STOCHASTIC STABILITY OF PHYSICAL MEASURES IN CONSERVATIVE SYSTEMS

WEIWEI QI, ZHONGWEI SHEN, AND YINGFEI YI

Abstract. Given the significance of physical measures in understanding the complexity of dynamical systems as well as the noisy nature of real-world systems, investigating the stability of physical measures under noise perturbations is undoubtedly a fundamental issue in both theory and practice.

The present paper is devoted to the stochastic stability of physical measures for conservative systems on a smooth, connected, and closed Riemannian manifold. It is assumed that a conservative system admits an invariant measure with a positive and mildly regular density. Our findings affirm, in particular, that such an invariant measure has strong stochastic stability whenever it is physical, that is, for a large class of small random perturbations, the density of this invariant measure is the zero-noise limit in L^1 of the densities of unique stationary measures of corresponding randomly perturbed systems. Stochastic stability in a stronger sense is obtained under additional assumptions. Examples are constructed to demonstrate that stochastic stability could occur even if the invariant measure is non-physical.

Our approach to establishing stochastic stability is rooted in the analysis of Fokker-Planck equations associated with randomly perturbed systems. The crucial element in our proof is the establishment of uniform-in-noise estimates in Sobolev spaces and positive lower and upper bounds for the densities of stationary measures, which are natural yet far-reaching consequences of the conservativeness of the unperturbed system. Not only do these results have stochastic stability as immediate results, but also they readily confirm the so-called sub-exponential large deviation principle of stationary measures. A distinguishing feature of our approach is that it does not rely on uniform, non-uniform, or partial hyperbolicity assumptions, which are often required in the existing literature when investigating stochastic stability. Consequently, our study opens up a new avenue for the exploration of stochastic stability and related issues.

Contents

2010 *Mathematics Subject Classification.* Primary 60J60, 60J70, 34F05; secondary 92D25, 60H10.

Key words and phrases. stochastic stability, conservative system, invariant measure, physical measure, ergodicity, stationary measure.

Z.S. was partially supported by a start-up grant from the University of Alberta, NSERC RGPIN-2018-04371 and NSERC DGECR-2018-00353. Y.Y. was partially supported by NSERC RGPIN-2020-04451, a faculty development grant from the University of Alberta, and a Scholarship from Jilin University.

1. Introduction

Invariant measures are fundamental objects in the study of statistical properties of dynamical systems and are especially powerful in the characterization of complex dynamics. Among these, physical measures [27, 52] – those that can be observed – are of notable interest and paramount relevance in numerous practical applications across various scientific and engineering disciplines. Statistical properties such as entropy, Lyapunov exponents, and mixing properties, quantified by physical measures, often provide valuable information about a system's predictability, sensitivity to initial conditions, and overall complexity. Since the pioneering work on Axiom A attractors [47, 43, 42, 21], finding physical measures that can capture the complexity of systems has attracted a lot of attention resulting in a substantial body of literature. Interested readers are referred to surveys [52, 53] and references therein for earlier developments, and to [49, 26, 10, 23, 5, 39, 7, 20, 24, 22], to name just a few, for more recent ones.

Given that real-world systems are intrinsically slightly noisy, investigating the stability of dynamical systems, particularly their key dynamical properties, under small noise perturbations is of both theoretical and practical significance [28, 37, 17]. This makes it imperative and fundamental to study the stochastic stability of physical measures, especially those related to complex dynamical behaviours, that is crucial for making reliable predictions about the long-term behaviours and helps in understanding how noises affect systems' dynamics. In the present paper, we focus on this issue for conservative systems.

1.1. **Setup.** We formulate the problem and present our findings in dimension $d \geq 2$; the one-dimensional case is special and treated separately in Section 4. From now on, we are going to use some special notations and direct the reader to Subsection 1.3 for their precise meanings.

Let (M, g) be a d-dimensional smooth, connected and closed manifold endowed with a Riemannian metric g, and denote by Vol the Riemannian volume of M. It is assumed that g is normalized so that $\mathrm{Vol}(M) = 1$ (see [38, Chapter 16]). Consider the following ordinary differential equation (ODE) over M:

$$\dot{x} = B(x), \tag{1.1}$$

where $B : M \to TM$ is a Lipschitz continuous vector field on M, ensuring the global well-posedness of (1.1). The system (1.1) is assumed to be conservative (or generalized volume-preserving) in the sense that (1.1) admits an invariant measure μ_0 with a positive density $u_0 \in W^{1,p_0}$ for some $p_0 > d$ with respect to Vol. Clearly, $\mathrm{div}(u_0 B) = 0$ Vol-a.e. (see Lemma 3.1).

Note that (1.1) may admit many and even infinitely many invariant measures like μ_0; typical examples are rotations on tori with rationally dependent frequencies. In which case, μ_0 is a representation of these invariant measures.

To investigate the stochastic stability of μ_0 or any similar invariant measures, we examine the following small random perturbation of (1.1):

$$dX_t^\epsilon = B(X_t^\epsilon)dt + \epsilon^2 A_0^\epsilon(X_t^\epsilon)dt + \epsilon \sum_{i=1}^m A_i^\epsilon(X_t^\epsilon) \circ dW_t^i, \tag{1.2}$$

where $0 < \epsilon \ll 1$ is the noise intensity, $m \geq d$, $A = \{A_i^\epsilon, i \in \{0, 1, \dots m\}, \ \epsilon\}$ is a collection of vector fields on M, $\{W_t^i\}$ are m independent and standard one-dimensional Brownian motions on some probability space, and the stochastic integrals are understood in the sense of Stratonovich. The vector fields A are chosen from the *admissible class* defined as follows.

Definition 1.1 (Admissible class)**.** *A collection A of vector fields on M is said to be in the* admissible class $\mathcal{A}$ *if $A = \{A_i^\epsilon, i \in \{0, \dots, m\}, \ \epsilon\}$ for some $m \geq d$ and the following conditions are satisfied:*

(A1) *there exists $p > d$ such that $A_0^\epsilon \in L^p$, $A_i^\epsilon \in W^{1,p}$ for $i \in \{1, \dots, m\}$, and*

$$\|A_0^\epsilon\|_p + \max_i \|A_i^\epsilon\|_{1,p} \lesssim 1;$$

(A2) *there exists $\lambda > 0$ such that*

$$\inf_\epsilon \sum_{i=1}^m |A_i^\epsilon f|^2 \geq \lambda |\nabla f|^2 \quad \text{Vol-}a.e., \quad \forall f \in W^{1,1}.$$

The condition **(A1)** is a mild integrability condition. The condition **(A2)** is a uniform-in-ϵ positivity condition, guaranteeing in particular the non-degeneracy of the stochastic differential equation (SDE) (1.2) when $A \in \mathcal{A}$.

It should be pointed out that for $A \in \mathcal{A}$, the SDE (1.2) may not be well-posed even in the weak sense, and therefore, transition probabilities and stationary distributions are hardly defined. However, it is quite convenient to work with the Fokker-Planck equation associated with the SDE (1.2):

$$\partial_t u = \mathcal{L}_\epsilon^* u, \tag{1.3}$$

where $\mathcal{L}_\epsilon^*$ is the Fokker-Planck operator, which is the formal L^2-adjoint operator of the generator $\mathcal{L}_\epsilon$ given by

$$\mathcal{L}_\epsilon := \frac{\epsilon^2}{2} \sum_{i=1}^m (A_i^\epsilon)^2 + \epsilon^2 A_0^\epsilon + B.$$

This weak formalism of the SDE (1.2) is often adopted when its coefficients have low regularity (see e.g. [19]). We are mostly interested in *stationary (probability) measures* of (1.2) (that is, stationary solutions of (1.3) in the class of probability measures) that generalize stationary distributions of SDEs. The reader is referred to Appendix A for the basics of stationary measures.

According to Theorem A.1, if $A \in \mathcal{A}$, then for each ϵ, the SDE (1.2) admits a unique stationary measure μ_ϵ having a positive density $u_\epsilon \in W^{1,p}$, where $p > d$ is the same as in **(A1)**. We have suppressed the dependence of μ_ϵ and u_ϵ on A; this shall cause no trouble. Moreover, the set $\mathcal{M}_A$ of all the limiting measures of μ_ϵ as $\epsilon \to 0$ under the weak*-topology is non-empty thanks to the compactness of M and Prokhorov's theorem, and each element in $\mathcal{M}_A$ must be an invariant measure of (1.1) [37].

The stochastic stability of an invariant measure of (1.1) is defined as follows.

Definition 1.2 (Stochastic stability). *An invariant measure μ of* (1.1) *is said to be* stochastically stable *with respect to $\mathcal{A}$ if $\mathcal{M}_A = \{\mu\}$ for all $A \in \mathcal{A}$.*

In the present paper, we mainly focus on addressing the stochastic stability issue of the invariant measure μ_0. The *strong admissible class* of vector fields is considered in pursuit of stronger results regarding the stochastic stability.

Definition 1.3 (Strong admissible class). *A collection A of vector fields on M is said to be in the* strong admissible class $\mathcal{SA}$ *if $A = \{A_i^\epsilon, i \in \{0, \dots, m\}, \ \epsilon\}$ for some $m \geq d$ and it satisfies* **(A2)** *and*

(SA1) *there exists $p > d$ such that*

$$\|A_0^\epsilon\|_{1,p} + \max_i \|A_i^\epsilon\|_{2,p} \lesssim 1.$$

1.2. **Statement of main results.** Our first result addresses uniform estimates of stationary measures $\{u_\epsilon\}_\epsilon$ as well as the stochastic stability of μ_0.

Recall that an invariant measure μ of (1.1) is called a *physical measure* (or *physical*) if $\mathrm{Vol}(B_\mu) > 0$, where

$$B_\mu := \left\{ x \in M : \lim_{t \to \infty} \frac{1}{t} \int_0^t \delta_{\varphi^s(x)} ds = \mu \quad \text{under the weak*-topology} \right\}$$

is referred to as the basin of μ, where φ^t denotes the flow generated by solutions of (1.1).

Theorem A. *The following statements hold.*

(1) *For any $A \in \mathcal{A}$, there hold*

$$\|u_\epsilon\|_{1,2} \lesssim 1 \quad \text{and} \quad 1 \lesssim \min u_\epsilon \leq \max u_\epsilon \lesssim 1.$$

In particular, any $\mu \in \mathcal{M}_A$ has a density u belonging to $W^{1,2}$ and satisfying $u, \frac{1}{u} \in L^\infty$.

(2) *If μ_0 is physical, then it is stochastically stable with respect to $\mathcal{A}$.*

(3) *If μ_0 is the only invariant measure of* (1.1) *with a density in $W^{1,2}$, then it is stochastically stable with respect to $\mathcal{A}$.*

In the case of either (2) or (3), the limit $\lim_{\epsilon \to 0} u_\epsilon = u_0$ holds weakly in $W^{1,2}$ and strongly in L^p for any $p \in [1, \frac{2d}{d-2})$.

Remark 1.1. *We make some comments about Theorem* A.

(i) *The uniform estimates of $\{u_\epsilon\}_\epsilon$, having conclusions in (2) and (3) as immediate consequences, are natural yet far-reaching consequences of the conservativeness of the unperturbed system* (1.1). *Our approach to establishing these uniform estimates builds on analyzing the stationary Fokker-Planck equation satisfied by u_ϵ, namely, $\mathcal{L}_\epsilon^* u_\epsilon = 0$ in the weak sense. Developing uniform-in-ϵ Harnack's estimate and Moser iteration for u_ϵ plays a crucial role in the proof.*

In addition to the implications for stochastic stability, these uniform estimates are key to the justification of the sub-exponential large deviation principle of $\{u_\epsilon\}_\epsilon$ (see Remark 1.2).

(ii) *It is shown in Corollary 3.1 that μ_0 is physical if and only if it is ergodic. The ergodicity of μ_0 clearly implies that it is the only invariant measure of* (1.1) *with a density in $W^{1,2}$, yielding that (2) is stronger than (3). In Theorem* B, *we construct examples that satisfy (3) while not fulfilling (2).*

(iii) *The limit $\lim_{\epsilon \to 0} u_\epsilon = u_0$ holds in L^1 for each $A \in \mathcal{A}$. In literature, such a result is often called the* strong *stochastic stability with respect to $\mathcal{A}$.*

(iv) *In the case where B is divergence free, the stochastic stability of the volume* Vol *with respect to homogeneous noises (that is, $A_0^\epsilon = 0$ and $A_i^\epsilon = A_i$ is a constant vector field for each $i \in \{1, \ldots, m\}$) was obtained in* [1]*. This result is straightforward as the unique stationary measure μ_ϵ coincides with* Vol*. It is generalized in Theorem 3.1 that sheds light on the issues of* invariant measure selection by noise *and* stochastic instability *when* (1.1) *admits multiple invariant measures like μ_0.*

Introduced by Kolmogorov and Sinai [47, 48], the stochastic stability of invariant measures, especially physical measures, has attracted considerable attention and received affirmative results in many instances. The reader is referred to [37] and references therein for earlier investigations concerning uniquely ergodic systems or systems having relatively simple dynamics. Significant advancements have been achieved in various settings, including uniformly hyperbolic systems [36, 51], systems that feature spectral gaps applicable to many exponentially mixing piecewise expanding maps [34, 12, 11, 15, 16, 35], and non-uniformly hyperbolic maps [33, 14, 40, 3, 2, 8, 13, 4, 6, 44, 45].

A distinguishing feature of our results (Theorem A and the subsequent Theorems B and C) is that they do not rely on uniform, non-uniform, or partial hyperbolicity assumptions, which are typically required in most previous works. While these hyperbolicity assumptions have become standard in the geometric or statistical theory of smooth dynamical systems, they are often not met or challenging to verify in the case of complex systems such as those related to complex fluids. Therefore, our study opens up a new avenue for the exploration of stochastic stability and related issues.

When μ_0 is a physical measure, Theorem A(2) asserts its stochastic stability with respect to $\mathcal{A}$. While there are an abundance of examples asserting the stochastic instability of μ_0 with respect to $\mathcal{A}$ when it fails to be physical (see Remark 3.1), the converse is generally wrong as shown in the following result. In which, we construct a volume-preserving system where the normalized volume is stochastically stable but non-physical.

Theorem B. *There exist a three-dimensional smooth, connected, and closed Riemannian manifold M and a smooth divergence-free vector field $B : M \to TM$ satisfying the following conditions:*

(1) *$M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial$, where M_1 and M_2 are three-dimensional smooth, connected and compact manifolds with a common boundary ∂,*
(2) *the interior M_i° of M_i for each $i = 1, 2$ and the boundary ∂ are invariant under φ^t,*
(3) $\mathrm{Vol}|_{M_i^\circ}$ *is strongly mixing for each $i = 1, 2$,*

where φ^t is the flow generated by B. Then, the following hold.

(i) *Any invariant measure of φ^t has the form of $\nu_1 + \nu_2 + \nu_\partial$, where ν_1, ν_2 and ν_∂ are invariant measures of φ^t when restricted to M_1°, M_2° and ∂, respectively.*
(ii) Vol *is neither ergodic nor physical, but it is the only invariant measure of φ^t admitting a density in $W^{1,2}$.*
(iii) Vol *is stochastically stable with respect to $\mathcal{A}$.*

Note that, in the context of the flow φ^t described in Theorem B, both $\frac{\mathrm{Vol}|_{M_1}}{\mathrm{Vol}(M_1)}$ and $\frac{\mathrm{Vol}|_{M_2}}{\mathrm{Vol}(M_2)}$ are physical measures of φ^t. However, it is important to note that neither of them is stochastically stable with respect to $\mathcal{A}$. This observation prompts an intriguing question since there is a prevailing belief that physical measures should demonstrate stability under small random perturbations to some extent. This issue can be attributed to a timescale problem, implying that a physical measure may exhibit stochastic stability only over certain finite timescales. The exploration of this matter will be the subject of future investigations.

The conclusions in Theorem A can be enhanced if additional conditions on u_0, A and B are imposed.

Theorem C. *Assume $u_0 \in W^{2,p_0}$. Suppose there are vector fields $X_i : M \to TM$, $i \in \{1, \dots, n\}$ with $n \geq d$, belonging to $W^{1,p}$ for some $p > d$ such that*

(i) *for each i, $\operatorname{div}(u_0 X_i) = 0$ and $[u_0 X_i, u_0 B] = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket;*
(ii) *$\{X_i\}_{i=1}^n$ spans the tangent bundle TM.*

Then, for each $A \in \mathcal{SA}$,

(1) *$\|u_\epsilon\|_{1,q} \lesssim 1$ for all $q \geq 1$;*
(2) *if μ_0 is the only invariant measure of (1.1) with a density in $W^{1,2}$, then*

$$\lim_{\epsilon \to 0} u_\epsilon = u_0 \text{ in } C^\alpha, \quad \forall \alpha \in (0, 1).$$

Theorem C applies particularly to rotations on tori. In fact, for a rotation on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, we can choose μ_0 as the normalized Lebesgue measure, set $n = d$, and define $X_i := \frac{\partial}{\partial x_i}$ for $i \in \{1, \dots, d\}$. Under these choices, the conditions in Theorem C are satisfied.

Consider the case when M is the 2-sphere $\mathbb{S}^2$ equipped with the spherical coordinate (ϕ, θ), where ϕ and θ stand for the longitude and latitude, respectively. Let $B := \frac{\partial}{\partial \theta}$ be the vector field generating a volume-preserving flow. Straightforward calculations show that if X is a divergence-free vector field satisfying $[X, B] = 0$, and B and X are linearly independent, then X must have a component $\frac{C}{\sin\phi}\frac{\partial}{\partial\theta}$ for some $C \in \mathbb{R} \setminus \{0\}$. Consequently, X does not belong to $W^{1,p}$ for any $p > 2$. Hence, the conditions in Theorem C cannot be satisfied in this case.

Remark 1.2. *It is worthwhile to highlight the significance of the conclusions drawn in Theorems A and C for justifying the sub-exponential large deviation principle (LDP), also known as the zeroth-order WKB expansion [29].*

For randomly perturbed conservative systems, the sub-exponential LDP of invariant measures $\{\mu_\epsilon\}_\epsilon$ concerns the rigorous justification of

$$u_\epsilon = R_\epsilon e^{-\frac{2}{\epsilon^2}V} \quad \text{with} \quad \min V = 0 \text{ and } R_\epsilon = R_0 + o(\epsilon^2),$$

where V and R_ϵ are called the quasi-potential function and prefactor, respectively. Theorems A and C assert that $V = 0$, and therefore, $R_\epsilon = u_\epsilon$ has relevant properties.

This outcome sheds light on rigorously justifying the sub-exponential LDP of invariant measures $\{\mu_\epsilon\}_\epsilon$ for randomly perturbed dissipative systems, particularly when the global attractor of the dissipative system is a Riemannian manifold M, and the dissipative system, when restricted to M, is a conservative system in the same sense as that of (1.1). In such instances, one needs to justify

$$u_\epsilon = \frac{R_\epsilon}{\epsilon^{N-d}} e^{-\frac{2}{\epsilon^2}V} \quad \text{with} \quad \min V = 0 \text{ and } R_\epsilon = R_0 + o(\epsilon^2),$$

where N is the system's dimension and d is the dimension of M. The dynamics of the dissipative system on M for sure plays a crucial role in quantifying the limit of u_ϵ.

The sub-exponential LDP for randomly perturbed dissipative systems has many applications, including classical first exit problems [25], stochastic bifurcations [54, 29], stochastic populations [9], and the landscape and flux theory of non-equilibrium systems [50]. However, thus far, it has been only established in scenarios where the global attractor is a non-degenerate equilibrium point [46, 25, 41], owing to certain essential difficulties.

We conclude this section by discussing our study in one dimension. The one-dimensional case is highly exceptional, as the conservative vector field B, which is identified as a function on M, must either vanish entirely or be sign-definite. We focus on the sign-definite scenario, which is evidently more interesting. This unique characteristic enables us to provide a significantly more concise proof of the one-dimensional counterpart of Theorem A(1) and to establish uniform estimates for stationary measures for a substantially broader range of random perturbations. The corresponding findings are detailed in Section 4.

1.3. **Organization and notation.** The rest of the paper is organized as follows. In Section 2, we establish uniform estimates for the stationary measures of (1.2) when the vector field B is divergence-free. Section 3 is dedicated to proving Theorems A-C. When the system (1.1) admits multiple invariant measures like μ_0, the problem of invariant measure selection by noise is studied in Subsection 3.3. In Section 4, we address the one-dimensional case. Appendix A contains a classical result regarding the existence and uniqueness of stationary measures for SDEs with less regular coefficients. Additionally, in Appendix B, we collect some commonly used formulas from calculus on manifolds.

Notation. The following list of notations are used throughout this paper.

- $d_* := \frac{d}{d-2}$ if $d \geq 3$ and fix any $d_* \in (1, \infty)$ if $d = 2$.
- $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.
- The (vector-valued) spaces $L^p(M)$, $W^{k,p}(M)$, and $C^\alpha(M)$ are written as L^p, $W^{k,p}$, and C^α, respectively.
- The usual L^p-norm and $W^{k,p}$-norm are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively.
- For constants $\alpha(\epsilon)$ and $\beta(\epsilon)$ indexed by ϵ, we write
  - $\alpha(\epsilon) \lesssim \beta(\epsilon)$ (resp. $\alpha(\epsilon) \gtrsim \beta(\epsilon)$) if there is a positive constant C, independent of ϵ, such that $\alpha(\epsilon) \leq C\beta(\epsilon)$ for all ϵ (resp. $\alpha(\epsilon) \geq C\beta(\epsilon)$ for all ϵ);
  - $\alpha(\epsilon) \approx \beta(\epsilon)$ if $\alpha(\epsilon) \lesssim \beta(\epsilon)$ and $\alpha(\epsilon) \gtrsim \beta(\epsilon)$.
- For $f \in C^0$, $\min_M f$ and $\max_M f$ are written as $\min f$ and $\max f$, respectively.
- The integral $\int_M f d\mathrm{Vol}$ is written as $\int f$, and the integral $\int_M f d\mu$, where μ is a measure on M, is written as $\int f d\mu$.
- Einstein's summation convention is used throughout the paper unless otherwise specified.

2. Uniform estimates in the divergence-free case

This section is devoted to uniform estimates for stationary measures of (1.2) under different assumptions on the vector field B. The corresponding result serves as a crucial step in the proof of our main results, and is of independent interest.

We make the following assumptions on the vector field B.

(A)$_B$ $B \in W^{1,p_0}$ for some $p_0 > \frac{d}{2}$ and $\mathrm{div} B = 0$ Vol-a.e.

That is, B is divergence-free and has weak regularity. It is well known that the flow generated by B, if exists, is volume-preserving. Unfortunately, $B \in W^{1,p_0}$ is insufficient for (1.1) to generate such a flow. But, this causes no trouble at all since the dynamical implication is off the table in this section.

Note that $B \in L^{p_1}$ with some $p_1 > d$ thanks to the Sobolev embedding theorem. Hence, Theorem A.1 applies and yields that if $A \in \mathcal{A}$, then for each ϵ, the SDE (1.2) admits a unique stationary measure μ_ϵ having a positive density $u_\epsilon \in W^{1,p_*}$, where $p_* := \min\{p_1, p\}$ and p is as assumed in **(A1)**. Moreover,

$$-\frac{\epsilon^2}{2}\int A_i^\epsilon f\left[A_i^\epsilon u_\epsilon + (\mathrm{div} A_i^\epsilon)u_\epsilon\right] + \int \left(\epsilon^2 A_0^\epsilon f + Bf\right)u_\epsilon = 0, \quad \forall f \in W^{1,2}. \tag{2.1}$$

The following two theorems are the main results in this section.

Theorem 2.1. *Assume* $(\mathbf{A})_B$. *For any* $A \in \mathcal{A}$, *there hold*

$$\|u_\epsilon\|_{1,2} \lesssim 1 \quad \text{and} \quad 1 \lesssim \min u_\epsilon \le \max u_\epsilon \lesssim 1.$$

Theorem 2.2. *Assume* $(\mathbf{A})_B$ *with* $p_0 > d$. *Suppose there are vector fields* $X_i : M \to TM$, $i \in \{1, \dots, n\}$ *with* $n \ge d$, *belonging to* $W^{1,p}$ *for some* $p > d$ *such that*

(i) *for each* i, $\mathrm{div} X_i = 0$ *and* $[X_i, B] = 0$;
(ii) $\{X_i\}_{i=1}^n$ *spans the tangent bundle* TM.

Then, for any $A \in \mathcal{SA}$, *there holds* $\|u_\epsilon\|_{1,q} \lesssim 1$ *for all* $q \ge 1$.

The rest of this subsection is devoted to the proof of Theorems 2.1 and 2.2. To proceed with the proof of Theorem 2.1, we need two lemmas. From now on, we will frequently use some formulas relevant to calculus on manifolds. The reader is directed to Appendix B for details.

Lemma 2.1. *Assume* $(\mathbf{A})_B$. *For any* $A \in \mathcal{A}$, *we have* $\|u_\epsilon\|_{1,2} \lesssim 1$.

Proof. Let $A \in \mathcal{A}$. Taking $f = u_\epsilon$ in (2.1) yields

$$-\frac{\epsilon^2}{2}\int A_i^\epsilon u_\epsilon[A_i^\epsilon u_\epsilon + (\mathrm{div} A_i^\epsilon)u_\epsilon] + \int (\epsilon^2 A_0^\epsilon u_\epsilon + Bu_\epsilon)u_\epsilon = 0. \tag{2.2}$$

Since $\mathrm{div} B = 0$ by $(\mathbf{A})_B$, we derive from the divergence theorem that

$$\int (Bu_\epsilon)u_\epsilon = \frac{1}{2}\int Bu_\epsilon^2 = -\frac{1}{2}\int (\mathrm{div} B)\, u_\epsilon^2 = 0,$$

which together with (2.2) leads to

$$\sum_{i=1}^m \int |A_i^\epsilon u_\epsilon|^2 + \int (A_i^\epsilon u_\epsilon)(\mathrm{div} A_i^\epsilon)u_\epsilon - 2\int (A_0^\epsilon u_\epsilon)u_\epsilon = 0.$$

The assumption **(A1)** and the Sobolev embedding theorem ensure $\max_i \|A_i^\epsilon\|_\infty \lesssim 1$ and $\|A_0^\epsilon\|_p + \max_i \|\mathrm{div} A_i^\epsilon\|_p \lesssim 1$. Hence, we apply Hölder's inequality to deduce that

$$\sum_{i=1}^m \int |A_i^\epsilon u_\epsilon|^2 \lesssim \|\nabla u_\epsilon\|_2 \|u_\epsilon\|_r \left(\sum_{i=1}^m \|\mathrm{div} A_i^\epsilon\|_p + \|A_0^\epsilon\|_p\right) \lesssim \|\nabla u_\epsilon\|_2 \|u_\epsilon\|_r,$$

where $r := \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} \in (2, 2d_*)$. As $\sum_{i=1}^m \int |A_i^\epsilon u_\epsilon|^2 \gtrsim \|\nabla u_\epsilon\|_2^2$ ensured by **(A2)**, we arrive at $\|\nabla u_\epsilon\|_2 \lesssim \|u_\epsilon\|_r$. The Sobolev embedding theorem then leads to

$$\|u_\epsilon\|_{2d_*} \lesssim \|u_\epsilon\|_2 + \|\nabla u_\epsilon\|_2 \lesssim \|u_\epsilon\|_r, \tag{2.3}$$

where we used $\|u_\epsilon\|_2 \lesssim \|u_\epsilon\|_r$ in the second inequality.

Note that $\|u_\epsilon\|_r \le \|u_\epsilon\|_1^\alpha \|u_\epsilon\|_{2d_*}^{1-\alpha}$ (by interpolation) with $\alpha := \left(\frac{1}{d} - \frac{1}{p}\right)\left(\frac{1}{2} + \frac{1}{d}\right)^{-1}$. The fact $\|u_\epsilon\|_1 = 1$ results in $\|u_\epsilon\|_r \le \|u_\epsilon\|_{2d_*}^{1-\alpha}$. It then follows from (2.3) that $\|u_\epsilon\|_{2d_*} \lesssim 1$, and thus, $\|u_\epsilon\|_r \lesssim 1$. The desired conclusion follows readily from the second inequality in (2.3). □

Lemma 2.2. *Assume* $\mathbf{(A)}_B$*. For any* $A\in\mathcal{A}$*, the following hold.*

(1) $\max u_\epsilon \lesssim 1$*;*
(2) *For any* $\gamma>0$*, we have* $\min u_\epsilon \gtrsim \|u_\epsilon^{-1}\|_\gamma^{-1}$*.*

Proof. We first establish some estimates for u_ϵ. Recall that $u_\epsilon\in W^{1,p_*}$ and u_ϵ, $u_\epsilon^{-1}\in L^\infty$ due to the Sobolev embedding theorem.

Let $q\in\mathbb{R}\setminus\{0,1\}$. In the following, for any two constants $\alpha(\epsilon,q)$ and $\beta(\epsilon,q)$ indexed by ϵ and q, we write $\alpha(\epsilon,q)\lesssim\beta(\epsilon,q)$ to imply the existence of C (independent of ϵ and q) such that $\alpha(\epsilon,q)\le C\beta(\epsilon,q)$ for all (ϵ,q). Setting $f=u_\epsilon^{q-1}\in W^{1,2}$ in (2.1) gives rise to

$$-\frac{\epsilon^2}{2}\int (A_i^\epsilon u_\epsilon^{q-1})\,[A_i^\epsilon u_\epsilon+(\mathrm{div}A_i^\epsilon)u_\epsilon]+\int(\epsilon^2A_0^\epsilon u_\epsilon^{q-1}+Bu_\epsilon^{q-1})u_\epsilon=0.$$

Since $\mathrm{div}B=0$ by $\mathbf{(A)}_B$, we apply the divergence theorem to find

$$\int(Bu_\epsilon^{q-1})u_\epsilon=\frac{1}{q}\int Bu_\epsilon^q=-\frac{1}{q}\int(\mathrm{div}B)u_\epsilon^q=0.$$

Therefore,

$$\begin{aligned}0&=\int (A_i^\epsilon u_\epsilon^{q-1})\,[A_i^\epsilon u_\epsilon+(\mathrm{div}A_i^\epsilon)u_\epsilon]-2\int(A_0^\epsilon u_\epsilon^{q-1})u_\epsilon\\&=(q-1)\sum_{i=1}^m\int u_\epsilon^{q-2}|A_i^\epsilon u_\epsilon|^2+(q-1)\int(\mathrm{div}A_i^\epsilon)u_\epsilon^{q-1}A_i^\epsilon u_\epsilon-2(q-1)\int u_\epsilon^{q-1}A_0^\epsilon u_\epsilon.\end{aligned}$$

Since $A\in\mathcal{A}$, we apply Hölder's inequality to derive

$$\begin{aligned}\int u_\epsilon^{q-2}|\nabla u_\epsilon|^2\lesssim\sum_{i=1}^m\int u_\epsilon^{q-2}|A_i^\epsilon u_\epsilon|^2&=-\int(\mathrm{div}A_i^\epsilon)u_\epsilon^{q-1}A_i^\epsilon u_\epsilon+2\int u_\epsilon^{q-1}A_0^\epsilon u_\epsilon\\&\lesssim\left(\int u_\epsilon^{q-2}|\nabla u_\epsilon|^2\right)^{\frac12}\|u_\epsilon^{\frac q2}\|_r\left(\sum_{i=1}^m\|\mathrm{div}A_i^\epsilon\|_p+\|A_0^\epsilon\|_p\right)\\&\lesssim\left(\int u_\epsilon^{q-2}|\nabla u_\epsilon|^2\right)^{\frac12}\|u_\epsilon^{\frac q2}\|_r,\end{aligned}$$

where $r:=\left(\frac12-\frac1p\right)^{-1}\in(2,2d_*)$. As a result, $\int|\nabla u_\epsilon^{\frac q2}|^2=\frac{q^2}{4}\int u_\epsilon^{q-2}|\nabla u_\epsilon|^2\lesssim q^2\|u_\epsilon^{\frac q2}\|_r^2$, which together with the Sobolev embedding theorem leads to

$$\|u_\epsilon^{\frac q2}\|_{2d_*}^2\lesssim\|u_\epsilon^{\frac q2}\|_2^2+\|\nabla u_\epsilon^{\frac q2}\|_2^2\lesssim\|u_\epsilon^{\frac q2}\|_2^2+q^2\|u_\epsilon^{\frac q2}\|_r^2.$$

Noting that $\|u_\epsilon^{\frac q2}\|_2^2\lesssim\|u_\epsilon^{\frac q2}\|_r^2$, we find $C_*>0$, independent of ϵ and q, such that $\|u_\epsilon^{\frac q2}\|_{2d_*}^2\le C_*q^2\|u_\epsilon^{\frac q2}\|_r^2$. As $\|u_\epsilon^{\frac q2}\|_r\le\|u_\epsilon^{\frac q2}\|_2^\alpha\|u_\epsilon^{\frac q2}\|_{2d_*}^{1-\alpha}$ (by interpolation) with $\alpha:=1-\frac dp$, we deduce that

$$\|u_\epsilon\|_{qd_*}\le C_*^{\frac{1}{\alpha q}}q^{\frac{2}{\alpha q}}\,\|u_\epsilon\|_q\quad\text{if }q>0,\ q\neq1,\tag{2.4}$$

and

$$\|u_\epsilon^{-1}\|_{|q|d_*}\le C_*^{\frac{1}{\alpha|q|}}|q|^{\frac{2}{\alpha|q|}}\|u_\epsilon^{-1}\|_{|q|}\quad\text{if }q<0.\tag{2.5}$$

Now, we prove the results.

(1) Setting $q = 2d_*^k$ in (2.4) for each $k \in \mathbb{N}_0$, we see that

$$\|u_\epsilon\|_{2d_*^{k+1}} \le C_*^{\frac{1}{2\alpha d_*^k}} (2d_*^k)^{\frac{1}{\alpha d_*^k}} \|u_\epsilon\|_{2d_*^k}, \quad \forall k \in \mathbb{N}_0.$$

By iteration,

$$\|u_\epsilon\|_\infty = \lim_{k\to\infty} \|u_\epsilon\|_{2d_*^k} \le (4C_*)^{\frac{1}{2\alpha}\sum_{k=0}^\infty \frac{1}{d_*^k}} d_*^{\frac{1}{\alpha}\sum_{k=0}^\infty \frac{k}{d_*^k}} \|u_\epsilon\|_2 \lesssim \|u_\epsilon\|_2 \lesssim \|u_\epsilon\|_\infty^{\frac{1}{2}} \|u_\epsilon\|_1^{\frac{1}{2}}.$$

The result then follows from the fact $\|u_\epsilon\|_1 = 1$.

(2) For $\gamma > 0$, we set $q = -\gamma d_*^k$ in (2.5) for each $k \in \mathbb{N}_0$ to find that

$$\left\|u_\epsilon^{-1}\right\|_{\gamma d_*^{k+1}} \le C_*^{\frac{1}{\alpha\gamma d_*^k}} \left(\gamma d_*^k\right)^{\frac{2}{\alpha\gamma d_*^k}} \left\|u_\epsilon^{-1}\right\|_{\gamma d_*^k}, \quad \forall k \in \mathbb{N}_0.$$

It follows that

$$\|u_\epsilon^{-1}\|_\infty \le (C_*\gamma^2)^{\frac{1}{\alpha\gamma}\sum_{k=0}^\infty \frac{1}{d_*^k}} d_*^{\frac{2}{\alpha\gamma}\sum_{k=0}^\infty \frac{k}{d_*^k}} \|u_\epsilon^{-1}\|_\gamma,$$

that is, $\min u_\epsilon \gtrsim \|u_\epsilon^{-1}\|_\gamma^{-1}$. This completes the proof. □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Given Lemmas 2.1 and 2.2, it remains to prove

$$\|u_\epsilon^{-1}\|_\gamma \lesssim 1 \quad \text{for some} \quad \gamma > 0. \tag{2.6}$$

To verify (2.6), we set $v_\epsilon := \ln u_\epsilon - \int \ln u_\epsilon$ and break the proof into four steps.

Step 1. We show that for each $\eta \in C^2(\mathbb{R})$,

$$\sum_{i=1}^m \int [\eta'(v_\epsilon) - \eta''(v_\epsilon)] |A_i^\epsilon v_\epsilon|^2 = \int [\eta''(v_\epsilon) - \eta'(v_\epsilon)] \left[(\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right]. \tag{2.7}$$

Recall that $u_\epsilon \in W^{1,p_*}$ and u_ϵ, $u_\epsilon^{-1} \in L^\infty$ due to the Sobolev embedding theorem. Then, for any $g \in W^{1,2}$, we are able to set $f = \frac{g}{u_\epsilon} \in W^{1,2}$ in (2.1) to derive

$$\begin{aligned}
0 &= -\frac{\epsilon^2}{2}\int A_i^\epsilon \frac{g}{u_\epsilon}[A_i^\epsilon u_\epsilon + (\mathrm{div} A_i^\epsilon) u_\epsilon] + \int \left(\epsilon^2 A_0^\epsilon \frac{g}{u_\epsilon} + B\frac{g}{u_\epsilon}\right) u_\epsilon \\
&= -\frac{\epsilon^2}{2}\int \left(\frac{A_i^\epsilon g}{u_\epsilon} - \frac{g}{u_\epsilon^2} A_i^\epsilon u_\epsilon\right)[A_i^\epsilon u_\epsilon + (\mathrm{div} A_i^\epsilon) u_\epsilon] + \epsilon^2 \int \left(\frac{A_0^\epsilon g}{u_\epsilon} - \frac{g}{u_\epsilon^2} A_0^\epsilon u_\epsilon\right) u_\epsilon \\
&\quad + \int \left(\frac{Bg}{u_\epsilon} - \frac{g}{u_\epsilon^2} B u_\epsilon\right) u_\epsilon \\
&= -\frac{\epsilon^2}{2}\int \left(A_i^\epsilon g A_i^\epsilon v_\epsilon - g\sum_{i=1}^d |A_i^\epsilon v_\epsilon|^2\right) - \frac{\epsilon^2}{2}\int (\mathrm{div} A_i^\epsilon)\left(A_i^\epsilon g - g A_i^\epsilon v_\epsilon\right) \\
&\quad + \epsilon^2 \int A_0^\epsilon g - g A_0^\epsilon v_\epsilon + \int Bg - gBv_\epsilon.
\end{aligned} \tag{2.8}$$

Clearly, $v_\epsilon \in W^{1,p_*}$. For each $\eta \in C^2(\mathbb{R})$, we take $g = \eta'(v_\epsilon) \in W^{1,2}$ and then apply the divergence theorem to find from $\mathrm{div} B = 0$ (by $\mathbf{(A)}_B$) that $\int B\eta'(v_\epsilon) = 0$ and $\int \eta'(v_\epsilon) B v_\epsilon = \int B\eta(v_\epsilon) = 0$. As a result, the equality (2.7) follows from (2.8).

Step 2. We show

$$D_0 := \sup_\epsilon \left(\|v_\epsilon\|_2^2 + \|\nabla v_\epsilon\|_2^2\right) < \infty. \tag{2.9}$$

Taking $\eta(t) = t$ in (2.7) yields $\sum_{i=1}^m \int |A_i^\epsilon v_\epsilon|^2 = -\int (\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon$. By **(A1)** and the Sobolev embedding theorem, there hold $\max_i \|A_i^\epsilon\|_\infty \lesssim 1$ and $\|A_0^\epsilon\|_p + \max_i \|\mathrm{div} A_i^\epsilon\|_p \lesssim 1$. We apply Hölder's inequality to find for $r := \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} \in (2, 2d_*)$ that

$$\sum_{i=1}^m \int |A_i^\epsilon v_\epsilon|^2 \lesssim \|\nabla v_\epsilon\|_2 \|1\|_r \left(\sum_{i=1}^m \|\mathrm{div} A_i^\epsilon\|_p + \|A_0^\epsilon\|_p\right) \lesssim \|\nabla v_\epsilon\|_2.$$

By **(A2)**, $\|\nabla v_\epsilon\|^2 \lesssim \sum_{i=1}^m \int |A_i^\epsilon v_\epsilon|^2 \lesssim \|\nabla v_\epsilon\|_2$, and thus, $\|\nabla v_\epsilon\|_2 \lesssim 1$. As $\int v_\epsilon = 0$, we apply Poincaré inequality to find $\|v_\epsilon\|_2 \lesssim 1$, and hence, (2.9).

Before proceeding with the proof, we mention that constants $C_1, \dots, C_6$, which appear in the following **Step 3** and **Step 4**, are independent of ϵ and $q \geq 1$ (to be included in **Step 3**). Moreover, for any two constants $\alpha(\epsilon, q)$ and $\beta(\epsilon, q)$ indexed by ϵ and q, we write $\alpha(\epsilon, q) \lesssim \beta(\epsilon, q)$ to imply the existence of C (independent of ϵ and q) such that $\alpha(\epsilon, q) \leq C\beta(\epsilon, q)$ for all (ϵ, q).

Step 3. We claim the existence of D_1, $D_2 > 0$, independent of ϵ, such that

$$\|v_\epsilon\|_{2qd_*} \leq D_1 q D_2^{\frac{1}{2q}} + D_2^{\frac{1}{2q}} q^{\frac{2}{\alpha q}} \|v_\epsilon\|_{2q}, \quad \forall q \geq 1, \tag{2.10}$$

where $\alpha := 1 - \frac{d}{p} \in (0, 1)$.

Let $q \geq 1$. Setting $\eta(t) = \frac{1}{2q+1} t^{2q+1}$ in (2.7) gives rise to

$$\begin{aligned} \sum_{i=1}^m \int v_\epsilon^{2q} |A_i^\epsilon v_\epsilon|^2 = 2q \sum_{i=1}^m \int v_\epsilon^{2q-1} |A_i^\epsilon v_\epsilon|^2 + 2q \int v_\epsilon^{2q-1} \left[(\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right] \\ - \int v_\epsilon^{2q} \left[(\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right]. \end{aligned} \tag{2.11}$$

Applying Young's inequality leads to

$$|v_\epsilon|^{2q-1} \leq \frac{2q-1}{2q} v_\epsilon^{2q} \delta^{\frac{2q}{2q-1}} + \frac{1}{2q}\delta^{-2q}, \quad \forall \delta > 0.$$

Setting $\delta = [2(2q-1)]^{\frac{1-2q}{2q}}$ yields $|v_\epsilon|^{2q-1} \leq \frac{1}{4q} v_\epsilon^{2q} + \frac{1}{2q}[2(2q-1)]^{2q-1}$, and hence,

$$\begin{aligned} 2q \sum_{i=1}^m \int v_\epsilon^{2q-1} |A_i^\epsilon v_\epsilon|^2 &\leq \frac{1}{2} \sum_{i=1}^m \int v_\epsilon^{2q} |A_i^\epsilon v_\epsilon|^2 + [2(2q-1)]^{2q-1} \|A_i^\epsilon\|_\infty \int |\nabla v_\epsilon|^2 \\ &\leq \frac{1}{2} \sum_{i=1}^m \int v_\epsilon^{2q} |A_i^\epsilon v_\epsilon|^2 + C_1 (4q)^{2q-1}, \end{aligned}$$

where we used (2.9) in the second inequality. Inserting this into (2.11) results in

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^m \int v_\epsilon^{2q} |A_i^\epsilon v_\epsilon|^2 \leq C_1 (4q)^{2q-1} + 2q \int v_\epsilon^{2q-1} \left|(\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right| \\ - \int v_\epsilon^{2q} \left|(\mathrm{div} A_i^\epsilon) A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right|. \end{aligned} \tag{2.12}$$

For the second and third terms on the right-hand side of (2.12), we apply Hölder's inequality and then Young's inequality to find

$$\begin{aligned} 2q\int v_\epsilon^{2q-1}\left|(\mathrm{div}A_i^\epsilon)A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right| &\le 2q(\max_i \|A_i^\epsilon\|_\infty + 1)\left(\sum_{i=1}^m \|\mathrm{div}A_i^\epsilon\|_p + 2\|A_0^\epsilon\|_p\right)\|v_\epsilon^{q-1}\|_r\|v_\epsilon^q\nabla v_\epsilon\|_2 \\ &\le \frac{\lambda}{8}\|v_\epsilon^q\nabla v_\epsilon\|_2^2 + C_2q^2\|v_\epsilon^{q-1}\|_r^2, \end{aligned} \tag{2.13}$$

where r is defined in **Step 2**, and

$$\begin{aligned} \int v_\epsilon^{2q}\left|(\mathrm{div}A_i^\epsilon)A_i^\epsilon v_\epsilon - 2A_0^\epsilon v_\epsilon\right| &\le (1+\max_i \|A_i^\epsilon\|_\infty)\left(\sum_{i=1}^m \|\mathrm{div}A_i^\epsilon\|_p + 2\|A_0^\epsilon\|_p\right)\|v_\epsilon^{q}\|_r\|v_\epsilon^q\nabla v_\epsilon\|_2 \\ &\le \frac{\lambda}{8}\|v_\epsilon^q\nabla v_\epsilon\|_2^2 + C_3\|v_\epsilon^{q}\|_r^2, \end{aligned} \tag{2.14}$$

where $\lambda>0$ is the constant appearing in **(A2)**. Inserting (2.13) and (2.14) back into (2.12), we derive from **(A2)** that

$$\|v_\epsilon^q\nabla v_\epsilon\|_2^2 \lesssim (4q)^{2q-1} + q^2\|v_\epsilon^{q-1}\|_r^2 + \|v_\epsilon^q\|_r^2.$$

According to Hölder's inequality and Young's inequality, we find

$$\|v_\epsilon^{q-1}\|_r^2 \lesssim \|v_\epsilon^q\|_r^{\frac{2(q-1)}{q}} \lesssim \frac{q-1}{q}\|v_\epsilon^q\|_r^2 + \frac{1}{q},$$

and hence,

$$\|v_\epsilon^q\nabla v_\epsilon\|_2^2 \lesssim (4q)^{2q-1} + (q^2-q+1)\|v_\epsilon^q\|_r^2 + q. \tag{2.15}$$

As $v_\epsilon^{2q-2} \le \frac{q-1}{q}v_\epsilon^{2q} + \frac{1}{q}$ by Young's inequality, we deduce

$$\|v_\epsilon^{q-1}\nabla v_\epsilon\|_2^2 \le \frac{q-1}{q}\|v_\epsilon^q\nabla v_\epsilon\|_2^2 + \frac{D_0}{q},$$

where D_0 is given in (2.9). This together with (2.15) leads to

$$\|v_\epsilon^{q-1}\nabla v_\epsilon\|_2^2 \lesssim (4q)^{2q} + q^2\|v_\epsilon\|_{qr}^{2q}.$$

It then follows from the equality $\nabla|v_\epsilon|^q = q|v_\epsilon|^{q-2}v_\epsilon\nabla v_\epsilon$ that

$$\|\nabla|v_\epsilon|^q\|_2^2 \lesssim (4q)^{2q+2} + q^4\|v_\epsilon\|_{qr}^{2q}.$$

Since $\|v_\epsilon\|_{2qd_*}^{2q} \lesssim \|\nabla|v_\epsilon|^q\|_2^2 + \|v_\epsilon\|_{2q}^{2q}$ by the Sobolev embedding theorem, we arrive at

$$\|v_\epsilon\|_{2qd_*}^{2q} \le C_4(4q)^{2q+2} + C_4q^4\|v_\epsilon\|_{qr}^{2q} + C_4\|v_\epsilon\|_{2q}^{2q}. \tag{2.16}$$

Noting that applications of the interpolation inequality and Young's inequality yield

$$C_4q^4\|v_\epsilon\|_{qr}^{2q} \le C_4q^4\|v_\epsilon\|_{2q}^{2q\alpha}\|v_\epsilon\|_{2qd_*}^{2q(1-\alpha)} \le \alpha\left(C_4q^4\right)^{\frac{1}{\alpha}}\|v_\epsilon\|_{2q}^{2q} + (1-\alpha)\|v_\epsilon\|_{2qd_*}^{2q},$$

we substitute the above inequality into (2.16) to find

$$\|v_\epsilon\|_{2qd_*}^{2q} \le \frac{1}{\alpha}C_4(4q)^{2q+2} + \frac{1}{\alpha}\left[C_4 + \alpha\left(C_4q^4\right)^{\frac{1}{\alpha}}\right]\|v_\epsilon\|_{2q}^{2q} \le C_5(4q)^{2q+2} + C_5q^{\frac{4}{\alpha}}\|v_\epsilon\|_{2q}^{2q}.$$

Taking the $2q$-th root of both sides results in

$$\|v_\epsilon\|_{2qd_*} \le (C_5)^{\frac{1}{2q}}(4q)^{\frac{q+1}{q}} + (C_5)^{\frac{1}{2q}} q^{\frac{2}{\alpha q}} \|v_\epsilon\|_{2q},$$

giving rise to (2.10).

Step 4. We finish the proof. Set $q = d_*^k$ and $I_k^\epsilon := \|v_\epsilon\|_{2d_*^k}$ for $k \in \mathbb{N}_0$. It follows from (2.10) that

$$I_{k+1}^\epsilon \le D_1 d_*^k D_2^{\frac{1}{2d_*^k}} + d_*^{\frac{2k}{\alpha d_*^k}} D_2^{\frac{1}{2d_*^k}} I_k^\epsilon.$$

By iteration,

$$\begin{aligned} I_{k+1}^\epsilon &\le D_1 d_*^k D_2^{\frac{1}{2d_*^k}} + D_1 d_*^{k-1} d_*^{\frac{2k}{\alpha d_*^k}} D_2^{\frac{1}{2d_*^k}+\frac{1}{2d_*^{k-1}}} + d_*^{\frac{2k}{\alpha d_*^k}+\frac{2(k-1)}{\alpha d_*^{k-1}}} D_2^{\frac{1}{2d_*^k}+\frac{1}{2d_*^{k-1}}} I_{k-1}^\epsilon \\ &\le \cdots \le D_1 D_2^{\frac{1}{2}\sum_{k=0}^\infty \frac{1}{d_*^k}} d_*^{\frac{2}{\alpha}\sum_{k=0}^\infty \frac{k}{d_*^k}} \left(\sum_{i=0}^k d_*^i + I_0^\epsilon\right) \lesssim \sum_{i=0}^k d_*^i + 1, \end{aligned}$$

where we used the fact $\sum_{k=0}^\infty \left(\frac{1}{d_*^k} + \frac{k}{d_*^k}\right) < \infty$ due to $d_* > 1$ and $I_0^\epsilon \lesssim 1$ due to (2.9). As a result, $I_{k+1}^\epsilon \lesssim d_*^k$ for each $k \in \mathbb{N}_0$. Let $n \ge 2$. Then, there exists a unique $k = k(n) \in \mathbb{N}_0$ such that $2d_*^k \le n < 2d_*^{k+1}$. Applying Hölder's inequality yields $\|v_\epsilon\|_n \lesssim \|v_\epsilon\|_{2d_*^{k+1}} \lesssim d_*^k \le \frac{n}{2}$, and hence, $\|v_\epsilon\|_n \le \frac{C_6 n}{2}$. Setting $\gamma := (eC_6)^{-1}$ results in

$$\frac{\gamma^n}{n!}\int |v_\epsilon|^n \le \frac{\gamma^n}{n!}\left(\frac{C_6 n}{2}\right)^n \le \frac{\gamma^n}{n!}\left(\frac{C_6}{2}\right)^n e^n n! = \frac{1}{2^n}, \quad \forall n \ge 2,$$

where we used $n^n \le e^n n!$ in the second inequality. In addition, we apply Hölder's inequality to find from (2.9) that $\|v_\epsilon\|_1 \lesssim D_0$. As a result,

$$\int e^{\gamma|v_\epsilon|} = 1 + \gamma\int |v_\epsilon| + \sum_{n=2}^\infty \frac{\gamma^n}{n!}\int |v_\epsilon|^n \lesssim 1 + \gamma D_0 + \frac{1}{2} \approx 1.$$

In particular,

$$\int u_\epsilon^{-\gamma}\int u_\epsilon^\gamma = \int e^{\gamma v_\epsilon}\int e^{-\gamma v_\epsilon} \lesssim 1. \tag{2.17}$$

Taking C_6 so large that $\gamma \in (0,1)$, we deduce $\int u_\epsilon^\gamma = \int \frac{u_\epsilon}{u_\epsilon^{1-\gamma}} \ge \frac{1}{\|u_\epsilon\|_\infty^{1-\gamma}}\int u_\epsilon \gtrsim 1$, where we used the fact $\int u_\epsilon = 1$ and Lemma 2.2 (1) in the last inequality. It follows from (2.17) that $\int u_\epsilon^{-\gamma} \lesssim 1$, proving (2.6), and hence, completing the proof. □

Now, we prove Theorem 2.2.

Proof of Theorem 2.2. Let $A \in \mathcal{SA}$. Without loss of generality, we may assume p is such that $A_i^\epsilon \in W^{2,p}$, for $i \in \{1,\dots,m\}$, $A_0^\epsilon \in W^{1,p}$ and $B \in W^{1,p}$. By Theorem A.1, $u_\epsilon \in W^{1,p}$ and (2.1) is satisfied. Moreover, the classical regularity theory for elliptic equations ensures $u_\epsilon \in W^{2,p}$.

The proof is divided into four steps.

Step 1. We show that for each $k\in\{1,\dots,n\}$ and $g\in W^{1,2}$,

$$\frac{\epsilon^2}{2}\int (A_i^\epsilon g)A_i^\epsilon X_k u_\epsilon - \int (Bg)X_k u_\epsilon = \frac{\epsilon^2}{2}(\mathrm{I}+\mathrm{II}), \tag{2.18}$$

where

$$\mathrm{I} := \int A_i^\epsilon g\,(-(\mathrm{div}A_i^\epsilon)X_k u_\epsilon - [X_k, A_i^\epsilon]u_\epsilon) + ([A_i^\epsilon, X_k]g)A_i^\epsilon u_\epsilon + 2(X_k g)A_0^\epsilon u_\epsilon$$

and

$$\mathrm{II} := \int -(A_i^\epsilon g)(X_k \mathrm{div}A_i^\epsilon)u_\epsilon + ([A_i^\epsilon, X_k]g)(\mathrm{div}A_i^\epsilon)u_\epsilon + 2(X_k g)(\mathrm{div}A_0^\epsilon)u_\epsilon.$$

To do so, we fix $k\in\{1,\dots,n\}$ and take $g\in W^{2,2}$. Setting $f=X_k g\in W^{1,2}$ in (2.1), we find

$$-\frac{\epsilon^2}{2}\int A_i^\epsilon X_k g\,[A_i^\epsilon u_\epsilon + (\mathrm{div}A_i^\epsilon)u_\epsilon] + \epsilon^2\int (A_0^\epsilon X_k g)u_\epsilon + \int (BX_k g)u_\epsilon = 0. \tag{2.19}$$

Straightforward calculations yield

$$\begin{aligned}
&\int A_i^\epsilon X_k g\,[A_i^\epsilon u_\epsilon + (\mathrm{div}A_i^\epsilon)u_\epsilon] - 2\int (A_0^\epsilon X_k g)u_\epsilon\\
&\quad= \int (X_k A_i^\epsilon g + [A_i^\epsilon, X_k]g)\,[A_i^\epsilon u_\epsilon + (\mathrm{div}A_i^\epsilon)u_\epsilon] - 2\int (A_0^\epsilon X_k g)u_\epsilon\\
&\quad= \int (X_k A_i^\epsilon g)A_i^\epsilon u_\epsilon + \int (X_k A_i^\epsilon g)(\mathrm{div}A_i^\epsilon)u_\epsilon + \int ([A_i^\epsilon, X_k]g)\,[A_i^\epsilon u_\epsilon + (\mathrm{div}A_i^\epsilon)u_\epsilon] - 2\int (A_0^\epsilon X_k g)u_\epsilon\\
&\quad=: \int (X_k A_i^\epsilon g)A_i^\epsilon u_\epsilon + \mathrm{I}' + \mathrm{II}' + \mathrm{III}'.
\end{aligned} \tag{2.20}$$

Since $\mathrm{div}X_k=0$ and $[X_k,B]=0$, we derive

$$\begin{aligned}
\int (X_k A_i^\epsilon g)A_i^\epsilon u_\epsilon &= -\int (\mathrm{div}X_k)A_i^\epsilon g A_i^\epsilon u_\epsilon - \int (A_i^\epsilon g)X_k A_i^\epsilon u_\epsilon\\
&= -\int (A_i^\epsilon g)A_i^\epsilon X_k u_\epsilon - \int (A_i^\epsilon g)[X_k, A_i^\epsilon]u_\epsilon
\end{aligned}$$

and

$$\begin{aligned}
\int (BX_k g)u_\epsilon &= \int (X_k Bg + [B, X_k]g)u_\epsilon = \int (X_k Bg)u_\epsilon\\
&= -\int (Bg)u_\epsilon(\mathrm{div}X_k) - \int (Bg)X_k u_\epsilon = -\int (Bg)X_k u_\epsilon,
\end{aligned}$$

which together with (2.19) and (2.20) lead to

$$\frac{\epsilon^2}{2}\int (A_i^\epsilon g)A_i^\epsilon X_k u_\epsilon - \int (Bg)X_k u_\epsilon = \frac{\epsilon^2}{2}(\mathrm{I}' + \mathrm{II}' + \mathrm{III}') - \frac{\epsilon^2}{2}\int (A_i^\epsilon g)[X_k, A_i^\epsilon]u_\epsilon. \tag{2.21}$$

Note that

$$\begin{aligned}
\mathrm{I}' &= -\int (A_i^\epsilon g)(\mathrm{div}A_i^\epsilon)u_\epsilon(\mathrm{div}X_k) - \int (A_i^\epsilon g)X_k(u_\epsilon \mathrm{div}A_i^\epsilon)\\
&= -\int (A_i^\epsilon g)(\mathrm{div}A_i^\epsilon)X_k u_\epsilon - \int (A_i^\epsilon g)(X_k \mathrm{div}A_i^\epsilon)u_\epsilon,
\end{aligned}$$

and $\mathrm{III}' = 2\int X_k g\,[(\mathrm{div}A_0^\epsilon)u_\epsilon + A_0^\epsilon u_\epsilon]$. Inserting them into (2.21) results in (2.18). Obviously, each term in (2.18) is well-defined even if $g\in W^{1,2}$. Since $W^{2,2}$ is dense in $W^{1,2}$, it follows from standard approximation arguments that (2.18) holds for any $g\in W^{1,2}$.

In **Step 2** and **Step 3**, for any two constants $\alpha(\epsilon,q)$ and $\beta(\epsilon,q)$ indexed by (ϵ,q) with q to be included in **Step 2**, we write $\alpha(\epsilon,q)\lesssim\beta(\epsilon,q)$ to imply the existence of C, which is independent of ϵ and q, such that $\alpha(\epsilon,q)\le C\beta(\epsilon,q)$ for any (ϵ,q).

Step 2. We show

$$\int |X_k u_\epsilon|^{2q}|\nabla X_k u_\epsilon|^2 \lesssim \sum_{k=1}^n \left(\|X_k u_\epsilon\|_{(q+1)r}^{2(q+1)} + \|X_k u_\epsilon\|_{qr}^{2q}\right), \quad \forall q\ge 0, \tag{2.22}$$

where $r := \left(\frac{1}{2}-\frac{1}{p}\right)^{-1} \in (2, 2d_*)$. We remind the reader of the use of Einstein's summation convention on the index k in the left-hand side of (2.22). In the following, we first prove inequalities for fixed $k\in\{1,\dots,n\}$ and then sum them up to achieve (2.22).

Let $q\ge 0$ and $k\in\{1,\dots,n\}$. Set $g := |X_k u_\epsilon|^{2q}X_k u_\epsilon \in W^{1,2}$ in (2.18). Note that

$$Yg = (2q+1)|X_k u_\epsilon|^{2q} Y X_k u_\epsilon, \tag{2.23}$$

for any vector field $Y : M\to TM$. Then, we see from $\mathrm{div}B=0$ that

$$-\int (Bg)X_k u_\epsilon = -(2q+1)\int |X_k u_\epsilon|^{2q}(BX_k u_\epsilon)X_k u_\epsilon = -\frac{2q+1}{2q+2}\int B|X_k u_\epsilon|^{2q+2} = 0.$$

Hence, it follows from (2.18) that

$$\int (A_i^\epsilon g)A_i^\epsilon X_k u_\epsilon = \mathrm{I} + \mathrm{II}. \tag{2.24}$$

Applying (2.23), we derive from **(A2)** that

$$\frac{\text{LHS of (2.24)}}{2q+1} = \sum_{i=1}^d \int |X_k u_\epsilon|^{2q}|A_i^\epsilon X_k u_\epsilon|^2 \gtrsim \int |X_k u_\epsilon|^{2q}|\nabla X_k u_\epsilon|^2 \tag{2.25}$$

and

$$\frac{\text{RHS of (2.24)}}{2q+1} = \frac{\mathrm{I}}{2q+1} + \frac{\mathrm{II}}{2q+1}, \tag{2.26}$$

with

$$\begin{aligned}\frac{\mathrm{I}}{2q+1} &= -\int |X_k u_\epsilon|^{2q}(A_i^\epsilon X_k u_\epsilon)\left((\mathrm{div}A_i^\epsilon)X_k u_\epsilon + [X_k, A_i^\epsilon]u_\epsilon\right)\\ &\quad + \int |X_k u_\epsilon|^{2q}\left(([A_i^\epsilon, X_k]X_k u_\epsilon)A_i^\epsilon u_\epsilon + 2(X_k^2 u_\epsilon)A_0^\epsilon u_\epsilon\right)\end{aligned}$$

and

$$\frac{\mathrm{II}}{2q+1} = \int |X_k u_\epsilon|^{2q}u_\epsilon\left(-(A_i^\epsilon X_k u_\epsilon)(X_k \mathrm{div}A_i^\epsilon) + ([A_i^\epsilon, X_k]X_k u_\epsilon)\mathrm{div}A_i^\epsilon + 2(X_k^2 u_\epsilon)\mathrm{div}A_0^\epsilon\right).$$

Since $A_i^\epsilon\in W^{2,p}$ for $i\in\{1,\dots,m\}$, and $A_0^\epsilon,\ X_k\in W^{1,p}$, we apply the Sobolev embedding theorem to find that $A_0^\epsilon,\ A_i^\epsilon,\ \mathrm{div}A_i^\epsilon,\ X_k\in C^0$ and $[X_k, A_i^\epsilon],\ X_k\mathrm{div}A_i^\epsilon,\ \mathrm{div}A_0^\epsilon\in L^p$. Hence, an application of Hölder's inequality yields

$$\frac{|\mathrm{I}|}{2q+1} \lesssim \left(\int |X_k u_\epsilon|^{2q}|\nabla X_k u_\epsilon|^2\right)^{\frac{1}{2}} \||X_k u_\epsilon|^q \nabla u_\epsilon\|_r\,,$$

and

$$\frac{|\mathrm{II}|}{2q+1} \lesssim \sup_\epsilon \|u_\epsilon\|_\infty \times \left(\int |X_k u_\epsilon|^{2q}|\nabla X_k u_\epsilon|^2\right)^{\frac{1}{2}} \||X_k u_\epsilon|^q\|_r \lesssim \left(\int |X_k u_\epsilon|^{2q}|\nabla X_k u_\epsilon|^2\right)^{\frac{1}{2}} \|X_k u_\epsilon\|_{qr}^q\,,$$

where we used the fact $\|u_\epsilon\|_\infty \lesssim 1$ due to Theorem 2.1 in the second inequality. This together with (2.24)-(2.26) gives

$$\int |X_k u_\epsilon|^{2q} |\nabla X_k u_\epsilon|^2 \lesssim \||X_k u_\epsilon|^q \nabla u_\epsilon\|_r^2 + \|X_k u_\epsilon\|_{qr}^{2q}.$$

Since $\{X_j\}_{j=1}^n$ spans the tangent bundle TM, we see from Lemma B.2 the existence of $D > 0$ (independent of ϵ) such that $\frac{1}{D}|\nabla u_\epsilon| \leq \sum_{j=1}^n |X_j u_\epsilon| \leq D|\nabla u_\epsilon|$ on M. That is, $|\nabla u_\epsilon| \approx \sum_{j=1}^n |X_j u_\epsilon|$ uniformly on M. Hence,

$$\int |X_k u_\epsilon|^{2q} |\nabla X_k u_\epsilon|^2 \lesssim \sum_{j=1}^n \|X_j u_\epsilon\|_{(q+1)r}^{2(q+1)} + \|X_k u_\epsilon\|_{qr}^{2q}.$$

Summarizing the above inequalities in k yields (2.22).

Step 3. We prove the existence of $C_* > 0$ independent of ϵ such that

$$\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*} \leq C_*(q+1)^{\frac{1}{\alpha(q+1)}} \sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)} + C_*(q+1)^{\frac{1}{\alpha(q+1)}}, \quad \forall q \geq 0, \tag{2.27}$$

where $\alpha = 1 - \frac{d}{p} \in (0,1)$.

According to Hölder's inequality and Young's inequality, we find

$$\|X_k u_\epsilon\|_{qr}^{2q} \lesssim \|X_k u_\epsilon\|_{(q+1)r}^{2q} \lesssim \frac{q}{q+1}\|X_k u_\epsilon\|_{(q+1)r}^{2(q+1)} + \frac{1}{q+1}.$$

Inserting this into (2.22) gives rise to

$$\int |X_k u_\epsilon|^{2q} |\nabla X_k u_\epsilon|^2 \lesssim \sum_{k=1}^n \|X_k u_\epsilon\|_{(q+1)r}^{2(q+1)} + 1. \tag{2.28}$$

Noting that

$$(q+1)^2 \int |X_k u_\epsilon|^{2q} |\nabla X_k u_\epsilon|^2 = \left\|\nabla |X_k u_\epsilon|^{q+1}\right\|_2^2,$$

and

$$\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(q+1)} \lesssim \sum_{k=1}^n \left(\|X_k u_\epsilon\|_{2(q+1)}^{2(q+1)} + \left\|\nabla |X_k u_\epsilon|^{q+1}\right\|_2^2 \right)$$

due to the Sobolev embedding theorem, we derive from (2.28) that

$$\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(q+1)} \leq C_1(q+1)^2 \sum_{k=1}^n \left(\|X_k u_\epsilon\|_{2(q+1)}^{2(q+1)} + \|X_k u_\epsilon\|_{(q+1)r}^{2(q+1)} \right) + C_1(q+1)^2. \tag{2.29}$$

Since applications of the interpolation inequality and then Young's inequality lead to

$$\begin{aligned}
\|X_k u_\epsilon\|_{(q+1)r}^{2(q+1)} &\leq \|X_k u_\epsilon\|_{2(q+1)}^{2\alpha(q+1)} \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(1-\alpha)(q+1)} \\
&\leq \alpha\delta^{-\frac{1}{\alpha}} \|X_k u_\epsilon\|_{2(q+1)}^{2(q+1)} + (1-\alpha)\delta^{\frac{1}{1-\alpha}} \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(q+1)},
\end{aligned}$$

where $\alpha := 1 - \frac{p}{d} \in (0,1)$, we set $\delta := [2(1-\alpha)C_1(q+1)^2]^{-(1-\alpha)}$ to derive from (2.29) that

$$\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(q+1)} \leq C_2(q+1)^{\frac{2}{\alpha}} \sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)}^{2(q+1)} + C_2(q+1)^{\frac{2}{\alpha}}.$$

Taking the $2(q+1)$-th root of both sides leads to

$$\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*} \lesssim \left(\sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)d_*}^{2(q+1)}\right)^{\frac{1}{2(q+1)}} \lesssim (q+1)^{\frac{1}{\alpha(q+1)}} \sum_{k=1}^n \|X_k u_\epsilon\|_{2(q+1)} + (q+1)^{\frac{1}{\alpha(q+1)}}$$

and thus, (2.27) holds.

Step 4. We finish the proof. Set $q+1 = d_*^k$ for $k \in \mathbb{N}_0$ and denote $I_k^\epsilon := \sum_{k=1}^n \|X_k u_\epsilon\|_{2d_*^k}$. It follows from the result in **Step 3** that

$$I_{k+1}^\epsilon \leq C_* d_*^{\frac{k}{\alpha d_*^k}} I_k^\epsilon + C_* d_*^{\frac{k}{\alpha d_*^k}}.$$

By iteration, we arrive at

$$I_{k+1}^\epsilon \leq C_*^2 d_*^{\frac{k}{\alpha d_*^k} + \frac{k}{\alpha d_*^{k-1}}} I_{k-1}^\epsilon + C_*^2 d_*^{\frac{k}{\alpha d_*^k} + \frac{k}{\alpha d_*^{k-1}}} \leq \cdots \leq C_*^{k+1} d_*^{\frac{1}{\alpha}\sum_{i=0}^\infty \frac{k}{d_*^i}} I_0^\epsilon + k C_*^{k+1} d_*^{\frac{1}{\alpha}\sum_{i=0}^\infty \frac{k}{d_*^i}}.$$

Recall from **Step 2** that $\sum_{k=1}^n |X_k u_\epsilon| \approx |\nabla u_\epsilon|$ uniformly on M. Since $\|\nabla u_\epsilon\|_2 \lesssim 1$ by Lemma 2.1, we find $I_0^\epsilon \approx \|\nabla u_\epsilon\|_2 \lesssim 1$, and thus, $\|\nabla u_\epsilon\|_{2d_*^k} \approx I_k^\epsilon \lesssim 1$ for all $k \in \mathbb{N}$. The interpolation inequality then ensures $\|\nabla u_\epsilon\|_{p'} \lesssim 1$ for any $p' > 1$. This completes the proof. □

3. Proof of main results

This section is devoted to the proof of our main results.

3.1. **Converting to divergence-free vector fields.** Recall that the system (1.1) is assumed to have an invariant measure μ_0 with a positive density $u_0 \in W^{1,p_0}$ for some $p_0 > d$. We introduce the invertible transformations to convert (1.2) into a system whose unperturbed part is divergence-free so that results obtained in Section 2 apply.

Set

$$\tilde{B} := u_0 B, \quad \tilde{u}_0 := 1 \quad \text{and} \quad d\tilde{\mu}_0 := d\mathrm{Vol}, \tag{3.1}$$

and for each ϵ,

$$\tilde{A}_0^\epsilon := u_0 A_0^\epsilon - \frac{1}{2}\sqrt{u_0}(A_j^\epsilon \sqrt{u_0}) A_j^\epsilon, \quad \tilde{A}_i^\epsilon := \sqrt{u_0} A_i^\epsilon \quad \text{for} \quad i \in \{1, \ldots, m\}, \tag{3.2}$$

$$\tilde{u}_\epsilon := \frac{u_\epsilon}{u_0} \quad \text{and} \quad d\tilde{\mu}_\epsilon := \tilde{u}_\epsilon d\mathrm{Vol}. \tag{3.3}$$

Denote $\tilde{A} := \left\{\tilde{A}_i^\epsilon,\ i \in \{0, \ldots, m\},\ \epsilon\right\}$. The following result is elementary.

Lemma 3.1. *Let $A \in \mathcal{A}$. Then, the following hold.*

(1) $\tilde{B} \in W^{1,p_0}$ *and* $\mathrm{div}\tilde{B} = 0$.
(2) $\tilde{A} \in \mathcal{A}$.
(3) $\tilde{\mu}_\epsilon$ *is a stationary measure of the SDE (1.2) with B and A replaced by $\tilde{B}$ and $\tilde{A}$, respectively.*

Proof. Recall that $u_0 \in W^{1,p_0}$ and $u_0 > 0$. It follows from the embedding $W^{1,p_0} \hookrightarrow C^0$ that $0 < \min u_0 \leq \max u_0 < \infty$.

(1) Clearly, $B \in W^{1,\infty}$ implies $\tilde{B} \in W^{1,p_0}$. It remains to prove $\mathrm{div}\tilde{B} = 0$. The fact that μ_0 is an invariant measure of (1.1) ensures $\int f u_0 = \int (f \circ \varphi^t) u_0$ for all $f \in C^0$, where φ^t is the flow generated

by (1.1) or the vector field B. If $f \in C^1$, we differentiate $\int (f\circ\varphi^t)u_0$ with respect to t and then set $t=0$ to derive

$$0=\int (Bf)u_0=\int (u_0B)f=\int \tilde{B}f.$$

Applying the divergence theorem then gives $\int f\mathrm{div}\tilde{B}=0$. Hence, we see from the arbitrariness of $f\in C^1$ that $\mathrm{div}\tilde{B}=0$.

(2) Since $A\in\mathcal{A}$, there exists $p>d$ such that $\|A_0^\epsilon\|_p+\max_i\|A_i^\epsilon\|_{1,p}\lesssim 1$. Since $u_0\in W^{1,p_0}$, straightforward calculations show that for $p_1:=\min\{p_0,p\}$, there holds $\|\tilde{A}_0^\epsilon\|_{p_1}+\max_i\|\tilde{A}_i^\epsilon\|_{1,p_1}\lesssim 1$. That is, $\tilde{A}$ satisfies **(A1)**. Obviously, $\tilde{A}$ satisfies **(A2)**, and hence, $\tilde{A}\in\mathcal{A}$.

(3) For $f\in C^2$, we calculate

$$\begin{aligned}\int (\mathcal{L}_\epsilon f)u_\epsilon &=\int\left[\frac{\epsilon^2}{2}u_0A_i^\epsilon A_i^\epsilon f+\epsilon^2u_0A_0^\epsilon f+u_0Bf\right]\tilde{u}_\epsilon\\ &=\int\left[\frac{\epsilon^2}{2}\sqrt{u_0}A_i^\epsilon(\sqrt{u_0}A_i^\epsilon f)-\frac{\epsilon^2}{2}\sqrt{u_0}(A_i^\epsilon\sqrt{u_0})A_i^\epsilon f+\epsilon^2u_0A_0^\epsilon f+u_0Bf\right]\tilde{u}_\epsilon=\int(\tilde{\mathcal{L}}_\epsilon f)\tilde{u}_\epsilon.\end{aligned}$$

Since $\int(\mathcal{L}_\epsilon f)u_\epsilon=0$, we conclude that $\tilde{\mu}_\epsilon$ is a stationary measure of the SDE (1.2) with B and A replaced by $\tilde{B}$ and $\tilde{A}$, respectively. □

3.2. **Proof of Theorems A-C.** For the proof of Theorem A, we need the following lemma. Recall the definition of a physical measure and its basin from Subsection 1.2.

Lemma 3.2. *If μ is a physical measure of (1.1), then $\mu=\frac{\mu_0|_{B_\mu}}{\mu_0(B_\mu)}$ and it is ergodic.*

Proof. Fix $f\in C^0$. The fact that μ is a physical measure implies

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\varphi^s(x))ds=\int fd\mu,\quad \forall x\in B_\mu. \tag{3.4}$$

It follows from Fubini's theorem and the dominated convergence theorem that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{B_\mu}f\circ\varphi^sd\mu_0ds=\lim_{t\to\infty}\int_{B_\mu}\left(\frac{1}{t}\int_0^t f\circ\varphi^sds\right)d\mu_0=\mu_0(B_\mu)\int fd\mu.$$

Note that

$$\int_{B_\mu}f\circ\varphi^sd\mu_0=\int(1_{B_\mu}f)\circ\varphi^sd\mu_0=\int 1_{B_\mu}fd\mu_0,$$

where we used the φ^t-invariance of B_μ in the first equality and the fact that μ_0 is an invariant measure of φ^t in the second one. Hence, we arrive at

$$\int 1_{B_\mu}fd\mu_0=\mu_0(B_\mu)\int fd\mu,\quad\forall f\in C^0,$$

leading to $\mu=\frac{\mu_0|_{B_\mu}}{\mu_0(B_\mu)}$. In particular, $\mu(B_\mu)=1$.

For the ergodicity of μ, we note that $\mu(B_\mu)=1$ implies that $\lim_{t\to\infty}\frac{1}{t}\int_0^t\delta_{\varphi^s(x)}ds=\mu$ under the weak*-topology for μ-a.e. $x\in M$. This is equivalent to the ergodicity of μ. □

Corollary 3.1. *The following statements are equivalent.*

(1) *μ_0 is physical.*
(2) *μ_0 is ergodic.*
(3) *There is a physical measure μ of (1.1) with $\mathrm{Vol}(B_\mu)=1$.*

Whenever these statements hold, μ_0 is the unique physical measure.

Proof. (1) $\Longrightarrow$ (2). It follows directly from Lemma 3.2.

(2) $\Longrightarrow$ (1) and (3). Since μ_0 has a positive density, Birkhoff's ergodic theorem ensures that μ_0 itself is physical and satisfies $\mathrm{Vol}(B_{\mu_0}) = 1$.

(3) $\Longrightarrow$ (2). If there is a physical measure μ of (1.1) with $\mathrm{Vol}(B_\mu) = 1$, then the equivalence between μ_0 and Vol yields $\mu_0(B_\mu) = \mu_0(M) = 1$, and hence, $\mu = \mu_0$ thanks to Lemma 3.2.

Now, suppose that (1)-(3) hold. If μ is a physical measure, then Lemma 3.2 yields $\mu \ll \mu_0$, and hence, $\mu = \mu_0$ thanks to the ergodicity of μ_0. □

Proof of Theorem A. (1) Let $A \in \mathcal{A}$ and $\tilde{A}$, $\tilde{B}$ and $\tilde{\mu}_\epsilon$ be as defined in (3.1)-(3.3). Given Lemma 3.1, we apply Theorem 2.1 to find $\|\tilde{u}_\epsilon\|_{1,2} \lesssim 1$ and $1 \lesssim \min \tilde{u}_\epsilon \le \max \tilde{u}_\epsilon \lesssim 1$. Since $u_\epsilon = u_0 \tilde{u}_\epsilon$ and $u_0 \in W^{1,p_0}$ for some $p_0 > d$, straightforward calculations yield $\|u_\epsilon\|_{1,2} \lesssim 1$ and $1 \lesssim \min u_\epsilon \le \max u_\epsilon \lesssim 1$. As a result, $\{u_\epsilon\}_\epsilon$ is precompact in $W^{1,2}$ under the weak topology. The "In particular" part follows readily.

(2) The conclusions in (1) guarantees that each element of $\mathcal{M}_A$ (must be an invariant measure of (1.1)) is equivalent to μ_0. Then, the ergodicity of μ_0 (by Corollary 3.1) asserts $\mathcal{M}_A = \{\mu_0\}$.

(3) In this case, $\mathcal{M}_A = \{\mu_0\}$ follows immediately from conclusions in (1).

Whenever either (2) or (3) is true, there holds $\mathcal{M}_A = \{\mu_0\}$. It then follows from the uniform estimates of $\{u_\epsilon\}_\epsilon$ in $W^{1,2}$ (by (1)) and the Rellich–Kondrachov theorem that $\lim_{\epsilon\to 0} u_\epsilon = u_0$ weakly in $W^{1,2}$ and strongly in L^p for any $p \in [1, \frac{2d}{d-2})$. This completes the proof. □

Theorem C follows readily.

Proof of Theorem C. (1) Let $A \in \mathcal{SA}$ and $\tilde{A}$, $\tilde{B}$ and $\tilde{\mu}_\epsilon$ be as defined in (3.1)-(3.3). Then, all the results in Lemma 3.1 hold. Moreover, since $u_0 \in W^{2,p_0}$, we can follow the proof of Lemma 3.1 to find $\tilde{B} \in W^{1,\infty}$ and $\tilde{A} \in \mathcal{SA}$.

Set $\tilde{X}_i := u_0 X_i$. Clearly, $[\tilde{X}_i, \tilde{B}] = 0$, $\mathrm{div}\tilde{X}_i = 0$, and $\{\tilde{X}_i\}_{i=1}^n$ spans the tangent bundle TM. We then apply Theorem 2.2 to conclude that $\|\tilde{u}_\epsilon\|_{W^{1,q}} \lesssim 1$ for all $q \ge 1$. Since $u_\epsilon = u_0\tilde{u}_\epsilon$ and $u_0 \in W^{2,p_0}$, the conclusion follows.

(2) Theorem A(3) asserts that $\mathcal{M}_A = \{\mu_0\}$. The conclusion then follows from (1) and the compact Sobolev embedding theorem. □

Finally, we prove Theorem B.

Proof of Theorem B. The construction of the manifold M and the vector field B satisfying (1)-(3) are done in **Step 1** and **Step 2**. We prove (i)-(iii) in **Step 3**.

Step 1. Set $D := \{x = (x_1, x_2) : x_1^2 + x_2^2 \le 1\}$. It is well-known [32] that there exists a smooth area-preserving Bernoulli diffeomorphism $F : D \to D$ such that $F - \mathrm{Id}$ is "infinitely flat" on ∂D, that is, both $F - \mathrm{Id}$ and all its derivatives of any order vanish on ∂D.

We construct a smooth suspension flow $\tilde{\psi}^t$ of F on the smooth, connected Riemannian suspension manifold $(\tilde{D}, \tilde{g})$ such that $\tilde{\psi}^t$ is volume-preserving and has a Poincaré map smoothly conjugate to F. Moreover, both $\tilde{D}^{\circ}$ and $\partial\tilde{D}$ are invariant under $\tilde{\psi}^t$.

To do so, we define a $\mathbb{Z}$-action on $D \times \mathbb{R}$ generated by the map

$$(x, t) \mapsto (F(x), t - 1) : D \times \mathbb{R} \to D \times \mathbb{R}, \tag{3.5}$$

which naturally induces an equivalence relation $\sim$ on $D \times \mathbb{R}$:

$$(x, t) \sim (x', t') \quad \text{iff} \quad (x', t') = (F^n(x), t - n) \text{ for some } n \in \mathbb{Z}.$$

$$\begin{array}{ccc} (D\times\mathbb{R}, g) & \xrightarrow{\pi_D} & (\tilde{D}, \tilde{g}) \\ \downarrow \psi^t & & \downarrow \tilde{\psi}^t \\ (D\times\mathbb{R}, g) & \xrightarrow{\pi_D} & (\tilde{D}, \tilde{g}) \end{array}$$

FIGURE 1. Construction of the suspension flow $\tilde{\psi}^t$.

Set $\tilde{D} := D\times\mathbb{R}/\sim$. It is a smooth, connected, and compact manifold with boundary $\pi_D(\partial D\times\mathbb{R})$, where $\pi_D : D\times\mathbb{R}\to\tilde{D}$ is the natural projection. To furnish $\tilde{D}$ with a Riemannian metric $\tilde{g}$, we first take the following Riemannian metric on $D\times[-1,0]$:

$$g = (dx_1, dx_2)\left[[DF(\gamma(x,t))]^t\right]^\top [DF(\gamma(x,t))]^t (dx_1, dx_2)^\top + dt^2 =: g_t + dt^2,$$

where $\gamma : D\times[-1,0]\to D$ is a smooth function satisfying

$$\gamma(x,t) = \begin{cases} F^{-1}(x), & t\in\left[-1,-\frac{3}{4}\right], \\ x, & t\in\left[-\frac{1}{4},0\right], \end{cases} \qquad x\in D.$$

It is easy to check that $g_0 = dx_1^2 + dx_2^2 = F^*g_{-1}$, where F^*g_{-1} denotes the pullback of g_{-1} under F. Therefore, through the smooth diffeomorphism $(x,t)\mapsto(F^n(x), t-n)$ on $D\times(n-1,n]$ for each $n\in\mathbb{Z}$, we can extend g to be a smooth Riemannian metric on $D\times\mathbb{R}$, stilled denoted by g. Since $F = \mathrm{Id}$ on ∂D and $F-\mathrm{Id}$ is infinitely smooth, we see that $DF = \mathrm{Id}$ on ∂D and $\gamma(x,t) = x$ for any $(x,t)\in\partial D\times[-1,0]$, yielding

$$g = g_0 + dt^2 \quad \text{on} \quad \partial D\times\mathbb{R}. \tag{3.6}$$

Note that the construction g through the extension ensures its invariance under the $\mathbb{Z}$-action. Hence, a Riemannian metric $\tilde{g}$ on $\tilde{D}$ is naturally induced.

Now, we construct the suspension flow $\tilde{\psi}^t$. Let ψ^t be the flow generated by $\frac{\partial}{\partial t}$ on $D\times\mathbb{R}$. Since F is area-preserving, there holds $|\det(DF)|\equiv 1$, and thus, $\det(g_t)\equiv 1$. Straightforward calculations then yield $\mathrm{div}\frac{\partial}{\partial t} = 0$, and hence, ψ^t is volume-preserving. Here, we used g_t with a slight abuse of notation to represent the matrix when defining the 2-form g_t. Since ψ^t preserves the equivalence relation $\sim$, it naturally induces a volume-preserving flow $\tilde{\psi}^t := \pi_D\circ\psi^t\circ(\pi_D)^{-1}$ on $\tilde{D}$ (see Figure 1 for an illustration). Clearly, $\tilde{\psi}^t$ is smooth, generated by the pushforward $(\pi_D)_*\frac{\partial}{\partial t}$, and satisfies

$$\tilde{\psi}^t(\tilde{D}^{\circ}) = \tilde{D}^{\circ}, \quad \text{and} \quad \tilde{\psi}^t(\pi_D(x,s)) = \pi_D(x, s+t), \quad \forall (x,s)\in\partial D\times\mathbb{R}. \tag{3.7}$$

In particular,

$$\tilde{\psi}^1(\pi_D(x,0)) = \pi_D(\psi^1(x,0)) = \pi_D(x,1) = \pi_D(F(x),0), \quad \forall x\in D.$$

Since $\pi_D|_{D\times\{0\}} : D\times\{0\}\to\pi_D(D\times\{0\})$ is a smooth diffeomorphism, the above equality asserts that $\tilde{\psi}^1|_{\pi_D(D\times\{0\})}$, which is a Poincaré map of $\tilde{\psi}^t$, is smoothly conjugate to F. Hence, $\tilde{\psi}^t$ is strongly mixing.

$$\begin{array}{ccccc} D\times\mathbb{R} & \xrightarrow{\pi_D} & \tilde{D} & \xrightarrow{\tilde{\psi}^t} & \tilde{D} \\ \uparrow{\scriptstyle (h_i,\mathrm{Id})} & & \uparrow{\scriptstyle \tilde{h}_i} & & \uparrow{\scriptstyle \tilde{h}_i} \\ \mathbb{S}^2_i\times\mathbb{R} & \xrightarrow{\pi_i} & M_i & \xrightarrow{\varphi_i^t} & M_i \end{array}$$

FIGURE 2. Construction of the suspension flow φ_i^t on M.

Step 2. We construct M and B.

Let $\mathbb{S}^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere and denote by

$$\mathbb{S}^2_1 := \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 \geq 0\} \quad \text{and} \quad \mathbb{S}^2_2 := \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 \leq 0\}$$

the upper hemisphere and lower hemisphere, respectively. Let $h_1 : \mathbb{S}^2_1 \to D$ and $h_2 : \mathbb{S}^2_2 \to D$ be the standard stereographic projections from the south pole $(0, 0, -1)$ and the north pole $(0, 0, 1)$, respectively. Through h_i, the equivalence relation $\sim$ on $D \times \mathbb{R}$ is naturally lifted to its counterpart $\sim_i$ on $\mathbb{S}^2_i \times \mathbb{R}$. Set $M_i := \mathbb{S}^2_i \times \mathbb{R}/\sim_i$ and let $\pi_i : \mathbb{S}^2_i \times \mathbb{R} \to M_i$ be the natural projection. Then,

$$\tilde{h}_i := \pi_D \circ (h_i, \mathrm{Id}) \circ \pi_i^{-1} : M_i \to \tilde{D}$$

defines a smooth diffeomorphism, allowing us to endow M_i with the Riemannian metric $g_i := (\tilde{h}_i)^*\tilde{g}$ and define a flow $\varphi_i^t := (\tilde{h}_i)^{-1} \circ \tilde{\psi}^t \circ \tilde{h}_i$ on M_i. Obviously, (M_i, g_i) is a smooth and connected Riemannian manifold with boundary $\partial := (\tilde{h}_i)^{-1}\partial\tilde{D}$ and φ_i^t is a smooth and volume-preserving flow, generated by the vector field $B_i := (\tilde{h}_i^{-1})_*(\pi_D)_*\frac{\partial}{\partial t}$. Since φ_i^t is smoothly conjugate to $\tilde{\psi}^t$, it follows that φ_i^t is as well strongly mixing. We refer the reader to Figure 2 for clarity.

Note that the map (3.5), which generates the equivalence relation $\sim$, preserves $\partial D \times \mathbb{R}$ since $F - \mathrm{Id}$ is infinitely smooth on ∂D. We see from the form (3.6) of g on $\partial D \times \mathbb{R}$ and the construction of (M_i, g_i), $i = 1, 2$ that $M := M_1 \cup M_2$ is a smooth, connected, and closed manifold and $g_M := g_i$ on M_i for $i = 1, 2$ is a well-defined smooth Riemannian metric on M.

Now, we construct a smooth flow φ^t on (M, g). By the definition of the flow ψ^t on $D \times \mathbb{R}$ that results in a particular form of $\tilde{\psi}^t$ on $\partial\tilde{D}$, namely, the second equality in (3.7), we see from the construction of φ_i^t, $i = 1, 2$ that $\varphi^t := \varphi_i^t$ on M_i for $i = 1, 2$ is a well-defined, smooth and volume-preserving flow on M. Setting $B := B_i$ on M_i for $i = 1, 2$, we conclude that B is smooth and divergence-free, and generates φ^t. The properties (1)-(3) follow readily.

Step 3. (i) is an immediate consequence of (1) and (2). (ii) follows readily from (3). (iii) follows directly from (ii) and Theorem A(3). □

3.3. **Invariant measure selection by noise.** Recall from Corollary 3.1 that μ_0 is physical if and only if it is ergodic. When μ_0 fails to be ergodic or physical, the system (1.1) could admit multiple invariant measures like μ_0. In this subsection, we show that any of them can be selected by noise.

Definition 3.1 (Symmetric admissible class)**.** *Let $u : M \to \mathbb{R}$ be positive. A collection of vector fields $A = \{A_i^\epsilon, i \in \{0, \dots, m\},\ \epsilon\}$ is said to be in the* symmetric admissible class $\mathcal{A}_u^{sym}$ *if for each ϵ, there exists a symmetric positive definite endomorphism $A_\epsilon^{sym} : M \to \mathrm{End}(TM)$ belonging to $W^{1,1}$ such* *that*

$$\frac{1}{2}\sum_{i=1}^{m}(A_i^\epsilon)^2 f + A_0^\epsilon f = \frac{1}{u}\text{div}(A_\epsilon^{sym}\nabla f), \quad \forall f \in C^2(M).$$

Theorem 3.1. *Let μ be an invariant measure of* (1.1) *and have a positive density $u \in W^{1,p}$ for some $p > d$. Then, $\mathcal{A} \cap \mathcal{A}_u^{sym} \neq \emptyset$, and for any $A \in \mathcal{A}_u^{sym}$ there holds $\mathcal{M}_A = \{\mu\}$.*

Proof of Theorem 3.1. First, we show $\mathcal{A}_u^{sym} \neq \emptyset$. Note that the Nash embedding theorem asserts M is isometrically embedded into $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Thus, there exist m smooth vector fields $\tilde{A}_i$, $i \in \{1, \dots, m\}$, on M so that $\sum_{i=1}^m (\tilde{A}_i)^2 = \Delta$ (see e.g. [31]). For each ϵ, we define

$$A_0^\epsilon := \frac{1}{4u^2}(\tilde{A}_i u)\tilde{A}_i \quad \text{and} \quad A_i^\epsilon := \frac{1}{\sqrt{u}}\tilde{A}_i, \quad i \in \{1, \dots, m\}.$$

As u is positive and belongs to $W^{1,p}$ for some $p > d$, it is easy to see that $A := \{A_i^\epsilon, i \in \{0, \dots, m\},\ \epsilon\} \in \mathcal{A}$. Straightforward calculations yield

$$\frac{1}{2}\sum_{i=1}^{m}(A_i^\epsilon)^2 f + A_0^\epsilon f = \frac{1}{2u}\sum_{i=1}^{m}(\tilde{A}_i)^2 f = \frac{1}{2u}\Delta f, \quad \forall f \in C^2(M),$$

resulting in $A \in \mathcal{A}_u^{sym} \cap \mathcal{A}$.

Now, we prove $\mathcal{M}_A = \{\mu\}$ for any $A \in \mathcal{A}_u^{sym}$. Fix such an A. Then, for each ϵ there exists a symmetric positive definite endomorphism $A_\epsilon^{sym} : M \to \text{End}(TM)$ belonging to $W^{1,1}$ such that

$$\frac{1}{2}\sum_{i=1}^{m}(A_i^\epsilon)^2 f + A_0^\epsilon f = \frac{1}{u}\text{div}(A_\epsilon^{sym}\nabla f), \quad \forall f \in C^2(M).$$

Since μ is an invariant measure of (1.1) and $u \in W^{1,p}$, we see from Lemma 3.1 (1) with u_0 replaced by u that $\text{div}(uB) = 0$. Consequently, an application of the divergence theorem yields

$$\int (\mathcal{L}_\epsilon f)u = \int \left(\frac{\epsilon^2}{u}\text{div}(A_\epsilon^{sym}\nabla f) + Bf\right) u = 0, \quad \forall f \in C^2(M).$$

This together with Theorem A.1 implies $\mu_\epsilon = \mu$, and hence, $\mathcal{M}_A = \{\mu\}$. □

Remark 3.1. *If the system* (1.1) *has multiple invariant measures similar to μ_0, then Theorem 3.1 says that they are all selectable, and therefore, none of them (and none of the invariant measures of* (1.1)*) are stochastically stable with respect to $\mathcal{A}$. Hence, Theorem 3.1 can be regarded as a result towards stochastic instability.*

4. The one-dimensional case

We study the one-dimensional case in this section. Given that any smooth, connected, and closed one-dimensional manifold is smoothly diffeomorphic to a circle (see e.g. [30]), we consider M to be the circle $\mathbb{S}^1$ for the sake of clarity.

4.1. Setup and results.

Consider the following one-dimensional ODE over $\mathbb{S}^1$:

$$\dot{x} = B(x), \tag{4.1}$$

where $B : \mathbb{S}^1 \to T\mathbb{S}^1$ is a Lipchitz continuous vector field. Note that a vector field over $\mathbb{S}^1$ is naturally identified with a function on $\mathbb{S}^1$. In the sequel, the same notation is used for a vector field over $\mathbb{S}^1$ and its identification as a function on $\mathbb{S}^1$. This shall cause no trouble. It is assumed that (4.1) is conservative or generalized volume-preserving in the sense that it admits an invariant measure μ_0 with a positive density function u_0. Consequently, it must hold that B is either equal to zero ($B \equiv 0$), positive ($B > 0$), or negative ($B < 0$). The $B \equiv 0$ case is of no interest, and the other two cases are essentially the same. To maintain the clarity, we here focus on the $B > 0$ case. It is then easy to see that (4.1) is uniquely ergodic and $u_0 = \frac{\gamma}{B} \in W^{1,\infty}$, where $\gamma = (\int \frac{1}{B})^{-1}$ is the normalizing constant.

Consider (4.1) under small random perturbations:

$$dX_t^\epsilon = B(X_t^\epsilon)dt + \epsilon^2 A_0^\epsilon(X_t^\epsilon) + \epsilon \sum_{i=1}^m A_i^\epsilon(X_t^\epsilon) \circ dW_t^i, \tag{4.2}$$

where $0 < \epsilon \ll 1$ is the noise intensity, $m \geq 1$, $A = \{A_i^\epsilon,\ i \in \{0, 1, \dots, m\},\ \epsilon\}$ is a collection of vector fields on $\mathbb{S}^1$, and $\{W_t^i\}$ are m independent and standard one-dimensional Brownian motions on some probability space. The stochastic integrals are understood in the sense of Stratonovich. The collection of vector fields A is taken from the admissible class defined as follows.

Definition 4.1 (Admissible class-1D)**.** *A collection A of vector fields on $\mathbb{S}^1$ is said to be in the* admissible class $\mathcal{A}_{1D}$ *if $A = \{A_i^\epsilon, i \in \{0, \dots, m\},\ \epsilon\}$ for some $m \geq 1$ and the following conditions are satisfied:*

(A1)$_{1D}$ *there exists $p > 2$ such that $A_0^\epsilon \in L^p$, $A_i^\epsilon \in W^{1,p}$ for $i \in \{1, \dots, m\}$, and*

$$\|A_0^\epsilon\|_p + \max_i \|A_i^\epsilon\|_{1,p} \lesssim 1;$$

(A2)$_{1D}$ $\min \sum_{i=1}^m |A_i^\epsilon|^2 \gtrsim 1$.

Remark 4.1. *The only point that Definition 4.1 is not consistent with Definition 1.1 lies in the requirement $p > 2$ instead of $p > d = 1$ in* **(A1)**$_{1D}$.

If $A \in \mathcal{A}_{1D}$, Theorem A.1 is applied to yield that for each ϵ, (4.2) admits a unique stationary measure μ_ϵ. Moreover, μ_ϵ admits a positive density $u_\epsilon \in W^{1,p}$, where $p > 2$ is the same as in **(A1)**$_{1D}$. In addition, u_ϵ satisfies the stationary Fokker-Planck equation:

$$\frac{\epsilon^2}{2}(a_\epsilon u_\epsilon)'' - [(B + \epsilon^2 b_\epsilon)u_\epsilon]' = 0 \quad \text{in the weak sense,} \tag{4.3}$$

or equivalently,

$$-\frac{\epsilon^2}{2} \int a_\epsilon u_\epsilon' f' + \int \left(-\frac{\epsilon^2}{2} a_\epsilon' + B + \epsilon^2 b_\epsilon \right) u_\epsilon f' = 0, \quad \forall f \in W^{1,2},$$

where $a_\epsilon := \sum_{i=1}^m |A_i^\epsilon|^2$ and $b_\epsilon := A_0^\epsilon + \frac{1}{2} \sum_{i=1}^m A_i^\epsilon (A_i^\epsilon)'$.

Since μ_0 is the unique invariant measure of (4.1), its stochastic stability with respect to $\mathcal{A}_{1D}$ follows readily. We are more interested in enhanced results, which are stated in the following two theorems.

Theorem 4.1. *For any $A \in \mathcal{A}_{1D}$, there hold*

$$\|u_\epsilon\|_{1,2} \lesssim 1 \quad \textit{and} \quad 1 \lesssim \min u_\epsilon \leq \max u_\epsilon \lesssim 1.$$

Consequently, the limit $\lim_{\epsilon \to 0} u_\epsilon = u_0$ holds weakly in $W^{1,2}$ and strongly in C^α for any $\alpha \in (0, \frac{1}{2})$.

Theorem 4.2. *Let $A = \{A_i^\epsilon,\ i \in \{0, 1, \dots, m\},\ \epsilon\}$ be a collection of vector fields on $\mathbb{S}^1$ and satisfy*

- $A_0^\epsilon \in C^0$, $A_i^\epsilon \in C^1$, $i \in \{1, \dots, m\}$, *and* $\lim_{\epsilon \to 0} \epsilon^2 \left(\|A_0^\epsilon\|_\infty + \sum_{i=1}^m \|A_i^\epsilon\|_{1,\infty} \right) = 0$;
- $\min \sum_{i=1}^m |A_i^\epsilon|^2 > 0$ *for each* ϵ.

Then, $1 \lesssim \min u_\epsilon \le \max u_\epsilon \lesssim 1$ *and the limit* $\lim_{\epsilon\to 0} u_\epsilon = u_0$ *holds weakly in* L^p *for any* $p > 1$.

If, in addition,

$$A_0^\epsilon \in C^1,\ A_i^\epsilon \in C^2,\ i\in\{1,\dots,m\},\ \text{and}\ \lim_{\epsilon\to 0}\epsilon^2\left(\|A_0^\epsilon\|_{1,\infty}+\sum_{i=1}^m \|A_i^\epsilon\|_{2,\infty}\right)=0,$$

then $\|u_\epsilon'\|_\infty \lesssim 1$ *and the limit* $\lim_{\epsilon\to 0} u_\epsilon = u_0$ *holds in* C^α *for any* $\alpha\in(0,1)$.

Remark 4.2. *Theorem 4.1 is a one-dimensional counterpart of Theorem A(1). Uniform-in-ϵ bounds of A_0^ϵ and A_i^ϵ in* **(A1)** *and* $\mathbf{(A1)}_{1D}$ *play important roles in the respective proofs of these two theorems.*

The aim of Theorem 4.2 is to extend the scope of Theorem 4.1 by relaxing the requirement for uniform-in-ϵ bounds of A_0^ϵ and A_i^ϵ. An essential factor that enables this relaxation is the positivity of B, which is a unique characteristic in dimension one.

4.2. **Proof of Theorems 4.1 and 4.2.** First, we prove Theorem 4.1, which naturally follows from the reasoning leading to Theorem A(A). Given the one-dimensional nature of the problem, we present a significantly more concise proof.

Proof of Theorem 4.1. It suffices to establish the uniform estimates for $\{u_\epsilon\}_\epsilon$. Set $w_\epsilon := Bu_\epsilon$. Since B is Lipschitz continuous and positive, it is equivalent to show

$$\|w_\epsilon\|_{1,2}\lesssim 1 \quad \text{and} \quad 1\lesssim \min w_\epsilon \le \max w_\epsilon \lesssim 1. \tag{4.4}$$

It is seen from (4.3) that w_ϵ satisfies $\frac{\epsilon^2}{2}\left(\frac{a_\epsilon}{B}w_\epsilon\right)'' - \left[w_\epsilon + \epsilon^2\frac{b_\epsilon}{B}w_\epsilon\right]' = 0$ in the weak sense. That is,

$$-\frac{\epsilon^2}{2}\int\left(\frac{a_\epsilon}{B}w_\epsilon\right)' f' + \int\left[w_\epsilon+\epsilon^2\frac{b_\epsilon}{B}w_\epsilon\right] f' = 0,\quad \forall f\in W^{1,2}. \tag{4.5}$$

The proof of (4.4) is broken into three steps.

Step 1. We claim $\|w_\epsilon\|_{1,2}\lesssim 1$ and $\max w_\epsilon \lesssim 1$.

Taking $f := w_\epsilon$ in (4.5) yields

$$-\frac{\epsilon^2}{2}\int\left(\frac{a_\epsilon}{B}w_\epsilon\right)' w_\epsilon' + \int\left[w_\epsilon+\epsilon^2\frac{b_\epsilon}{B}w_\epsilon\right] w_\epsilon' = 0.$$

Using the fact $\int w_\epsilon' w_\epsilon = \frac{1}{2}\int (w_\epsilon^2)' = 0$, we find

$$\int \frac{a_\epsilon}{B}|w_\epsilon'|^2 = \int\left[-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right] w_\epsilon w_\epsilon'. \tag{4.6}$$

It follows from $\mathbf{(A1)}_{1D}$, $\mathbf{(A2)}_{1D}$ and the Lipschitz continuity and positivity of B that

$$\left\|\frac{a_\epsilon}{B}\right\|_{1,p} + \left\|\frac{b_\epsilon}{B}\right\|_p \lesssim 1 \quad \text{and} \quad \min\frac{a_\epsilon}{B}\gtrsim 1, \tag{4.7}$$

for some $p>2$. Applying Hölder's inequality to the right-hand side of (4.6) results in

$$\|w_\epsilon'\|_2^2 \lesssim \int\frac{a_\epsilon}{B}|w_\epsilon'|^2 \le \left\|-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right\|_p \|w_\epsilon\|_r\|w_\epsilon'\|_2 \lesssim \|w_\epsilon\|_r\|w_\epsilon'\|_2,$$

where $r := \left(\frac{1}{2}-\frac{1}{p}\right)^{-1}$. Therefore,

$$\|w_\epsilon'\|_2 \lesssim \|w_\epsilon\|_r \tag{4.8}$$

An application of the Sobolev embedding theorem and the interpolation inequality then leads to

$$\|w_\epsilon\|_\infty \lesssim \|w_\epsilon\|_2 + \|w_\epsilon'\|_2 \lesssim \|w_\epsilon\|_r \lesssim \|w_\epsilon\|_\infty^{1-\frac{1}{r}}\|w_\epsilon\|_1^{\frac{1}{r}},$$

that is, $\|w_\epsilon\|_\infty \lesssim \|w_\epsilon\|_1$. Since $\|w_\epsilon\|_1 \le \|u_\epsilon\|_1 \max B = \max B$, we find $\|w_\epsilon\|_\infty \lesssim 1$, which together with (4.8) yields $\|w_\epsilon'\|_2 \lesssim 1$.

Step 2. Set $v_\epsilon := \ln w_\epsilon - \frac{1}{2\pi}\int \ln w_\epsilon$. We prove $\|v_\epsilon\|_\infty \lesssim 1$.

Setting $f := \frac{1}{w_\epsilon}$ in (4.5) results in

$$\frac{\epsilon^2}{2}\int \left(\frac{a_\epsilon}{B}w_\epsilon\right)' \frac{w_\epsilon'}{w_\epsilon^2} - \int \left[w_\epsilon + \epsilon^2 \frac{b_\epsilon}{B} w_\epsilon\right] \frac{w_\epsilon'}{w_\epsilon^2} = 0,$$

which together with $\int \frac{w_\epsilon'}{w_\epsilon} = \int (\ln w_\epsilon)' = 0$ yields

$$\int \frac{a_\epsilon}{B}\frac{|w_\epsilon'|^2}{w_\epsilon^2} = \int \left[-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right] \frac{w_\epsilon'}{w_\epsilon}.$$

Noting that $v_\epsilon' = \frac{w_\epsilon'}{w_\epsilon}$, we apply Hölder's inequality to find

$$\int \frac{a_\epsilon}{B}|v_\epsilon'|^2 = \int \left[-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right] v_\epsilon' \le \left\|-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right\|_p \|1\|_r \|v_\epsilon'\|_2.$$

This together with (4.7) leads to $\|v_\epsilon'\|_2 \lesssim 1$. A further application of the Sobolev embedding theorem and Poincaré inequality results in $\|v_\epsilon\|_\infty \lesssim \|v_\epsilon\|_2 + \|v_\epsilon'\|_2 \lesssim \|v_\epsilon'\|_2 \lesssim 1$.

Step 3. We show $\min w_\epsilon \gtrsim 1$.

Letting $f := \frac{1}{w_\epsilon^3}$ in (4.5) yields

$$\frac{\epsilon^2}{2}\int \left(\frac{a_\epsilon}{B}w_\epsilon\right)' \frac{3w_\epsilon'}{w_\epsilon^4} - \int \left[w_\epsilon + \epsilon^2 \frac{b_\epsilon}{B} w_\epsilon\right] \frac{3w_\epsilon'}{w_\epsilon^4} = 0.$$

Since $\int \frac{w_\epsilon'}{w_\epsilon^3} = -\frac{1}{2}\int (\frac{1}{w_\epsilon^2})' = 0$, we arrive at

$$\int \frac{a_\epsilon}{B}\frac{|w_\epsilon'|^2}{w_\epsilon^4} = \int \left[-\left(\frac{a_\epsilon}{B}\right)' + 2\frac{b_\epsilon}{B}\right] \frac{w_\epsilon'}{w_\epsilon^3},$$

and hence, the fact $(\frac{1}{w_\epsilon})' = -\frac{w_\epsilon'}{w_\epsilon^2}$ and Hölder's inequality imply

$$\int \frac{a_\epsilon}{B}\left|\left(\frac{1}{w_\epsilon}\right)'\right|^2 = \int \left[\left(\frac{a_\epsilon}{B}\right)' - 2\frac{b_\epsilon}{B}\right]\frac{1}{w_\epsilon}\left(\frac{1}{w_\epsilon}\right)' \le \left\|\left(\frac{a_\epsilon}{B}\right)' - 2\frac{b_\epsilon}{B}\right\|_p \left\|\frac{1}{w_\epsilon}\right\|_r \left\|\left(\frac{1}{w_\epsilon}\right)'\right\|_2.$$

It then follows from (4.7) that $\left\|\left(\frac{1}{w_\epsilon}\right)'\right\| \lesssim \left\|\frac{1}{w_\epsilon}\right\|_r$, which together with the Sobolev embedding theorem and interpolation inequality yields

$$\left\|\frac{1}{w_\epsilon}\right\|_\infty \lesssim \left\|\frac{1}{w_\epsilon}\right\|_2 + \left\|\left(\frac{1}{w_\epsilon}\right)'\right\|_2 \lesssim \left\|\frac{1}{w_\epsilon}\right\|_r. \tag{4.9}$$

Since $\|w_\epsilon\|_r \left\|\frac{1}{w_\epsilon}\right\|_r = \left(\int e^{rv_\epsilon}\int e^{-rv_\epsilon}\right)^{\frac{1}{r}} \lesssim 1$ by **Step 2**, we find

$$\left\|\frac{1}{w_\epsilon}\right\|_r \lesssim \frac{1}{\|w_\epsilon\|_r} \lesssim \frac{1}{(\min w_\epsilon)^{1-\frac{1}{r}}\|w_\epsilon\|_1^{\frac{1}{r}}} \lesssim \left\|\frac{1}{w_\epsilon}\right\|_\infty^{1-\frac{1}{r}},$$

where we used $\frac{1}{(\min w_\epsilon)^{1-\frac{1}{r}}} = \|\frac{1}{w_\epsilon}\|_\infty^{1-\frac{1}{r}}$ and $\|w_\epsilon\|_1 \ge \|u_\epsilon\|_1 \min B = \min B$ in the third inequality. Applying (4.9) then results in $\|\frac{1}{w_\epsilon}\|_\infty \lesssim 1$. Hence, $\min w_\epsilon = \|\frac{1}{w_\epsilon}\|_\infty^{-1} \gtrsim 1$.

Consequently, (4.4) follows from **Step 1** and **Step 3**, completing the proof. □

Next, we prove Theorem 4.2 by means of Berstein-type estimates.

Proof of Theorem 4.2. Thanks to Theorem A.1, u_ϵ belongs to $W^{1,p}$ for any $p>2$. Hence, $u_\epsilon\in C^0$ by the Sobolev embedding theorem.

We claim $u_\epsilon\in C^1$. Indeed, (4.3) says particularly that $\frac{\epsilon^2}{2}(a_\epsilon u_\epsilon)'-(B+\epsilon^2 b_\epsilon)u_\epsilon$ has weak derivative 0. Thus, $\frac{\epsilon^2}{2}(a_\epsilon u_\epsilon)'-(B+\epsilon^2 b_\epsilon)u_\epsilon$ is absolutely continuous (up to a set of zero Lebesgue measure) and there exists $C_\epsilon\in\mathbb{R}$ such that

$$\frac{\epsilon^2}{2}(a_\epsilon u_\epsilon)'-(B+\epsilon^2 b_\epsilon)u_\epsilon=C_\epsilon\quad\text{a.e. on}\quad\mathbb{S}^1. \tag{4.10}$$

Obviously, $a_\epsilon\in C^1$ and $b_\epsilon\in C^0$. Given $\min a_\epsilon>0$ (by assumption), the continuity of B and $u_\epsilon\in C^0$, we find from (4.10) that u_ϵ' is a.e. equal to a continuous function. Thus, we may assume, without loss of generality, that $u_\epsilon\in C^1$.

Integrating (4.10) over $\mathbb{S}^1$ yields

$$-\int(B+\epsilon^2 b_\epsilon)u_\epsilon=C_\epsilon. \tag{4.11}$$

Note that the assumptions ensure that

$$\lim_{\epsilon\to 0}\epsilon^2\left(\|a_\epsilon\|_\infty+\|b_\epsilon\|_\infty\right)=0. \tag{4.12}$$

Then, we see from (4.11), $B>0$ and $\int u_\epsilon=1$ that

$$C_\epsilon\approx -1 \tag{4.13}$$

Suppose u_ϵ attains its maximum and minimum on $\mathbb{S}^1$ at x_ϵ and y_ϵ, respectively. Then, $u_\epsilon'(x_\epsilon)=0=u_\epsilon'(y_\epsilon)$. Substituting them into (4.10) yields that the equality $\frac{\epsilon^2}{2}a_\epsilon' u_\epsilon-(B+\epsilon^2 b_\epsilon)u_\epsilon=C_\epsilon$ holds at x_ϵ and y_ϵ, which together with (4.12) and (4.13) gives

$$u_\epsilon(x_\epsilon)=\frac{C_\epsilon}{\frac{\epsilon^2}{2}a_\epsilon'(x_\epsilon)-(B+\epsilon^2 b_\epsilon)(x_\epsilon)}\lesssim 1\quad\text{and}\quad u_\epsilon(y_\epsilon)=\frac{C_\epsilon}{\frac{\epsilon^2}{2}a_\epsilon'(x_\epsilon)-(B+\epsilon^2 b_\epsilon)(x_\epsilon)}\gtrsim 1.$$

It remains to show the convergence result for $\{u_\epsilon\}_\epsilon$. Recall that μ_0 is the unique invariant measure of (4.1). Thanks to the compactness of $\mathbb{S}^1$ and Prokhorov's theorem, the limit $\lim_{\epsilon\to 0}\mu_\epsilon=\mu_0$ holds under the weak*-topology. This together with $\|u_\epsilon\|_\infty\lesssim 1$ yields $\lim_{\epsilon\to 0}u_\epsilon=u_0$ weakly in L^p for any $p>1$.

Now, we prove the results under additional assumptions. Note that $u_\epsilon\in C^1$ in this case. Since $\|u_\epsilon\|_\infty\lesssim 1$, the classical theory for elliptic equations guarantees that $u_\epsilon\in W^{2,p}$ for any $p>2$. Obviously, $a_\epsilon\in C^2$, $b_\epsilon\in C^1$ and $\min a_\epsilon>0$. Since B is Lipschitz continuous and $u_\epsilon\in C^1$, we see from (4.3) that $u_\epsilon''\in C^0$.

Suppose $(u_\epsilon')^2$ attains its maximum at $x_\epsilon\in\mathbb{S}^1$. Then, $(u_\epsilon' u_\epsilon'')(x_\epsilon)=0$. Multiplying (4.3) by u_ϵ' and evaluating at x_ϵ, we deduce

$$\begin{aligned}0&=\frac{\epsilon^2}{2}(a_\epsilon u_\epsilon)'' u_\epsilon'-[(B+\epsilon^2 b_\epsilon)u_\epsilon]' u_\epsilon'\\&=\frac{\epsilon^2}{2}\left(a_\epsilon u_\epsilon'' u_\epsilon'+2a_\epsilon'(u_\epsilon')^2+a_\epsilon'' u_\epsilon u_\epsilon'\right)-(B+\epsilon^2 b_\epsilon)(u_\epsilon')^2-(B+\epsilon^2 b_\epsilon)' u_\epsilon u_\epsilon'\\&=\left[\epsilon^2 a_\epsilon'-(B+\epsilon^2 b_\epsilon)\right](u_\epsilon')^2+\left(\frac{\epsilon^2}{2}a_\epsilon''-(B+\epsilon^2 b_\epsilon)'\right)u_\epsilon u_\epsilon'\quad\text{at }x_\epsilon.\end{aligned} \tag{4.14}$$

Since the assumptions ensure $\lim_{\epsilon\to 0}\epsilon^2(\|a'_\epsilon\|_\infty+\|a''_\epsilon\|_\infty+\|b_\epsilon\|_\infty+\|b'_\epsilon\|_\infty)=0$, we conclude from $B>0$, (4.14) that $\|u'_\epsilon\|_\infty=|u'_\epsilon(x_\epsilon)|\lesssim\|u_\epsilon\|_\infty\lesssim 1$. It then follows from the Arzelà-Ascoli theorem that the limit $\lim_{\epsilon\to 0}u_\epsilon=u_0$ holds in C^α for any $\alpha\in(0,1)$. ☐

Acknowledgement

Z.S. would like to express gratitude to Professor Jianyu Chen at Soochow University for insightful discussions regarding Theorem B.

Appendix A. Stationary measure

In this appendix, we include the definition of *stationary measures* for SDEs with less regular coefficients, and present classical results on their existence, uniqueness and regularity.

Let $d\geq 1$ be an integer. Recall that M is a d-dimensional smooth, connected, and closed Riemannian manifold. Consider the following SDE on M:

$$dX_t=A_0(X_t)dt+\sum_{i=1}^m A_i(X_t)\circ dW_t^i, \tag{A.1}$$

where $m\geq d$, A_i, $i\in\{0,\ldots,m\}$ are vector fields on M, W_t^i, $i\in\{1,\ldots,m\}$ are independent and standard one-dimensional Brownian motions on some probability space, and the stochastic integrals are understood in the sense of Stratonovich.

We assume that $A_0\in L^p$ and $A_i\in W^{1,p}$, $i\in\{1,\ldots,m\}$ for some $p>\max\{d,2\}$, and there exists $\lambda>0$ such that

$$\sum_{i=1}^m|A_if|^2\geq\lambda|\nabla f|^2\quad\text{Vol-a.e.},\quad\forall f\in W^{1,1}.$$

Denote by $\mathcal{L}$ the generator of (A.1), that is, $\mathcal{L}:=\frac{1}{2}\sum_{i=1}^m A_i^2+A_0$. Its formal L^2-adjoint operator, namely, the Fokker-Planck operator, is denoted by $\mathcal{L}^*$.

Definition A.1 (Stationary measure)**.** *A probability measure μ_ϵ on M is called a* stationary measure *of* (A.1) *if $\mathcal{L}^*\mu=0$ in the sense that $\int\mathcal{L}f\,d\mu=0$ for all $f\in C^2$.*

Theorem A.1 ([54, 18])**.** *The SDE* (A.1) *admits a unique stationary measure μ, which has a positive density $u\in W^{1,p}$. Moreover, there holds*

$$-\frac{1}{2}\int A_if\,[A_iu+(\mathrm{div}A_i)u]+\int(A_0f)\,u=0,\quad\forall f\in W^{1,2}.$$

Appendix B. Some formulas

We collect some of the frequently used formulas for calculus on manifolds.

Lemma B.1. *Let $f,h:M\to\mathbb{R}$ belong to $W^{1,2}$ and $X:M\to TM$ be a vector field in $W^{1,2}$. Then, the following hold:*

(1) $X(fh)=(Xf)h+fXh$;
(2) $\mathrm{div}(fX)=Xf+f\mathrm{div}X$;
(3) $\int Xf=-\int f\mathrm{div}X$;
(4) *if* $X\in W^{1,p}$ *for some* $p>\frac{d}{2}$, *then* $\int hXf=-\int fh(\mathrm{div}X)-\int fXh$.

Proof. (1)-(2) are obvious. Note that $\int Xf=-\int\mathrm{div}(fX)-\int f\mathrm{div}X$. Since M has no boundary, we apply the divergence theorem to find $\int\mathrm{div}(fX)=0$. Hence, (3) holds. Similarly, (4) also holds. ☐

The next elementary result concerns vector fields spanning the tangent bundle TM.

Lemma B.2. *Let $X_i : M \to TM$, $i \in \{1, \dots, n\}$ for some $n \geq d$, be continuous vectors fields on M. Assume $\{X_i\}_{i=1}^n$ spans the tangent bundle TM. Then, there is $C > 1$ such that for any $f \in W^{1,1}$, there holds*

$$\frac{1}{C}|\nabla f| \leq \sum_{i=1}^n |X_i f| \leq C|\nabla f| \quad \text{Vol-a.e. on } M.$$

Proof. We only prove the lemma when $f \in C^1$; the general case follows from standard approximation arguments. Let $(U, (x_1, \dots, x_d))$ be any smooth local chart of M. It is known that $\nabla f = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$ in U, where (g^{ij}) is the inverse of the Riemannian metric $g := (g_{ij})$. Hence, there exist positive constants C_1 and C_2, independent of f, such that

$$|\nabla f|^2 \leq C_1 \sum_{i=1}^d \left| g^{ij} \frac{\partial f}{\partial x_j} \right|^2 \leq C_2 \sum_{i=1}^d \left| \frac{\partial f}{\partial x_j} \right|^2 \quad \text{in} \quad U. \tag{B.1}$$

Since $\{X_i\}_{i=1}^n$ spans the tangent bundle TM, we find a continuous function $(h_{ij}) : U \to \mathbb{R}^{d \times n}$ with full rank such that $\frac{\partial}{\partial x_i} = h_{ij} X_j$ in U for all $i \in \{1, \dots, d\}$, and thus, there is $C_3 > 0$ such that

$$\left| \frac{\partial f}{\partial x_i} \right|^2 = |h_{ij} X_j f|^2 \leq C_3 \sum_{j=1}^n |X_j f|^2 \quad \text{in} \quad U, \quad \forall i \in \{1, \dots, d\}.$$

This together with (B.1) leads to $|\nabla f|^2 \leq dC_2 C_3 \sum_{j=1}^n |X_j f|^2$ in U.

Conversely, as $X_i = X_{ij} \frac{\partial}{\partial x_j}$ for continuous functions $X_{ij} : U \to \mathbb{R}$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$, we see that $\sum_{i=1}^n |X_i f|^2 \leq C_4 |\nabla f|^2$ in U for some $C_4 > 0$. As a result, there is $C_5 > 1$ such that

$$\frac{1}{C_5}|\nabla f| \leq \sum_{i=1}^n |X_i f| \leq C_5 |\nabla f| \quad \text{in} \quad U.$$

Since $(U, (x_1, \dots, x_d))$ is any local chart, the desired result follows immediately. □

References

[1] S. Aizicovici and T. Young. Stochastic stability for flows with smooth invariant measures. *Libertas Math.*, 30:71–79, 2010.

[2] J. F. Alves. Strong statistical stability of non-uniformly expanding maps. *Nonlinearity*, 17(4):1193–1215, 2004.

[3] J. F. Alves and V. Araújo. Random perturbations of nonuniformly expanding maps. *Astérisque*, (286):xvii, 25–62, 2003.

[4] J. F. Alves, V. Araújo, and C. H. Vásquez. Stochastic stability of non-uniformly hyperbolic diffeomorphisms. *Stoch. Dyn.*, 7(3):299–333, 2007.

[5] J. F. Alves, C. L. Dias, S. Luzzatto, and V. Pinheiro. SRB measures for partially hyperbolic systems whose central direction is weakly expanding. *J. Eur. Math. Soc. (JEMS)*, 19(10):2911–2946, 2017.

[6] J. F. Alves and H. Vilarinho. Strong stochastic stability for non-uniformly expanding maps. *Ergodic Theory Dynam. Systems*, 33(3):647–692, 2013.

[7] M. Andersson and C. H. Vásquez. On mostly expanding diffeomorphisms. *Ergodic Theory Dynam. Systems*, 38(8):2838–2859, 2018.

[8] V. Araújo and A. Tahzibi. Stochastic stability at the boundary of expanding maps. *Nonlinearity*, 18(3):939–958, 2005.

[9] M. Assaf and B. Meerson. WKB theory of large deviations in stochastic populations. *J. Phys. A: Math. Theor.*, 50(26):263001, 2017.

[10] V. Baladi, M. Benedicks, and D. Schnellmann. Whitney-Hölder continuity of the SRB measure for transversal families of smooth unimodal maps. *Invent. Math.*, 201(3):773–844, 2015.

[11] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Sci. École Norm. Sup. (4)*, 29(4):483–517, 1996.
[12] V. Baladi and L.-S. Young. On the spectra of randomly perturbed expanding maps. *Comm. Math. Phys.*, 156(2):355–385, 1993.
[13] M. Benedicks and M. Viana. Random perturbations and statistical properties of Hénon-like maps. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 23(5):713–752, 2006.
[14] M. Benedicks and L.-S. Young. Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. *Ergodic Theory Dynam. Systems*, 12(1):13–37, 1992.
[15] M. Blank and G. Keller. Stochastic stability versus localization in one-dimensional chaotic dynamical systems. *Nonlinearity*, 10(1):81–107, 1997.
[16] M. Blank and G. Keller. Random perturbations of chaotic dynamical systems: stability of the spectrum. *Nonlinearity*, 11(5):1351–1364, 1998.
[17] M. L. Blank. Small perturbations of chaotic dynamical systems. *Uspekhi Mat. Nauk*, 44(6(270)):3–28, 203, 1989.
[18] V. I. Bogachev, N. V. Krylov, and M. Röckner. On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. *Comm. Partial Differential Equations*, 26(11-12):2037–2080, 2001.
[19] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov. *Fokker-Planck-Kolmogorov equations*, volume 207 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
[20] R. T. Bortolotti. Physical measures for certain partially hyperbolic attractors on 3-manifolds. *Ergodic Theory Dynam. Systems*, 39(1):74–104, 2019.
[21] R. Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.
[22] Y. Cao, Z. Mi, and D. Yang. On the abundance of Sinai-Ruelle-Bowen measures. *Comm. Math. Phys.*, 391(3):1271–1306, 2022.
[23] V. Climenhaga, S. Luzzatto, and Y. Pesin. The geometric approach for constructing Sinai-Ruelle-Bowen measures. *J. Stat. Phys.*, 166(3-4):467–493, 2017.
[24] V. Climenhaga, S. Luzzatto, and Y. Pesin. SRB measures and Young towers for surface diffeomorphisms. *Ann. Henri Poincaré*, 23(3):973–1059, 2022.
[25] M. V. Day. Recent progress on the small parameter exit problem. *Stochastics*, 20(2):121–150, 1987.
[26] G. Del Magno, J. Lopes Dias, P. Duarte, J. P. Gaivão, and D. Pinheiro. SRB measures for polygonal billiards with contracting reflection laws. *Comm. Math. Phys.*, 329(2):687–723, 2014.
[27] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Rev. Modern Phys.*, 57(3):617–656, 1985.
[28] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.
[29] R. Graham. *Macroscopic potentials, bifurcations and noise in dissipative systems*, page 225–278. Cambridge University Press, 1989.
[30] V. Guillemin and A. Pollack. *Differential topology*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1974.
[31] E. P. Hsu. *A brief introduction to Brownian motion on a Riemannian manifold*.
[32] A. Katok. Bernoulli diffeomorphisms on surfaces. *Ann. of Math. (2)*, 110(3):529–547, 1979.
[33] A. Katok and Y. Kifer. Random perturbations of transformations of an interval. *J. Analyse Math.*, 47:193–237, 1986.
[34] G. Keller. Stochastic stability in some chaotic dynamical systems. *Monatsh. Math.*, 94(4):313–333, 1982.
[35] G. Keller and C. Liverani. Stability of the spectrum for transfer operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(1):141–152, 1999.
[36] Y. Kifer. Small random perturbations of certain smooth dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:1091–1115, 1974.
[37] Y. Kifer. *Random perturbations of dynamical systems*, volume 16 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1988.
[38] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.
[39] R. Leplaideur and D. Yang. SRB measures for higher dimensional singular partially hyperbolic attractors. *Ann. Inst. Fourier (Grenoble)*, 67(6):2703–2717, 2017.

[40] R. J. Metzger. Stochastic stability for contracting Lorenz maps and flows. *Comm. Math. Phys.*, 212(2):277–296, 2000.
[41] T. Mikami. Asymptotic analysis of invariant density of randomly perturbed dynamical systems. *Ann. Probab.*, 18(2):524–536, 1990.
[42] D. Ruelle. A measure associated with axiom-A attractors. *Amer. J. Math.*, 98(3):619–654, 1976.
[43] D. Ruelle. *Thermodynamic formalism*, volume 5 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, MA, 1978. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota.
[44] W. Shen. On stochastic stability of non-uniformly expanding interval maps. *Proc. Lond. Math. Soc. (3)*, 107(5):1091–1134, 2013.
[45] W. Shen and S. van Strien. On stochastic stability of expanding circle maps with neutral fixed points. *Dyn. Syst.*, 28(3):423–452, 2013.
[46] S. J. Sheu. Asymptotic behavior of the invariant density of a diffusion Markov process with small diffusion. *SIAM J. Math. Anal.*, 17(2):451–460, 1986.
[47] Ja. G. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
[48] Ya. G. Sinaĭ. Kolmogorov's work on ergodic theory. *Ann. Probab.*, 17(3):833–839, 1989.
[49] M. Viana and J. Yang. Physical measures and absolute continuity for one-dimensional center direction. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 30(5):845–877, 2013.
[50] J. Wang. Landscape and flux theory of non-equilibrium dynamical systems with application to biology. *Advances in Physics*, 64(1):1–137, 2015.
[51] L.-S. Young. Stochastic stability of hyperbolic attractors. *Ergodic Theory Dynam. Systems*, 6(2):311–319, 1986.
[52] L.-S. Young. What are SRB measures, and which dynamical systems have them? *J. Statist. Phys.*, 108(5-6):733–754, 2002.
[53] L.-S. Young. Understanding chaotic dynamical systems. *Comm. Pure Appl. Math.*, 66(9):1439–1463, 2013.
[54] E. C. Zeeman. Stability of dynamical systems. *Nonlinearity*, 1(1):115–155, 1988.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: wqi2@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: zhongwei@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada, and School of Mathematics, Jilin University, Changchun 130012, PRC
Email address: yingfei@ualberta.ca