compact invariant set of φ^t in Ω and by (a) that $\mathcal{J}_\mathcal{A}^\Omega \subset \mathcal{J}$. It follows from Proposition 3.2, Lemma 3.1(a) and Corollary 3.1 that both $\mathcal{J}_\mathcal{A}^\Omega$ and $\mathcal{J}$ are relatively $\mathcal{A}$-stable sets in Ω with $\mathcal{J}_\mathcal{A}^\Omega$ being the smallest relatively $\mathcal{A}$-stable set in Ω. Since the maximal attractor $\mathcal{J}$ is the largest compact invariant set of φ^t in Ω, the proof is complete. $\square$

Note that Theorem 3.2(a) immediately implies Theorem B (b) when $\Omega = \mathcal{U}$.

Corollary 3.2. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. If (1.1) admits a C^2 Lyapunov function whose second derivatives are bounded on $\mathcal{U}$, then the global attractor $\mathcal{J}$ in $\mathcal{U}$ is $\mathcal{A}$-stable with respect to any invariant, bounded null family $\mathcal{A}$.*

Proof. It follows from Proposition 2.5, Remark 2.3(1), and Theorem 3.2 with $\Omega = \mathcal{U}$. □

3.4. Local concentration of limit measures

For a given admissible null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, if there exists a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$, then when $\Omega = \mathcal{U}$ Theorem 3.2 asserts that any limit measure $\mu \in \mathcal{M}_{\mathcal{A}}$ is an invariant measure of φ^t supported on the global attractor $\mathcal{J}$ of φ^t. In this case, a possible local concentration of μ is clear because any invariant measure of a flow defined on a compact metric space is supported on the closure of the recurrent set of the flow [43].

In case that a uniform Lyapunov function is not known on $\mathcal{U}$, information may still be obtained about possible local concentration of a limit measure $\mu \in \mathcal{M}_{\mathcal{A}}$ on a compact invariant set $\mathcal{E} \subset \mathcal{U}$ of φ^t, i.e., there is a neighborhood $\mathcal{W}$ of $\mathcal{E}$ in $\mathcal{U}$ such that $\mu(\mathcal{W} \setminus \mathcal{E}) = 0$, provided that a uniform Lyapunov or anti-Lyapunov function is known on $\mathcal{W}$.

We first consider a pre-compact set $\Omega \subset \mathcal{U}$ in which a uniform anti-Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ exists. We note that $\mathcal{M}_{\mathcal{A}}^{\bar{\Omega}}$ is always non-empty (see Remark 2.4) even if $\mathcal{M}_{\mathcal{A}}^{\Omega}$ is empty. We also note by Proposition 2.4 that, with the existence of a uniform anti-Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$, φ^t must be anti-dissipative in Ω. In particular, Ω contains a maximal repeller of φ^t.

Lemma 3.2. *Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family. Suppose that Ω is a connected, open, and pre-compact subset of $\mathcal{U}$ and that there is a uniform anti-Lyapunov function U with respect to $\{\mathcal{L}_{A_\alpha}\}$ in Ω. Then the following holds.*

(a) *The family $\{\mu_\alpha^\Omega\}$ of normalized stationary measures in Ω corresponding to $\{\mathcal{L}_{A_\alpha}\}$ is relatively sequentially compact in $M(\bar{\Omega})$.*
(b) *For any $\mu \in \mathcal{M}_{\mathcal{A}}^{\bar{\Omega}}$, $\text{supp}(\mu) \subset \bar{\Omega}_{\rho_m} \cup \partial\Omega$, where ρ_m is the essential lower bound of U and Ω_{ρ_m} is the ρ_m-sublevel set of U.*
(c) *If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$, then for any $\mu \in \mathcal{M}_{\mathcal{A}}^{\bar{\Omega}}$, $\text{supp}(\mu) \subset \mathcal{R} \cup \partial\Omega$, where $\mathcal{R}$ is the maximal repeller of φ^t in Ω. Moreover, if $\mu(\mathcal{R}) \neq 0$, then $\frac{\mu|_{\mathcal{R}}}{\mu(\mathcal{R})}$ is an invariant measure of φ^t on $\mathcal{R}$, and consequently, μ is an invariant measure of φ^t in Ω if $\mu(\partial\Omega) = 0$.*

Proof. (a) holds because $\{\mu_\alpha^\Omega\}$, as a family of Borel probability measures on the compact set $\bar{\Omega}$, is tight.

To prove (b), we let $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ and $\{\mu_l := \mu_l^\Omega\}$ be sequences such that $A_l \to 0$ and $\mu_l \to \mu \in M(\bar{\Omega})$ as $l \to \infty$. Denote ρ_M as the essential upper bound of U, γ as the uniform anti-Lyapunov constant of U, and Ω_ρ as the ρ-sublevel set of U in Ω for each $\rho \in [\rho_m, \rho_M)$.

For each $\rho \in [\rho_m, \rho_M)$, we have

$$a_l^{ij}(x)\partial_i U(x)\partial_j U(x) \leq |A_l|_{C(\bar{\Omega})} |\nabla U|_{C(\bar{\Omega}_\rho)}^2, \qquad x \in \partial\Omega_\rho.$$

Thus, for a given $\rho_0 \in (\rho_m, \rho_M)$, it follows from Proposition 2.3 that

$$\mu_l(\Omega_{\rho_0} \setminus \Omega_{\rho_m}^*) \leq \mu_l(\Omega_\rho \setminus \Omega_{\rho_m}^*)e^{-\frac{\gamma C_\rho}{a_l}} \leq e^{-\frac{\gamma C_\rho}{a_l}}, \qquad (3.6)$$

where $\Omega_{\rho_m}^* = \Omega_{\rho_m} \cup U^{-1}(\rho_m)$, $a_l = |A_l|_{C(\bar{\Omega})}$, and $C_\rho = \frac{\rho - \rho_0}{|\nabla U|_{C(\bar{\Omega}_\rho)}^2}$, $\rho \in (\rho_0, \rho_M)$.

Since $A_l \to 0$ and $\mathcal{L}_{A_l}U(x) \geq \gamma$ for all l and $x \in \tilde{\Omega} = \Omega \setminus \bar{\Omega}_{\rho_m}$, taking limit $l \to \infty$ yields that $V(x) \cdot \nabla U(x) \geq \gamma$ in $\tilde{\Omega}$. Hence $V(x) \cdot \nabla U(x) \geq \gamma$ on the closure of $\tilde{\Omega}$ in Ω, i.e., on the set $\{x \in \Omega : U(x) \geq \rho_m\}$. For any $\rho \in [\rho_m, \rho_M)$ and $x \in \partial\Omega_\rho$, the Implicit Function Theorem implies that in the vicinity of x, $\partial\Omega_\rho$ is a local C^2 hypersurface which coincides with a portion of the level surface $U^{-1}(\rho)$. Since $\partial\Omega_\rho$ is a compact set, it consists of finitely many C^2 components, each of which is oriented. Thus, Ω_ρ is a C^2 domain and

$$\partial\Omega_\rho = U^{-1}(\rho), \qquad \text{for any } \rho \in [\rho_m, \rho_M). \qquad (3.7)$$

In particular, $\Omega_{\rho_m}^* = \bar{\Omega}_{\rho_m}$. Since $a_l \to 0$, an application of Proposition 2.6 to (3.6) yields that $\mu(\Omega_{\rho_0} \setminus \bar{\Omega}_{\rho_m}) = 0$ for any $\rho_0 \in (\rho_m, \rho_M)$. It follows that $\mu(\Omega \setminus \bar{\Omega}_{\rho_m}) = 0$. Hence $\text{supp}(\mu) \subset \bar{\Omega}_{\rho_m} \cup \partial\Omega$.

To prove (c), we observe that if $\mu(\bar{\Omega}_{\rho_m}) = 0$, then by (b), $\text{supp}(\mu) \subset \partial\Omega$. Now suppose $\mu(\bar{\Omega}_{\rho_m}) \neq 0$ and let $\rho_*, \rho_0 \in (\rho_m, \rho_M)$ be fixed such that $\rho_* < \rho_0$. We have by (b) that $\mu(\partial\Omega_{\rho_0}) = 0$. It follows from Proposition 2.6 that $\lim_{l \to \infty} \mu_l(\Omega_{\rho_0}) = \mu(\Omega_{\rho_0})$, and therefore

$$\tilde{\mu}_l =: \frac{\mu_l|_{\Omega_{\rho_0}}}{\mu_l(\Omega_{\rho_0})}$$

is a sequence of stationary measures corresponding to $\{\mathcal{L}_{A_l}\}$ in Ω_{ρ_0} which converges to

$$\tilde{\mu} =: \frac{\mu|_{\Omega_{\rho_0}}}{\mu(\Omega_{\rho_0})}$$

as $l \to \infty$. We clearly have $\text{supp}(\tilde{\mu}) \subset \Omega_{\rho_*}$ because $\rho_m < \rho_*$. Since $V(x) \cdot \nabla U(x) \geq \gamma$ in $\{x \in \Omega : U(x) \geq \rho_m\}$, Ω_{ρ_*} is a negatively invariant, pre-compact open subset of Ω_{ρ_0}. It follows from Theorem 3.1(c) with Ω in place of $\mathcal{U}$ and Ω_{ρ_0} in place of Ω that $\tilde{\mu}$ is an invariant measure of φ^t on Ω_{ρ_0}. Moreover, by Proposition A.5, $\tilde{\mu}$ is supported on $\alpha(\Omega_{\rho_0}) = \mathcal{R}$—the maximal repeller of φ^t in Ω. Therefore, $\text{supp}(\tilde{\mu}) \subset \mathcal{R}$. By (b), $\text{supp}(\mu) \subset \mathcal{R} \cup \partial\Omega$. Since $\frac{\mu|_{\mathcal{R}}}{\mu(\mathcal{R})} = \tilde{\mu}|_{\mathcal{R}}$, the proof is complete. □

Theorem 3.3. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family, $\mu \in \mathcal{M}_{\mathcal{A}}$, and $\Omega \subset \mathcal{U}$ be a connected open set. Then the following holds.*

(a) *If there exists a uniform Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$, then $\mu(\Omega \setminus \mathcal{J}) = 0$, where $\mathcal{J}$ is the maximal attractor of φ^t in Ω.*
(b) *If Ω is pre-compact in $\mathcal{U}$ and there exists a uniform anti-Lyapunov function in Ω with respect to $\{\mathcal{L}_{A_\alpha}\}$, then $\mu(\Omega \setminus \mathcal{R}) = 0$, where $\mathcal{R}$ is the maximal repeller of φ^t in Ω.*

Proof. Let $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ be a sequence such that $A_l \to 0$ and $\{\mu_l\}$ be a sequence of stationary measures corresponding to $\{\mathcal{L}_{A_l}\}$ such that $\mu_l \to \mu \in M(\mathcal{U})$ as $l \to \infty$. Also let U be a uniform Lyapunov function in Ω in the case (a) and a uniform anti-Lyapunov function in Ω in the case (b), with essential lower bound ρ_m and upper bound ρ_M. For each $\rho \in [\rho_m, \rho_M)$, denote $\bar{\Omega}_\rho$ as the ρ-sublevel set of U.

If $\mu(\Omega) = 0$, then the theorem holds automatically. We now suppose that $\mu(\Omega) \neq 0$. For any $\rho_0 \in (\rho_m, \rho_M)$, applications of (3.4), (3.6), (3.7) to

$$\tilde{\mu}_l =: \frac{\mu_l|_\Omega}{\mu_l(\Omega)}$$

in place of μ_l for the case (a), (b) respectively yield that

$$\mu(\Omega \setminus \bar{\Omega}_{\rho_0}) = 0, \quad \text{in the case (a)}, \tag{3.8}$$

$$\mu(\Omega_{\rho_0} \setminus \bar{\Omega}_{\rho_m}) = 0, \quad \text{in the case (b)}, \tag{3.9}$$

for any $\rho_0 \in (\rho_m, \rho_M)$. Now let $\rho_* \in (\rho_m, \rho_M)$ be fixed. Then (3.8) and (3.9) imply that

$$\mu(\partial \Omega_{\rho_*}) = 0.$$

Hence it follows from Proposition 2.6 that

$$\mu_l^* =: \frac{\mu_l|_{\Omega_{\rho_*}}}{\mu_l(\Omega_{\rho_*})}$$

is a sequence of stationary measures corresponding to $\{\mathcal{L}_{A_l}\}$ in Ω_{ρ_*} which converges to

$$\mu^* =: \frac{\mu|_{\Omega_{\rho_*}}}{\mu(\Omega_{\rho_*})}.$$

We note that U is also a uniform Lyapunov function in Ω_{ρ_*} in the case (a) and a uniform anti-Lyapunov function in Ω_{ρ_*} in the case (b), with respect to $\{\mathcal{L}_{A_\alpha}\}$.

In the case (a), it follows from Theorem 3.2 with Ω in place of $\mathcal{U}$ and Ω_{ρ_*} in place of Ω that $\mu^*(\Omega_{\rho_*} \setminus \mathcal{J}) = 0$, implying that $\mu(\Omega_{\rho_*} \setminus \mathcal{J}) = 0$. In the case (b), since $\mu^*(\partial \Omega_{\rho_*}) = 0$, we have by Lemma 3.2 (c) with Ω in place of $\mathcal{U}$ and Ω_{ρ_*} in place of Ω that $\mu^*(\Omega_{\rho_*} \setminus \mathcal{R}) = 0$, implying that $\mu(\Omega_{\rho_*} \setminus \mathcal{R}) = 0$. This completes the proof because $\rho_* \in (\rho_m, \rho_M)$ is arbitrary. $\quad\square$

Part (a) of the following result is precisely part (c) of Theorem A.

Corollary 3.3. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and φ^t admits a strong local attractor $\mathcal{J}_0$ (resp. strong local repeller $\mathcal{R}_0$) with an isolating neighborhood $\mathcal{W}_{\mathcal{J}_0}$ (resp. $\mathcal{W}_{\mathcal{R}_0}$). Then the following holds for any admissible null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$.*

(a) *For any $\mu \in \mathcal{M}_{\mathcal{A}}$, $\mu(\mathcal{W}_{\mathcal{J}_0} \setminus \mathcal{J}_0) = 0$ (resp. $\mu(\mathcal{W}_{\mathcal{R}_0} \setminus \mathcal{R}_0) = 0$).*
(b) *$\mathcal{J}_0$ is relatively $\mathcal{A}$-stable in $\mathcal{W}_{\mathcal{J}_0}$ if $\mathcal{A}$ is invariant.*

Proof. We note by the proof of Proposition A.4 that (1.1) admits a C^2 Lyapunov (resp. anti-Lyapunov) function U in $\mathcal{W} =: \mathcal{W}_{\mathcal{J}_0}, \mathcal{W}_{\mathcal{R}_0}$. Since $\mathcal{W}$ is a pre-compact subset of $\mathcal{U}$, all second derivatives of U are bounded in $\mathcal{W}$. Let $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be any null family. Without loss of generality, we assume $d(A_\alpha, 0) \ll 1$. It follows from Proposition 2.5 that U is a uniform Lyapunov (resp. anti-Lyapunov) function in $\mathcal{W}$ with respect to $\{\mathcal{L}_{A_\alpha}\}$. By Theorem 3.3, $\mu(\mathcal{W} \setminus \mathcal{J}_0) = 0$ (resp. $\mu(\mathcal{W} \setminus \mathcal{R}_0) = 0$). This proves (a).

Using the uniform Lyapunov function U in $\mathcal{W}_{\mathcal{J}_0}$ with respect to $\{\mathcal{L}_{A_\alpha}\}$ and the fact that $\mathcal{J}_0$ is the maximal attractor of φ^t in $\mathcal{W}_{\mathcal{J}_0}$, (b) follows from Theorem 3.2 with $\Omega = \mathcal{W}_{\mathcal{J}_0}$. $\quad\square$

4. Stabilization and de-stabilization via multiplicative noises

Throughout the section, we assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a C^2 Lyapunov function with bounded second derivatives on $\mathcal{U}$. For any bounded null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, it follows from Proposition 2.5, Remark 2.3(1), and Theorem 3.2 that $\mathcal{A}$ is admissible, $\mathcal{M}_{\mathcal{A}} \neq \emptyset$, and any limit measure $\mu \in \mathcal{M}_{\mathcal{A}}$ is an invariant measure of φ^t supported on the global attractor $\mathcal{J}$ of φ^t, i.e., $\mathcal{J}$ is $\mathcal{A}$-stable if $\mathcal{A}$ is also invariant. Moreover, by Corollary 3.3, if $\mathcal{J}$ contains a strong local attractor $\mathcal{J}_0$ (resp. repeller $\mathcal{R}_0$), then μ may be locally concentrated on $\mathcal{J}_0$ (resp. $\mathcal{R}_0$). In fact, we believe that local concentration on a strong local repeller can be expected only when it further contains a sub-local attractor, as suggested by Theorem 4.2.

In this section, we will demonstrate an important role played by multiplicative noise perturbations. We will show that if $\mathcal{J}$ contains a strong local attractor $\mathcal{J}_0$ of φ^t, then one can design a particular invariant, bounded null family $\mathcal{A} =: \{A_\alpha\} \subset \tilde{\mathcal{A}}$ by adding stronger noise away from the local attractor such that any $\mu \in \mathcal{M}_{\mathcal{A}}$ is (globally) concentrated on $\mathcal{J}_0$, i.e., $\mathcal{J}_0$ is $\mathcal{A}$-stable. On the contrary, if $\mathcal{J}$ contains a strong local repeller $\mathcal{R}_0$ of φ^t, then one can design a particular invariant, bounded null family $\mathcal{A} =: \{A_\alpha\} \subset \tilde{\mathcal{A}}$ by adding stronger noise on and near the repeller such that any $\mu \in \mathcal{M}_{\mathcal{A}}$ is (globally) concentrated away from $\mathcal{R}_0$, i.e., $\mathcal{R}_0$ is strongly $\mathcal{A}$-unstable.

In the case that $\mathcal{R}_0$ is a strongly repelling equilibrium, we will show that it is strongly $\mathcal{A}$-unstable with respect to any invariant, normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$.

4.1. Stabilizing a local attractor and de-stabilizing a local repeller

Let $\mathcal{E}_0$ be either a strong local attractor or a strong local repeller of φ^t in $\mathcal{U}$. We denote the isolating neighborhood of $\mathcal{E}_0$ by $\mathcal{W}_0$. Then by the proof of Proposition A.4 there is a positive function $U_0 \in C^2(\mathcal{U})$ and constants $\gamma_0, \tilde{\rho} > 0$ such that $\partial \mathcal{W}_0 = \{U_0(x) = \tilde{\rho}\}$ and

$$|V(x) \cdot \nabla U_0(x)| > \gamma_0 |\nabla U_0(x)|, \qquad x \in \partial \mathcal{W}_0 = \partial \Omega_{\tilde{\rho}}^0, \tag{4.1}$$

where Ω_ρ^0 denotes the ρ-sublevel set of U_0 for each $\rho \geq 0$. By modifying U_0 away from $\partial \mathcal{W}_0$, we can assume without loss of generality that $U_0(x) < \tilde{\rho}$ for all $x \in \mathcal{W}_0$, so that U_0 becomes a compact function in $\mathcal{W}_0$. We note that $\bar{\mathcal{W}}_0 = \bar{\Omega}_{\tilde{\rho}}^0 = \Omega_{\tilde{\rho}}^0 \cup U_0^{-1}(\tilde{\rho})$.

Lemma 4.1. *Let $\mathcal{E}_0, \mathcal{W}_0, U_0, \tilde{\rho}, \gamma_0, \Omega_{\tilde{\rho}}^0$ be as in the above, and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family on $\mathcal{U}$. Then there are positive constants ρ_*, ρ^*, depending only on $\mathcal{W}_0, U_0$, with $\rho_* < \tilde{\rho} < \rho^*$, such that as $d(A_\alpha, 0) \ll 1$,*

$$\mu_\alpha(\Omega_{\rho^*}^0 \setminus \mathcal{W}_0) = \mu_\alpha(\bar{\Omega}_{\rho^*}^0 \setminus \mathcal{W}_0) \leq e^{-\frac{C_1}{a_1(\alpha)}},$$
when $\mathcal{E}_0$ is a local attractor,

$$\mu_\alpha(\mathcal{W}_0 \setminus \bar{\Omega}_{\rho_*}^0) = \mu_\alpha(\bar{\mathcal{W}}_0 \setminus \Omega_{\rho_*}^0) \leq e^{-\frac{C_2}{a_2(\alpha)}},$$
when $\mathcal{E}_0$ is a local repeller

for all stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$, where

$$C_1 = \frac{\gamma_0(\tilde{\rho} - \rho_*) \min_{x \in \partial \Omega_{\tilde{\rho}}^0} |\nabla U_0(x)|}{2 \max_{\rho_* \leq U_0(x) \leq \tilde{\rho}} |\nabla U_0(x)|^2},$$
$$a_1(\alpha) = \max_{\rho_* \leq U_0(x) \leq \tilde{\rho}} |A_\alpha(x)|,$$
$$C_2 = \frac{\gamma_0(\rho^* - \tilde{\rho}) \min_{x \in \partial \Omega_{\tilde{\rho}}^0} |\nabla U_0(x)|}{2 \max_{\tilde{\rho} \leq U_0(x) \leq \rho^*} |\nabla U_0(x)|^2},$$
$$a_2(\alpha) = \max_{\tilde{\rho} \leq U_0(x) \leq \rho^*} |A_\alpha(x)|.$$

Proof. By (4.1), there are ρ_*, ρ^* with $\rho_* < \tilde{\rho} < \rho^*$ such that, for all $x \in \bar{\Omega}_{\rho^*}^0 \setminus \Omega_{\rho_*}^0, \nabla U_0(x) \neq 0$ and

$$|\mathcal{L}_{A_\alpha} U_0(x)| = |a_\alpha^{ij} \partial_{ij}^2 U_0(x) + V(x) \cdot \nabla U_0(x)| \geq \frac{\gamma_0 \min_{x \in \partial \Omega_{\tilde{\rho}}^0} |\nabla U_0(x)|}{2},$$

as $d(A_\alpha, 0) \ll 1$. Since $\mathcal{W}_0$ is connected, by making ρ_*, ρ^* sufficiently close to $\tilde{\rho}$ if necessary, we may assume without loss of generality that Ω_ρ^0 is a connected open set for each $\rho \in (\rho_*, \rho^*)$. As in the proof of Lemma 3.2, the Implicit Function Theorem implies that Ω_ρ^0 is a C^2 domain and $\bar{\Omega}_\rho^0 = \Omega_\rho^0 \cup U_0^{-1}(\rho)$ for each $\rho \in (\rho_*, \rho^*)$.

In the case that $\mathcal{E}_0$ is a strong local attractor, the remaining part of the lemma follows from Proposition 2.2(b) with $\Omega_{\rho^*}^0$ in place of

Ω and

$$\rho_M =: \rho^*, \quad \rho_m = \rho_*, \quad \rho =: \tilde{\rho}, \quad \gamma =: \frac{\gamma_0 \min_{x \in \partial \Omega_{\tilde{\rho}}^0} |\nabla U_0(x)|}{2},$$

$$H = a_1(\alpha) \max_{\rho_* \leq U_0(x) \leq \tilde{\rho}} |\nabla U_0(x)|^2.$$

In the case that $\mathcal{E}_0$ is a strong local repeller, the remaining part of the lemma follows from Proposition 2.3 with $\mathcal{W}_0$ in place of Ω and

$$\rho =: \rho^*, \quad \rho_m = \rho_*, \quad \rho_0 =: \tilde{\rho}, \quad \gamma =: \frac{\gamma_0 \min_{x \in \partial \Omega_{\tilde{\rho}}^0} |\nabla U_0(x)|}{2},$$

$$H = a_2(\alpha) \max_{\tilde{\rho} \leq U_0(x) \leq \rho^*} |\nabla U_0(x)|^2. \quad \square$$

Theorem C(a), (b) follow from the following result.

Theorem 4.1. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a C^2 Lyapunov function with bounded second derivatives on $\mathcal{U}$. Then the following holds.*

(a) *For any strong local attractor $\mathcal{J}_0$ lying in $\mathcal{U}$, there is a normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$ on $\mathcal{U}$ which is both invariant and admissible such that $\mathcal{M}_\mathcal{A} \neq \emptyset$ and each limit measure $\mu \in \mathcal{M}_\mathcal{A}$ is an invariant measure of φ^t supported on $\mathcal{J}_0$. Consequently, $\mathcal{J}_0$ is $\mathcal{A}$-stable.*

(b) *For any strong local repeller $\mathcal{R}_0$ lying in $\mathcal{U}$, there is a normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$ which is both invariant and admissible such that $\mathcal{M}_\mathcal{A} \neq \emptyset$ and each limit measure $\mu \in \mathcal{M}_\mathcal{A}$ is an invariant measure of φ^t supported on the maximal attractor $\mathcal{J}_*$ of φ^t in $\mathcal{J} \setminus \mathcal{W}_0$, where $\mathcal{J}$ is the global attractor of φ^t and $\mathcal{W}_0$ is an isolating neighborhood of $\mathcal{R}_0$. Consequently, $\mathcal{J}_*$ is $\mathcal{A}$-stable and $\mathcal{R}_0$ is strongly $\mathcal{A}$-unstable.*

Proof. Let $\mathcal{E}_0$ denote either a local attractor $\mathcal{J}_0$ or a local repeller $\mathcal{R}_0$, and $\mathcal{W}_0$ be an isolating neighborhood of $\mathcal{E}_0$. We note that since $\mathcal{U}$ is dissipative, $\mathcal{E}_0$ must lie in the global attractor $\mathcal{J}$ of φ^t.

First of all, we fix a pre-compact, connected open neighborhood Ω_0 of $\mathcal{J}$ in $\mathcal{U}$. Let $\mathcal{A} = \{A_\alpha\} = \{(a_\alpha^{ij})\} \subset \tilde{\mathcal{A}}$ be any bounded null family. We have by Proposition 2.5 that there is a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ in $\mathcal{U}$. Hence by Remark 2.3(1) $\mathcal{A}$ is admissible. Denote $\{\mu_\alpha\}$ as the set of all stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$. It follows from Lemma 3.1(a) that $\{\mu_\alpha\}$ is $\mathcal{A}$-sequentially null compact in $M(\mathcal{U})$ and hence $\mathcal{M}_\mathcal{A} \neq \emptyset$. By Theorem 3.2, any $\mu \in \mathcal{M}_\mathcal{A}$ is an invariant measure of φ^t supported on the global attractor $\mathcal{J}$.

If $\mathcal{U} = \mathbb{R}^n$, then $\mathcal{A}$ is clearly invariant. Otherwise, one can modify each $A \in \mathcal{A}$ near $\partial \mathcal{U}$ such that $A(x) \to 0$ as $x \to \partial \mathcal{U}$. Since (1.1) admits a Lyapunov function in $\mathcal{U}$, it is not hard to see that the modified normal null family $\mathcal{A}$ becomes invariant.

Let $\rho_*, \rho^*, \tilde{\rho}, \Omega_\rho^0, \rho \in [\rho_*, \rho^*]$, be as in Lemma 4.1. Recall that $\rho_* < \tilde{\rho} < \rho^*$ and $\bar{\mathcal{W}}_0 = \bar{\Omega}_{\tilde{\rho}}^0$. By taking ρ^* further small if necessary, we can assume $\bar{\Omega}_{\rho^*}^0 \subset \Omega_0$.

For fixed $\tilde{\rho}^* \in (\tilde{\rho}, \rho^*)$, $\tilde{\rho}_* \in (\rho_*, \tilde{\rho})$, we consider sets

$$D = \begin{cases} \bar{\Omega}_0 \setminus \Omega_{\tilde{\rho}^*}^0, & \text{when } \mathcal{E}_0 = \mathcal{J}_0, \\ \bar{\Omega}_{\tilde{\rho}_*}^0, & \text{when } \mathcal{E}_0 = \mathcal{R}_0, \end{cases}$$

$$D^* = \begin{cases} \bar{\Omega}_{\rho^*}^0 \setminus \Omega_{\tilde{\rho}}^0, & \text{when } \mathcal{E}_0 = \mathcal{J}_0, \\ \bar{\Omega}_{\tilde{\rho}}^0 \setminus \Omega_{\rho_*}^0, & \text{when } \mathcal{E}_0 = \mathcal{R}_0, \end{cases}$$

$$D_* = \begin{cases} \bar{\Omega}_{\tilde{\rho}}^0 \setminus \Omega_{\rho_*}^0, & \text{when } \mathcal{E}_0 = \mathcal{J}_0, \\ \bar{\Omega}_{\rho^*}^0 \setminus \Omega_{\tilde{\rho}}^0, & \text{when } \mathcal{E}_0 = \mathcal{R}_0, \end{cases}$$

and fix a fixed small neighborhood Ω of D in $\mathcal{U}$ with $\bar{\Omega} \cap D_* = \emptyset$. Positions of the sets $\bar{\mathcal{W}}_0, D, D^*, D_*, \Omega, \mathcal{E}_0$ in $\mathcal{U}$ are illustrated in Figs. 1 and 2.
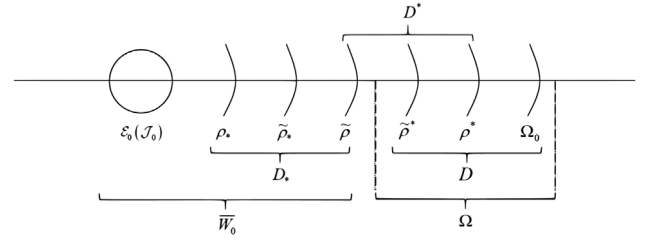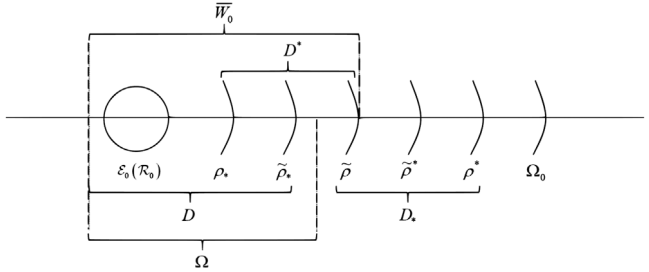

Fig. 1. The attractor case.

Fig. 2. The repeller case.

Let $\{B_R^k(y_k)\}_{k=1}^N$, for some integer N and small real number $R > 0$, be balls of radius R centered at $y_k \in D, k = 1, \ldots, N$, such that their union covers D and

$$\bigcup_{k=1}^N B_{4R}^k(y_k) \subset \Omega.$$

Using the facts that $\bar{\Omega}_{\rho^*}^0 \subset \Omega_0$, Ω_0 is connected, and Ω_ρ^0 is a connected open set with oriented C^2 boundary for each $\rho \in (\rho_*, \rho^*)$, we see that D has finitely many components, say, $D_1, D_2, \ldots, D_I$. Since $\partial \Omega_{\tilde{\rho}^*}^0 \subset \text{int}(\bar{\Omega}_{\rho^*}^0 \setminus \Omega_{\tilde{\rho}}^0)$ and $\partial \Omega_{\tilde{\rho}}^0 \subset \text{int}(\Omega_{\tilde{\rho}}^0 \setminus \bar{\Omega}_{\rho_*}^0)$, it is not hard to see that $|D_i \cap D^*| > 0$ for all $i = 1, 2, \ldots, I$.

We now restrict $\{A_\alpha\}$ to be a normal null family satisfying $A_\alpha \in C^1(\mathcal{U}, GL(n, \mathbb{R}))$ for each α and

$$\sup_\alpha \tilde{\Lambda}_\alpha(\Omega) < 1, \tag{4.2}$$

where, for each α,

$$\tilde{\Lambda}_\alpha(\Omega) = \sup_{x \in \Omega} |\partial A_\alpha(x)|.$$

For each α, denote u_α as the weak stationary solution corresponding to $\mathcal{L}_{A_\alpha}$ that is associated with the measure μ_α. We also let $\Lambda_\alpha, \lambda_\alpha$ be the quantities defined as in Definition 2.2 on each pre-compact open subset of $\mathcal{U}$ for the restricted family $\{A_\alpha\}$.

Note that the above neighborhood Ω of D can be chosen such that the number of components of Ω is at most I. Therefore, applying Proposition 2.7 in Ω with $b^i = \partial_j a_\alpha^{ij} - V^i$, $c^i = d \equiv 0$, $i = 1, \ldots, n$, we have

$$\sup_{B_R(y_i)} u_\alpha(x) \leq C_0^{(C_1 + \frac{C_2 R}{\lambda_\alpha(\Omega)})} \inf_{B_R(y_i)} u_\alpha(x), \quad i = 1, \ldots, N,$$

where $C_0 \geq 1$ is a positive constant depending only on n and Ω, $C_1 = \sup_\alpha \frac{\Lambda_\alpha(\Omega)}{\lambda_\alpha(\Omega)} < \infty$, and $C_2 = 2n(\sup_\alpha \tilde{\Lambda}_\alpha(\Omega) + \sup_{x \in \Omega} |V(x)|) \leq 2n(1 + \sup_{x \in \Omega} |V(x)|) < \infty$.

Let $C_3 = C_0^{NC_1}$ and $C_* = NC_2R\ln C_0$. Then for each $i = 1, 2, \ldots, I$,

$$\sup_{D_i} u_\alpha(x) \le C_3 e^{\frac{C_*}{\lambda_\alpha(\Omega)}} \inf_{D_i} u_\alpha(x) \le C_3 e^{\frac{C_*}{\lambda_\alpha(\Omega)}} \inf_{D_i \cap D^*} u_\alpha(x)$$

$$\le \frac{C_3}{|D_i \cap D^*|} e^{\frac{C_*}{\lambda_\alpha(\Omega)}} \mu_\alpha(D_i \cap D^*)$$

$$\le \frac{C_3}{|D_i \cap D^*|} e^{\frac{C_*}{\lambda_\alpha(\Omega)}} \mu_\alpha(D^*). \tag{4.3}$$

By Lemma 4.1, there is a positive constant C^* depending only on U_0 and D_* such that

$$\mu_\alpha(D^*) \le e^{-\frac{C^*}{\Lambda_\alpha(D_*)}}, \tag{4.4}$$

as $|A_\alpha| \ll 1$.

It now follows from (4.3) and (4.4) that

$$\mu_\alpha(D) \le \sum_{i=1}^{I} |D_i| \sup_{D_i} u_\alpha(x) \le \sum_{i=1}^{I} \frac{|D_i|C_3}{|D_i \cap D^*|} e^{-\frac{C^*}{\Lambda_\alpha(D_*)} + \frac{C_*}{\lambda_\alpha(\Omega)}}.$$

Therefore, if we further restrict $\{A_\alpha\}$ to be such that

$$\frac{\lambda_\alpha(\Omega)}{\Lambda_\alpha(D_*)} > \frac{C_* + 1}{C^*} \tag{4.5}$$

for all α, then $\mu_\alpha(D) \to 0$ as $A_\alpha \to 0$. Since $D = \bar{\Omega}_0 \setminus \Omega_{\tilde{\rho}_*}^0$ when $\mathcal{E}_0$ is a local attractor and $D = \bar{\Omega}_{\tilde{\rho}_*}^0$ when $\mathcal{E}_0$ is a local repeller, it follows that

$$\begin{cases} \mu_\alpha(\Omega_0 \setminus \bar{\Omega}_{\tilde{\rho}^*}^0) \to 0, & \text{when } \mathcal{E}_0 \text{ is a local attractor}, \\ \mu_\alpha(\Omega_{\tilde{\rho}_*}^0) \to 0, & \text{when } \mathcal{E}_0 \text{ is a local repeller}, \end{cases} \tag{4.6}$$

as $A_\alpha \to 0$. As Ω and D_* are separated by a positive distance, it is easy to construct a null families, still denoted by $\mathcal{A}$, such that both (4.2) and (4.5) hold simultaneously.

Let $\mu \in \mathcal{M}_\mathcal{A}$. Then it follows from Proposition 2.6 and (4.6) that μ is supported on $\bar{\Omega}_{\tilde{\rho}^*}^0$ in the case (a) and on $\Omega_0 \setminus \Omega_{\tilde{\rho}_*}^0$ in the case (b), both of which are positively invariant set of φ^t. It follows from Proposition A.5 that μ is actually supported on $\omega(\bar{\Omega}_{\tilde{\rho}^*}^0)$ which equals $\mathcal{J}_0$ in the case (a) and on $\omega(\Omega_0 \setminus \Omega_{\tilde{\rho}_*}^0)$ which equals $\mathcal{J}_*$ in the case (b). By Proposition 3.2, both $\mathcal{J}_0$ and $\mathcal{J}_*$ are $\mathcal{A}$-stable. □

The above theorem naturally applies to families of the form $\{\epsilon A\} \subset \mathcal{A}$, where $\epsilon > 0$ is a small parameter and A is a bounded, everywhere positive definite, $n \times n$ matrix-valued C^1 function on $\mathcal{U}$.

Corollary 4.1. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a C^2 Lyapunov function with bounded second derivatives on $\mathcal{U}$. Then there are bounded, everywhere positive definite, $n \times n$ matrix-valued C^1 functions A_i, $i = 1, 2$, on $\mathcal{U}$, such that (a), (b) of Theorem 4.1 hold respectively for the family $\{\epsilon A_i\}$, $i = 1, 2$, as $\epsilon \to 0$.*

Proof. For any bounded, everywhere positive definite, $n \times n$ matrix-valued C^1 function A on $\mathcal{U}$, $\{\epsilon A\}$ is clearly a normal null family. In the case that $\mathcal{U} \ne \mathbb{R}^n$, the invariance of $\{\epsilon A\}$ is guaranteed if $A(x) \to 0$ as $x \to \partial\mathcal{U}$. The corollary now follows from Theorem 4.1.

In fact, by the proof of Theorem 4.1, we just need to choose $A = (a^{ij})$ such that

$$\frac{\inf_\Omega \lambda_A(x)}{\max_{D_*} \Lambda_A(x)} > \frac{C_* + 1}{C^*},$$

where Ω, D_*, C_*, C^* are as in the proof of Theorem 4.1, $\Lambda_A(x) = \sqrt{\sum_{i,j}|a^{ij}(x)|^2}$, and $\lambda_A(x)$ is the smallest eigenvalue of $A(x)$ for each $x \in \mathcal{U}$. □

4.2. Uniform de-stabilization of a locally repelling equilibrium

We note that if $\mathcal{R}_0$ is a strong local repeller, then in general not every normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$ can de-stabilize $\mathcal{R}_0$. This is because $\mathcal{R}_0$ may be further decomposed to contain a strong local attractor for which Theorem 4.1(a) is applicable.

We now show that if $\mathcal{R}_0$ is a strongly repelling equilibrium (see Definition A.4), then it is de-stabilized by any normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$. This particularly implies part (c) of Theorem C.

Theorem 4.2. *Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a C^2 Lyapunov function with bounded second derivatives on $\mathcal{U}$. Let x_0 be a strongly repelling equilibrium of φ^t contained in the global attractor $\mathcal{J}$ of φ^t and $\mathcal{W}$ be an isolating neighborhood of x_0. Then with respect to any normal null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, $\mathcal{M}_\mathcal{A} \ne \emptyset$, and moreover, each $\mu \in \mathcal{M}_\mathcal{A}$ is supported on the maximal attractor $\mathcal{J}_*$ of φ^t in $\mathcal{J} \setminus \mathcal{W}$. Consequently, with respect to any invariant, normal null family $\mathcal{A}$, $\mathcal{J}_*$ is $\mathcal{A}$-stable and $\{x_0\}$ is strongly $\mathcal{A}$-unstable.*

Proof. Let $\mathcal{A} = \{A_\alpha\} = \{(a_\alpha^{ij})\} \subset \tilde{\mathcal{A}}$ be any bounded null family such that $|A_\alpha| \ll 1$. By Proposition 2.5, $\{\mathcal{L}_{A_\alpha}\}$ admits a uniform Lyapunov function in $\mathcal{U}$. By Proposition 2.1, for each α, there exists a stationary measure μ_α corresponding to $\{\mathcal{L}_{A_\alpha}\}$ which is also regular, i.e., $d\mu_\alpha(x) = u_\alpha(x)dx$ for some stationary solution $u_\alpha \in W_{loc}^{1,p}(\mathcal{U})$ corresponding to $\mathcal{L}_{A_\alpha}$ in $\mathcal{U}$.

Without loss of generality, we assume $x_0 = 0$. In virtue of Definition A.4, let U be an entire weak anti-Lyapunov function associated with $\mathcal{W}$ that satisfies (A.3)–(A.5). We note that since $\mathcal{W} \subset \mathcal{J}$, it is a pre-compact set in $\mathcal{U}$. For each $\rho \in [0, \rho_M)$, where ρ_M is the essential upper bound of U, we denote Ω_ρ as the ρ-sublevel set of U. We note by (A.5) that $\partial\Omega_\rho = U^{-1}(\rho)$ and hence $\bar{\Omega}_\rho = \{x \in \mathcal{W} : U(x) \le \rho\}$ for any $\rho \in (0, \rho_M)$.

Fix $\bar{\rho} \in (0, \rho_M)$. Let $\lambda_\alpha^{\bar{\rho}} := \inf_{x \in \Omega_{\bar{\rho}}} \lambda_\alpha(x)$, $\lambda_U^{\bar{\rho}} := \inf_{x \in \Omega_{\bar{\rho}}} \lambda_U(x)$, where $\lambda_\alpha(x)$, $\lambda_U(x)$ denote the smallest eigenvalues of $A_\alpha(x)$, $D^2U(x)$, respectively. Then for each parameter α,

$$\mathcal{L}_{A_\alpha} U(x) = a_\alpha^{ij}(x)\partial_{ij}^2 U(x) + V(x) \cdot \nabla U(x)$$

$$\ge a_\alpha^{ij}(x)\partial_{ij}^2 U(x) = \text{tr}\left(A_\alpha(x)D^2U(x)\right)$$

$$\ge \lambda_U^{\bar{\rho}} \lambda_\alpha^{\bar{\rho}} =: \gamma_\alpha^{\bar{\rho}}, \qquad x \in \Omega_{\bar{\rho}},$$

i.e., U is an anti-Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ on $\Omega_{\bar{\rho}}$ with the essential lower bound $\rho_m = 0$.

By (A.3), (A.4) and the Taylor's expansion of U at the point 0, it is easy to see that there is a constant $C_2 > 0$ and $\rho_* < \bar{\rho}$ such that

$$|\nabla U(x)|^2 \le C_2 U(x), \qquad x \in \bar{\Omega}_{\rho_*}.$$

Since U is of the class C^1 and $U(x) > 0$ on the compact set $\bar{\Omega}_{\bar{\rho}} \setminus \Omega_{\rho_*}$, we can make C_2 larger if necessary so that

$$|\nabla U(x)|^2 \le C_2 U(x), \qquad x \in \bar{\Omega}_{\bar{\rho}} \setminus \Omega_{\rho_*},$$

i.e., there exists a constant $C_2 = C_2(\bar{\rho}) > 0$ such that

$$|\nabla U(x)|^2 \le C_2 U(x), \qquad x \in \bar{\Omega}_{\bar{\rho}}.$$

It follows that

$$a_\alpha^{ij}(x)\partial_i U(x)\partial_j U(x) \le C_2 \Lambda_\alpha \rho =: H^\alpha(\rho), \qquad x \in \Omega_\rho, \ \rho \in [0, \bar{\rho}],$$

where $\Lambda_\alpha = \Lambda_\alpha(\bar{\rho}) := \sup_{x \in \Omega_{\bar{\rho}}} |A_\alpha(x)|$. We note that, since each μ_α is a regular stationary measure and $\bar{\Omega}_{\rho_m} = \{0\}$, $\mu_\alpha(\bar{\Omega}_{\rho_m}) = 0$. Applying Proposition 2.3 with respect to the stationary measure

$$\tilde{\mu}_\alpha = \frac{\mu_\alpha|_{\Omega_{\bar{\rho}}}}{\mu_\alpha(\Omega_{\bar{\rho}})}$$

corresponding to $\mathcal{L}_{A_\alpha}$ in $\mathcal{W}$, we have

$$\mu_\alpha(\Omega_{\rho_0}) = \mu_\alpha(\Omega_{\rho_0} \setminus \bar{\Omega}_{\rho_m}) \le e^{-C\int_{\rho_0}^{\bar{\rho}} \frac{1}{\rho}d\rho}, \qquad \rho_0 \in (0, \bar{\rho}), \tag{4.7}$$

where $C = C(\bar{\rho}) = \frac{\lambda_U^{\bar{\rho}}}{C_2(\bar{\rho})} \inf_\alpha \frac{\lambda_\alpha^{\bar{\rho}}}{A_\alpha(\bar{\rho})}$, which is a positive constant since $\mathcal{A} = \{A_\alpha\}$ is a normal null family.

It follows from Theorem 3.2 that $\mathcal{M}_\mathcal{A} \neq \emptyset$, and that any $\mathcal{A}$-limit measure is an invariant measure of φ^t supported on the global attractor $\mathcal{J}$. Let $\mu \in \mathcal{M}_\mathcal{A}$. Since Ω_{ρ_0} is open, we have by applying Proposition 2.6 to (4.7) and taking limit $A_\alpha \to 0$ that

$$\mu(\Omega_{\rho_0}) \leq e^{-C \int_{\rho_0}^{\bar{\rho}} \frac{1}{\rho} d\rho}, \qquad \rho_0 \in (0, \bar{\rho}).$$

By letting $\rho_0 \to 0$ in the above, we have

$$\mu(\{0\}) = 0. \tag{4.8}$$

For any given $\rho \in (0, \bar{\rho})$, we have by (A.5) and Proposition 2.5 that $\{\mathcal{L}_{A_\alpha}\}$ admits a uniform anti-Lyapunov function in Ω_ρ. It also follows from (A.5) that $\{0\}$ is the maximal repeller of φ^t in Ω_ρ. Applying Theorem 3.3(b) and (4.8), we have

$$\mu(\Omega_\rho) = \mu(\Omega_\rho \setminus \{0\}) = 0.$$

Since ρ is arbitrary, $\mu(\Omega_{\bar{\rho}}) = 0$. Since $\bar{\rho} \in (0, \rho_M)$ is also arbitrary, $\mu(\mathcal{W}) = 0$.

Since $\text{supp}(\mu) \subset \mathcal{J}$ and $\mathcal{J} \setminus \mathcal{W}$ is a positively invariant set of φ^t, it follows from Proposition A.5 that μ is actually supported on the ω-limit set $\mathcal{J}_* = \omega(\mathcal{J} \setminus \mathcal{W})$. The $\mathcal{A}$-stability of $\mathcal{J}_*$ follows from Proposition 3.2, Remark 2.3(1), and Lemma 3.1(a). □

Corollary 4.2. *Assume* $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ *and that* (1.1) *admits a* C^2 *Lyapunov function with bounded second derivatives on* $\mathcal{U}$. *Let* $x_0 \in \mathcal{U}$ *be an equilibrium of* φ^t *such that all eigenvalues of* $DV(x_0)$ *have positive real parts. Then* x_0 *is a strongly repelling equilibrium, and consequently the conclusion of Theorem 4.2 holds for* x_0.

Proof. By standard theory of dissipative dynamical systems [44, Sections 17 and 18], x_0 is a strongly repelling equilibrium. In fact, there is a positive definite $n \times n$ matrix B such that

$$(DV(x_0))^\top B + B(DV(x_0)) = I.$$

Let $U(x) = (x - x_0)^\top B(x - x_0)$. Then U is a desired entire weak anti-Lyapunov function satisfying (A.3)–(A.5) in the vicinity of x_0. □

Corollary 4.3. *Assume* $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ *and that* (1.1) *admits a* C^2 *Lyapunov function with bounded second derivatives on* $\mathcal{U}$. *Then Theorem 4.2 and Corollary 4.2 hold for any family of the form* $\{\epsilon A\}$ *when* $\epsilon \to 0$, *where* $A \in \tilde{\mathcal{A}}$ *is such that* $\sup_{x \in \mathcal{U}} |A(x)| < \infty$ *and* $A(x) \to 0$ *as* $x \to \partial\mathcal{U}$ *when* $\mathcal{U} \neq \mathbb{R}^n$.

Proof. Theorem 4.2 and Corollary 4.2 clearly hold because $\{\epsilon A\}$ is an invariant normal null family. □

Remark 4.1. In Theorems 4.1 and 4.2, we have assumed that (1.1) admits a C^2 Lyapunov function with bounded second derivatives on $\mathcal{U}$, which is easy to understand and to use since we only make restriction to the unperturbed system.

(1) By observing the proof of Theorem 4.1, for any given invariant null family $\mathcal{A} = \{A_\alpha\}$ (need not to be bounded) satisfying (2.4), if $\{\mathcal{L}_{A_\alpha}\}$ admits a uniform Lyapunov function U (need not to have bounded second derivatives on $\mathcal{U}$) and (4.2) and (4.5) hold, then the results of Theorem 4.1 remain valid, i.e., (a) the strong local attractor $\mathcal{J}_0$ is $\mathcal{A}$-stable; (b) the strong local repeller $\mathcal{R}_0$ is strongly $\mathcal{A}$-unstable.

(2) By observing the proof of Theorem 4.2, for any given null family $\mathcal{A} = \{A_\alpha\}$ (need not to be bounded) satisfying (2.4), if $\{\mathcal{L}_{A_\alpha}\}$ admits a uniform Lyapunov function U (need not to have bounded second derivatives on $\mathcal{U}$), then the results of Theorem 4.2 remain valid, i.e., $\mathcal{M}_\mathcal{A} \neq \emptyset$, each $\mu \in \mathcal{M}_\mathcal{A}$ is supported on the maximal

attractor $\mathcal{J}_*$ of φ^t in $\mathcal{J} \setminus \mathcal{W}$, and consequently $\mathcal{J}_*$ is $\mathcal{A}$-stable and $\{x_0\}$ is strongly $\mathcal{A}$-unstable if $\mathcal{A}$ is invariant.

These observations allow more flexible applications of our results, see e.g., the examples in Section 5.

5. Examples

In the presence of bounded multiplicative noise coefficients, the application of our concentration and de-concentration results to stochastic stability problems is rather straightforward because many dissipative deterministic systems are known to have Lyapunov functions which then become uniform Lyapunov functions for the noised perturbed systems.

In this section, we demonstrate by two examples the applications of our concentration and de-concentrations results to stochastic stability problems under multiplicative white noises with non-Lipschitz, super-linearly growing coefficients, to which the large derivation theory does not seem to apply at least not directly.

5.1. An example of stochastic Hope bifurcation

Consider stochastic differential equations

$$\begin{cases} dx = (bx - y - x(x^2 + y^2))dt + \sqrt{\epsilon}g^{11}(x, y)dW_1 \\ \quad + \sqrt{\epsilon}g^{12}(x, y)dW_2, \\ dy = (x + by - y(x^2 + y^2))dt + \sqrt{\epsilon}g^{21}(x, y)dW_1 \\ \quad + \sqrt{\epsilon}g^{22}(x, y)dW_2, \end{cases} \tag{5.1}$$

where b, ϵ are parameters, $g^{ij} \in W_{loc}^{1,p}(\mathbb{R}^2)$ for some $p > 2$, $|g^{ij}(x, y)| \leq c(x^2 + y^2)$ for some positive constant c outside some ball in $\mathbb{R}^2$, $i, j = 1, 2$, and $G(x, y) = (g^{ij}(x, y))$ is everywhere non-singular. Denote $V_b(x, y) = (bx - y - x(x^2 + y^2), x + by - y(x^2 + y^2))^\top$ and $A(x, y) = \frac{G(x,y)G^\top(x,y)}{2}$. Also denote the adjoint Fokker–Planck operators associated with (5.1) by $\mathcal{L}_{V_b, \epsilon A}$.

Let $U(x, y) = x^2 + y^2$. Then it is clear that

$$V_b \cdot \nabla U = 2U(b - U) \tag{5.2}$$

which approaches to $-\infty$ as $x^2 + y^2 \to \infty$. It follows that, for each b, the unperturbed system of (5.1) is dissipative; in particular, it generates a (positive) semiflow φ_b^t on $\mathbb{R}^2$ which admits a global attractor $\mathcal{J}_b$. Moreover, it follows from (5.2) and the assumption $|g^{ij}| \leq cU$, $i, j = 1, 2$, that U is in fact a uniform Lyapunov function in $\mathbb{R}^2$ with respect to the family $\{\mathcal{L}_{V_b, \epsilon A}\}$ when $0 < \epsilon \ll 1$. We have by Proposition 2.1 that each $\mathcal{L}_{V_b, \epsilon A}$ admits a unique stationary measure $\mu_{b, \epsilon A}$, by Lemma 3.1(a) that the set of limit measures of $\{\mu_{b, \epsilon A}\}$ as $\epsilon \to 0$ is non-empty, and by Theorem 3.2 that each limit measure is an invariant measure of φ_b^t supported on $\mathcal{J}_b$.

When $b \leq 0$, $\mathcal{J}_b = \{0\}$. Hence $\mu_{b, \epsilon A} \to \delta_0$ as $\epsilon \to 0$. Consequently, $\{0\}$ and δ_0 are $\{\epsilon A\}$-stable.

When $b > 0$, $\mathcal{J}_b = \bar{\Omega}_b$—the closed disk of radius $\sqrt{b}$ centered at the origin. We note that U is also an entire weak anti-Lyapunov function in Ω_b satisfying (A.3)–(A.5) for $x_0 = 0$. An application of Remark 4.1(2) in the open disk Ω_b yields that each limit measure of $\{\mu_{b, \epsilon A}\}$ is supported on $\partial\Omega_b =: C_b$—the circle with radius $\sqrt{b}$, as $\epsilon \to 0$. Since C_b is a periodic invariant cycle of φ_b^t, it is uniquely ergodic with the Haar measure μ_b as the only invariant measure. Therefore, $\mu_{b, \epsilon A} \to \mu_b$ as $\epsilon \to 0$. Consequently, C_b and μ_b become $\{\epsilon A\}$-stable in this case.

This example thus demonstrates an interesting phenomenon of stochastic Hopf bifurcation in which the repelling equilibrium $\{0\}$ is invisible when $b > 0$ under noise perturbations.

5.2. An example with non-decomposable ω-limit sets

Let $f \in C^2(\mathbb{R})$ be such that

$$f(x) = \begin{cases} 0, & \frac{1}{2} \leq |x| \leq \frac{3}{2}, \\ x^4, & |x| \geq 2, \end{cases}$$

$f(0) > 0$ is a local maximum, $f(x) > 0$ when $|x| < \frac{1}{2}$ and $|x| > \frac{3}{2}$, and $f'(x) \neq 0$ when $0 < |x| < \frac{1}{2}$ and $|x| > \frac{3}{2}$.

We consider the gradient system

$$\dot{x} = -f'(x) \tag{5.3}$$

and its noise perturbed system

$$\mathrm{d}x = -f'(x)\mathrm{d}t + \sqrt{\epsilon}\, g(x)\mathrm{d}W, \tag{5.4}$$

where $g \in W_{loc}^{1,p}(\mathbb{R})$ for some $p > 1$, $g(x) \neq 0$ everywhere and $|g(x)| \leq cx^2$ for some positive constant c when $|x| \gg 1$. We note that in this example the ω-limit sets of the unperturbed system cannot be decomposed into a finite number of equivalent classes as assumed in the Freidlin–Wentzell theory (see [13, Chapter 6] for details).

Denote $A(x) = \frac{g^2(x)}{2}$. It is immediate to check that the unperturbed system (5.3) admits the global attractor $\mathcal{J} = [-\frac{3}{2}, \frac{3}{2}]$, the function $U = x^2$ is an entire weak Lyapunov function for (5.3) and is a uniform Lyapunov function with respect to the family $\{\mathcal{L}_{\epsilon A}\}$ when $0 < \epsilon \ll 1$. By Proposition 2.1 each $\mathcal{L}_{\epsilon A}$ admits a unique stationary measure $\mu_{\epsilon A}$, by Lemma 3.1(a) the set of limit measures of $\{\mu_{\epsilon A}\}$ as $\epsilon \to 0$ is non-empty, and by Proposition 3.1 each limit measure is an invariant measure of (5.3) supported on the set $S_1 = \{x : -f'(x) \cdot U'(x) = 0\} = \{0\} \cup [-\frac{3}{2}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{2}]$. However, as $\{0\}$ is strongly repelling, we have by Remark 4.1(2) that all the limit measures are only supported on the set $S = [-\frac{3}{2}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{2}]$.

Note that both $[-\frac{3}{2}, -\frac{1}{2}]$ and $[\frac{1}{2}, \frac{3}{2}]$ are strong local attractors of (5.3), so if we decrease the amplitude of g in a small neighborhood of $[\frac{1}{2}, \frac{3}{2}]$ such that (4.5) holds, then by Remark 4.1(1) all the limit measures of $\{\mu_{\epsilon A}\}$ will be supported only on $[\frac{1}{2}, \frac{3}{2}]$. The same holds when we decrease the amplitude of g in a small neighborhood of $[-\frac{3}{2}, -\frac{1}{2}]$.

Remark 5.1. In general, knowing where the limits of stationary measures will concentrate on, additional information is still needed to determine what the limit measures are, even for systems as simple as gradient ones (see [45] for details).

Acknowledgments

Appendix

In this section, we will summarize some general notions and fundamental properties of dynamics of ordinary differential equations including dissipation and invariant measures. Some materials concerning dissipation are taken from the Appendix of [34] for a general domain $\mathcal{U} \subset \mathbb{R}^n$.

Consider (1.1) for a continuous V. Throughout the section, we assume that solutions of (1.1) are locally unique (e.g., V is locally Lipschitz continuous). Then (1.1) generates a (continuous) *local flow* $\varphi^t : \mathcal{U} \to \mathcal{U}$, where for each $\xi \in \mathcal{U}$, $\varphi^t(\xi)$ is the solution of (1.1) for t lying in the maximal interval $\mathcal{I}_\xi$ of existence in $\mathcal{U}$. A set $B \subset \mathcal{U}$ is called an *invariant set* (resp. *positively invariant set*, resp. *negatively invariant set*) of φ^t if $\mathcal{I}_\xi = \mathbb{R}$ (resp. $\mathcal{I}_\xi \supset \mathbb{R}_+$, resp. $\mathcal{I}_\xi \supset \mathbb{R}_-$) for all $\xi \in B$, and $\varphi^t(B) = B$ for all $t \in \mathbb{R}$ (resp. $\varphi^t(B) \subset B$ for all $t \in \mathbb{R}_+$, resp. $\varphi^t(B) \subset B$ for all $t \in \mathbb{R}_-$). If $\mathcal{U}$ itself is an *invariant set* (resp. *positively invariant set*, resp. *negatively invariant set*) of φ^t, φ^t is called a *flow* (resp. *positive semiflow*, resp. *negative semiflow*). We note that if φ^t is a flow, then for each $t \in \mathbb{R}$, $\varphi^t : \mathcal{U} \to \mathcal{U}$ is a homeomorphism.

A.1. Dissipation vs. anti-dissipation

For any set $B \subset \mathcal{U}$ such that $\cup_{t \in \mathbb{R}_+}\varphi^t(B)$ (resp. $\cup_{t \in \mathbb{R}_-}\varphi^t(B)$) exists and is pre-compact in $\mathcal{U}$, we recall that the ω-*limit set* of B (resp. α-*limit set of* B) is defined as

$$\omega(B) = \cap_{\tau \geq 0}\overline{\cup_{t \geq \tau}\varphi^t(B)} \qquad (\text{resp. } \alpha(B) = \cap_{\tau \leq 0}\overline{\cup_{t \leq \tau}\varphi^t(B)}).$$

Definition A.1. Let $\Omega \subset \mathcal{U}$ be a connected open set.

(1) φ^t is said to be *dissipative* (resp. *anti-dissipative*) *in* Ω if Ω is a positively (resp. negatively) invariant set of φ^t and there exists a compact subset B of Ω with the property that for any $\xi \in \Omega$ there exists a $t_0(\xi) > 0$ such that $\varphi^t(\xi) \in B$ as $t \geq t_0(\xi)$ (resp. $t \leq -t_0(\xi)$).

(2) When Ω is a positively (resp. negatively) invariant set of φ^t, the *maximal attractor* $\mathcal{J}$ (resp. *maximal repeller* $\mathcal{R}$) of φ^t *in* Ω is a compact invariant set of φ^t which attracts (resp. repels) any pre-compact subset K of Ω, i.e., $\omega(K) \subset \mathcal{J}$ (resp. $\alpha(K) \subset \mathcal{R}$), or equivalently, $\lim_{t \to +\infty}\mathrm{dist}(\varphi^t(K), \mathcal{J}) = 0$ (resp. $\lim_{t \to -\infty}\mathrm{dist}(\varphi^t(K), \mathcal{R}) = 0$), where for any two subsets A, B of $\mathbb{R}^n$, $\mathrm{dist}(A, B)$ denotes the Hausdorff semi-distance from A to B.

(3) Suppose that the maximal attractor $\mathcal{J}$ (resp. maximal repeller $\mathcal{R}$) of φ^t exists in Ω. If $\Omega = \mathcal{U}$, then we call $\mathcal{J}$ (resp. $\mathcal{R}$) the *global attractor* (resp. *global repeller*) of φ^t. Otherwise, it is called a *local attractor* (resp. *local repeller*) of φ^t.

It is clear that maximal attractor or repeller of φ^t in Ω, if exists, must be unique. Consequently, global attractor or repeller of φ^t is unique.

Proposition A.1. *Let* $\Omega \subset \mathcal{U}$ *be a connected open set.*

(1) φ^t *is dissipative (resp. anti-dissipative) in* Ω *if and only if* Ω *is a positively (resp. negatively) invariant set and admits a maximal attractor (resp. repeller) of* φ^t.

(2) If φ^t *is dissipative (resp. anti-dissipative) in* Ω, *then the maximal attractor* $\mathcal{J}$ *(resp. maximal repeller* $\mathcal{R}$*) of* φ^t *in* Ω *equals*

$$\mathcal{J} = \bigcup_{B \subset \Omega \text{ pre-compact}} \omega(B) \quad (\text{resp. } \mathcal{R} = \bigcup_{B \subset \Omega \text{ pre-compact}} \alpha(B)).$$

Proof. See Proposition 5.2 in [34]. $\square$

Definition A.2. Let $\Omega \subset \mathcal{U}$ be a connected open set and $U \in C^1(\Omega)$ be a compact function with essential upper bound ρ_M. For each $\rho \in [0, \rho_M)$, denote Ω_ρ as the ρ-sublevel set of U.

(1) U is a *weak Lyapunov function* (resp. *weak anti-Lyapunov function*) of (1.1) *in* Ω if

$$V(x) \cdot \nabla U(x) \leq 0 \qquad (\text{resp. } \geq 0), \quad x \in \tilde{\Omega} = \Omega \setminus \bar{\Omega}_{\rho_m}, \tag{A.1}$$

where $\rho_m \in (0, \rho_M)$ is a constant, called *essential lower bound of* U, and $\tilde{\Omega}$ is called *essential domain of* U *in* Ω.
If (A.1) holds for all $x \in \Omega$ instead of $\tilde{\Omega}$, then U is called an *entire weak Lyapunov function (resp. entire weak anti-Lyapunov function) of* (1.1) *in* Ω.

(2) U is a *Lyapunov function* (resp. *anti-Lyapunov function*) *of* (1.1) *in* Ω if there exists a constant $\gamma > 0$, called a *Lyapunov constant of* U, such that

$$V(x) \cdot \nabla U(x) \le -\gamma \qquad (\text{resp.} \ge \gamma),$$
$$x \in \tilde{\Omega} = \Omega \setminus \bar{\Omega}_{\rho_m}, \tag{A.2}$$

where $\rho_m \in (0, \rho_M)$ is a constant, called *essential lower bound of* U, and $\tilde{\Omega}$ is called *essential domain of* U *in* Ω.

Proposition A.2. *Let* $\Omega \subset \mathcal{U}$ *be a connected open set.*

(a) If (1.1) *admits a weak Lyapunov (resp. anti-Lyapunov) function in* Ω, *then* Ω *must be a positively (resp. negatively) invariant set of* φ^t.

(b) If (1.1) *admits a Lyapunov (resp. anti-Lyapunov) function in* Ω, *then it must be dissipative (resp. anti-dissipative) in* Ω, *with the maximal attractor (resp. repeller) in* Ω *being* $\omega(\Omega_{\rho_m})$ *(resp.* $\alpha(\Omega_{\rho_m})$), *where* ρ_m *is the essential lower bound and* Ω_{ρ_m} *is the* ρ_m-*sublevel set of the Lyapunov (resp. anti-Lyapunov) function.*

Proof. See Proposition 5.3 in [34]. □

The following is a strong version of the well-known LaSalle invariance principle ([46, Chapter 2, Theorem 6.4]) for locating ω-limit sets in a domain which admits an entire weak Lyapunov function.

Proposition A.3. *Let* $\Omega \subset \mathcal{U}$ *be a connected open set. If* (1.1) *admits an entire weak Lyapunov function* U *in* Ω, *then for any* $x_0 \in \Omega$,

$$\omega(x_0) \subset S,$$

where $S = \{x \in \Omega : V(x) \cdot \nabla U(x) = 0\}$.

We note that the original LaSalle invariance principle does not require U be a compact function (in this case, the conclusion of Proposition A.3 only holds for points which have bounded positive orbits).

Definition A.3. A compact invariant set $\mathcal{J} \subset \mathcal{U}$ (resp. $\mathcal{R} \subset \mathcal{U}$) of φ^t is called a *strong local attractor* (resp. *strong local repeller*) of φ^t if there is a connected, positively (resp. negatively) invariant neighborhood $\mathcal{W}$ of $\mathcal{J}$ (resp. $\mathcal{R}$) in $\mathcal{U}$, called an *isolating neighborhood*, with oriented C^2 boundary such that

(1) $\mathcal{J}$ attracts $\mathcal{W}$ (resp. $\mathcal{R}$ repels $\mathcal{W}$);
(2) $V(x) \cdot \nu(x) < 0$ (resp. $V(x) \cdot \nu(x) > 0$), $x \in \partial\mathcal{W}$, where $\nu(x)$ denotes the outward unit normal vector of $\partial\mathcal{W}$ at $x \in \partial\mathcal{W}$.

Remark A.1. The neighborhood $\mathcal{W}$ in Definition A.3 is indeed an isolating neighborhood in the usual sense, i.e., any entire orbits of φ^t lying in $\mathcal{W}$ must lie in $\mathcal{J}$ (resp. $\mathcal{R}$).

Proposition A.4. *A local attractor (resp. repeller) of* φ^t *is a strong local attractor (resp. repeller* $\mathcal{R}$) *if and only if it has an isolating neighborhood on which* (1.1) *admits a Lyapunov (resp. anti-Lyapunov) function.*

Proof. We only treat the case of a local attractor $\mathcal{J}$ of φ^t.

To show the necessity, let $\mathcal{W}$ be an isolating neighborhood of $\mathcal{J}$ satisfying properties described in Definition A.3. Using smooth partition of unity, we can parametrize the boundary $\partial\mathcal{W}$ by a positive C^2 function U in the vicinity of $\partial\mathcal{W}$ so that $\partial\mathcal{W} = \{U(x) = \rho_0\}$ for some $\rho_0 > 0$. By modifying U away from $\partial\mathcal{W}$, we can

assume without loss of generality that U is defined everywhere in the interior of $\mathcal{W}$ and $U(x) < \rho_0$ for all $x \in \mathcal{W}$, i,e, U is a compact function in $\mathcal{W}$.

Let $\gamma = \min_{x \in \partial\mathcal{W}}|V(x) \cdot \nu(x)|$. Then $\gamma > 0$. Since $\nu(x) = \nabla U(x)/|\nabla U(x)|$,

$$V(x) \cdot \nabla U(x) \le -\gamma|\nabla U(x)| \le -\gamma \min_{x \in \partial\mathcal{W}}|\nabla U(x)| =: -\tilde{\gamma}$$

for all $x \in \partial\mathcal{W}$. It follows that there is a neighborhood $\tilde{\mathcal{W}}$ of $\partial\mathcal{W}$ in $\mathcal{W}$ such that

$$V(x) \cdot \nabla U(x) < -\frac{\tilde{\gamma}}{2}, \qquad x \in \tilde{\mathcal{W}},$$

i.e., U is a Lyapunov function of (1.1) in $\mathcal{W}$.

To prove the sufficiency, we let $\mathcal{W}$ be an isolating neighborhood of $\mathcal{J}$ on which a Lyapunov function U of (1.1) exists. Using C^2 approximation, we may assume without loss of generality that U is of the class C^2 in a connected open set $\tilde{\mathcal{W}} \supset \mathcal{W}$ on which U remains as a Lyapunov function of φ^t. Then it is clear that Definition A.3(2) is satisfied on $\partial\mathcal{W}$. □

Definition A.4. An equilibrium x_0 of (1.1) is called a (local) *strongly repelling equilibrium* if there is a connected open set $\mathcal{W}$ containing x_0, called an *isolating neighborhood of* x_0, and a compact function $U \in C^2(\mathcal{W})$ such that

the Hessian matrix D^2U is everywhere positive
  definite in $\mathcal{W}$, (A.3)

$$U(x_0) = 0, \quad \nabla U(x_0) = 0; \quad U(x) > 0, \ x \in \mathcal{W} \setminus \{x_0\},$$
and (A.4)

$$V(x) \cdot \nabla U(x) > 0, \ x \in \mathcal{W} \setminus \{x_0\}. \tag{A.5}$$

Remark A.2. (1) We note that a strongly repelling equilibrium x_0 of (1.1) is necessarily a strong local repeller.

(2) By a time reversing application of Converse Lyapunov Theorem (see e.g., [47, Theorem 4.2.1], [48, Theorem V.2.12]), if x_0 is a (local) repelling equilibrium in the usual sense, then there is a Lipschitz continuous function U in a neighborhood of x_0 which is an almost everywhere entire weak anti-Lyapunov function and satisfies (A.4), (A.5) almost everywhere.

A.2. Invariant measures

For a Borel set $\Omega \subset \mathbb{R}^n$, we denote by $M(\Omega)$ the set of Borel probability measures on Ω furnished with the weak*-topology.

For an invariant (resp. positively invariant, negatively invariant) Borel set $\Omega \subset \mathcal{U}$ of φ^t, $\mu \in M(\Omega)$ is said to be an *invariant measure* (resp. *positively invariant measure, negatively invariant measure*) of φ^t on Ω if for any Borel set $B \subset \Omega$,

$$\mu\left(\varphi^{-t}(B)\right) = \mu(B), \qquad t \in \mathbb{R} \ (\text{resp. } t \in \mathbb{R}_+, \ t \in \mathbb{R}_-), \tag{A.6}$$

where $\varphi^{-t}(B) = \{x \in \Omega : \varphi^t(x) \in B\}$.

We note that if Ω is an invariant Borel set, then it follows from the flow property that any positively or negatively invariant measure on Ω must be invariant.

An invariant measure μ is an *ergodic measure* if any invariant Borel set has either full μ-measure or zero μ-measure. A compact invariant set admits at least one ergodic measure. It is said to be *uniquely ergodic* if it admits only one invariant measure.

Proposition A.5. *Let* $\Omega \subset \mathcal{U}$ *be a positively (resp. negatively) invariant Borel set in* $\mathcal{U}$ *which admits a positively (resp. negatively) invariant measure* μ *of* φ^t. *Then the following holds.*

(1) *If Ω is pre-compact in $\mathcal{U}$, then μ is an invariant measure of φ^t supported on $\omega(\Omega)$ (resp. $\alpha(\Omega)$).*

(2) *If Ω is a connected open subset of $\mathcal{U}$ and φ^t is dissipative (resp. anti-dissipative) in Ω, then μ is supported on the maximal attractor (resp. repeller) of φ^t in Ω.*

Proof. (1) can be easily seen from the definitions of invariant measures, positively (resp. negatively) invariant measures, ω-limit (resp. α-limit) sets.

To prove (2), we first consider the case that φ^t is dissipative in Ω. Let $\mathcal{J}$ denote the maximal attractor of φ^t in Ω. Since $\mathcal{J}$ is a compact invariant set, it is sufficient to show that $\mu(\mathcal{J}) = 1$.

For any given $\epsilon > 0$, we have by the Borel regularity of μ that there exist a compact subset B of Ω and an open neighborhood W of $\mathcal{J}$ lying in Ω such that $\mu(B) > 1 - \epsilon$ and $\mu(W \setminus \mathcal{J}) < \epsilon$. Since $\mathcal{J}$ attracts B, there exists $t > 0$ such that $\varphi^t(B) \subseteq W$. Thus,

$$\mu(\mathcal{J}) \geq \mu(W) - \epsilon \geq \mu(\varphi^t(B)) - \epsilon$$
$$= \mu\left((\varphi^t)^{-1}(\varphi^t(B))\right) - \epsilon \geq \mu(B) - \epsilon \geq 1 - 2\epsilon.$$

Since ϵ is arbitrary, $\mu(\mathcal{J}) = 1$.

The case when φ^t is anti-dissipative in Ω is similar. □

References

[1] L. Arnold, P. Kloeden, Discretization of a random dynamical system near a hyperbolic point, Math. Nachr. 181 (1996) 43–72.

[2] J. Duan, K. Lu, B. Schmalfuss, Invariant manifolds for stochastic partial differential equations, Ann. Probab. 31 (2003) 2109–2135.

[3] J. Duan, K. Lu, B. Schmalfuss, Smooth stable and unstable manifolds for stochastic evolutionary equations, J. Dynam. Differential Equations 16 (2004) 949–972.

[4] T. Wanner, Linearization of random dynamical systems, in: C. Jones, U. Kirchgraber, H.O. Walther (Eds.), Dynamics Reported, Vol. 4, 1995, pp. 203–269.

[5] L. Arnold, Random Dynamical Systems, in: Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, p. xvi+586.

[6] L. Arnold, P. Imkeller, Normal forms for stochastic differential equations, Probab. Theory Related Fields 110 (1998) 559–588.

[7] L. Arnold, K. Xu, Normal forms for random diffeomorphisms, Dynam. Differential Equations 4 (1992) 445–483.

[8] W. Li, K. Lu, Sternberg theorems for random dynamical systems, Comm. Pure Appl. Math. 58 (2005) 941–988.

[9] R.Z. Has'minskiĭ, Stochastic Stability of Differential Equations, in: Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, vol. 7, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, p. xvi+344. Translated from the Russian by D. Louvish.

[10] K. Liu, Stability of Infinite Dimensional Stochastic Differential Equations with Applications, in: Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 135, Chapman & Hall/CRC, Boca Raton, FL, 2006, p. xii+298.

[11] R.Z. Has'minskiĭ, Principle of averaging for parabolic and elliptic differential equations and for Markov processes with small diffusion, Theory Probab. Appl. 8 (1963) 1–21.

[12] M.B. Nevel'son, The behavior of an invariant measure of a diffusion process with small diffusion on a circle, Theory Probab. Appl. 9 (1964) 125–130.

[13] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, second ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 260, Springer-Verlag, New York, 1998, p. xii+430. Translated from the 1979 Russian original by Joseph Szücs.

[14] W. Cowieson, L.-S. Young, SRB measures as zero-limits, Ergodic Theory Dynam. Systems 25 (2005) 1115–1138.

[15] Y. Kifer, Random Perturbations of Dynamical Systems, in: Progress in Probability and Statistics, vol. 16, Birkhäuser Boston, Inc., Boston, MA, 1988, p. vi+294.

[16] Y. Kifer, On small random perturbations of some smooth dynamical systems, Izv. Akad. Nauk SSSR 38 (1974) 1083–1107.

[17] L.-S. Young, Stochastic stability of hyperbolic attractors, Ergodic Theory Dynam. Syst. 6 (1986) 311–319.

[18] E.C. Zeeman, Stability of dynamical systems, Nonlinearity 1 (1988) 115–155.

[19] W. Just, H. Kantz, C. Rödenbeck, M. Helm, Stochastic modelling: replacing fast degrees of freedom by noise, J. Phys. A: Math. Gen. 34 (2001) 3199–3213.

[20] R.S. Mackay, Langevin equation for slow degrees of freedom of Hamiltonian systems, in: Nonlinear Dynamics and Chaos: Advances and Perspectives UnderstandIng Complex Systems, Springer, 2010, pp. 89–102.

[21] J. De Simoi, C. Liverani, The martingale approach after Varadhan and Dolgopyat, in: Hyperbolic Dynamics, Fluctuations and Large Deviations, in: Proc. Sympos. Pure Math., vol. 89, Amer. Math. Soc., Providence, RI, 2015.

[22] J. De Simoi, C. Liverani, C. Poquet, D. Volk, Fast-slow partially hyperbolic systems versus Freidlin–Wentzell random systems, J. Stat. Phys. 166 (3–4) (2017) 650–679.

[23] D. Kelly, I. Melbourne, Deterministic homogenization for fast-slow systems with chaotic noise, J. Funct. Anal. 272 (10) (2017) 4063–4102.

[24] H. Qian, Stochastic physics, complex systems and biology, Quant. Biol. 1 (2013) 50–53.

[25] V.I. Bogachev, N.V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, Comm. Partial Differential Equations 26 (2001) 2037–2080.

[26] V.I. Bogachev, N.V. Krylov, M. Röckner, Elliptic and parabolic equations for measures, Russian Math. Surveys 64 (2009) 973–1078.

[27] V.I. Bogachev, M. Röckner, V. Stannat, Uniqueness of invariant measures and essential m-dissipativity of diffusion operators on L^1, in: Infinite Dimensional Stochastic Analysis (Amsterdam, 1999), 39-54, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52, R. Neth. Acad. Arts Sci., Amsterdam, 2000.

[28] V.I. Bogachev, M. Röckner, V. Stannat, Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions, (Russian) Mat. Sb. 193 (7) (2002) 3–36 translation in Sb. Math. 193 2002, 7-8, 945–976.

[29] A. Arapostathis, V.S. Borkar, M.K. Ghosh, Ergodic Control of Diffusion Processes, Cambridge University Press, Cambridge, 2012.

[30] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, On positive and probability solutions of the stationary Fokker–Planck-Kolmogorov equation, Dokl. Math. 85 (2012) 350–354.

[31] A. Yu. Veretennikov, On polynomial mixing bounds for stochastic differential equations, Stochastic Process. Appl. 70 (1997) 115–127.

[32] W. Huang, M. Ji, Z. Liu, Y. Yi, Steady states of Fokker–Planck equations: I. Eixstence, J. Dynam. Differential Equations 27 (2015) 721–742.

[33] W. Huang, M. Ji, Z. Liu, Y. Yi, Steady States of Fokker–Planck equations: III. Degenerate Diffusion, J. Dynam. Differential Equations 28 (2016) 127–141.

[34] W. Huang, M. Ji, Z. Liu, Y. Yi, Steady States of Fokker–Planck equations: II. Non-Eixstence, J. Dynam. Differential Equations 27 (2015) 743–762.

[35] W. Huang, M. Ji, Z. Liu, Y. Yi, Integral identity and measure estimates for stationary Fokker–Planck equations, Ann. Probab. 43 (2015) 1712–1730.

[36] W. Feller, The parabolic differential equations and associated semi-groups of transformations, Ann. of Math. (2) 55 (1952) 468–519.

[37] W. Feller, Diffusion processes in one dimension, Trans. Amer. Math. Soc. 77 (1954) 1–31.

[38] A.D. Wentzell, On boundary conditions for multi-dimensional diffusion processes, Theory Probab. Appl. 4 (1959) 164–177.

[39] V.I. Bogachev, M. Röckner, A generalization of Khasminskii's theorem on the existence of invariant measures for locally integrable drifts, Theory Probab. Appl. 45 (2001) 363–378.

[40] K.R. Parthasarathy, Probability Measures on Metric Spaces, in: Probability and Mathematical Statistics, vol. 3, Academic Press, Inc., New York-London, 1967, p. xi+276.

[41] P. Billingsley, Convergence of Probability Measures, second ed., in: Wiley Series in Probability and Statistics: Probability and Statistics, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, p. x+277.

[42] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, in: Classics in Mathematics, Springer-Verlag, Berlin, 2001, p. xiv+517. Reprint of the 1998 edition.

[43] P. Walter, An Introduction To Ergodic Theory, in: Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982, p. ix+250.

[44] J.P. LaSalle, S. Lefschetz, Stability By Liapunov's Direct Method with Applications, in: Mathematics in Science and Engineering, vol. 4, Academic Press, New York-London, 1961, p. vi+134.

[45] W. Huang, M. Ji, Z. Liu, Y. Yi, Stochastic stability of measures in gradient systems, Physica D 314 (2016) 9–17.

[46] J.P. LaSalle, The Stability of Dynamical Systems, in: CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 25, SIAM, Philadelphia, PA, 1976.

[47] X. Liao, L. Wang, P. Yu, Stability of Dynamical Systems, in: Monograph Series on Nonlinear Science and Complexity, vol. 5, Elsevier B.V., Amsterdam, 2007, p. xii+706.

[48] N.P. Bhatia, G.P. Szegö, Stability Theory of Dynamical Systems, in: Classics in Mathematics, Springer-Verlag, Berlin, 2002, p. xii+225. Reprint of the 1970 edition.