String Modes

Normal Modes & Earth's Free Oscillations

<u>In a nutshell</u>: Normal modes are simply another solution for the wave equation.

Start with wave equation:

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

<u>Known solution so far:</u>

(Generic) $u(x,t) = f(x \pm vt)$

(Special) $u(x,t) = A\cos(\omega t - kx)$

<u>Mode Solution to Wave Equation:</u> A solution that separates out time and space dependence (note: the spatial locations of nodes are independent of time!).

Mode-wave duality----represent traveling wave as a weighted mode sum.

$$u(x,t) = \sum_{0}^{\infty} A_n U_n(x,\omega_n)\cos(\omega_n t)$$

Let's investigate this

$$y(x,t) = A_n U_n(x,\omega_n)\cos(\omega_n t)$$

Plug into Wave Eqn. $$\frac{\partial^2 y(x,t)}{\partial x^2} = A_n U_n''(x,\omega_n)\cos(\omega_n t)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -A_n \omega_n{}^2 U_n(x,\omega_n)\cos(\omega_n t)$$

Wave Eqn.

$$A_n U_n''(x,\omega_n)\cos(\omega_n t) = -A_n \frac{\omega^2}{v^2} U_n(x,\omega_n)\cos(\omega_n t)$$

$$U_n''(x,\omega_n) = -\frac{\omega^2}{v^2} U_n(x,\omega_n)$$

This is an **Eigenvalue problem**, i.e., **A mathematical 'machine' acts on a function ends up a scalar multiply by the function.**

Eigenfunction
(i.e., a guess solution)

$$U_n(x,\omega) = \sin(\omega_n x/v)$$

$$-\frac{\omega_n{}^2}{v^2}\sin(\omega_n x/v) = -\frac{\omega_n{}^2}{v^2} U_n(x,\omega_n)$$

Now, the string example tells us that the length of string L = n half cycles

λ=2L/n

→

ω_n=vk=2πv/λ

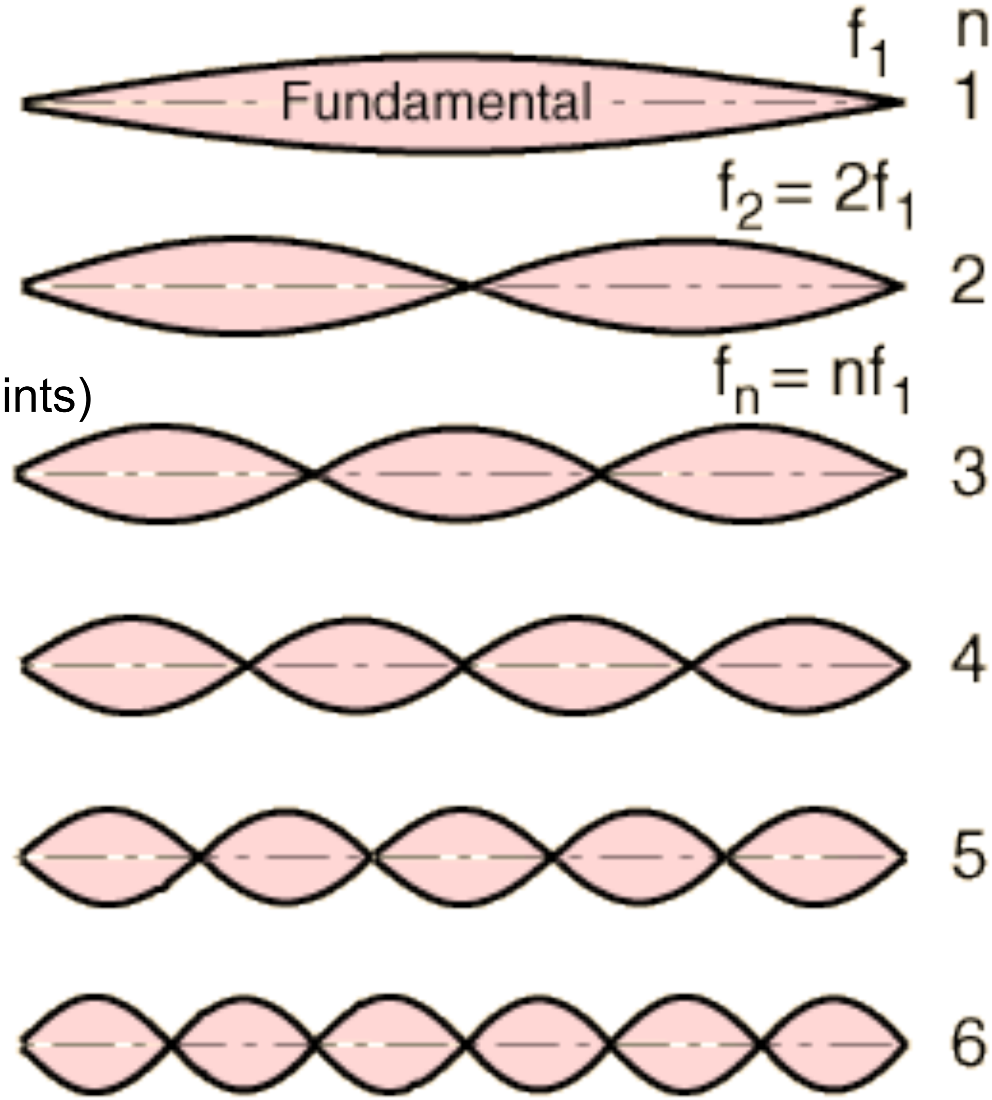
$$U_n = \sin(n\pi x/L)$$

Dependencies of A_n: (1) source characteristics of vibration
(2) source time function

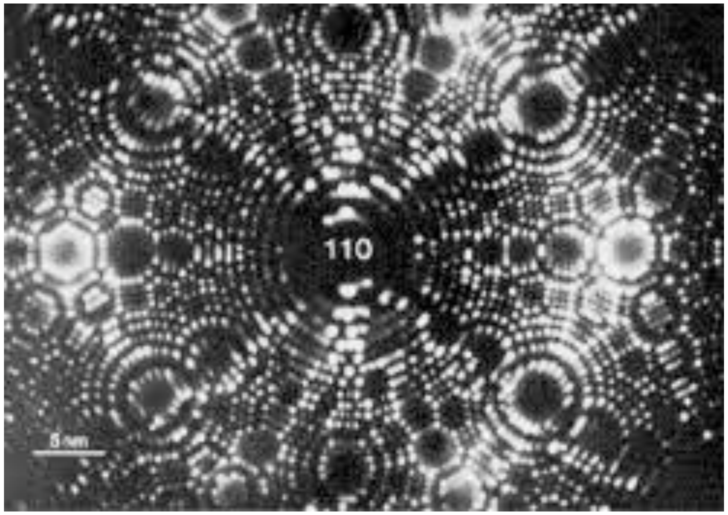
1D string modes

n = mode number (start with 1)
= number of half cycles
= number of peaks

n-1 = number of nodes (stationary points)



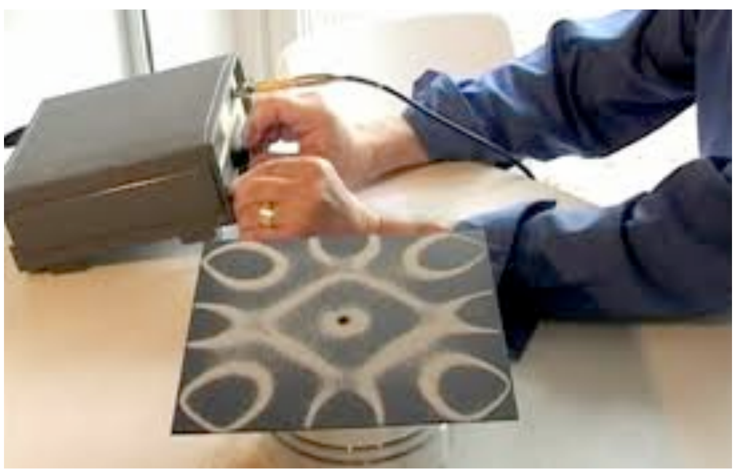
Standing Waves in 2-3 D

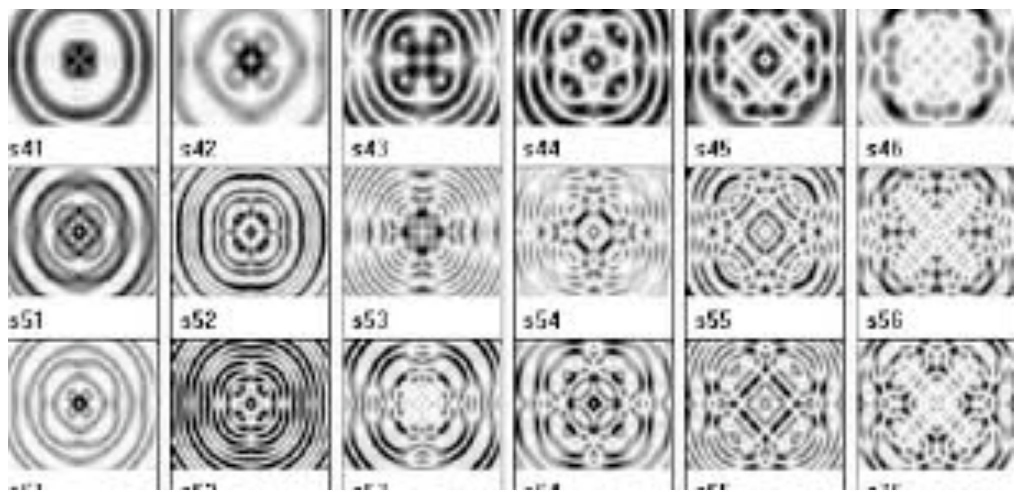


Left: Vertically shaking a cylinder of water with a constant frequency.

Figure taken from http://www.blazelabs.com/f-p-wave.asp

Below: Sand patterns on a “shake table”. Set vibration to a frequency and wait until sand patterns form. Like strings, this requires multiple tries to excite interesting-looking modes.. Mode summation = simulating movements by adding all the small patterns with variable weights.





Normal modes (standing waves) of the earth

The earth "rings like a bell" after a big earthquake. This ringing will often last for days, even weeks. This process can be called an excitation of Earth's *normal modes*, or *free oscillations*.

Earthquake

$$\mathbf{u}(r,\ \theta,\ \phi) = \sum_{\mathrm{n}} \sum_{\mathrm{l}} \sum_{m} {}_nA_l^m \ \underbrace{{}_{\mathrm{n}}\mathrm{f}_{\mathrm{l}}(\mathrm{r})\ \mathbf{x}_{\mathrm{l}}^{\mathrm{m}}(\theta,\ \phi)}_{\mathrm{U}}\ e^{i\,{}_n\omega_l^m t}$$

${}_nA_l^m$ = weights of eigenfunctions, depends on source

${}_{\mathrm{n}}\mathrm{f}_{\mathrm{l}}(\mathrm{r})$ = radial eigenfunction; shows variation with depth

$\mathrm{x}_{\mathrm{l}}^{\mathrm{m}}(\theta,\ \phi)$ = surface vector eigenfunction

${}_n\omega_l^m$ = eigenfrequency, m is not needed for laterally homogeneous earth

The surface eigenfunctions are related to **spherical harmonics:**

$$Y_l^m(\theta,\phi) = (-1)^m \left[\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta) e^{im\phi}$$

Y_l^m is complex

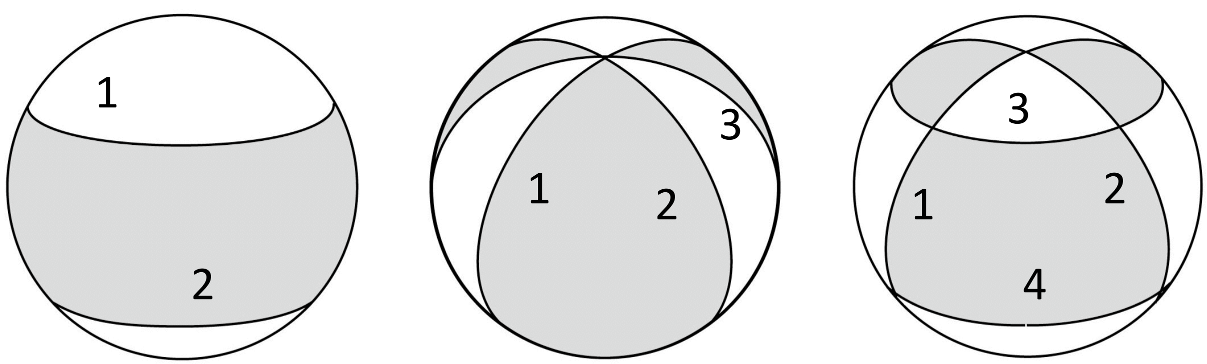
where P_l^m is **associated Legendre Polynomial**

$$P_l^m(x) = \frac{(1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

and the **azimuthal order**, **m**, varies over $-l \le m \le l$ (degeneracy)

Symmetric spherical harmonics

Figure 2.9-4: Examples of spherical harmonics.



Y_2^0 — zonal

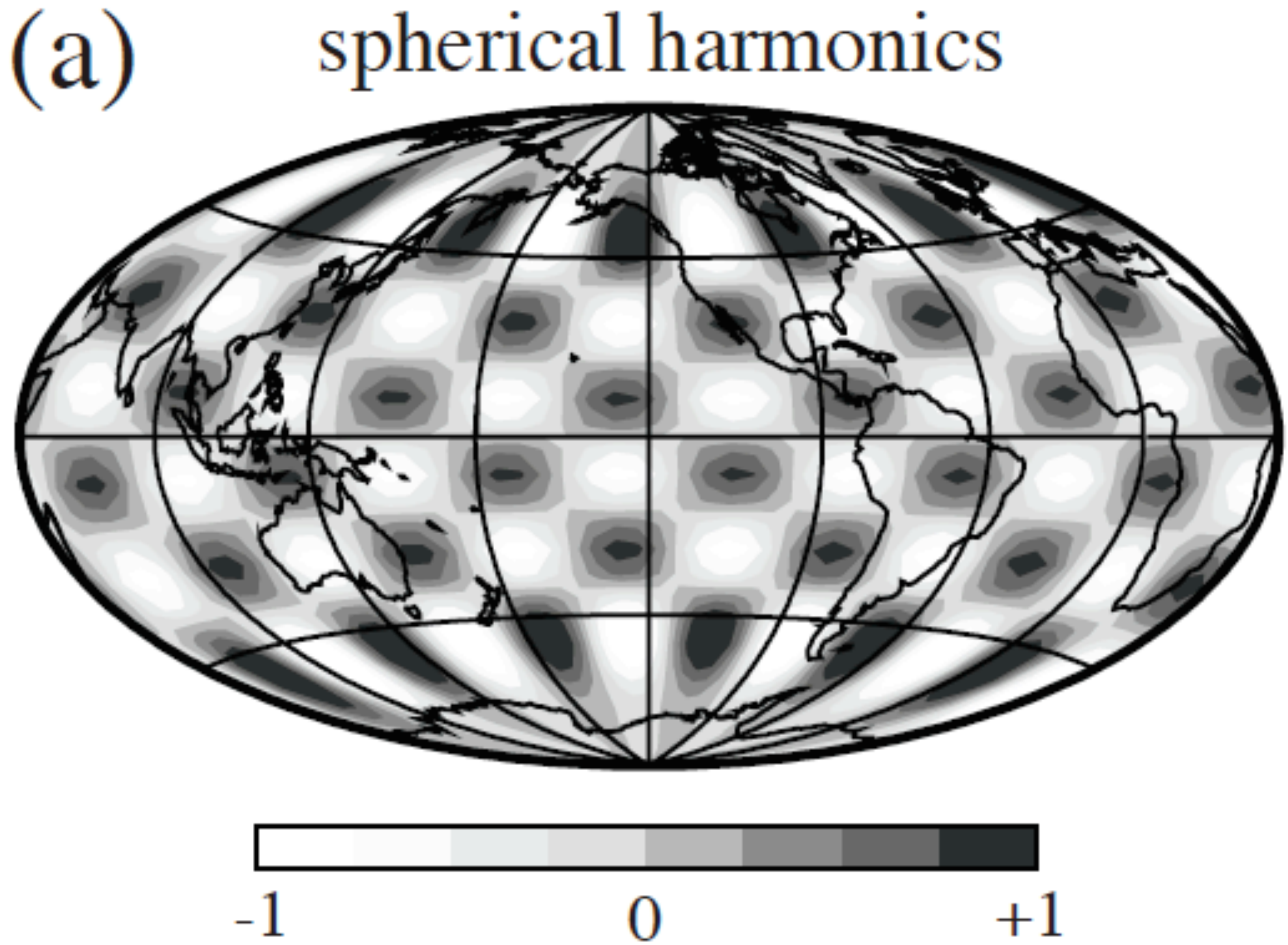
$\mathrm{Re}(Y_3^3)$ — sectoral

$\mathrm{Re}(Y_4^2)$ — tesseral

Angular order *l* gives number of nodal lines

m=0 ---> small circle about pole

l=m ---> goes through pole



L = 12 example

This was the resolution limit of seismic tomography in the early 1990s

Spherical harmonics give a nice set of basis functions that are Orthogonal (linearly independent) that spans the spherical surface, Precisely the reason why it is so useful in earth sciences.

Orthogonal because

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta Y_{l'}^{m'*}(\theta,\phi) Y_l^m(\theta,\phi)\, d\theta d\phi = \delta_{l'l}\delta_{m'm}$$

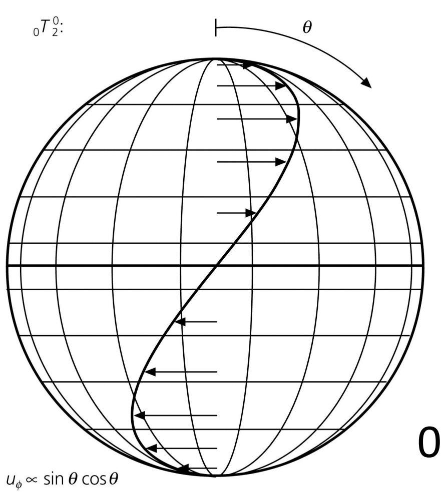
Two types of Normal Modes: (1) Toroidal Mode (SH waves)
(2) Spheroidal Mode (P-SV motion)

Toroidal Modes:

$$\mathbf{x}_l^m = \mathbf{T}_l^m = \left(0,\ \frac{1}{\sin\theta}\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi},\ \frac{-\partial Y_l^m(\theta,\phi)}{\partial\theta}\right)$$

This is where spherical harm is tied to modes

Figure 2.9-5: Displacement associated with torsional mode $_0T_2$.



0 T 2

For **u=(u$_r$, u$_\theta$, u$_\phi$)**

$$\mathbf{u}^T(r,\ \theta,\ \phi) = \sum_n \sum_l \sum_m {}_nA_l^m\ {}_n\mathrm{f}_l(\mathrm{r})\ \mathbf{T}_l^m(\theta,\ \phi) e^{i_n\omega_l^m t}$$

Motions are purely SH,

n = radial order

l = angular order

m = azimuthal order

Spheroidal modes: normal modes that describe the P-SV motions in the earth.

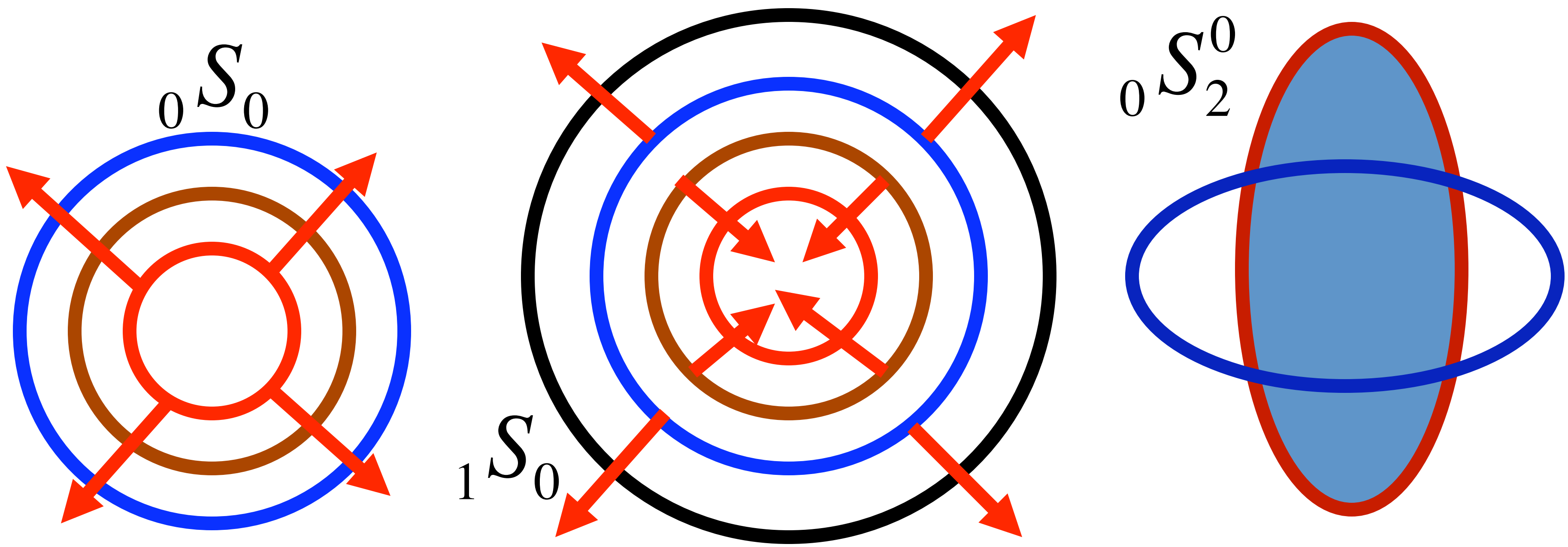
$$\mathbf{u}^S(r,\ \theta,\ \phi) = \sum_{n} \sum_{l} \sum_{m=-l \text{ to } l} {}_nA_l^m \left[{}_n\mathrm{U}_l(\mathrm{r})\ \mathbf{R}_l^m(\theta,\ \phi) + {}_n\mathrm{V}_l(\mathrm{r})\ \mathbf{S}_l^m(\theta,\ \phi) \right] e^{i\,{}_n\omega_l^m t}$$

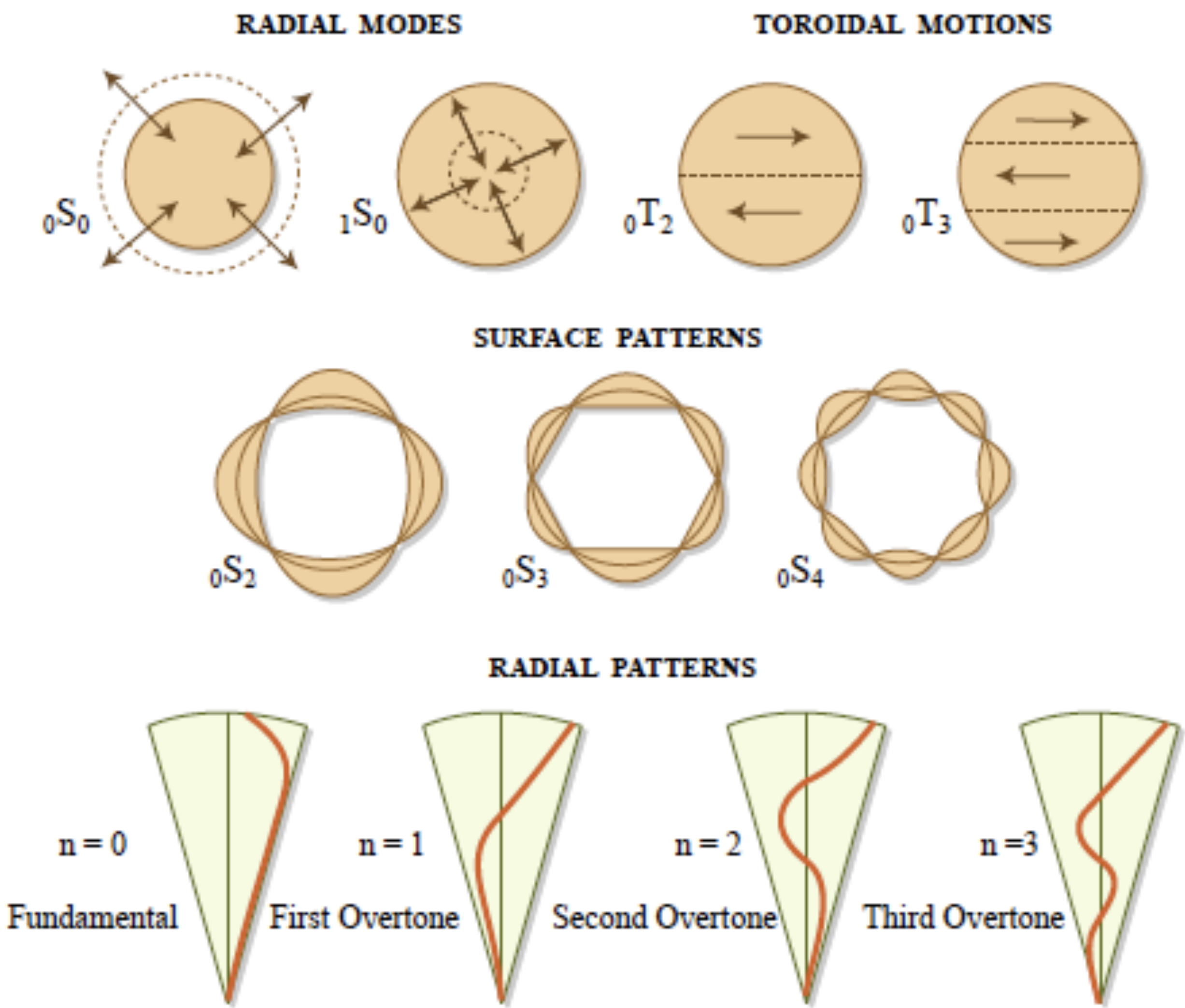
Where,

$$\mathbf{R}_l^m = \left(Y_l^m(\theta,\phi),\ 0,0 \right)$$ (radial motion)

$$\mathbf{S}_l^m = \left(0,\ \frac{\partial Y_l^m(\theta,\phi)}{\partial \phi},\ \frac{1}{\sin\theta} \frac{\partial Y_l^m(\theta,\phi)}{\partial \theta} \right)$$ (horizontal)

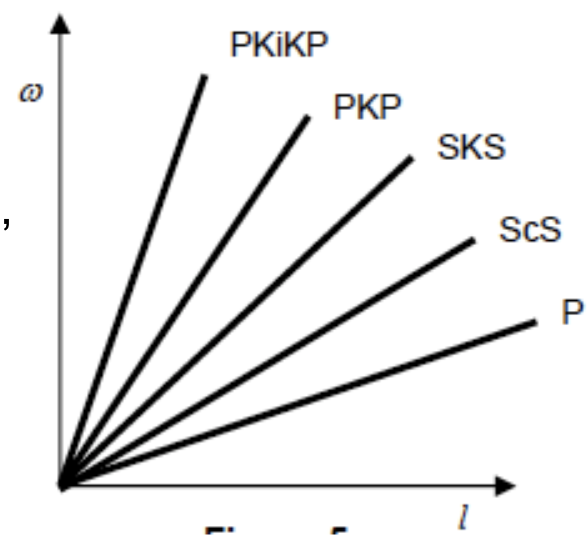
l=number of surface nodel lines, *l*=0 is radial mode.





These are simple modes in 3D. The fundamental mode has no zero-crossing in amplitude (note: looks a bit like sensitivity function of surface waves!). In fact, normal modes and surface wave modes are related.

Mode Ray Duality and ω-l diagram (similar in to f-k diagram in 1D)

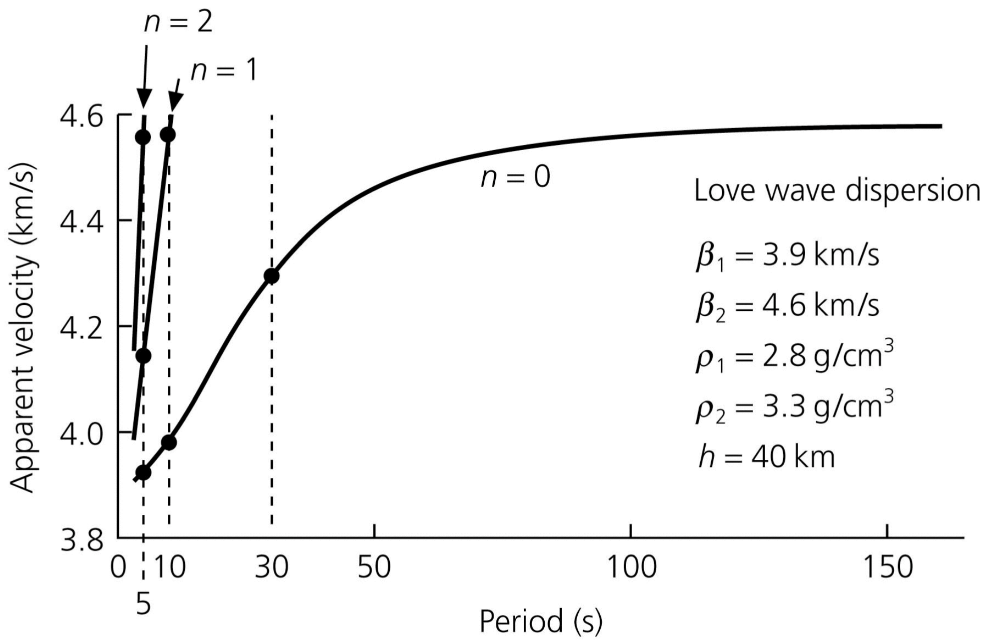


Right picture: Note that the deeper the phase goes, the higher the frequency and the smaller the L.

Check out animation at (REALLY COOL!!)

http://icb.u-bourgogne.fr/nano/manapi/saviot/terre/index.en.html

Figure 2.7-9: Dispersion curves for Love waves in a layer over a halfspace.



Mode Wave Duality (all observable surface wave modes)

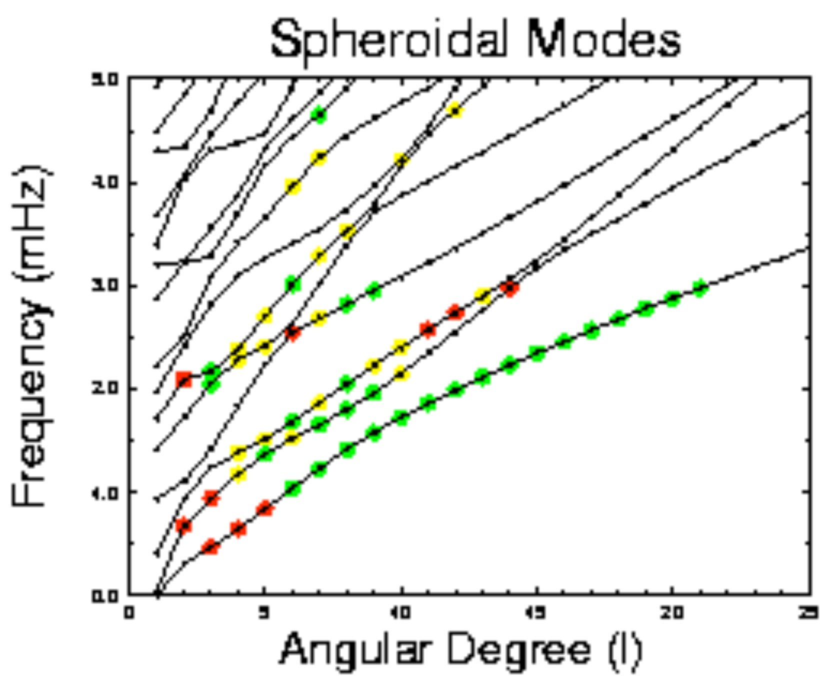
$$\lambda_x = 2\pi / |\mathbf{k}_x| = 2\pi a / (l + 1/2)$$

$$c_x = {}_n\omega_l / |\mathbf{k}_x| = {}_n\omega_l a / (l + 1/2)$$

$$(a = \text{earth radius})$$

Top Left: Surface wave mode branch (note how broad fundamental mode is, precisely why it is most useful observationally)

Free oscillations



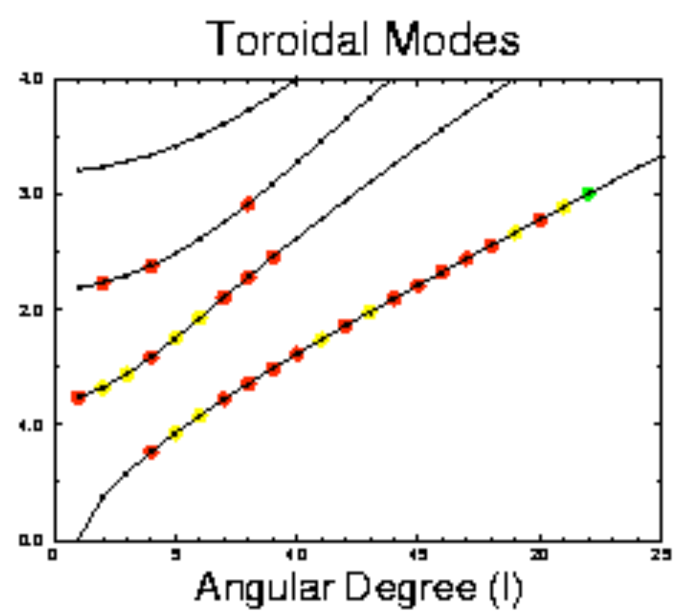
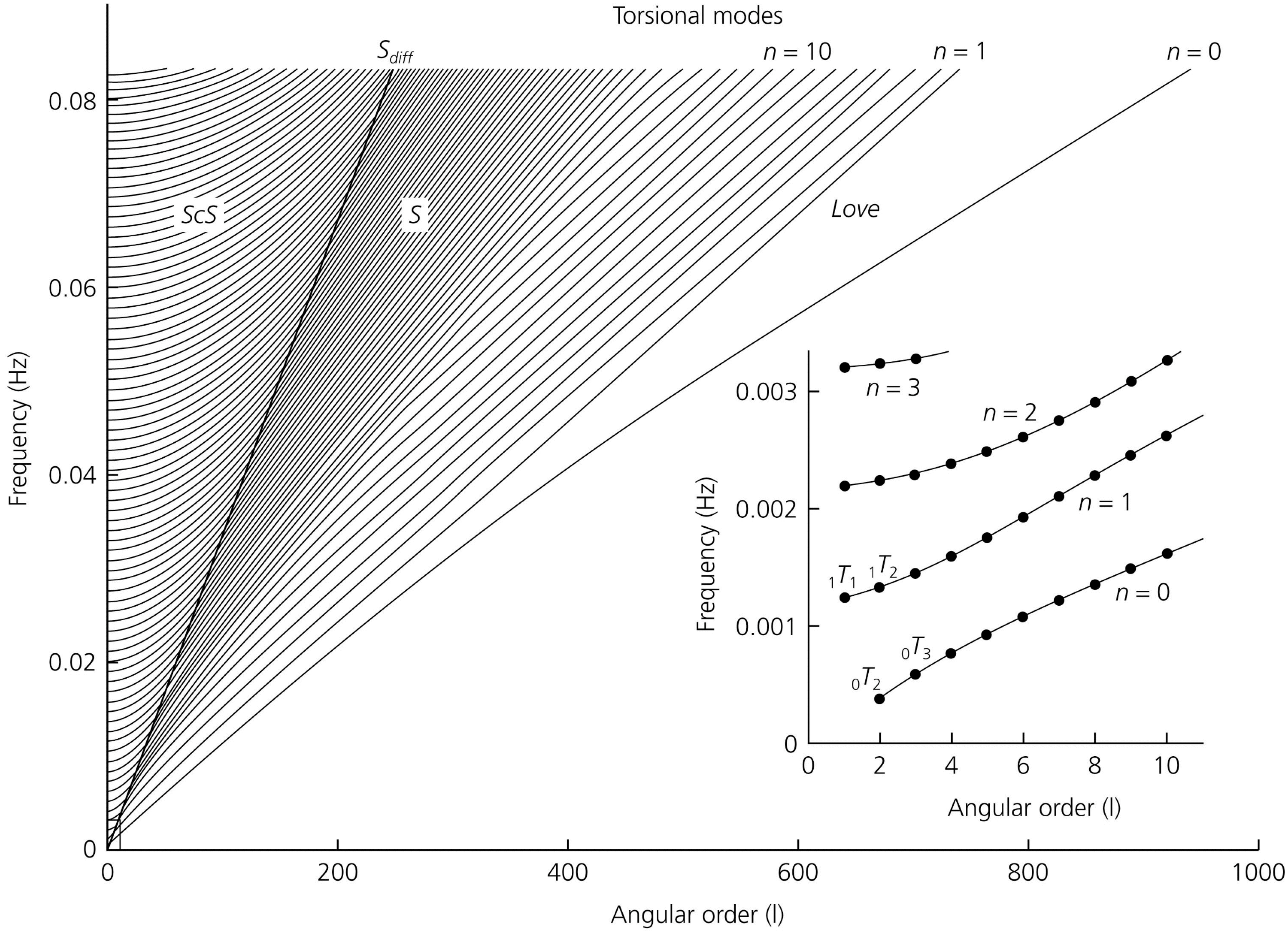
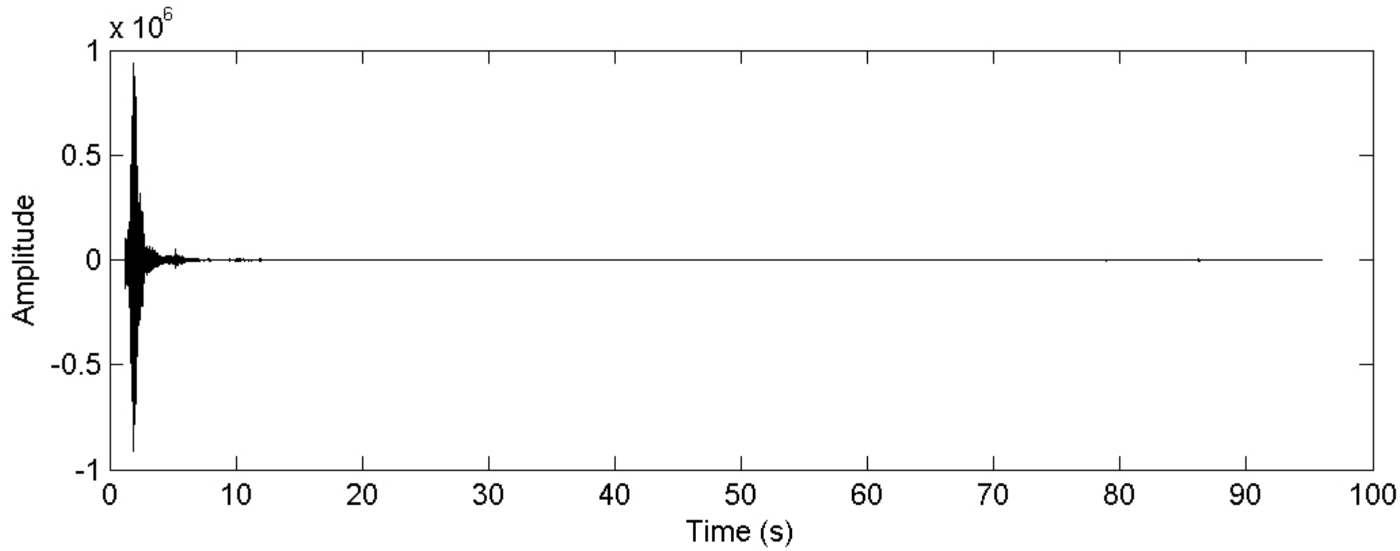
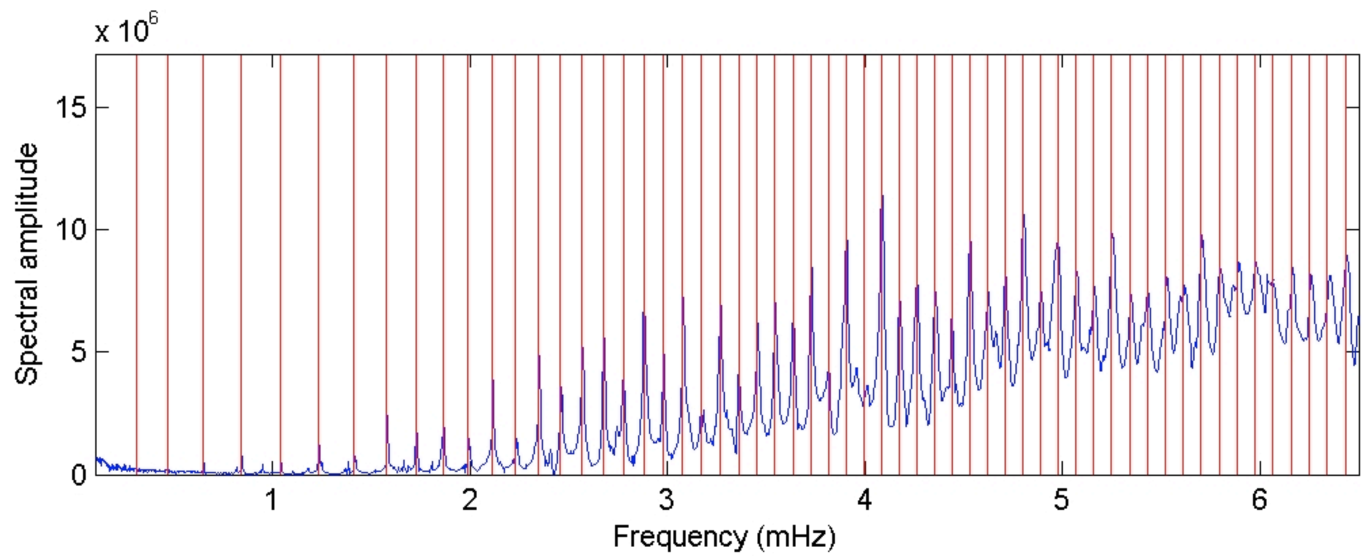


Figure 2.9-10: Dispersion plot for torsional modes.

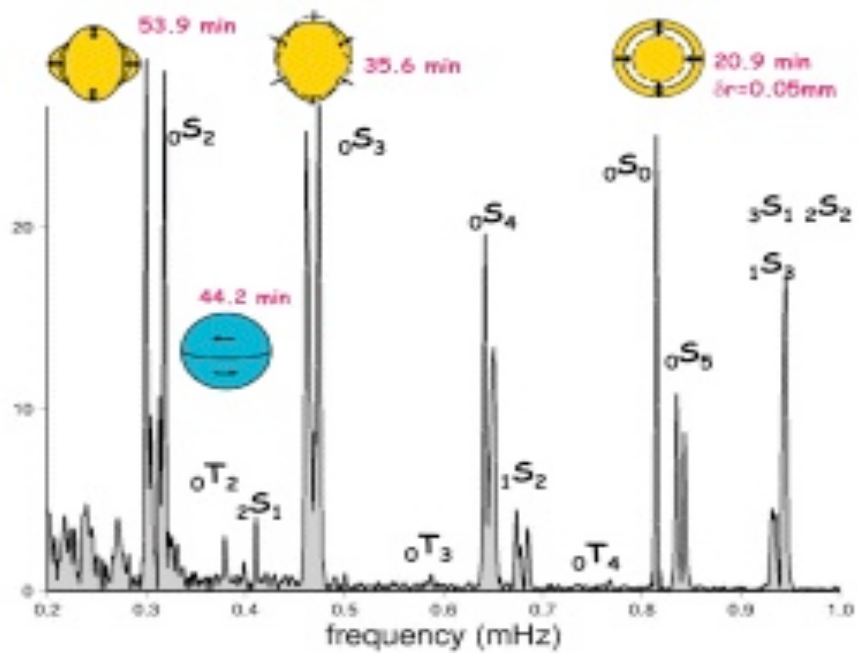


Sumatra Earthquake Normal Mode Observations (Dec 26, 2004)





Mode Observations & Usage

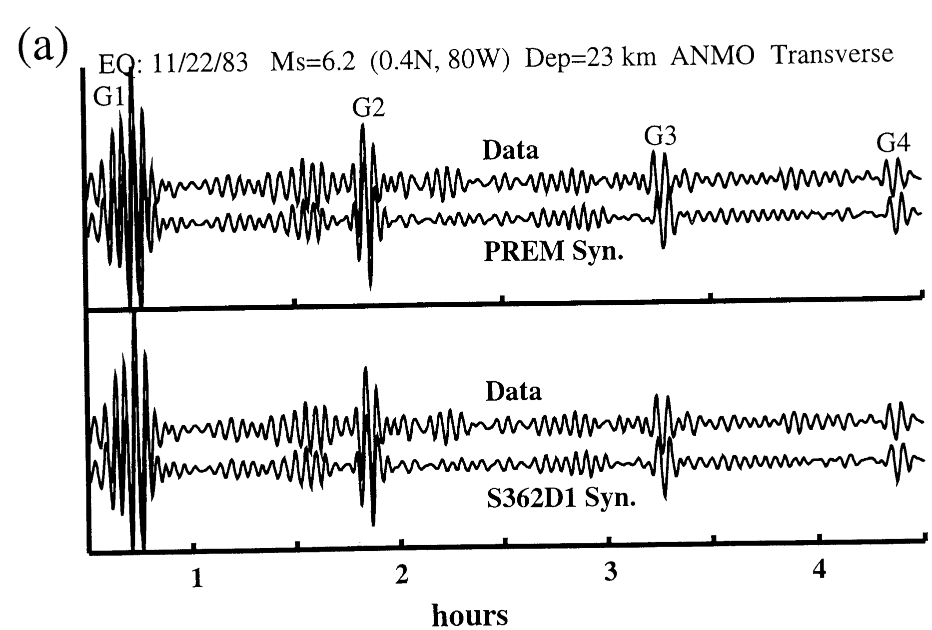


Top: Mode observations by Fourier transforms of the time-domain recordings. Modes are usually very long periods and only comes in certain discrete frequencies.

Bottom plot: Mode Summations in the simulation of Love waves. This differs from a Reflectivity Method approach, but get the same results for simulating traveling waves.

(Pro: Very fast if a mode catalogue exists, just pick and add.
Con: Mode catalogue is HUGE for high-frequency simulations!

GU ET AL.: DISCONTINUITY IN THE PATTERN OF HETEROGENEITI



Singlets and multiplets:

Each azimuthal order represents a spectral Peak. Without lateral velocity and density variations, also without consideration of earth's ellipticity, they all should come at the same frequency. The fact they all have the *Same frequency* is called "***degeneracy***" of mode.

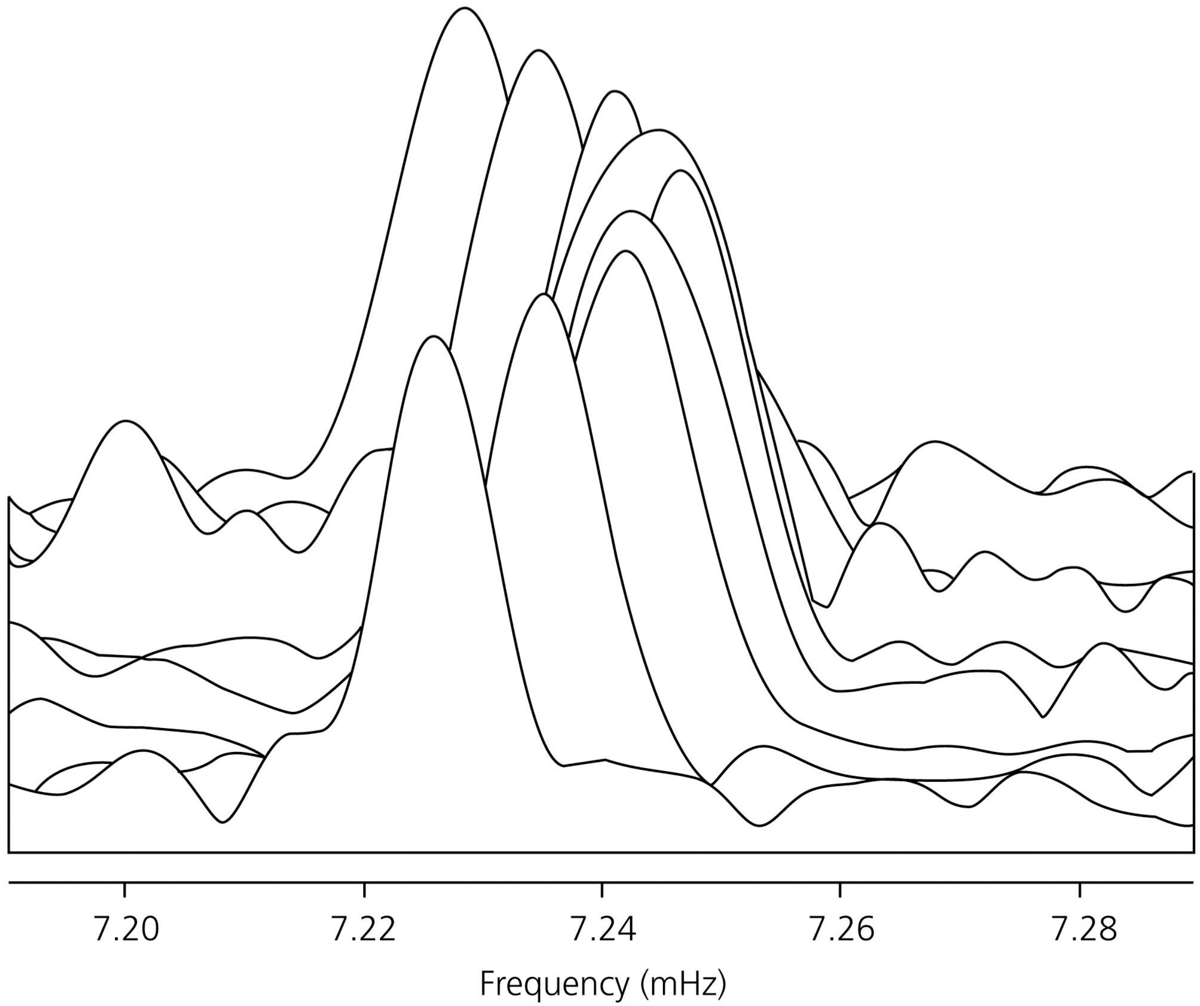
Each of them is called a "**Singlet**".

Their superposition forms one spec

But with those above factors preser

"***mode splitting***".

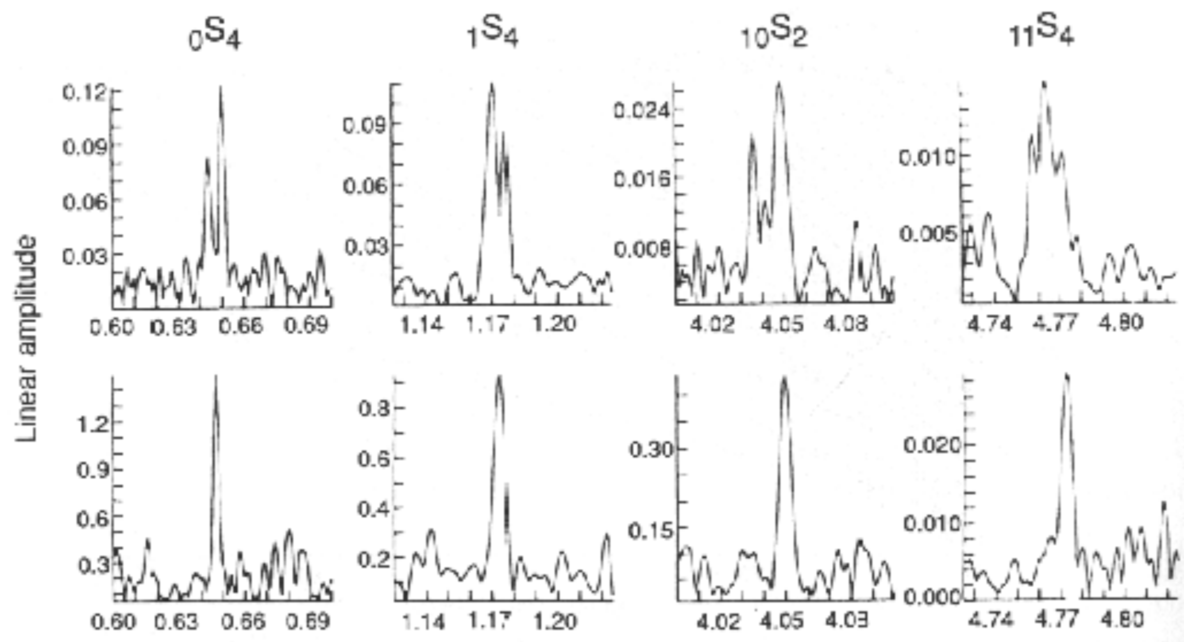
Figure 2.9-15: Example of singlets for a split spheroidal mode multiplet.



Earth's Rotation Effect

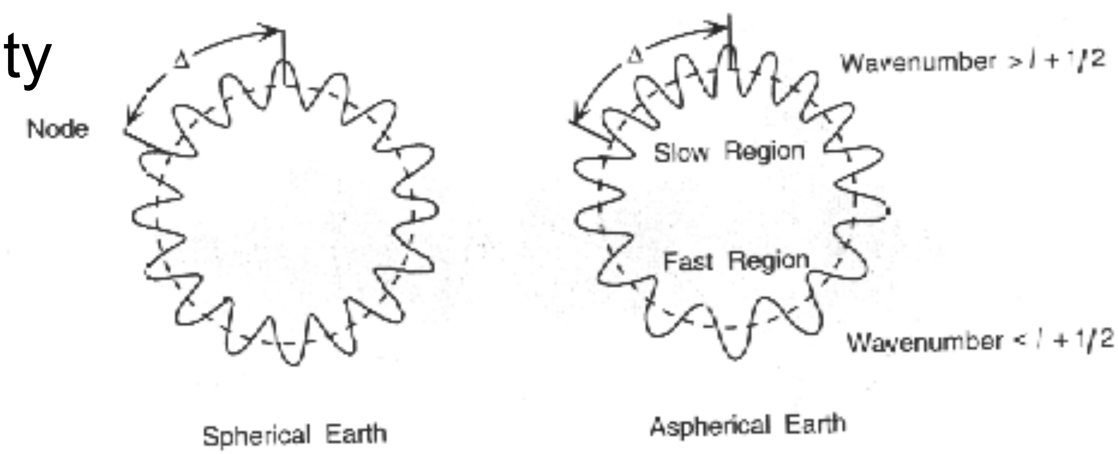
Non-polar latitude

Polar latitude



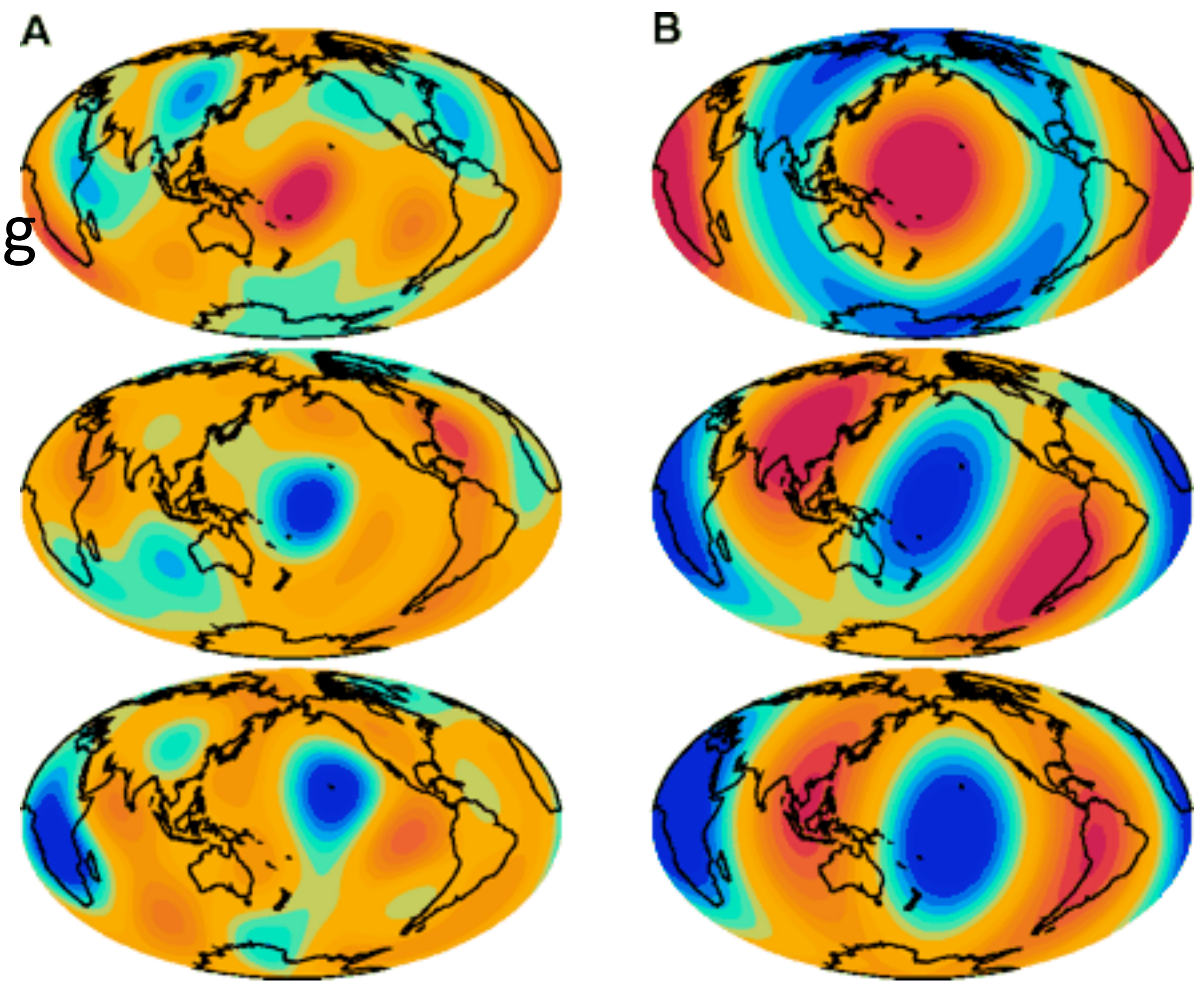
Rotational effect: more significant at longer periods, high speeds, and is larger than effect of heterogeneity.

Effect of Heterogeneity



Shear and Bulk-sound speeds of the Core-Mantle-Boundary $v\_bs = \sqrt{\frac{k}{\rho}}$

This is a density map obtained from measurements of normal mode splitting



Good thing about Normal modes:

Depends on the particle Motions with depth (radial eigenfunctions), Some modes have motions as deep as the core. That Gives us information about the "Very Deep Earth".

$$\delta\omega = \int_V (\delta\alpha K_\alpha + \delta_\beta K_\beta + \delta\rho K_\rho) dV$$

Summary of Surface Wave and Modes

Rayleigh waves are solutions to the elastic wave equation given a half space and a free surface. Their amplitude decays exponentially with depth. The particle motion is elliptical (larger vertical than horizontal, retrograde near surface) and consists of motion in the plane through source and receiver.

SH-type surface waves do not exist in a half space. However in layered media, particularly if there is a low-velocity surface layer.
Love waves exist which are dispersive, propagate along the surface. Their amplitude also decays exponentially with depth.

Free oscillations are standing waves which form after big earthquakes inside the Earth (useful ones are Mw 8-9.5, unfortunately).
Spheroidal and toroidal eigenmodes correspond are analogous concepts to P and shear waves. Modes sum up to produce both body waves and surface waves (*not the other way round*).