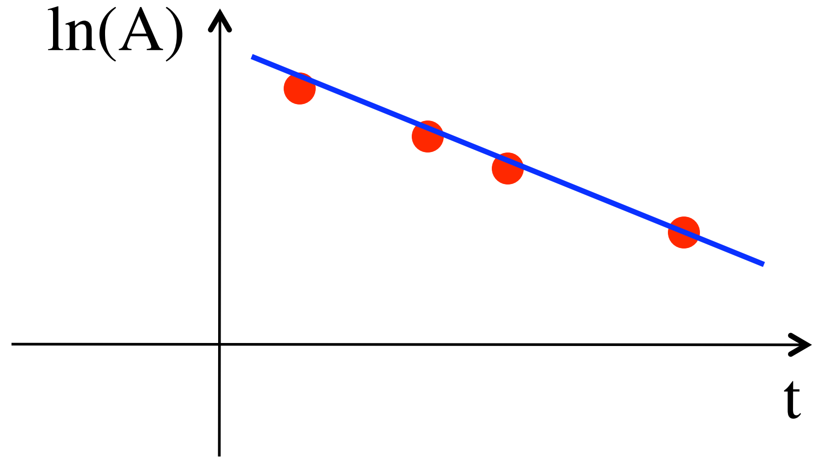
What is Q?

Interpretation 1: Suppose A_0 represents wave *amplitudes*, then

$$A = A_0 e^{-bt} = A_0 e^{-\omega_0 t/(2Q)}$$

$$\ln(A) = \ln(A_0) - \left[\frac{\omega 0}{2Q}\right] t$$

intercept — slope



Interpretation 2: Suppose u represents *displacement*, then

$$u(t) = A_0 e^{i(a+ib)t} = A_0 e^{i\omega_0 t} e^{\omega_0 t/(2Q)}$$

$a = \omega_0 \sqrt{1 - 1/4Q^2}$ (real) ⟶ **ω** = "modified" or "instantaneous freq"

$b = \dfrac{\omega_0}{2Q}$ (imaginary)

Suppose: small attenuation, the $a \approx \omega \approx \omega_0$ $\quad b \approx \dfrac{\omega}{2Q}$

We can define b = ω*, where ω* ---> 0 as Q increases (imaginary freq due to attenuation),

$\Longrightarrow$ $\omega^* = \dfrac{\omega}{2Q}$ $\Longrightarrow$ $Q^{-1} = 2\omega^*/\omega$

Relation with velocity: $c + ic^* = \dfrac{\omega}{k} + i\dfrac{\omega^*}{k} = \dfrac{\omega}{k} + i\dfrac{Q^{-1}}{2k}$

$\Longrightarrow$ Imaginary velocity due to attenuation

$$c^* = \frac{\omega}{2k}Q^{-1} \Rightarrow Q^{-1} = 2\frac{c^*}{c}$$

So Q is a quantity that defines the relationship between real and imaginary frequency (or velocity) under the influence of attenuation.

Interpretation 3: Q is the number of cycles the oscillations take to decay to a certain amplitude level. $n = t/T = t(\omega/2\pi)$

if $Q \to \infty$ then $n \approx t(\omega_0/2\pi) \rightarrow t_n = \dfrac{n \cdot 2\pi}{\omega_0}$

So amplitude at time t_n (after n cycles)

$$A(t_n) \approx A_0 e^{-w_0 t_n/2Q} = A_0 e^{-w_0 n 2\pi/(\omega_0 2Q)} = A_0 e^{-n\pi/Q}$$

$$if\ n = Q,\ \text{then A} = \text{A}_0\text{e}^{-\pi} \approx 0.04 A_0$$

Attenuation and Physical Dispersion (continued…)

Different interpretations of Q (quality factor):

(1) As a damping term $Q = \omega_0/\gamma$

(2) As a fraction between imaginary and real frequencies (or imaginary velocity to real velocity)

$$Q^{-1} = \frac{2\omega^*}{\omega} \quad \text{or} \quad Q^{-1} = \frac{2c^*}{c}$$

(3) As the number of cycles for a wave to decay to a certain amplitude. If $n = Q$, then

$$A(t_n) = A_0 e^{-\omega_0 t/2Q} = A_0 e^{-(\omega_0/2Q)(n*2\pi/\omega_0)} \approx 0.04 A_0$$

(4) Connection with t* (for body wave).

$$t^* = \int_{path} \frac{dt}{Q(\mathbf{r})} \approx \sum_{i=1}^{N} \frac{\Delta t_i}{Q_i} \quad (\text{N layers, } \mathbf{r} = \text{location})$$

1 number that describes Q structure of several layers

(5) Energy formula

$$\frac{1}{Q(\omega)} = -\frac{\Delta E}{2\pi E} \quad (-\Delta E = \text{energy loss per cycle})$$

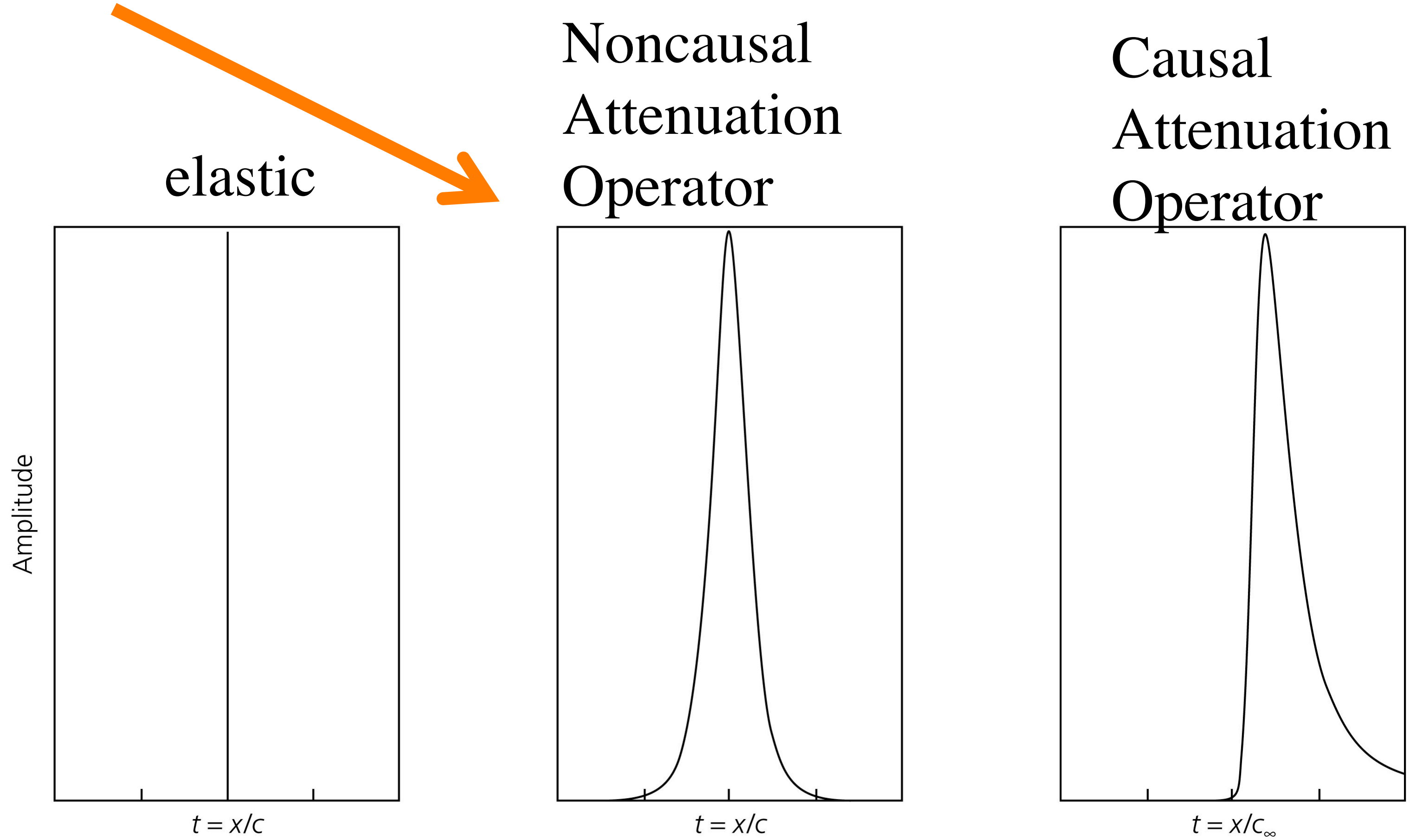
Effects of Q (assume the SAME Q value)

(1) $$A(\omega) = A_0(\omega)e^{-\omega_0 t/2Q} \approx A_0(\omega)e^{-\omega x/(2Qv)}$$

dependencies

- large distance x ----> more amplitude decay
- large velocity v -----> the less amplitude decay
- large frequency ω ----> more amplitude decay

Physical Dispersion



Observation: pulse is "spread out" which means dispersive (different frequencies arrive at different times!)

Problem: envelope of the function is nonzero before t=x/c (it is like receiving earthquake energy before the rupture, not physical!)

How to make this process causal?

One of the often-cited solutions: ***Azimi's Attenuation Law***:

$$c(\omega) = c_0\left[1 + \frac{1}{\pi Q}\ln\left(\frac{\omega}{\omega_0}\right)\right]$$

c_0 = reference speed for frequency ω_0

if $Q = \infty$, then $c = c_0$ (no dispersion)

if $\omega = \omega_0$, then $c = c_0$ (no dispersion)

if $\omega = \infty$, then $c \approx c_0$ (means high freq arrives first)

<u>Where does it come from?</u>

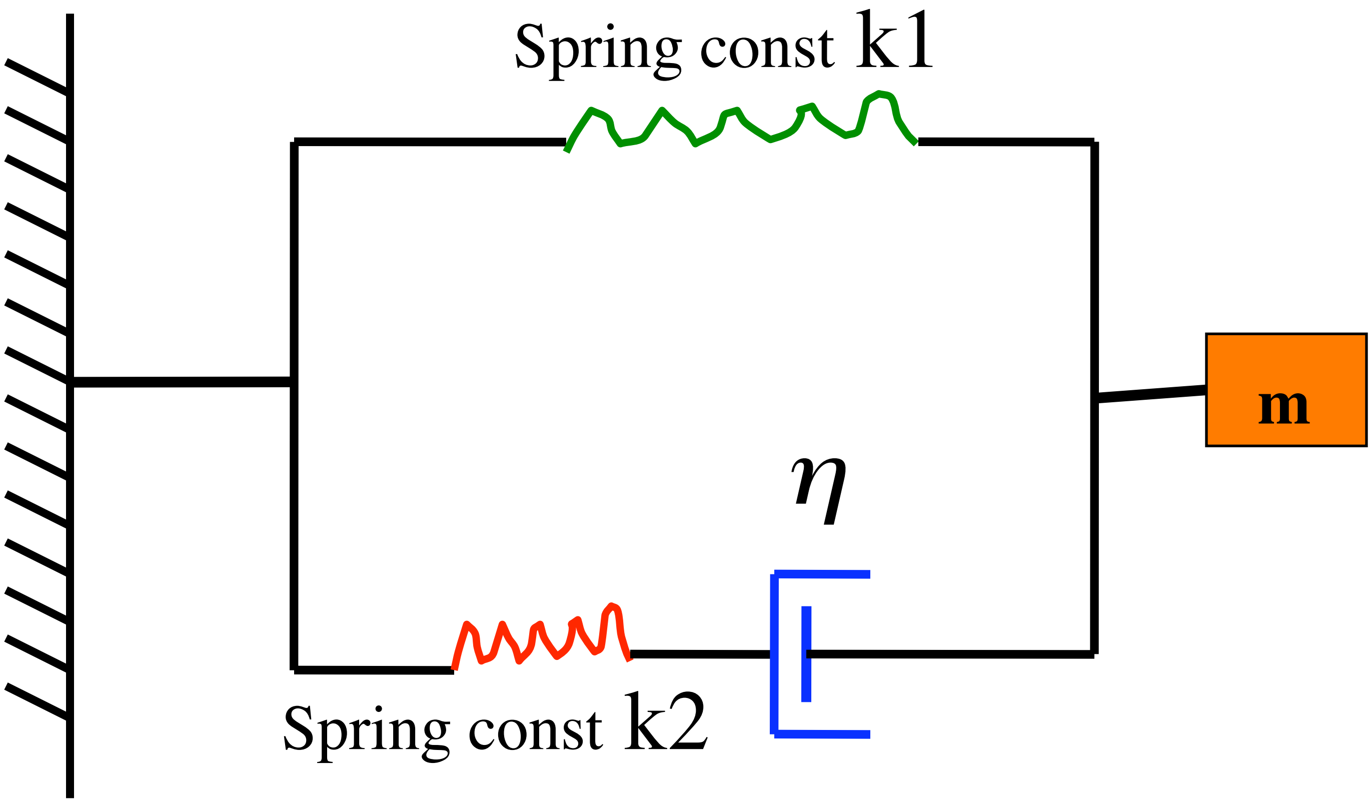
Answer: Derived under the following ***<u>causality condition:</u>***

$u(x,t) = 0$ for all $t < x/c(\infty)$, where $c(\infty)$ is the highest (infinite) frequency that arrives first.

Physical Models of Anelasticity

In an nutshell, Earth is composed of lots of ***Viscoelastic (or Standard Linear) Solids***

Standard Linear Solid (or Viscoelastic Solid)



Specification: (1) consist of two springs and a dashpot

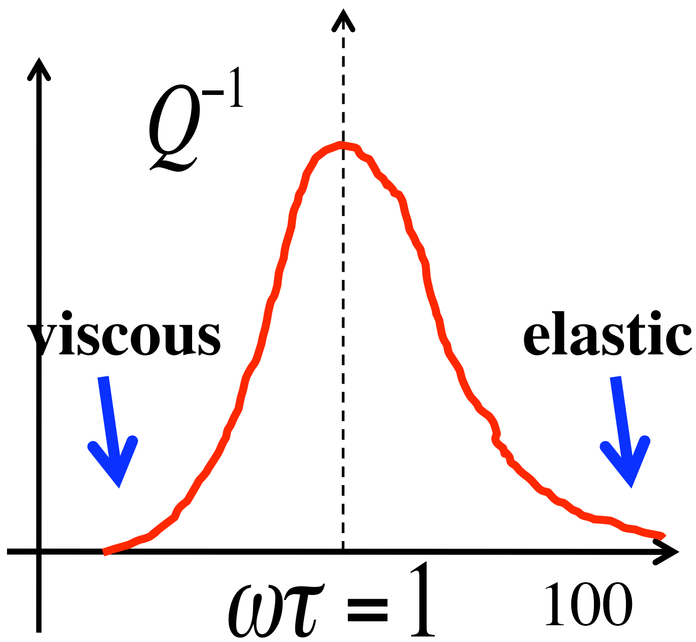
(2) η = viscosity of fluid inside dashpot

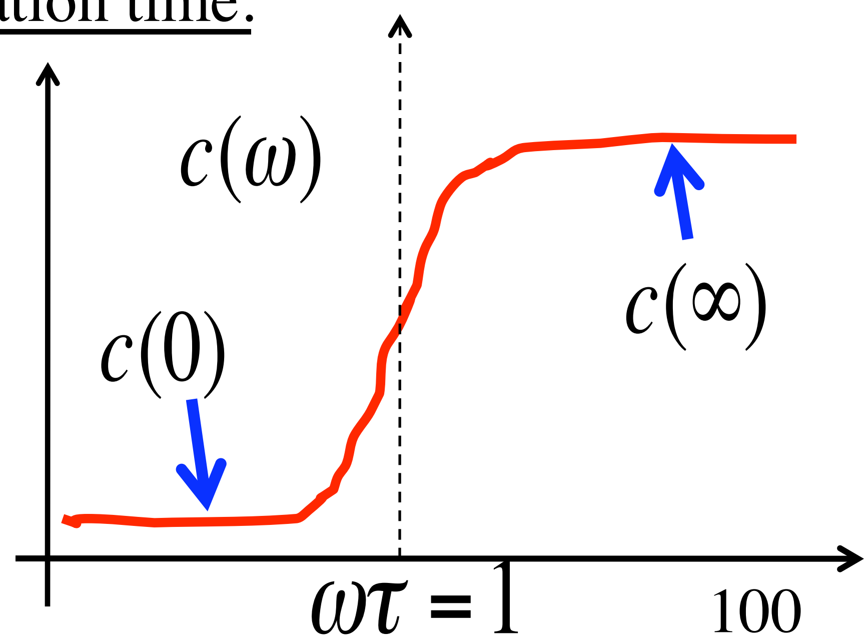
Governing Equation (stress): $\sigma(t) = k_1 H(t) + k_2 e^{-t/\tau}$

where $H(t)$ = step (or Heavyside function)

and $\tau = \eta/k_2$ (relaxation time)

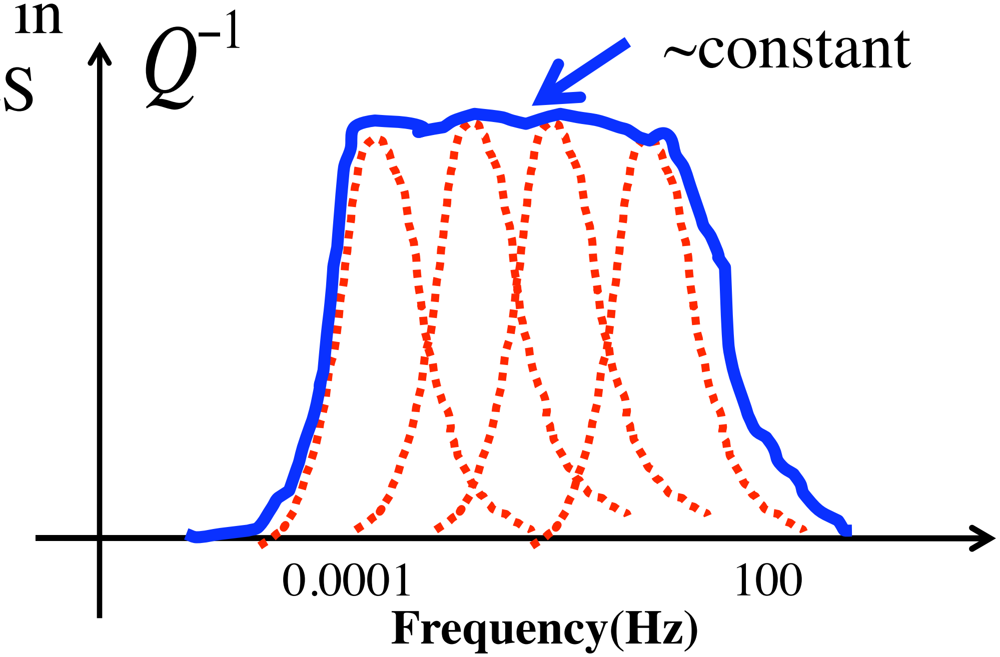
<u>The response to a harmonic wave depends on the product of the angular frequency and the relaxation time.</u>





The left-hand figure is the absorption function. The absorption is small at both very small and very large frequencies. It is the max at $\omega\tau = 1$!

A given polycrystalline material in the Earth is formed of many SLS superimposed together. So the final Frequency dependent Q is constant for many seismic frequencies ----→

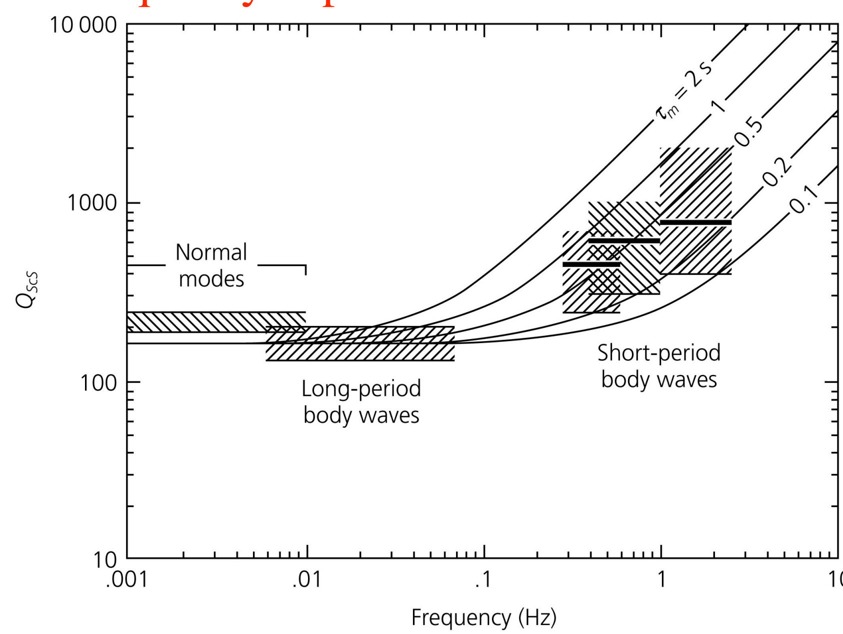


<u>Question:</u> Is the fact that high frequencies are attenuated more a contradiction to this flat Q observation??

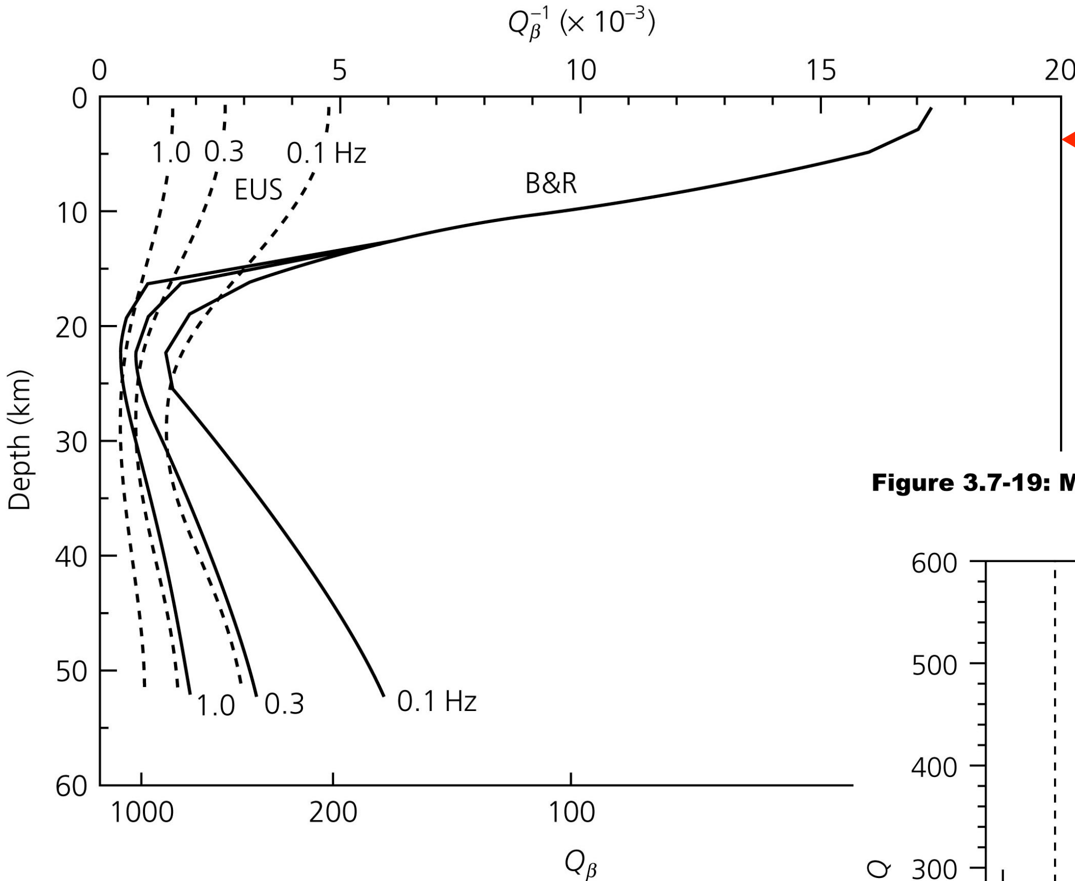
<u>Answer:</u> No. High frequencies are attenuated more due to the following equation that works with the same Q. So it is really "frequency dependent amplitude", NOT "frequency dependent Q".

$$A(\omega) \approx A_0(\omega) e^{-\omega x/(2Qv)}$$

Frequency dependence of mantle attenuation

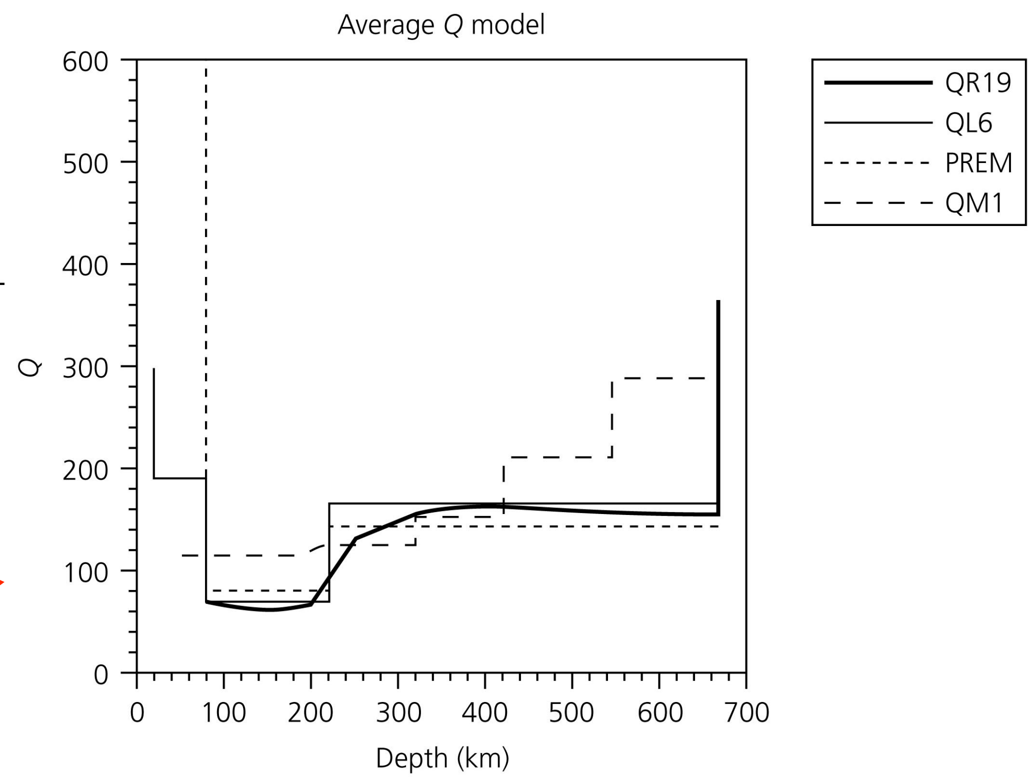


Eastern USA (EUS) and Basin-and-Range Attenuation



Mitchell 1995 (lower attenuation occurring at high frequencies, tells us that frequency independency is not always true

Figure 3.7-19: Models of upper mantle attenuation.

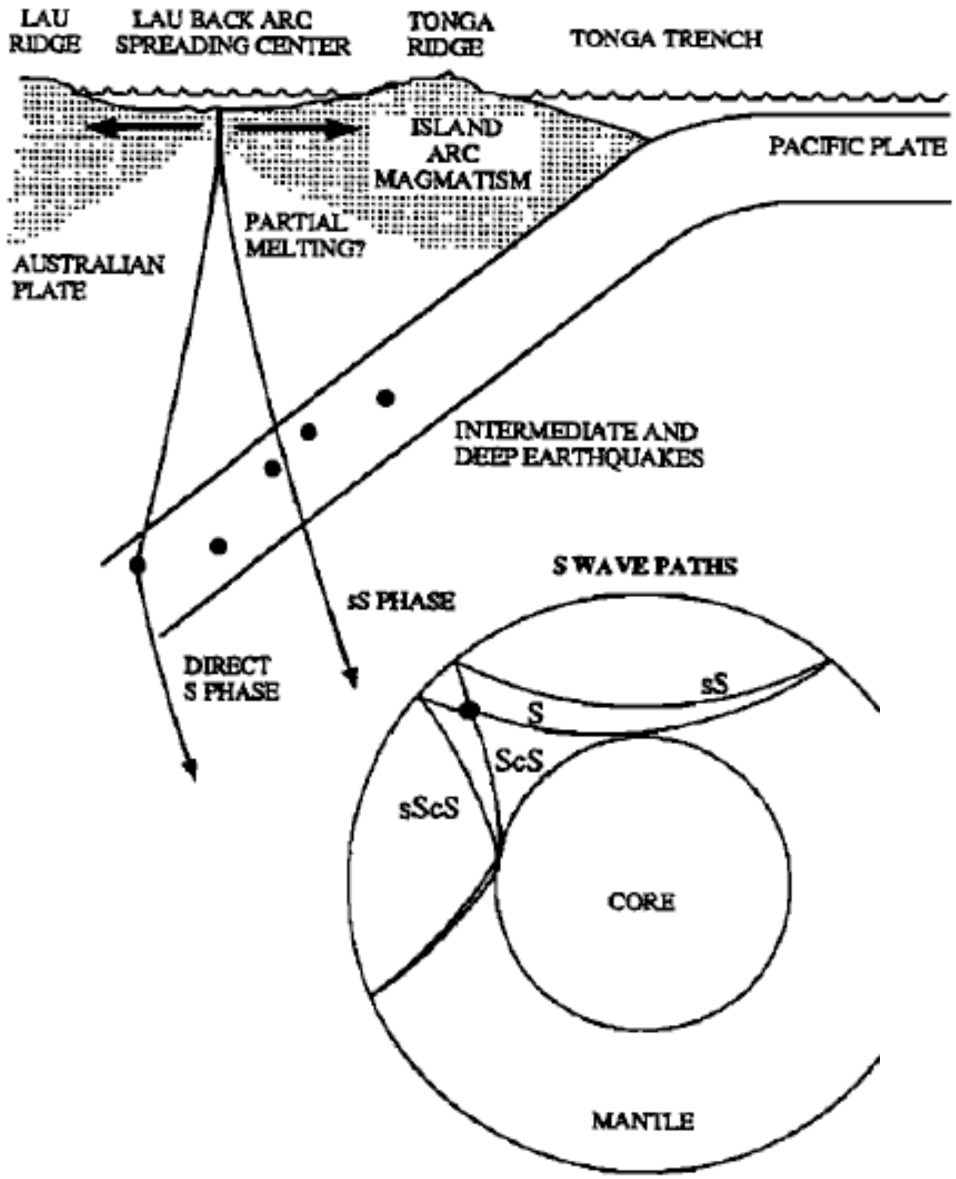


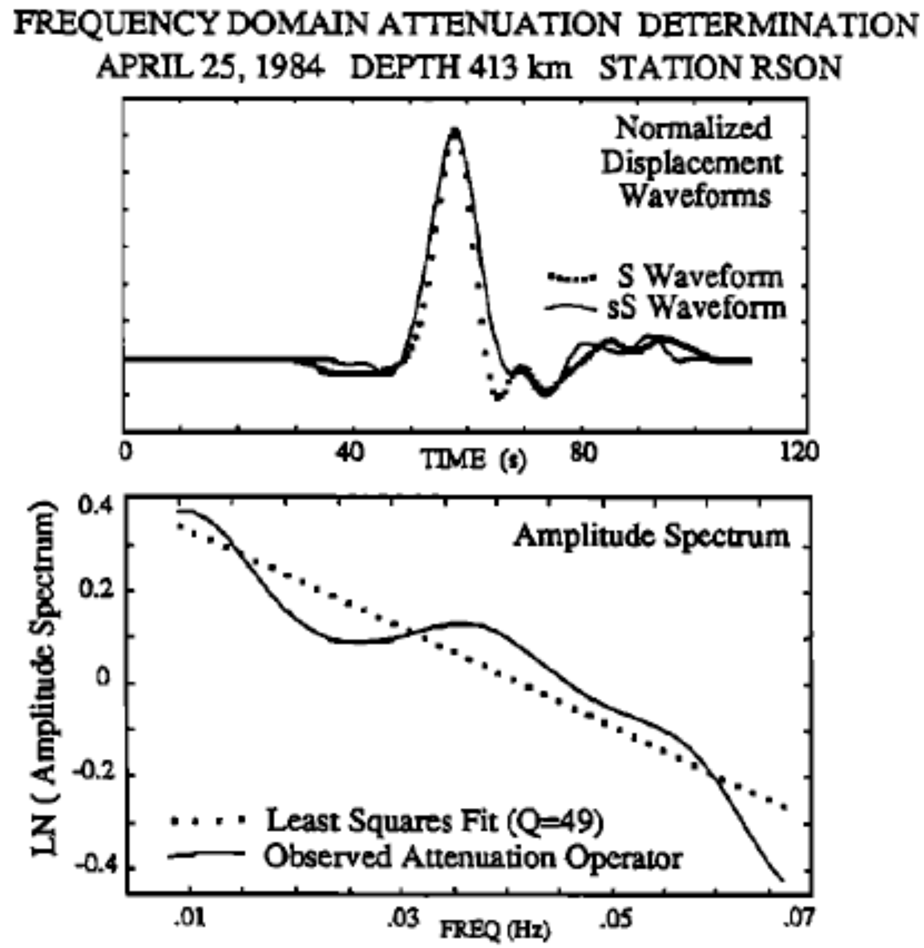
Low Q in the asthenosphere, Romanowicz, 1995

An example of Q extraction from differential waveforms.

Two phases of interest:

sS-S
sScS - ScS





Flanagan & Wiens (1990)

Key Realization: The two waveforms are similar in nature, mainly differing by the segment in the above source (the small depth phase segment)

$$sS(\omega) \approx S(\omega)R(\omega)A(\omega)$$

$sS(\omega) =$ frequency spec of sS

$S(\omega) =$ frequency spec of S

$R(\omega) =$ crustal operator

$A(\omega) =$ attenuation operator of interest

Approach: Spectral division of S from SS, then divide out the Crustal operator (a function in freq that accounts for the of the additional propagation in a normal crust)

Spectral dividing S and R will leave the attenuation term A(w)

$$|A(\omega)| = e^{-\omega t/2Q} \Rightarrow \log(|A(\omega)|) = -\frac{t}{2Q}\omega$$

(slope = =t/2Q,
t = time diff of sS-S)