RELAXATION OSCILLATIONS IN PREDATOR-PREY SYSTEMS

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Dedicated to the memory of Professor Pavol Brunosky

Abstract. We characterize a criterion for the existence of relaxation oscillations in planar systems of the form

$$\frac{du}{dt} = u^{k+1}g(u, v, \varepsilon), \qquad \frac{dv}{dt} = \varepsilon f(u, v, \varepsilon) + u^{k+1}h(u, v, \varepsilon),$$

where $k \geq 0$ is a given integer and $\varepsilon > 0$ is a sufficiently small parameter. Taking into account of possible degeneracy of the "discriminant" function occurred when $k > 0$, this criterion generalizes those for the case $k = 0$ obtained by Hsu [10] and Hsu and Wolkowicz [12]. Differing from the case of $k = 0$, our proof of the criterion is based on the construction of an invariant, thin annular region in an arbitrarily prescribed small neighborhood of a singular closed orbit and the establishment of an asymptotic formula for solutions near the v-axis.

As applications of this criterion, we will give concrete conditions ensuring the existence of relaxation oscillations in general predator-prey systems, as well as spatially homogenous relaxation oscillations and relaxed periodic traveling waves in a class of diffusive predator-prey systems.

1. Introduction

This paper is devoted to the study of relaxation oscillations in planar systems of the form

$$\begin{cases} \dfrac{du}{dt} = u^{k+1}g(u, v, \varepsilon), \\ \dfrac{dv}{dt} = \varepsilon f(u, v, \varepsilon) + u^{k+1}h(u, v, \varepsilon), \end{cases} \tag{1.1}$$

where $k \geq 0$ is a given integer, $\varepsilon > 0$ is a small parameter, and f, g, h are smooth functions. Under suitable conditions on f, g, h, we will establish a criterion for the existence of relaxation oscillations of (1.1) for any k.

A prototype of (1.1) with $k = 0$ is the predator-prey system

$$\begin{cases} u' = u\left(1 - u - \frac{v}{1+u}\right), \\ v' = v\left(-\varepsilon + \frac{u}{1+u}\right), \end{cases} \tag{1.2}$$

or more generally, the system

$$\begin{cases} u' = p(u)(g(u) - v), \\ v' = v(-\varepsilon + p(u)), \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ denotes the small death rate of the predator population $v(t)$, and, with $K > 0$ being the carrying capacity for the prey population $u(t)$, $p(u)$ and $g(u)$ are smooth functions

2010 *Mathematics Subject Classification.* 34C26, 34C25,34D15, 92D25.

Key words and phrases. Relaxation oscillations, Periodic traveling waves, Singular and regular perturbations, Predator-prey systems.

The second author was partially supported by NSERC discovery grants 1257749 and RGPIN-2020-04451, a faculty development grant from University of Alberta, and a Scholarship from Jilin University.

satisfying $p(0) = 0$, $p'(0) > 0$, $p(u) > 0$ for $u \in (0, K]$, $g(K) = 0$, and $g(u) > 0$ for $u \in [0, K)$. An example of (1.1) with $k = 1$ is the predator-prey system

$$\begin{cases} u' = u^2 \left(1 - u - \frac{\alpha v}{a+bu+u^2}\right), \\ v' = v\left(-\varepsilon + \frac{\beta u^2}{a+bu+u^2}\right), \end{cases} \tag{1.4}$$

where the term $u^2(1-u)$ describes weak Allee effect on the growth of the prey population [2] and the term $u^2/(a + bu + u^2)$ is the generalized Holling-type IV functional response.

As a significant application of the criterion to be established, we will give concrete conditions which guarantee the existence of relaxation oscillations in general predator-prey systems of the form

$$\begin{cases} \dfrac{du}{dt} = u^{k+1} p_0(u, \varepsilon)[g(u) - q(v) + R(u, v, \varepsilon)], \\ \dfrac{dv}{dt} = h_1(v, \varepsilon)[-\varepsilon f(u, \varepsilon) + u^{k+1} p_1(u, \varepsilon)]. \end{cases}$$

We will also verify these conditions in diffusive predator-prey systems of the form

$$\begin{cases} u_t = D_u u_{xx} + u^{k+1}[g(u) - p_0(u)v], \\ v_t = D_v v_{xx} + v\Big(-\varepsilon + \beta u^{k+1} p_1(u)\Big) \end{cases}$$

to obtain spatially homogeneous relaxation oscillations and relaxed periodic traveling waves whose profiles exhibit relaxation behaviors.

Among multi-scale oscillatory phenomena arising in many problems of science and engineering, relaxation oscillations were first discovered in slow-fast planar systems of the form

$$\begin{cases} \dfrac{du}{dt} = g(u, v, \varepsilon), \\ \dfrac{dv}{dt} = \varepsilon f(u, v, \varepsilon), \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is small parameter. These oscillations typically exhibit a "relaxed" behavior characterized by alternating processes on different time scales, i.e., a long relaxation period during which the solutions approach a branch of equilibrium points of (1.5) with $\varepsilon = 0$, followed by a short impulsive period in which the solutions jump to a different branch of equilibrium points. The periods of such relaxation oscillations are mainly determined by the relaxation time which go to infinity as $\varepsilon \to 0$. A classical example of (1.5) is the van der Pol's equation

$$\begin{cases} \dfrac{du}{dt} = v - (\dfrac{1}{3}u^3 - u), \\ \dfrac{dv}{dt} = -\varepsilon u \end{cases}$$

in which the relaxation oscillation is a limit cycle, called *relaxation cycle*, approaching, as $\varepsilon \to 0$, to a piecewise smooth closed curve, called *singular closed orbit*, that consists of the left and right arcs on the cubic curve $v = u^3/3 - u$ and two horizontal line segments joining these arcs. We refer the reader to books [7, 27] for the rigorous study of the existence, orbital stability, and asymptotic profiles of relaxation oscillations in the van der Pol's equation, and to [8, 17, 18, 19, 26] and the references therein for the investigations of relaxation oscillations in nonlinear systems of the form (1.5) arising in the studies of networks of firing nerve cells, coupled chemical reactions, the beating human heart, earthquakes, gene activation systems, etc.

Motivated by applications arising in populations dynamics, many recent studies (see e.g., [1, 3, 4, 6, 9, 21, 24] and the references therein) are made towards relaxation oscillations in planar systems of the form

$$\begin{cases} \dfrac{du}{dt} = u^{k+1} g(u, v, \varepsilon), \\ \dfrac{dv}{dt} = \varepsilon f(u, v, \varepsilon), \end{cases} \tag{1.6}$$

where $k \geq 0$ is a given integer. A new feature of such a system is that the limit of its relaxation cycles, as $\varepsilon \to 0$, as the singular closed orbit Γ_0, now consists of an arc on the curve $\{g(u, v, 0) = 0\}$, a segment on the curve $\{u = 0\}$, and two horizontal line segments joining these pieces. The entry-exit function that determines the piece of Γ_0 on the v-axis was introduced, and several methods such as the combination of geometric singular perturbation theory and the exchange lemma, blow up techniques, and classical analysis arguments, etc. have been used to study the entry-exit function and the existence of the relaxation oscillations of (1.6).

The system (1.1) in the case of $k = 0$ has been extensively studied (see e.g., [10, 11, 12, 13, 14, 20, 22, 29]). In particular, using the geometric theory of singular perturbations [5, 15] and the Exchange Lemma [15, 16, 25], general criteria for the existence of relaxation oscillations of (1.1) in the case of $k = 0$ were established in [10, 11, 12]. These works have suggested that relaxation oscillations in system (1.1) appear to be more complicated than the ones in the slow-fast systems (1.5) and (1.6) in the sense that they approach a singular closed orbit Γ_0, as $\varepsilon \to 0$, that consists of a heteroclinic orbit in the first quadrant of their limiting system

$$\begin{cases} \dfrac{du}{dt} = u^{k+1} g(u, v, 0), \\ \dfrac{dv}{dt} = u^{k+1} h(u, v, 0), \end{cases}$$

and a segment on the v-axis between the α- and ω- limit points of this heteroclinic orbit (see the blue curves in Figure 1 for a demonstration of Γ_0). An essential feature of such Γ_0 is that its segment on the v-axis is equal to an interval determined by the entry-exit function of the slow-fast system (1.6) (see Section 2 for more explanations).

In this paper, we extend the results of [10, 11, 12] by establishing a criterion for the existence of relaxation oscillations of (1.1) for any k. Differing from the geometric method used in these works, our approach is more constructive following the classical method of finding invariant regions (see e.g. [1, 7, 24, 27]). More precisely, we will construct a thin invariant annular region $\mathcal{U}_\varepsilon$ in an arbitrarily prescribed small neighborhood of Γ_0, with $\mathcal{U}_\varepsilon$ being enclosed by two orbits of (1.1) and two segments connecting their own starting and ending points respectively in a vertical transversal segment of (1.1) (see Figure 1 for an illustration), together with the establishment of an asymptotic formula for solutions near the v-axis. Not only does this approach yield a better estimate on the locations of relaxation cycles than those in the case $k = 0$ but also it allows one to handle the possible degenerate situations occurred in the "discriminant" function when $k > 0$ (see Remark 2.1 for more detail).

We remark that our approach can also be used to find multiple relaxation oscillations for (1.1) with any $k \geq 0$, as in [10, 12] for the case $k = 0$. We focus on the construction of one relaxation orbit in this paper for the sake of simplicity.

The paper is organized as follows. In Section 2, we describe and states two results, i.e., Theorem 2.1 concerning the approximation of the orbits of (1.1) by their "limiting orbits", and Theorem 2.2, which is our main result, stating the existence of relaxation oscillations. Various terminologies including singular orbit and entry-exit and"discriminant" functions are introduced in this section. In Section 3, we present the complete proofs of Theorems 2.1 and 2.2. A main ingredient of the proofs is the establishment of an asymptotic formula for the solutions of (1.1) near the v-axis, using an argument similar to the one made in [1] for

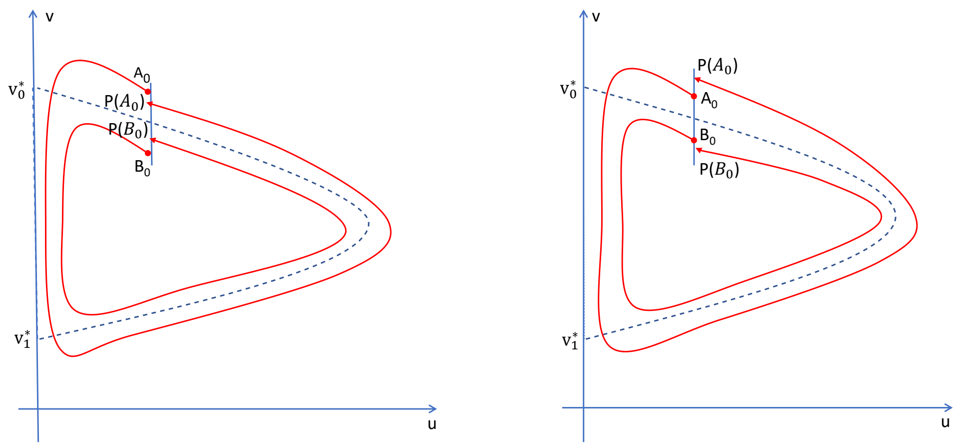

FIGURE 1. *The dotted blue curve is the singular closed orbit Γ_0 that consists of the heteroclinic orbit $\widehat{v_1^* v_0^*}$ and the segment $\overline{v_0^* v_1^*}$ on the v-axis; two red curves $\widehat{A_0 P(A_0)}$ and $\widehat{B_0 P(B_0)}$ are two orbits of (1.1); the region bounded by the outer boundary $\widehat{A_0 P(A_0)} \cup \overline{P(A_0) A_0}$ and the inner boundary $\widehat{B_0 P(B_0)} \cup \overline{P(B_0) B_0}$ is $\mathcal{U}_\varepsilon$, which is positively (resp. negatively) invariant for (1.1) in the left (resp. right) figure. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)*

system (1.6). In Section 4, we apply Theorem 2.2 to two classes of predator-prey models, one is a general class of planar models admitting relaxation oscillations and the other one is a class of diffusive systems admitting spatially homogeneous relaxation oscillations and relaxed periodic traveling waves.

2. Description and statement of main result

In this section, we make general assumptions, define the singular orbit Γ_0, introduce the entry-exit function and a "discriminant" function $\Delta(v_0)$ whose sign determines the relative locations of the "Poincare return map" for the limiting orbits of (1.1), and construct the annular region $\mathcal{U}_\varepsilon$. Our main result will be stated at the end.

2.1. **Singular closed orbit.** We note that outside any given neighborhood of the v-axis, system (1.1), as $\varepsilon \to 0$, has the limiting system

$$\begin{cases} \dfrac{du}{dt} = u^{k+1} g(u, v, 0), \\ \dfrac{dv}{dt} = u^{k+1} h(u, v, 0) \end{cases} \tag{2.1}$$

which has equilibira consisting of the points on the v-axis. As in [11], we assume the following conditions:

$$(\mathcal{H}_1) \quad \begin{cases} \exists\, 0 < v_1^* < v_0^* < \infty \text{ such that: } (i)\ g(0, v_0^*, 0) < 0, \quad g(0, v_1^*, 0) > 0; \text{ and} \\ (ii)\ (2.1) \text{ has a heteroclini orbit } \gamma(v_0^*) \text{ lying strictly in the first quadrant} \\ \qquad \text{of } (u, v)\text{-plane connecting } (0, v_1^*) \text{ at } t = -\infty \text{ and } (0, v_0^*) \text{ at } t = \infty. \end{cases}$$

Since $|g(u, v, 0)|$ is strictly positive in small neighborhoods of $(0, v_0^*)$ and $(0, v_1^*)$ respectively, the system (2.1) can be written as the smooth scalar equation

$$\frac{dv}{du} = \frac{h(u, v, 0)}{g(u, v, 0)}$$

in these neighborhoods. This together with $(\mathcal{H}_1)$ and the continuous dependence of solutions of (2.1) on initial data imply that

$$(\mathcal{H}_1')\quad \begin{cases} \text{there exists a small } \delta^*>0 \text{ and a smooth decreasing function} \\ a:(v_0^*-\delta^*, v_0^*+\delta^*)\to(a(v_0^*+\delta^*), a(v_0^*-\delta^*)) \text{ such that } a(v_0^*)=v_1^*\ , \\ \text{and, for each } v_0\in(v_0^*-\delta^*, v_0^*+\delta^*), \text{ the system (2.1) has a heteroclinic orbit} \\ \gamma(v_0) \text{ connecting } (0,a(v_0)) \text{ and } (0,v_0) \text{ at } t=-\infty \text{ and } t=\infty, \text{ respectively.} \end{cases}$$

We refer the reader to Figure 2 (a) for an illustration of three of these heteroclinic orbits.

On one hand, we will show in Section 3.3 that, for $v_0\in(v_0^*-\delta^*, v_0^*+\delta^*)$,

$$a'(v_0)=\frac{g(0,v_0,0)}{g(0,a(v_0),0)}\exp\left(-\int_{\gamma(v_0)}\frac{g_u(u,v,0)}{g(u,v,0)}\,du+\frac{h_v(u,v,0)}{h(u,v,0)}\,dv\right). \tag{2.2}$$

On the other hand, in the region $0<u^{k+1}\ll\varepsilon$ and $v>0$, system (1.1) can be written as

$$\begin{cases} \dfrac{du}{dt}=u^{k+1}g(u,v,\varepsilon), \\ \dfrac{dv}{dt}=\varepsilon[f(u,v,\varepsilon)+o(1)], \end{cases} \tag{2.3}$$

which is a slow-fast system with u being the slow variable and u' changing the sign in this region (at least once) because of $(\mathcal{H}_1)$ (i). We assume as in [11] that

$$(\mathcal{H}_2)\quad \begin{cases} (i) \quad f(0,v,0)<0, \qquad \forall\, v\in[v_1^*,v_0^*]; \\ (ii) \quad \int_{v_1^*}^{v_0^*}\frac{g(0,s,0)}{f(0,s,0)}\,ds=0, \qquad \int_{v_1}^{v_0^*}\frac{g(0,s,0)}{f(0,s,0)}\,ds<0, \qquad \forall\, v_1\in(v_1^*,v_0^*). \end{cases}$$

The assumptions $(\mathcal{H}_1)$ (i) and $(\mathcal{H}_2)$ together with the implicit function theorem yield that there is a smooth function $b(v_0)$, $v_0\in(v_0^*-\delta^*, v_0^*+\delta^*)$, such that $b(v_0^*)=v_1^*$ and

$$\int_{b(v_0)}^{v_0}\frac{g(0,s,0)}{f(0,s,0)}\,ds=0, \qquad \int_{v_1}^{v_0}\frac{g(0,s,0)}{f(0,s,0)}\,ds<0, \qquad \forall\, v_1\in(b(v_0),v_0). \tag{2.4}$$

We note that if $g(0,v,0)$ changes the sign only once in (v_1^*,v_0^*), then the inequalities in $(\mathcal{H}_2)$ (ii) and (2.4) are automatically satisfied. We will show in Section 3 that given any $(u_0,\tilde{v}_0)$ near $(0,v_0^*)$, for sufficiently small $\varepsilon>0$ the orbit of (1.1) starting from this point moves downward slowly near the v-axis until reaching a point near $(0,b(v_0))$ where it exits this region (see Figure 2 (b)). The function $b(v_0)$ is referred to as the *entry-exit function* in the literature. Differentiating the equation in (2.4) with respect to v_0 gives

$$b'(v_0)=\frac{g(0,v_0,0)f(0,b(v_0),0)}{f(0,v_0,0)g(0,b(v_0),0)}<0. \tag{2.5}$$

For each $v_0\in(v_0^*-\delta^*, v_0^*+\delta^*)$, we denote the segment on the v-axis between v_0 and $b(v_0)$ by $\sigma(v_0)$ and let $\Gamma_0(v_0):=\sigma(v_0)\cup\gamma(v_0)$. It is clear that $\Gamma_0(v_0)$ is a closed curve if and only if $a(v_0)=b(v_0)$. Since $a(v_0^*)=b(v_0^*)=v_1^*$, $\Gamma_0:=\Gamma_0(v_0^*)$ is closed, which we refer to as a *singular closed orbit* of (1.1).

Consider the *"discriminant" function*

$$\Delta(v_0):=a(v_0)-b(v_0), \qquad v_0\in(v_0^*-\delta^*, v_0^*+\delta^*).$$

It follows that $\Delta(v_0^*)=0$. If Δ changes the sign exactly once in $(v_0^*-\delta^*, v_0^*+\delta^*)$, i.e., either

$$(v_0-v_0^*)\Delta(v_0)<0, \qquad \forall\, v_0\in(v_0^*-\delta^*, v_0^*+\delta^*)\setminus\{v_0^*\}, \tag{2.6}$$

or

$$(v_0-v_0^*)\Delta(v_0)>0, \qquad \forall\, v_0\in(v_0^*-\delta^*, v_0^*+\delta^*)\setminus\{v_0^*\}, \tag{2.7}$$

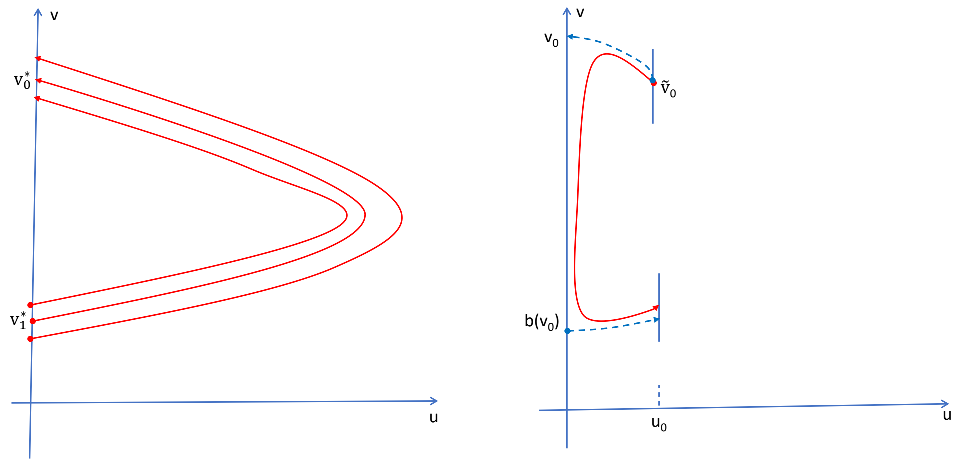

FIGURE 2. *In the left figure, three read curves are heteroclinic orbits of* (2.1)*. In the right figure, two dotted blue curves are orbits of* (2.1) *starting from* $(u_0, \tilde{v}_0)$ *and* $(0, b(v_0))$ *respectively, while the red curve is the orbit of* (1.1) *starting from the same point* $(u_0, \tilde{v}_0)$ *which as* $\varepsilon \to 0$ *approaches the union of the two blue orbits and the segment on the* v*-axis from* v_0 *to* $b(v_0)$*.*

then Γ_0 is an isolated singular closed orbit. Clearly, it follows from the mean-value theorem that (2.7) holds if $\Delta'(v_0^*) > 0$ and (2.6) holds if $\Delta'(v_0^*) < 0$. We will show in Section 3.3 that

$$\Delta'(v_0^*) = \frac{g(0, v_0^*, 0) f(0, v_1^*, 0)}{g(0, v_1^*, 0) f(0, v_0^*, 0)} \Big(e^{-\lambda(v_0^*)} - 1 \Big), \tag{2.8}$$

where

$$\lambda(v_0^*) = \ln \frac{f(0, v_1^*, 0)}{f(0, v_0^*, 0)} + \int_{\gamma(v_0^*)} \frac{g_u(u, v, 0)}{g(u, v, 0)}\, du + \frac{h_v(u, v, 0)}{h(u, v, 0)}\, dv. \tag{2.9}$$

Note that, as the integral in the above formula is a line integral along the heteroclinic orbit $\gamma(v_0^*)$, if $(u^*(t), v^*(t))$ is a solution of (2.1) whose orbit is $\gamma(v_0^*)$, then

$$\lambda(v_0^*) = \ln \frac{f(0, v_1^*), 0)}{f(0, v_0^*, 0)} + \int_{-\infty}^{\infty} (u^*(t))^{k+1} \Big[g_u(u^*(t), v^*(t), 0) + h_v(u^*(t), v^*(t), 0) \Big]\, dt. \tag{2.10}$$

From (2.8) and the signs of f and g, we have $\Delta'(v_0^*) < 0$ if and only if $\lambda(v_0^*) < 0$.

Let $\delta_0 > 0$ be sufficiently small and also fix $u_0 = u_0(\delta) > 0$ sufficiently small which satisfies the requirements to be specified at the beginning of Section 3. For a given $(u_0, \tilde{v}_0^*) \in \gamma(v_0^*)$ near $(0, v_0^*)$, we consider the short vertical segment

$$L_{\delta_0} := \{(u_0, \tilde{v}_0) : \tilde{v}_0 \in [\tilde{v}_0^* - \delta_0, \tilde{v}_0^* + \delta_0]\} \tag{2.11}$$

centered at the point $(u_0, \tilde{v}_0^*)$. Since $g(u, v, \varepsilon) < 0$ for (u, v) in a neighborhood of $(0, v_0^*)$ and $\varepsilon \geq 0$ is sufficiently small, it follows that L_{δ_0} is transversal to the flows of (1.1). Denote $\gamma_1(\tilde{v}_0)$ as the piece of orbit of (2.1) from $(u_0, \tilde{v}_0)$ to $(0, v_0)$, where $v_0 := v_0(\tilde{v}_0)$ is a smooth bijection with $v_0(\tilde{v}_0^*) = v_0^*$, and $\gamma_2(v_1)$ as the piece of orbit of (2.1) from $(0, v_1)$ with $v_1 := b(v_0)$ to a point $(u_0, \tilde{\tilde{v}}_0) \in L_{\delta_0}$. We note that $\gamma_1(\tilde{v}_0)$, respectively $\gamma_2(v_1)$, is a portion of the heteroclinic orbit of (2.1) connecting the point $(0, v_0)$ at $t = \infty$, respectively connecting the point $(0, v_1)$ at $t = -\infty$. It is clear that if $b(v_0) \neq a(v_0)$, then $\gamma_1(\tilde{v}_0)$ and $\gamma_2(v_1)$ do not lie in a same heteroclinic orbit of (2.1). For $v_0 := v_0(\tilde{v}_0)$, $v_1 := b(v_0)$, and $\delta \in (0, \delta_0)$ sufficiently small such that $L_\delta := \{u_0\} \times (\tilde{v}_0^* + \delta, \tilde{v}_0^* + \delta) \subset L_{\delta_0}$, consider the curve

$$\tilde{\Gamma}_0(\tilde{v}_0) = \gamma_1(\tilde{v}_0) \cup \sigma(v_0) \cup \gamma_2(v_1), \qquad \tilde{v}_0 \in (\tilde{v}_0^* + \delta, \tilde{v}_0^* + \delta).$$

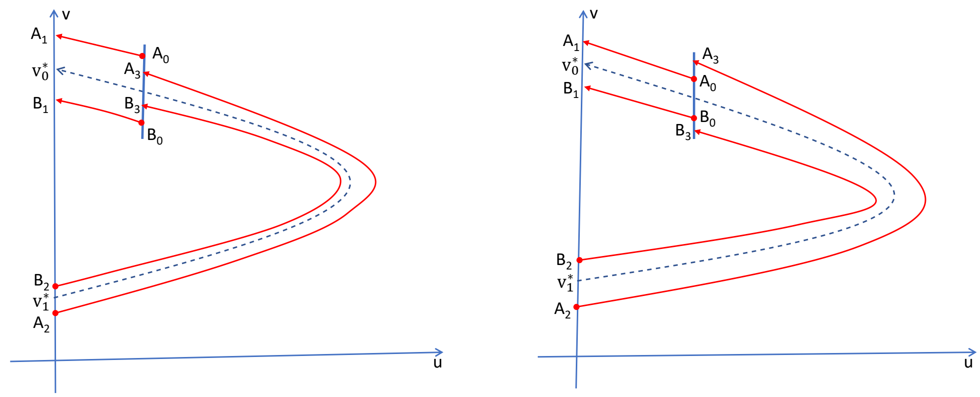

FIGURE 3. *The four red curves in above figures are four orbit curves of* (2.1), *the piecewise smooth curves* $\widetilde{\Gamma}_0(A_0) = \widehat{A_0A_1} \cup \overline{A_1A_2} \cup \widehat{A_2A_3}$ *and* $\widetilde{\Gamma}_0(B_0) = \widehat{B_0B_1} \cup \overline{B_1B_2} \cup \widehat{B_2B_3}$ *are the limits of the outer and inner boundaries of* $\mathcal{U}_\varepsilon$ *in* Figure 1, *respectively. We assume* (2.6) *in the left figure and* (2.7) *in the right one.*

Note that $\widetilde{\Gamma}_0(\tilde{v}_0) = \Gamma_0(v_0)$ only when it is closed. Therefore, under the assumption (2.7) or (2.6), $\widetilde{\Gamma}_0(\tilde{v}_0)$ is not closed for any $\tilde{v}_0 \neq \tilde{v}_0^*$, and moreover, $\tilde{v}_0$ and $\tilde{\tilde{v}}_0$ satisfy

$$(2.12) \qquad \begin{cases} (\tilde{v}_0 - \tilde{v}_0^*)(\tilde{\tilde{v}}_0 - \tilde{v}_0) > 0, & \text{if (2.7) holds,} \\ (\tilde{v}_0 - \tilde{v}_0^*)(\tilde{\tilde{v}}_0 - \tilde{v}_0) < 0, & \text{if (2.6) holds.} \end{cases}$$

Inequalities in (2.12) characterize four possibilities of the curve $\Gamma_0(\tilde{v}_0)$, as illustrated in Figure 3. For instance, in the case that (2.7) holds, (2.12) implies that if $\tilde{v}_0 < \tilde{v}_0^*$ (resp. $\tilde{v}_0 > \tilde{v}_0^*$), then $\Delta(v_0) = a(v_0) - b(v_0) > 0$ (resp. < 0), i.e., $(u_0, \tilde{\tilde{v}}_0)$ lies below (resp. above) $(u_0, \tilde{v}_0)$.

The following result shows that, as $\varepsilon \to 0$, $\widetilde{\Gamma}_0(\tilde{v}_0)$ is the limiting orbit of that of (1.1) that passes through the same starting point $(u_0, \tilde{v}_0) \in L_\delta$.

Theorem 2.1. *Assume* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, *and either* (2.7) *or* (2.6). *In the case* $k > 0$, *let* σ *be an arbitrary number such that* $\frac{2}{2k+1} < \sigma < \frac{1}{k}$. *Then given an arbitrarily small* $\delta_0 > 0$, *there exist* $u_0 \in (0, \delta_0)$, $\delta \in (0, \delta_0)$ *and constant* $M_0 > 0$ *such that for sufficiently small* $\varepsilon > 0$, *if* $(u_\varepsilon, v_\varepsilon)$ *is the solution of* (1.1) *with* $(u_\varepsilon(0), v_\varepsilon(0)) = (u_0, \tilde{v}_0) \in L_\delta$, *then there is a smallest* $T(\varepsilon) > 0$ *such that* $(u_\varepsilon(t), v_\varepsilon(t))$ *is defined for all* $t \in [0, T(\varepsilon)]$ *with* $(u_\varepsilon(T(\varepsilon)), v_\varepsilon(T(\varepsilon)) \in L_{\delta_0}$ *and satisfies, for* $t \in [0, T(\varepsilon)]$,

$$(2.13) \qquad dist((u_\varepsilon(t), v_\varepsilon(t)), \widetilde{\Gamma}_0(\tilde{v}_0)) \leq M_0\phi(\varepsilon) := M_0 \begin{cases} \varepsilon|\ln \varepsilon|, & \text{if } k = 0, \\ \varepsilon^{1-k\sigma}, & \text{if } k > 0. \end{cases}$$

2.2. **The annulus region.** We are now in the position to define the annular region $\mathcal{U}_\varepsilon$ which contains an orbit of relaxation oscillations. Let $(u_\varepsilon^\pm(t), v_\varepsilon^\pm(t))$ be the solutions of (1.1) with $(u_\varepsilon^\pm(0), v_\varepsilon^\pm(0)) = (u_0, \tilde{v}_0^* \pm \delta) \in L_\delta$ and $T_\pm > 0$ be their first return times $T(\varepsilon)$ to L_{δ_0} respectively as stated in Theorem 2.1. It follows from (2.12) and (2.13) that

$$v_\varepsilon^-(T_-) < v_\varepsilon^-(0) \quad \text{and} \quad v_\varepsilon^+(T_+) > v_\varepsilon^+(0), \qquad \text{if (2.7) holds,}$$

and

$$v_\varepsilon^-(T_-) > v_\varepsilon^-(0) \quad \text{and} \quad v_\varepsilon^+(T_+) < v_\varepsilon^+(0), \qquad \text{if (2.6) holds.}$$

As illustrated in Figure 1, we let $\mathcal{U}_\varepsilon$ be the annular region such that its outer boundary consists of the orbit of $(u_\varepsilon^+(t), v_\varepsilon^+(t))$ with $0 \leq t \leq T_+$ and the segment lying in L_{δ_0} between the points $(u_0, v_\varepsilon^+(0))$ and $(u_0, v_\varepsilon^+(T_+))$ and its inner boundary consists of the orbit of

$(u_\varepsilon^-(t), v_\varepsilon^-(t))$ with $0 \le t \le T_-$ and the segment lying in L_{δ_0} between the points $(u_0, v_\varepsilon^-(0))$ and $(u_0, v_\varepsilon^-(T_-))$. We have from (2.12) and (2.13) that $\mathcal{U}_\varepsilon$ is positively (resp. negatively) invariant under the assumption (2.6) (resp. (2.7)), and moreover, the smallness of δ ensures that $dist(\mathcal{U}_\varepsilon, \Gamma_0(v_0^*)) < \delta_0$.

We note that for sufficiently small $\varepsilon > 0$, only near the points $(0, \bar{v}_0)$ where $g(0, \bar{v}_0, 0) = 0$ and $h(0, \bar{v}_0, 0) \ge 0$, the system (1.1) may admit equilibria $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$, which must satisfy that

$$\hat{v}_\varepsilon = \bar{v}_0 + o(1), \quad \begin{cases} \hat{u}_\varepsilon = \left(\frac{-f(0,\bar{v}_0,0)}{h(0,\bar{v}_0,0)}\varepsilon\right)^{1/(k+1)}\Big[1 + o(1)\Big], & \text{if } h(0, \bar{v}_0, 0) < 0, \\ \hat{u}_\varepsilon \gg \varepsilon^{1/(k+1)}, & \text{if } h(0, \bar{v}_0, 0) = 0. \end{cases} \tag{2.14}$$

For example, if $g(0, v, 0)$ changes the sign for $v - \bar{v}_0$ small, then the existence of at least one $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ is guaranteed. In fact, we will show later that all these equilibria lie outside of $\mathcal{U}_\varepsilon$.

2.3. **Main result.** With above preparations and the Poincare-Bendixson theorem, we have the following general criterion on the existence of relaxation oscillations of (1.1).

Theorem 2.2. *Assume* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, *and either* (2.7) *or* (2.6). *Then the followings hold.*

(i) *For an arbitrarily small* $\delta_0 > 0$, *if* $\varepsilon > 0$ *is sufficiently small, then system* (1.1) *admits a relaxed closed orbit* Γ_ε *of* (2.6) *in* $\mathcal{U}_\varepsilon$ *satisfying* $dist(\Gamma_\varepsilon, \Gamma_0) < \delta_0$.

(ii) *If, in addition,* $\Delta'(v_0^*) \neq 0$, *then* Γ_ε *is the unique relaxation cycle in the* δ_0*-neighborhood of* Γ_0, *which is asymptotically orbitally stable if* $\Delta'(v_0^*) < 0$ *and orbitally unstable if* $\Delta'(v_0^*) > 0$.

Remark 2.1. (i) In the case $k = 0$ and $\Delta'(v_0^*) \neq 0$, Theorem 2.2 was proved in [10, 11, 12]. It is clear that the condition (2.7) or (2.6) include some degenerate situations such as $\Delta(v_0^*) = \Delta'(v_0^*) = \Delta''(v_0^*) = 0$ and $\Delta'''(v_0^*) \neq 0$.

(ii) Using the estimate (2.13) in Theorem 2.1, it can be shown that the period of Γ_ε has the asymptotic formula $\frac{1}{\varepsilon}\int_{v_1^*}^{v_0^*} \frac{1}{|f(0,v,0)|}\, dv\, (1 + o(1))$ as $\varepsilon \to 0$.

(iii) In the definition of $\tilde{\Gamma}_0(\tilde{v}_0)$, if we take $\tilde{v}_0 := \tilde{v}_0^* + M\phi(\varepsilon)$ for a much larger constant $M > M_0$, where M_0 is given in (2.13), then from (3.3), (3.5), and the smoothness of $a(v_0)$ and $b(v_0)$ we conclude that the ending point $(u_0, \tilde{\tilde{v}}_0)$ of $\tilde{\Gamma}(\tilde{v}_0)$ satisfies $\tilde{\tilde{v}}_0 > \tilde{v}_0 + M_0\tilde{v}_0^*$. Let $(u_\varepsilon^+(t), v_\varepsilon^+(t))$, $0 \le t \le T(\varepsilon)$, be the solution of (1.1) with $(u_\varepsilon^+(0), v_\varepsilon^+(0)) = (0, \tilde{v}_0^* + M\phi(\varepsilon))$, where $T(\varepsilon)$ is defined in Theorem 2.1. We conclude from (2.13) that $\tilde{v}_0 < v_\varepsilon^+(T(\varepsilon)) < \tilde{v}_0 + O(\phi(\varepsilon))$. We now use the orbit $(u_\varepsilon^+(t), v_\varepsilon^+(t))$, $0 \le t \le T(\varepsilon)$, as the part of outer boundary of the new $\mathcal{U}_\varepsilon$. Similarly, we use the orbit $(u_\varepsilon^-(t), v_\varepsilon^-(t))$, $0 \le t \le T(\varepsilon)$, as the part of inner boundary of the new $\mathcal{U}_\varepsilon$, where $(u_\varepsilon^-(t), v_\varepsilon^-(t)))$ is the solution of (1.1) with $(u_\varepsilon^-(0), v_\varepsilon^-(0)) = (u_0, \tilde{v}_0^* - M\phi(\varepsilon))$ and $\tilde{v}_0 - O(\phi(\varepsilon)) < v_\varepsilon^-(T(\varepsilon)) < \tilde{v}_0$. Applying the Poincare-Bendixson theorem on this new $\mathcal{U}_\varepsilon$ we can improve the conclusion of Theorem 2.2 (i) as follows: For sufficiently small $\varepsilon > 0$, the system (1.1) has an relaxation cycle Γ_ε in $\mathcal{U}_\varepsilon$ that satisfies

$$dist(\Gamma_\varepsilon, \Gamma_0) = O(\phi(\varepsilon)) = \begin{cases} O(\varepsilon|\ln\varepsilon|), & \text{if } k = 0, \\ O(\varepsilon^{1-k\sigma}), & \text{if } k > 0. \end{cases}$$

3. Proofs of Theorems 2.1 and 2.2

Let $\delta_0 > 0$ be sufficiently small and $\mathcal{R} := R_{\delta_0}$ the union of two rectangular boxes in the (u, v)-plane near the points $(0, v_0^*)$ and $(0, v_1^*)$ defined by

$$\mathcal{R}_{\delta_0} = [0, \delta_0] \times ([v_0^* - 4\delta_0, v_0 + 4\delta_0] \cup [v_1^* - 4\delta_0, v_1^* + 4\delta_0]).$$

Denote

$$F(u, v, \varepsilon) = \frac{f(u, v, \varepsilon)}{g(u, v, \varepsilon)}, \qquad H(u, v, \varepsilon) = \frac{h(u, v, \varepsilon)}{g(u, v, \varepsilon)}, \qquad (u, v) \in \mathcal{R},\ \varepsilon \in [0, \delta_0].$$

It follows from the assumption $(\mathcal{H}_1)$ that F and H are smooth on $\mathcal{R}$ and there are constants m and M with $0 < m < 1 < M$ such that for all $(u, v) \in \mathcal{R}$ and $\varepsilon \in [0, \delta_0]$,

$$m \le |F(u, v, \varepsilon)| \le M, \ |H(u, v, \varepsilon)| \le M, \ |H_v(u, v, \varepsilon)| \le M, \ |H_\varepsilon(u, v, \varepsilon)| \le M. \tag{3.1}$$

Let $u_0 > 0$ be such that $Mu_0 < \delta_0$ and $e^{Mu_0} \le 2$. Also let L_{δ_0} be defined as in (2.11).

3.1. **Orbits of the limiting system (2.1).** Let $\gamma_1(\tilde{v}_0)$, respectively $\gamma_2(v_1)$, be the orbit of the limiting system (2.1) that passes through a given point $(u_0, \tilde{v}_0) \in L_{\delta_0}$, respectively a given point $(0, v_1)$, near $(0, v_1^*)$. We give some estimates to these orbits below.

Let $(u(t, \tilde{v}_0), v(t, \tilde{v}_0))$, $0 \le t < \infty$, be the solution of (2.1) with $(u(0, \tilde{v}_0), v(0, \tilde{v}_0)) = (u_0, \tilde{v}_0)$ whose orbit is $\gamma_1(\tilde{v}_0)$. Since $u'(t, \tilde{v}_0) < 0$ and $(u(t, \tilde{v}_0), v(t, \tilde{v}_0)) \to (0, v_0)$ as $t \to \infty$ for some $v_0 := v_0(\tilde{v}_0)$ close to v_0^*, we can parameterize $\gamma_1(\tilde{v}_0)$ in term of the variable u as the graph of $V_0(u, \tilde{v}_0) := v(t, \tilde{v}_0)$, $t \in [0, \infty)$. Then $V_0(u, \tilde{v}_0)$ satisfies

$$\frac{dV_0}{du} = H(u, V_0, 0), \qquad \forall\, u \in [0, u_0], \qquad V_0(u_0, \tilde{v}_0) = \tilde{v}_0, \quad V_0(0, \tilde{v}_0) = v_0. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$|V_0(u, \tilde{v}_0) - v_0| \le Mu, \quad |V_0(u, \tilde{v}_0) - V_0(u, \tilde{v}_0^*)| \le e^{Mu_0}|\tilde{v}_0 - \tilde{v}_0^*|, \quad \forall\, u \in [0, u_0], \tag{3.3}$$

where the last inequality follows from the estimate

$$\begin{aligned} |\frac{d}{du}[V_0(u, \tilde{v}_0) - V_0(u, \tilde{v}_0^*)]| &= |H(u, V_0(u, \tilde{v}_0), 0) - H(u, V_0(u, \tilde{v}_0^*), 0)| \\ &\le M|V_0(u, \tilde{v}_0) - V_0(u, \tilde{v}_0^*)| \end{aligned}$$

and the Gronwall's inequality. The estimates in (3.3) and (3.1) then yield

$$\begin{aligned} &|v_0 - v_0^*| \le e^{Mu_0}|\tilde{v}_0 - \tilde{v}_0^*| \le 2|\tilde{v}_0 - \tilde{v}_0^*| \le 2\delta_0, \\ &|V_0(u, \tilde{v}_0) - v_0^*| \le |V_0(u, \tilde{v}_0) - V_0(u, \tilde{v}_0^*)| + |V_0(u, \tilde{v}_0^*) - v_0^*| \le 2|\tilde{v}_0 - \tilde{v}_0^*| + \delta_0 \le 3\delta_0, \end{aligned}$$

implying that the orbits $\gamma_1(\tilde{v}_0)$ and $\gamma_1(\tilde{v}_0^*)$ lie entirely in $\mathcal{R}$.

Similarly, let $|v_1 - v_1^*| > 0$ be small. Let $(u(t, v_1), v(t, v_1))$ be the solution of (2.1) connecting $(0, v_1)$ at $t = -\infty$ and a point $(u_0, \tilde{\tilde{v}}_0) \in L_{\delta_0}$ at some time $T_1 < \infty$, whose orbit for $t \in (-\infty, T_1)$ is $\gamma_2(v_1)$. Since $u'(t, v_1) > 0$ on the part of $\gamma_2(v_1)$ with $u(t, v_1) \le u_0$, we can parameterize this part of $\gamma_2(v_1)$ as the graph of $V_0(u, v_1) := v(t, v_1)$ which satisfies

$$\frac{dV_0}{du} = H(u, V_0, 0), \qquad \forall\, u \in [0, u_0], \qquad V_0(0, v_1) = v_1. \tag{3.4}$$

This yields the estimates

$$|V_0(u, v_1) - v_1| \le Mu, \quad |V_0(u, v_1) - V_0(u, v_1^*)| \le e^{Mu_0}|v_1 - v_1^*|, \quad \forall\, u \in [0, u_0]. \tag{3.5}$$

It follows from the estimates in (3.3) and (3.5), the continuity of b, and the continuous dependence of solutions on initial data that if $\delta > 0$ is sufficiently small, then

$$dist(\tilde{\Gamma}_0(\tilde{v}_0)), \Gamma_0(\tilde{v}_0^*)) < \delta_0/2, \qquad \forall \tilde{v}_0 \in [\tilde{v}_0^* - \delta, \tilde{v}_0^* + \delta].$$

Using the relationship between $a(v_0)$ and $b(v_0)$ and the relative positions of the corresponding heteroclinic orbits of (2.1) starting from $(0, a(v_0))$ and $(0, b(v_0))$, we easily conclude the properties of $\tilde{v}_0$ and $\tilde{\tilde{v}}_0$ stated in (2.12), as illustrated in Figure 3.

3.2. **Proofs of Theorem 2.1 and Theorem 2.2 (i).** We first prove Theorem 2.1 with four lemmas below by assuming $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and either (2.7) or (2.6). Recall the function

$$\phi(\varepsilon) = \begin{cases} \varepsilon|\ln\varepsilon|, & \text{if } k = 0, \\ \varepsilon^{1-k\sigma}, & \text{if } k > 0, \end{cases}$$

from (2.13) and define

$$\alpha = \begin{cases} 2, & \text{if } k = 0, \\ \sigma, & \text{if } k > 0, \end{cases}$$

where for each $k > 0$, σ is an arbitrarily chosen number in $(\frac{2}{2k+1}, \frac{1}{k})$.

For given $u_0 = u_0(\delta) > 0$ sufficiently small and $\tilde{v}_0 \in (\tilde{v}_0^* + \delta, \tilde{v}_0^* + \delta)$ with $(u_0, \tilde{v}_0) \in L_\delta$, let $(u_\varepsilon, v_\varepsilon)$ be the solution of (1.1) satisfying $(u_\varepsilon(0), v_\varepsilon(0)) = (u_0, \tilde{v}_0)$. We prove in these lemmas that there are $0 < t_0 < t_1 < t_2 < T(\varepsilon)$ such that $u_\varepsilon(t_0) = \varepsilon^\alpha$ and $(u_\varepsilon(t), v_\varepsilon(t))$ is close to $\gamma_1(\tilde{v}_0)$ for $t \in [0, t_0]$, $u_\varepsilon(t_1) = \varepsilon^\alpha$ and $(u_\varepsilon(t), v_\varepsilon(t))$ is close to $\sigma(v_0)$ for $t \in [t_0, t_1]$, $u_\varepsilon(t_2) = u_0$, $u_\varepsilon(T(\varepsilon)) = u_0$ and $(u_\varepsilon(t), v_\varepsilon(t))$ is close to $\gamma_2(v_1)$, with $v_1 = b(v_0(\tilde{v}_0))$, for all $t \in [t_1, T(\varepsilon)]$.

Denote $[0, t_*)$ as the maximal interval such that $u_\varepsilon(t) \le u_0$ and $v_1^* - 4\delta_0 < v_\varepsilon(t) < v_0^* + 4\delta_0$ for all $t \in [0, t_*)$. Then as long as $(u_\varepsilon(t), v_\varepsilon(t)) \in \mathcal{R}$ for $t \in [0, t_*)$, $u_\varepsilon'(t) < 0$, so the function $u = u_\varepsilon(t)$ admits an inverse $t = t_\varepsilon(u)$ in $[0, t_*)$. Let $V_\varepsilon(u, \tilde{v}_0) := v_\varepsilon(t_\varepsilon(u))$ whenever $v_0^* - 4\delta_0 < v_\varepsilon(t) < v_0^* + 4\delta_0$ for $t \in [0, t_*)$. We note that V_ε satisfies $V_\varepsilon(u_0, \tilde{v}_0) = \tilde{v}_0$ and

$$\frac{dV_\varepsilon}{du} = \frac{\varepsilon f(u, V_\varepsilon, \varepsilon)}{u^{k+1} g(u, V_\varepsilon, \varepsilon)} + \frac{h(u, V_\varepsilon, \varepsilon)}{g(u, V_\varepsilon, \varepsilon)} = \frac{\varepsilon}{u^{k+1}} F(u, V_\varepsilon, \varepsilon) + H(u, V_\varepsilon, \varepsilon). \tag{3.6}$$

Lemma 3.1. *There is $t_0 \in (0, t_*)$ such that*

$$\begin{aligned} & u_\varepsilon(t_0) = \varepsilon^\alpha, \\ & \varepsilon^\alpha < u_\varepsilon(t) < u_0, \quad \forall t \in (0, t_0), \\ & v_0 < v_\varepsilon(t) < v_0^* + \delta_0, \quad \forall\, t \in (0, t_0], \\ & |V_\varepsilon(u, \tilde{v}_0) - V_0(u, \tilde{v}_0)| \le M_1 \phi(\varepsilon), \qquad \forall\, u \in [\varepsilon^\alpha, u_0], \end{aligned} \tag{3.7}$$

where $M_1 > 0$ is a generic constant independent of the solution $(u_\varepsilon, v_\varepsilon)$.

Proof. Let $t_0 = \inf\{t > 0 : u_\varepsilon > \varepsilon^\alpha, v_0^* + \delta_0 < v_\varepsilon < v_0^* + \delta_0 \text{ on } (0, t)\}$. It follows that $(u_\varepsilon(t), v_\varepsilon(t)) \in \mathcal{R}$ for all $t \in (0, t_0)$. Hence $-m_\varepsilon < u_\varepsilon'(t) < 0$ for all $t \in (0, t_0)$, where $m_\varepsilon = \min\{u^{k+1} g(u, v, \varepsilon) : \varepsilon^\alpha \le u \le u_0,\ v_0^* + \delta_0 \le v \le v_0^* + \delta_0\}$. This implies that t_0 is finite and $t_0 < t_*$.

For simplicity, denote $V(u) := V_\varepsilon(u, \tilde{v}_0)$, $V_0(u) := V_0(u, \tilde{v}_0)$, and $Z(u) = V(u) - V_0(u)$, $u \in [u_\varepsilon(t_0), u_0]$. It follows from (3.2) and (3.6) that

$$\frac{dZ}{du} = \frac{\varepsilon}{u^{k+1}} F(u, V(u), \varepsilon) + H(u, V(u), \varepsilon) - H(u, V_0(u), 0), \qquad u \in [u_\varepsilon(t_0), u_0].$$

Note by the mean value theorem that

$$H(u, V(u), \varepsilon) - H(u, V_0(u), 0) = H_1(u, \varepsilon) Z + H_2(u, \varepsilon)\varepsilon,$$

where, for $i = 1, 2$, $w_1 = v$ and $w_2 = \varepsilon$,

$$H_i(u, \varepsilon) = \int_0^1 \frac{\partial H}{\partial w_i}(u, \theta V(u) + (1 - \theta) V_0(u), \theta\varepsilon)\, d\theta.$$

Hence

$$\frac{dZ}{du} = \frac{\varepsilon}{u^{k+1}} F(u, V(u), \varepsilon) + H_1(u, \varepsilon) Z + H_2(u, \varepsilon)\varepsilon.$$

Since both $(u, V(u))$ and $(u, V_0(u))$ lie in $\mathcal{R}$ for all $u \in [u_\varepsilon(t_0), u_0]$, $|F|$ and $|H_2|$ are bounded from the above, in $\mathcal{R} \times [0, \delta_0]$ by a constant independent of the solution. Hence, there ia a constant M_1 independent of the solution such that

$$|\frac{dZ}{du} - H_1(u,\varepsilon)Z| \le M_1\varepsilon\left(\frac{1}{u^{k+1}}+1\right).$$

An application of the variation of parameters formula to the above, together with the fact that $Z(u_0)=0$, yields that

$$|Z(u)| \le M_1\varepsilon\int_u^{u_0}\left(\frac{1}{\eta^{k+1}}+1\right)e^{-\int_u^\eta H_1(\xi,\varepsilon)\,d\xi}\,d\eta, \qquad u\in[u_\varepsilon(t_0),u_0]. \tag{3.8}$$

Consequently, when $k>0$, using the fact that $u>\varepsilon^\sigma$ in (3.8) gives

$$|Z(u)| \le M_1\varepsilon\left(\frac{1}{u^k}-\frac{1}{u_0^k}+u_0-u\right)\le M_1\varepsilon^{1-k\sigma},$$

and when $k=0$, using the fact that $u>\varepsilon^2$ in (3.8) gives

$$|Z(u)| \le M_1\varepsilon\left(\ln u_0-\ln u+u_0-u\right)\le M_1\varepsilon|\ln\varepsilon|.$$

These estimates together with the fact that $v_0^*-\delta_0<V_0(u)<v_0^*+\delta_0$ for all $u\in[0,u_0]$ and the definition of t_0 imply that $u_\varepsilon(t_0)=\varepsilon^\alpha$ and (3.7) holds. This proves the lemma. □

Lemma 3.2. *There is $t_1\in(t_0,t_*)$ such that*

$$\begin{aligned}&u_\varepsilon(t)<\varepsilon^\alpha,\quad \forall t\in(t_0,t_1),\quad u_\varepsilon(t_1)=\varepsilon^\alpha,\\ &v_\varepsilon'(t)<0,\quad \forall t\in(t_0,t_1],\\ &|v_\varepsilon(t_1)-V_0(\varepsilon^\alpha,b(v_0))|\le M_2\phi(\varepsilon),\end{aligned}\tag{3.9}$$

where $V_0(u,b(v_0))$ is the solution of (3.4) *with $v_1=b(v_0)$ and $M_2>0$ is a generic constant independent of the solution $(u_\varepsilon,v_\varepsilon)$.*

Proof. Let $t_1:=\sup\{t>t_0: u_\varepsilon<\varepsilon^\alpha,\ v_1^*-\delta_0<v_\varepsilon \text{ on } (t_0,t)\}$, $M_f:=\max\{f(u,v,\varepsilon):(u,v,\varepsilon)\in\mathcal{R}\times[0,\delta_0]\}$, and $M_h:=\max\{|h(u,v,\varepsilon)|:(u,v,\varepsilon)\in\mathcal{R}\times[0,\delta_0]\}$. Since $M_f<0$ and $\alpha(k+1)-1>0$, we have on $[t_0,t_1)$ that

$$v_\varepsilon'<M_f\varepsilon+M_h\varepsilon^{\alpha(k+1)}=\varepsilon(M_f+M_h\varepsilon^{\alpha(k+1)-1})<\frac{1}{2}M_f\varepsilon<0.$$

It follows from the definition of t_1 that $t_1<\infty$ and $v_\varepsilon'<0$ on $[t_0,t_1]$. Thus, we can parameterize the orbit of $(u_\varepsilon,v_\varepsilon)$ for $t\in[t_0,t_1]$ as the graph of $U_\varepsilon(v):=u_\varepsilon(t_\varepsilon(v))$ for $v\in[v_\varepsilon(t_1),v_\varepsilon(t_0)]$, where $t=t_\varepsilon(v)$ is the inverse function of $v=v_\varepsilon(t)$ for $t\in[t_0,t_1]$. We note that $U:=U_\varepsilon$ satisfies

$$\frac{dU}{dv}=\frac{U^{k+1}g(U,v,\varepsilon)}{\varepsilon f(U,v,\varepsilon)+U^{k+1}h(U,v,\varepsilon)},\qquad U(v_\varepsilon(t_0))=\varepsilon^\alpha.\tag{3.10}$$

Let $v\in[v_\varepsilon(t_1),v_\varepsilon(t_0)]$. We first consider the case $k>0$. Then $\alpha=\sigma$. Since $0<(k+1)\sigma-1<\sigma$, $(U(v),v)\in\mathcal{R}$, and $U(v)\le\varepsilon^\sigma$, we have

$$\begin{aligned}g(U(v),v,\varepsilon)&=g(0,v,0)+O(U(v)+\varepsilon)=g(0,v,0)+O(\varepsilon^\sigma),\\ \varepsilon f(U(v),v,\varepsilon)&+u^{k+1}h(U(v),v,\varepsilon)=\varepsilon f(0,v,0)[1+O(U(v)+\varepsilon)]+O(u^{k+1})\\ &=\varepsilon f(0,v,0)[1+O(\varepsilon^\sigma)]+O(\varepsilon^{(k+1)\sigma})=\varepsilon f(0,v,0)[1+O(\varepsilon^{(k+1)\sigma-1})].\end{aligned}$$

Hence, letting $G(v):=\frac{g(0,v,0)}{f(0,v,0)}$, the differential equation in (3.10) can be written as

$$\frac{1}{U^{k+1}}\frac{dU}{dv}=\frac{g(0,v,0)+O(\varepsilon^\sigma)}{\varepsilon f(0,v,0)[1+O(\varepsilon^{(k+1)\sigma-1})]}=\frac{1}{\varepsilon}\Big[G(v)+O(\varepsilon^{(k+1)\sigma-1})\Big],$$

whose integrating over $[v, v_\varepsilon(t_0)]$, by noting $\int_{b(v_\varepsilon(t_0))}^{v_\varepsilon(t_0)} G(s)\,ds = 0$, yields that

$$U^k(v) = \frac{\varepsilon^{k\sigma}}{1 + k\left(\varepsilon^{k\sigma-1}\int_{b(v_\varepsilon(t_0))}^{v} -G(s)\,ds + O(\varepsilon^{(2k+1)\sigma-2})\right)}, \quad \text{if } k > 0 \tag{3.11}$$

for all $v \in [v_\varepsilon(t_1), v_\varepsilon(t_0))$. For the case $k = 0$ using $\alpha = 2$ and the above argument we obtain for all $v \in [v_\varepsilon(t_1), v_\varepsilon(t_0))$

$$U(v) = \frac{\varepsilon^2}{e^{-\frac{1}{\varepsilon}\int_{b(v_\varepsilon(t_0))}^{v} G(s)\,ds + O(1)}}, \quad \text{if } k = 0. \tag{3.12}$$

When $k > 0$, since $(2k+1)\sigma > 2$, by making δ_0 further small if necessary, we can assume without loss of generality that the $O(\varepsilon^{(2k+1)\sigma-2})$ term in the above is bounded, in absolute value, from the above by $1/4k$. When $k = 0$, we let $K_0 > 0$ be the upper bound of the absolute value of the $O(1)$ term above. We let $m > 0$ such that $|G(v)| > m$ for $v \in [v_1^* - \delta_0, v_1^* - \delta_0]$. Denote

$$v_\pm := b(v_\varepsilon(t_0)) \pm \begin{cases} \frac{1}{2km}\varepsilon^{1-k\sigma}, & \text{if } k > 0, \\ \frac{2K_0}{m}\varepsilon, & \text{if } k = 0. \end{cases}$$

We claim that $v_- < v_\varepsilon(t_1) < v_+$. If $v_\varepsilon(t_1) \geq v_+$, then

$$\int_{b(v_\varepsilon(t_0))}^{v_\varepsilon(t_1)} -G(s)\,ds \geq \int_{b(v_\varepsilon(t_0))}^{v_+} -G(s)\,ds \geq m(v_+ - b(v_\varepsilon(t_0))) = \begin{cases} \frac{\varepsilon^{1-k\sigma}}{2k}, & k > 0, \\ 2K_0\varepsilon, & k = 0, \end{cases}$$

which, together with (3.11) and and (3.12), implies that $U^k(v_\varepsilon(t_1)) \leq \frac{4}{5}\varepsilon^{k\sigma} < \varepsilon^{k\sigma}$ when $k > 0$ and $U(v_\varepsilon(t_1)) \leq \varepsilon^2 e^{-K_0} < \varepsilon^2$ when $k = 0$. If $v_\varepsilon(t_1) \leq v_-$, then

$$\int_{b(v_\varepsilon(t_0))}^{v_\varepsilon(t_1)} -G(s)\,ds \leq \int_{v_-}^{b(v_\varepsilon(t_0))} G(s)\,ds \leq -m(b(v_\varepsilon(t_0)) - v_-) = \begin{cases} -\frac{\varepsilon^{1-k\sigma}}{2k}, & k > 0, \\ -2K_0\varepsilon, & k = 0, \end{cases}$$

which, together with (3.11) and (3.12), implies that $U^k(v_\varepsilon(t_1)) \geq \frac{4}{3}\varepsilon^{k\sigma} > \varepsilon^{k\sigma}$ when $k > 0$ and $U(v_\varepsilon(t_1)) \geq e^{K_0}\varepsilon^2 > \varepsilon^2$ when $k = 0$. In both cases, we obtain a contradiction to the fact that $U(v_\varepsilon(t_1)) = u_\varepsilon(t_1) = \varepsilon^\alpha$, according to the definition of t_1.

It follows from this claim and the definition of t_1 that $U(v_\varepsilon(t_1)) = \varepsilon^\alpha$ and $v_\varepsilon(t_1) \in (v_-, v_+)$, i.e., $u_\varepsilon(t_1) = \varepsilon^\alpha$ and

$$|v_\varepsilon(t_1) - b(v_\varepsilon(t_0))| \leq \begin{cases} \frac{1}{2km}\varepsilon^{1-k\sigma}, & \text{if } k > 0, \\ \frac{2K_0}{m}\varepsilon, & \text{if } k = 0. \end{cases} \tag{3.13}$$

It remains to show (3.9). Below, constants involved in the big "O" asymptotic notations are all independent of the solution. Since, for $v_0 \in [v_0^* - \delta, v_0^* + \delta]$, $b(v_0)$ is close to v_1^*, $g(0, b(v_0), 0)$ is bounded away from zero, which, according to the formula (2.5), implies that $|b'(v_0)|$ is bounded from the above by a constant independent of the solution. Thus, $b(v_\varepsilon(t_0)) - b(v_0) = O(|v_\varepsilon(t_0) - v_0|)$. Applying Lemma 3.1 and (3.3) and $\varepsilon^\sigma < \varepsilon^{1-k\sigma}$ when $k > 0$, we have

$$\begin{aligned} |v_\varepsilon(t_0) - v_0| &= |v_\varepsilon(t_0) - V_0(\varepsilon^\alpha, \tilde{v}_0)| + |V_0(\varepsilon^\alpha, \tilde{v}_0) - v_0| \\ &= |V_\varepsilon(\varepsilon^\alpha, \tilde{v}_0) - V_0(\varepsilon^\alpha, \tilde{v}_0)| + |V_0(\varepsilon^\alpha, \tilde{v}_0) - v_0| \\ &= \begin{cases} O(\varepsilon^{1-k\sigma}), & \text{if } k > 0, \\ O(\varepsilon|\ln\varepsilon|), & \text{if } k = 0. \end{cases} \end{aligned}$$

Hence

$$|b(v_\varepsilon(t_0)) - V_0(\varepsilon^\alpha, b(v_0))| = \begin{cases} O(\varepsilon^{1-k\sigma}), \text{ if } k > 0, \\ O(\varepsilon|\ln\varepsilon|), \text{ if } k = 0. \end{cases}$$

This, together with (3.13) leads to (3.9).

□

Lemma 3.3. *There is $t_2 \in (t_1, t_*)$ such that*

$$\begin{aligned} &\varepsilon^\alpha < u_\varepsilon(t) < u_0, \quad \forall t \in (t_1, t_2), \quad u_\varepsilon(t_2) = u_0, \\ &v_1^* - \delta_0 < v_\varepsilon(t) < v_1^* + \delta_0, \quad \forall t \in [t_1, t_2], \\ &|V_\varepsilon(u, v_\varepsilon(t_1)) - V_0(u, b(v_0))| \le M_3\phi(\varepsilon), \qquad \forall\, u \in [\varepsilon^\alpha, u_0], \end{aligned} \tag{3.14}$$

where $V_\varepsilon(u, v_\varepsilon(t_1))$ is the portion of $V_\varepsilon(u, \tilde{v}_0)$ for $u \in [u_\varepsilon(t_1), u_\varepsilon(t_2)] = [\varepsilon^\alpha, u_0]$, $V_0(u, b(v_0))$ is the solution of (3.4) with $v_1 = b(v_0)$, and $M_3 > 0$ is a generic constant independent of $(u_\varepsilon, v_\varepsilon)$.

Proof. Let $t_2 = \inf\{t > t_1 : \varepsilon^\alpha < u_\varepsilon < u_0,\ v_1^* - \delta_0 < v_\varepsilon < v_1^* + \delta_0 \text{ on } (t_1, t)\}$. Then $(u_\varepsilon(t), v_\varepsilon(t)) \in \mathcal{R}$ for all $t \in (t_1, t_2)$. It follows that $u_\varepsilon'(t) > m_\varepsilon > 0$, for all $t \in (t_1, t_2)$, where $m_\varepsilon = \min\{u^{k+1}g(u, v, \varepsilon) : \varepsilon^\alpha \le u \le u_0, v_1^* - \delta_0 \le v \le v_1^* + \delta_0\}$. Hence t_2 is finite and $t_2 < t_*$.

We note that $V_\varepsilon(u, v_\varepsilon(t_1))$ is the solution of (3.6) with $V_\varepsilon(\varepsilon^\alpha, v_\varepsilon(t_1)) = v_\varepsilon(t_1)$. By an argument similar to that in the proof of Lemma 3.1 and the estimate of $|v_\varepsilon(t_1) - V_0(\varepsilon^\alpha, b(v_0))|$ in Lemma 3.2, we see that there is a constant $M_3 > 0$ independent of the solution such that

$$\begin{aligned} &|V_\varepsilon(u, v_\varepsilon(t_1)) - V_0(u, b(v_0))| \\ &\le |v_\varepsilon(t_1) - V_0(\varepsilon^\alpha, b(v_0))| e^{\int_{\varepsilon^\alpha}^u H_1(\xi,\varepsilon)\,d\xi}\, d\eta + M_3\varepsilon \int_{\varepsilon^\alpha}^u \left(\frac{1}{\eta^{k+1}} + 1\right) e^{\int_\eta^u H_1(\xi,\varepsilon)\,d\xi}\, d\eta, \end{aligned}$$

for all $u \in [\varepsilon^\alpha, u_\varepsilon(t_2)]$. In the case that $k > 0$, we use the fact that $u > \varepsilon^\sigma$ to obtain

$$|V_\varepsilon(u, v_\varepsilon(t_1)) - V_0(u, b(v_0))| \le M_3\varepsilon^{1-k\sigma} + M_3\varepsilon\left(\frac{1}{\varepsilon^{k\sigma}} - \frac{1}{u^k} + u - \varepsilon^\sigma\right) \le M_3\varepsilon^{1-k\sigma},$$

and in the case of $k = 0$, we use the fact that $u > \varepsilon^2$ to obtain

$$|V_\varepsilon(u, v_\varepsilon(t_1)) - V_0(u, b(v_0))| \le M_3\varepsilon|\ln\varepsilon| + M_3\varepsilon\left(\ln u - 2\ln\varepsilon + u - \varepsilon^2\right) \le M_3\varepsilon|\ln\varepsilon|.$$

These estimates, together with the fact that $v_0^* - \delta_0 < V_0(u, b(v_0)) < v_0^* + \delta_0$, $u \in [0, u_0]$, and the definition of t_2, imply that $u_\varepsilon(t_2) = u_0$ and (3.14) holds. This completes the proof.

□

Lemma 3.4. *Let $(\bar{u}(t), \bar{v}(t))$ be the solution of (2.1) with $(\bar{u}(0), \bar{v}(0)) = (u_0, V_0(u_0, b(v_0)))$, where $V_0(u, b(v_0))$ is the solution of (3.4) with $v_1 = b(v_0)$. Denote $T_0 > 0$ as the first time when this solution orbit intersects the segment L_{δ_0}. Then there is $T(\varepsilon) = t_2 + T_0 + O(\varepsilon)$ such that $(u_\varepsilon(t), v_\varepsilon(t))$ is defined for all $t \in [t_2, T(\varepsilon)]$ and satisfies $(u_\varepsilon(T(\varepsilon)), v_\varepsilon(T(\varepsilon))) \in L_{\delta_0}$ and*

$$|(u_\varepsilon(t_2 + t), v_\varepsilon(t_2 + t)) - (\bar{u}(t), \bar{v}(t))| \le M_4\phi(\varepsilon), \qquad t \in [0, T(\varepsilon) - t_2], \tag{3.15}$$

where $M_4 > 0$ is a generic constant independent of $(u_\varepsilon, v_\varepsilon)$.

Proof. Fix a small number $\mu > 0$ and denote $(u(t), v(t)) := (u_\varepsilon(t_2 + t), v_\varepsilon(t_2 + t))$. Since $u(0) = u_\varepsilon(t_2) = \bar{u}(0) = u_0$ and $|v(0) - \bar{v}(0)| = |v_\varepsilon(t_2) - V_0(u_0, b(v_0))| \le M_3\phi(\varepsilon)$ by Lemma 3.3, the continuous dependence of solutions on initial data implies that $(u(t), v(t))$ is defined for all $t \in [0, T_0 + \mu]$, and there is $T_0(\varepsilon) = T_0 + o(1)$ such that $u(T_0(\varepsilon)) = u_0$. On the interval $[0, T_0 + \mu]$, it follows from the mean value theorem that there is a generic constant M_4 independent of the solution such that

$$\begin{aligned} |u'(t) - \bar{u}'(t)| &= |u^{k+1}(t)g(u(t), v(t), \varepsilon) - \bar{u}^{k+1}(t)g(\bar{u}(t), \bar{v}(t), 0)| \\ &\le M_4\varepsilon + M_4(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|), \qquad \forall t \in [0, T_0 + \mu], \end{aligned}$$

and

$$\begin{aligned}|v'(t)-\bar v'(t)| &= |\varepsilon f(u(t),v(t),\varepsilon)+u^{k+1}(t)h(u(t),v(t),\varepsilon)-\bar u^{k+1}(t)h(\bar u(t),\bar v(t),0)|\\ &\le M_4\varepsilon + M_4(|u(t)-\bar u(t)|+|v(t)-\bar v(t)|), \qquad \forall t\in[0,T_0+\mu].\end{aligned}$$

Using the fact that $\phi(\varepsilon)>\varepsilon$, direct integrations of the above yield that

$$|(u(t),v(t))-(\bar u(t),\bar v(t))| \le M_4\phi(\varepsilon)+M_4\int_0^t (|u(s)-\bar u(s)|+|v(s)-\bar v(s)|)\,ds,$$

which concludes (3.15) by applying the Gronwall's inequality.

Using the fact that $u(T_0(\varepsilon),\varepsilon)=u_0$ and the implicit function theorem, we have

$$T_0'(\varepsilon) = -\frac{\partial u(T_0(\varepsilon),\varepsilon)}{\partial\varepsilon}\Big/ u'(T_0(\varepsilon),\varepsilon).$$

Since $u'(T_0(\varepsilon),\varepsilon) = u_0^{k+1}g(u_0,v(T_0(\varepsilon),\varepsilon),\varepsilon)$ is bounded away from zero, we see from the above identity that $T_0'(\varepsilon)=O(1)$, which, together with the mean-value theorem, yields that $T_0(\varepsilon)=T_0+O(\varepsilon)$. The proof is now completed by setting $T(\varepsilon) := t_2+T_0(\varepsilon)$.

□

Proof of Theorem 2.1. Theorem 2.1 follows directly from Lemmas 3.1-3.4 and the definition of $\tilde\Gamma_0(\tilde v_0)$. □

Proof of Theorem 2.2 (i). From Theorem 2.1 and the construction of $\mathcal{U}_\varepsilon$ in Section 2, we see that $dist(\mathcal{U}_\varepsilon,\Gamma_0(v_0^*))<\delta_0$, and moreover, $\mathcal{U}_\varepsilon$ is negatively invariant if (2.7) holds and positively invariant if (2.6) holds.

Let $(u_\varepsilon^-(t),u_\varepsilon^-(t))$ the the solution defined in Section 2.2 which is a portion of the inner boundary of $\mathcal{U}_\varepsilon$. If $(\hat u_\varepsilon,\hat v_\varepsilon)$ contains an equilibrium, say $(\hat u_\varepsilon,\hat v_\varepsilon)$, of (1.1) in $\mathcal{U}_\varepsilon$, then it must satisfies properties described in (2.14). Let $\hat t\in(t_1,t_2)$ be such that $v_\varepsilon^-(\hat t)=\hat v_\varepsilon$. We have by (3.11) that $u_\varepsilon^-(\hat t)=O(\varepsilon^{1/k})$ in the case $k>0$ and by (3.12) that $u_\varepsilon^-(\hat t)=O(\varepsilon^2e^{-1/\varepsilon})$ in the case $k=0$, which, together with (2.14), imply that $u_\varepsilon^-(\hat t)\ll\hat u_\varepsilon$ in both cases. This is a contradiction because $(u_\varepsilon^-(t),u_\varepsilon^-(t))$ lies in the inner boundary of $\mathcal{U}_\varepsilon$. Hence $\mathcal{U}_\varepsilon$ does not contain any equilibrium point of (1.1).

Now, an application of the Poincare-Bendixson theorem shows the existence of a closed orbit Γ_ε of (1.1) in $\mathcal{U}_\varepsilon$ which is relaxed due to its closeness to Γ_0. This proves Theorem 2.2 (i). □

3.3. **Derivative of the "discriminant" function.** We show below that the derivative $\Delta'(v_0^*)$ of the "discriminant" function satisfies the formula (2.8). The major part of the proof is devoted to deriving the formula $a'(v_0)$ given in (2.2).

For convenience, we consider the time-reversed system (2.1), i.e.,

$$\begin{cases} u' = -u^{k+1}g(u,v,0),\\ v' = -u^{k+1}h(u,v,0).\end{cases} \tag{3.16}$$

For each $v_0\in(v_0^*-\delta^*,v_0^*+\delta^*)$, we use $-\gamma(v_0)$ to denote the heteroclinic orbit of (3.16) connecting $(0,v_0)$ and $(0,a(v_0))$, at $t=-\infty$ and $t=\infty$, respectively, and let $(u(t,v_0),v(t,v_0))$ be a solution of (3.16) that parameterizes $-\gamma(v_0)$. For a fixed small $u_0>0$, since $g(u,v,0)<0$ near $(0,v_0^*)$, there is $T_1\in(-\infty,\infty)$ such that $u(T_1,v_0)=u_0$ and $u'=-u^{k+1}g>0$ along $-\gamma(v_0)$ for all $t\in(-\infty,T_1]$. Similarly, there is $T_2\in(T_1,\infty)$ such that $u(T_2,v_0)=u_0$ and $u'=-u^{k+1}g<0$ along $\gamma(v_0)$ for all $t\in[T_2,\infty)$. To make separate treatments on the analysis of $(u(t,v_0),v(t,v_0))$ near $t=\pm\infty$, we divide $-\gamma(v_0)$ into the following three parts:

$$\begin{aligned}\tilde\gamma_1(v_0) &:= \{(u(t,v_0),v(t,v_0)) : -\infty<t\le T_1\},\\ \tilde\gamma_2(v_0) &:= \{(u(t,v_0),v(t,v_0)) : T_1\le t\le T_2\},\end{aligned}$$

$$\tilde{\gamma}_3(v_0) := \{(u(t, v_0), v(t, v_0)) : T_2 \le t < \infty\}.$$

On the part $\tilde{\gamma}_1(v_0)$, since $u' > 0$ on $[0, u_0]$, we can parameterize its orbit as the graph of $V(u, v_0) := v(t, v_0)$ which satisfies

$$\frac{dV}{du} = \frac{h(u, V, 0)}{g(u, V, 0)}, \qquad u \in [0, u_0], \tag{3.17}$$

with the initial condition $V(0, v_0) = v_0$. The smoothness of the system (3.17) implies that $V(u, v_0)$ is differentiable on both u and v_0. By differentiating both hand sides of (3.17) and the initial condition with respect to v_0, we obtain an initial value problem for $\frac{\partial V(u,v_0)}{\partial v_0}$ whose solution is simply $\frac{\partial V(u,v_0)}{\partial v_0} = \exp\left(\int_0^u \frac{\partial}{\partial v}\left(\frac{h(s,V(s,v_0),0)}{g(s,V(s,v_0),0)}\right) ds\right)$, $u \in (0, u_0]$. Hence, letting $v_1(v_0) := v(T_1, v_0) = V(u_0, v_0)$, we have

$$v_1'(v_0) = \exp\left(\int_0^{u_0} \frac{\partial}{\partial v}\left(\frac{h(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)}\right) ds\right).$$

Since, by a straightforward calculation,

$$\begin{aligned}\int_0^{u_0} \frac{\partial}{\partial v}\left(\frac{h(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)}\right) ds = &-\ln\frac{g(u_0, v_1(v_0), 0)}{g(0, v_0, 0)} \\ &+ \int_0^{u_0}\left(\frac{h_v(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} + \frac{g_u(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)}\right) ds,\end{aligned}$$

we have

$$\begin{aligned} v_1'(v_0) &= \frac{g(0, v_0, 0)}{g(u_0, v_1(v_0), 0)} \exp\left(\int_0^{u_0}\left(\frac{h_v(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} + \frac{g_u(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)}\right) ds\right) \\ &= \frac{g(0, v_0, 0)}{g(u_0, v_1(v_0), 0)} e^{-\int_{-\infty}^{T_1} u^{k+1}[g_u + h_v]\, dt}. \end{aligned} \tag{3.18}$$

On the part $\tilde{\gamma}_2(v_0)$, we note that $(u(t, v_0), v(t, v_0))$ satisfies (3.16) with

$$u(T_1, v_0) = u_0, \quad v(T_1, v_0) = v_1(v_0), \qquad u(T_2, v_0) = u_0, \quad v(T_2, v_0) =: v_2(v_0).$$

Differentiating these equations with respect to v_0 yields

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial u}{\partial v_0}\Big) &= -\Big[(k+1)u^k g + u^{k+1} g_u\Big]\frac{\partial u}{\partial v_0} - u^{k+1} g_v \frac{\partial v}{\partial v_0}, \\ \frac{d}{dt}\Big(\frac{\partial v}{\partial v_0}\Big) &= -\Big[(k+1)u^k h + u^{k+1} h_u\Big]\frac{\partial u}{\partial v_0} - u^{k+1} h_v \frac{\partial v}{\partial v_0}, \end{aligned}$$

and

$$\begin{aligned} u'(T_1, v_0)T_1'(v_0) + \frac{\partial u}{\partial v_0}(T_1, v_0) = 0, &\qquad v'(T_1, v_0)T_1'(v_0) + \frac{\partial v}{\partial v_0}(T_1, v_0) = v_1'(v_0), \\ u'(T_2, v_0)T_2'(v_0) + \frac{\partial u}{\partial v_0}(T_2, v_0) = 0, &\qquad v'(T_2, v_0)T_2'(v_0) + \frac{\partial v}{\partial v_0}(T_2, v_0) = v_2'(v_0). \end{aligned}$$

Using these equations and the Abel's formula for the Wronskian, we obtain

$$\begin{aligned} v_2'(v_0) &= -v'(T_2, v_0)\frac{\frac{\partial u}{\partial v_0}(T_2, v_0)}{u'(T_2, v_0)} + \frac{\partial v}{\partial v_0}(T_2, v_0) \\ &= \frac{1}{u'(t_2, v_0)}\Big[u'(T_2, v_0)\frac{\partial v}{\partial v_0}(T_2, v_0) - v'(T_2, v_0)\frac{\partial u}{\partial v_0}(T_2, v_0)\Big] \\ &= \frac{1}{u'(T_2, v_0)}\begin{vmatrix} u'(T_2, v_0) & \frac{\partial u}{\partial v_0}(T_2, v_0) \\ v'(T_2, v_0) & \frac{\partial v}{\partial v_0}(T_2, v_0) \end{vmatrix} \end{aligned}$$

$$= \frac{1}{u'(T_2, v_0)} \begin{vmatrix} u'(T_1, v_0) & \frac{\partial u}{\partial v_0}(T_1, v_0) \\ v'(T_1, v_0) & \frac{\partial v}{\partial v_0}(T_1, v_0) \end{vmatrix} e^{-\int_{T_1}^{T_2} [(k+1)u^k g + u^{k+1} g_u + u^{k+1} h_v] \, dt}$$

$$= \frac{u'(T_1, v_0)}{u'(T_2, v_0)} v_1'(v_0) e^{-\int_{T_1}^{T_2} [(k+1)u^k g + u^{k+1} g_u + u^{k+1} h_v] \, dt}.$$

Using (3.16) and the fact that $u(T_1) = u(T_2) = u_0$, we see that that

$$-\int_{T_1}^{T_2} (k+1)u^k g \, dt = \int_{T_1}^{T_2} (k+1) \frac{u'}{u} \, dt = \ln \frac{u^{k+1}(T_2)}{u^{k+1}(T_1)} = 0.$$

It follows from (3.18) that

$$v_2'(v_0) = \frac{u'(T_1, v_0)}{u'(T_2, v_0)} v_1'(v_0) e^{-\int_{T_1}^{T_2} u^{k+1}[g_u + h_v] \, dt} = \frac{g(0, v_0, 0)}{g(u_0, v_2(v_0), 0)} e^{-\int_{-\infty}^{T_2} u^{k+1}[g_u + h_v] \, dt}. \tag{3.19}$$

On the part $\tilde{\gamma}_3(v_0)$, since $u' < 0$ on $(0, u_0]$, we can parameterize its orbit as the graph of $V(u, v_0) := v(t, v_0)$. Similar to the case for $\tilde{\gamma}_1(v_0)$, $V(u, v_0)$ satisfies (3.17) with the initial condition $V(u_0, v_0) = v_2(v_0)$, and, differentiating both sides of this equation and the initial condition with respect to v_0 yields an initial value problem. By noting that $a(v_0) := V(0, v_0)$ and $\frac{\partial V(u_0, v_0)}{\partial v_0} = v_2'(v_0)$, solving this initial value problem gives

$$a'(v_0) = v_2'(v_0) \exp\left(\int_{u_0}^{0} \frac{\partial}{\partial v} \left(\frac{h(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} \right) ds \right).$$

Since

$$\int_{0}^{u_0} \frac{\partial}{\partial v} \left(\frac{h(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} \right) ds = -\ln \frac{g(u_0, v_2(v_0), 0)}{g(0, a(v_0), 0)} + \int_{0}^{u_0} \left(\frac{h_v(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} + \frac{g_u(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} \right) ds,$$

we have by (3.19) that

$$a'(v_0) = v_2'(v_0) \frac{g(u_0, v_2(v_0), 0)}{g(0, a(v_0), 0)} \exp\left[\int_{u_0}^{0} \left(\frac{h_v(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} + \frac{g_u(s, V(s, v_0), 0)}{g(s, V(s, v_0), 0)} \right) ds \right]$$

$$= v_2'(v_0) \frac{g(u_0, v_2(v_0), 0)}{g(0, a(v_0), 0)} e^{-\int_{T_2}^{\infty} u^{k+1}[g_u + h_v] \, dt} = \frac{g(0, v_0, 0)}{g(0, a(v_0), 0)} e^{-\int_{-\infty}^{\infty} u^{k+1}[g_u + h_v] \, dt},$$

which is the desired formula stated in (2.2).

Now, by using the fact that $b(v_0^*) = a(v_0^*) = v_1^*$, (2.2), (2.5) and the definition of $\lambda(v_0^*)$ in (2.9), we have

$$[a'(v_0^*) - b'(v_0^*)] \frac{g(0, v_1^*, 0)}{g(0, v_0^*, 0)} = -\frac{f(0, v_1^*, 0)}{f(0, v_0^*, 0)} + \exp\left(-\int_{\gamma(v_0^*)} \frac{g_u(u, v, 0)}{g(u, v, 0)} \, du + \frac{h_v(u, v, 0)}{h(u, v, 0)} \, dv \right)$$

$$= \left(1 - e^{\lambda(v_0^*)}\right) \exp\left(-\int_{\gamma(v_0^*)} \frac{g_u(u, v, 0)}{g(u, v, 0)} \, du + \frac{h_v(u, v, 0)}{h(u, v, 0)} \, dv \right),$$

which leads to the desired formula of $\Delta'(v_0^*)$ stated in (2.8).

3.4. **Proof of Theorem 2.2 (ii).** In order to establish the local uniqueness and orbital stability of the closed orbit of (1.1) in a small neighborhood Γ_0 as guaranteed by Theorem 2.2 (i), we have the following result concerning its non-trivial Floquet exponent.

Lemma 3.5. *Let Γ_ε be a closed orbit of* (1.1) *in $\mathcal{U}_\varepsilon$. Then its non-trivial Floquet exponent $\lambda(\Gamma_\varepsilon)$ satisfies*

$$\lambda(\Gamma_\varepsilon) = \lambda(v_0^*) + O(\delta_0),$$

where $|O(\delta_0)| \le \widetilde{M}\delta_0$ for some constant $\widetilde{M} > 0$ independent of δ_0 and ε.

Proof. We fix a small $\delta_0 > 0$ and $u_0 \in (0, \delta_0)$ such that $e^{Mu_0} \le 2$ where M is defined at the beginning of Section 3. Let L_{δ_0} and $\mathcal{U}_\varepsilon$ be constructed as in Section 2 and $(u(t), v(t)) := (u_\varepsilon(t), v_\varepsilon(t))$ with $(u(0), v(0) \in L_{\delta_0}$ be the periodic solution of (1.1) with orbit Γ_ε and the lease period $T := T(\varepsilon)$. Since, by the fact that $u(t) > 0$ and (1.1), $\int_0^T u^k(t)g(u(t), v(t), \varepsilon)\,dt = \int_0^T \frac{u'(t)}{u(t)}\,dt = \ln u(T) - \ln u(0) = 0$, it is easy to see that

$$\begin{aligned}\lambda(\Gamma_\varepsilon) = &\int_0^T [(k+1)u^k(t)g(u(t), v(t), \varepsilon) + u^{k+1}(t)g_u(u(t), v(t), \varepsilon)\\ &+ \varepsilon f_v(u(t), v(t), \varepsilon) + u^{k+1}h_v(u(t), v(t), \varepsilon)]\,dt\\ = &\int_0^T [u^{k+1}(t)(g_u(u(t), v(t), \varepsilon) + h_v(u(t), v(t), \varepsilon)) + \varepsilon f_v(u(t), v(t), \varepsilon)]\,dt. \qquad (3.20)\end{aligned}$$

To estimate $\lambda(\Gamma_\varepsilon)$, we recall from the proofs of Lemmas 3.1-3.4 that there are $0 < t_0 < t_1 < t_2 < T$ such that $u(t_0) = u(t_1) = \varepsilon^\alpha$, $u(0) = u(t_2) = u(T) = u_0$. These times divide Γ_ε into the following four parts:

$$\begin{aligned}&\Gamma_1 := \{(u(t), v(t)) : 0 \le t \le t_0\}, \quad \Gamma_2 := \{(u(t), v(t)) : t_0 \le t \le t_1\},\\ &\Gamma_3 := \{(u(t), v(t)) : t_1 \le t \le t_2\}, \quad \Gamma_4 := \{(u(t), v(t)) : t_2 \le t \le T\}.\end{aligned}$$

We also recall from the proofs of Lemmas 3.1-3.3 that there are constants $0 < m < M$ independent of ε such that

$$\begin{cases} g \le -m, \quad f \le -m, \quad |v - v_0^*| \le M(u_0 + \delta_0) \le 2M\delta_0, & \text{on } \Gamma_1,\\ f \le -m, \quad v' < 0 \quad \text{on } \Gamma_2,\\ g \ge m, \quad f \le -m, \quad |v - v_1^*| \le M(u_0 + \delta_0) \le 2M\delta_0, & \text{on } \Gamma_3.\end{cases}$$

On Γ_1, since $\varepsilon^\sigma \le u(t) \le u_0$ and $u' = u^{k+1}g < -mu^{k+1}$, we have

$$\int_0^{t_0} u^{k+1}(g_u + h_v)\,dt = \int_{u_0}^{u(t_0)} \frac{g_u + h_v}{g}\,du = O(u_0) = O(\delta_0), \qquad (3.21)$$

and

$$\int_0^{t_0} \varepsilon f_v\,dt = \int_{u_0}^{u(t_0)} \frac{\varepsilon f_v}{u^{k+1}g}\,du = O(\varepsilon)\int_{\varepsilon^\sigma}^{u_0} \frac{1}{u^{k+1}}\,du = \frac{O(\varepsilon)}{\varepsilon^{k\sigma}} = \varepsilon^{1-k\sigma} = O(\delta_0). \qquad (3.22)$$

On Γ_2, since $u < \varepsilon^\sigma$, $u^{k+1}h < \varepsilon^{(k+1)\sigma} \ll \varepsilon$, and

$$v' = \varepsilon f + u^{k+1}h = \varepsilon f[1 + O(\varepsilon^{(k+1)\sigma-1})],$$

we have

$$\int_{t_0}^{t_1} u^{k+1}(g_u + h_v)\,dt \le \frac{1}{m}\int_{v(t_1)}^{v(t_0)} \frac{u^{k+1}|g_u + h_v|}{\varepsilon}\,dv = O(\varepsilon^{(k+1)\sigma-1}) = O(\delta_0), \qquad (3.23)$$

and

$$\int_{t_0}^{t_1} \varepsilon f_v\,dt = \int_{v(t_0)}^{v(t_1)} \frac{\varepsilon f_v}{\varepsilon f[1 + O(\varepsilon^{(k+1)\sigma-1})]}\,dv.$$

Note that, since $u(t) < \varepsilon^\sigma$ and $(k+1)\sigma - 1 < \sigma$, we have

$$f(u(t), v(t), \varepsilon) = f(0, v, 0) + O(u(t) + \varepsilon) = f(0, v, 0) + O(\varepsilon^{(k+1)\sigma-1}),$$

$$f_v(u(t), v(t), \varepsilon) = f_v(0, v, 0) + O(u(t) + \varepsilon) = f_v(0, v, 0) + O(\varepsilon^{(k+1)\sigma-1}).$$

It follows from the facts $v(t_0) = v_0^* + O(\delta_0)$ and $v(t_1) = v_1^* + O(\delta_0)$ that

$$\int_{t_0}^{t_1} \varepsilon f_v\, dt = \int_{v(t_0)}^{v(t_1)} \frac{f_v(0, v, 0)}{f(0, v, 0)}\, dv + O(\varepsilon^{(k+1)\sigma-1}) = \ln \frac{f(0, v(t_1), \varepsilon)}{f(0, v(t_0), \varepsilon)} + O(\delta_0)$$

$$= \ln \frac{f(0, v_1^*, \varepsilon)}{f(0, v_0^*, \varepsilon)} + O(\delta_0). \tag{3.24}$$

On Γ_3, a similar argument as that for Γ_1 yields

$$\int_{t_1}^{t_2} u^{k+1}(g_u + h_v)\, dt = O(\delta_0), \qquad \int_{t_1}^{t_2} \varepsilon f_v\, dt = O(\delta_0). \tag{3.25}$$

On Γ_4, we note from the definition of t_2 that $|v(t_2) - v_{02}^*| \le 2\delta$, where v_{02}^* is such that (u_0, v_{02}^*) lies in $\Gamma_0(v_0^*)$ near $(0, v_1^*)$. Let $(u^*(t), v^*(t))$ be a heteroclinic solution of (2.1) with $(u^*(0), v^*(0)) = (u_0, v_{02}^*)$ whose orbit is $\gamma(v_0^*)$. Then

$$\int_{-\infty}^{0} (u^*(t))^{k+1}[g_u(u^*(t), v^*(t), 0) + h_v(u^*(t), v^*(t), 0)]\, dt$$

$$= \int_0^{u_0} \frac{g_u(\eta, v^*(t), 0) + h_v(\eta, v^*(t), 0)}{g(\eta, v^*(t), 0)}\, d\eta = O(u_0) = O(\delta_0),$$

and

$$\int_{T-t_2}^{\infty} (u^*(t))^{k+1}[g_u(u^*(t), v^*(t), 0) + h_v(u^*(t), v^*(t), 0)]\, dt$$

$$= \int_{u_0(T-t_2)}^{0} \frac{g_u(\eta, v^*(t), 0) + h_v(\eta, v^*(t), 0)}{g(\eta, v^*(t), 0)}\, d\eta = O(u_0 + \delta) = O(\delta_0).$$

Since the same argument as in the proof of Lemma 3.4 yields that

$$|(u(t + t_2), v(t + t_2)) - (u^*(t), v^*(t))| \le M(\varepsilon + \delta) = O(\delta_0), \qquad t \in [0, T - t_2],$$

we have by also noting the fact $u^*(t) \le u_0 + \delta$, $t \in (-\infty, 0) \cup (T - t_2, \infty)$, that

$$\int_{t_2}^{T} [u^{k+1}(t)(g_u(u(t), v(t), \varepsilon) + h_v(u(t), v(t), \varepsilon)) + \varepsilon f_v(u(t), v(t), \varepsilon)]\, dt$$

$$= \int_0^{T-t_2} [u^{k+1}(t + t_2)(g_u(u(t + t_2), v(t + t_2), \varepsilon) + h_v(u(t + t_2), v(t + t_2), \varepsilon)]\, dt$$

$$= \int_0^{T-t_2} [(u^*(t))^{k+1}(g_u(u^*(t), v^*(t), 0) + h_v(u^*(t), v^*(t), 0)]\, dt + O(\delta_0)$$

$$= \int_{-\infty}^{\infty} [(u^*(t))^{k+1}(g_u(u^*(t), v^*(t), 0) + h_v(u^*(t), v^*(t), 0)]\, dt + O(\delta_0). \tag{3.26}$$

Now, the proof is completed by substituting estimates (3.21)-(3.26) into (3.20).

□

Proof of Theorem 2.2 (ii): If $\Delta'(v_0^*) < 0$ (resp. > 0), then Lemma 3.5 implies that $\lambda(\Gamma_\varepsilon) < 0$ (resp. > 0) by taking δ_0 sufficiently small, and consequently, $\lambda(\Gamma_\varepsilon)$ is asymptotically orbitally stable (resp. unstable). In both cases, the stability and instability imply the local uniqueness of Γ_ε. □

4. Predator-prey systems

In this section, we apply our main result Theorem 2.2 to obtain relaxation oscillations in diffusive predator-prey systems of the form

$$\begin{cases} u_t = D_u u_{xx} + u^{k+1}[g(u) - p_0(u)v], \\ v_t = D_v v_{xx} + v\Big(-\varepsilon + \beta u^{k+1} p_1(u)\Big), \end{cases} \tag{4.1}$$

where t is the time variable, $x \in \mathbb{R}$ is the spatial location, u, v are related to population densities of the prey and the predator populations respectively, D_u, D_v are positive diffusive coefficients of the two populations, $k \geq 0$ is an integer, and $0 < \varepsilon \ll 1$ is a parameter. Denote $K > 0$ as the carrying capacity of the system. We assume that g, p_0 and p_1 are C^1 functions on $[0, K]$ satisfying that $g(K) = 0$, $g(u) > 0$ in $[0, K)$, $p_0(u) > 0$ in $[0, K]$, and $p_1(u) > 0$ in $[0, K)$. We note that when $k > 0$, the growth rate $u^{k+1}g(u)$ of the prey population and the functional response $u^{k+1}p_0(u)$ of the predator represent the weak Aleen effect.

Under certain explicit conditions, we will show the existence of spatially homogenous relaxation oscillations and relaxed periodic traveling waves in (4.1). By extending the arguments contained in [10, 12], our analysis is based on the verification of conditions of Theorem 2.2 (ii) with respect to a general class of planar predator-prey systems.

4.1. **Relaxation oscillations in a general class of planar predator-prey systems.** The verification of the conditions $\Delta(v_0^*) = 0$, $\Delta'(v_0^*) \neq 0$ in Theorem 2.2 (ii) can be a nontrivial task in general. However, we would like to show that these conditions can be replaced by some easily verifiable conditions for the following general class of planar predator-prey systems:

$$\begin{cases} u' = u^{k+1} p_0(u, \varepsilon)[g(u) - q(v) + R(u, v, \varepsilon)], \\ v' = h_1(v, \varepsilon)[-\varepsilon f(u, \varepsilon) + u^{k+1} p_1(u, \varepsilon)], \end{cases} \tag{4.2}$$

where $k \geq 0$ is an integer, p_0, g, q, R, h_1, f and p_1 are C^1 functions satisfying

$$(\mathcal{H}_3) \quad \begin{cases} R(u, v, 0) \equiv 0,\ q(0) = 0, g(K) = 0, \text{ for some } K > 0, \\ g(0) > 0,\ f(0, 0) > 0,\ p_0(0, 0) > 0,\ p_1(0, 0) > 0, \\ p_0(u, 0) > 0,\ p_1(u, 0) > 0,\ g(u) > 0, \quad \forall u \in (0, K), \\ q \text{ is strictly increasing on } [0, \infty), \\ h_1(0, 0) = 0,\ h_1(v, 0) > 0 \text{ for } v > 0. \end{cases}$$

The limiting system of (4.2) simply reads

$$\begin{cases} u' = u^{k+1} p_0(u, 0)[g(u) - q(v)], \\ v' = h_1(v, 0) u^{k+1} p_1(u, 0). \end{cases} \tag{4.3}$$

The nontrivial u-nullcline of (4.3) is $g(u) - q(v) = 0$, i.e. the graph of $v = q^{-1}(g(u))$ for $u \in [0, K]$, which intersects the u-axis and v-axis at the points $(K, 0)$ and $(0, \bar{v}_0)$ respectively, where $q(\bar{v}_0) = g(0)$ (see Figure 4). It is clear that $u' > 0$ below this graph, $u' < 0$ above this graph, and $v' > 0$ as long as $v > 0$. One can show that for any $u_0 \in (0, K)$ the orbit of (4.3) through the point $(u_0, q^{-1}(g(u_0)))$ is a heteroclinic orbit that connects a pair of equilibrium points $(0, a(v_0))$ and $(0, v_0)$ for some $v_0 > \bar{v}_0$ and $a(v_0) \in (0, \bar{v}_0)$, and moreover, $v_0 \to L$ and $a(v_0) \to 0$ as $u_0 \to K$ for some finite number $L > 0$, and $v_0 \to \bar{v}_0$ and $a(v_0) \to \bar{v}_0$ as $u_0 \to 0$. Since u_0 and v_0 have one to one correspondence, the function $v_0 := v_0(u_0)$ defines a bijection, so that we can still denote the heteroclinic orbit through $(u_0, q^{-1}(g(u_0)))$ by $\gamma(v_0)$. Also denote the arcs of $\gamma(v_0)$ above and below the graph of $v = q^{-1}(g(u))$ by $v = V_+(u, u_0)$ and $v = V_-(u, u_0)$, $u \in (0, u_0]$, respectively. We note that $V_+(u, u_0)$ is decreasing and $V_-(u, u_0)$ is increasing with $q(V_-(u, u_0)) < g(u) < q(V_+(u, u_0))$ for $u \in (0, u_0)$.

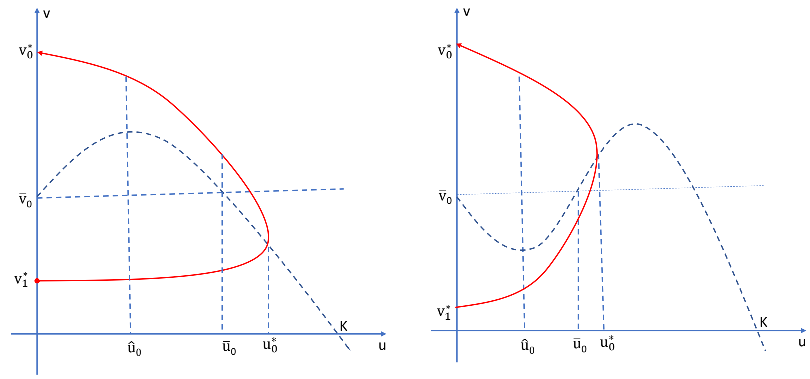

FIGURE 4. The dotted lines are the curves $g(u)-q(v)=0$ and the red curves are the heterolinic orbits of (4.3) with $\chi(u_0^*)=0$.

Consider the function

$$\chi(u_0) := \int_0^{u_0} \frac{p_1(u,0)}{p_0(u,0)}\Big[g(u)-g(0)\Big]\Big[\frac{1}{q(V_+(u,u_0))-g(u)} + \frac{1}{g(u)-q(V_-(u,u_0))}\Big]\,du,$$

$u_0 \in (0,K)$ and denote $\Gamma_0(v_0) := \sigma(v_0)\cup\gamma(v_0)$, where $\sigma(v_0)$ is the segment on the v-axis between v_0 and $b(v_0)$ with b being the entry-exit function associated to (4.2). We have the following result.

Theorem 4.1. *Under the conditions in* $(\mathcal{H}_3)$*, the following hold for system* (4.2)*:*

(i) *For some* $u_0^* \in (0,K)$*,* $\Delta(v_0^*)=0$*, where* $v_0^* := v_0(u_0^*)$*, is equivalent to* $\chi(u_0^*)=0$*. Moreover, if* $\chi(u_0^*)=0$*, then* $\lambda(v_0^*)$ *defined in* (2.9) *can be expressed as*

$$\lambda(v_0^*) = \int_0^{u_0^*} g'(u)\Big[\frac{1}{q(V_+(u,u_0^*))-g(u)} + \frac{1}{g(u)-q(V_-(u,u_0^*))}\Big]\,du.$$

(ii) *Assume that there is* $\hat u \in (0,K)$ *such that* $g'(u)>0$ *on* $(0,\hat u)$ *and* $g'(u)<0$ *on* $(\hat u,K)$*. Let* $\bar u_0 \in (\hat u,K)$ *be such that* $g(\bar u_0)=g(0)>0$*. Then there exists* $u_0^*\in(\bar u_0,K)$ *such that* $\chi(u_0^*)=0$*,* $\lambda(v_0^*)<0$*, where* $v_0^*=v_0(u_0^*)$*, and consequently, as* $\varepsilon>0$ *sufficiently small, the system* (4.2) *admits a unique relaxation cycle in a small neighborhood of* $\Gamma_0(v_0^*)$ *which is asymptotically orbitally stable* (*see* Figure 4 (a)).

(iii) *Assume that there are* $0<\hat u<\bar u_0<u_0^*<K$*, where* $g(\bar u_0)=g(0)$*, such that* $g'(u)<0$ *on* $(0,\hat u)$ *and* $g'(u)>0$ *on* $(\hat u,u_0^*]$*. If* $\chi(u_0^*)=0$*, then* $\lambda(v_0^*)>0$*, where* $v_0^*=v_0(u_0^*)$*, and consequently, as* ε *sufficiently small, the system* (4.2) *admits a unique relaxation cycle in a small neighborhood of* $\Gamma_0(v_0^*)$ *which is orbitally unstable* (*see* Figure 4 (b)).

Proof. (i) It follows from (4.2) that the delay of stability loss along the v-axis is characterized by the equations

$$\begin{cases} u' = u^{k+1}p_0(u,0)[g(u)-q(v)], \\ v' = -\varepsilon f(u,0)h_1(v,0). \end{cases}$$

This motivates us to consider the function

$$H(v) := -\frac{p_0(0,0)}{f(0,0)}\int_{\bar v_0}^{v} \frac{g(0)-q(\eta)}{h_1(\eta,0)}\,d\eta.$$

It is clear that $H(v) = \frac{p_0(0,0)}{f(0,0)}\Big(v - \bar{v}_0 - \bar{v}_0 \ln \frac{v}{\bar{v}_0}\Big)$ when $h_1(v,0) = q(v) = v$. By the definition of $b(v_0)$, we have

$$H(b(v_0)) - H(v_0) = \frac{p_0(0,0)}{f(0,0)} \int_{b(v_0)}^{v_0} \frac{g(0) - q(\eta)}{h_1(\eta,0)}\, d\eta = 0, \qquad \forall v_0 \in (\bar{v}_0, L).$$

For given $u_0 \in (0,K)$, let $(u(t), v(t))$ be a solution of (4.3) on the heteroclinic orbit $\gamma(v_0)$, where $v_0 = v_0(u_0)$. Then, along $\gamma(v_0)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(H(v(t)) + \frac{p_0(0,0)}{f(0,0)} \int_{\bar{u}}^{u(t)} \frac{p_1(s,0)}{p_0(s,0)}\, ds\right) \\
&= -\frac{p_0(0,\varepsilon)}{f(0,0)}[g(0) - q(v)]u^{k+1}\frac{v'}{h_1(v,0)} + \frac{p_0(0,0)}{f(0,0)}\frac{p_1(u,0)}{p_0(u,0)}u' \\
&= \frac{p_0(0,0)}{f(0,0)}[g(0) - q(v)]u^{k+1}p_1(u,0) + \frac{p_0(0,0)}{f(0,0)}[g(u) - q(v)]u^{k+1}p_1(u,0) \\
&= \frac{p_0(0,0)}{f(0,0)}u^{k+1}p_1(u,0)[g(u) - g(0)],
\end{aligned}$$

whose integration over $(-\infty, \infty)$ yields

$$\begin{aligned}
&H(v_0) - H(a(v_0)) = \frac{p_0(0,0)}{f(0,0)} \int_{-\infty}^{\infty} u^{k+1}(t)p_1(u(t),0)[g(u(t)) - g(0)]\, dt \\
&= \frac{p_0(0,0)}{f(0,0)} \int_0^{u_0} \frac{p_1(u,0)}{p_0(u,0)}\Big[g(u) - g(0)\Big]\Big[\frac{1}{q(V_+(u,u_0)) - g(u)} + \frac{1}{g(u) - q(Y_-(u,u_0))}\Big]\, du \\
&= \frac{p_0(0,0)}{f(0,0)}\chi(v_0).
\end{aligned}$$

Since $H(b(v_0)) = H(v_0)$, we have $H(b(v_0)) - H(a(v_0)) = \chi(v_0)$. It follows from the strict monotonicity of $H(v)$ on $(0, \bar{v}_0)$ that, for any $u_0^* \in (0,K)$ and $v_0^* = v_0(u_0^*)$,

$$\Delta(v_0^*) = b(v_0^*) - a(v_0^*) = 0 \iff H(a(v_0^*)) - H(b(v_0^*)) = 0 \iff \chi(u_0^*) = 0.$$

For some $u_0^* \in (0,K)$, assume $\Delta(v_0^*) = 0$, where $v_0^* := v_0(u_0^*)$, and let $v_1^* = a(v_0^*)$. Since

$$\int_{\gamma(v_0^*)} \frac{\partial_v(h_1(v,0)p_1(u,0))}{h_1(v,0)p_1(u,0)}dv = \int_{\gamma(v_0^*)} \frac{\partial_v h_1(v,0)}{h_1(v,0)}\, dv = \ln \frac{h_1(v_0^*,0)}{h_1(v_1^*,0)},$$

$$\begin{aligned}
\int_{\gamma(v_0^*)} \frac{\partial_u(p_0(u,0)[g(u) - q(v)])}{p_0(u,0)[g(u) - q(v)]}du &= \int_{\gamma(v_0^*)} \frac{p_0'(u,0)}{p_0(u,0)}du + \int_{\gamma(v_0^*)} \frac{g'(u)}{g(u) - q(v)}du \\
&= \int_{\gamma(v_0^*)} \frac{g'(u)}{g(u) - q(v)}du,
\end{aligned}$$

$$\ln \frac{-h_1(v_1^*,0)f(0,0)}{-h_1(v_0^*,0)f(0,0)} = \ln \frac{h_1(v_1^*,0)}{h_1(v_0^*,0)} = -\ln \frac{h_1(v_0^*,0)}{h_1(v_1^*,0)},$$

we have by (2.9) that

$$\lambda(v_0^*) = \int_{\gamma(v_0^*)} \frac{g'(u)}{g(u) - q(v)}du = \int_0^{u_0^*} g'(u)\Big[\frac{1}{q(V_+(u,u_0^*)) - g(u)} + \frac{1}{g(u) - q(V_-(u,u_0^*))}\Big].$$

(ii) Since $g(u) > g(0)$, $u \in (0, \bar{u}_0)$, the expression of $\chi(u_0)$ implies that $\chi(u_0) > 0$ for all $u_0 \in (0, \bar{u}_0)$. Since $h_1(v,0) = [\frac{\partial h_1}{\partial v}(0,0) + o(1)]v = O(v)$ as $v \to 0$, $g(0) > 0$, $q(0) = 0$, $v_0 \to L$

and $a(v_0) \to 0$ as $u_0 \to K$, we have $H(v_0) \to L$ as $u_0 \to K$, and

$$H(a(v_0)) := \frac{p_0(0,0)}{f(0,0)} \int_{a(v_0)}^{\bar{v}_0} \frac{g(0) - q(\eta)}{h_1(\eta, 0)}\, d\eta \to \infty, \quad \text{as } v_0 \to L.$$

It follows that

$$\chi(u_0) = \frac{f(0,0)}{p_0(0,0)} \left(H(v_0) - H(a(v_0))\right) \to -\infty, \qquad \text{as } u_0 \to K.$$

Hence there is a first $u_0^* \in (\bar{u}_0, K)$ such that $\chi(u_0^*) = 0$.

We now show that $\lambda(v_0^*) < 0$. Since $V_+(u) := V_+(u, u_0^*)$ is decreasing, $V_-(u) = V_-(u, , u_0^*)$ is increasing, $\left[\frac{1}{q(V_+(u)) - g(u)} + \frac{1}{g(u) - q(V_-(u))}\right] > 0$, and $g'(u) < 0$ in $(\hat{u}, u_0^*]$, we have

$$\lambda(v_0^*) < \int_0^{\bar{u}_0} g'(u) \Big[\frac{1}{q(V_+(u)) - g(u)} + \frac{1}{g(u) - q(V_-(u))}\Big]\, du. \tag{4.4}$$

The monotonicity of q, V_+, V_- and constant signs of $g'(u)$ on intervals $[0, \hat{u}]$ and $[\hat{u}, \bar{u}_0]$ yield that for $u \in [0, \bar{u}_0]$,

$$\begin{aligned} & g'(u) \Big[\frac{1}{q(V_+(u)) - g(u)} + \frac{1}{g(u) - q(V_-(u))}\Big] \\ & \le g'(u) \Big[\frac{1}{q(V_+(\hat{u})) - g(u)} + \frac{1}{g(u) - q(V_-(\hat{u}))}\Big]. \end{aligned} \tag{4.5}$$

Using the fact that $g(0) = g(\bar{u}_0)$, we then have

$$\begin{aligned} \lambda(v_0^*) &< \int_0^{\bar{u}_0} g'(u) \Big[\frac{1}{q(V_+(\hat{u})) - g(u)} + \frac{1}{g(u) - q(V_-(\hat{u}))}\Big]\, du \\ &= \ln \frac{g(u) - q(V_-(\hat{u}))}{q(V_+(\hat{u})) - g(u)}\Big|_0^{\bar{u}_0} = 0. \end{aligned}$$

(iii) The proof follows from that of (ii) by reversing the inequalities (4.4) and (4.5).

□

4.2. **Spatially homogeneous relaxation oscillations of** (4.1)**.** It is clear that spatially homogeneous solutions of (4.1) satisfy the planar predator-prey system

$$\begin{cases} u' = u^{k+1}[g(u) - p_0(u)v], \\ v' = v[-\varepsilon + \beta u^{k+1} p_1(u)]. \end{cases} \tag{4.6}$$

We assume the following condition:

$(\mathcal{H}_4)$ There is a $\hat{u} \in (0, K)$ such that $\frac{g(u)}{p_0(u)}$ is increasing on $(0, \hat{u})$ and decreasing on $(\hat{u}, K)$.

Upon re-writing the first equation of (4.6) as

$$u' = u^{k+1} p_0(u) \Big[\frac{g(u)}{p_0(u)} - v\Big],$$

an application of Theorem 4.1 (ii) concludes the following result.

Theorem 4.2. *Assume* $(\mathcal{H}_4)$*. Then for sufficiently small* $\varepsilon > 0$ *the system* (4.6) *has a unique orbitally stable, relaxation cycle near the singular orbit.*

We note that Theorem 4.2 with $k = 0$ is one of the main results in [10] which has been applied to several concrete models in [10, 12]. Below, we give two new examples of such application for both cases of $k = 0$ and $k = 1$.

Example 4.1. Consider the following predator-prey model contained in [29] with Holling type II functional response and weak Allee effect growth in prey:

$$\begin{cases} u' = u\Big[(u-K)(u+c) - \frac{mv}{u+n}\Big], \\ v' = v\Big(-\varepsilon + \frac{mu}{u+n}\Big), \end{cases} \tag{4.7}$$

where K, m, n, c are positive constants. This system is a special case of (4.6) with $k = 0$, $g(u) = (u-K)(u+c)$, and $p_1(u) = p_0(u) = \frac{m}{u+n}$. Let $G(u) = g(u)/p_0(u) = \frac{1}{m}(u-K)(u+c)(u+n)$. A straightforward computation gives $G'(u) = -3u^2 + 2(K-n-c)u + [(n+c)K - ce]$, and solving $G'(u) = 0$ yields that

$$u_\pm = \frac{1}{3}\Big[(K-n-c) \pm \sqrt{(K-n-c)^2 + 3[(n+c)K - nc]}\Big].$$

Hence, if $(n+c)K > nc$, then $G'(u) = 0$ has a unique positive root u_+ which clearly lies in $(0, K)$. By setting $\hat{u} := u_+$, it follows from Theorem 4.2 that for sufficiently small $\varepsilon > 0$, (4.7) has a relaxation cycle which is asymptotically orbitally stable. We note that in [29], except the condition $(n+c)K > nc$, an additional condition $K < n + c$ is required for the same conclusion.

Example 4.2. Consider the system (1.4) which is also a special case of (4.6) with $k = 1$, $K = 1$, $g(u) = 1 - u$, $p_0(u) = \frac{\alpha}{a+bu+u^2}$, and $p_1(u) = \frac{\beta}{a+bu+u^2}$. Let $G(u) := g(u)/p_0(u) = (1-u)(a + bu + u^2)$. Then $G'(u) = -3u^2 + 2(1-b)u + b - a = 0$ has roots $u_\pm = \frac{1}{3}\Big(\sqrt{(1-b)^2 + 3(b-a)} + 1 - b\Big)$. Hence, if $b > a > 0$, then $G'(u) = 0$ has a unique positive root u_+ which clearly lies in $(0, 1)$. By setting $\hat{u} := u_+$, it follows from Theorem 4.2 that for sufficiently small $\varepsilon > 0$, (1.4) has a relaxation cycle which is asymptotically orbitally stable.

4.3. **Relaxed periodic traveling waves of** (4.1)**.** Let $\mu = D_u/c^2$, $\theta = D_v/D_u$. We assume that

$(\mathcal{H}_5)$ $0 < \mu \ll 1$ and θ is a fixed real number independent of μ.

Replacing x in (4.1) by the moving coordinate $\zeta = x - ct$, we obtain

$$\begin{cases} D_u u_{\zeta\zeta} + cu_\zeta + u^{k+1}[g(u) - p_0(u)v] = 0, \\ D_v v_{\zeta\zeta} + cv_\zeta + v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big) = 0. \end{cases} \tag{4.8}$$

Then the existence of relaxed periodic traveling waves of (4.1) is equivalent to that of relaxation cycles of (4.8). We note that, since the system (4.8) is invariant under the transformation $(\zeta, c) \to (-\zeta, -c)$, it suffices to only consider the case $c > 0$.

Under the change of coordinate $z = \zeta/c$, (4.8) is transformed to

$$\begin{cases} \mu u_{zz} + u_z + u^{k+1}[g(u) - p_0(u)v] = 0, \\ \mu\theta v_{zz} + v_z + v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big) = 0 \end{cases} \tag{4.9}$$

whose first order reduction is the following slow-fast system

$$\begin{cases} \frac{du}{dz} = w_1, \quad \mu\frac{dw_1}{dz} = -w_1 - u^{k+1}[g(u) - p_0(u)v], \\ \frac{dv}{dz} = w_2, \quad \mu\frac{dw_2}{dz} = -\frac{1}{\theta}w_2 - \frac{1}{\theta}v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big) \end{cases} \tag{4.10}$$

when treating μ as the singular parameter. In term of the fast time variable $s = z/\mu$, (4.9) becomes

$$\begin{cases} \frac{du}{ds} = \mu w_1, \quad \frac{dw_1}{ds} = -w_1 - u^{k+1}[g(u) - p_0(u)v], \\ \frac{dv}{ds} = \mu w_2, \quad \frac{dw_2}{ds} = -\frac{1}{\theta}w_2 - \frac{1}{\theta}v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big). \end{cases} \tag{4.11}$$

When $\mu = 0$, (4.10) admits a two-dimensional critical manifold

$$\mathcal{M}_0 = \Big\{(u, w_1, v, w_2) : w_1 = -u^{k+1}[g(u) - p_0(u)v], w_2 = -v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big)\Big\}$$

whose dynamics is described by the system

$$\begin{cases} \frac{du}{dz} = -u^{k+1}[g(u) - p_0(u)v], \\ \frac{dv}{dz} = -v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big). \end{cases} \tag{4.12}$$

With respect to the fast system (4.11) when $\mu = 0$, each point of $\mathcal{M}_0$ is an equilibrium whose linearization has two zero eigenvalues and two negative eigenvalues $-1, -1/\theta$. Therefore, $\mathcal{M}_0$ is a normally hyperbolically stable manifold of system (4.11) when $\mu = 0$. It follows from the geometric theory of singular perturbations [5] that $\mathcal{M}_0$ is persistent under the perturbation, i.e., there exist locally invariant manifolds $\mathcal{M}_\mu$, $0 < \mu \ll 1$, which are diffeomorphic to $\mathcal{M}_0$ such that $\mathcal{M}_\mu = \mathcal{M}_0 + O(\mu)$ as $\mu \to 0$. Hence the restriction of (4.11) to $\mathcal{M}_\mu$, up to its lowest order, has the form

$$\begin{aligned} \frac{du}{dz} &= -u^{k+1}[g(u) - p_0(u)v] + O(\mu), \\ \frac{dv}{dz} &= -v\Big(-\varepsilon + \beta u^{k+1}p_1(u)\Big) + O(\mu). \end{aligned} \tag{4.13}$$

Since system (4.12) is just the time-reversed form of (4.6), it follows from Theorem 4.2 that if $(\mathcal{H}_4)$ holds then it admits an asymptotically orbitally stable relaxation cycle, which we denote by $\Gamma^0(\varepsilon)$. Then by the persistence theory of normally hyperbolic invariant manifolds (see also argument given in [1]), we conclude that for a fixed sufficiently small $\varepsilon > 0$, if $\mu > 0$ is sufficiently small, then the system (4.13) has a unique relaxation cycle $\Gamma^\mu(\varepsilon)$ such that $dist(\Gamma^\mu(\varepsilon), \Gamma^0(\varepsilon)) \to 0$ as $\mu \to 0$. We note that $\mu_0(\varepsilon)$ may go to 0 as $\varepsilon \to 0$.

Summarizing up, we have the following result.

Theorem 4.3. *Assume $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$. Then for given ε sufficiently small, there is a sufficiently small $\mu_0 := \mu_0(\varepsilon) > 0$ such that for every $0 < \mu := D_\mu/c < \mu_0$, system (4.1) has a periodic traveling wave solution $(u(x - ct), v(x - ct))$ whose profile exhibits relaxation behaviors.*

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