Theoretical Seismology (GEOP 523); Surface Waves in Layered Media

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March 7, 2011

We have already encountered boundary-associated elastic wave disturbances (evanescent or inhomogeneous waves) in layered media and in a homogeneous medium bounded by a free surface. However, all of these disturbances were excited by plane waves in the system and could be thought of as a postcritical disturbance associated with other waves in the system which had real incidence angles.

The next type of wave that we will discuss, the *Rayleigh wave*, can be thought of as consisting entirely of $\mathrm{P-S_V}$ system evanescent waves. It is also the only surface wave that propagates in an homogeneous elastic half space. If we consider such a half space, we can derive the Raleigh wave by considering a P displacement potential

$$\Phi = Ae^{\imath(\omega t - k_x x - k_x r_\alpha z)} \tag{1}$$

and a S displacement potential

$$\mathbf{\Psi} = Be^{\imath(\omega t - k_x x - k_x r_\beta z)}\hat{y} \equiv \Psi_y \hat{y} \ . \tag{2}$$

Note that these are reflected-type potentials (traveling towards increasing z for real values of r_α and r_β). When the horizontal phase velocity, c_x, is less than the shear velocity, β, both potentials will be evanescent. In this case we have the imaginary cotangents

$$r_\alpha = -\imath\sqrt{1 - \frac{c_x^2}{\alpha^2}} \tag{3}$$

and

$$r_\beta = -\imath\sqrt{1 - \frac{c_x^2}{\beta^2}} \tag{4}$$

where we have taken the $-\imath$ square root so that the disturbances will go to zero amplitude at $z = \infty$. The traction boundary condition for the free surface inplies that σ_{xz} and σ_{zz} both be zero at $z = 0$. As in our previous derivation of the reflection coefficients in the $\mathrm{P-S_V}$ system, we have

$$\sigma_{xz} = 2\mu\epsilon_{xz} = \mu\left(2\frac{\partial^2\Phi}{\partial x \partial z} - \frac{\partial^2\Psi_y}{\partial z^2} + \frac{\partial^2\Psi_y}{\partial x^2}\right) \tag{5}$$

which implies that

$$0 = -2Ar_\alpha + Br_\beta^2 - B \ , \tag{6}$$

and

$$\sigma_{zz} = 2\mu\epsilon_{zz} + \lambda\Theta = 2\mu\left(\frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Psi_y}{\partial z\partial x}\right) + \lambda\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2}\right) \tag{7}$$

which implies that

$$0 = 2\mu(Ar_\alpha^2 + Br_\beta) + \lambda(A(1 + r_\alpha^2)) \ . \tag{8}$$

Noting that

$$r_\alpha^2 = \frac{c_x^2}{\alpha^2} - 1 \tag{9}$$

and

$$r_\beta^2 = \frac{c_x^2}{\beta^2} - 1 \ . \tag{10}$$

allows us to rewrite the constraint equations in terms of the velocities

$$0 = 2A\sqrt{\frac{c_x^2}{\alpha^2} - 1} + B\left(2 - \frac{c_x^2}{\beta^2}\right) \tag{11}$$

$$0 = 2\mu\left(A\left(\frac{c_x^2}{\alpha^2} - 1\right) + B\left(\sqrt{\frac{c_x^2}{\beta^2} - 1}\right)\right) + \lambda A\frac{c_x^2}{\alpha^2} \ . \tag{12}$$

Dividing (12) through by the density and using

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{13}$$

$$\beta = \sqrt{\frac{\mu}{\rho}} \tag{14}$$

gives

$$0 = A(c_x^2 - 2\beta^2) + 2B\sqrt{(c_x^2\beta^2 - \beta^4)} \ , \tag{15}$$

and dividing by β^2 finally gives

$$0 = A\left(\frac{c_x^2}{\beta^2} - 2\right) + 2B\sqrt{\frac{c_x^2}{\beta^2} - 1} \ . \tag{16}$$

We thus have the coupled system of equations (11) and (16) for the two evanescent wave displacement potential amplitudes

$$\begin{pmatrix} 2\sqrt{\frac{c_x^2}{\alpha^2} - 1} & 2 - \frac{c_x^2}{\beta^2} \\ \frac{c_x^2}{\beta^2} - 2 & 2\sqrt{\frac{c_x^2}{\beta^2} - 1} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \ . \tag{17}$$

(17) will have a trivial solution for $A = B = 0$ (no waves), but will also have nontrivial solutions when the system matrix is singular, or where

$$\begin{vmatrix} 2\sqrt{\frac{c_x^2}{\alpha^2} - 1} & 2 - \frac{c_x^2}{\beta^2} \\ \frac{c_x^2}{\beta^2} - 2 & 2\sqrt{\frac{c_x^2}{\beta^2} - 1} \end{vmatrix} = 0 \ . \tag{18}$$

(18) is called the *Rayleigh wave equation*. It has nontrivial solutions when

$$4\sqrt{\frac{c_x^2}{\alpha^2} - 1}\sqrt{\frac{c_x^2}{\beta^2} - 1} + \left(2 - \frac{c_x^2}{\beta^2}\right)^2 = 0 \ . \tag{19}$$

Moving the second term to the right-hand side, squaring both sides, and factoring out β^2/α^2 on the left-hand side gives

$$16\frac{\beta^2}{\alpha^2}\left(\frac{c_x^2}{\beta^2} - \frac{\alpha^2}{\beta^2}\right)\left(\frac{c_x^2}{\beta^2} - 1\right) = \left(2 - \frac{c_x^2}{\beta^2}\right)^4 \ . \tag{20}$$

Multiplying out terms and defining $\xi = c_x^2/\beta^2$ gives

$$16\frac{\beta^2}{\alpha^2}\left(\xi - \frac{\alpha^2}{\beta^2}\right)(\xi - 1) = 16 - 32\xi + 24\xi^2 - 8\xi^3 + \xi^4 \tag{21}$$

For a Poisson solid, $\alpha^2/\beta^2 = 3$, and

$$16(\xi - 3)(\xi - 1) = 48 - 96\xi + 72\xi^2 - 24\xi^3 + 3\xi^4 \tag{22}$$

which combines to the quartic equation

$$-32\xi + 56\xi^2 - 24\xi^3 + 3\xi^4 = 0 \ . \tag{23}$$

The $\xi = 0$ ($c_x = 0$) root is physically uninteresting, so we are left with the cubic equation

$$-32 + 56\xi - 24\xi^2 + 3\xi^3 = (\xi - 4)(3\xi^2 - 12\xi + 8). \tag{24}$$

which has roots at $\xi = 4, 2 \pm 2/\sqrt{3}$. The only nonzero root that meets the requirement that c_x be less than β is thus

$$\xi = \frac{c_x^2}{\beta^2} \equiv \frac{c_R^2}{\beta^2} = 2 - \frac{2}{\sqrt{3}} \ , \tag{25}$$

which defines the *Rayleigh wave velocity* for a Poisson solid half space,

$$c_R = \beta\sqrt{2 - \frac{2}{\sqrt{3}}} \approx 0.91940\beta \ . \tag{26}$$

The potential amplitude ratio for the inhomogeneous constituents of the Rayleigh wave can be found from the coefficients A and B in (17). Substituting c_R for c_x in the second constraint equation gives

$$B = \left.\frac{(2 - c_x^2/\beta^2)A}{2\sqrt{(c_x^2/\beta^2 - 1)}}\right|_{c_x^2/\beta^2 = 2 - 2/\sqrt{3}} \equiv -\imath CA \tag{27}$$

where $C \approx 1.468$. We can now examine the displacement field derived from the potential expressions (1) and (2)

$$u_x = (\bigtriangledown\Phi + \bigtriangledown \times \mathbf{\Psi})_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi_y}{\partial z} \tag{28}$$

$$= -\imath k_x A(\Phi + \imath r_\beta C \Psi_y) = -\imath k_x A \left(e^{\imath(\omega t - k_x x - k_x r_\alpha z)} + \imath C r_\beta e^{\imath(\omega t - k_x x - k_x r_\beta z)} \right) .$$

For $c_x = c_R$, we have

$$r_\alpha = \sqrt{c_x^2/\alpha^2 - 1} \approx \pm 0.847\imath \tag{29}$$

and

$$r_\beta = \sqrt{c_x^2/\beta^2 - 1} \approx \pm 0.393\imath . \tag{30}$$

We take the negative roots for the physically reasonable case where amplitude decreases with increasing z to obtain

$$u_x = -\imath k_x A e^{\imath(\omega t - k_x x)} \left(e^{-0.847 k_x z} - 0.577 e^{-0.393 k_x z} \right) \tag{31}$$

which has a real part given by

$$Re(u_x) = k_x A \sin(\omega t - k_x x) \left(e^{-0.847 k_x z} - 0.577 e^{-0.393 k_x z} \right) . \tag{32}$$

Similarly,

$$u_z = (\bigtriangledown\Phi + \bigtriangledown \times \mathbf{\Psi})_z = \frac{\partial \Phi}{\partial z} - \frac{\partial \Psi_y}{\partial x} \tag{33}$$

$$= -\imath k_x A(r_\alpha \Phi - C \Psi_y) = -\imath k_x A \left(r_\alpha e^{\imath(\omega t - k_x x - k_x r_\alpha z)} + \imath C e^{\imath(\omega t - k_x x - k_x r_\beta z)} \right) .$$

$$= -\imath k_x A e^{\imath(\omega t - k_x x)} \left(-0.847\imath e^{-0.847 k_x z} + 1.47\imath e^{-0.393 k_x z} \right)$$

which has a real part

$$Re(u_z) = k_x A \cos(\omega t - k_x x) \left(-0.847 e^{-0.847 k_x z} + 1.47 e^{-0.393 k_x z} \right) . \tag{34}$$

We thus see that the displacement field falls off with depth and is controlled by two decay constants which are each proportional to the horizontal wavenumber, k_x. At the surface, we have

$$u_x = 0.423 k_x A \sin(\omega t - k_x x) \tag{35}$$

and

$$u_z = 0.623 k_x A \cos(\omega t - k_x x) \tag{36}$$

which is a retrograde elliptical particle motion (Figure 1). Because the amplitude of u_x changes sign when

$$e^{-0.847 \cdot 2\pi z/\lambda_x} - 0.577 e^{-0.393 \cdot 2\pi z/\lambda_x} = 0 , \tag{37}$$

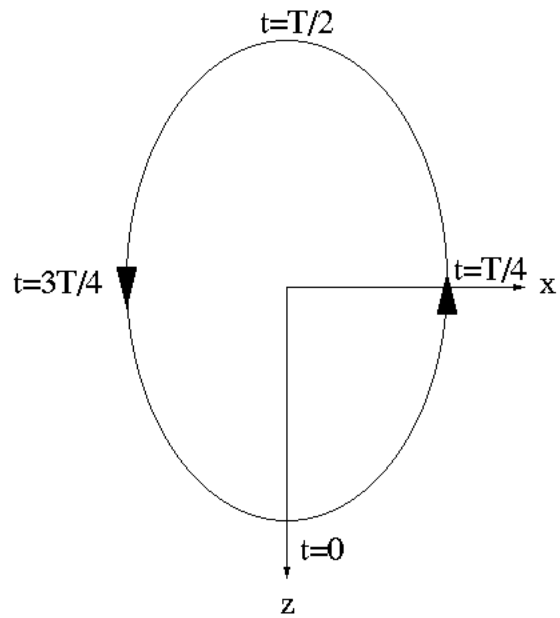

Figure 1: Surface Particle Motion for a Rayleigh Wave traveling in the $+\hat{x}$ Direction.

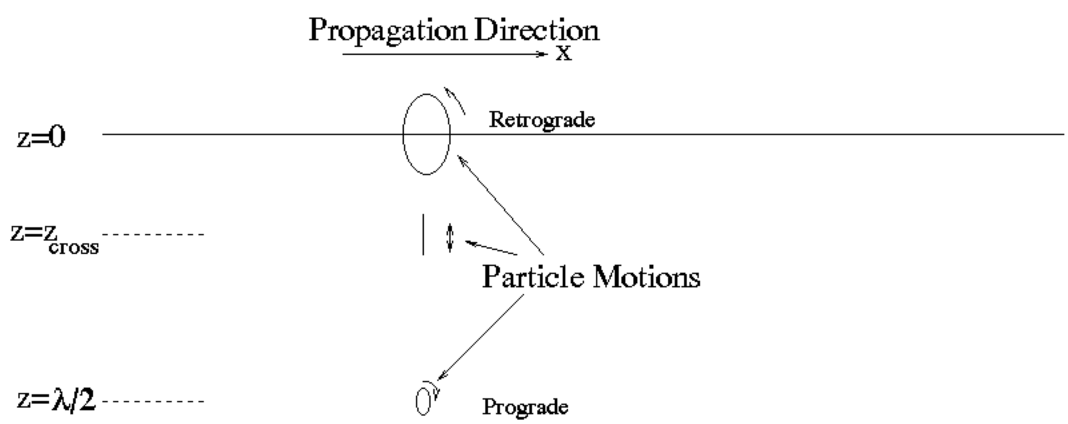

Figure 2: Particle Motions for a Rayleigh Wave traveling in the $+\hat{x}$ Direction.

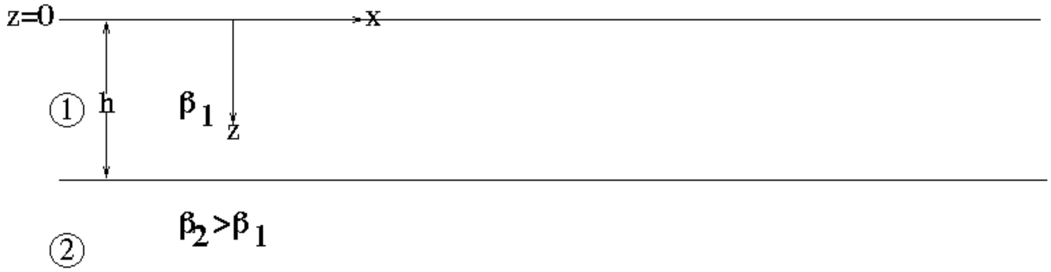

Figure 3: A Low Velocity Layer atop a Half Space.

there will be a depth

$$z_{cross} \approx \frac{\lambda_x \ln(0.577)}{2\pi(0.393 - 0.847)} \approx 0.193\lambda_x \tag{38}$$

where the particle motion is purely vertical (Figure 2). At depths greater than z_{cross}, the change in sign for u_x makes the particle motion prograde. Note that the particle motion ellipses are always taller than they are wide, as $u_z > u_x$ for all $z > 0$. Another cartoon of the Rayleigh wave displacement field is in Figure 6 of the Seismic Wave Equation portion of the notes.

A Rayleigh wave is the only surface wave that will propagate in a homogeneous half space. However, both Rayleigh and *Love* waves (the S_H system counterpart) will propagate in a layered half space where seismic velocities increase with depth. To examine the Love wave system, consider a low velocity surface layer on top of a higher velocity half space

Note immediately that the structure of Figure 3 has a scale length defined by the thickness of the low velocity layer, h. We can thus suspect immediately that surface waves in this system will be dispersive, as the manner in which seismic waves of different wavelengths propagate in the medium will be controlled in some way by h. For wavelengths much greater than h, we the system will approach a homogeneous half-space characterized by the shear wave velocity β_2, while for wavelengths much less than h the near-surface behavior of the system will become increasingly controlled by the top layer with shear wave velocity β_1.

We will derive Love waves in this system as a coupled set of upward- and downward-traveling S_H plane waves in the top layer

$$u_y^{(-)} = B_1 e^{\imath(\omega t - k_x x - k_x r_{\beta_1} z)} + B_2 e^{\imath(\omega t - k_x x + k_x r_{\beta_1} z)} \tag{39}$$

and an evanescent S_H wave in the lower half space

$$u_y^{(+)} = B_3 e^{\imath(\omega t - k_x x - k_x r_{\beta_2} z)} \ . \tag{40}$$

for $u_y^{(+)}$ to be evanescent, we must have the horizontal apparent velocity of waves in the system be less than the shear velocity in layer 2, i.e., $c_x < \beta_2$, so that r_{β_2} is pure imaginary

$$r_{\beta_2} = \pm \imath \sqrt{1 - \frac{c_x^2}{\beta_2^2}} \ . \tag{41}$$

At the free surface, all tractions must be zero. As we have previously demonstrated when considering S_H waves interacting with a free surface, this dynamic boundary condition is equivalent to

$$\sigma_{yz}|_{z=0} = 0 = \mu_1 \frac{\partial u_y^{(-)}}{\partial z}|_{z=0} = \mu_1 \left(-\imath k_x r_{\beta_1} (B_1 - B_2)\right) e^{\imath(\omega t - k_x x)} \tag{42}$$

which reduces to the free-surface S_H reflection solution $B_1 = B_2$.

In this system, we have other matching conditions, namely that displacement and traction must be continuous across the welded contact at $z = h$

$$u_y|_{z=h^{(-)}} = u_y|_{z=h^{(+)}} \tag{43}$$

and

$$\sigma_{yz}|_{z=h^{(-)}} = \sigma_{yz}|_{z=h^{(+)}} \ . \tag{44}$$

The displacement condition gives, after dividing out the usual common exponential terms,

$$B_1 e^{-\imath k_x r_{\beta_1} h} + B_1 e^{\imath k_x r_{\beta_1} h} = B_3 e^{-\imath k_x r_{\beta_2} h} \tag{45}$$

or

$$B_1 \cos(k_x r_{\beta_1} h) = \frac{B_3}{2} e^{-\imath k_x r_{\beta_2} h} \ . \tag{46}$$

The traction matching condition at $z = h$ gives

$$\sigma_{yz}|_{z=h^{(+)}} = \sigma_{yz}|_{z=h^{(-)}} = \mu_1 \frac{\partial u_y^{(-)}}{\partial z}|_{z=h} = \mu_2 \frac{\partial u_y^{(+)}}{\partial z}|_{z=h} \tag{47}$$

or

$$\mu_1 B_1 (-\imath k_x r_{\beta_1}) \left(e^{-\imath k_x r_{\beta_1} h} - e^{\imath k_x r_{\beta_1} h}\right) = -\imath \mu_2 B_3 k_x r_{\beta_2} e^{-\imath k_x r_{\beta_2} h} \tag{48}$$

or

$$B_1 \sin(k_x r_{\beta_1} h) = \frac{B_3 \mu_2 r_{\beta_2}}{2i \mu_1 r_{\beta_1}} e^{-\imath k_x r_{\beta_2} h} \tag{49}$$

Dividing (49) by (46) gives

$$\tan(k_x r_{\beta_1} h) = \tan\left(\omega h \sqrt{\frac{1}{\beta_1^2} - \frac{1}{c_x^2}}\right) = \frac{\mu_2 \sqrt{1 - \frac{c_x^2}{\beta_2^2}}}{\mu_1 \sqrt{\frac{c_x^2}{\beta_1^2} - 1}} \ . \tag{50}$$

(50) is called the *period equation* or *dispersion equation* for Love waves in a two-layer half space. It relates the horizontal velocity, c_x and the frequency ω.

To study the values of c_x and ω for which (50) is satisfied, we rewrite the expression as

$$\tan(\omega\zeta) = \frac{\mu_2\sqrt{1-\frac{c_x^2}{\beta_2^2}}}{\mu_1\sqrt{\frac{c_x^2}{\beta_1^2}-1}} \ . \tag{51}$$

where

$$\zeta = h\sqrt{\frac{1}{\beta_1^2}-\frac{1}{c_x^2}} \ . \tag{52}$$

The right-hand side of the period equation is real-valued only for $\beta_1 < c_x < \beta_2$, the range of apparent velocities where we have real plane waves in the top layer and evanescent waves in the lower half-space. For a given value of ω, we can plot the left-hand side, $\tan(\omega\zeta)$, for this range of c_x. Similarly, we can also plot the right hand side of (50) for this range of c_x. Where the two loci intersect, we will have period equation solutions. Specifically, for a given frequency ω_0, $\tan(\omega\zeta)$ ranges from zero at $c_x = \beta_1$ to

$$\tan(\zeta_{max}) = \tan\left(h\omega_0\sqrt{1/\beta_1^2 - 1/\beta_2^2}\right) \tag{53}$$

at $c_x = \beta_2$. This left-hand side expression will have zeros at $\zeta = n\pi/\omega$ and will be infinite for $\zeta = (2n+1)\pi/(2\omega)$, where n is an integer.

The right-hand side of the period equation is

$$\frac{\mu_2\sqrt{1-\frac{c_x^2}{\beta_2^2}}}{\mu_1\sqrt{\frac{c_x^2}{\beta_1^2}-1}} \ . \tag{54}$$

which decreases monotonically with apparent velocity from infinity at $c_x = \beta_1$ to zero at $c_x = \beta_2$. For a given frequency (or period, $2\pi/\omega$), the solution to the period equation with the smallest value of c_x is called the *fundamental mode.* Other solutions that may exist with greater apparent velocities are called *higher modes* or *overtones.*

To obtain a deeper insight into the nature of Love wave solutions in the system of Figure 3, we note that a harmonic Love wave disturbance is a reinforcing superposition of harmonic postcritical S_H waves in the top layer, coupled with evanescent waves in the underlying half space (Figure 4).

If the phase change over the ray length ABQ is a multiple of 2π, then the down-going wavefront at A will be in phase with the up-going wavefront at Q. The phase change over ABQ is the sum of a propagation shift and a reflection shift at the bottom (the reflection coefficient at the free surface is just 1, so it introduces no phase shift). The propagation phase shift for a harmonic wave of the form

$$Be^{\imath(\omega t - k_x x \pm r_{\beta_1} k_x z)} \tag{55}$$

is

$$\phi_{prop} = -ABQ \cdot |k| = -\frac{(AB+BQ)\cdot 2\pi}{\lambda} \tag{56}$$

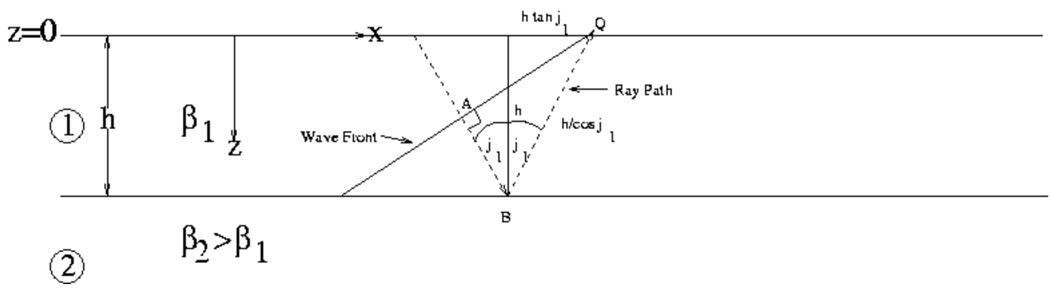

Figure 4: Reinforcing Plane waves in a Layer atop a Half Space.

where λ is the wavelength, where

$$AB + BQ = (h/\cos j_1) \cdot \cos 2j_1 + h/\cos j_1 \tag{57}$$

$$h(1 + \cos 2j_1)/\cos j_1 = h(1 + 2\cos^2 j_1 - 1)/\cos j_1 = 2h\cos j_1 \ .$$

so that

$$\phi_{prop} = -\frac{4\pi h \cos j_1}{\lambda} \ . \tag{58}$$

The wavelength is

$$\lambda = \frac{2\pi}{|\mathbf{k}|} = \frac{2\pi}{\sqrt{k_x^2(1 + r_{\beta_1}^2)}} = \frac{2\pi}{k_{\beta_1}} \tag{59}$$

where $k_{\beta_1} \equiv k_x\sqrt{1 + r_{\beta_1}^2}$, so that

$$\phi_{prop} = -2hk_{\beta_1}\cos j_1 \ . \tag{60}$$

Note, however, that

$$k_{\beta_1}\cos j_1 = k_{\beta_1}\sin j_1 \cdot \frac{\cos j_1}{\sin j_1} = k_x \cot j_1 = k_x r_{\beta_1} \ . \tag{61}$$

The phase shift from the post-critical reflection is

$$\phi_{refl} = 2\tan^{-1}\left(\frac{\imath r_{\beta_2}}{r_{\beta_1}}\right) = 2\tan^{-1}\left(\frac{\mu_2\sqrt{1 - c_x^2/\beta_2^2}}{\mu_1\sqrt{c_x^2/\beta_1^2 - 1}}\right) \ . \tag{62}$$

The condition that the total phase change over ABQ be equal to an integral number of wavelengths is thus

$$\phi = \phi_{prop} + \phi_{refl} = 2\tan^{-1}\left(\frac{\mu_2\sqrt{1 - c_x^2/\beta_2^2}}{\mu_1\sqrt{c_x^2/\beta_1^2 - 1}}\right) - 2hk_x r_{\beta_1} = 2\pi n \ . \tag{63}$$

Dividing by 2, substituting $k_x = \omega/c_x$ and $r_{\beta_1} = \sqrt{c_x^2/\beta_1^2 - 1}$, taking the tangent of both sides, and finally noting that $\tan(x + \pi n) = x$, gives the period equation (50) again! We thus see that Love waves are just self-reinforcing constructively interfering post-critical waves in the upper layer.

The lowest Love wave mode corresponds to trapped waves with the shallowest angle of incidence (and hence the lowest apparent velocity, c_x). As we make ABQ longer with less normal ray paths in the upper medium, we can still always squeeze in a fundamental Love wave mode, even for the longest wavelengths, and c_x will approach β_1. Conversely, if the body wave wavelength in the upper layer is sufficiently short, there will be multiple values of c_x where the constructive interference condition (63) is satisfied, and hence higher modes will be allowed.

Consider a Love wave propagating in a simple continental crustal model, where a 40-km thick crust with properties $\beta_1 = 3.9$ km/s, $\mu_1 = 4.26 \times 10^{10}$ Pa, and $\rho_1 = 2800$ km/m^3 is underlain by a mantle with $\beta_2 = 4.6$ km/s, $\mu_2 = 6.98 \times 10^{10}$ Pa, and $\rho_2 = 3300$ km/m^3. All Love waves are confined to the range of velocities from β_1 to β_2. For arbitrarily long periods, T, the fundamental mode exists, but the first overtone doesn't exist until there exists a mode with a phase velocity of β_2. In general, the lowest frequency where the n^{th} mode exists will be where

$$\omega\zeta|_{c_x=\beta_2} = n\pi \tag{64}$$

or

$$\omega h\sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}} = n\pi \tag{65}$$

so that

$$\omega = \frac{n\pi}{h\sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}}} \equiv \omega_{c_n} \ . \tag{66}$$

For our simple crustal model, then, the first overtone ($n = 1$ mode) won't be allowed until a sufficiently short period of

$$T_{c_1} = \frac{2\pi}{\omega_{c_1}} = \frac{2h}{n}\sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}} \tag{67}$$

which is about 10.9 seconds (Figure 5).

The fundamental ($n = 0$ mode), on the other hand, always exists, and its phase velocity approaches (but never makes it to) β_2 at very long periods, so that the disturbance remains evanescent in the lower medium.

The relationship between c_x and period is called the *dispersion curve.* Within each mode, the apparent velocity of the Love wave increases with increasing period. To obtain the functional form of the dispersion curve, we rewrite the period equation as

$$\frac{\omega}{c_x} = k_{xn}(c_x) = \frac{1}{h\sqrt{\frac{c_x^2}{\beta_1^2} - 1}}\left(\tan^{-1}\left(\frac{\mu_2\sqrt{1 - c_x^2/\beta_2^2}}{\mu_1\sqrt{c_x^2/\beta_1^2 - 1}}\right) + n\pi\right) \tag{68}$$

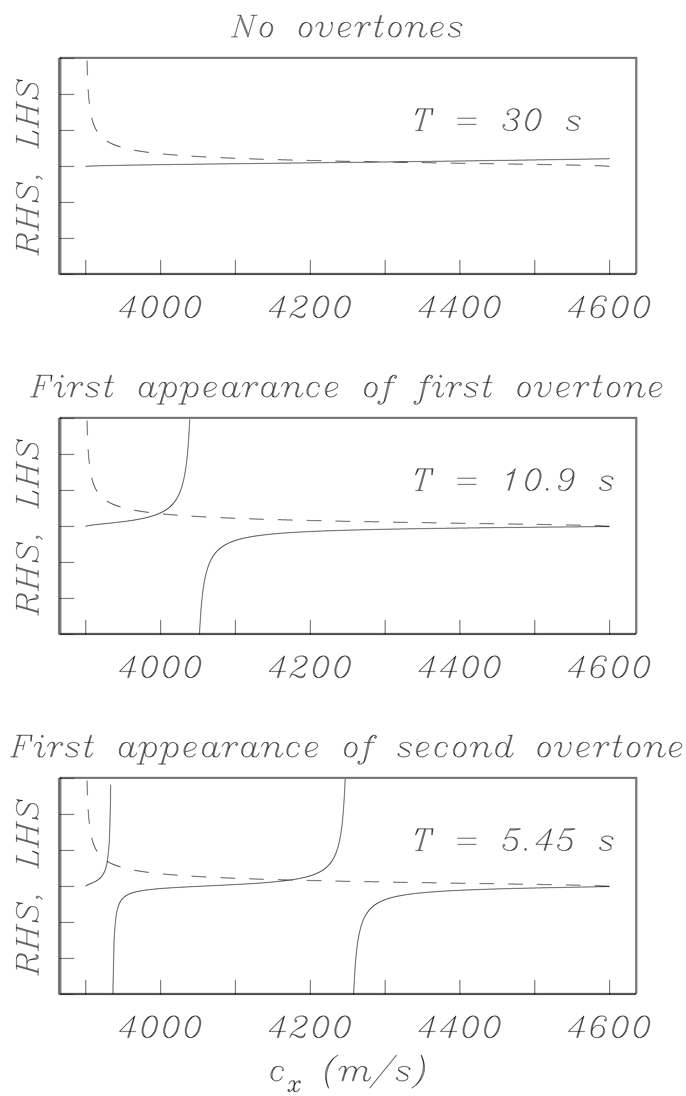

Figure 5: Appearance of successive Love Wave Modes at Shorter Periods; Two-Layer Structure

where we take the fundamental branch of the arctangent in evaluating the expression. The relationship between period and c_x is thus

$$T_n(c_x) = \frac{2\pi}{c_x k_{xn}(c_x)} \ . \tag{69}$$

As the period increases, successive modes disappear, as we previously showed, when $\omega < \omega_{c_n}$ (Figure 6).

What about particle motions for the Love wave?

In the upper medium, we have

$$u_y^{(-)} = B_1 e^{\imath(\omega t - k_x x - k_x r_{\beta_1} z)} + B_2 e^{\imath(\omega t - k_x x + k_x r_{\beta_1} z)} \tag{70}$$

As $B_1 = B_2$ when we have a Love wave, we can write

$$u_y^{(-)} = 2B_1 e^{\imath(\omega t - k_x x)} \cos(k_x r_{\beta_1} z) \ . \tag{71}$$

In the lower half space, we have

$$u_y^{(+)} = B_3 e^{\imath(\omega t - k_x x - k_x r_{\beta_1} z)} \ . \tag{72}$$

As r_{β_2} is pure imaginary for Love waves, we can thus write this as the evanescent disturbance

$$u_y^{(+)} = B_3 e^{\imath(\omega t - k_x x)} e^{-k_x \sqrt{1 - c_x^2/\beta_2^2} z)} \ . \tag{73}$$

Continuity of displacement across the contact at $z = h$ gives the relationship between the amplitudes B_1 and B_3

$$2B_1 e^{\imath(\omega t - k_x x)} \cos(k_x r_{\beta_1} h) = B_3 e^{\imath(\omega t - k_x x} \cdot e^{-k_x \sqrt{1 - c_x^2/\beta_2^2} h)} \ . \tag{74}$$

or

$$B_3 = \frac{2B_1 \cos(k_x r_{\beta_1} h)}{e^{-k_x \sqrt{1 - c_x^2/\beta_2^2} h)}} \tag{75}$$

Aside from the usual propagating term in the $\hat{x}$ direction, we thus have amplitudes that modulate to produce oscillations in the $\hat{z}$ direction in the upper medium, and exponential decay in the lower medium. The n^{th} mode will have n zero crossings (nodes) in the upper medium. Notice that there is also always an antinode (displacement maximum) at the free surface (Figure 7).

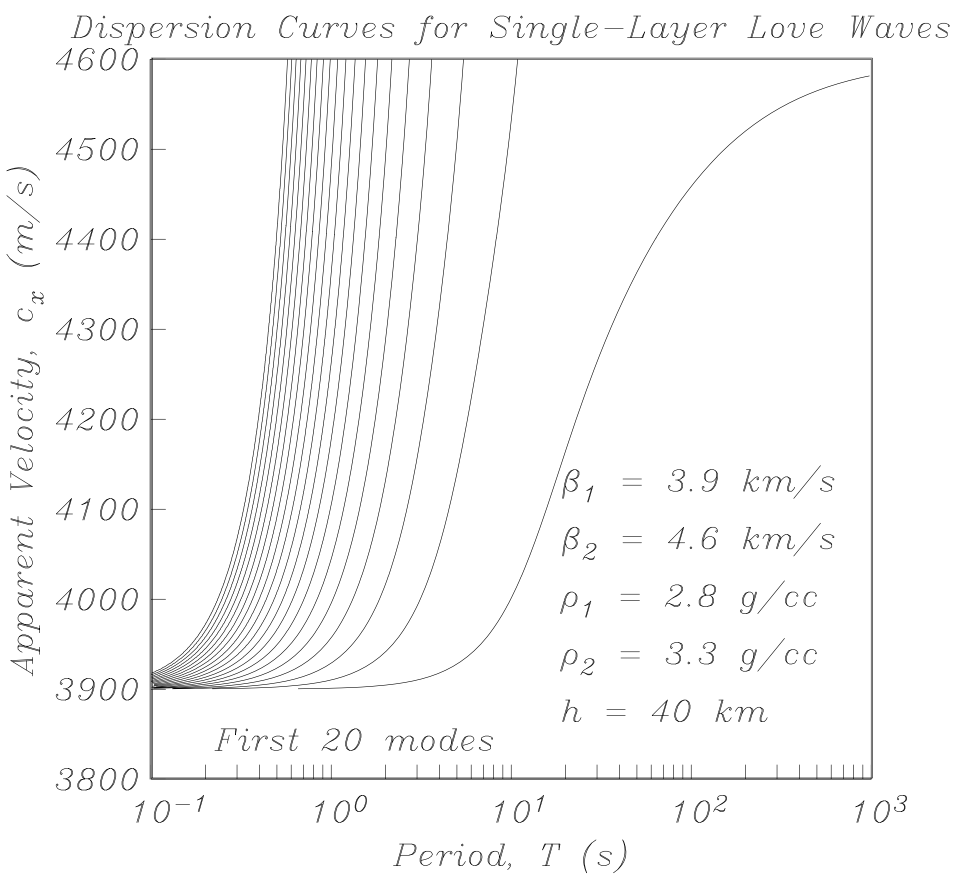

Figure 6: Dispersion curve for Love Waves; Two-Layer Structure

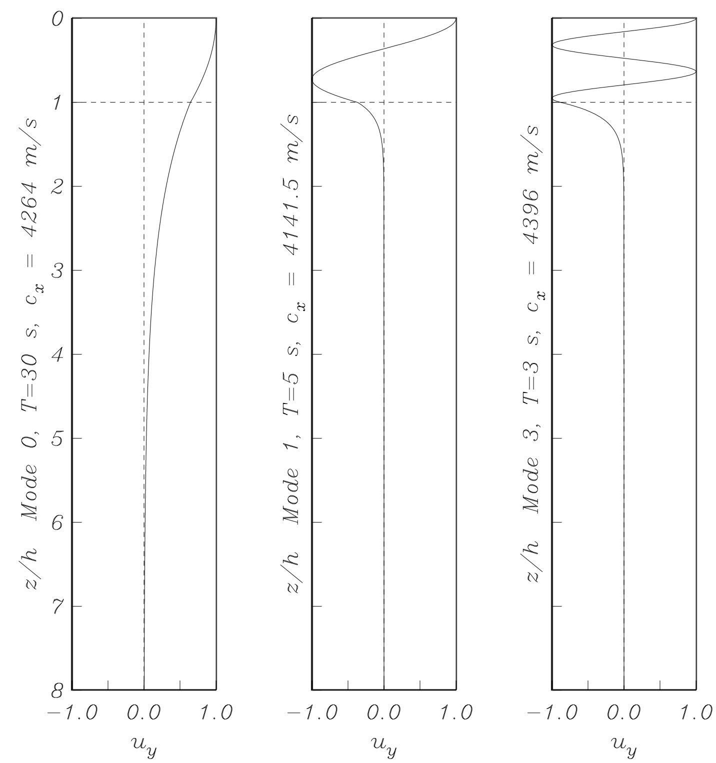

Figure 7: Example Love Wave Particle Motions; Two-Layer Structure