Strings and 1D wave equation

Important Concepts/Assumptions:

(1) **Taylor expansion**

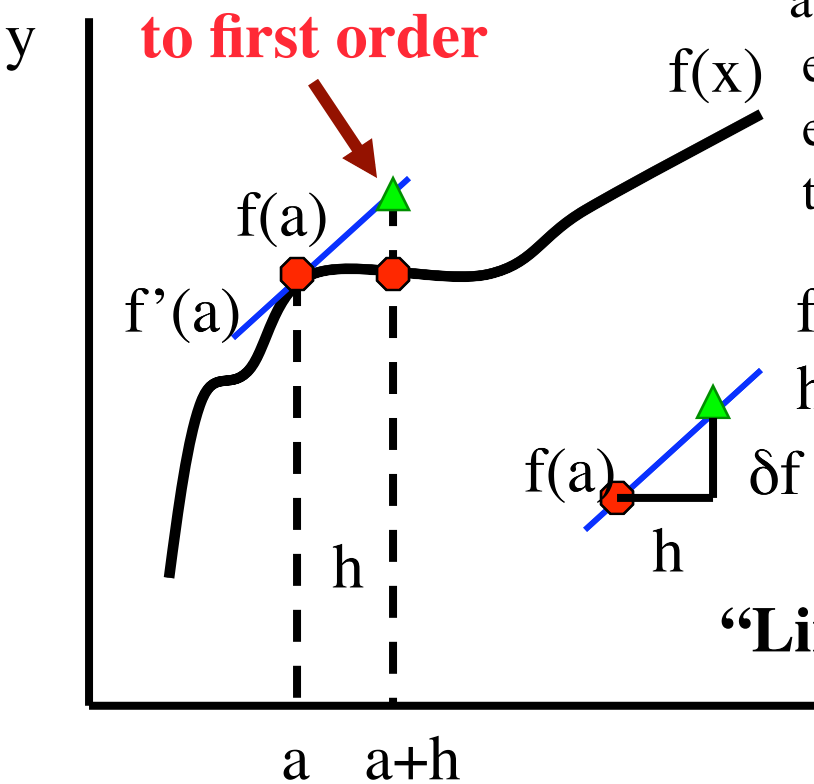
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \mathcal{O}(h^4).$$

Accurcy ∝ function smoothness + #terms

What it is:

An approximation for the function f at a given value (a+h) from function f(a). For example, see the left fig for expansions up to second term.

Expansion f(a+h) to first order



f'(a)=δf/h

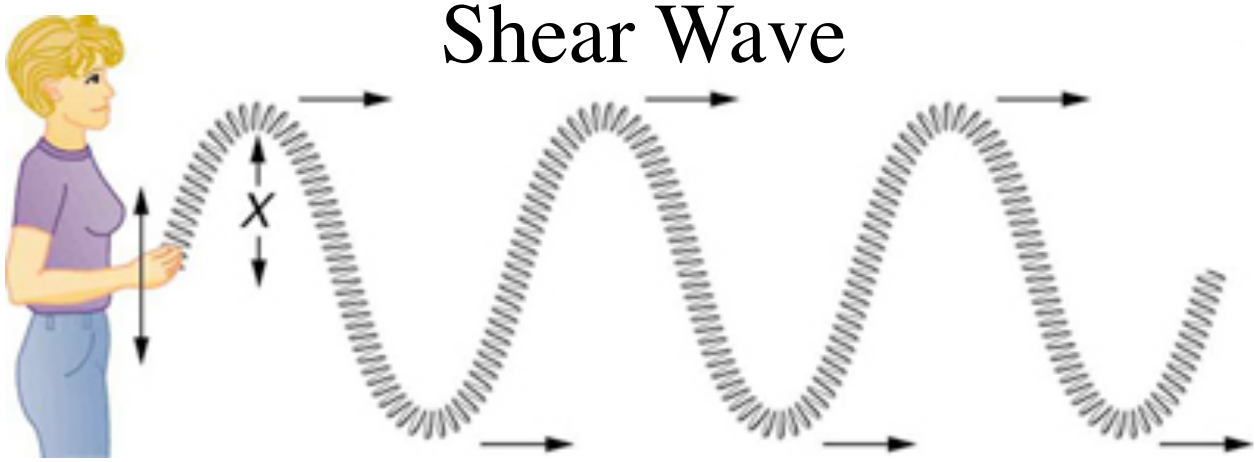
h*f'(a)=δf

"Linear Approximation"

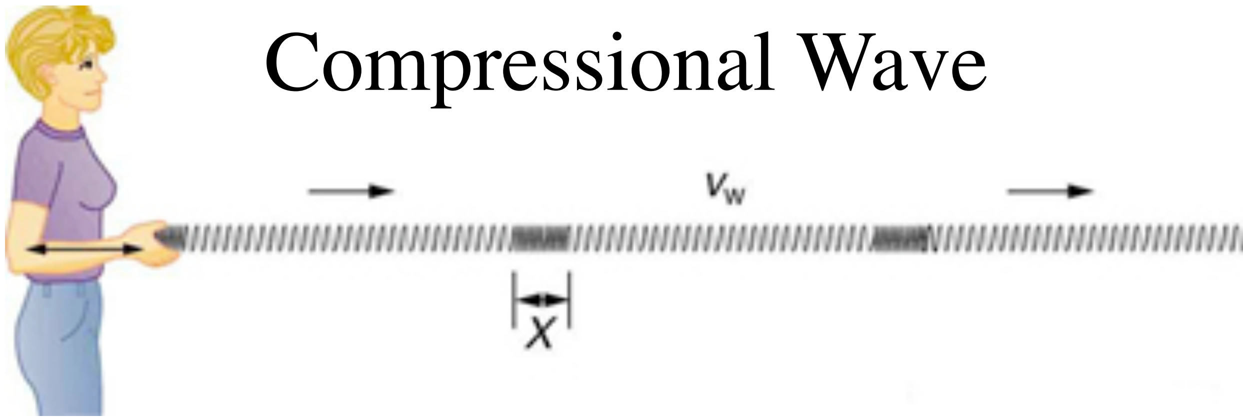
(2) Continuum: continuous distribution of material in space.

(3) Lagrangian description: follows a particle and "feel" the action, perfect for seismic instrument that senses particle motion!

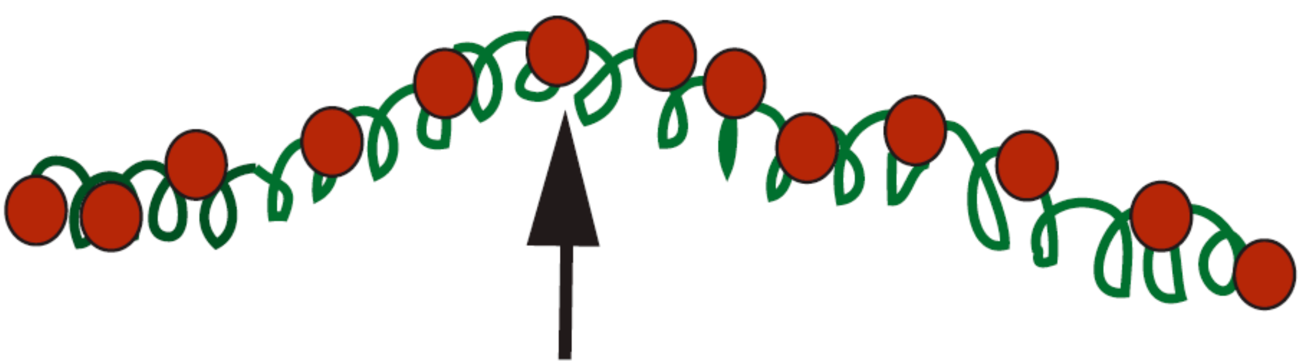
Shear Wave



Compressional Wave



Particle motion of shear wave



Simple Proof of Wave Equation :

An idealized mathematical string extends in the x direction, initially straight with tension force **τ** along it.

Initial condition: ***u***(x, t=0) =0 in y direction, where u= displacement. Now plucking the string.

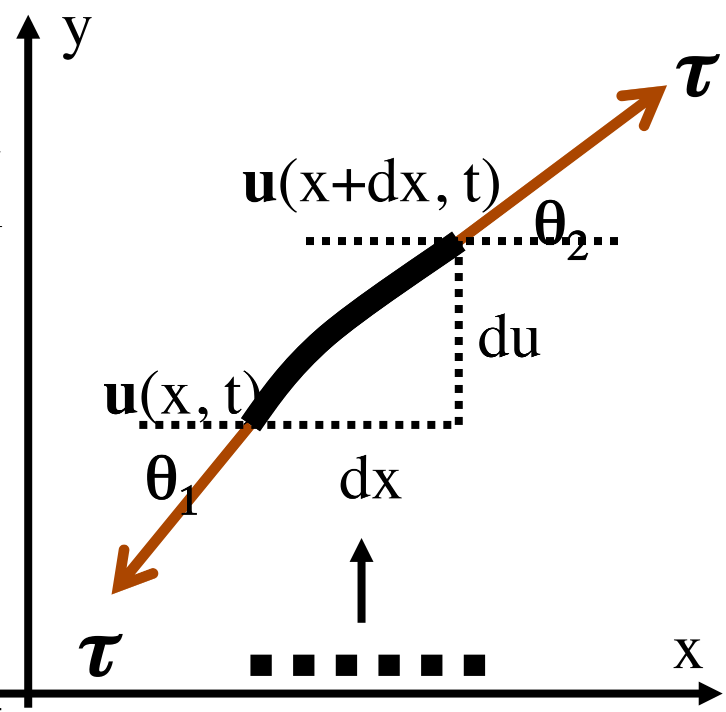
Problem: **Describe the displacement *u*(x, t)**

Solution:

Condition: **u** is vertical displacement, dx is small

Vertical tension along y:

$$\tau \sin\theta_2 - \tau \sin\theta_1$$



Newton II:

$$F = ma = \rho dx \frac{\partial^2 u}{\partial t^2}$$

$$\tau \sin\theta_2 - \tau \sin\theta_1 = \rho dx \frac{\partial^2 u}{\partial t^2}$$

Taylor expansion about 0:

$\sin(\theta)=\theta-\theta^3/3!+\theta^5/5!-\ldots\sim\theta$ (coef = 0 1 0 -1 …)

$\tan(\theta)= \theta+\theta^3/3+2\theta^5/15 \ldots\sim\theta$

Hence $\sin\theta=\tan\theta=\theta$ Another way to derive sin(x)=tan(x)

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$ (used L'Hopitals rule)

$$\lim_{x\to o}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sec^2 x}{1} = \lim_{x\to o}\frac{1/\cos^2 x}{1} = 1$$

3

Tension along y = $\tau \sin\theta_2 - \tau \sin\theta_1$

----> $\tau(\tan\theta_2 - \tan\theta_1) = \rho dx \frac{\partial^2 u}{\partial t^2}$

$$\tau\left(\frac{\partial u(x+dx,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) = \rho dx \frac{\partial^2 u(x,t)}{\partial t^2}$$

Taylor expand u(x+dx, t) with respect to x, remove higher order terms,

$$u(x+dx,t) = u(x,t) + \frac{\partial u(x,t)}{\partial x} dx + \frac{\partial u^2(x,t)}{2\partial x^2} dx^2 + \ldots$$

$$\approx u(x,t) + \frac{\partial u(x,t)}{\partial x} dx$$

$$\tau\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} dx - \frac{\partial u(x,t)}{\partial x}\right) = \rho dx \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\tau \frac{\partial^2 u(x,t)}{\partial x^2} dx = \rho dx \frac{\partial^2 u(x,t)}{\partial t^2}$$

remove dx terms, reorganize,

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{(\sqrt{\tau/\rho})^2} \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 u(x,t)}{\partial t^2}$$

where $v = \sqrt{\tau/\rho}$

or commonly written as a homogeneous solution,

$$\nabla^2 u - \frac{1}{v^2} \partial^2{}_t u = 0$$

Wave equation!

Note

$v = \sqrt{\tau/\rho}$ ----> tension increases, v increases (**in x**)

density increases, velocity decreases

To verify the formula, can work out units,

v = (force/density)$^{1/2}$= [(density*m*m/s^2)/density]$^{1/2}$=m/s

Rays and Body Waves

Wave equation: A second-order differential equation that describes a how a wave propagates in a medium with velocity v. In 1D, lets assume u is the displacement of a particle at a given time.

$$\nabla^2 u - \frac{1}{v^2}\partial^2{}_t u = 0$$

Note: this is a relationship between 2nd time derivative of displacement/potential with 2nd spatial derivative. time *u* and *v* do not have to be in the same direction,

Solutions:

Any function of the type $u(x,t) = f(x \pm vt)$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = f''(x \pm vt) \quad \text{and} \quad \frac{\partial^2 u(x,t)}{\partial t^2} = v^2 f''(x \pm vt)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

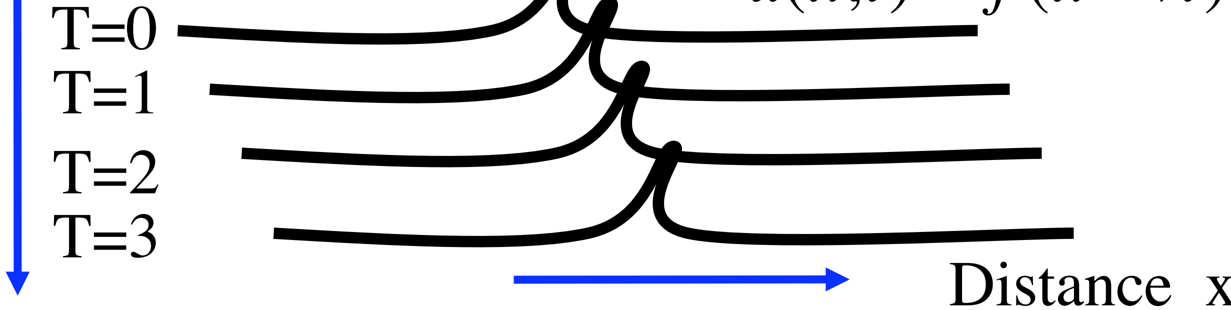
Example: $f(x, t) = 2(x-vt)^3$

$$\frac{\partial u(x,t)}{\partial x} = 6(x - vt)^2 \qquad \frac{\partial u^2(x,t)}{\partial x^2} = 12(x - vt)$$

$$\frac{\partial u(x,t)}{\partial t} = -6v(x - vt)^2 \qquad \frac{\partial u^2(x,t)}{\partial t^2} = 12v^2(x - vt)$$

In fact, "-" shows a pulse moving in the positive x direction.

$$u(x,t) = f(x - vt)$$



A particularly useful class of solutions:

$$u(x,t) = Ae^{i(\omega t \pm kx)} = A[\cos(\omega t \pm kx) + i\sin(\omega t \pm kx)]$$

A is amplitude, real part of the exponent is phase that describes the current "state" within its cycle, i.e., if a cycle is 360 degrees, the current angle.

Now differentiate:

$$\frac{\partial^2 u}{\partial x^2} = Ak^2 e^{i(\omega t \pm kx)} \quad \text{and} \quad \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = A\frac{\omega^2}{v^2}e^{i(\omega t \pm kx)}$$

Equate them, $k^2 = \frac{\omega^2}{v^2} \Rightarrow k = \frac{\omega}{v}$

k = wave number, v = speed, ω=angular frequency

To be physical, we usually use

$$u(x,t) = A\cos(\omega t - kx) = \text{real}[Ae^{i(\omega t - kx)}]$$

(since displacement is real, "-" is harmonic wave moving in positive x direction)

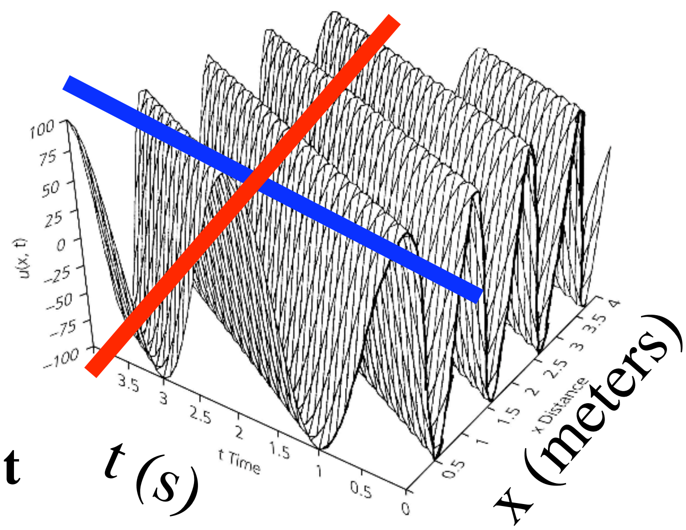
Displacement field (1D)

A cut through time

A cut through space

Figure 2.2-3: Displacement field of $u = A\cos(\pi t - 2\pi x)$.

Assume constant values of w and k, u will look like ----->



<u>Note</u>: the vertical cut in x or t will yield a periodic function

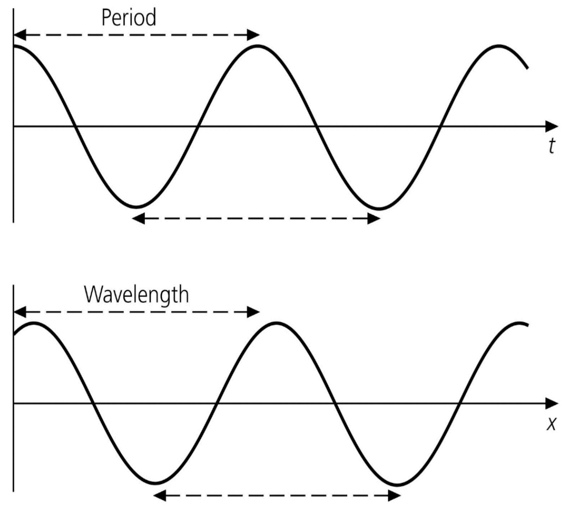
Periodicity

$$T = 2\pi / \omega$$

$$\lambda = 2\pi / k$$

$$v = \omega / k = \lambda / T$$

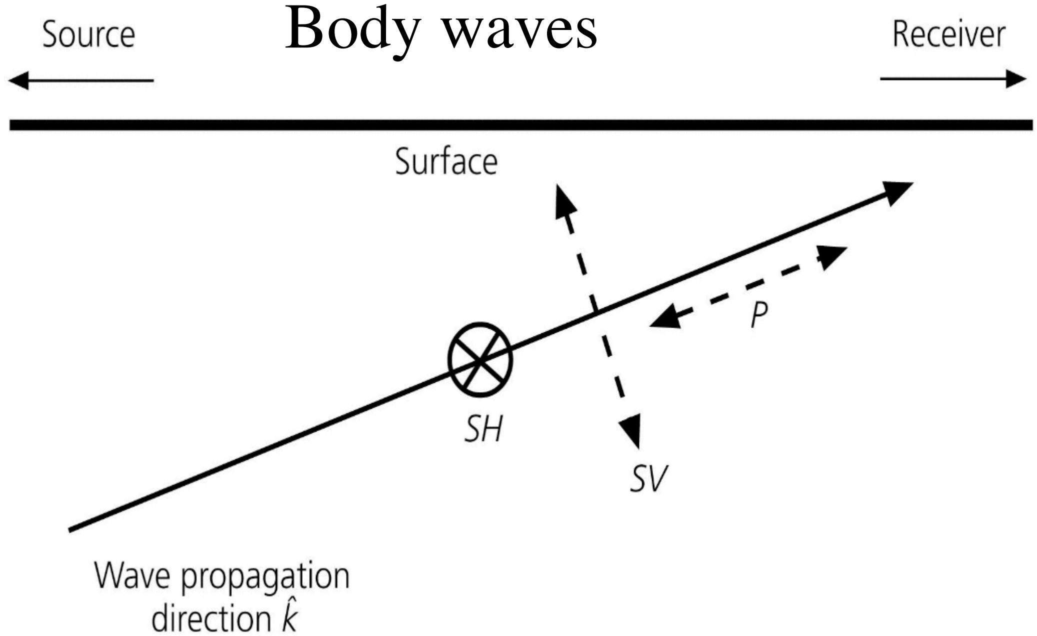
Figure 2.2-4: Harmonic wave, $u = A \cos(\omega t - kx)$.



Note: going from one medium to another, T is almost constant, wavelength (which depends on k) will change!

Figure 2.4-4: Displacements for *P*, *SV*, and *SH*.

Body waves



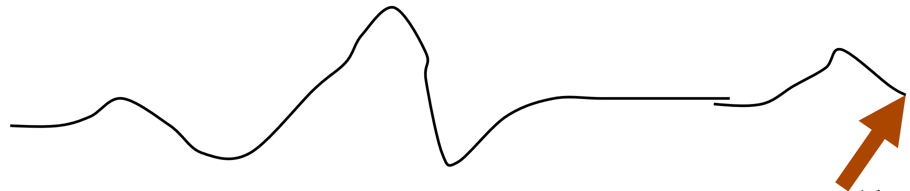
SH waves: transverse particle motion, normal to P-SV plane

SV waves: normal to wave propagation direction, within the P-SV (blackboard) plane

P waves: parallel to wave propagation direction

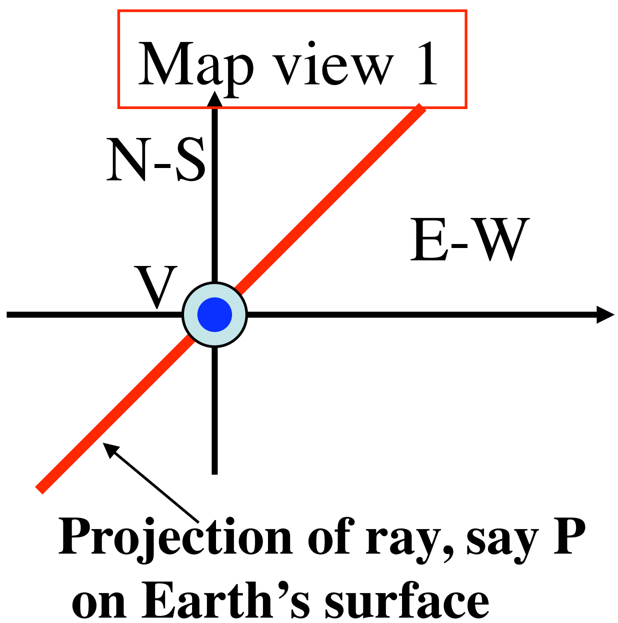
Components and Particle Motion

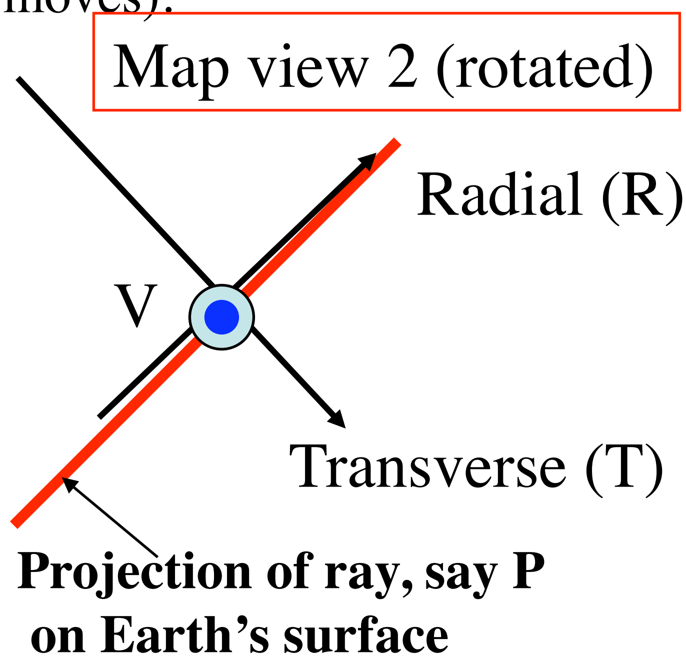
Imagine a seismometer needle:

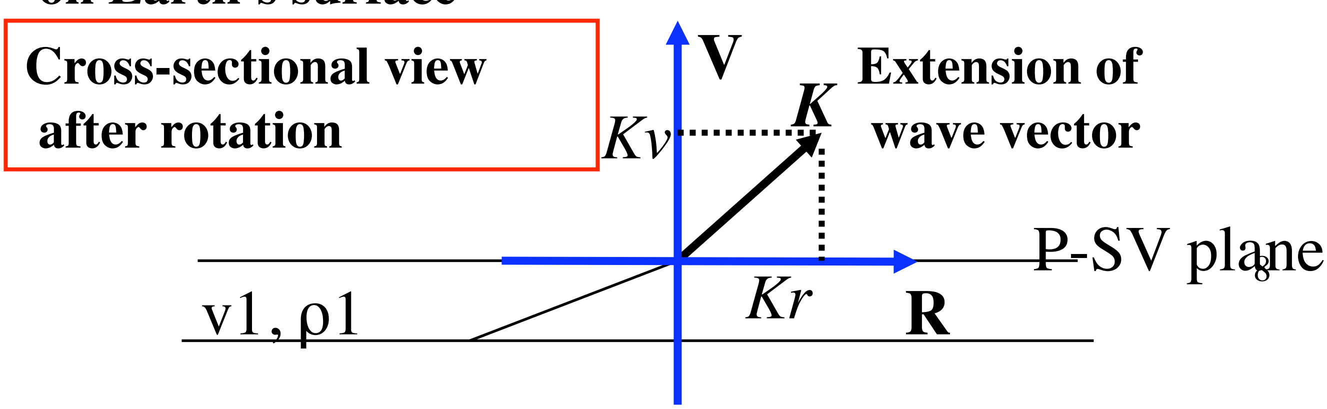


Suppose there is 3 cm displacement in east-west component, what does that really mean?

Answer: That means a particle positioned at the needle tip moves by 3 cm, i.e., particles on the ground/rock moves by 3 cm due to seismic wave. Seismometer is a Lagrangian motion recorder (meaning observing motion by sitting on a particle that moves).



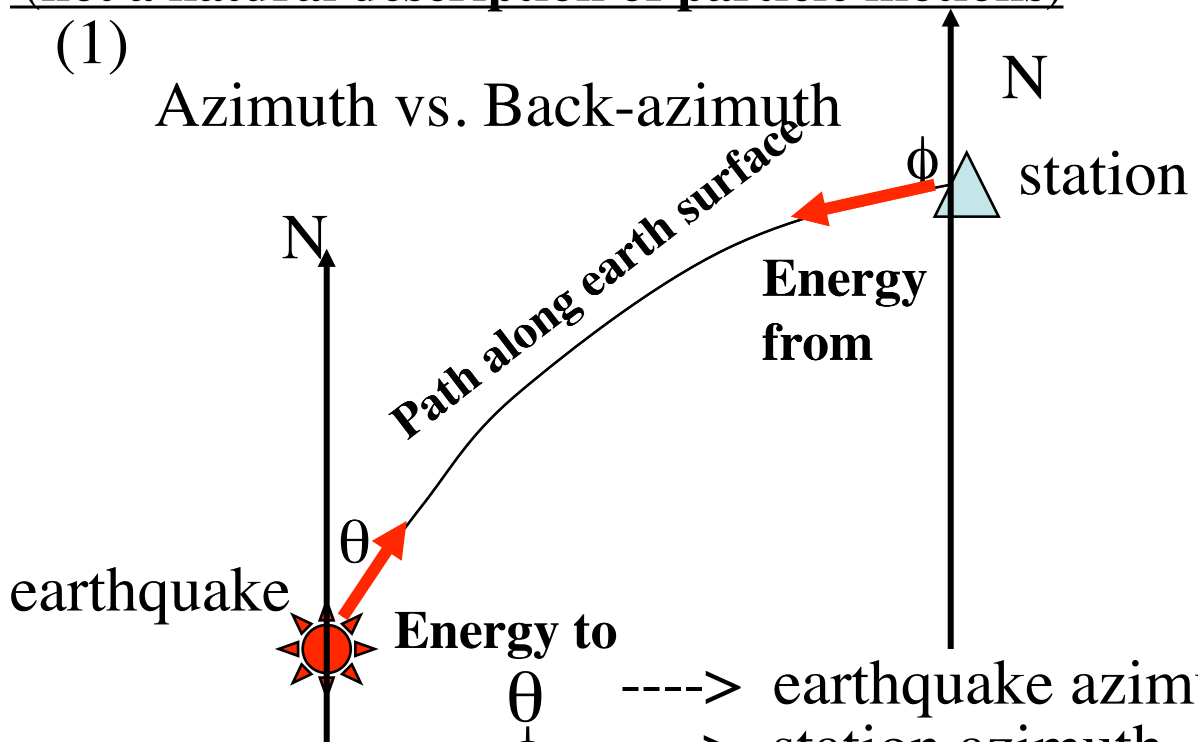




Typical Observations: E-W, N-S, Vertical ground motions
(not a natural description of particle motions)

(1)

Azimuth vs. Back-azimuth

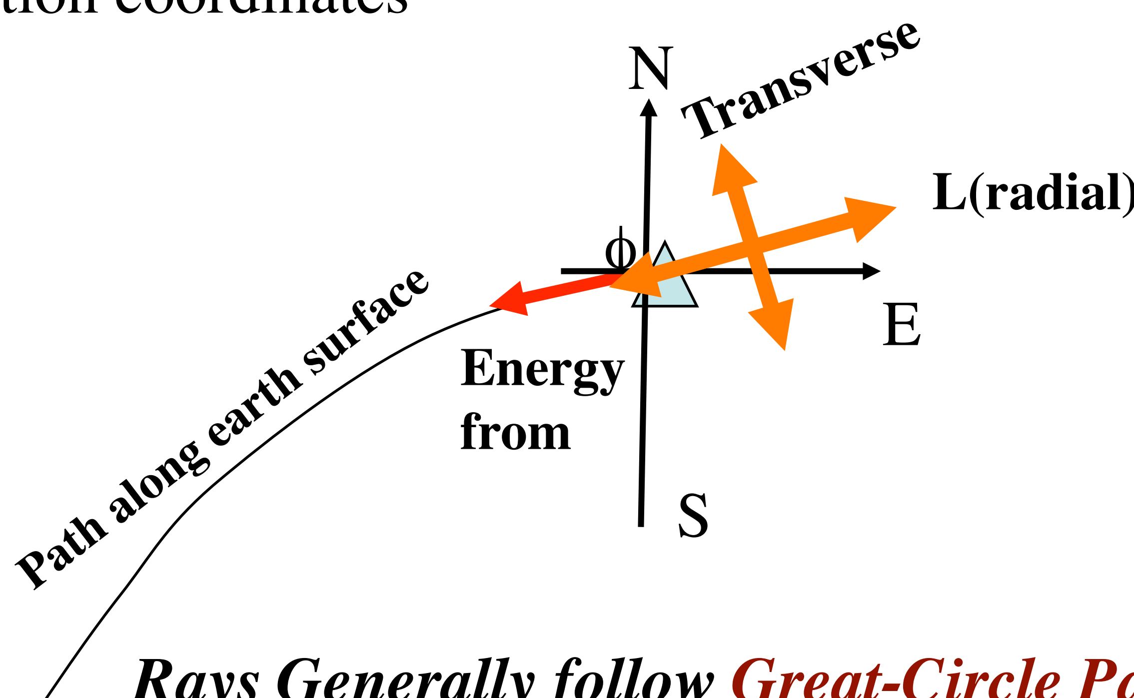


θ ----> earthquake azimuth

φ ----> station azimuth **(Back-azimuth)**

So does $\theta + \phi = 360$?? **NO.**

(2) Rotate by back-azimuth to natural particle motion coordinates

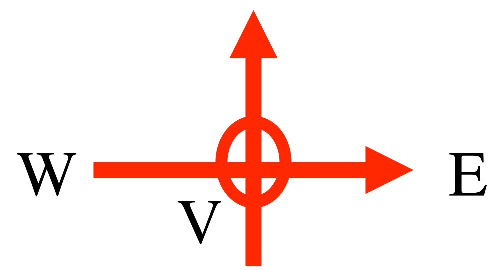


Rays Generally follow Great-Circle Paths

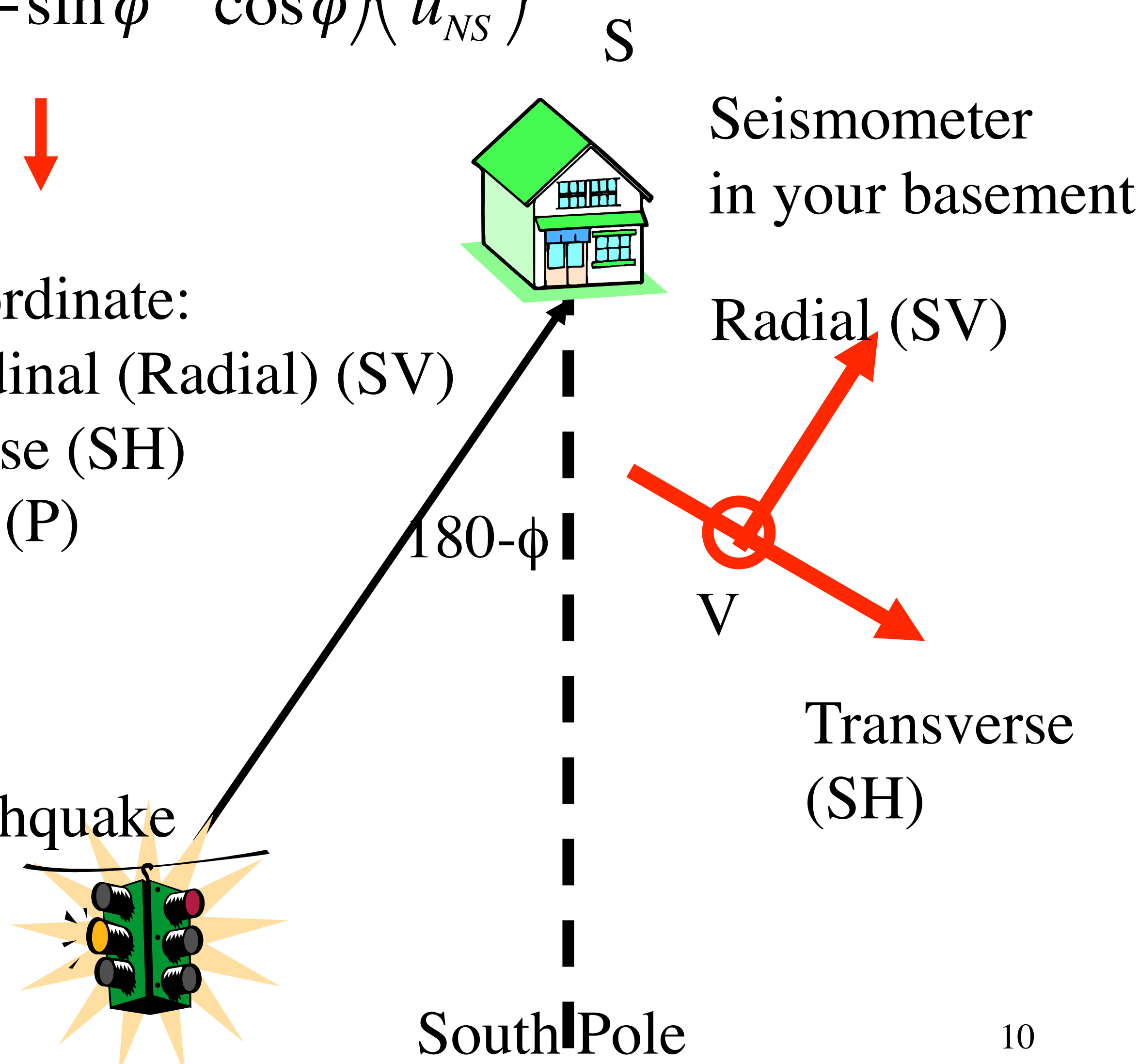
Recording: **3 component displacement/velocity:**
North-South
East-west
Vertical

Rotation around vertical

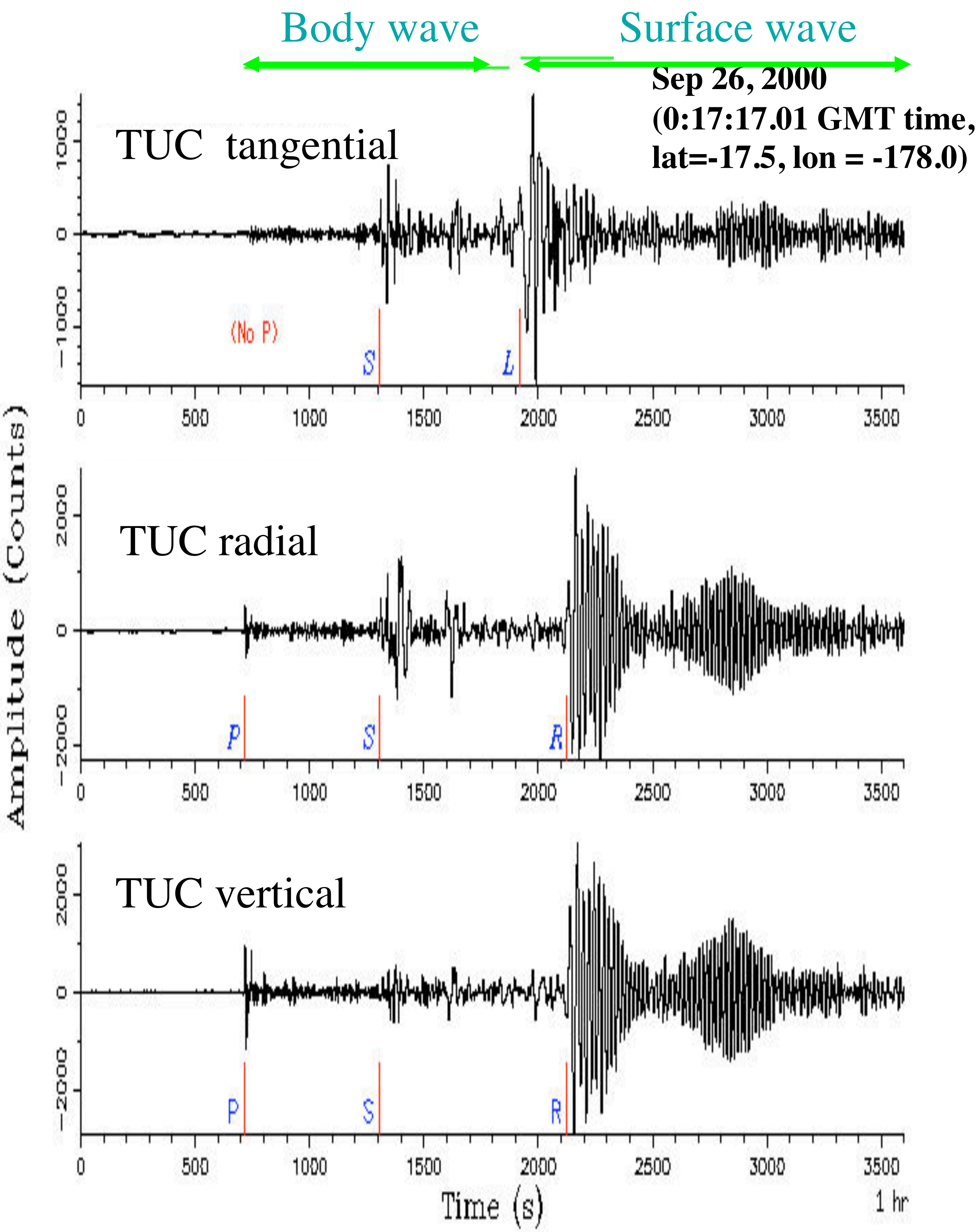
$$\begin{pmatrix} u_R \\ u_T \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} u_{EW} \\ u_{NS} \end{pmatrix}$$



New coordinate:
Longitudinal (Radial) (SV)
Transverse (SH)
Vertical (P)



Seismic Observations *Rotated, a rough view*



Polarization/particle motion

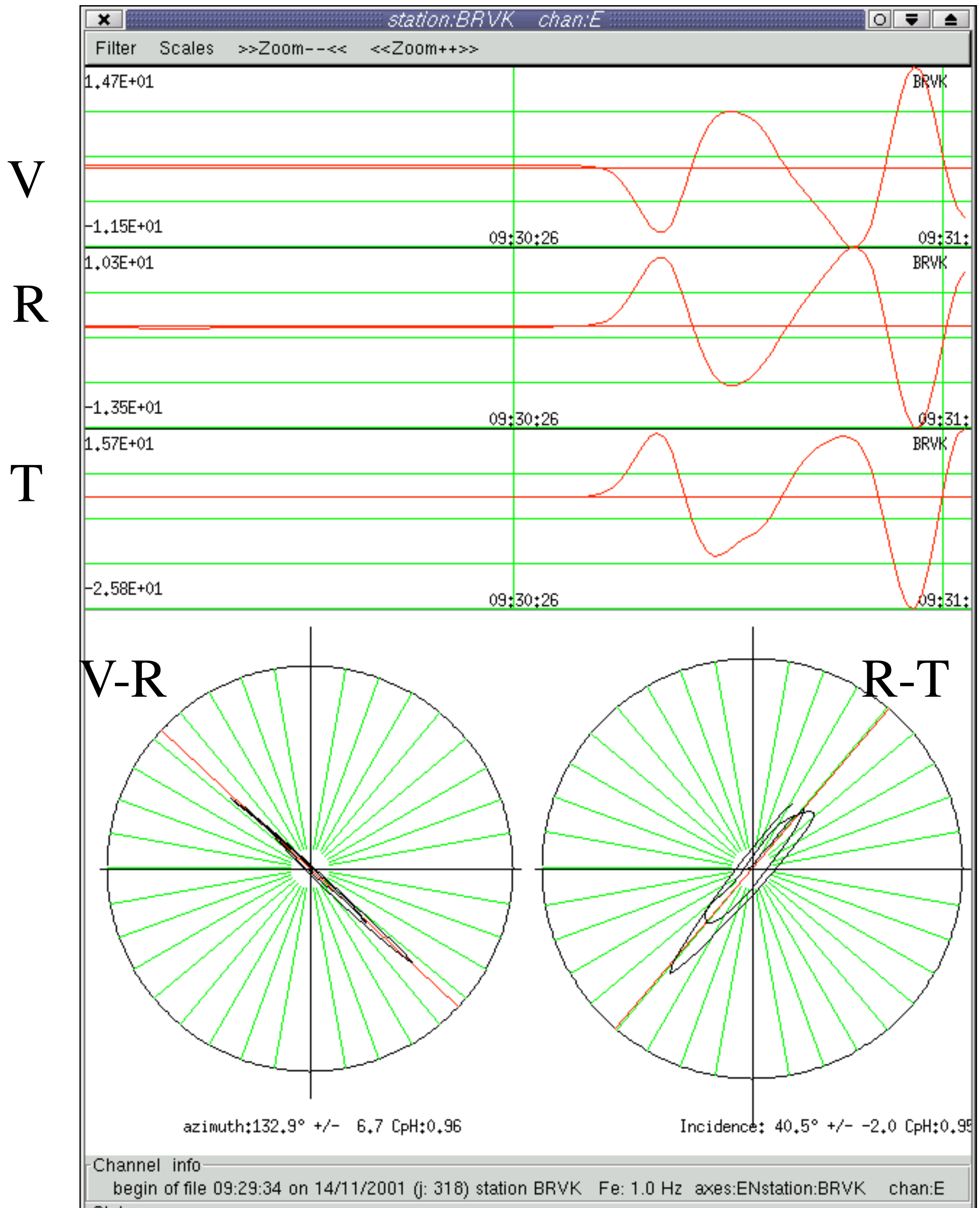
What is it: Plotting the two recorded components of a given seismic phase, say P, or SS, etc.

Why useful:

(1) V-R (vertical-radial) plot give first motion + arrival angle + strength of vertical vs. radial components.

(2) R-T (radial vs. transverse) plot gives presence of anisotropy

(3) E-N (eastwest vs. northsouth) gives back-azimuth!



(1) <u>Notational confusion:</u>

P wave, S wave --- related to particle motions

PcP, ScS --- these are called *Seismic phases* or *seismic arrivals*, they belong to either P or S types of particle motions

(2) <u>Many S-type seismic phases come in preferred polarizations</u>,

S --- Strong on both Radial and Transverse

SS --- Strong on Transverse

ScS --- Strong on Transverse

SP, PKP, or anything with P or K or I in name --> SV type --> Radial

(3) P waves travel faster than S waves because

$$Vp = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad Vs = \sqrt{\frac{\mu}{\rho}}$$

Where λ and μ are Lame parameters, λ is related to bulk modulus κ of materials (coefficient on how easily a piece of material can compressed) and μ is shear modulus that tells how easily something can be "torn" or sheared.

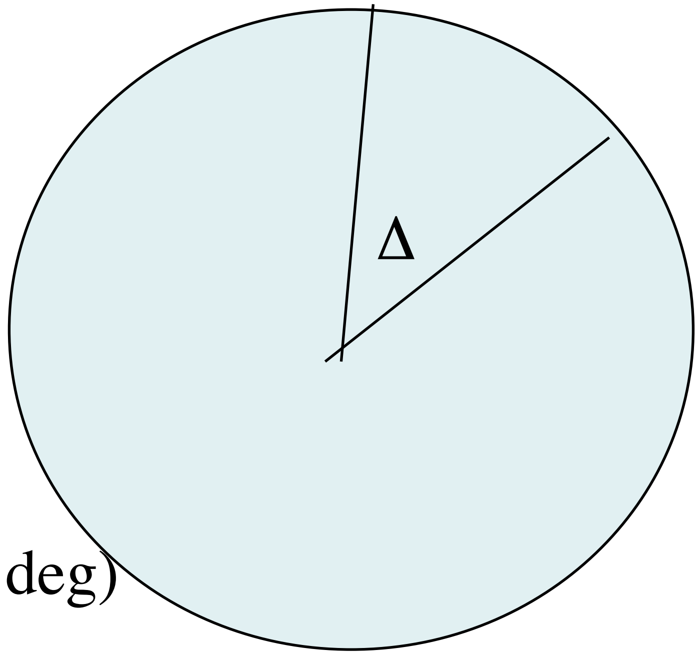
$\lambda > 0$ and $\mu > 0$ -----> Vp > Vs for most materials

(4) Most Earth materials can be approximated by what is called a "Poisson Solid" where $\lambda \sim \mu$ -------> $Vp \sim \sqrt{3}\, Vs$

Distance Calculations in Global Earth Problems

Known:

Radius of Earth = 6371 km



<u>Distance conversion to degrees:</u>

2*3.141593*6371 km/(360 deg)
= 111.195 km/deg

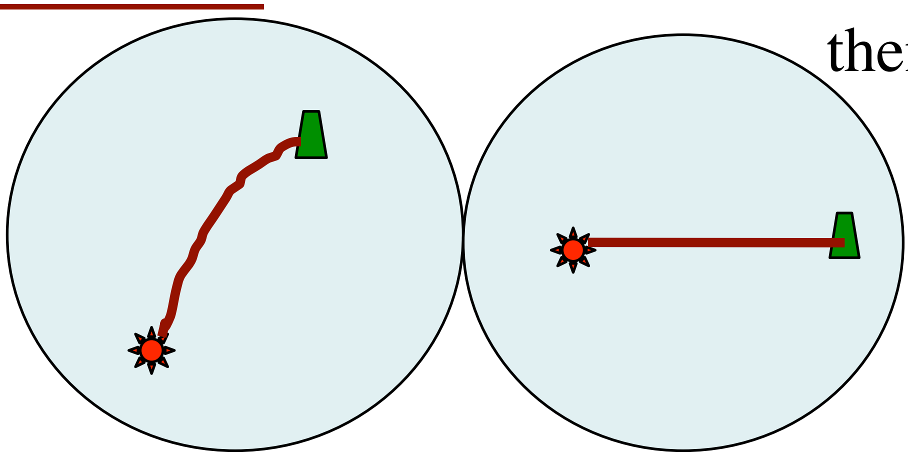
A seismic arrival that travels an ***epicentral distance*** (projection of raypath on the surface, same as surface wave distance) of 100 deg ~ 11119.5 km

Local distance: <10 deg, Regional distance < 30 deg, Teleseismic distance: > 30 deg

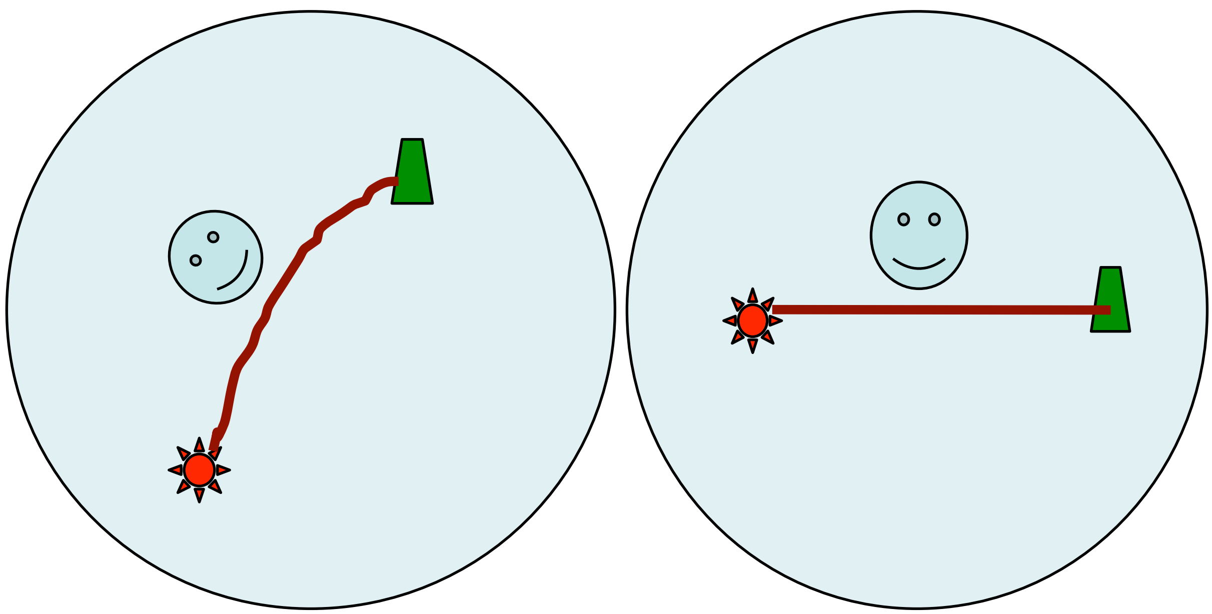
Question: How to compute distance for a random source and a given station?

<u>Procedure:</u>

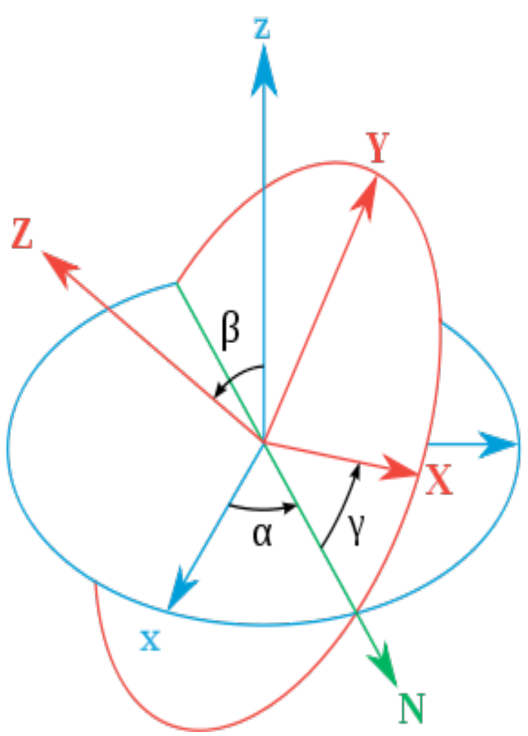
Rotate to Equator, read off distance, then rotate back.



Advantage: Suppose we want to find the perpendicular distance from the Smiley to the line, by rotating to the Equatorial system, the latitude of the rotated Smiley in the new Equatorial framework IS the perpendicular distance we want.



Difficulty: Rotation (need Euler angles & poles)



Euler angles · The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red.

gimbal

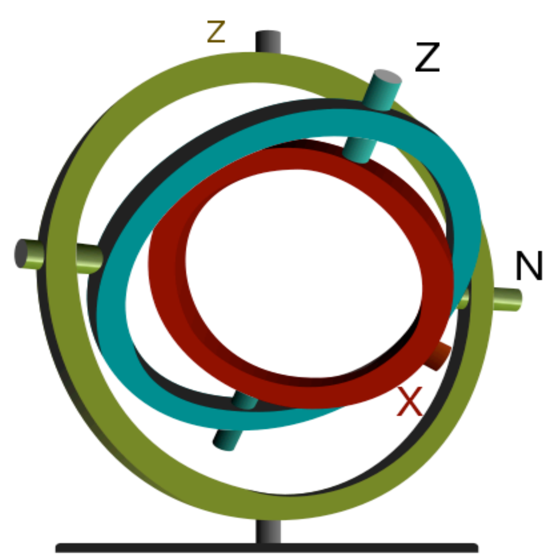
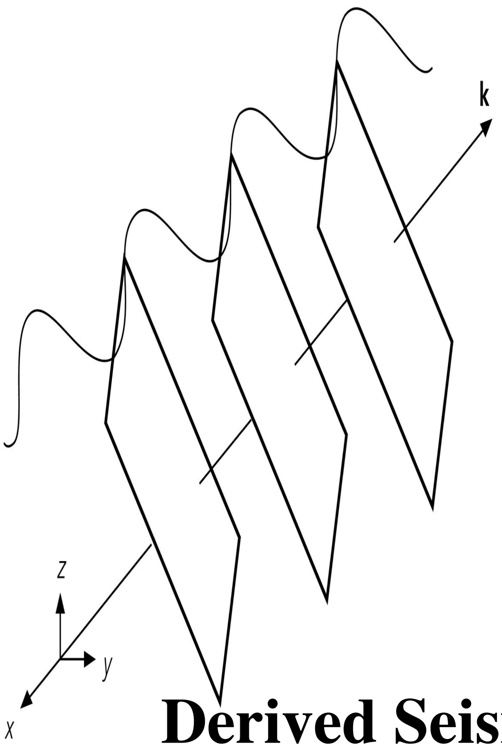


Figure 2.4-1: Plane wave fronts.



Dissecting Seismic Waves

When people say: seismic waves are solutions to the wave equation, what they really refer to is "seismic potentials". This is different from Strings (*<u>except for SH component</u>*)!

P-SV displacement

$$\mathbf{u}(\mathbf{x},t) = \overline{\nabla}\phi(\mathbf{x},t) + \overline{\nabla} \times \mathbf{\Psi}(\mathbf{x},t)$$

Derived Seismic Wave Eqn from Newton's II:

$$(\overline{\nabla}^2\phi) = \frac{1}{v_p^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} \qquad \text{(P waves)}$$

$$\overline{\nabla}^2\mathbf{\Psi} = \frac{1}{v_s^2}\frac{\partial^2\mathbf{\Psi}}{\partial t^2} \qquad \text{(SV wave, not SH)}$$

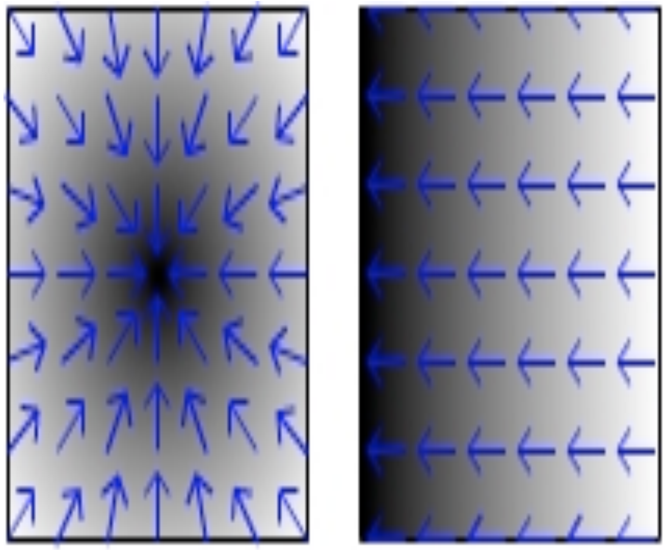
Get harmonic solutions in plane waves:

$$\phi(\mathbf{x},t) = Ae^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})} = Ae^{i(\omega t \pm k_x x \pm k_y y \pm k_z z)}$$

$$\mathbf{\Psi}(\mathbf{x},t) = \mathbf{A}e^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})} = \mathbf{A}e^{i(\omega t \pm k_x x \pm k_y y \pm k_z z)}$$

Understand Potentials "Physically"

Assume a scalar field $\phi(\mathbf{x})$

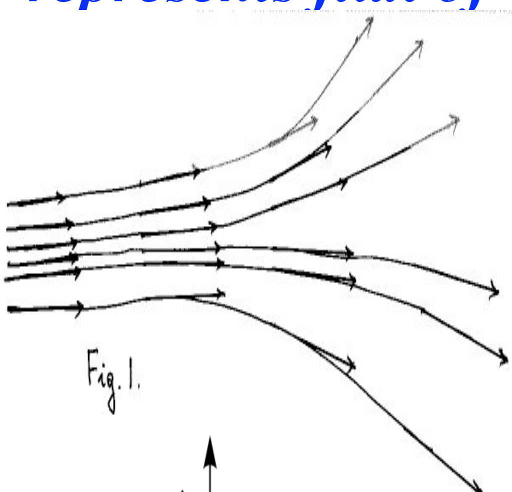


$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\hat{\mathbf{x}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{x}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{x}}_3$$

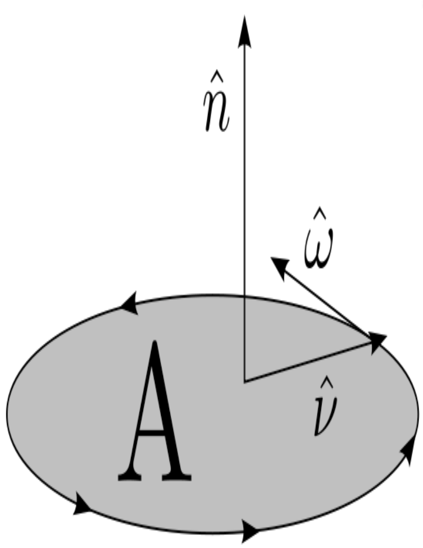
Gradient of scalar field, which makes sense to be related to P.

P direction = wave direction, compression

For comparison, divergence of a vector field Ψ, represents flux of vector field through a unit



$$\nabla \cdot \mathbf{\Psi} = \frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2} + \frac{\partial\psi_3}{\partial x_3} = \psi_{i,i}$$



Curl of a vector field Ψ, represents rate of rotation (in this case, it makes sense that shear waves are "rotated" and "perpendicular" to direction of P.

in the case of SV waves, $\mathbf{u} = \nabla \times \mathbf{\Psi}$