A marble moves along the x-axis. Its potential energy function \( U(x) \) is shown in the figure.

a) Where is the force on the marble zero?
b) Stable equilibri(um)(a)?
c) Unstable equilibri(um)a?

**Answers:**

a) b and d since force is negative of the x derivative
b) b since potential energy increases around it, means kinetic energy decreases, tends to stay “calm”.
c) d due to increase of Kinetic energy, hence can easily run away.

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**CURVEBALL on the Definition of Conservative Forces**

Recall: *IF* a force is conservative, *THEN* work done by the force along a closed path in space is zero,

The contrapositive of this statement (*), which *is* logically equivalent to it is that

*IF* the work done by a force around a closed path is NOT zero *THEN* the force is NON-conservative.

The reverse of this statement is NOT TRUE!

*IF* work done by the force along a closed path in space is zero, *THEN* a force is conservative

Moral: Only if the work done around every possible closed path is zero could we say that a force is conservative (impossible to demonstrate!).
An example which serves as a warning (i.e. an example where the converse is false)

Suppose a variable force acting on an object moving in the plane is given by: \( \vec{F} = (xy, 2x) = xy \hat{x} + 2x \hat{y} \)

The work done by \( \vec{F} \) in moving it around the closed loop \((0,0)\rightarrow(3,0)\rightarrow(3,2)\rightarrow(0,0)\) is

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^3 (0,2x) \cdot (dx,0) + \int_0^2 (3y,6) \cdot (0,dy) + \int_0^3 (x \cdot \frac{2}{3} x, 2x) \cdot (dx, \frac{2}{3} dx)
\]

\[
= 0 + \int_0^2 6 dy + \int_0^3 \left( \frac{2}{3} x^2 + \frac{4}{3} x \right) dx = 12 + \left( \frac{2}{9} x^3 + \frac{2}{3} x^2 \right) \bigg|_{x=0}^{x=3} = 12 - (6 + 6) = 0
\]

But this force is in fact non-conservative: move it around the unit square: \((0,0)\rightarrow(1,0)\rightarrow(1,1)\rightarrow(0,1)\rightarrow(0,0)\)

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (0,2x) \cdot (dx,0) + \int_0^1 (y,2) \cdot (0,dy) + \int_0^1 (x,2x) \cdot (dx,0) + \int_0^1 (0,0) \cdot (0,dy)
\]

\[
= 0 + 2 \cdot \frac{1}{2} + 0 = \frac{3}{2} \neq 0
\]

Power under Constant Force
(below, amazing horsepower!)

- Consider a constant force, \( F \), acting on a body
- The power of this force is:
  \[
  power = \frac{\text{Work Done}}{\Delta t} = \frac{F \Delta s}{\Delta t} = F \bar{v} \quad (\bar{v} \text{ is mean velocity})
  \]
Implications:

(a) Power increases when the velocity of the object it acts on increases.

(b) Suppose your car maxes out at 550 Horsepower in your new BMW, by this relationship, the faster you travel (average velocity), the smaller the force it generate ---→ smaller acceleration --→ it is hard to increase velocity further (i.e., hard to accelerate more on a high way when you are already fast!)

SI Unit: J/s = W (Watt)

Instantaneous Power

• changing forces
  ■ Consider a force, F, which acts on a body. During a short time $\delta t$ the body moves a short distance $\delta x$. If the time interval is short enough F is approximately constant for the period

  ![Diagram](image)

  $\delta W = F \delta x \cos \theta$

  ■ Now the power of the force during this period is simply the amount of work done divided by the time it takes to do it so...

  $$P = \frac{\delta W}{\delta t} = \frac{F \delta x \cos \theta}{\delta t}$$
In the limit that the time change shrinks to Zero, we have

\[
P = F \cos \theta \frac{\delta x}{\delta t} = F \cos \theta \frac{dx}{dt} = Fv \cos \theta
\]

**Instantaneous Power**

\[
P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}
\]

**Example:** A army tank has a power of 2000,000 W. If the mass of the tank is 100 tons. Neglect friction, what is the acceleration of the tank when it is travelling at (1) 36 km/hour (2) 72 km/h?

**Solution:**

(a) Start with a power equation which is related to force

\[
P = Fv \cos \theta
\]

Now use Newton II to solve for acceleration

\[
F = ma
\]

(b) Carry out similar calculation, for \( v = 72 \text{ km/h} \), we get \( a = 1 \text{ m/s}^2 \)

**Moral:**

(a) Without friction, acceleration decreases as speed increases
(b) Without friction the acceleration is never 0!

We should be thankful that friction keeps us from getting that speeding ticket!

**Why is Power a limiting factor??**

- Need to remember that the work done by the force is converted into Kinetic Energy where:

\[
\text{Kinetic Energy} = \frac{1}{2}mv^2
\]

  • energy is proportional to \( v^2 \), not \( v \)

- Consider the work done by the force

\[
W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m(v + v_0)(v - v_0)
\]

\[
W = m \frac{(v + v_0)(v - v_0)}{2} \quad \Rightarrow \quad W = m \vec{v} \Delta v
\]
What does the following equation tell us? \( W = m\bar{v}\Delta v \)

**Suppose** mass of a remote-control toy car is 1 kg, moving with an average speed of 10 m/s on the road, to make a 2 m/s change in speed, what is the required work by the engine? What happens when the average speed is 20 m/s?

**Case 1** \( W = m\bar{v}\Delta v = 1 \times 10 \times 2 = 20 J \)

**Case 2** \( W = m\bar{v}\Delta v = 1 \times 20 \times 2 = 40 J \)

A lot more work needs to be spent to bring the same speed change when the toy car moves with faster average speed. Power is work over time, so need more power as well!

**Energy Conversion:** Energy cannot be created or destroyed in a closed system (careful: mechanical energy of an object can be lost due to frictional heating), but can be converted from one form to another.

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**Energy Related Calculations**

- **The fastest steam engine in the world is the Mallard which achieved a speed of 126mph (202 km/h) on 3\textsuperscript{rd} July 1938 over a 5000 m stretch of track**

  - The current electric trains operating on the same line today have a maximum speed of 225 km/h!

- **Based on the following info, what was the power of the engine and the amount of coal burnt during the speed trial?**

  - Coal contains 30MJ/kg chemical energy
  - The burning and heat extraction efficiencies were 80% and 50% respectively
  - The six coaches pulled gave a frictional force of 10000 N
Power of the engine:

Power = Force × velocity = \(10,000 \times 202 \times \frac{1000}{3600}\) = 561 kW

To calculate the coal burned we need to know the energy required

\[
W = \text{Power} \times \text{time} = 561,000 \times \frac{5000}{202,000} \times 3600 = 50 \text{ MJ}
\]

Now the conversion efficiency of the heat energy into mechanical work is 50%

Heat Energy Needed = \(\frac{50\text{MJ}}{50\%}\) = 100 MJ

The conversion efficiency of coal’s chemical energy into heat is 80%

Chem Energy Needed = \(\frac{100\text{MJ}}{80\%}\) = 125 MJ

So the total number of kg of coal needed:

Chem Energy Needed = \(\frac{125\text{MJ}}{30\text{MJ/kg}}\) = 4.2 kg

Forces and Potential Energy in 2D & 3D (and the differential criterion for determining whether a 2D/3D force is conservative)

Partial derivatives

Suppose we have a function \(U\) of two variables \(x\) and \(y\). Then we can ask “how fast” does \(U\) change in the \(x\)-direction and how fast does \(U\) change in the \(y\)-direction. So there are two natural rates of change, or two partial derivatives we can take:

\[
\frac{\partial U(x, y)}{\partial x} = \lim_{h \to 0} \frac{U(x + h, y) - U(x, y)}{h} \quad \frac{\partial U(x, y)}{\partial y} = \lim_{k \to 0} \frac{U(x, y + k) - U(x, y)}{k}
\]

In practice, we compute partial derivatives exactly as you would an ordinary derivative except that you treat every variable as a constant except the one you are differentiating with respect to.

1) \(U(x, y) = \alpha x^3 + \beta y^2\)

\[
\frac{\partial U}{\partial x} = 3\alpha x^2 \quad \frac{\partial U}{\partial y} = 2\beta y
\]

2) \(U(x, y) = \gamma x^7 y^5 + 10^{100}\)

\[
\frac{\partial U}{\partial x} = 7\gamma x^6 y^5 \quad \frac{\partial U}{\partial y} = 5\gamma x^7 y^4
\]

3) \(U(x, y) = U_0 \sin\left(\frac{x^2 y}{L^3}\right)\)

\[
\frac{\partial U}{\partial x} = U_0 \cos\left(\frac{x^2 y}{L^3}\right) \frac{\partial^2 y}{\partial x^2} = U_0 \cos\left(\frac{x^2 y}{L^3}\right) \cdot 2\frac{x y}{L^2}
\]

\[
\frac{\partial U}{\partial y} = U_0 \cos\left(\frac{x^2 y}{L^3}\right) \frac{\partial(x^2 y)}{\partial y} = U_0 \cos\left(\frac{x^2 y}{L^3}\right) \cdot \frac{x^2}{L^2}
\]