Math 201 (Fall 2009)
Differential Equations

Solution #5

1. Solve the following initial value problems with the help of the Laplace transform

   (a) $y'' + 4y = 4t^2 - 4t + 10, \quad y(0) = 0,\ y'(0) = 3;$

   (b) $w'' + w = t^2 + 2, \quad w(0) = 1,\ w'(0) = -1;$

   (c) $y'' + y = t, \quad y(\pi) = 0,\ y'(\pi) = 0.$

   Solution:

   (a) Applying the Laplace transform to both sides of the differential equation, we obtain

   $$(s^2Y(s) - 3) + 4Y(s) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s},$$

   so that

   $$\begin{aligned} Y(s) &= \frac{1}{s^2+4}\left(3 + \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s}\right) \\ &= \frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2+4)} \\ &= \frac{2}{s^3} - \frac{1}{s^2} + \frac{2}{s} - 2\frac{s}{s^2+4} + 2\frac{2}{s^2+2}. \end{aligned}$$

   Applying the inverse Laplace transform yields

   $$y(t) = t^2 - t + 2 - 2\cos 2t + 2\sin 2t.$$

   (b) Applying the Laplace transform to both sides of the differential equation yields

   $$s^2W(s) + s - 1 + W(s) = \frac{2}{s^3} + \frac{2}{s} = 2\frac{s^2+1}{s^3}$$

   and

   $$(s^2+1)W(s) = -s + 1 + 2\frac{s^2+1}{s^3}$$

   and finally

   $$W(s) = -\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{s^3}.$$

   Hence, we have

   $$w(t) = -\cos t + \sin t + t^2.$$

(c) Define $z(t) := y(t+\pi)$. Then the initial value problem becomes

$$z'' + z = t + \pi, \qquad z(0) = 0,\ z'(0) = 0.$$

Applying the Laplace transform, we obtain

$$s^2 Z(s) + Z(s) = \frac{1}{s^2} + \frac{\pi}{s},$$

so that

$$Z(s) = \frac{1}{s^2+1}\left(\frac{1}{s^2} + \frac{\pi}{s}\right) = \frac{1}{s^2} + \frac{\pi}{s} - \frac{\pi s}{s^2+1} - \frac{1}{s^2+1}$$

and

$$z(t) = t + \pi - \pi\cos t - \sin t.$$

Shifting back the argument, we obtain finally

$$y(t) = z(t-\pi) = t - \pi + \pi - \pi\cos(t-\pi) - \sin(t-\pi) = t + \pi\cos t + \sin t.$$

2. Solve the initial value problem

$$y''' - y'' + y' - y = 0, \quad y(0) = 1,\ y'(0) = 1,\ y''(0) = 3.$$

Solution: Note that

$$\mathcal{L}\{y'\}(s) = sY(s) - 1, \qquad \mathcal{L}\{y''\}(s) = s^2Y(s) - s - 1,$$
$$\text{and} \qquad \mathcal{L}\{y'''\}(s) = s^3Y(s) - s^2 - s - 3,$$

so that

$$s^3Y(s) - s^2 - s - 3 - s^2Y(s) + s + 1 + sY(s) - 1 - Y(s) = 0,$$

i.e.,

$$(s^3 - s^2 + s - 1)Y(s) = s^2 + 3.$$

Partial fractions yield

$$Y(s) = \frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2-1},$$

so that

$$y(t) = e^t - \sin t - \cos t.$$

3. Solve the initial value problem

$$ty'' - ty' + y = 2, \quad y(0) = 2,\ y'(0) = -1.$$

Solution: Apply the Laplace transform to both sides of the differential equation:

$$\mathcal{L}\{ty''\}(s) - \mathcal{L}\{ty'\}(s) + \mathcal{L}\{y\}(s) = \frac{2}{s}.$$

Note that

$$\mathcal{L}\{y'\}(s) = sY(s) - 2 \qquad \text{and} \qquad \mathcal{L}\{y''\}(s) = s^2Y(s) - 2s + 1,$$

so that

$$\mathcal{L}\{ty'\}(s) = -\frac{d}{ds}\mathcal{L}\{y'\}(s) = -\frac{d}{ds}(sY(s) - 2) = -sY'(s) - Y(s)$$

and

$$\mathcal{L}\{ty''\}(s) = -\frac{d}{ds}\mathcal{L}\{y''\}(s) = -\frac{d}{ds}(s^2Y(s) - 2s + 1) = -s^2Y'(s) - 2sY(s) + 2.$$

Altogether, we obtain

$$(-s^2Y'(s) - 2sY(s) + 2) - (-sY'(s) - Y(s)) + Y(s) = \frac{2}{s}$$

or, equivalently,

$$s(1-s)Y'(s) + 2(1-s)Y(s) = \frac{2(1-s)}{s}.$$

Division by $s(1-s)$ yields the fird order linear differential equation

$$Y'(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}.$$

Its integrating factor is

$$\mu(s) = \exp\int \frac{2}{s}\,ds = e^{2\ln|s|} = s^2,$$

so that

$$Y(s) = \frac{1}{\mu(s)}\int \mu(s)\frac{2}{s^2}\,ds = \frac{1}{s^2}\int 2\,ds = \frac{2}{s} + \frac{C}{s^2}$$

for some constant C. It follows that

$$y(t) = 2 + Ct.$$

The initial condition $y'(0) = -1$ yields $C = -1$, so that $y(t) = 2 - t$.

4. The current $I(t)$ in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t), \quad I(0) = 1,\ I'(0) = 3$$

where

$$g(t) := \begin{cases} 3\sin t, & 0 \le t \le 2\pi, \\ 0, & 2\pi < t \end{cases}$$

Determine the current as a function of time.

Solution: Since

$$\mathcal{L}\{I''\}(s) = s\mathcal{L}\{I\}(s) - s - 3,$$

we obtain

$$\mathcal{L}\{I''\}(s) + 4\mathcal{L}\{I\}(s) =$$
$$s\mathcal{L}\{I\}(s) - s - 3 + 4\mathcal{L}\{I\}(s) = (s^2+4)\mathcal{L}\{I\}(s) - (s+3) = \mathcal{L}\{g\}(s).$$

To compute $\mathcal{L}\{g\}(s)$, note that

$$g(t) = 3(\sin t - (\sin t)u(t-2\pi)),$$

so that

$$\mathcal{L}\{g\}(s) = 3\left(\frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}\right) = \frac{3(1-e^{-2\pi s})}{s^2+1}.$$

Solving for $\mathcal{L}\{I\}(s)$ yields

$$\mathcal{L}\{I\}(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4} + \frac{3(1-e^{-2\pi s})}{(s^2+1)(s^2+4)}.$$

Since

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} - \frac{1}{s^2+4},$$

we obtain

$$\mathcal{L}\{I\}(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4} + (1-e^{-2\pi s})\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

and thus

$$\begin{aligned} I(t) &= \cos 2t + \frac{3}{2}\sin 2t + \sin t - \frac{1}{2}\sin 2t - \left(\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)\right)u(t-2\pi) \\ &= \sin t + \sin 2t + \cos 2t + \left(\frac{1}{2}\sin 2t - \sin t\right)u(t-2\pi). \end{aligned}$$

5. Let $f(t)$ be the function of period two which is defined as

$$f(t) := \begin{cases} t, & 0 < t < 1, \\ 1-t, & 1 < t < 2. \end{cases}$$

Determine $\mathcal{L}\{f\}$.

Solution: The windowed version f_T of f is

$$f_T(t) := \begin{cases} t, & 0 < t < 1, \\ 1-t, & 1 < t < 2, \\ 0, & 2 < t. \end{cases}$$

Its Laplace transform F_T is computed as (integration by parts)

$$\begin{aligned} F_T(s) &= \int_0^\infty e^{-st} f_T(t)\,dt \\ &= \int_0^1 e^{-st} t\,dt + \int_1^2 e^{-st}(1-t) \\ &= \frac{1 - 2e^{-s} - se^{-s} + e^{-2s} + se^{-2s}}{s^2} \end{aligned}$$

It follows that

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-2s}} = \frac{1 - 2e^{-s} - se^{-s} + e^{-2s} + se^{-2s}}{s^2(1-e^{-2s})}.$$

6. Use convolution to determine a formula for the solution of the initial value problem

$$y'' + 9y = g(t), \quad y(0) = 1,\, y'(0) = 0$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

Solution: Applying the Laplace Transform yields

$$s^2 Y(s) - s + 9Y(s) = G(s),$$

so that

$$(s^2 + 9)Y(s) = s + G(s)$$

and thus

$$Y(s) = \frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+1}G(s).$$

Applying the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}(t) + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+1}G(s)\right\} \\ &= \cos 3t + \frac{1}{3}(\sin 3t) * g(t) \\ &= \cos 3t + \int_0^t \sin(3(t-\tau))g(\tau)\,d\tau. \end{aligned}$$

7. Use convolution to find the inverse Laplace transforms of the following functions:

(a) $\dfrac{1}{(s+1)(s+2)}$; (b) $\dfrac{1}{s^3(s^2+1)}$; (c) $\dfrac{s+1}{(s^2+1)^2}$.

Solution:

(a) Since $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(t) = e^{at}$, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\frac{1}{s+2}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&= e^{-t} * e^{-2t} \\
&= \int_0^t e^{-(t-\tau)}e^{-2\tau}\,d\tau \\
&= e^{-t}\int_0^t e^{-\tau}\,d\tau \\
&= e^{-t}(1-e^{-t}) \\
&= e^{-t} - e^{-2t}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\frac{1}{s^2+1}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \\
&= \frac{t^2}{2} * \sin t \\
&= \frac{1}{2}\int_0^t (t-\tau)^2 \sin\tau\,d\tau \\
&= \frac{1}{2}\left(-(\tau-t)^2\cos\tau\Big|_0^t + 2\int_0^t (\tau-t)\cos\tau\,d\tau\right) \\
&= \frac{1}{2}\left(t^2 + 2(\tau-t)\sin\tau\Big|_0^t - 2\int_0^t \sin\tau\,d\tau\right) \\
&= \frac{t^2}{2} + \cos t - 1.
\end{aligned}$$

(c) First note that

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t)$$

The second summand is evaluated in Example 2 of Section 7.7:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) = \frac{\sin t - t\cos t}{2}.$$

For the first summand, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \\
&= \cos t * \sin t \\
&= \int_0^t \cos(t-\tau)\sin\tau\,\tau \\
&= \frac{1}{2}\int_0^t \sin t + \sin(t-2\tau)\,d\tau \\
&= \frac{1}{2}\left(\tau\sin t + \frac{\cos(t-2\tau)}{2}\Bigg|_0^t\right) \\
&= \frac{t\sin t}{2}.
\end{aligned}$$

Altogether, we obtain

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\}(t) = \frac{t\sin t}{2} + \frac{\sin t - t\cos t}{2} = \frac{t\sin t + \sin t - t\cos t}{2}.$$

8. Solve the integro-differential equation

$$y'(t) - 2\int_0^t e^{t-\tau}y(\tau)\,d\tau = t, \quad y(0) = 2.$$

Solution: We rewrite the equation as

$$y'(t) - 2e^t * y(t) = t$$

and apply the Laplace transform to both sides. We obtain

$$sY(s) - 2 - \frac{2}{s-1}Y(s) = \frac{1}{s^2}.$$

It follows that

$$Y(s) = \frac{(2s^2+1)(s-1)}{s^2(s+1)(s-2)} = \frac{1}{2}\frac{1}{s^2} - \frac{3}{4}\frac{1}{s} + 2\frac{1}{s+1} + \frac{3}{4}\frac{1}{s-2},$$

so that

$$y(t) = \frac{t}{2} - \frac{3}{4} + 2e^{-t} + \frac{3e^{2t}}{4}.$$

9. A linear system is governed by the initial value problem

$$y'' + 2y' - 15y = g(t), \quad y(0) = 0,\ y'(0) = 8.$$

Find the transfer function $H(s)$ of the system, the impulse response function $h(t)$, and give a formulate for the solution to the initial value problem.

Solution: The transfer function is

$$H(s) = \frac{1}{s^2 + 2s - 15} = \frac{1}{(s-3)(s55)}.$$

Consequently, the impulse response function is computed as

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}(t) \\ &= e^{3t} * e^{-5t} \\ &= \int_0^t e^{3(t-\tau)} e^{-5\tau}\, d\tau \\ &= e^{3t}\left(-\frac{e^{-8\tau}}{8}\Big|_0^t\right) \\ &= \frac{e^{3t} - e^{-5t}}{8}. \end{aligned}$$

To solve the initial value problem, we need the solution $y_k(t)$ for the *homogeneous* problem

$$y'' + 2y' - 15y = 0, \quad y(0) = 0,\, y'(0) = 8.$$

Applying the Laplace transform yields

$$s^2 Y_k(s) - 8 + 2sY_k(s) - 15Y_k(s) = 0,$$

so that

$$Y_k(s) = \frac{8}{s^2 + 2s - 15}.$$

Since $Y_k(s) = 8H(s)$, it follows that $y_k(t) = 8h(t) = e^{3t} - e^{-5t}$. All in all, we obtain

$$y(t) = (h * g)(t) + y_k(t) = \frac{1}{8}\int_0^t (e^{3(t-\tau)} - e^{-5(t-\tau)})g(\tau)\, d\tau + e^{3t} - e^{-5t}.$$

10. Use convolution to show that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(\tau)\, d\tau$$

where $F(s) = \mathcal{L}\{f\}(s)$.

Solution: Simply note that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) * \mathcal{L}^{-1}\{F(s)\}(t) \\ &= (1 * f)(t) \\ &= \int_0^t f(\tau)\, d\tau. \end{aligned}$$

11. Solve the symbolic initial value problem

$$y'' + 5y' + 6y = e^{-t}\delta(t-2), \quad y(0) = 2,\, y'(0) = -5.$$

Solution: Applying the Laplace transform to both sides of the equation, we obtain

$$s^2 Y(s) - 2s + 5 + 5(sY(s) - 2) + 6Y(s) = \mathcal{L}\{e^{-t}\delta(t-2)\}(s).$$

By the translation property of the Laplace trasnform, we have

$$\mathcal{L}\{e^{-t}\delta(t-2)\}(s) = \mathcal{L}\{\delta(t-2)\}(s+1) = e^{-2(s+1)}.$$

Solving for $Y(s)$, we obtain

$$\begin{aligned} Y(s) &= \frac{2s+5+e^{-2(s+1)}}{s^2+5s+6} \\ &= \frac{2s+5+e^{-2(s+1)}}{(s+2)(s+3)} \\ &= \frac{1}{s+2} + \frac{1}{s+3} + e^{-2}e^{-2s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right). \end{aligned}$$

Applying the inverse Laplace transform finally yields

$$y(t) = e^{-2t} + e^{-3t} + e^{-2}\left(e^{-2(t-2)} - e^{3(t-2)}\right)u(t-2).$$

12. A mass attached to a spring is released from rest 1 meter below the equilibrum position for the mass-spring system and begins to vibrate. After $\frac{\pi}{2}$ seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{d^2x}{dt^2} + 9x = -3\delta\left(t - \frac{\pi}{2}\right), \quad x(0) = 1,\, \frac{dx}{dt}(0) = 0$$

where $x(t)$ denotes the displacement from equilibrum at time t. What happens to the mass after it has been struck?

Solution: Applying the Laplace transform, we obtain

$$s^2 X(s) - s + 9X(s) = -3e^{-\frac{\pi}{2}s},$$

so that

$$X(s) = \frac{s}{s^2+9} - e^{-\frac{\pi}{2}s}\frac{3}{s^2+9}.$$

It follows that

$$x(t) = \cos 3t - \sin 3\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right) = \cos 3t - \cos 3t\, u\left(t - \frac{\pi}{2}\right),$$

i.e.,

$$x(t) = \begin{cases} \cos t, & 0 < t < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < t \end{cases}$$

The blow of the hammer thus stops the mass in its equilibrum position.

13. Use Laplace transforms to solve the initial value problem

$$x' + y = x, \quad 2x' + y'' = u(t-3), \qquad x(0) = 0,\ y(0) = 1,\ y'(0) = -1.$$

Solution: Applying the Laplace transform to the two differential equations yields

$$sX(s) + Y(s) = X(s) \qquad \text{and} \qquad 2sX(s) + s^2Y(s) - s + 1 = \frac{e^{-3s}}{s}$$

i.e.,

$$(s-1)X(s) + Y(s) = 0 \qquad \text{and} \qquad 2sX(s) + s^2Y(s) = s - 1 + \frac{e^{-3s}}{s}.$$

Solving for $X(s)$ yields

$$\begin{aligned} X(s) &= \frac{1 - s - e^{-3s}}{s(s^2 - s - 2)} \\ &= \frac{1-s}{s(s+1)(s-2)} - e^{-3s}\frac{1}{s(s+1)(s-2)} \\ &= -\frac{1}{2}\frac{1}{s} + \frac{2}{3}\frac{1}{s+1} - \frac{1}{6}\frac{1}{s-2} - e^{-3s}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} - \frac{1}{3}\frac{1}{s+1} + \frac{1}{12}\frac{1}{s-2}\right) \end{aligned}$$

and thus

$$x(t) = -\frac{1}{2} + \frac{2e^{-t}}{3} - \frac{e^{2t}}{6} - \left(\frac{1}{4} - \frac{t-3}{2} - \frac{e^{3-t}}{3} + \frac{e^{2t-6}}{12}\right)u(t-3).$$

Since $y = x - x'$, we finally obtain

$$y(t) = -\frac{1}{2} + \frac{4e^{-t}}{3} + \frac{e^{2t}}{6} - \left(\frac{3}{4} - \frac{t-3}{2} - \frac{2e^{3-t}}{3} - \frac{e^{2t-6}}{12}\right)u(t-3).$$