LARGE DEVIATION PRINCIPLE FOR QUASI-STATIONARY DISTRIBUTIONS AND MULTISCALE DYNAMICS OF ABSORBED SINGULAR DIFFUSIONS

WEIWEI QI, ZHONGWEI SHEN, AND YINGFEI YI

Abstract. The present paper is devoted to the investigation of an important family of absorbed singular diffusion processes exhibiting long transient dynamics, namely, interesting and important dynamical behaviours over long but finite time scales. We explore the multiscale dynamics by establishing the asymptotic distribution of the normalized extinction time, the asymptotic reciprocal relationship between the mean extinction time and the principal eigenvalue of the generator, and a sophisticated multiscale estimate of solutions. While information about the extinction time and mean extinction time uncovers fundamental principles quantifying in particular the lifespan of interesting dynamical behaviours combined and its natural connection with the principal eigenvalue, the multiscale estimate characterizes the dynamics over different time scales. These are achieved by examining quasi-stationary distributions (QSDs) that govern the dynamics before the eventual absorption happens, and establishing the powerful sub-exponential large deviation principle (LDP) for QSDs, which determines the quasi-potential function and prefactor in the WKB expansion, and therefore, provides very fine asymptotic or concentration properties of QSDs. To the best of our knowledge, this is the first time that the sub-exponential LDP for QSDs is established for absorbed singular diffusion processes. Our approach is analytic and elementary. As byproducts or consequences, new results about QSDs near the absorbing state and infinity, the sub-exponential asymptotic of the principal eigenvalue, and the asymptotic of the principal eigenfunction are obtained. The sub-exponential LDP for QSDs is of independent interest and expected to have more far-reaching consequences. Applications to logistic diffusion processes arising from chemical reactions and population dynamics are discussed. In particular, Keizer's paradox concerning the long-term dynamical disagreement between a deterministic model and its stochastic counterpart, and diffusion approximation of QSDs are rigorously justified.

Contents

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1. Introduction

A large number of experimental and numerical evidences show that complex processes in biology, chemistry, fluids, etc. often exhibit *transient dynamics*, namely, intriguing and important dynamical behaviours over a relatively long but finite time period. For instance, species in a community usually coexist for a long period that may span dozens or even hundreds of generations before the extinction of at least one species (see e.g. [37, 38, 67]). In a closed chemical reaction system, chemical oscillations could last for a long period before the system eventually relaxes to the thermal equilibrium due to the inevitable heat dissipation (see e.g. [69, 78]). In an open flow, transiently chaotic advective dynamics can be generated to impact the spreading of pollutants, the population dynamics of plankton and larvae, biological and chemical reactions and so on (see e.g. [50, 74]). In the dynamics of decision making, the course of thinking or discussion could be complex and last for a long period before a decision is reached (see e.g. [26, 73, 74]). The treatment of such dynamical behaviours is out of the scope of traditional dynamical system theories focusing on long-term dynamics. Addressing long but finite-time dynamical behaviors, transient dynamics has demonstrated its significance in many scientific areas and been attracting an increasing amount of attention. Given more and more results from experiments and numerical studies (see e.g. [50, 65, 38]), rigorous mathematical frameworks are expected to classify transient dynamics of different mechanisms and further stimulate the investigation towards a better understanding of transient dynamics.

In the present paper, we continue to study the transient dynamics and related properties of a family of absorbed singular diffusion processes arising from chemical reactions and population dynamics initiated in the works [70, 45]. More precisely, we consider the following randomly perturbed dynamical systems:

$$dx = b(x)dt + \epsilon\sqrt{a(x)}dW_t, \quad x \in [0,\infty), \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a parameter, $b : [0,\infty) \to \mathbb{R}$, $a : [0,\infty) \to [0,\infty)$ and W_t is the standard one-dimensional Wiener process on some probability space. The equation (1.1) is often derived as the diffusion approximation [49] of re-scaled birth-and-death processes modelling the evolution of some species in a community or some type of molecules in a chemical reaction system (see e.g. [27, 47, 3]). The reader is referred to Subsections 6.1 and 6.2 for a brief exposition. In this circumstance, the unperturbed ordinary differential equation (ODE)

$$\dot{x} = b(x), \quad x \in [0,\infty) \tag{1.2}$$

is the classical mean-field approximation [48] and the small noise $\epsilon\sqrt{a(x)}W_t$ is often interpreted as the demographic or internal noise.

We make the following standard assumptions on the coefficients a and b throughout this paper.

(H) The functions $b : [0,\infty) \to \mathbb{R}$ and $a : [0,\infty) \to [0,\infty)$ satisfy the following conditions:
(1) $b \in C^1([0,\infty)) \cap C^2((0,\infty))$, $b(0) = 0$, $b'(0) > 0$, and $\limsup_{x\to\infty} b(x) < 0$;
(2) $a \in C^2([0,\infty)) \cap C^3((0,\infty))$, $a(0) = 0$, $a'(0) > 0$, and $a > 0$ on $(0,\infty)$;
(3) $\lim_{x\to\infty} \frac{b^2(x)}{a(x)} = \infty$, $\limsup_{x\to\infty} \frac{\max\left\{a(x),|a'(x)|,|a''(x)|,|b'(x)|\right\}}{|b(x)|} < \infty$, and there is $m > 0$ such that $\frac{|b(x)|}{a(x)} \le \left|\int_0^x \frac{b}{a} ds\right|^m$ for $x \gg 1$.

(H)(1) says that b is a logistic-type growth rate function that plays important roles in especially biological and ecological applications. **(H)**(2) assumes that a is degenerate at 0 and behaves like $a'(0)x$ near 0. In particular, $\sqrt{a}$ vanishes and is singular at 0, causing the *non-integrability* of the Gibbs density near 0 that leads to substantial difficulties in the analysis of (1.1). The assumptions $\limsup_{x\to\infty} b(x) < 0$ in **(H)**(1) and $\lim_{x\to\infty} \frac{b^2(x)}{a(x)} = \infty$ in **(H)**(3) ensure the dissipativity of (1.1). Other conditions in **(H)**(3) restricting the behaviours of a, b and the ratio $\frac{b}{a}$ near ∞ are mild technical assumptions, and they are sufficiently general for applications (see Section 6). For the time being, it is beneficial to keep in mind the typical example:

$$dx = x(1-x)dt + \epsilon\sqrt{x}dW_t, \quad x \in [0,\infty),$$

and to point out that (1.1) has two unpleasant features: (i) the vector field vanishes on the boundary, and (ii) the noise is degenerate, that are often kept away from in the study of randomly perturbed dynamical systems and known to cause essential difficulties in the analysis.

1.1. **Quasi-stationary distributions.** Let X_t^ϵ be the stochastic process on $[0,\infty)$ generated by solutions of (1.1). For singular diffusion processes like (1.1), the strong uniqueness is ensured by the well-known Yamada-Watanabe theory [80, 79]. Clearly, 0 is an absorbing state of X_t^ϵ, and is often called the *extinction state* in especially biology and ecology. Under **(H)**, sample paths or trajectories of X_t^ϵ reach the extinction state 0 in finite time almost surely [10, 44]. This is mainly due to the demographic noise, which drives a species to extinction when its density becomes low. Therefore, the long-term behavior of X_t^ϵ tells nothing interesting, driving us to look at the dynamics of X_t^ϵ before hitting 0. Since the ODE (1.2) may contain multiple (local) attractors in $(0,\infty)$, the sample path *large deviation principle* (LDP) [31] indicates with probability almost one that trajectories of X_t^ϵ sojourn around these attractors for a long period before going to extinction, demonstrating fascinating transient dynamics. As in [70, 45], we adopt a distribution/observable-based viewpoint and use *quasi-stationary distributions* (QSDs) (see e.g. [62, 18]), being initial distributions on $(0,\infty)$ such that X_t^ϵ

conditioned on the non-extinction or survival up to time t is independent of $t \geq 0$, to capture the transient dynamics of X_t^ϵ.

We mention that the theory of QSDs for absorbed Markov processes has a long history [68] and finds numerous applications in especially population biology and chemical reactions (see e.g. [62, 18, 22]). However, even for one-dimensional absorbed singular diffusion processes like (1.1), the fundamental theory (i.e., the existence and uniqueness of QSDs and the exponential convergence to QSDs) is only developed recently in the breakthrough [10] and subsequent works [56, 66, 40]. We refer the reader to [11, 14, 15, 16, 41, 35, 30] and references therein for significant developments in higher dimensions.

Denote by T_0^ϵ the *extinction time* of X_t^ϵ, namely, the first time that X_t^ϵ hits 0. More precisely,

$$T_0^\epsilon = \inf\left\{t \geq 0 : X_t^\epsilon = 0\right\}.$$

Then, $\mathbb{P}_\mu^\epsilon[T_0^\epsilon < \infty] = 1$ as mentioned above (see also [44, Chapter VI, Section 3]), where $\mathbb{P}_\mu^\epsilon$ denotes the law of X_t^ϵ with initial distribution μ. The expectation associated with $\mathbb{P}_\mu^\epsilon$ is written as $\mathbb{E}_\mu^\epsilon$. When $\mu = \delta_x$ is the Dirac measure at x, we simply write $\mathbb{P}_{\delta_x}^\epsilon$ and $\mathbb{E}_{\delta_x}^\epsilon$ as $\mathbb{P}_x^\epsilon$ and $\mathbb{E}_x^\epsilon$, respectively.

Definition 1.1 (Quasi-stationary distribution)**.** *A Borel probability measure μ_ϵ on $(0,\infty)$ is called a* quasi-stationary distribution (QSD) *of X_t^ϵ if*

$$\mathbb{P}_{\mu_\epsilon}^\epsilon\left[X_t^\epsilon \in B | t < T_0^\epsilon\right] = \mu_\epsilon(B), \quad \forall t \geq 0, \quad B \in \mathcal{B}((0,\infty)),$$

where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra of $(0,\infty)$.

It is known from the general theory of QSDs (see e.g. [62, 18]) that if μ_ϵ is a QSD of X_t^ϵ, then there is a unique positive number $\lambda_{\epsilon,1}$ such that $T_0^\epsilon \sim \exp(\lambda_{\epsilon,1})$ provided $X_0^\epsilon \sim \mu_\epsilon$. For this reason, $\lambda_{\epsilon,1}$ is often referred to as the *extinction rate*.

We state in Proposition 2.1 the existence of a unique QSD μ_ϵ of X_t^ϵ with a positive and continuously differentiable density u_ϵ. Moreover, the associated extinction rate $\lambda_{\epsilon,1}$ is exactly the principal or first eigenvalue of $-\mathcal{L}_\epsilon$, where $\mathcal{L}_\epsilon$ denotes an appropriate closed extension of the generator or diffusion operator $\phi \mapsto \frac{\epsilon^2}{2}a\phi'' + b\phi'$ of (1.1) (see Subsections 2.1 and 6.1 for details). In addition, the density u_ϵ is a positive and integrable eigenfunction of the Fokker-Planck operator $\phi \mapsto \frac{\epsilon^2}{2}(a\phi)'' - (b\phi)'$ associated with the eigenvalue $-\lambda_{\epsilon,1}$ (see (2.3)).

In previous works [70, 45], the authors study the tightness and rough concentration estimates of $\{\mu_\epsilon\}_\epsilon$, as well as the exponential asymptotic of the first two eigenvalues of $-\mathcal{L}_\epsilon$ in order to characterize the transient dynamics of X_t^ϵ. The main purpose of the present paper is to investigate the multiscale dynamics of X_t^ϵ by establishing the asymptotic distribution of the normalized extinction time, the asymptotic reciprocal relationship between the mean extinction time $\mathbb{E}_\bullet^\epsilon[T_0^\epsilon]$ and the principal eigenvalue $\lambda_{\epsilon,1}$, and a multiscale estimate of the dynamics of X_t^ϵ. While information about the extinction time and mean extinction time uncovers fundamental principles quantifying in particular the lifespan of interesting dynamical behaviours combined and its natural connection with the principal eigenvalue, the multiscale estimate characterizes the dynamics over different time scales. These are achieved mainly by establishing the powerful sub-exponential LDP for the QSD μ_ϵ or its density u_ϵ, which captures very fine asymptotic or concentration properties of μ_ϵ as $\epsilon \to 0$. That is, we rigorously justify the Wentzel-Kramers-Brillouin (WKB) expansion (see e.g. [34, 2])

$$u_\epsilon = \frac{1}{\epsilon a} e^{-\frac{2}{\epsilon^2}v}\left[R_0 + \epsilon^2 R_1 + \cdots + \epsilon^{2n} R_n + o(\epsilon^{2n})\right] \quad \text{in} \quad (0,\infty) \tag{1.3}$$

in the case $n=0$, so that

$$u_\epsilon = \frac{R_\epsilon}{\epsilon a} e^{-\frac{2}{\epsilon^2} v} \quad \text{and} \quad R_\epsilon = R_0 + o(1) \quad \text{in} \quad (0,\infty),$$

where v is often called the *quasi-potential function* or *rate function*, and the sub-exponential term $\frac{R_\epsilon}{\epsilon a}$ is often referred to as the *prefactor* in physics literature. Determining the quasi-potential function v via studying the limit $\lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \ln u_\epsilon$ is the purpose of the LDP. We point out that the WKB expansion (1.3) in the case $n=1$ could *fail* (see Remark 1.1 (4) below for detailed comments). The sub-exponential LDP for μ_ϵ or u_ϵ is of independent interest and expected to have more far-reaching consequences. Not only do results proven in this paper greatly improve many of those contained in [70, 45], but also they widely broaden the scope of the study.

We state our main results in the following Subsections 1.2-1.4. In Subsection 1.5, we briefly discuss about their applications to logistic diffusion processes.

1.2. **Large deviation principle for QSDs.** Consider the potential function:

$$V(x) = -\int_0^x \frac{b}{a} ds, \quad x \in (0,\infty). \tag{1.4}$$

We follow [60] to define *valleys* of V, which reveal certain geometric properties of V.

Definition 1.2. *An open interval $I \subset (0,\infty)$ is called a* valley *(of V) if it is one of the connected components of the sublevel set $\{x \in (0,\infty) : V(x) < \rho\}$ and satisfies $V(\partial I) = \rho$ for some $\rho \in \mathbb{R}$. We say $I \subset (0,\infty)$ a d-*valley *if it is a valley of depth d, namely, $\sup_I V - \inf_I V = d$.*

Set

$$d_1 := \sup_{x\in(0,\infty)} \left[\sup_{(0,x)} V - V(x) \right] > 0, \tag{1.5}$$

which is the depth of the deepest valleys of V. Since $V(0+) = 0$ and $V(\infty) = \infty$ by **(H)**, there exist finitely many d_1-valleys and no d-valley with depth $d > d_1$.

The LDP is proven when there exists a unique d_1-valley, which is a generic case. Recall that u_ϵ is the positive and continuously differentiable density of the unique QSD μ_ϵ of X_t^ϵ.

Theorem A. *Assume* **(H)** *and the existence of a unique d_1-valley (α, β). Then,*

$$\lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \ln u_\epsilon = -v \quad \textit{locally uniformly in} \quad (0,\infty),$$

where v is a locally Lipschitz viscosity solution of the following Hamilton-Jacobi equation

$$(v')^2 + \frac{b}{a} v' = 0 \quad \textit{in} \quad (0,\infty), \tag{1.6}$$

and is given as follows:

- *if $\alpha = 0$, then $v = d_1 + V$;*
- *if $\alpha > 0$, then*

$$v(x) = \begin{cases} d_1 + V(x) - \sup_{(0,x)} V, & x \in (0,\alpha], \\ d_1 + V(x) - V(\alpha), & x \in (\alpha,\infty). \end{cases}$$

Obviously, in the case of a unique d_1-valley (α, β) with $\alpha > 0$, the *quasi-potential function* v obtained in Theorem A is not continuously differentiable everywhere and could be non-differentiable at many points depending on the geometry of V on $(0, \alpha)$. See Figure 1 for an illustration of V and v in the case $\alpha > 0$.

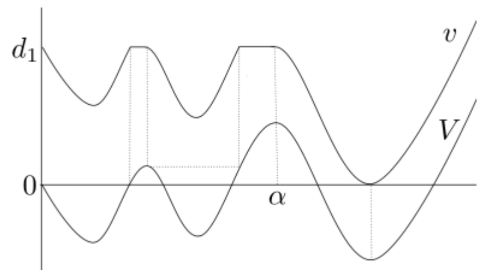

FIGURE 1. Illustration of a potential function V and the associated quasi-potential function v in the case $\alpha > 0$.

We point out that the Hamilton-Jacobi equation (HJE) (1.6) admits infinitely many locally Lipschitz viscosity solutions, giving rise to major difficulties in determining the quasi-potential function v. It is the combined effect of the dynamics of (1.2) and the noise that allows us to uniquely select the solution of (1.6), and therefore, determine v.

It turns out that the quasi-potential function v is a Lyapunov function for (1.2) in the sense of the following result.

Corollary A. *Assume* **(H)** *and the existence of a unique d_1-valley. Let v be the quasi-potential function as in Theorem A. Then, for any solution $x(t)$ of (1.2) with $x(0) \in (0, \infty)$, the function $t \mapsto v(x(t))$ is non-increasing on $[0, \infty)$.*

Proof. As v is locally Lipschitz by Theorem A, so is $v(x(t))$. It is known that $\frac{d}{dt}v(x(t)) = v'(x(t))x'(t)$ for a.e. $t \in \mathbb{R}$ with the understanding that $v'(x(t))x'(t) = 0$ when $x'(t) = 0$ even if v is not differentiable at $x(t)$. It follows from the equations satisfied by v and $x(t)$ that for $0 \leq t_1 < t_2 < \infty$,

$$v(x(t_2)) - v(x(t_1)) = \int_{t_1}^{t_2} v'(x(t))x'(t)dt = -\int_{t_1}^{t_2} a(x(t)) \left[v'(x(t))\right]^2 dt \leq 0,$$

completing the proof. $\square$

Let v be as in Theorem A. To establish the sub-exponential asymptotic of u_ϵ as $\epsilon \to 0$ (or to determine the prefactor in the WKB expansion of u_ϵ), we set

$$R_\epsilon := \epsilon a u_\epsilon e^{\frac{2}{\epsilon^2} v} \tag{1.7}$$

and examine the asymptotic of R_ϵ as $\epsilon \to 0$.

Assuming **(H)** and the existence of a unique d_1-valley (α, β), we see that the set

$$\mathcal{M} := \left\{ x \in (\alpha, \beta) : V(x) = \min_{(\alpha,\beta)} V \right\}$$

is closed and contained in (α, β). Note that $\mathcal{M}$ is exactly the set of global minima of V when $\alpha = 0$, and it may not be when $\alpha > 0$. For fixed $0 < \delta_0 \ll 1$ so that $\mathcal{M}_{\delta_0} := \{x \in (0, \infty) : d(x, \mathcal{M}) < \delta_0\} \subset (\alpha, \beta)$, we set

$$M_\epsilon := \left(\frac{1}{\epsilon} \int_{\mathcal{M}_{\delta_0}} \frac{1}{a} e^{-\frac{2}{\epsilon^2} v} dx \right)^{-1}. \tag{1.8}$$

Clearly, by Laplace's method and the expression of v given in Theorem A, the asymptotic of M_ϵ is determined by $V|_{\mathcal{M}}$, and hence, is independent of the choice of $0 < \delta_0 \ll 1$.

Throughout this paper, for positive numbers A_ϵ and B_ϵ indexed by ϵ, we write

$$A_\epsilon \approx_\epsilon B_\epsilon, \quad A_\epsilon \lesssim_\epsilon B_\epsilon \quad \text{and} \quad A_\epsilon \gtrsim_\epsilon B_\epsilon$$

if $\lim_{\epsilon \to 0} \frac{A_\epsilon}{B_\epsilon} = 1$, $\limsup_{\epsilon \to 0} \frac{A_\epsilon}{B_\epsilon} \leq 1$ and $\liminf_{\epsilon \to 0} \frac{A_\epsilon}{B_\epsilon} \geq 1$, respectively.

Theorem B. *Assume* **(H)** *and the existence of a unique d_1-valley (α, β).*

1. *If $\alpha = 0$, then $R_\epsilon \approx_\epsilon M_\epsilon$ locally uniformly in $(0, \infty)$.*
2. *If $\alpha > 0$, $V(\alpha) > V$ in $(0, \alpha)$ and $b'(\alpha) > 0$, then*

$$\frac{R_\epsilon}{\epsilon} \approx_\epsilon -\frac{M_\epsilon}{2V'(0+)} \sqrt{\frac{-V''(\alpha)}{\pi}} \text{ locally uniformly in } (0, x_0),$$

$$R_\epsilon(x) \approx_\epsilon \frac{M_\epsilon}{\epsilon} \sqrt{\frac{-V''(\alpha)}{\pi}} \int_0^x e^{\frac{2}{\epsilon^2}[V - \sup_{(0,x)} V]} dz, \quad x \in [x_0, \alpha),$$

$$R_\epsilon(x) \approx_\epsilon \begin{cases} \frac{M_\epsilon}{2}, & x = \alpha, \\ M_\epsilon \text{ locally uniformly in } x \in (\alpha, \infty), \end{cases}$$

where $x_0 \in (0, \alpha]$ is the smallest positive zero of V.

To determine the asymptotic of M_ϵ and obtain finer results about the asymptotic of R_ϵ, we impose the following stronger but generic assumption on $\mathcal{M}$. Denote by $\mathbb{N}$ the set of positive integers.

$(\mathbf{H}_V)$ There are $x_1, \dots, x_N \in (0, \infty)$ for some $N \in \mathbb{N}$ such that $\mathcal{M} = \{x_1, \dots, x_N\}$ and $b'(x_i) < 0$ for each $i \in \{1, \dots, N\}$.

It says that $V|_{(\alpha,\beta)}$ attains its minimal value at only finitely many points and they are non-degenerate equilibria of (1.2). Whenever $(\mathbf{H}_V)$ is assumed, we denote

$$M_0 := \left(\sum_{i=1}^N \frac{1}{a(x_i)} \sqrt{\frac{\pi}{V''(x_i)}} \right)^{-1}.$$

Under $(\mathbf{H}_V)$, we readily see from Laplace's method that $\lim_{\epsilon \to 0} M_\epsilon = M_0$, leading to the next result.

Corollary B. *Assume* **(H)**, *the existence of a unique d_1-valley (α, β) and* $(\mathbf{H}_V)$.

1. *If $\alpha = 0$, then $\lim_{\epsilon \to 0} R_\epsilon = M_0$ locally uniformly in $(0, \infty)$.*
2. *If $\alpha > 0$, $V(\alpha) > V$ in $(0, \alpha)$ and $b'(\alpha) > 0$, then*

$$\lim_{\epsilon \to 0} \frac{R_\epsilon}{\epsilon} = -\frac{M_0}{2V'(0+)} \sqrt{\frac{-V''(\alpha)}{\pi}} \text{ locally uniformly in } (0, x_0),$$

$$R_\epsilon(x) \approx_\epsilon \frac{M_0}{\epsilon} \sqrt{\frac{-V''(\alpha)}{\pi}} \int_0^x e^{\frac{2}{\epsilon^2}[V - \sup_{(0,x)} V]} dz, \quad x \in [x_0, \alpha),$$

$$\lim_{\epsilon \to 0} R_\epsilon(x) = \begin{cases} \frac{M_0}{2}, & x = \alpha, \\ M_0 \text{ locally uniformly in } x \in (\alpha, \infty), \end{cases}$$

where $x_0 \in (0,\alpha]$ is the smallest positive zero of V.

Remark 1.1. *We make some comments about Theorem B (2) and Corollary B (2) regarding additional assumptions and the asymptotic of R_ϵ in $(0,\alpha)$.*

1. *As (α,β) is the unique d_1-valley, there must hold that $V(\alpha)\geq V$ in $(0,\alpha)$ with strict inequality in $(0,\delta)$ for some $\delta\in(0,\alpha)$. If there is $x_*\in(0,\alpha)$ such that $V(\alpha)=V(x_*)$, we need to impose additional conditions on V at such a x_* in order to determine the asymptotic. While it is certainly doable, the statement would be messy. That is why we assume $V(\alpha)>V$ in $(0,\alpha)$.*
2. *The condition $b'(\alpha)>0$ is not a strong restriction, and can be replaced by a higher order derivative condition at α if a and b, so V, have enough differentiability near α.*
3. *Note that $V(\alpha)\geq 0$. When $x_0=\alpha$ (if and only if $V(\alpha)=0$), the asymptotic of R_ϵ as $\epsilon\to 0$ is explicitly characterized. When $x_0<\alpha$ (if and only if $V(\alpha)>0$), it is theoretically possible to establish the explicit asymptotic of $R_\epsilon(x)$ for $x\in[x_0,\alpha)$ by means of Laplace's method. But, it is hard to state the result in a concise way because the asymptotic of $\int_0^x e^{\frac{2}{\epsilon^2}[V-\sup_{(0,x)}V]}dz$ for $x\in[x_0,\alpha)$ depends heavily on the geometry of V on $[x_0,\alpha)$. As the explicit asymptotic is not of much use, we do not pursue here.*
4. *Setting $R_{\frac{1}{2}}:=-\frac{M_0}{2V'(0+)}\sqrt{\frac{-V''(\alpha)}{\pi}}$, we see that $R_\epsilon=\epsilon R_{\frac{1}{2}}+o(\epsilon)$ in $(0,x_0)$, where $o(\epsilon)$ is locally uniformly in $(0,x_0)$. Therefore,*

$$u_\epsilon=\frac{1}{\epsilon a}e^{-\frac{2}{\epsilon^2}v}\left[\epsilon R_{\frac{1}{2}}+o(\epsilon)\right]\quad in\quad (0,x_0),$$

giving the "half-order" WKB expansion of u_ϵ, and hence, saying in particular the failure *of the first-order WKB expansion of u_ϵ (i.e., (1.3) in the case $n=1$) in $(0,x_0)$.*

It should be pointed out that establishing the sub-exponential LDP for stationary distributions or QSDs is generally a very challenging problem as it relies heavily on the dynamical structure of the unperturbed system, and the mathematical treatment often needs to solve badly behaved Hamilton-Jacobi equations (HJEs) and singularly perturbed equations. To be more specific and for clarity, let f be a smooth vector field on an open domain $\mathcal{U}\subset\mathbb{R}^d$ generating the flow φ^t and consider

$$dx=f(x)dt+\epsilon dW_t\quad\text{in}\quad\mathcal{U}. \tag{1.9}$$

Suppose u_ϵ is the smooth density of a stationary distribution or QSD in $\mathcal{U}$ of (1.9). Then, there is $\lambda_\epsilon\geq 0$ such that

$$\frac{\epsilon^2}{2}\Delta u_\epsilon-\nabla\cdot(fu_\epsilon)=-\lambda_\epsilon u_\epsilon\quad\text{in}\quad\mathcal{U}. \tag{1.10}$$

Assume further that $\mathcal{U}$ is contained in the basin of attraction of a normally hyperbolic and attractive compact invariant manifold $\mathcal{M}$ of φ^t with dimension $m\leq d-1$. Then, $\lambda_\epsilon=o(e^{-\frac{\gamma}{\epsilon^2}})$ for some $\gamma>0$. We look for v (the quasi-potential function) and R_ϵ such that $u_\epsilon=\frac{R_\epsilon}{\epsilon^{d-m}}e^{-\frac{2}{\epsilon^2}v}$ and $R_\epsilon=O(1)$. Inserting this ansatz into (1.10), we find that v satisfies the HJE

$$|\nabla v|^2+f\cdot\nabla v=0, \tag{1.11}$$

and R_ϵ solves the following singularly perturbed equation

$$\frac{\epsilon^2}{2}\Delta R_\epsilon-(f+2\nabla v)\cdot\nabla R_\epsilon-(\nabla\cdot f-\lambda_\epsilon+\Delta v)R_\epsilon=0. \tag{1.12}$$

There are essential difficulties in solving (1.11) and (1.12).

(i) The quasi-potential function v, if exists, must be a viscosity solution of the HJE (1.11), which however admits infinitely many viscosity solutions. Therefore, one has to determine v from a different perspective. This often requires additional dynamical assumptions on $\mathcal{M}$, say, φ^t being transitive and uniquely ergodic on $\mathcal{M}$.

(ii) Observe that $v \in C^2(\mathcal{U})$ is highly expected to studying R_ϵ, but this is not the case in general even if $\mathcal{M}$ is a singleton set (see [21]). The regularity in a small neighborhood $\mathcal{O}$ of $\mathcal{M}$ is possible thanks to the dynamical structure of φ^t in $\mathcal{O}$.

(iii) Establishing $R_\epsilon = O(1)$ in $\mathcal{O}$ is greatly challenged by the sign-indefiniteness of coefficients. In fact, under the additional dynamical assumption mentioned in (i), ∇v vanishes on $\mathcal{M}$ and Δv only vanishes along $\mathcal{M}$. Hence, components of $f + 2\nabla v$ and the term $\nabla \cdot f + \Delta v$ are generally sign-indefinite, causing substantial troubles in deriving uniform-in-ϵ estimates for R_ϵ.

Given these difficulties, the sub-exponential LDP for stationary distributions or QSDs is only known when φ^t has very simple dynamics, saying that $\mathcal{M}$ is a linearly stable equilibrium of the flow φ^t, and $\overline{\mathcal{U}}$ is contained in the basin of attraction of $\mathcal{M}$.

The situation is certainly much more complex if $\mathcal{M}$ is just a local or global attractor, or the additive noise in (1.9) is replaced by a multiplicative noise that becomes degenerate and singular in part of $\partial\mathcal{U}$. Unfortunately, we run into such issues. In fact, in our case, $\mathcal{M}$ is just the global attractor of the unperturbed ODE (1.2) in $(0, \infty)$ (generally consisting of equilibria and their connecting orbits) and the noise is singular and degenerate at 0.

Now, we mention relevant works about the LDP for stationary distributions and QSDs, and compare our approach with those contained in literature. For stationary distributions of randomly perturbed dynamical systems of the form

$$dx = f(x)dt + \epsilon\sigma(x)dW_t, \quad x \in \mathbb{R}^d,$$

where the unperturbed ODE $\dot{x} = f(x)$ admits a non-degenerate globally asymptotically stable equilibrium and the diffusion matrix $\sigma\sigma^\top$ is uniformly positive definite, the LDP as in Theorem A has been studied in [31, 72], the sub-exponential LDP as in Theorem B has been established in [71, 21, 7], and the WKB asymptotic expansion in a small neighbourhood of the equilibrium has been justified in [63, 64]. All of them build on the sample path LDP due to Freidlin and Wentzell [31], except the work [7] in which the authors tackle the problem from a control theoretic viewpoint and are able to prove the LDP for vector fields admitting finitely many asymptotically stable equilibria and no other ω-limit sets. In [72], the author replaces the positive definiteness of $\sigma\sigma^\top$ by some conditions on the controlled trajectories (see the condition (A4) in [72, Theorem 1] for details), and therefore, are capable of treating some degenerate cases. In [58], the authors study a family of continuous-time symmetric random walks on the unit circle and establish the LDP for stationary distributions by means of the Aubry-Mather theory (see e.g. [6, 29]).

As for the LDP for QSDs, there exist a few results [60, 12, 9, 17]. Consider the following reversible diffusion processes or overdamped Langevin equation:

$$dx = -\nabla f(x)dt + \epsilon dW_t, \quad x \in \mathbb{R}^d,$$

which is restricted on a smooth, open, bounded and connected domain Ω and killed on its boundary $\partial\Omega$. The density of the unique QSD is given by $\frac{\phi_\epsilon\gamma_\epsilon}{\int_\Omega \phi_\epsilon\gamma_\epsilon dx}$ in Ω, where $\gamma_\epsilon = \frac{e^{-\frac{2}{\epsilon^2}f}}{\int_\Omega e^{-\frac{2}{\epsilon^2}f}dx}$ and $\phi_\epsilon > 0$ in Ω is the principal eigenfunction of the generator and is normalized to satisfy $\int_\Omega \phi_\epsilon^2\gamma_\epsilon dx = 1$. In [60], assuming the existence of a unique deepest valley D contained in Ω, the author shows by a functional analytic approach that $\lim_{\epsilon\to 0}\phi_\epsilon = 1$ locally uniformly in D, leading to the sub-exponential LDP for the QSD in D. More precise asymptotic of the principal eigenfunction ϕ_ϵ in a neighbourhood of a non-degenerate local minimal point of the potential function f is obtained in [9] by a potential theoretic approach exploiting the deep connection between capacities and exit times. Our sub-exponential LDP for u_ϵ is relevant to the ones in [12, 17], where one-dimensional re-scaled absorbed birth-and-death processes whose diffusion approximation has the form (1.1) are investigated. In [12], the LDP for the QSD is established. The sub-exponential LDP is obtained in [17] only when the mean field ODE has a unique asymptotically stable equilibrium. Both works heavily use the recursive formula satisfied by the QSD. Therefore, not only can our sub-exponential LDP for u_ϵ be regarded as an extension and improvement of those contained in [12, 17] as we allow the ODE (1.2) to have multiple stable equilibria, but also it can be seen as a global version of those contained in [60]. To the best of our knowledge, this is the first time that the sub-exponential LDP for QSDs is established for absorbed singular diffusion processes like (1.1).

Our two-step approach is different from those contained in literature. The first step studying the vanishing viscosity limit of the logarithmic transform $v_\epsilon := -\frac{\epsilon^2}{2}\ln(au_\epsilon)$ is somewhat standard. Establishing the local uniform boundedness of $\{v_\epsilon\}_\epsilon$ and $\{v'_\epsilon\}_\epsilon$, we find candidates for the quasi-potential function who are viscosity solutions of the HJE (1.6). Previous studies on the tightness and rough concentration estimates of QSDs [70] give basic properties of the candidates (see Section 3 for details). Due to the non-uniqueness of viscosity solutions of (1.6) (although some properties of the candidates have been established), an approach to the determination of the quasi-potential function is needed. This is the purpose of the second step. In literature, methods based on the Freidlin-Wentzell theory, control theory, Aubry-Mather theory, etc. have been used to achieve this goal as mentioned earlier. However, none of them can be easily adapted to treat out problem because we aim at establishing the sub-exponential LDP in the whole half line $(0,\infty)$, where the unperturbed ODE (1.2) could admit all types of equilibria. We tackle the problem from a completely different perspective that takes full advantage of the one-dimensional structure and avoids studying the singularly perturbed equation (1.12). More precisely, exploring the properties of u_ϵ near 0 and ∞, we are able to establish integral identities for u_ϵ, v_ϵ and v'_ϵ (see Proposition 4.1 for details). Elementary analysis based on these identities and Laplace's method then allows us to establish the LDP as stated in Theorems A and B.

As byproducts of the proof and consequences of Theorems A and B, we obtain new results about uniform-in-ϵ estimates of μ_ϵ or u_ϵ near 0 and ∞, the sub-exponential asymptotic of the principal eigenvalue $\lambda_{\epsilon,1}$, and the asymptotic of the positive eigenfunction $\phi_{\epsilon,1}$ of $-\mathcal{L}_\epsilon$ corresponding to the principal eigenvalue $\lambda_{\epsilon,1}$ and satisfying the normalization $\|\phi_{\epsilon,1}\|_{L^2(u_\epsilon^G)} = 1$, where $u_\epsilon^G := \frac{1}{a}e^{-\frac{2}{\epsilon^2}V}$ is the non-integrable Gibbs density. Note from (2.2) and Proposition 2.1 that u_ϵ and $\phi_{\epsilon,1}$ are related by

$$u_\epsilon = \frac{\phi_{\epsilon,1}u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}}.$$

Theorem C. *Assume* **(H)**. *The following hold.*

(1) *There exist $L \gg 1$, $C > 0$ and $0 < \epsilon_* \ll 1$ such that*
$$u_\epsilon \le \frac{C}{a^{\frac{3}{4}}} e^{\frac{1}{\epsilon^2}\int_L^\bullet \frac{b}{a}ds} \quad in \quad [L,\infty), \quad \forall \epsilon \in (0,\epsilon_*).$$

(2) *For each $0 < \delta \ll 1$, there are $0 < x_\delta \ll 1$ and $0 < \epsilon_\delta \ll 1$ such that*
$$e^{-\frac{2}{\epsilon^2}(d_1+\delta)} \le u_\epsilon \le e^{-\frac{2}{\epsilon^2}(d_1-\delta)} \quad in \quad (0,x_\delta), \quad \forall \epsilon \in (0,\epsilon_\delta).$$

(3) *Suppose in addition the existence of a unique* d_1*-valley* (α,β).
   (i) *If* $\alpha=0$, *then* $\epsilon\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}\approx_\epsilon \frac{b'(0)}{a'(0)}M_\epsilon$ *and*
$$\lim_{\epsilon\to 0}\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1}=1 \textit{ locally uniformly in } (0,\infty).$$
   (ii) *If* $\alpha>0$, $V(\alpha)>V$ *in* $(0,\alpha)$ *and* $b'(\alpha)>0$, *then* $\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}\approx_\epsilon \frac{M_\epsilon}{2}\sqrt{\frac{-V''(\alpha)}{\pi}}$ *and*
$$\lim_{\epsilon\to 0}\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1}(x)=\begin{cases}0, & \textit{uniformly in } x\in(0,\tilde\alpha) \textit{ for each } \tilde\alpha\in(0,\alpha),\\ \frac{1}{2}, & x=\alpha,\\ 1, & \textit{locally uniformly in } x\in(\alpha,\infty).\end{cases}$$

Corollary C. *Assume* **(H)**, *the existence of a unique* d_1*-valley* (α,β) *and* $(\mathbf{H}_V)$.

(1) *If* $\alpha=0$, *then* $\lim_{\epsilon\to0}\epsilon\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}=\frac{b'(0)}{a'(0)}M_0$.
(2) *If* $\alpha>0$, $V(\alpha)>V$ *in* $(0,\alpha)$ *and* $b'(\alpha)>0$, *then* $\lim_{\epsilon\to0}\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}=\frac{M_0}{2}\sqrt{\frac{-V''(\alpha)}{\pi}}$.

Conclusions like those in Theorems A, B and C (3) have fruitful and far-reaching consequences, and have profound influences on the study of randomly perturbed dynamical systems. For instance, in [21], the author used the sub-exponential LDP for stationary measures to rigorously justify an important formula concerning the asymptotic exit distribution originally derived in [61]. In the works [23, 24, 25] (see [55] for an exposition) studying exit events and the Eyring-Kramers formula on the basis of QSDs for the overdamped Langevin equation, the sub-exponential asymptotic of the principal eigenvalue plays a significant role in computing the asymptotic of transition rates and determining the asymptotic exit distribution.

Here, we use them to establish the asymptotic distribution of the normalized extinction time, the asymptotic reciprocal relationship between the mean extinction time $\mathbb{E}^\epsilon_\bullet[T^\epsilon_0]$ and the principal eigenvalue $\lambda_{\epsilon,1}$, and the multiscale estimate of the dynamics of X^ϵ_t. These results greatly benefit from the limit $\lim_{\epsilon\to0}\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1}$ in Theorem C (3), which does not require $(\mathbf{H}_V)$.

We introduce some notations that are frequently used in the sequel. Let $\mathcal{P}((0,\infty))$ be the set of Borel probability measures on $(0,\infty)$. In the case that (α,β) is the unique d_1-valley with $\alpha>0$, we set for $\mu\in\mathcal{P}((0,\infty))$,
$$p_\mu:=\frac{1}{2}\mu(\{\alpha\})+\mu((\alpha,\infty)).$$

1.3. **Asymptotic reciprocal relationship.** We state results concerning the asymptotic distribution of the normalized extinction time and the asymptotic reciprocal relationship between $\mathbb{E}^\epsilon_\bullet[T^\epsilon_0]$ and $\lambda_{\epsilon,1}$, generalizing respectively the fact that $T^\epsilon_0\sim\exp(\lambda_{\epsilon,1})$ if $X^\epsilon_0\sim\mu_\epsilon$, and its consequence $\lambda_{\epsilon,1}\mathbb{E}^\epsilon_{\mu_\epsilon}[T^\epsilon_0]=1$.

Theorem D. *Assume* **(H)** *and the existence of a unique* d_1*-valley* (α,β). *Let* $\mu\in\mathcal{P}((0,\infty))$ *have compact support in* $(0,\infty)$.

(1) *If* $\alpha=0$, *then* $\lim_{\epsilon\to0}\mathbb{P}^\epsilon_\mu[\lambda_{\epsilon,1}T^\epsilon_0>t]=e^{-t}$ *for all* $t>0$, *and* $\lim_{\epsilon\to0}\lambda_{\epsilon,1}\mathbb{E}^\epsilon_\mu[T^\epsilon_0]=1$. *In particular,* $\lim_{\epsilon\to0}\mathbb{P}^\epsilon_\mu\left[\frac{T^\epsilon_0}{\mathbb{E}^\epsilon_\mu[T^\epsilon_0]}>t\right]=e^{-t}$ *for all* $t>0$.
(2) *If* $\alpha>0$, $V(\alpha)>V$ *in* $(0,\alpha)$ *and* $b'(\alpha)>0$, *then* $\lim_{\epsilon\to0}\mathbb{P}^\epsilon_\mu[\lambda_{\epsilon,1}T^\epsilon_0>t]=p_\mu e^{-t}$ *for all* $t>0$, *and* $\lim_{\epsilon\to0}\lambda_{\epsilon,1}\mathbb{E}^\epsilon_\mu[T^\epsilon_0]=p_\mu$. *In particular, if* $p_\mu>0$, *then* $\lim_{\epsilon\to0}\mathbb{P}^\epsilon_\mu\left[\frac{T^\epsilon_0}{\mathbb{E}^\epsilon_\mu[T^\epsilon_0]}>t\right]=p_\mu e^{-p_\mu t}$ *for all* $t>0$.

Theorem D shows that as $\epsilon \to 0$, the normalized extinction time $\frac{T_0^\epsilon}{\mathbb{E}_\mu^\epsilon[T_0^\epsilon]}$ weakly converges to an exponential random variable with parameter 1 when $\alpha = 0$ and p_μ when $\alpha > 0$. It also uncovers a fundamental principle connecting the mean extinction time $\mathbb{E}_\bullet^\epsilon[T_0^\epsilon]$ and the principal eigenvalue $\lambda_{\epsilon,1}$. One of its importance is that it allows using information about one of them to analyze the other one. In particular, given Theorem D and the asymptotic of $\lambda_{\epsilon,1}$ in Theorem C, we readily obtain the asymptotic of $\mathbb{E}_\bullet^\epsilon[T_0^\epsilon]$ in terms of the quantity M_ϵ. More precise asymptotic can be derived under the additional assumption $(\mathbf{H}_V)$.

Corollary D. *Assume* $(\mathbf{H})$*, the existence of a unique* d_1*-valley* (α, β) *and* $(\mathbf{H}_V)$*. Let* $\mu \in \mathcal{P}((0,\infty))$ *have compact support in* $(0,\infty)$*.*

1. *If* $\alpha = 0$*, then* $\mathbb{E}_\mu^\epsilon[T_0^\epsilon] \approx_\epsilon \frac{\epsilon a'(0)}{M_0 b'(0)} e^{\frac{2}{\epsilon^2} d_1}$.
2. *If* $\alpha > 0$*,* $V(\alpha) > V$ *in* $(0,\alpha)$ *and* $b'(\alpha) > 0$*, then* $\mathbb{E}_\mu^\epsilon[T_0^\epsilon] \approx_\epsilon \frac{2p_\mu}{M_0}\sqrt{\frac{\pi}{-V''(\alpha)}} e^{\frac{2}{\epsilon^2} d_1}$ *provided* $p_\mu > 0$.

The mean extinction time $\mathbb{E}_\bullet[T_0^\epsilon]$ is a special mean exit time, whose asymptotic reciprocal relationship with the principal eigenvalue is widely acknowledged and established in many situations (see e.g. [59, 61, 20, 8, 9, 39, 42, 53]). In [59, 61], such a relationship is formally derived by means of the asymptotic expansion. The first rigorous proof is provided in [20] dealing with randomly perturbed dynamical systems exiting from a bounded domain containing a unique asymptotically stable equilibrium. In the case that V has multiple wells, the mean extinction time is closely related to the transition rate among local minima. The sub-exponential asymptotic of the transition rate, often called the Eyring-Kramers formula (or law), and the principal eigenvalue are proven for regular reversible diffusion processes in [8, 9, 39, 42, 53], leading directly to their asymptotic reciprocal relationship. We refer the reader to surveys [5, 55] for more details. Very recently, the Eyring-Kramers formula is justified in [51, 54, 57] for irreversible diffusion processes having the Gibbs measure as the unique stationary measure, and in [52] for irreversible random walks in a potential field.

1.4. Multiscale estimate.

We introduce some notations before stating the multiscale estimate of the dynamics of X_t^ϵ. For $d > 0$, let $N(d)$ be the number of d-valleys. It is easy to see that $d \mapsto N(d)$ is a non-negative, non-increasing and left-continuous function on $(0,\infty)$. For each $i \in \mathbb{N}$, we define

$$d_i := \inf\{d > 0 : N(d) < i\}. \tag{1.13}$$

Since $V(0+) = 0$ and $V'(x) > 0$ for $x \gg 1$, for each $i \in \mathbb{N}$ there always exists $d \in (0,\infty)$ such that $N(d) < i$, and hence, d_i is well-defined. Intuitively, d_i, $i \in \mathbb{N}$ are the points where $N(d)$ has jump discontinuities. This definition of d_1 coincides with the one given in (1.5). Clearly, $d_1 > 0$ and $d_1 \geq d_2 \geq d_3 \geq \cdots \geq 0$. Moreover, if there is only one d_1-valley (the generic case that we focus on), then $d_1 > d_2$. It is shown in Lemma 2.3 that d_i is exactly the exponential asymptotic rate of the i-th eigenvalue $\lambda_{\epsilon,i}$ of $-\mathcal{L}_\epsilon$.

Our result regarding the multiscale estimate of the dynamics of X_t^ϵ is stated as follows. Denote by $\|\cdot\|_{TV}$ the total variation distance.

Theorem E. *Assume* $(\mathbf{H})$ *and the existence of a unique* d_1*-valley* (α,β)*. If* $k \in \mathbb{N}$ *is such that* $d_1 > d_2 > \cdots > d_k > d_{k+1}$*, then for each compact* $K \subset (0,\infty)$*, there are positive constants* $\gamma = \gamma(k,K)$*,* $C = C(k,K)$ *and* $\epsilon_* = \epsilon_*(k,K)$ *such that the following hold.*

(1) *If* $\alpha=0$, *then*

$$\sup_{\substack{\mu\in\mathcal{P}((0,\infty))\\ \mathrm{supp}(\mu)\subset K}}\left\|\mathbb{P}^{\epsilon}_{\mu}[X^{\epsilon}_t\in\bullet]-\left[\sum_{i=1}^{k}e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle\mu_{\epsilon,i}+\left(1-\sum_{i=1}^{k}e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle\right)\delta_0\right]\right\|_{TV}$$
$$\leq e^{\frac{\gamma}{\epsilon^2}-\lambda_{\epsilon,k+1}t},\quad \forall t>2 \text{ and } 0<\epsilon<\epsilon_*,$$

where $\alpha_{\epsilon,i}:=\langle\mathbb{1},\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\phi_{\epsilon,i}$, $\langle\mu,\alpha_{\epsilon,i}\rangle:=\int_0^\infty\alpha_{\epsilon,i}d\mu$ *satisfies* $\sup_{0<\epsilon<\epsilon_*}|\langle\mu,\alpha_{\epsilon,i}\rangle|\leq C$, *and* $\mu_{\epsilon,i}$ *is defined by* $d\mu_{\epsilon,i}:=\frac{\phi_{\epsilon,i}u^G_\epsilon}{\langle\mathbb{1},\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}}dx$ *and satisfies* $\sup_{0<\epsilon<\epsilon_*}\|\langle\mu,\alpha_{\epsilon,i}\rangle\mu_{\epsilon,i}\|_{TV}\leq C$. *Moreover,* $\mu_{\epsilon,1}=\mu_\epsilon$ *and* $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=1$.

(2) *If* $\alpha>0$, $V(\alpha)>V$ *in* $(0,\alpha)$ *and* $b'(\alpha)>0$, *then the same conclusion as in (1) holds except* $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=p_\mu$.

Since $d_1>d_2$ under **(H)**, we immediately have the following result that is of particular interest.

Corollary E. *Assume* **(H)** *and the existence of a unique* d_1*-valley* (α,β). *Then for each compact* $K\subset(0,\infty)$, *there are positive constants* $\gamma=\gamma(K)$ *and* $\epsilon_*=\epsilon_*(K)$ *such that the following hold.*

(1) *If* $\alpha=0$, *then*

$$\sup_{\substack{\mu\in\mathcal{P}((0,\infty))\\ \mathrm{supp}(\mu)\subset K}}\left\|\mathbb{P}^{\epsilon}_{\mu}[X^{\epsilon}_t\in\bullet]-\left[e^{-\lambda_{\epsilon,1}t}\langle\mu,\alpha_{\epsilon,1}\rangle\mu_\epsilon+\left(1-e^{-\lambda_{\epsilon,1}t}\langle\mu,\alpha_{\epsilon,1}\rangle\right)\delta_0\right]\right\|_{TV}$$
$$\leq e^{\frac{\gamma}{\epsilon^2}-\lambda_{\epsilon,2}t},\quad \forall t>2 \text{ and } 0<\epsilon<\epsilon_*,$$

where $\alpha_{\epsilon,1}:=\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1}$ *and* $\langle\mu,\alpha_{\epsilon,1}\rangle:=\int_0^\infty\alpha_{\epsilon,1}d\mu$ *satisfies* $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=1$.

(2) *If* $\alpha>0$ *and* $V(\alpha)>V$ *in* $(0,\alpha)$ *and* $b'(\alpha)>0$, *then the same conclusion as in (1) holds except* $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=p_\mu$.

Remark 1.2. *We make some comments about Theorem E and Corollary E.*

- *Theorem E builds on the eigenfunction expansion of the semigroup associated with* X^ϵ_t *before it reaches the extinction state* 0 *(see Lemma 2.1). The primary achievements of this theorem include the tail estimate* $e^{\frac{\gamma}{\epsilon^2}-\lambda_{\epsilon,k+1}t}$, *uniform-in-*$\epsilon$ *bounds of the coefficients* $\langle\mu,\alpha_{\epsilon,i}\rangle$ *and* $\langle\mu,\alpha_{\epsilon,i}\rangle\mu_{\epsilon,i}$, *and the limit* $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle$, *making the dynamical estimate meaningful. To obtain these results, it is necessary to extract information from the expansion and involved eigenfunctions, which only have natural meanings in the weighted space* $L^2(u^G_\epsilon)$. *The degeneracy and singularity of the noise results in the non-integrable singularity of the weight* u^G_ϵ *near* 0, *complicating the situation.*
- *We comment on the definition of* $\mu_{\epsilon,i}$ *for* $i\geq2$ *in Theorem E. If* $\langle\mathbb{1},\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\neq0$, *then* $\mu_{\epsilon,i}$ *is a signed measure satisfying* $\mu_{\epsilon,i}((0,\infty))=1$. *Otherwise, we can set* $\mu_{\epsilon,i}$ *to be any fixed measure satisfying* $\mu_{\epsilon,i}((0,\infty))=1$ *since* $\langle\mu,\alpha_{\epsilon,i}\rangle=0$. *We choose to use* $\mu_{\epsilon,i}$ *for that the conclusion then quantifies at least formally the total variation distance between* $\mathbb{P}^\epsilon_\mu[X^\epsilon_t\in\bullet]$ *and the convex combination of the measures* $\mu_{\epsilon,i}$, $i\in\{1,\dots,k\}$ *and* δ_0.
- *Lemma 2.3 and conditions in Theorem E ensure that the eigenvalues* $\lambda_{\epsilon,i}$, $i=1,\dots,k$ *are exponentially small, and* $\lambda_{\epsilon,i}$ *is exponentially smaller than* $\lambda_{\epsilon,i+1}$ *for* $i=1,\dots,k$. *The reciprocal of these eigenvalues gives rise to multiple time scales, which together with the estimate established in Theorem E characterize the multiscale dynamics of* X^ϵ_t *governed by the measures* $\mu_{\epsilon,i}$, $i\in\{1,\dots,k\}$ *and* δ_0.

- *The QSD μ_ϵ plays a special role in characterizing the dynamics of X_t^ϵ. Whenever involved (depending on the initial distribution in the case $\alpha > 0$), X_t^ϵ spends most of the time with it before reaching the extinction. The limiting behavior of $\alpha_{\epsilon,1}$ (addressed in Theorem C) allows us to describe this in a more precise way as follows:*
  - *if t is such that $t \gg \frac{1}{\lambda_{\epsilon,1}}$, then $\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] \sim \delta_0$;*
  - *if t is such that $\frac{1}{\epsilon^2\lambda_{\epsilon,2}} \ll t \ll \frac{1}{\lambda_{\epsilon,1}}$, then $\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] \sim \mu_\epsilon$ under conditions in (1), and $\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] \sim p_\mu \mu_\epsilon + (1-p_\mu)\delta_0$ under conditions in (2), that is, the probability that X_t^ϵ experiences transient dynamics captured by μ_ϵ during the period $\left[\frac{1}{\epsilon^2\lambda_{\epsilon,2}}, \frac{1}{\lambda_{\epsilon,1}}\right]$ is approximately p_μ.*

  It is interesting to see that under the conditions in Corollary E (2), the QSD μ_ϵ plays no role in describing the dynamics of X_t^ϵ if the initial distribution μ is supported in $(0,\alpha)$ so that $p_\mu = 0$. The reason is that trajectories are more likely to exit from $(0,\alpha)$ through 0 instead of α, that is, $\lim_{\epsilon\to 0}\mathbb{P}_\mu^\epsilon\left[T_{(0,\alpha)}^\epsilon = 0\right] = 1$, where $T_{(0,\alpha)}^\epsilon := \inf\{t \geq 0 : X_t^\epsilon \notin (0,\alpha)\}$ is the first time that X_t^ϵ exits from $(0,\alpha)$, while the QSD μ_ϵ is mainly concentrated in a neighborhood of the set of global minima of $V|_{(\alpha,\beta)}$ (see Theorem A). This actually is a delicate issue when $V(\alpha) = 0$ $(= V(0+))$, in which case, $(0,\alpha)$ is a valley. Exiting from $(0,\alpha)$ through 0 is then a result of the fact that $V'(0+) < 0 = V'(\alpha)$.

Theorem E (1) or Corollary E (1) covers a fundamentally important case in biology and ecology that b is a standard logistic growth rate function, namely, $b(x) = b_1 x - b_2 x^2$ for some $b_1, b_2 > 0$. In this case, V is a single-well potential function with the unique global minimal point non-degenerate, and the second eigenvalue $\lambda_{\epsilon,2}$ satisfies $\lim_{\epsilon\to 0}\lambda_{\epsilon,2} = b_1$ (see [45, Theorem B]). The solution X_t^ϵ conditioned on survival $[t < T_0^\epsilon]$ converges exponentially fast with rate $\lambda_{\epsilon,2} - \lambda_{\epsilon,1} (\approx_\epsilon \lambda_{\epsilon,2})$ to the QSD μ_ϵ as $t \to \infty$. Therefore, X_t^ϵ stays very close to μ_ϵ over a time scale that the conditioned process has been staying with the QSD and most trajectories are alive. Such dynamics with sharp time scales is stated in the next result for a vector field that is slightly more general than the standard logistic growth rate function.

Recall that a function $w : [0,\infty) \to [0,\infty)$ is called a *modulus of continuity* if w is increasing and continuous at 0 with $w(0) = 0$. For any $x_0 \in (0,\infty)$, we denote by $w[x_0]$ the set of all continuous functions $f : [0,\infty) \to \mathbb{R}$ having w as the modulus of continuity at x_0, namely, $|f(x) - f(x_0)| \leq w(|x - x_0|)$ for all x in a neighbourhood of x_0.

Theorem F. *Assume* **(H)**, $\{x \in (0,\infty) : b(x) = 0\} = \{x_*\}$ *and* $b'(x_*) < 0$. *Let* $w : [0,\infty) \to [0,\infty)$ *be a modulus of continuity. Then, for each compact* $K \subset (0,\infty)$, $M > 0$ *and sequences* $\{\underline{t}_\epsilon\}_\epsilon$, $\{\bar{t}_\epsilon\}_\epsilon$ *in* $(0,\infty)$ *satisfying* $\underline{t}_\epsilon < \bar{t}_\epsilon$ *for each* ϵ, $\lim_{\epsilon\to 0}\underline{t}_\epsilon = \infty$ *and* $\lim_{\epsilon\to 0}\frac{\bar{t}_\epsilon}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds} = 0$, *there holds*

$$\lim_{\epsilon\to 0}\sup_{\mathrm{supp}(\mu)\subset K}\sup_{\underline{t}_\epsilon \leq t \leq \bar{t}_\epsilon}\sup_{\substack{f\in w[x_*]\\ \|f\|_\infty \leq M}}\left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] - \int_0^\infty f d\mu_\epsilon\right| = 0.$$

We highlight that time scales $\underline{t}_\epsilon$ and $\bar{t}_\epsilon$ appearing in Theorem F are sharp in the following sense. Since the spectral gap $\lambda_{\epsilon,2} - \lambda_{\epsilon,1}$ satisfies $\lim_{\epsilon\to 0}(\lambda_{\epsilon,2} - \lambda_{\epsilon,1}) = b_1 > 0$ (see [45, Theorem B]), a time scale $\underline{t}_\epsilon$ satisfying $\lim_{\epsilon\to 0}\underline{t}_\epsilon = \infty$ is required to observe the QSD μ_ϵ. Recall that $\lambda_{\epsilon,1}$ is the extinction rate and its reciprocal $\frac{1}{\lambda_{\epsilon,1}}$ is essentially the mean extinction time (see Theorem D (1)). Under conditions on the vector field b in Theorem F, we see from Corollary C (1) that $\lambda_{\epsilon,1} \approx_\epsilon \frac{C}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds}$, where $C = \frac{b'(0)}{a'(0)}\sqrt{-\frac{b(x_*)}{\pi a(x_*)}}$. If the time scale $\bar{t}_\epsilon$ is such that $\lim_{\epsilon\to 0}\bar{t}_\epsilon\lambda_\epsilon = 0$ (equivalent to $\lim_{\epsilon\to 0}\frac{\bar{t}_\epsilon}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds}=0$ thanks to the asymptotic of $\lambda_{\epsilon,1}$), then most trajectories of X_t^ϵ are alive by $\bar{t}_\epsilon$.

1.5. **Applications.** Results in Subsections 1.2-1.4 are applied to logistic diffusion processes:

$$dx=(b_1x-b_2x^2)dt+\epsilon\sqrt{a_1x+a_2x^2}dW_t,\quad x\in[0,\infty), \tag{1.14}$$

where $0<\epsilon\ll 1$ is a parameter, b_1, b_2 and a_1 are positive constants, and $a_2\geq 0$. Such an equation arises for instance from chemical reactions and population dynamics and can be derived as diffusion approximations of relevant birth-and-death processes (BDPs). See Section 6 for details.

On the basis of Theorems A-F, we obtain in particular the following.

- The unique QSD of (1.14) tends to concentrate on the Dirac measure at $\frac{b_1}{b_2}$ as $\epsilon\to 0$ in a Gaussian manner under both the total variation distance and Wasserstein distances (see Theorem 6.1 (6)-(7)).
- As aforementioned, (1.14) can be derived as diffusion approximations of BDPs, which however are valid only over finite time intervals in general. In order for the validity over longer time intervals, it is necessary to verify the diffusion approximation for special dynamical states, especially the QSD in the current context. This is shown to be the case in Theorem 6.2 for a class of logistic BDPs.
- We resolve *Keizer's paradox* [46] regarding the long-term dynamical disagreement between deterministic and stochastic models modelling the same process. In terms of (1.14) and its unperturbed ODE $\dot{x}=b_1x-b_2x^2$, we show their dynamical agreement from observables' point of view over a "maximal" time horizon. Details are given in Remark 6.1.

1.6. **Organization of the rest of the paper.** The rest of the paper is organized as follows. In Section 2, we collect some preliminary results, including diffusion approximations, spectral theory of $\mathcal{L}_\epsilon$, Liouville-type transform of $\mathcal{L}_\epsilon$ and the resulting semi-classical Schrödinger operators, and concentration estimates for QSDs. As mentioned earlier, our approach to establishing the sub-exponential LDP for $\{\mu_\epsilon\}$ consists of two steps. The first step addressing the vanishing viscosity limits of $v_\epsilon=-\frac{\epsilon^2}{2}\ln(au_\epsilon)$ is contained in Section 3. The second step including proving the crucial integral identities for u_ϵ, v_ϵ and v_ϵ' and completing the proof (of Theorems A, B and C) is presented in Section 4. Section 5 is devoted to the multiscale dynamics of X_t^ϵ. In Subsection 5.1, we establish the asymptotic distribution of the normalized extinction time and the asymptotic reciprocal relationship between $\mathbb{E}_\bullet^\epsilon[T_0^\epsilon]$ and $\lambda_{\epsilon,1}$. In particular, we prove Theorem D. In Subsection 5.2, we establish the multiscale estimate of the dynamics of X_t^ϵ and prove Theorems E and F. Applications to logistic diffusion processes are discussed in Section 6.

2. **Preliminary**

In this section, we recall and establish some preliminary results for later purposes. We assume **(H)** throughout this section. Subsection 2.1 is devoted to the rigorous formalism of the generator $\mathcal{L}_\epsilon$ of X_t^ϵ, the spectral theory of $\mathcal{L}_\epsilon$ and the stochastic representation and dynamics of the semigroup generated by $\mathcal{L}_\epsilon$. In Subsection 2.2, we derive the Schrödinger operator that is unitarily equivalent to $\mathcal{L}_\epsilon$. In Subsection 2.3, we present basic results about QSDs of X_t^ϵ including the existence and uniqueness, previous concentration estimates away from ∞ and new ones near ∞.

2.1. **Generator, spectral theory and dynamics.** In this subsection, we discuss the spectral theory of the generator of X_t^ϵ, and the dynamics of the Markov semigroup associated with X_t^ϵ.

Consider the symmetric quadratic form $\mathcal{E}_\epsilon : C_0^\infty((0,\infty)) \times C_0^\infty((0,\infty)) \to \mathbb{R}$ defined by

$$\mathcal{E}_\epsilon(\phi,\psi) = \frac{\epsilon^2}{2}\int_0^\infty a\phi'\psi' u_\epsilon^G dx, \quad \forall \phi,\psi \in C_0^\infty((0,\infty)),$$

where $u_\epsilon^G := \frac{1}{a}e^{-\frac{2}{\epsilon^2}V}$ is the non-integrable Gibbs density and the potential function V is defined in (1.4). That is, u_ϵ^G is the unique (up to constant multiplication) solution to $\frac{\epsilon^2}{2}(au)' - bu = 0$ in $(0,\infty)$. In particular, it solves the stationary Fokker-Planck equation

$$\frac{\epsilon^2}{2}(au)'' - (bu)' = 0 \quad \text{in} \quad (0,\infty).$$

The quadratic form $\mathcal{E}_\epsilon$ is Markovian and closable [32]. Its smallest closed extension, again denoted by $\mathcal{E}_\epsilon$, is a Dirichlet form with domain $D(\mathcal{E}_\epsilon)$ being the closure of $C_0^\infty((0,\infty))$ under the norm $\|\phi\|^2_{D(\mathcal{E}_\epsilon)} := \|\phi\|^2_{L^2(u_\epsilon^G)} + \mathcal{E}_\epsilon(\phi,\phi)$, where $L^2(u_\epsilon^G) := L^2((0,\infty), u_\epsilon^G dx)$. Denote by $\mathcal{L}_\epsilon$ the non-positive self-adjoint operator in the weighted space $L^2(u_\epsilon^G)$ associated with $\mathcal{E}_\epsilon$ such that

$$\mathcal{E}_\epsilon(\phi,\psi) = \langle -\mathcal{L}_\epsilon\phi, \psi\rangle_{L^2(u_\epsilon^G)}, \quad \forall \phi \in D(\mathcal{L}_\epsilon),\ \psi \in D(\mathcal{E}_\epsilon),$$

where

$$D(\mathcal{L}_\epsilon) := \left\{u \in D(\mathcal{E}_\epsilon) : \exists f \in L^2(u_\epsilon^G) \text{ s.t. } \mathcal{E}_\epsilon(u,\phi) = \langle f,\phi\rangle_{L^2(u_\epsilon^G)}, \forall \phi \in D(\mathcal{E}_\epsilon)\right\}$$

is the domain of $\mathcal{L}_\epsilon$ and contained in particular in $L^2(u_\epsilon^G)$. Note that

$$\mathcal{L}_\epsilon\phi = \frac{\epsilon^2}{2}a\phi'' + b\phi' \quad \text{for} \quad \phi \in C_0^\infty((0,\infty)),$$

that is, $\mathcal{L}_\epsilon$ is a self-adjoint extension of the generator of (1.1).

We present the following results about the spectrum of $-\mathcal{L}_\epsilon$ and the semigroup generated by $\mathcal{L}_\epsilon$.

Lemma 2.1 ([10, 45])**.** *For each $0 < \epsilon \ll 1$, the following hold.*

(1) *$-\mathcal{L}_\epsilon$ has purely discrete spectrum contained in $(0,\infty)$ and listed as follows:*

$$\lambda_{\epsilon,1} < \lambda_{\epsilon,2} < \lambda_{\epsilon,3} < \cdots \to \infty.$$

(2) *Each $\lambda_{\epsilon,i}$ is associated with a unique eigenfunction $\phi_{\epsilon,i} \in D(\mathcal{L}_\epsilon) \cap L^1(u_\epsilon^G) \cap C^3((0,\infty))$ subject to the normalization $\|\phi_{\epsilon,i}\|_{L^2(u_\epsilon^G)} = 1$. Moreover, $\phi_{\epsilon,1}$ is positive on $(0,\infty)$.*

(3) *The set $\{\phi_{\epsilon,i}, i \in \mathbb{N}\}$ is an orthonormal basis of $L^2(u_\epsilon^G)$.*

(4) *$\mathcal{L}_\epsilon$ generates a positive analytic semigroup $(P_t^\epsilon)_{t\ge 0}$ of contractions on $L^2(u_\epsilon^G)$ having the stochastic representation $P_t^\epsilon f = \mathbb{E}^\epsilon_\bullet[f(X_t^\epsilon)\mathbb{1}_{t<T_0^\epsilon}]$ for all $f \in L^2(u_\epsilon^G) \cap C_b([0,\infty))$ and $t \ge 0$.*

(5) *For each $k \in \mathbb{N}$, $f \in L^2(u_\epsilon^G)$ and $t > 0$,*

$$P_t^\epsilon f = \sum_{i=1}^{k-1} e^{-\lambda_{\epsilon,i}t}\langle f, \phi_{\epsilon,i}\rangle_{L^2(u_\epsilon^G)}\phi_{\epsilon,i} + P_t^\epsilon Q_k^\epsilon f, \tag{2.1}$$

where Q_k^ϵ is the spectral projection of $\mathcal{L}_\epsilon$ corresponding to the eigenvalues $\{-\lambda_{\epsilon,j}\}_{j\ge k}$. Moreover,

$$\|P_t^\epsilon Q_k^\epsilon\|_{L^2(u_\epsilon^G)\to L^2(u_\epsilon^G)} \le e^{-\lambda_{\epsilon,k}t}, \quad t \ge 0.$$

(6) *For each $f \in C_b([0,\infty))$, the stochastic representation in (4) and (2.1) hold pointwisely.*

The next result concerning L^∞ estimates of $(P_t^\epsilon)_{t\ge 0}$ is proven in [45].

Lemma 2.2 ([45, Lemma 6.1])**.** *For each $k\in\mathbb{N}$, the following statements hold.*

(1) *There exists $C>0$ such that for each $0<\epsilon\ll 1$,*

$$|P_t^\epsilon Q_k^\epsilon f|\leq \frac{C}{\epsilon}a^{\frac{1}{4}}e^{\frac{V}{\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_{L^2(u_\epsilon^G)}\quad in\quad (0,\infty),\quad \forall f\in L^2(u_\epsilon^G)\text{ and }t>1.$$

(2) *There exists $\gamma>0$ such that for each $0<\epsilon\ll 1$,*

$$|P_t^\epsilon Q_k^\epsilon f|\leq a^{\frac{1}{4}}e^{\frac{V+\gamma}{\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_\infty\quad in\quad (0,\infty),\quad \forall f\in C_b([0,\infty))\text{ and }t>2.$$

The following result regarding the exponential asymptotic of the eigenvalues $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$ is proven in [45]. Recall from (1.13) the definition of d_i, $i\in\mathbb{N}$.

Lemma 2.3 ([45, Theorem A])**.** *For each $i\in\mathbb{N}$, $\lim_{\epsilon\to 0^+}\frac{\epsilon^2}{2}\ln\lambda_{\epsilon,i}=-d_i$.*

We point out that the notations r_i, $i\in\mathbb{N}$ used in [45] correspond to $2d_i$, $i\in\mathbb{N}$ used in the present paper. Since $d_1>0$, Lemma 2.3 says that $\lambda_{\epsilon,1}$ is exponentially small in ϵ.

2.2. **Semi-classical Schrödinger operators.** We derive the semi-classical Schrödinger operator that is unitarily equivalent to the generator $\mathcal{L}_\epsilon$ of X_t^ϵ. It plays important technical roles that we comment at the end of this subsection.

Consider the transform $y=\xi(x)=\int_0^x\frac{1}{\sqrt{a}}dz$ for $x\in(0,\infty)$. Assumptions on a ensure that $\xi'>0$ on $(0,\infty)$ and $\xi(0+)=0$. Set $y_\infty:=\xi(\infty)\in(0,\infty]$. In particular, $\xi:(0,\infty)\to(0,y_\infty)$ is invertible. This transform converts the SDE (1.1) to the following SDE with constant noise coefficient:

$$dy=-q_\epsilon(y)dt+\epsilon dW_t,\quad y\in[0,y_\infty),$$

where $q_\epsilon=-(\mathcal{L}_\epsilon\xi)\circ\xi^{-1}$.

Let $v_\epsilon^G(y):=\frac{u_\epsilon^G(x)}{\xi'(x)}=\sqrt{a(x)}u_\epsilon^G(x)$ and set $L^2(v_\epsilon^G):=L^2((0,y_\infty),v_\epsilon^G dy)$. Define

$$\mathcal{L}_\epsilon^Y:=\frac{\epsilon^2}{2}\frac{d^2}{dy^2}-q_\epsilon(y)\frac{d}{dy}\quad\text{in}\quad L^2(v_\epsilon^G).$$

It is not hard to check that $U_\epsilon\mathcal{L}_\epsilon=\mathcal{L}_\epsilon^Y U_\epsilon$, where $U_\epsilon:L^2(u_\epsilon^G)\to L^2(v_\epsilon^G)$, $f\mapsto f\circ\xi^{-1}$ is a unitary transform. Consider the semi-classical Schrödinger operator

$$\mathcal{L}_\epsilon^S:=\frac{\epsilon^2}{2}\frac{d^2}{dy^2}-\frac{1}{2}\left[\frac{q_\epsilon^2(y)}{\epsilon^2}-q_\epsilon'(y)\right]\quad\text{in}\quad L^2((0,y_\infty)).$$

It is easy to verify that $\tilde{U}_\epsilon\mathcal{L}_\epsilon^Y=\mathcal{L}_\epsilon^S\tilde{U}_\epsilon$, where $\tilde{U}_\epsilon:L^2(v_\epsilon)\to L^2((0,y_\infty))$, $f\to f\sqrt{v_\epsilon^G}$ is a unitary transform. Hence, $\tilde{U}_\epsilon U_\epsilon\mathcal{L}_\epsilon=\mathcal{L}_\epsilon^S\tilde{U}_\epsilon U_\epsilon$, that is, $\mathcal{L}_\epsilon$ is unitarily equivalent to $\mathcal{L}_\epsilon^S$.

We include the following commutative diagram for readers' convenience:

$$\begin{array}{ccccc}
L^2(u_\epsilon^G) & \xrightarrow{U_\epsilon} & L^2(v_\epsilon^G) & \xrightarrow{\tilde{U}_\epsilon} & L^2((0,y_\infty))\\
\downarrow{\scriptstyle\mathcal{L}_\epsilon} & & \downarrow{\scriptstyle\mathcal{L}_\epsilon^Y} & & \downarrow{\scriptstyle\mathcal{L}_\epsilon^S}\\
L^2(u_\epsilon^G) & \xrightarrow{U_\epsilon} & L^2(v_\epsilon^G) & \xrightarrow{\tilde{U}_\epsilon} & L^2((0,y_\infty))
\end{array}$$

We mention that the rigorous definition of $\mathcal{L}_\epsilon^Y$ and $\mathcal{L}_\epsilon^S$ can be done using quadratic forms as it is done for $\mathcal{L}_\epsilon$ in Subsection 2.1.

Denote by V_ϵ the potential of the Schrödinger operator $\mathcal{L}_\epsilon^S$, namely, $V_\epsilon=\frac{1}{2}\left(\frac{q_\epsilon^2}{\epsilon^2}-q_\epsilon'\right)$.

Lemma 2.4. *The following hold.*

(1) *There exist $C_1>0$ and $y_1\in(0,y_\infty)$ such that*

$$V_\epsilon \geq \frac{C_1\epsilon^2}{\xi^{-1}} \quad \text{in} \quad (0,y_1], \quad \forall 0<\epsilon\ll 1 \quad \text{and} \quad \inf_{\epsilon}\inf_{(0,y_1]} V_\epsilon>0;$$

(2) *For each $y_2\in(0,y_\infty)$ with $\xi^{-1}(y_2)\gg 1$, there exists $C_2=C_2(y_2)>0$ such that*

$$V_\epsilon \geq \frac{C_2}{\epsilon^2}\frac{b^2\circ\xi^{-1}}{a\circ\xi^{-1}} \quad \text{in} \quad [y_2,y_\infty), \quad \forall 0<\epsilon\ll 1.$$

(3) *The family $\{V_\epsilon\}_\epsilon$ is uniformly lower bounded, that is, $\inf_\epsilon \min_{(0,y_\infty)} V_\epsilon>-\infty$.*

Proof. The proof of this lemma is given in [45, Lemma 2.2]. The only difference is that in (2), we fixed a $y_2\in(0,y_\infty)$ there, while we do not fix it here. □

Remark 2.1. *The semi-classical Schrödinger operator $\mathcal{L}_\epsilon^S$ plays important technical roles. Due to its unitary equivalence to $\mathcal{L}_\epsilon$, properties of $\mathcal{L}_\epsilon^S$ can be easily passed on to that of $\mathcal{L}_\epsilon$. These include in particular the following.*

- *In [10], the authors established the spectral theory of $\mathcal{L}_\epsilon$ as stated in Lemma 2.1 (1)-(3) appealing to the well-known spectral theory of $\mathcal{L}_\epsilon^S$ (see e.g. [4]).*
- *The semigroup estimates in Lemma 2.2 is established in [45, Lemma 6.1] by exploring solutions of $u_t=\mathcal{L}_\epsilon^S u$.*
- *In Lemma 2.6 below, we prove tail estimates of u_ϵ by means of the classical decaying properties of eigenfunctions of $\mathcal{L}_\epsilon^S$.*

2.3. **Concentration estimates and tightness of QSDs.** Recall from Definition 1.1 the definition of QSDs of X_t^ϵ, and from Lemma 2.1 the positive eigenfunction $\phi_{\epsilon,1}$ of $-\mathcal{L}_\epsilon$ associated with $\lambda_{\epsilon,1}$. Set

$$u_\epsilon := \frac{\phi_{\epsilon,1}u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}} \quad \text{and} \quad d\mu_\epsilon := u_\epsilon dx. \tag{2.2}$$

Lemma 2.1 (2) ensures $\mu_\epsilon\in\mathcal{P}((0,\infty))$, where $\mathcal{P}((0,\infty))$ is the set of Borel probability measures on $(0,\infty)$.

Proposition 2.1 ([10])**.** *For each ϵ, μ_ϵ is the unique QSD of X_t^ϵ with extinction rate $\lambda_{\epsilon,1}$.*

We point out that μ_ϵ being a QSD of X_t^ϵ follows directly from Lemma 2.1. Moreover, it is straightforward to check that the density u_ϵ satisfies

$$\frac{\epsilon^2}{2}(au_\epsilon)''-(bu_\epsilon)'=-\lambda_{\epsilon,1}u_\epsilon \quad \text{in} \quad (0,\infty), \tag{2.3}$$

that is, u_ϵ is a positive and integrable eigenfunction of the Fokker-Planck operator $\phi\mapsto\frac{\epsilon^2}{2}(a\phi)''-(b\phi)'$ in $(0,\infty)$ associated with the eigenvalue $-\lambda_{\epsilon,1}$.

Proving the uniqueness result in Proposition 2.1 is however much more involved. In [10], the authors achieve this by exploring the so-called "coming down from infinity" saying that ∞ is an entrance boundary for X_t^ϵ, and obtain a necessary and sufficient condition. As a result, they show that for any $\mu\in\mathcal{P}((0,\infty))$ the conditioned dynamics $\mathbb{P}_\mu^\epsilon[X_t^\epsilon\in\bullet|t<T_0^\epsilon]$ converges to the QSD μ_ϵ as $t\to\infty$. This can be improved to exponential convergence with rate $\lambda_{\epsilon,2}-\lambda_{\epsilon,1}$ if μ is compactly supported in $(0,\infty)$. More precisely, it is proven in [10, Proposition 5.5] that the following holds for

each ϵ: for each $\mu\in\mathcal{P}((0,\infty))$ with compact support in $(0,\infty)$,

$$\lim_{t\to\infty} e^{(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})t}\left(\mathbb{P}^\epsilon_\mu[X^\epsilon_t\in B|t<T^\epsilon_0]-\mu_\epsilon(B)\right)$$
$$=\frac{\int_0^\infty \phi_{\epsilon,2}d\mu}{\int_0^\infty \phi_{\epsilon,1}d\mu}\left(\frac{\langle \mathbb{1}_B,\phi_{\epsilon,2}\rangle_{L^2(u^G_\epsilon)}}{\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}}-\frac{\langle \mathbb{1}_B,\phi_{\epsilon,1}\rangle_{L^2(u^G_\epsilon)}\langle \mathbb{1},\phi_{\epsilon,2}\rangle_{L^2(u^G_\epsilon)}}{\|\phi_{\epsilon,1}\|^2_{L^1(u^G_\epsilon)}}\right),\quad \forall B\in\mathcal{B}((0,\infty)),$$

where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra of $(0,\infty)$. It is worthwhile to mention that acquiring information about the dynamics of X^ϵ_t from the conditioned dynamics $\mathbb{P}^\epsilon_\mu[X^\epsilon_t\in\bullet|t<T^\epsilon_0]$ is not straightforward as it is generally hard to study the survival event $[t<T^\epsilon_0]$ for an arbitrarily given initial distribution.

Under the assumptions on b, the ODE (1.2) restricted on $(0,\infty)$ is dissipative, and therefore, admits the global attractor $\mathcal{A}$. By definition (see e.g. [36, 75]), $\mathcal{A}$ is the largest compact invariant set of the flow φ^t generated by solutions of (1.2) and has bounded dissipation property in the sense that

$$\lim_{t\to\infty}\operatorname{dist}_H\left(\varphi^t(B),\mathcal{A}\right)=0,\quad \forall B\subset\subset(0,\infty),$$

where dist_H denotes the Hausdorff semi-distance on $(0,\infty)$. In the current one-dimensional case, it is easy to check that $\mathcal{A}$ is just a closed interval (being possibly a singleton set) with its left endpoint and right endpoint being respectively the smallest positive zero and largest zero of b. The structure of $\mathcal{A}$ is fairly simple. It consists of either a single point, or equilibria, or equilibria and their connecting orbits.

We recall from [70] concentration estimates of $\{u_\epsilon\}_\epsilon$ away from $\mathcal{A}$ and ∞.

Lemma 2.5 ([70]). *The following hold.*

(1) *For each $\mathcal{O}\subset\subset(0,\infty)\setminus\mathcal{A}$, there are $\gamma_{\mathcal{O}}>0$ and $0<\epsilon_{\mathcal{O}}\ll 1$ such that*

$$\sup_{\mathcal{O}} u_\epsilon\le e^{-\frac{\gamma_{\mathcal{O}}}{\epsilon^2}},\quad \forall\epsilon\in(0,\epsilon_{\mathcal{O}}).$$

(2) *For each $\kappa\in(0,1)$, there are $x_\kappa\in(0,1)$ and $0<\epsilon_\kappa\ll1$ such that*

$$u_\epsilon(x)\le\frac{1}{x^\kappa},\quad \forall x\in(0,x_\kappa),\ \epsilon\in(0,\epsilon_\kappa).$$

The proof of Lemma 2.5 (1) in [70] is based on the sub-level set approach developed in [43] and the construction of uniform-in-noise Lyapunov functions. Lemma 2.5 (2) is the most important result in [70]. It addresses the tightness of $\{u_\epsilon\}_\epsilon$ near 0 by circumventing the difficulties caused by the degeneracy and singularity of the noise at 0.

In the rest of this subsection, we establish concentration estimates of $\{u_\epsilon\}_\epsilon$ near ∞ that turn out to be very useful in the sequel. Recall from Subsection 2.2 that $y=\xi(x)$ and $y_\infty=\xi(\infty)$.

Lemma 2.6. *Let $L\gg1$. The following hold for each $0<\epsilon\ll1$.*

(1) *If $y_\infty=\infty$, then*

$$u_\epsilon\le u_\epsilon(L)\left[\frac{a(L)}{a}\right]^{\frac34}e^{-\gamma_{\epsilon,L}[\xi-\xi(L)]}e^{\frac{1}{\epsilon^2}\int_L^\bullet\frac{b}{a}ds}\quad in\quad [L,\infty),$$

where $\gamma_{\epsilon,L}=\sqrt{\frac{2}{\epsilon^2}(\frac{C_L}{\epsilon^2}-\lambda_{\epsilon,1})}$. In which, $C_L:=C_2\inf_{[\xi(L),y_\infty)}\frac{b^2\circ\xi^{-1}}{a\circ\xi^{-1}}$, where $C_2=C_2(\xi(L))$ is given in Lemma 2.4 (2).

(2) *If $y_\infty<\infty$, then*

$$u_\epsilon \le u_\epsilon(L)\left[\frac{a(L)}{a}\right]^{\frac{3}{4}} \frac{e^{\gamma_{\epsilon,L}[\xi-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-\xi]}}{e^{\gamma_{\epsilon,L}[\xi(L)-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-\xi(L)]}} e^{\frac{1}{\epsilon^2}\int_L^{\bullet}\frac{b}{a}ds} \quad in \quad [L,\infty),$$

where $\gamma_{\epsilon,L}$ is as in (1).

In particular, if $L\gg\sup\mathcal{A}$, then $u_\epsilon\le\left[\frac{a(L)}{a}\right]^{\frac{3}{4}}e^{\frac{1}{\epsilon^2}\int_L^{\bullet}\frac{b}{a}ds}$ in $[L,\infty)$.

Proof. The "In particular" part follows directly from (1), (2) and Lemma 2.5 (1). We prove (1) and (2) by exploiting decaying properties of eigenfunctions of the Schrödinger operator $\mathcal{L}_\epsilon^S$, which is unitarily equivalent to the generator $\mathcal{L}_\epsilon$ (see Subsection 2.2).

Note that $w_\epsilon:=\frac{u_\epsilon}{u_\epsilon^G}$ satisfies $\int_0^\infty w_\epsilon^2u_\epsilon^Gdx<\infty$ and $\mathcal{L}_\epsilon w_\epsilon=-\lambda_{\epsilon,1}w_\epsilon$, and $\tilde{w}_\epsilon:=\tilde{U}_\epsilon U_\epsilon w_\epsilon$ satisfies $\int_0^{y_\infty}(V_\epsilon+M)\tilde{w}_\epsilon^2dy<\infty$ and $\mathcal{L}_\epsilon^S\tilde{w}_\epsilon=-\lambda_{\epsilon,1}\tilde{w}_\epsilon$, where $M=|\inf_\epsilon\inf V_\epsilon|<\infty$ due to Lemma 2.4 (3). We readily check that $\tilde{w}_\epsilon(y)=w_\epsilon(x)\sqrt{v_\epsilon^G(y)}=\sqrt[4]{a(x)}w_\epsilon(x)\sqrt{u_\epsilon^G(x)}$. Thus,

$$\frac{\tilde{w}_\epsilon(y)\sqrt{u_\epsilon^G(x)}}{\sqrt[4]{a(x)}}=w_\epsilon(x)u_\epsilon^G(x)=u_\epsilon(x). \tag{2.4}$$

Fix $L\gg1$ and set $y_L:=\xi(L)$. We distinguish between the cases $y_\infty=\infty$ and $y_\infty<\infty$.

Case $y_\infty=\infty$. Consider the following problem:

$$\begin{cases}\frac{\epsilon^2}{2}\tilde{W}_\epsilon''-\frac{C_L}{\epsilon^2}\tilde{W}_\epsilon=-\lambda_{\epsilon,1}\tilde{W}_\epsilon \quad \text{in} \quad (y_L,\infty),\\ \tilde{W}_\epsilon(y_L)=\tilde{w}_\epsilon(y_L), \quad \tilde{W}_\epsilon(\infty)=0,\end{cases}$$

where C_L is as in the statement. The unique solution is given by $\tilde{W}_\epsilon(y)=\tilde{w}_\epsilon(y_L)e^{-\gamma_{\epsilon,L}(y-y_L)}$ for $y\in[y_L,\infty)$, where $\gamma_{\epsilon,L}$ is given in the statement. Since $V_\epsilon\ge\frac{C_L}{\epsilon^2}\ge\lambda_{\epsilon,1}$ on $[y_L,\infty)$ ensured by Lemma 2.4 (2), we find from the comparison principle (see e.g. [4, Chapter 2, Section 2.3]) that $\tilde{w}_\epsilon\le\tilde{W}_\epsilon$ in $[y_L,\infty)$. This together with (2.4) implies that for $x\in[L,\infty)$,

$$u_\epsilon(x)\le\frac{\tilde{W}_\epsilon(y)\sqrt{u_\epsilon^G(x)}}{\sqrt[4]{a(x)}}=u_\epsilon(L)\frac{[a(L)]^{\frac{3}{4}}}{[a(x)]^{\frac{3}{4}}}e^{-\gamma_{\epsilon,L}[\xi(x)-\xi(L)]}e^{\frac{1}{\epsilon^2}\int_L^x\frac{b}{a}ds}.$$

Case $y_\infty<\infty$. Consider the following problem:

$$\begin{cases}\frac{\epsilon^2}{2}\tilde{W}_\epsilon''-\frac{C_L}{\epsilon^2}\tilde{W}_\epsilon=-\lambda_{\epsilon,1}\tilde{W}_\epsilon \quad \text{in} \quad (y_L,y_\infty),\\ \tilde{W}_\epsilon(y_L)=\tilde{w}_\epsilon(y_L), \quad \tilde{W}_\epsilon(y_\infty)=0.\end{cases}$$

The unique solution is given by

$$\tilde{W}_\epsilon(y)=\tilde{w}_\epsilon(y_L)\frac{e^{\gamma_{\epsilon,L}(y-y_\infty)}-e^{\gamma_{\epsilon,L}(y_\infty-y)}}{e^{\gamma_{\epsilon,L}(y_L-y_\infty)}-e^{\gamma_{\epsilon,L}(y_\infty-y_L)}}, \quad y\in[y_L,y_\infty).$$

To apply the comparison principle, we verify

$$\tilde{w}_\epsilon(y_\infty):=\lim_{y\to y_\infty^-}\tilde{w}_\epsilon(y)=0. \tag{2.5}$$

To see this, we first claim for fixed $K\gg1$,

$$\int_{\xi(K)}^{y_\infty}V_\epsilon dy=\infty. \tag{2.6}$$

Indeed, we see from Lemma 2.4 (2) that $\int_{\xi(K)}^{y_\infty} V_\epsilon dy \geq \frac{C_2}{\epsilon^2}\int_{\xi(K)}^{y_\infty} \frac{b^2\circ\xi^{-1}}{a\circ\xi^{-1}}dy = \frac{C_2}{\epsilon^2}\int_K^\infty \frac{b^2}{a\sqrt{a}}dx$, where $C_2 = C_2(\xi(K))$. Since $\lim_{x\to\infty}\frac{b^2(x)}{a(x)} = \infty$ by **(H)**(3), there is $c_1 > 0$ such that

$$\int_{\{x\in(K,\infty):a(x)\leq 1\}} \frac{b^2}{a\sqrt{a}}dx \geq c_1 \left|\{x\in(K,\infty): a(x)\leq 1\}\right|. \tag{2.7}$$

As $\limsup_{x\to\infty}\frac{a(x)}{|b(x)|} < \infty$ by **(H)**(3), there is $c_2 > 0$ such that $\frac{a(x)}{|b(x)|} \leq \frac{1}{c_2}$ for all $x > K$ (making K larger if necessary). It follows that

$$\int_{\{x\in(K,\infty):a(x)>1\}} \frac{b^2}{a\sqrt{a}}dx \geq \int_{\{x\in(K,\infty):a(x)>1\}} \sqrt{a}\left(\frac{b}{a}\right)^2 dx \geq c_2^2\left|\{x\in(K,\infty): a(x)>1\}\right|,$$

which together with (2.7) yields $\int_K^\infty \frac{b^2}{a\sqrt{a}}dx = \infty$ and thus, (2.6) holds.

Now, we show (2.5). It follows from $\int_0^{y_\infty}(V_\epsilon + M)\tilde{w}_\epsilon^2 dy < \infty$, (2.6) and the positivity of $\tilde{w}_\epsilon$ that $\liminf_{y\to y_\infty^-}\tilde{w}_\epsilon(y) = 0$. Suppose for contradiction that $\limsup_{y\to y_\infty^-}\tilde{w}_\epsilon(y) > 0$. Then, there exists y_* (which can be chosen to be arbitrary close to y_∞) such that $\tilde{w}_\epsilon$ has a local maximum at y_*. In particular, $\tilde{w}_\epsilon''(y_*) \leq 0$. This together with $\frac{\epsilon^2}{2}\tilde{w}_\epsilon''(y_*) - V_\epsilon(y_*)\tilde{w}_\epsilon(y_*) = -\lambda_{\epsilon,1}\tilde{w}_\epsilon(y_*)$ implies that $V_\epsilon(y_*) \leq \lambda_{\epsilon,1}$. Since $V_\epsilon(y)\to\infty$ as $y\to y_\infty^-$, we arrive at a contradiction. Hence, (2.5) is true.

Due to (2.5) and $V_\epsilon \geq \frac{C_L}{\epsilon^2} \geq \lambda_{\epsilon,1}$ on $[y_L,\infty)$, we apply the comparison principle to conclude that $\tilde{w}_\epsilon(y) \leq \tilde{W}_\epsilon(y)$ for all $y\in[y_L,\infty)$. This together with (2.4) implies

$$u_\epsilon(x) \leq \frac{\tilde{W}_\epsilon(y)\sqrt{u_\epsilon^G(x)}}{\sqrt[4]{a(x)}} = u_\epsilon(L)\frac{[a(L)]^{\frac{3}{4}}}{[a(x)]^{\frac{3}{4}}}\frac{e^{\gamma_{\epsilon,L}[\xi(x)-y_\infty]} - e^{\gamma_{\epsilon,L}[y_\infty-\xi(x)]}}{e^{\gamma_{\epsilon,L}[\xi(L)-y_\infty]} - e^{\gamma_{\epsilon,L}[y_\infty-\xi(L)]}}e^{\frac{1}{\epsilon^2}\int_L^x\frac{b}{a}ds}$$

for all $x\in[L,\infty)$. This completes the proof. □

The following result is a direct consequence of Lemma 2.5 and Lemma 2.6.

Corollary 2.1. *For any open set $\mathcal{O}$ containing $\mathcal{A}$, there holds $\lim_{\epsilon\to 0}\mu_\epsilon(\mathcal{O}) = 1$. In particular, the family of QSDs $\{\mu_\epsilon\}_\epsilon$ is tight.*

We end this section by pointing out the difference between [70] and the present paper in treating the tightness of $\{\mu_\epsilon\}_\epsilon$ near infinity. Assuming the existence of a uniform-in-noise Lyapunov function near ∞, the authors proved in [70] the exponential smallness in ϵ of the tail estimate appealing to the sub-level set approach put forward in [43]. Here, explicit assumptions on a and b allow us to use decaying properties of eigenfunctions of the Schrödinger operator $\mathcal{L}_\epsilon^S$ (which is unitarily equivalent to $\mathcal{L}_\epsilon$) to establish exponential decaying estimates for the density u_ϵ. Corresponding results, presented in Lemma 2.6, turn out to be crucial in applying the identities in Proposition 4.1 to derive sharp asymptotic of $\{u_\epsilon\}_\epsilon$.

3. Vanishing viscosity limits

To study the exponential asymptotic of u_ϵ as $\epsilon\to 0$, we introduce the logarithmic transform:

$$v_\epsilon = -\frac{\epsilon^2}{2}\ln(au_\epsilon) \quad \text{in} \quad (0,\infty). \tag{3.1}$$

It is well-defined as both a and u_ϵ are positive on $(0,\infty)$. Moreover, since $a, u_\epsilon \in C^3((0,\infty))$, there holds $v_\epsilon\in C^3((0,\infty))$. Clearly, the local uniform convergence of $-\frac{\epsilon^2}{2}\ln u_\epsilon$ to some $v\in C((0,\infty))$ as $\epsilon\to 0$ is equivalent to the local uniform convergence of v_ϵ to v as $\epsilon\to 0$.

It is straightforward to check that v_ϵ satisfies the following singularly perturbed equation:

$$-\frac{\epsilon^2}{2}v_\epsilon'' + (v_\epsilon')^2 + \frac{b}{a}v_\epsilon' = \frac{\epsilon^2}{2}\left[\left(\frac{b}{a}\right)' - \frac{\lambda_{\epsilon,1}}{a}\right] \quad \text{in} \quad (0,\infty). \tag{3.2}$$

The next result addresses the local uniform boundedness of $\{v_\epsilon\}_\epsilon$ and $\{v_\epsilon'\}_\epsilon$. Its proof is postponed to the end of this section. Recall from Subsection 2.3 that $\mathcal{A}$ is the global attractor of $\dot{x} = b(x)$ in $(0,\infty)$.

Lemma 3.1. *The following hold.*

(1) *For each $\mathcal{O} \subset\subset (0,\infty)$, there exist $\gamma_{\mathcal{O}}^1 \in \mathbb{R}$, $\gamma_{\mathcal{O}}^2 > 0$ and $0 < \epsilon_{\mathcal{O}} \ll 1$ such that*

$$\gamma_{\mathcal{O}}^1 \le \inf_{\mathcal{O}} v_\epsilon \le \sup_{\mathcal{O}} v_\epsilon \le \gamma_{\mathcal{O}}^2, \quad \forall \epsilon \in (0,\epsilon_{\mathcal{O}}).$$

Moreover, if $\mathcal{O} \subset\subset (0,\infty) \setminus \mathcal{A} = \emptyset$, then $\gamma_{\mathcal{O}}^1 > 0$.

(2) *For each $\mathcal{O} \subset\subset (0,\infty)$, there exist $\Gamma_{\mathcal{O}} > 0$ and $0 < \epsilon_{\mathcal{O}} \ll 1$ such that $\sup_{\mathcal{O}} |v_\epsilon'| \le \Gamma_{\mathcal{O}}$ for all $\epsilon \in (0,\epsilon_{\mathcal{O}})$.*

Denote by $\mathcal{V}$ the set of limit points of $\{v_\epsilon\}_\epsilon$ under the topology of locally uniform convergence in $(0,\infty)$ as $\epsilon \to 0$. By Lemma 3.1, we apply the Arzelá-Ascoli theorem and standard diagonal argument to conclude $\mathcal{V} \neq \emptyset$ and $\mathcal{V} \subset C((0,\infty))$. Moreover, the well-known result on the stability of viscosity solutions (see e.g. [19]) ensures that each element of $\mathcal{V}$ is a viscosity solution of the following Hamilton-Jacobi equation:

$$(v')^2 + \frac{b}{a}v' = 0 \quad \text{in} \quad (0,\infty). \tag{3.3}$$

Unfortunately, (3.3) admits infinitely many viscosity solutions.

We prove some properties of functions in $\mathcal{V}$.

Proposition 3.1. *Each $v \in \mathcal{V}$ is locally Lipschitz continuous and satisfies*

$$(v')^2 + \frac{b}{a}v' = 0 \quad \textit{a.e. in} \quad (0,\infty).$$

Moreover, $v > 0$ on $(0,\infty) \setminus \mathcal{A}$, $v(0+) \in (0,\infty)$, $v(\infty) = \infty$ and $\min_{\mathcal{A}} v = 0$.

Proof. Let $v \in \mathcal{V}$. By Lemma 3.1 (2), v is locally Lipschitz continuous. Since v is a viscosity solution of (3.3), it is well-known (see e.g. [19]) that if v is differentiable at $x_0 \in (0,\infty)$, then $(v')^2 + \frac{b}{a}v' = 0$ holds at x_0. Hence, v satisfies $(v')^2 + \frac{b}{a}v' = 0$ a.e. in $(0,\infty)$.

Lemma 3.1 (1) ensures that $v > 0$ on $(0,\infty) \setminus \mathcal{A}$. Since $b > 0$ in $(0, \inf \mathcal{A})$, we see from the equation that $v' \le 0$ a.e. in $(0,\inf\mathcal{A})$, and thus, v is non-increasing on $(0,\inf\mathcal{A})$. It follows that $v(0+) \in (0,\infty]$. Since $v' \ge -\frac{b}{a}$ a.e. in $(0,\inf\mathcal{A})$, $v(0+)$ must be finite, and hence, $v(0+) \in (0,\infty)$.

We see from Lemma 2.6 that $v(x) \ge -\frac{1}{2}\int_L^x \frac{b(s)}{a(s)}ds$ for all $x \ge L$. Since $\int_L^x \frac{b(s)}{a(s)}ds \to -\infty$ as $x \to \infty$, we conclude $v(\infty) = \infty$.

It remains to show $\min_{\mathcal{A}} v = 0$. Let I be an open interval such that $\mathcal{A} \subset I \subset\subset (0,\infty)$. Corollary 2.1 ensures that $\lim_{\epsilon\to 0}\int_I u_\epsilon dx = 1$, or $\lim_{\epsilon\to 0}\int_I \frac{1}{a}e^{-\frac{2}{\epsilon^2}v_\epsilon}dx = 1$. This together with the uniform convergence of v_ϵ (up to a subsequence) to v on I as $\epsilon \to 0$ implies that $\inf_I v = 0$, and hence, $\min_{\mathcal{A}} v = 0$. □

The rest of this section is devoted to the proof of Lemma 3.1.

Proof of Lemma 3.1. (1) Let $\mathcal{O}\subset\subset(0,\infty)$ be open. It follows from the classical interior estimates for elliptic equations (see e.g. [33]) that there exists $\gamma_{\mathcal{O}}>0$ such that $\sup_{\mathcal{O}}u_\epsilon\leq e^{\frac{\gamma_{\mathcal{O}}}{\epsilon^2}}$, which together with (3.1) leads to $\inf_{\mathcal{O}}v_\epsilon\geq-\frac{\gamma_{\mathcal{O}}}{2}$.

To see the upper bound of v_ϵ on $\mathcal{O}$, we let $\mathcal{O}_1$ be an open interval satisfying $\mathcal{A}\cup\mathcal{O}\subset\mathcal{O}_1\subset\subset(0,\infty)$. Fix $0<\delta\ll 1$. Since Corollary 2.1 ensures $|\mathcal{O}_1|\sup_{\mathcal{O}_1}u_\epsilon\geq\int_{\mathcal{O}_1}u_\epsilon dx\geq 1-\delta$, we see from Harnack's inequality that there exists $\gamma_{\mathcal{O}_1}>0$ such that $\inf_{\mathcal{O}_1}u_\epsilon\geq e^{-\frac{\gamma_{\mathcal{O}_1}}{\epsilon^2}}$, which together with (3.1) yields $\sup_{\mathcal{O}}v_\epsilon\leq\sup_{\mathcal{O}_1}v_\epsilon\leq\frac{\gamma_{\mathcal{O}_1}}{2}$.

For the "Moreover" part, we let $\mathcal{O}\subset\subset(0,\infty)\setminus\mathcal{A}$. Then, Lemma 2.5 (1) yields the existence of $\hat{\gamma}_{\mathcal{O}}>0$ such that $\sup_{\mathcal{O}}u_\epsilon\leq e^{-\frac{\hat{\gamma}_{\mathcal{O}}}{\epsilon^2}}$, leading to $\inf_{\mathcal{O}}v_\epsilon\geq\frac{\hat{\gamma}_{\mathcal{O}}}{2}>0$. This completes the proof of (1).

(2) The proof is inspired by the Bernstein-type estimate in [28, Lemma 2.2]. The key point here lies in the non-negativeness of the term $(v_\epsilon')^2$ in (3.2). Let I_1, I_2 be open intervals and satisfy $\emptyset\neq I_1\subset\subset I_2\subset\subset(0,\infty)$. Let $\eta:(0,\infty)\to[0,1]$ be smooth and satisfy $\eta=1$ in I_1 and $\eta=0$ in $(0,\infty)\setminus I_2$.

Consider the auxiliary function $z_\epsilon=\eta^4(v_\epsilon')^2$. We claim that

$$\sup_\epsilon\max z_\epsilon<\infty. \tag{3.4}$$

If this is the case, then $\sup_\epsilon\sup_{I_1}|v_\epsilon'|<\infty$, leading to the conclusion.

It remains to show (3.4). Since z_ϵ is continuous and compactly supported in $\overline{I}_2$, there exists $x_\epsilon\in I_2$ such that $z_\epsilon(x_\epsilon)=\max z_\epsilon$. We may assume, without loss of generality, that $\max z_\epsilon>0$. Then, $\eta(x_\epsilon)>0$ and $v_\epsilon'(x_\epsilon)\neq 0$.

We calculate

$$z_\epsilon'=4\eta^3\eta'(v_\epsilon')^2+2\eta^4v_\epsilon'v_\epsilon'',\quad z_\epsilon''=(\eta^4)''(v_\epsilon')^2+16\eta^3\eta'v_\epsilon'v_\epsilon''+2\eta^4(v_\epsilon'')^2+2\eta^4v_\epsilon'v_\epsilon'''.$$

Multiplying the expression of z_ϵ'' by $-\frac{\epsilon^2}{2}$ and setting $c_\epsilon:=\frac{\epsilon^2}{2}\left[\left(\frac{b}{a}\right)'-\frac{\lambda_{\epsilon,1}}{a}\right]$ (i.e., the right hand side of (3.2)), we find from (3.2) and straightforward calculations that

$$\begin{aligned}-\frac{\epsilon^2}{2}z_\epsilon''&=-\frac{\epsilon^2}{2}(\eta^4)''(v_\epsilon')^2-8\epsilon^2\eta^3\eta'v_\epsilon'v_\epsilon''-\epsilon^2\eta^4(v_\epsilon'')^2+2\eta^4v_\epsilon'c_\epsilon'\\&\quad-4\eta^4(v_\epsilon')^2v_\epsilon''-2\eta^4\left(\frac{b}{a}\right)'(v_\epsilon')^2-2\eta^4\frac{b}{a}v_\epsilon'v_\epsilon''.\end{aligned}\tag{3.5}$$

At the point x_ϵ, there holds $z_\epsilon'=0$, namely, $4\eta^3\eta'(v_\epsilon')^2+2\eta^4v_\epsilon'v_\epsilon''=0$. Since $\eta(x_\epsilon)>0$ and $v_\epsilon'(x_\epsilon)\neq 0$, we find

$$\eta v_\epsilon''=-2\eta'v_\epsilon'\quad\text{at}\quad x_\epsilon. \tag{3.6}$$

As $z_\epsilon''(x_\epsilon)\leq 0$, we find from (3.5) and (3.6) that at the point x_ϵ there holds

$$\begin{aligned}\epsilon^2\eta^4(v_\epsilon'')^2&\leq-\frac{\epsilon^2}{2}(\eta^4)''(v_\epsilon')^2-8\epsilon^2\eta^3\eta'v_\epsilon'v_\epsilon''+2\eta^4v_\epsilon'c_\epsilon'-4\eta^4(v_\epsilon')^2v_\epsilon''-2\eta^4\left(\frac{b}{a}\right)'(v_\epsilon')^2-2\eta^4\frac{b}{a}v_\epsilon'v_\epsilon''\\&\leq-\frac{\epsilon^2}{2}(\eta^4)''(v_\epsilon')^2-8\epsilon^2\eta^2\eta'v_\epsilon'(-2\eta'v_\epsilon')+\eta^4(v_\epsilon')^2+\eta^4(c_\epsilon')^2\\&\quad-4\eta^3(v_\epsilon')^2(-2\eta'v_\epsilon')-2\eta^4\left(\frac{b}{a}\right)'(v_\epsilon')^2-2\eta^3\frac{b}{a}v_\epsilon'(-2\eta'v_\epsilon')\\&=8\eta^3\eta'(v_\epsilon')^3+\zeta_\epsilon\eta^2(v_\epsilon')^2+\eta^4(c_\epsilon')^2,\end{aligned}$$

where ζ_ϵ in the equality is given by

$$\zeta_\epsilon = -\frac{\epsilon^2}{2}(12|\eta'|^2 + 4\eta\eta'') + 16\epsilon^2(\eta')^2 + \eta^2 - 2\eta^2\left(\frac{b}{a}\right)' + 4\eta'\eta\frac{b}{a}.$$

Thus, setting $C_1 := 8\max|\eta'|$, $C_2 := \sup_\epsilon \max|\zeta_\epsilon|$ and $C_3 := \sup_\epsilon \max\left[\epsilon^2\eta^4(c'_\epsilon)^2\right]$, we find

$$\epsilon^2\eta^4(v''_\epsilon)^2 \le C_1\eta^3|v'_\epsilon|^3 + C_2\eta^2(v'_\epsilon)^2 + C_3 \quad \text{at} \quad x_\epsilon.$$

Since $C_2\eta^2(v'_\epsilon)^2 \le \frac{C_2^3}{3} + \frac{2}{3}\eta^3|v'_\epsilon|^3$ by Young's inequality, we arrive at

$$\epsilon^2\eta^4(v''_\epsilon)^2 \le C_4\eta^3|v'_\epsilon|^3 + C_5 \quad \text{at} \quad x_\epsilon, \tag{3.7}$$

where $C_4 = C_1 + \frac{2}{3}$ and $C_5 = \frac{C_2^3}{3} + C_3$.

As (3.2) gives $(v'_\epsilon)^2 = c_\epsilon - \frac{b}{a}v'_\epsilon + \frac{\epsilon^2}{2}v''_\epsilon$ and Hölder's inequality gives $|\frac{b}{a}v'_\epsilon| \le \frac{1}{2}\left(\frac{b}{a}\right)^2 + \frac{1}{2}(v'_\epsilon)^2$, we deduce $\frac{1}{2}(v'_\epsilon)^2 \le \frac{\epsilon^2}{2}v''_\epsilon + c_\epsilon + \frac{1}{2}\left(\frac{b}{a}\right)^2$. Thus,

$$\eta^4(v'_\epsilon)^4 \le \eta^4\left[\epsilon^2 v''_\epsilon + 2c_\epsilon + \left(\frac{b}{a}\right)^2\right]^2 \le 2\epsilon^4\eta^4(v''_\epsilon)^2 + 2\eta^4\left[2c_\epsilon + \left(\frac{b}{a}\right)^2\right]^2.$$

This together with (3.7) implies that $\eta^4(v'_\epsilon)^4 \le 2\epsilon^2C_4\eta^3|v'_\epsilon|^3 + C_6$ at x_ϵ, where

$$C_6 = \sup_{\epsilon\in(0,\epsilon_*)}\left\{2\epsilon^2C_5 + \max 2\eta^4\left[2c_\epsilon + \left(\frac{b}{a}\right)^2\right]^2\right\}.$$

Let $\kappa > 0$ be such that $\frac{3}{4}\kappa^{\frac{4}{3}} = \frac{1}{2}$. Applying Young's inequality, we find

$$\eta^4(v'_\epsilon)^4 \le \frac{2\epsilon^2C_4}{\kappa}\kappa\eta^3|v'_\epsilon|^3 + C_6 \le \frac{1}{4}\frac{16\epsilon^8C_4^4}{\kappa^4} + \frac{1}{2}\eta^4(v'_\epsilon)^4 + C_6 \quad \text{at} \quad x_\epsilon,$$

leading to $\eta^4(v'_\epsilon)^4 \le C_7 := \frac{8\epsilon_*^8C_4^4}{\kappa^4} + 2C_6$ at x_ϵ. It follows that $\max z_\epsilon = \eta^4(x_\epsilon)(v'_\epsilon)^2(x_\epsilon) \le \sqrt{C_7}\max\eta^2$. As the right hand side of this estimate is independent of ϵ, we conclude (3.4), and hence, complete the proof. □

4. **Large deviation principle for QSDs**

In this section, we study the LDP for QSDs $\{\mu_\epsilon\}_\epsilon$. In Subsection 4.1, we derive important identities for u_ϵ, v_ϵ and v'_ϵ. Subsections 4.2, 4.3 and 4.4 are respectively devoted to the proof of Theorems A, B and C.

4.1. **Identities.**

Recall u_ϵ and v_ϵ from (2.2) and (3.1), respectively. We derive identities for u_ϵ, v_ϵ and v'_ϵ that play crucial roles in proving the LDP for $\{\mu_\epsilon\}_\epsilon$.

Proposition 4.1. *Assume* **(H)**. *For each* ϵ,

$$\begin{aligned}
u_\epsilon &= \frac{2\lambda_{\epsilon,1}}{\epsilon^2 a}e^{-\frac{2}{\epsilon^2}V}\int_0^\bullet e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz,\\
v_\epsilon &= -\frac{\epsilon^2}{2}\ln\frac{2}{\epsilon^2} - \frac{\epsilon^2}{2}\ln\lambda_{\epsilon,1} - \frac{\epsilon^2}{2}\ln\int_0^\bullet e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz + V,\\
v'_\epsilon &= -\frac{b}{a} - \lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}v_\epsilon}\int_\bullet^\infty \frac{1}{a}e^{-\frac{2}{\epsilon^2}v_\epsilon}dz.
\end{aligned}$$

We establish several lemmas before proving Proposition 4.1. Recall from the proof of Lemma 2.6 that $w_\epsilon = \frac{u_\epsilon}{u_\epsilon^G}$.

Lemma 4.1. *For each ϵ, $\lim_{x\to\infty} \frac{w_\epsilon(x)}{\sqrt[4]{a(x)}} e^{-\frac{V(x)}{\epsilon^2}} = 0$.*

Proof. It is a byproduct of the proof of Lemma 2.6. Indeed, since

$$\frac{w_\epsilon(x)}{\sqrt[4]{a(x)}} e^{-\frac{V(x)}{\epsilon^2}} = \sqrt[4]{a(x)} w_\epsilon(x) \sqrt{u_\epsilon^G(x)} = \tilde{w}_\epsilon(\xi(x)) \leq \tilde{W}_\epsilon(\xi(x)), \quad \forall x \gg 1,$$

the lemma follows immediately from $\lim_{y\to y_\infty} \tilde{W}_\epsilon(y) = 0$ and $y = \xi(x)$. □

Lemma 4.2. *For each ϵ, $\frac{\epsilon^2}{2}\left(w_\epsilon' e^{-\frac{2}{\epsilon^2}V}\right)' = -\lambda_{\epsilon,1} u_\epsilon$ in $(0,\infty)$. In particular, $w_\epsilon' > 0$ in $(0,\infty)$ and $w_\epsilon(0+) = 0$.*

Proof. Note that w_ϵ satisfies $\mathcal{L}_\epsilon w_\epsilon = -\lambda_{\epsilon,1} w_\epsilon$, namely, $\frac{\epsilon^2}{2} a w_\epsilon'' + b w_\epsilon' = -\lambda_{\epsilon,1} w_\epsilon$. Multiplying this equation by u_ϵ^G, we readily derive the identity as in the statement.

We show $w_\epsilon' > 0$. Suppose for contradiction that there is $x_* \in (0,\infty)$ such that $w_\epsilon'(x_*) \leq 0$. Fix $x_{**} > x_*$. Integrating the identity over $[x_*, x_{**}]$ yields $w_\epsilon'(x_{**}) < 0$. We then integrate the identity over $[x_{**}, x]$ to find

$$w_\epsilon'(x) e^{-\frac{2}{\epsilon^2}V(x)} < -C_1 := w_\epsilon'(x_{**}) e^{-\frac{2}{\epsilon^2}V(x_{**})} < 0 \quad \text{for} \quad x > x_{**}.$$

It follows that $w_\epsilon'(x) < -C_1 e^{\frac{2}{\epsilon^2}V(x)}$ for $x > x_{**}$. Since $V(x) \to \infty$ as $x \to \infty$, there exists $C_2 > 0$ such that $w_\epsilon'(x) \leq -C_2$ for all $x \gg 1$, which implies that $w_\epsilon < 0$ for all $x \gg 1$, leading to a contradiction.

It remains to show $w_\epsilon(0+) = 0$. Since $w_\epsilon u_\epsilon^G = u_\epsilon \in L^1((0,\infty))$, we conclude from the behavior of $u_\epsilon^G(x)$ near $x = 0$ and the monotonicity of w_ϵ that $w_\epsilon(0+) = 0$. □

Lemma 4.3. *For each ϵ, $\lim_{x\to\infty} w_\epsilon'(x) e^{-\frac{2}{\epsilon^2}V(x)} = 0$.*

Proof. By Lemma 4.2, $w_\epsilon' e^{-\frac{2}{\epsilon^2}V}$ is positive and decreasing. So, $C := \lim_{x\to\infty} w_\epsilon'(x) e^{-\frac{2}{\epsilon^2}V(x)} \geq 0$. It suffices to show $C = 0$.

Suppose on the contrary that $C > 0$. Then, there is $x_* \gg 1$ such that $w_\epsilon' e^{-\frac{2}{\epsilon^2}V} \geq \frac{C}{2}$ in (x_*, ∞), and hence,

$$w_\epsilon(x) = w_\epsilon(x_*) + \int_{x_*}^x w_\epsilon'(s)ds \geq w_\epsilon(x_*) + \frac{C}{2}\int_{x_*}^x e^{\frac{2}{\epsilon^2}V(s)}ds, \quad \forall x > x_*. \tag{4.1}$$

Since **(H)**(3) ensures $V'(x) \leq V^m(x)$ for $x \gg 1$, we derive

$$\frac{\frac{d}{dx}\int_{x_*}^x e^{\frac{2}{\epsilon^2}V(s)}ds}{\frac{d}{dx} e^{\frac{3}{2\epsilon^2}V(x)}} = \frac{2\epsilon^2 e^{\frac{1}{2\epsilon^2}V(x)}}{3V'(x)} \geq \frac{2\epsilon^2 e^{\frac{1}{2\epsilon^2}V(x)}}{3V^m(x)} \to \infty \quad \text{as} \quad x \to \infty,$$

where we used $\lim_{x\to\infty} V(x) = \infty$ in the limit. It follows that $\lim_{x\to\infty} \frac{\int_{x_*}^x e^{\frac{2}{\epsilon^2}V(s)}ds}{e^{\frac{3}{2\epsilon^2}V(x)}} = \infty$, which together with (4.1) yields

$$w_\epsilon(x) \geq w_\epsilon(x_*) + \frac{C}{2} e^{\frac{3}{2\epsilon^2}V(x)}, \quad \forall x \gg 1. \tag{4.2}$$

Thanks to Lemma 2.6 and $w_\epsilon = a u_\epsilon e^{\frac{2}{\epsilon^2}V}$, we find $C_1 > 0$ such that $w_\epsilon(x) \leq C_1 a^{\frac{1}{4}} e^{\frac{1}{\epsilon^2}V(x)}$ for $x \gg 1$. By **(H)**(3), there is $C_2 > 0$ such that $a^{\frac{1}{4}}(x) \leq e^{C_2 x}$ and $V(x) \geq C_2 x$ for all $x \gg 1$. As a result,

$$w_\epsilon(x) \leq C_1 e^{C_2 x} e^{\frac{1}{\epsilon^2}V(x)} \leq C_1 e^{\frac{4}{3\epsilon^2}V(x)}, \quad \forall x \gg 1.$$

This contradicts (4.2) due to $\lim_{x\to\infty} V(x) = \infty$. Hence, $C = 0$. □

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Integrating the identity in Lemma 4.2 over $[x, \tilde{x}] \subset (0,\infty)$ yields

$$\frac{\epsilon^2}{2} w'_\epsilon(\tilde{x}) e^{-\frac{2}{\epsilon^2}V(\tilde{x})} - \frac{\epsilon^2}{2} w'_\epsilon(x) e^{-\frac{2}{\epsilon^2}V(x)} = -\lambda_{\epsilon,1} \int_x^{\tilde{x}} u_\epsilon dz.$$

Passing to the limit $\tilde{x} \to \infty$, we deduce from Lemma 4.3 that $\frac{\epsilon^2}{2} w'_\epsilon e^{-\frac{2}{\epsilon^2}V} = \lambda_{\epsilon,1} \int_\bullet^\infty u_\epsilon dz$, which together with $w_\epsilon(0+) = 0$ (by Lemma 4.2) gives $w_\epsilon = \frac{2\lambda_{\epsilon,1}}{\epsilon^2} \int_0^\bullet e^{\frac{2}{\epsilon^2}V(z)} \left(\int_z^\infty u_\epsilon d\tilde{z}\right) dz$. As $u_\epsilon = w_\epsilon u_\epsilon^G$, we derive the formula for u_ϵ. The formula for v_ϵ then is a direct consequence of its definition.

Note $v'_\epsilon = -\frac{\epsilon^2}{2} \frac{(au_\epsilon)'}{au_\epsilon}$. Integrating $\frac{\epsilon^2}{2}(au_\epsilon)'' - (bu_\epsilon)' = -\lambda_{\epsilon,1} u_\epsilon$ gives $\frac{\epsilon^2}{2}(au_\epsilon)' - bu_\epsilon = \lambda_{\epsilon,1} \int_\bullet^\infty u_\epsilon dz$, leading to $v'_\epsilon = -\frac{b}{a} + \frac{\lambda_{\epsilon,1}}{au_\epsilon} \int_\bullet^\infty u_\epsilon dz$. The conclusion follows readily from $u_\epsilon = \frac{1}{a} e^{-\frac{2}{\epsilon^2}v_\epsilon}$. □

The formula for u_ϵ in Proposition 4.1 leads to refined estimates of $\{u_\epsilon\}_\epsilon$ near 0 in comparison to those given in Lemma 2.5 (2).

Lemma 4.4. *Assume* **(H)**. *For each* $0 < \delta \ll 1$, *there are* $0 < x_\delta \ll 1$ *and* $0 < \epsilon_\delta \ll 1$ *such that*

$$e^{-\frac{2}{\epsilon^2}(d_1+\delta)} \leq u_\epsilon(x) \leq e^{-\frac{2}{\epsilon^2}(d_1-\delta)}, \quad \forall x \in (0, x_\delta),\ \epsilon \in (0, \epsilon_\delta).$$

Proof. We only establish the lower bound; the upper bound follows in a similar manner. Consider the formula for u_ϵ in Proposition 4.1. Note that for each $0 < \delta \ll 1$, there is $0 < x_\delta \ll 1$ such that $|V(x) - V(y)| \leq \frac{\delta}{2}$ for all $x, y \in (0, x_\delta)$. By Corollary 2.1, there exists $0 < \epsilon_\delta \ll 1$ such that $\int_{x_\delta}^\infty u_\epsilon d\tilde{z} \geq 1 - \delta$ for all $\epsilon \in (0, \epsilon_\delta)$. Then, for each $x \in (0, x_\delta)$ and $\epsilon \in (0, \epsilon_\delta)$,

$$u_\epsilon(x) \geq \frac{2(1-\delta)\lambda_{\epsilon,1}}{\epsilon^2 a(x)} \int_0^x e^{\frac{2}{\epsilon^2}[V(x)-V(z)]} dz = \frac{2(1-\delta)\lambda_{\epsilon,1}}{\epsilon^2 a(x)} x e^{\frac{2}{\epsilon^2}[V(x)-V(\xi)]},$$

where we used the mean value theorem in the equality and $\xi \in (0, x)$. The desired inequality then follows from Lemma 2.3 and the facts that $a(0) = 0$ and $a'(0) > 0$. □

The next result, improving Corollary 2.1, is a simple consequence of Lemma 2.5 (1), Lemma 2.6 and Lemma 4.4.

Corollary 4.1. *Assume* **(H)**. *For each open set* $\mathcal{O}$ *satisfying* $\mathcal{A} \subset \mathcal{O} \subset\subset (0,\infty)$, *there exist* $\gamma_{\mathcal{O}} > 0$ *and* $0 < \epsilon_{\mathcal{O}} \ll 1$ *such that* $\mu_\epsilon((0,\infty) \setminus \mathcal{O}) \leq e^{-\frac{\gamma_{\mathcal{O}}}{\epsilon^2}}$ *for all* $\epsilon \in (0, \epsilon_{\mathcal{O}})$.

4.2. **Proof of Theorem A.** Let $(\alpha, \beta) \subset (0,\infty)$ be the unique d_1-valley. We focus on the case $\alpha > 0$; the case $\alpha = 0$ can be treated in the same way and is easier.

Up to a subsequence, we may assume without loss of generality that $\lim_{\epsilon\to 0} v_\epsilon = v$ locally uniformly in $(0,\infty)$. We determine v within three steps.

Step 1. Let x_0 be the smallest zero of v. By Proposition 3.1, x_0 exists and belongs to $[\inf \mathcal{A}, \sup \mathcal{A}]$. We show

$$v(x) = d_1 + V(x) - \sup_{(0,x)} V = \begin{cases} d_1 + V(x) - \sup_{(0,x)} V, & x \in (0, \alpha], \\ d_1 + V(x) - V(\alpha), & x \in (\alpha, x_0], \end{cases} \tag{4.3}$$

and

$$x_0 = \min \left\{ x \in (\alpha, \beta) : V(x) = \min_{(\alpha,\beta)} V \right\}. \tag{4.4}$$

Fix $x \in (0, x_0)$. Note that Proposition 3.1 and the definition of x_0 ensure $\min_{(0,x]} v > 0$. The locally uniform convergence of v_ϵ to v as $\epsilon \to 0$ and Lemma 4.4 then imply $\lim_{\epsilon\to 0} \inf_{z\in(0,x)} \int_z^\infty u_\epsilon d\tilde{z} = 1$, and hence,

$$\lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \ln \int_0^x e^{\frac{2}{\epsilon^2} V(z)} \left(\int_z^\infty u_\epsilon d\tilde{z} \right) dz = \sup_{(0,x)} V.$$

This together with the formula for v_ϵ in Proposition 4.1 and Lemma 2.3 yields $\lim_{\epsilon\to 0} v_\epsilon(x) = d_1 + V(x) - \sup_{(0,x)} V$. From which and the continuity of v, the first equality in (4.3) follows readily.

Since $v(x_0) = 0$ by the definition of x_0, we see from the first equality in (4.3) that $\sup_{(0,x_0)} V - V(x_0) = d_1$. As (α, β) is the unique d_1-valley, there must hold $x_0 \in \left\{x \in (\alpha, \beta) : V(x) = \min_{(\alpha,\beta)} V\right\}$ and (4.4), otherwise v attains 0 in $(0, x_0)$.

Observing that $V(\alpha) = \max_{(0,\beta)} V$, we deduce the second equality in (4.3).

Step 2. We prove that for any $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$, there holds

$$\lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \ln \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(z)} \left(\int_z^\infty \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} \right) dz = -d_1 + \gamma(x_1, x_2), \tag{4.5}$$

where

$$\gamma(x_1, x_2) = \sup_{z\in[x_1,x_2]} \sup_{\tilde{z}\in(z,\infty)} [v(z) - v(\tilde{z})]. \tag{4.6}$$

To see this, we fix $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$ and split the integral

$$\int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(z)} \left(\int_z^\infty \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} \right) dz = \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(z)} \left(\int_z^{z_1} \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} + \int_{z_1}^\infty \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} \right) dz,$$

where $z_1 \gg x_2$ is such that $\inf_\epsilon \inf_{(z_1,\infty)} v_\epsilon > \sup_\epsilon \sup_{(x_1,x_2)} v_\epsilon$. Such an z_1 exists due to Lemma 2.6 and the locally uniform convergence of v_ϵ to v as $\epsilon \to 0$. It is then easy to see from the dominated convergence theorem that $\lim_{\epsilon\to 0} \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(z)} \left(\int_{z_1}^\infty \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} \right) dz = 0$.

Since $z_1 \gg 1$ and $\lim_{z\to\infty} v(z) = \infty$ by Proposition 3.1, we may assume without loss of generality that $\gamma(x_1, x_2) = \sup_{z\in[x_1,x_2]} \sup_{\tilde{z}\in(z,z_1)} [v(z) - v(\tilde{z})]$. Thanks to the locally uniform convergence of v_ϵ to v as $\epsilon \to 0$, we find for any $\delta' > 0$, there exists $0 < \epsilon' \ll 1$ such that

$$\begin{aligned} \lambda_{\epsilon,1} e^{-\frac{\delta'}{\epsilon^2}} &\int_{x_1}^{x_2} \int_z^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon^2}[v(z)-v(\tilde{z})]} d\tilde{z} dz \\ &\leq \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(z)} \left(\int_z^{z_1} \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} d\tilde{z} \right) dz \\ &\qquad \leq \lambda_{\epsilon,1} e^{\frac{\delta'}{\epsilon^2}} \int_{x_1}^{x_2} \int_z^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon^2}[v(z)-v(\tilde{z})]} d\tilde{z} dz, \quad \forall 0 < \epsilon < \epsilon'. \end{aligned}$$

Note that Laplace's method yields $\lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \ln \int_{x_1}^{x_2} \int_z^{z_1} \frac{1}{a} e^{\frac{2}{\epsilon^2}[v(z)-v(\tilde{z})]} d\tilde{z} dz = \gamma(x_1, x_2)$, which together with the above two-sided inequalities, Lemma 2.3 and the arbitrariness of $\delta' > 0$ leads to (4.5).

Step 3. We finish the proof by showing

$$v = d_1 + V - V(\alpha) \quad \text{in} \quad (x_0, \infty). \tag{4.7}$$

Integrating the formula for v_ϵ' in Proposition 4.1 over $(x_1, x_2) \subset\subset (\alpha, \infty)$ yields

$$v_\epsilon(x_2) - v_\epsilon(x_1) = V(x_2) - V(x_1) - \int_{x_1}^{x_2} \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2} v_\epsilon(\tilde{z})} \left(\int_{\tilde{z}}^\infty \frac{1}{a} e^{-\frac{2}{\epsilon^2} v_\epsilon} dz \right) d\tilde{z}. \tag{4.8}$$

Since the above inequality holds for any $x_1, x_2 \in (\alpha,\infty)$ with $x_1 < x_2$, the definition of γ in Definition (4.6) ensures that $\gamma(x_1,x_2) < d_1$. As a result, we let $\epsilon \to 0$ in (4.8) and apply (4.5) to conclude that $v(x_2) = v(x_1) + V(x_2) - V(x_1)$. Letting $x_1 \to \alpha^+$ and setting $x_2 = x \in (x_0,\infty)$, we conclude (4.7) from (4.3) and the continuity of v.

4.3. **Proof of Theorem B.** Recall M_ϵ from (1.8) as well as the set $\mathcal{M}_{\delta_0}$ appearing in the definition of M_ϵ.

(1) The formula for u_ϵ in Proposition 4.1 and the definition of R_ϵ (see (1.7)) give

$$R_\epsilon = \epsilon\lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2}d_1} \frac{2}{\epsilon^2}\int_0^\bullet e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz. \tag{4.9}$$

We claim

$$\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_0^\bullet e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz = -\frac{1}{V'(0+)} \text{ locally uniformly in } (0,\infty), \tag{4.10}$$

and

$$\epsilon\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1} \approx_\epsilon \frac{b'(0)}{a'(0)}M_\epsilon \tag{4.11}$$

These together with (4.9) lead to the conclusion.

We prove (4.10). Let $[\ell_1,\ell_2] \subset (0,\infty)$ and fix $x_* \in (0,\min\{\inf\mathcal{M}_{\delta_0},\ell_1\})$. For $x \in [\ell_1,\ell_2]$, there holds

$$\begin{aligned} C_1(\epsilon)\frac{2}{\epsilon^2}\int_0^{x_*} e^{\frac{2}{\epsilon^2}V}dz &\le \frac{2}{\epsilon^2}\int_0^x e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz \\ &\le \frac{2}{\epsilon^2}\int_0^{x_*} e^{\frac{2}{\epsilon^2}V}dz + \frac{2}{\epsilon^2}\int_{x_*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz, \end{aligned}$$

where $C_1(\epsilon) := \inf_{z\in(0,x_*)}\int_z^\infty u_\epsilon d\tilde{z} \to 1$ as $\epsilon \to 0$ thanks to Corollary 4.1 and Theorem A. Since $V|_{[0,x_*]}$ has the maximum value 0 attained only at $x = 0$ and $V'(0+) = -\frac{b'(0)}{a'(0)} < 0$, we apply Laplace's method to find $\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_0^{x_*} e^{\frac{2}{\epsilon^2}V}dz = -\frac{1}{V'(0+)}$. Therefore, (4.10) follows immediately if we show

$$\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_{x_*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz = 0. \tag{4.12}$$

Since the integral is increasing in ℓ_2, we assume without loss of generality that $\ell_2 > \beta$. Take $x^* \in (\sup\mathcal{M}_{\delta_0},\beta)$. Then,

$$\frac{2}{\epsilon^2}\int_{x_*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz \le \frac{2}{\epsilon^2}\int_{x_*}^{x^*} e^{\frac{2}{\epsilon^2}V(z)}dz + \frac{2}{\epsilon^2}\int_{x^*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz. \tag{4.13}$$

Since $\sup_{[x_*,x^*]}V(x) < 0$, we find $\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_{x_*}^{x^*} e^{\frac{2}{\epsilon^2}V(z)}dz = 0$. Note that Lemma 2.6 ensures the existence of some $z_* \gg \ell_2$ such that $\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_{x^*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_{z_*}^\infty u_\epsilon d\tilde{z}\right)dz = 0$. Moreover, Theorem A and the fact that $V(z) - \min_{[z,z_*]}V - d_1 < 0$ for $z \in [x^*,z_*]$ (otherwise, there are more than one d_1-valleys) yield

$$\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_{x^*}^{\ell_2} e^{\frac{2}{\epsilon^2}V(z)}\left(\int_z^{z_*} u_\epsilon d\tilde{z}\right)dz = \lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\int_{x^*}^{\ell_2}\int_z^{z_*} e^{\frac{2}{\epsilon^2}[V(z)-v_\epsilon(\tilde{z})]}d\tilde{z}dz = 0.$$

Then, (4.12) follows from (4.13). This proves (4.10).

Now, we show (4.11). Corollary 4.1 and Theorem A ensure the existence of $\gamma>0$ such that $\int_{\mathcal{M}_{\delta_0}} u_\epsilon dx = 1-o(e^{-\frac{\gamma}{\epsilon^2}})$. It follows from the formula for u_ϵ in Proposition 4.1 and (4.10) that

$$\begin{aligned}1-o(e^{-\frac{\gamma}{\epsilon^2}}) &= \frac{2\lambda_{\epsilon,1}}{\epsilon^2}\int_{\mathcal{M}_{\delta_0}} \frac{1}{a(x)} e^{-\frac{2}{\epsilon^2}V(x)} \int_0^x e^{\frac{2}{\epsilon^2}V(z)} \int_z^\infty u_\epsilon d\tilde{z}dzdx \\ &\approx_\epsilon \lambda_{\epsilon,1} e^{\frac{2}{\epsilon^2}d_1} \frac{a'(0)}{b'(0)} \int_{\mathcal{M}_{\delta_0}} \frac{1}{a} e^{-\frac{2}{\epsilon^2}(d_1+V)} dx,\end{aligned}$$

resulting in (4.11).

(2) The proof follows from similar arguments, but the mechanism is slightly different. We break the proof into three steps. Set $C_\alpha := \sqrt{\frac{\pi}{-V''(\alpha)}}$ for convenience.

Step 1. We prove

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^\bullet e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz = C_\alpha \text{ locally uniformly in } (\alpha,\infty). \tag{4.14}$$

Let $[\ell_1,\ell_2]\subset(\alpha,\infty)$ satisfy $\mathcal{M}_{\delta_0}\subset(\ell_1,\ell_2)$ and $\ell_2>\beta$. Fix $x_*\in(\alpha,\ell_1)$. Then, for each $x\in[\ell_1,\ell_2]$,

$$\begin{aligned}&C_4(\epsilon)\frac{1}{\epsilon}\int_0^{x_*} e^{\frac{2}{\epsilon^2}[V-V(\alpha)]}dz \\ &\quad\le \frac{1}{\epsilon}\int_0^x e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz \\ &\quad\quad\le \frac{1}{\epsilon}\int_0^{x_*} e^{\frac{2}{\epsilon^2}[V-V(\alpha)]}dz + \frac{1}{\epsilon}\int_{x_*}^{\ell_2} e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\left(\int_z^\infty u_\epsilon d\tilde{z}\right)dz,\end{aligned} \tag{4.15}$$

where $C_4(\epsilon):=\inf_{z\in(0,x_*)}\int_z^\infty u_\epsilon d\tilde{z}\to 1$ as $\epsilon\to 0$ thanks to Theorem A and Corollary 4.1.

We claim

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{x_*} e^{\frac{2}{\epsilon^2}[V-V(\alpha)]}dz = C_\alpha. \tag{4.16}$$

Recall the assumption $V(\alpha)>V$ in $(0,\alpha)$. In particular, $V(\alpha)\ge 0$. If $V(\alpha)>0$, then the function $z\mapsto V(z)-V(\alpha)$ on $[0,x_*]$ has the maximum value 0 attained only at $z=\alpha$. Moreover, $V''(\alpha)=-\frac{b'(\alpha)}{a(\alpha)}<0$ by assumption. Laplace's method then yields (4.16). If $V(\alpha)=0$, then the function $z\mapsto V(z)-V(\alpha)$ on $[0,x_*]$ has the maximum value 0 attained only at $z=0$ and $z=\alpha$. Since $V'(0)<0$, we find from Laplace's method that the integral $\int_0^{x_*} e^{\frac{2}{\epsilon^2}[V-V(\alpha)]}dz$ is dominated by $\int_{\alpha-\delta}^{\alpha+\delta} e^{\frac{2}{\epsilon^2}[V-V(\alpha)]}dz$ for any fixed $0<\delta\ll 1$. Hence, (4.16) holds.

Arguing as in (1), we deduce that the second integral in the last line of (4.15) tends to 0 as $\epsilon\to 0$. Then, (4.14) follows readily.

Step 2. We show

$$2\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}\approx_\epsilon \frac{M_\epsilon}{C_\alpha}. \tag{4.17}$$

By Corollary 4.1 and Theorem A, there exists $\gamma>0$ such that $\int_{\mathcal{M}_{\delta_0}} u_\epsilon dx = 1-o(e^{-\frac{\gamma}{\epsilon^2}})$. Given the formula for u_ϵ in Proposition 4.1, we derive

$$\begin{aligned}1-o(e^{-\frac{\gamma}{\epsilon^2}}) &= \frac{2\lambda_{\epsilon,1}}{\epsilon^2}\int_{\mathcal{M}_{\delta_0}}\frac{1}{a(x)}e^{-\frac{2}{\epsilon^2}[V(x)-V(\alpha)]}\int_0^x e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\int_z^\infty u_\epsilon(\tilde{z})d\tilde{z}dzdx \\ &\approx_\epsilon \frac{2C_\alpha\lambda_{\epsilon,1}}{\epsilon}\int_{\mathcal{M}_{\delta_0}}\frac{1}{a}e^{-\frac{2}{\epsilon^2}[V-V(\alpha)]}dx = 2C_\alpha\lambda_{\epsilon,1}e^{\frac{2}{\epsilon^2}d_1}\frac{1}{\epsilon}\int_{\mathcal{M}_{\delta_0}}\frac{1}{a}e^{-\frac{2}{\epsilon^2}[d_1+V-V(\alpha)]}dx,\end{aligned}$$

where we used (4.14) in the approximating equality. (4.17) follows readily.

Step 3. We prove the limit for R_ϵ. By the formula for u_ϵ in Proposition 4.1 and the definition of R_ϵ,

$$R_\epsilon(x)=\begin{cases}\dfrac{2\lambda_{\epsilon,1}}{\epsilon}e^{\frac{2}{\epsilon^2}d_1}\displaystyle\int_0^x e^{\frac{2}{\epsilon^2}[V(z)-\sup_{(0,x)}V]}\left(\int_z^\infty u_\epsilon d\tilde z\right)dz, & x\in(0,\alpha),\\ \dfrac{2\lambda_{\epsilon,1}}{\epsilon}e^{\frac{2}{\epsilon^2}d_1}\displaystyle\int_0^x e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\left(\int_z^\infty u_\epsilon d\tilde z\right)dz, & x\in[\alpha,\infty).\end{cases}\tag{4.18}$$

By (4.14) and (4.17), we find $R_\epsilon\approx_\epsilon M_\epsilon$ locally uniformly in (α,∞). Obviously, $\lim_{\epsilon\to0}\int_z^\infty u_\epsilon d\tilde z=1$ uniformly in $z\in(0,\alpha]$. Arguing as in **Step 1** yields $\lim_{\epsilon\to0}\frac{1}{\epsilon}\int_0^\alpha e^{\frac{2}{\epsilon^2}[V(z)-V(\alpha)]}\left(\int_z^\infty u_\epsilon d\tilde z\right)dz=\frac{C_\alpha}{2}$. This together with (4.17) leads to $R_\epsilon(\alpha)\approx_\epsilon\frac12 M_\epsilon$.

For $x\in(0,\alpha)$, we see from (4.17) and $\lim_{\epsilon\to0}\int_z^\infty u_\epsilon d\tilde z=1$ uniformly in $z\in(0,\alpha)$ that

$$R_\epsilon(x)\approx_\epsilon\frac{M_\epsilon}{\epsilon C_\alpha}\int_0^x e^{\frac{2}{\epsilon^2}[V(z)-\sup_{(0,x)}V]}dz.$$

Let x_0 be as in the statement. Clearly, $x_0\in(0,\alpha)$. It remains to show that

$$\frac{R_\epsilon}{\epsilon}\approx_\epsilon-\frac{M_\epsilon}{2C_\alpha V'(0+)}\quad\text{locally uniformly in}\quad(0,x_0).$$

Indeed, given (4.17) and the fact that $\lim_{\epsilon\to0}\int_z^\infty u_\epsilon d\tilde z=1$ uniformly in $z\in(0,\alpha)$, it suffices to study the asymptotic of the integral $\int_0^x e^{\frac{2}{\epsilon^2}[V-\sup_{(0,x)}V]}dz$ as $\epsilon\to0$. Clearly, $\sup_{(0,x)}V=0$. Since $V(0+)>V(z)$ for all $z\in(0,x]$ and $V'(0+)=-\frac{b'(0)}{a'(0)}<0$, Laplace's method yields $\lim_{\epsilon\to0}\frac{2}{\epsilon^2}\int_0^x e^{\frac{2}{\epsilon^2}V}dz=-\frac{1}{V'(0+)}$, which is locally uniformly in $x\in(0,x_0)$. The limit follows.

4.4. **Proof of Theorem C.** (1) It follows from Lemma 2.6. (2) It follows from Lemma 4.4.

(3) The limits concerning $\lambda_{\epsilon,1}$ in (i) and (ii) follow from (4.11) and (4.17), respectively. It remains to show the limit of $\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$. Recall that $u_\epsilon=\frac{\phi_{\epsilon,1}u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}}$ and $u_\epsilon^G=\frac1a e^{-\frac{2}{\epsilon^2}V}$.

(i) As $\phi_{\epsilon,1}=\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\frac{u_\epsilon}{u_\epsilon^G}$ and $\|\phi_{\epsilon,1}\|_{L^2(u_\epsilon^G)}=1$, we see $\|\phi_{\epsilon,1}\|^2_{L^1(u_\epsilon^G)}=\left(\int_0^\infty\frac{u_\epsilon^2}{u_\epsilon^G}dx\right)^{-1}$, and thus,

$$\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}=\left(\int_0^\infty\frac{u_\epsilon^2}{u_\epsilon^G}dx\right)^{-1}\frac{u_\epsilon}{u_\epsilon^G}=\left(\int_0^\infty R_\epsilon u_\epsilon dx\right)^{-1}R_\epsilon,$$

where we used the fact $u_\epsilon=\frac{R_\epsilon}{\epsilon a}e^{-\frac{2}{\epsilon^2}v}=\frac{R_\epsilon}{\epsilon}e^{-\frac{2}{\epsilon^2}d_1}u_\epsilon^G$ ensured by the definition of R_ϵ and Theorem A. Since $R_\epsilon\approx_\epsilon M_\epsilon$ locally uniformly in $(0,\infty)$ by Theorem B (1), the result follows if we can show

$$\int_0^\infty R_\epsilon u_\epsilon dx\approx_\epsilon M_\epsilon.\tag{4.19}$$

Fix $K\gg1$. Corollary 4.1 ensures the existence of $\gamma>0$ such that $\int_{(0,\frac1K)\cup(K,\infty)}u_\epsilon dx\le e^{-\frac{\gamma}{\epsilon^2}}$. Hence, Theorem B (1) yields $\int_{\frac1K}^K R_\epsilon u_\epsilon dx\approx_\epsilon M_\epsilon\int_{\frac1K}^K u_\epsilon dx\approx_\epsilon M_\epsilon$. It is obvious from (4.9) that R_ϵ is increasing, leading to $\int_0^{\frac1K}R_\epsilon u_\epsilon dx\lesssim_\epsilon M_\epsilon\int_0^{\frac1K}u_\epsilon dx\lesssim_\epsilon M_\epsilon e^{-\frac{\gamma}{\epsilon^2}}$.

We claim that there is $\gamma'>0$ such that $\int_K^\infty R_\epsilon u_\epsilon dx\lesssim M_\epsilon e^{-\frac{\gamma'}{\epsilon^2}}$. Then, (4.19) follows.

It remains to prove the claim. Fix $1\ll L<K$. We distinguish between $y_\infty=\infty$ and $y_\infty<\infty$.

- If $y_\infty=\infty$, then Lemma 2.6 and $u_\epsilon(L)=\frac{R_\epsilon(L)}{\epsilon a(L)}e^{-\frac{2}{\epsilon^2}v(L)}$ give

$$u_\epsilon\le\frac{R_\epsilon(L)}{\epsilon[a(L)]^{\frac14}a^{\frac34}}e^{-\frac{2}{\epsilon^2}v(L)}e^{-\gamma_{\epsilon,L}[\xi-\xi(L)]}e^{\frac{1}{\epsilon^2}\int_L^\bullet\frac{b}{a}ds}\quad\text{in}\quad[L,\infty),$$

where $\gamma_{\epsilon,L}=\sqrt{\frac{2}{\epsilon^2}(\frac{C_L}{\epsilon^2}-\lambda_{\epsilon,1})}$. In which, $C_L:=C_2\inf_{[\xi(L),y_\infty)}\frac{b^2\circ\xi^{-1}}{a\circ\xi^{-1}}$, where $C_2=C_2(\xi(L))$ is given in Lemma 2.4 (2). It follows that

$$\begin{aligned}\int_K^\infty R_\epsilon u_\epsilon dx&\leq\int_K^\infty\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}\sqrt{a(x)}}e^{-\frac{4}{\epsilon^2}v(L)}e^{-2\gamma_{\epsilon,L}[\xi(x)-\xi(L)]}e^{\frac{2}{\epsilon^2}\int_L^x\frac{b}{a}ds}e^{\frac{2}{\epsilon^2}v(x)}dx\\&=\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}}e^{-\frac{4}{\epsilon^2}v(L)}e^{\frac{2}{\epsilon^2}v(L)}\int_K^\infty\frac{1}{\sqrt{a}}e^{-2\gamma_{\epsilon,L}[\xi-\xi(L)]}dx\\&=\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\int_{\xi(K)}^\infty e^{-2\gamma_{\epsilon,L}[y-\xi(L)]}dy\lesssim_\epsilon\frac{M_\epsilon^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\frac{1}{2\gamma_{\epsilon,L}}e^{-2\gamma_{\epsilon,L}[\xi(K)-\xi(L)]}.\end{aligned}$$

- If $y_\infty<\infty$, then Lemma 2.6 and $u_\epsilon(L)=\frac{R_\epsilon(L)}{\epsilon a(L)}e^{-\frac{2}{\epsilon^2}v(L)}$ give

$$u_\epsilon\leq\frac{R_\epsilon(L)}{\epsilon[a(L)]^{\frac14}a^{\frac34}}e^{-\frac{2}{\epsilon^2}v(L)}\frac{e^{\gamma_{\epsilon,L}[\xi-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-\xi]}}{e^{\gamma_{\epsilon,L}[\xi(L)-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-\xi(L)]}}e^{\frac{1}{\epsilon^2}\int_L^\bullet\frac{b}{a}ds}\quad\text{in}\quad[L,\infty),$$

where $\gamma_{\epsilon,L}$ is as above. It follows that

$$\begin{aligned}\int_K^\infty R_\epsilon u_\epsilon dx&\leq\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\int_{\xi(K)}^{y_\infty}\left(\frac{e^{\gamma_{\epsilon,L}[y-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-y]}}{e^{\gamma_{\epsilon,L}[\xi(L)-y_\infty]}-e^{\gamma_{\epsilon,L}[y_\infty-\xi(L)]}}\right)^2dy\\&=\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\int_{\xi(K)}^{y_\infty}\frac{e^{2\gamma_{\epsilon,L}[y_\infty-y]}}{e^{2\gamma_{\epsilon,L}[y_\infty-\xi(L)]}}\left(\frac{e^{2\gamma_{\epsilon,L}[y-y_\infty]}-1}{e^{2\gamma_{\epsilon,L}[\xi(L)-y_\infty]}-1}\right)^2dy\\&\leq\frac{[R_\epsilon(L)]^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\int_{\xi(K)}^{y_\infty}e^{-2\gamma_{\epsilon,L}[y-\xi(L)]}dy\lesssim_\epsilon\frac{M_\epsilon^2}{\epsilon\sqrt{a(L)}}e^{-\frac{2}{\epsilon^2}v(L)}\frac{1}{2\gamma_{\epsilon,L}}e^{-2\gamma_{\epsilon,L}[\xi(K)-\xi(L)]}.\end{aligned}$$

By the definition of M_ϵ, it is clear that $M_\epsilon\lesssim_\epsilon e^{\frac{\gamma}{\epsilon^2}}$ for any $\gamma>0$. Hence, $\int_K^\infty R_\epsilon u_\epsilon dx\lesssim_\epsilon M_\epsilon e^{-\frac{\gamma'}{\epsilon^2}}$ for some $\gamma'>0$, proving the claim.

(ii) As in the proof of (i), we calculate

$$\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}=\left(\int_0^\infty\frac{u_\epsilon^2}{u_\epsilon^G}dx\right)^{-1}\frac{u_\epsilon}{u_\epsilon^G}=\left(\int_0^\infty\frac{R_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}(-v+V)}u_\epsilon dx\right)^{-1}\frac{R_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}(-v+V)},\tag{4.20}$$

where we used the fact $u_\epsilon=\frac{R_\epsilon}{\epsilon a}e^{-\frac{2}{\epsilon^2}v}$ ensured by the definition of R_ϵ and Theorem A. Fix $K\gg1$. We claim that

$$\int_0^\infty\frac{R_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}(-v+V)}u_\epsilon dx\approx_\epsilon\int_{\alpha+\frac1K}^K\frac{R_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}(-v+V)}u_\epsilon dx.\tag{4.21}$$

Suppose (4.21) for the moment. Theorem A and Corollary 4.1 ensure

$$\int_0^{\alpha+\frac1K}u_\epsilon dx\leq e^{-\frac{2\gamma}{\epsilon^2}}\text{ for some }\gamma>0\quad\text{and}\quad\lim_{\epsilon\to0}\int_{\alpha+\frac1K}^K u_\epsilon=1,\tag{4.22}$$

and

$$-v(x)+V(x)=\begin{cases}-d_1+\sup_{(0,x)}V, & x\in(0,\alpha],\\-d_1+V(\alpha), & x\in[\alpha,\infty),\end{cases}\tag{4.23}$$

It follows from (4.21) and Theorem B (2) that

$$\int_0^\infty\frac{R_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}(-v+V)}u_\epsilon dx\approx_\epsilon\frac{M_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}[-d_1+V(\alpha)]}\int_{\alpha+\frac1K}^K u_\epsilon dx\approx_\epsilon\frac{M_\epsilon}{\epsilon}e^{\frac{2}{\epsilon^2}[-d_1+V(\alpha)]},$$

which together with (4.20) leads to

$$\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}(x) \approx_\epsilon \frac{R_\epsilon}{M_\epsilon} e^{\frac{2}{\epsilon^2}[d_1 - V(\alpha)]} e^{\frac{2}{\epsilon^2}[-v(x)+V(x)]} \approx_\epsilon \begin{cases} \frac{R_\epsilon}{M_\epsilon}, & x \in [\alpha,\infty), \\ \frac{R_\epsilon}{M_\epsilon} e^{\frac{2}{\epsilon^2}[-V(\alpha)+\sup_{(0,x)} V]}, & x \in (0,\alpha). \end{cases}$$

An application of Theorem B (2) then yields the desired result.

It remains to prove (4.21). Since $\lim_{\epsilon\to 0}\int_{\alpha+\frac{1}{K}}^K u_\epsilon dx = 1$, we see from Theorem B and (4.23) that

$$\int_{\alpha+\frac{1}{K}}^K \frac{R_\epsilon}{\epsilon} e^{\frac{2}{\epsilon^2}(-v+V)} u_\epsilon dx \approx_\epsilon \frac{M_\epsilon}{\epsilon} e^{\frac{2}{\epsilon^2}[-d_1+V(\alpha)]}. \tag{4.24}$$

Note that $\int_K^\infty \frac{R_\epsilon}{\epsilon} e^{\frac{2}{\epsilon^2}(-v+V)} u_\epsilon dx = \frac{1}{\epsilon} e^{\frac{2}{\epsilon^2}[-d_1+V(\alpha)]} \int_K^\infty R_\epsilon u_\epsilon dx$. Arguing as in (i) yields

$$\int_K^\infty R_\epsilon u_\epsilon dx \lesssim_\epsilon M_\epsilon e^{-\frac{\gamma'}{\epsilon^2}} \quad \text{for some } \gamma' > 0. \tag{4.25}$$

Given (4.17), we deduce from (4.18) and the fact $\sup_{(0,x)} V = V(\alpha)$ for $x \in [\alpha, \alpha+\frac{1}{K}]$ that for any fixed $0<\gamma_1<\gamma$,

$$R_\epsilon(x) \le \frac{2\lambda_{\epsilon,1}}{\epsilon} e^{\frac{2}{\epsilon^2}d_1} \int_0^x e^{\frac{2}{\epsilon^2}[V(z)-\sup_{(0,x)} V]} dz \lesssim_\epsilon \frac{M_\epsilon}{\epsilon C_\alpha} e^{\frac{2\gamma_1}{\epsilon^2}} \text{ uniformly in } x \in [0,\alpha+\delta],$$

which together with (4.23) and (4.22) leads to

$$\begin{aligned} \int_0^{\alpha+\frac{1}{K}} \frac{R_\epsilon}{\epsilon} e^{\frac{2}{\epsilon^2}(-v+V)} u_\epsilon dx &\lesssim_\epsilon \frac{M_\epsilon}{\epsilon^2 C_\alpha} e^{\frac{2\gamma_1}{\epsilon^2}} \int_0^{\alpha+\frac{1}{K}} e^{\frac{2}{\epsilon^2}[-d_1+\sup_{(0,x)} V]} dz \\ &\lesssim_\epsilon \frac{M_\epsilon}{\epsilon^2 C_\alpha} e^{\frac{2}{\epsilon^2}[\gamma_1 - d_1 + V(\alpha)]} \int_0^{\alpha+\frac{1}{K}} u_\epsilon dx \le \frac{M_\epsilon}{\epsilon^2 C_\alpha} e^{\frac{2}{\epsilon^2}[-d_1+V(\alpha)]} e^{-\frac{2}{\epsilon^2}(\gamma-\gamma_1)}. \end{aligned}$$

This together with (4.24) and (4.25) leads to (4.21).

5. Multiscale dynamics

This section is devoted to the investigation of the multiscale dynamics of X_t^ϵ. We prove Theorem D in Subsection 5.1, and Theorems E and F in Subsection 5.2.

5.1. Asymptotic reciprocal relationship.

In this subsection, we establish the asymptotic distribution of the normalized extinction time and the asymptotic reciprocal relationship between the mean extinction time $\mathbb{E}_\bullet^\epsilon[T_0^\epsilon]$ and the principal eigenvalue $\lambda_{\epsilon,1}$. That is, we prove Theorem D.

Proof of Theorem D. By Lemma 2.1 (6),

$$\mathbb{P}_\mu^\epsilon[t' < T_0^\epsilon] = e^{-\lambda_{\epsilon,1}t'}\langle \mu, \alpha_{\epsilon,1}\rangle + \int_0^\infty P_{t'}^\epsilon Q_2^\epsilon \mathbb{1} d\mu, \quad \forall t' > 0,$$

where $\alpha_{\epsilon,1} := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$ and $\langle \mu, \alpha_{\epsilon,1}\rangle := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_0^\infty \phi_{\epsilon,1} d\mu$. In particular, setting $t' = t\lambda_{\epsilon,1}^{-1}$ for $t>0$ yields

$$\mathbb{P}_\mu^\epsilon[t\lambda_{\epsilon,1}^{-1} < T_0^\epsilon] = e^{-t}\langle \mu, \alpha_{\epsilon,1}\rangle + \int_0^\infty P_{t\lambda_{\epsilon,1}^{-1}}^\epsilon Q_2^\epsilon \mathbb{1} d\mu. \tag{5.1}$$

Thanks to Lemma 2.2 (2), we find $\gamma > 0$ (independent of ϵ),

$$\left| \int_0^\infty P_{t\lambda_{\epsilon,1}^{-1}}^\epsilon Q_2^\epsilon \mathbb{1} d\mu \right| \le e^{-\frac{\lambda_{\epsilon,2}}{\lambda_{\epsilon,1}}t} \int_0^\infty a^{\frac{1}{4}} e^{\frac{V+\gamma}{\epsilon^2}} d\mu \le C_\mu e^{-\frac{\lambda_{\epsilon,2}}{\lambda_{\epsilon,1}}t} e^{\frac{1}{\epsilon^2}(\sup_{\mathrm{supp}(\mu)} V + \gamma)}, \tag{5.2}$$

where $C_\mu = \sup_{\text{supp}(\mu)} a^{\frac{1}{4}}$. The uniqueness of d_1-vallys and Lemma 2.3 guarantee $\lim_{\epsilon\to 0}\ln\frac{\lambda_{\epsilon,2}}{\lambda_{\epsilon,1}} = d_1 - d_2 > 0$, resulting in $\lim_{\epsilon\to 0}\int_0^\infty P^\epsilon_{t\lambda^{-1}_{\epsilon,1}} Q^\epsilon_2 \mathbb{1} d\mu = 0$. Letting $\epsilon \to 0$ in (5.1), we derive the limit $\lim_{\epsilon\to 0}\mathbb{P}^\epsilon_\mu[t\lambda^{-1}_{\epsilon,1} < T^\epsilon_0]$ from Theorem C.

Now, we study $\lim_{\epsilon\to 0}\lambda_{\epsilon,1}\mathbb{E}^\epsilon_\mu[T^\epsilon_0]$. Note that for $t_0 > 0$,

$$\lambda_{\epsilon,1}\mathbb{E}^\epsilon_\mu[T^\epsilon_0] = \lambda_{\epsilon,1}\int_0^{t_0\lambda^{-1}_{\epsilon,1}} \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]dt + \lambda_{\epsilon,1}\int_{t_0\lambda^{-1}_{\epsilon,1}}^\infty \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]dt. \tag{5.3}$$

Obviously, $\lim_{t_0\to 0}\sup_\epsilon \lambda_{\epsilon,1}\int_0^{t_0\lambda^{-1}_{\epsilon,1}} \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]dt = 0$. Integrating (5.1) over (t_0,∞) yields

$$\lambda_{\epsilon,1}\int_{t_0\lambda^{-1}_{\epsilon,1}}^\infty \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]dt = \int_{t_0}^\infty \mathbb{P}^\epsilon_\mu[t\lambda^{-1}_{\epsilon,1} < T^\epsilon_0]dt = e^{-t_0}\langle\mu,\alpha_{\epsilon,1}\rangle + \int_{t_0}^\infty\int_0^\infty P^\epsilon_{t\lambda^{-1}_{\epsilon,1}}Q^\epsilon_2\mathbb{1}d\mu dt,$$

which together with (5.2) leads to

$$\left|\lambda_{\epsilon,1}\int_{t_0\lambda^{-1}_{\epsilon,1}}^\infty \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]dt - e^{-t_0}\langle\mu,\alpha_{\epsilon,1}\rangle\right| \le C_\mu\frac{\lambda_{\epsilon,1}}{\lambda_{\epsilon,2}}e^{-\frac{\lambda_{\epsilon,2}}{\lambda_{\epsilon,1}}t_0}e^{\frac{1}{\epsilon^2}(\sup_{\text{supp}(\mu)}V+\gamma)} \to 0 \quad \text{as} \quad \epsilon\to 0.$$

Letting $\epsilon \to 0$ and then $t_0 \to 0$ in (5.3), we conclude $\lim_{\epsilon\to 0}\lambda_{\epsilon,1}\mathbb{E}^\epsilon_\mu[T^\epsilon_0] = \lim_{\epsilon\to 0}\langle\mu,\alpha_{\epsilon,1}\rangle$. The results follow readily from Theorem C. □

5.2. **Multiscale estimate.** In this subsection, we establish the multiscale estimate of the dynamics of X^ϵ_t. In particular, we prove Theorems E and F.

The proof of Theorem E is in need of the following lemma regarding the boundedness of coefficients appearing in the expansion of semigroup P^ϵ_t given in (2.1).

Lemma 5.1. *If $k \in \mathbb{N}$ is such that $d_1 > d_2 > \cdots > d_k > d_{k+1}$, then for each compact $K \subset (0,\infty)$, there exist $C = C(k,K) > 0$ and $\epsilon_* = \epsilon_*(k,K) > 0$ such that*

$$\sup_{\substack{\mu\in\mathcal{P}((0,\infty))\\ \text{supp}(\mu)\subset K}}\ \sup_{0<\epsilon<\epsilon_*}\left|\langle f,\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\int_0^\infty \phi_{\epsilon,i}d\mu\right| \le C\|f\|_\infty$$

for all $f \in C_b([0,\infty))$ and $i \in \{1,\dots,k\}$.

Proof. Set $\Lambda_{\epsilon,i}(f,\mu) := \langle f,\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\int_0^\infty \phi_{\epsilon,i}d\mu$. Let k and K be as in the statement, $\mu \in \mathcal{P}((0,\infty))$ satisfy $\text{supp}(\mu)\subset K$, and $f \in C_b([0,\infty))$. By Lemma 2.1 (6), we have for $\ell \in \{1,\dots,k\}$,

$$\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}] = \sum_{i=1}^\ell e^{-\lambda_{\epsilon,i}t}\Lambda_{\epsilon,i}(f,\mu) + \int_0^\infty P^\epsilon_t Q^\epsilon_{\ell+1}f d\mu, \quad \forall t > 0. \tag{5.4}$$

We first estimate $\Lambda_{\epsilon,1}(f,\mu)$. Setting $\ell = 1$ in (5.4) gives

$$|\Lambda_{\epsilon,1}(f,\mu)| = e^{\lambda_{\epsilon,1}t}\left|\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}] - \int_0^\infty P^\epsilon_t Q^\epsilon_2 f d\mu\right| \le e^{\lambda_{\epsilon,1}t}\left(\|f\|_\infty + \int_0^\infty |P^\epsilon_t Q^\epsilon_2 f|d\mu\right), \quad \forall t > 0.$$

An application of Lemma 2.2 (2) yields the existence of $\gamma'_2 > 0$ such that

$$\int_0^\infty |P^\epsilon_t Q^\epsilon_2 f|d\mu \le e^{-\lambda_{\epsilon,2}t}\|f\|_\infty\int_0^\infty a^{\frac{1}{4}}e^{\frac{1}{\epsilon^2}\left(V+\gamma'_2\right)}d\mu \le e^{-\lambda_{\epsilon,2}t+\frac{\gamma_2}{\epsilon^2}}\|f\|_\infty, \quad \forall t > 2,$$

where $\gamma_2 := \max_K V + \gamma'_2 + 1$. Thus, $|\Lambda_{\epsilon,1}(f,\mu)| \le \left[e^{\lambda_{\epsilon,1}t} + e^{-(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})t+\frac{\gamma_2}{\epsilon^2}}\right]\|f\|_\infty$ for all $t > 2$. Setting $t = \frac{\gamma_2+1}{(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})\epsilon^2}$ yields $|\Lambda_{\epsilon,1}(f,\mu)| \le \left[\exp\left\{\frac{\lambda_{\epsilon,1}(\gamma_2+1)}{(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})\epsilon^2}\right\} + e^{-\frac{1}{\epsilon^2}}\right]\|f\|_\infty$. Note that Lemma 2.3

and $d_1 > d_2$ ensure $\lim_{\epsilon\to 0}\frac{\lambda_{\epsilon,1}(\gamma_2+1)}{(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})\epsilon^2}=0$. We then see the existence of $\epsilon_1=\epsilon_1(K)>0$ such that $|\Lambda_{\epsilon,1}(f,\mu)|\leq 2\|f\|_\infty$ for all $0<\epsilon<\epsilon_1$.

Next, we treat $\Lambda_{\epsilon,2}(f,\mu)$ if $k\geq 2$. It follows from (5.4) with $\ell=2$ and $|\Lambda_{\epsilon,1}(f,\mu)|\leq 2\|f\|_\infty$ that

$$|\Lambda_{\epsilon,2}(f,\mu)|\leq e^{\lambda_{\epsilon,2}t}\left(3\|f\|_\infty+\int_0^\infty |P_t^\epsilon Q_3^\epsilon f|d\mu\right),\quad \forall t>2.$$

Since $d_2>d_3$ implies that $\lambda_{\epsilon,2}$ is exponentially smaller than $\lambda_{\epsilon,3}$ (by Lemma 2.3), we can argue as above to conclude the existence of $\epsilon_2=\epsilon_2(K)\in(0,\epsilon_1)$ such that $\Lambda_{\epsilon,2}(f,\mu)\leq 4\|f\|_\infty$ for all $0<\epsilon<\epsilon_2$.

Following the above arguments, we see that for each $i\in\{2,\dots,k\}$, establishing the upper bound for $|\Lambda_{\epsilon,i}(f,\mu)|$ requires the condition $d_i>d_{i+1}$ and the upper bound for $|\Lambda_{\epsilon,i-1}(f,\mu)|$. Hence, we conclude the lemma by repeating the above procedure. □

Now, we prove Theorem E.

Proof of Theorem E. (1) Let k and K be as in the statement, $\mu\in\mathcal{P}((0,\infty))$ satisfy $\mathrm{supp}(\mu)\subset K$, and $f\in C_b([0,\infty))$. We pretend $\langle \mathbb{1},\phi_{\epsilon,i}\rangle_{L^2(u_\epsilon^G)}\neq 0$ so that $\mu_{\epsilon,i}$ is well-defined. By Lemma 2.1 (6),

$$\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)\mathbb{1}_{t<T_0^\epsilon}]=\sum_{i=1}^k e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle\int_0^\infty f d\mu_{\epsilon,i}+\int_0^\infty P_t^\epsilon Q_{k+1}^\epsilon f d\mu,\quad \forall t>0, \tag{5.5}$$

where $\alpha_{\epsilon,i}$, $\langle\mu,\alpha_{\epsilon,i}\rangle$ and $\mu_{\epsilon,i}$ are as in the statement. In particular, $\mu_{\epsilon,1}=\mu_\epsilon$. Noting that $\alpha_{\epsilon,1}=\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$, we apply Theorem C (1) to find $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=1$.

The bound for $\langle\mu,\alpha_{\epsilon,i}\rangle$ is proven in Lemma 5.1, which also yields the existence of $C=C(k,K)>0$ and $\epsilon_1=\epsilon_1(k,K)>0$ such that $\sup_{0<\epsilon<\epsilon_1}\left|\langle\mu,\alpha_{\epsilon,i}\rangle\int_0^\infty f d\mu_{\epsilon,i}\right|\leq C\|f\|_\infty$, leading to $\|\langle\mu,\alpha_{\epsilon,i}\rangle\mu_{\epsilon,i}\|_{TV}\leq C$ for all $i\in\{1,\dots,k\}$.

Lemma 2.2 (2) yields the existence of $\gamma'_{k+1}>0$ such that

$$\left|\int_0^\infty P_t^\epsilon Q_{k+1}^\epsilon f d\mu\right|\leq e^{-\lambda_{\epsilon,k+1}t}\|f\|_\infty\int_0^\infty a^{\frac14}e^{\frac{1}{\epsilon^2}(V+\gamma'_{k+1})}d\mu,\quad\forall t>2.$$

Set $\gamma_{k+1}:=\max_K V+\gamma'_{k+1}+1$. Clearly, there is $\epsilon_2=\epsilon_2(k,K)>0$ such that for $0<\epsilon<\epsilon_2$,

$$\left|\int_0^\infty P_t^\epsilon Q_{k+1}^\epsilon f d\mu\right|\leq e^{-\lambda_{\epsilon,k+1}t+\frac{\gamma_{k+1}}{\epsilon^2}}\|f\|_\infty,\quad\forall t>2. \tag{5.6}$$

Setting $f\equiv 1$ in (5.5) yields $\mathbb{P}_\mu^\epsilon[t<T_0^\epsilon]=\sum_{i=1}^k e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle+\int_0^\infty P_t^\epsilon Q_{k+1}^\epsilon\mathbb{1}d\mu$, which together with (5.5) gives for $t>0$,

$$\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)]=\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)\mathbb{1}_{t<T_0^\epsilon}]+f(0)(1-\mathbb{P}_\mu^\epsilon[t<T_0^\epsilon])=\mu_*(f)+\int_0^\infty P_t^\epsilon Q_{k+1}^\epsilon\left[f-f(0)\right]d\mu,$$

where $\mu_*(f):=\sum_{i=1}^k e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle\int_0^\infty fd\mu_{\epsilon,i}+\left(1-\sum_{i=1}^k e^{-\lambda_{\epsilon,i}t}\langle\mu,\alpha_{\epsilon,i}\rangle\right)f(0)$. It then follows from (5.6) that $\left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)]-\mu_*(f)\right|\leq 2e^{-\lambda_{\epsilon,k+1}t+\frac{\gamma_{k+1}}{\epsilon^2}}\|f\|_\infty$ for all $t>2$ and $0<\epsilon<\epsilon_*:=\min\{\epsilon_1,\epsilon_2\}$. The conclusion follows.

(2) We only need to show $\lim_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle=p_\mu$. Recall that $\alpha_{\epsilon,1}=\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$. Since μ is compactly supported in $(0,\infty)$, Theorem C (2) implies that for any $0<\delta\ll1$,

$$\lim_{\epsilon\to0}\int_{(0,\alpha-\delta]}\alpha_{\epsilon,1}d\mu=0,\quad \lim_{\epsilon\to0}\alpha_{\epsilon,1}(\alpha)=\frac12,\quad \lim_{\epsilon\to0}\int_{[\alpha+\delta,\infty)}\alpha_{\epsilon,1}d\mu=\mu([\alpha+\delta,\infty)). \tag{5.7}$$

Thus, $\liminf_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle\geq\frac12\mu(\{\alpha\})+\mu([\alpha+\delta,\infty))$ for $0<\delta\ll1$, leading to $\liminf_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle\geq p_\mu$.

To show the reverse inequality, we fix $0<\delta_*\ll 1$ and set for each $0<\delta<\delta_*$,

$$\mu_\delta:=\begin{cases}\frac{\mu|_{(\alpha-\delta,\alpha+\delta)\setminus\{\alpha\}}}{\mu((\alpha-\delta,\alpha+\delta)\setminus\{\alpha\})}, & \text{if}\quad \mu((\alpha-\delta,\alpha+\delta)\setminus\{\alpha\})\neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, either $\operatorname{supp}(\mu_\delta)\subset(\alpha-\delta_*,\alpha+\delta_*)$ or $\operatorname{supp}(\mu_\delta)=\emptyset$ for each $0<\delta<\delta_*$. Hence, an application of Lemma 5.1 yields the existence of $M=M(\delta_*)>0$ such that $\sup_\epsilon\sup_{0<\delta<\delta_*}\langle\mu_\delta,\alpha_{\epsilon,1}\rangle\leq M$. Therefore,

$$\sup_\epsilon\int_{(\alpha-\delta,\alpha+\delta)\setminus\{\alpha\}}\alpha_{\epsilon,1}d\mu\leq M\mu((\alpha-\delta,\alpha+\delta)\setminus\{\alpha\}),\quad\forall 0<\delta<\delta_*,$$

which together with (5.7) results in

$$\begin{aligned}\limsup_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle&=\limsup_{\epsilon\to0}\left[\left(\int_{(0,\alpha-\delta]}+\int_{(\alpha-\delta,\alpha+\delta)\setminus\{\alpha\}}+\int_{[\alpha+\delta,\infty)}\right)\alpha_{\epsilon,1}d\mu+\mu(\{\alpha\})\alpha_{\epsilon,1}(\alpha)\right]\\&\leq M\mu((\alpha-\delta,\alpha+\delta)\setminus\{\alpha\})+\mu([\alpha+\delta,\infty))+\frac{1}{2}\mu(\{\alpha\}),\quad\forall 0<\delta<\delta_*.\end{aligned}$$

Letting $\delta\to0^+$ yields $\limsup_{\epsilon\to0}\langle\mu,\alpha_{\epsilon,1}\rangle\leq p_\mu$. This completes the proof. □

We prove Theorem F in the rest of this section. The following result, confirming Theorem F over a shorter time scale, is needed.

Proposition 5.1. *Assume* **(H)**, $\{x\in(0,\infty):b(x)=0\}=\{x_*\}$ *and* $b'(x_*)<0$. *Let* $w:[0,\infty)\to[0,\infty)$ *be a modulus of continuity. Then, for each compact* $K\subset(0,\infty)$, $M>0$ *and sequences* $\{\underline{t}_\epsilon\}_\epsilon$, $\{t_\epsilon\}_\epsilon$ *in* $(0,\infty)$ *satisfying* $\underline{t}_\epsilon<t_\epsilon$ *for each* ϵ, $\lim_{\epsilon\to0}\underline{t}_\epsilon=\infty$ *and* $\lim_{\epsilon\to0}t_\epsilon e^{-\frac{\gamma}{\epsilon^2}}=0$ *for each* $\gamma>0$, *there holds*

$$\lim_{\epsilon\to0}\sup_{\operatorname{supp}(\mu)\subset K}\sup_{\underline{t}_\epsilon\leq t\leq t_\epsilon}\sup_{\substack{f\in w[x_*]\\ \|f\|_\infty\leq M}}\left|\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)]-\int_0^\infty fd\mu_\epsilon\right|=0.$$

Proof. Let K, $\underline{t}_\epsilon$, t_ϵ and w be as in the statement. Let $\mu\in\mathcal{P}((0,\infty))$ have compact support in K. Fix $0<\delta\ll1$. Let $f\in C_b([0,\infty))$ have modulus of continuity w at x_*. We write

$$\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)]=\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}]+f(0)\left(1-\mathbb{P}^\epsilon_\mu[t<T^\epsilon_0]\right)=:E^\epsilon_1(t)+E^\epsilon_2(t),\quad\forall t\geq0.$$

We first treat $E^\epsilon_1(t)$. Obviously, x_* is the global asymptotic stable equilibrium of $\dot{x}=b(x)$ in $(0,\infty)$, generating the flow φ^t. Hence, $\varphi^t(K)\subset\mathcal{A}_{\frac{\delta}{2}}:=(x_*-\frac{\delta}{2},x_*+\frac{\delta}{2})$ for all $t\geq\underline{t}_\epsilon$. An application of the sample path LDP (see e.g. [31]) yields the existence of $\gamma_1>0$ such that

$$\inf_{x\in K}\mathbb{P}^\epsilon_x[X^\epsilon_{\underline{t}_\epsilon}\in\mathcal{A}_\delta]\geq1-e^{-\frac{\gamma_1}{\epsilon^2}}.\tag{5.8}$$

Denote by $\mu^\epsilon_{\underline{t}_\epsilon}$ the distribution of $X^\epsilon_{\underline{t}_\epsilon}$ with $X^\epsilon_0\sim\mu$. Then, the strong Markov property and time-homogeneity of X^ϵ_t ensure that

$$\begin{aligned}E^\epsilon_1(t)&=\mathbb{E}^\epsilon_\mu\left[\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}]|X^\epsilon_{\underline{t}_\epsilon}]\right]\\&=\int_{\mathcal{A}_\delta}\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_{t-\underline{t}_\epsilon})\mathbb{1}_{t-\underline{t}_\epsilon<T^\epsilon_0}]d\mu^\epsilon_{\underline{t}_\epsilon}+\int_{(0,\infty)\setminus\mathcal{A}_\delta}\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_{t-\underline{t}_\epsilon})\mathbb{1}_{t-\underline{t}_\epsilon<T^\epsilon_0}]d\mu^\epsilon_{\underline{t}_\epsilon}\\&=:E^\epsilon_{11}(t)+E^\epsilon_{12}(t),\quad\forall t\geq\underline{t}_\epsilon.\end{aligned}$$

It follows from (5.8) that

$$|E^\epsilon_{12}(t)|\leq e^{-\frac{\gamma_1}{\epsilon^2}}\|f\|_\infty\leq\delta\|f\|_\infty,\quad\forall t\geq\underline{t}_\epsilon.\tag{5.9}$$

Applying [31, Theorem 4.4.2], we find $\gamma_2, \gamma_3 > 0$ such that $\lim_{\epsilon\to 0}\sup_{x\in\mathcal{A}_\delta}\mathbb{P}^\epsilon_x[e^{\frac{\gamma_2}{\epsilon^2}} < T^\epsilon_{\mathcal{A}_{2\delta}} < e^{\frac{\gamma_3}{\epsilon^2}}] = 1$, where $T^\epsilon_{\mathcal{A}_{2\delta}} := \inf\{t \geq 0 : X^\epsilon_t \notin \mathcal{A}_{2\delta}\}$ is the first exit time from $\mathcal{A}_{2\delta}$. As t_ϵ grows sub-exponentially in ϵ^2 to ∞ as $\epsilon \to 0$, we find

$$\sup_\epsilon \sup_{\underline{t}_\epsilon \leq t \leq t_\epsilon} \sup_{x\in\mathcal{A}_\delta} \mathbb{P}^\epsilon_x[t \geq T^\epsilon_{\mathcal{A}_{2\delta}}] \leq \delta. \tag{5.10}$$

Note that for any $t \geq \underline{t}_\epsilon$,

$$\begin{aligned} E^\epsilon_{11}(t) - f(x_*) &= \int_{\mathcal{A}_\delta} E^\epsilon_\bullet[f(X^\epsilon_{t-\underline{t}_\epsilon})\mathbb{1}_{t-\underline{t}_\epsilon < T^\epsilon_{\mathcal{A}_{2\delta}}}]d\mu^\epsilon_{\underline{t}_\epsilon} + \int_{\mathcal{A}_\delta} E^\epsilon_\bullet[f(X^\epsilon_{t-\underline{t}_\epsilon})\mathbb{1}_{T^\epsilon_{\mathcal{A}_{2\delta}}\leq t-\underline{t}_\epsilon < T^\epsilon_0}]d\mu^\epsilon_{\underline{t}_\epsilon} - f(x_*) \\ &= \int_{\mathcal{A}_\delta} E^\epsilon_\bullet\left[(f(X^\epsilon_{t-\underline{t}_\epsilon}) - f(x_*))\mathbb{1}_{t-\underline{t}_\epsilon < T^\epsilon_{\mathcal{A}_{2\delta}}}\right]d\mu^\epsilon_{\underline{t}_\epsilon} + \int_{\mathcal{A}_\delta} E^\epsilon_\bullet[f(X^\epsilon_{t-\underline{t}_\epsilon})\mathbb{1}_{T^\epsilon_{\mathcal{A}_{2\delta}}\leq t-\underline{t}_\epsilon < T^\epsilon_0}]d\mu^\epsilon_{\underline{t}_\epsilon} \\ &\quad - f(x_*)\int_{\mathcal{A}_\delta} \mathbb{P}^\epsilon_\bullet[t-\underline{t}_\epsilon \geq T^\epsilon_{\mathcal{A}_{2\delta}}]d\mu^\epsilon_{\underline{t}_\epsilon} - f(x_*)\mu^\epsilon_{\underline{t}_\epsilon}((0,\infty)\setminus\mathcal{A}_\delta). \end{aligned}$$

We find from (5.8) and (5.10) that

$$\begin{aligned} |E^\epsilon_{11}(t) - f(x_*)| &\leq w(2\delta) + (\|f\|_\infty + |f(x_*)|)\int_{\mathcal{A}_\delta} \mathbb{P}^\epsilon_\bullet[t \geq T^\epsilon_{\mathcal{A}_{2\delta}}]d\mu^\epsilon_{\underline{t}_\epsilon} + |f(x_*)|e^{-\frac{\gamma_1}{\epsilon^2}} \\ &\leq w(2\delta) + 3\|f\|_\infty\delta, \quad \forall \underline{t}_\epsilon \leq t \leq t_\epsilon. \end{aligned} \tag{5.11}$$

Thanks to Corollary 4.1,

$$\left|f(x_*) - \int_0^\infty f d\mu_\epsilon\right| \leq \int_{\mathcal{A}_\delta} |f(x_*) - f(x)|d\mu_\epsilon + \int_{(0,\infty)\setminus\mathcal{A}_\delta} |f(x_*) - f(x)|d\mu_\epsilon \leq w(\delta) + 2\|f\|_\infty\delta,$$

which together with (5.9) and (5.11) leads to

$$\begin{aligned} \left|E^\epsilon_1(t) - \int_0^\infty f d\mu_\epsilon\right| &\leq |E^\epsilon_{11}(t) - f(x_*)| + \left|f(x_*) - \int_0^\infty f d\mu_\epsilon\right| + |E^\epsilon_{12}(t)| \\ &\leq 2w(2\delta) + 6\|f\|_\infty\delta, \quad \forall \underline{t}_\epsilon \leq t \leq t_\epsilon. \end{aligned} \tag{5.12}$$

Now, we treat $E_2(t)$. The proof of Theorem E implies the existence of $\gamma_4 > 0$ such that

$$\left|\mathbb{P}^\epsilon_\mu[t < T^\epsilon_0] - e^{-\lambda_{\epsilon,1}t}\langle\mu, \alpha_{\epsilon,1}\rangle\right| \leq e^{\frac{\gamma_4}{\epsilon^2} - \lambda_{\epsilon,2}t}, \quad \forall t > 2. \tag{5.13}$$

Since $\lim_{\epsilon\to 0}\frac{\epsilon^2}{2}\ln\lambda_{\epsilon,1} = -d_1 < 0$ (see Lemma 2.3), $\lim_{\epsilon\to 0}\lambda_{\epsilon,2} = -b'(x_*) > 0$ (see [45, Theorem B]) and $\lim_{\epsilon\to 0}\langle\mu, \alpha_{\epsilon,1}\rangle = 1$ (see Theorem E), we deduce $\mathbb{P}^\epsilon_\mu[t < T^\epsilon_0] \geq 1 - 2\delta$ for all $\frac{1}{\epsilon^2} \ll t \ll e^{\frac{2}{\epsilon^2}d_1}$. It follows from the monotonicity of $t \mapsto \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]$ that $\mathbb{P}^\epsilon_\mu[t < T^\epsilon_0] \geq 1 - 2\delta$ for all $0 \leq t \leq t_\epsilon$. As a result, $|E^\epsilon_2(t)| \leq |f(0)|\left(1 - \mathbb{P}^\epsilon_\mu[t < T^\epsilon_0]\right) \leq 2\delta\|f\|_\infty$ for all $0 \leq t \leq t_\epsilon$. This together with (5.12) yields

$$\left|\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)] - \int_0^\infty f d\mu_\epsilon\right| \leq 2w(2\delta) + 8\|f\|_\infty\delta, \quad \forall \underline{t}_\epsilon \leq t \leq t_\epsilon.$$

The desired result then follows from the arbitrariness of $0 < \delta \ll 1$. □

Now, Theorem F is almost a direct consequence of Corollary E (1) and Proposition 5.1.

Proof of Theorem F. Let K, $\underline{t}_\epsilon$, $\bar{t}_\epsilon$ and w be as in the statement. Let t_ϵ satisfy $\lim_{\epsilon\to 0}\epsilon^2 t_\epsilon = \infty$ and $\lim_{\epsilon\to 0} t_\epsilon e^{-\frac{\gamma}{\epsilon^2}} = 0$ for each $\gamma > 0$. We may assume without loss of generality that $\underline{t}_\epsilon < t_\epsilon < \bar{t}_\epsilon$.

A direct application of Proposition 5.1 yields the conclusion over the time scale $[\underline{t}_\epsilon, t_\epsilon]$. It follows from Corollary C (1) that $\lambda_{\epsilon,1} \approx_\epsilon \frac{C}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds}$, where $C = \frac{b'(0)}{a'(0)}\sqrt{-\frac{b(x_*)}{\pi a(x_*)}}$, and from [45, Theorem B] that $\lim_{\epsilon\to 0}\lambda_{\epsilon,2} = -b'(x_*) > 0$. We then apply Corollary E (1) to arrive at the conclusion over the time scale $[t_\epsilon, \bar{t}_\epsilon]$. The theorem follows. □

6. Applications

This section is devoted to some applications of our main results. In Subsection 6.1, we roughly discuss about diffusion approximations leading to SDEs of the form (1.1). In Subsection 6.2, we study logistic diffusion processes arising particularly from chemical reactions and population dynamics. Subsection 6.3 is devoted the diffusion approximation of QSDs.

6.1. Diffusion approximation and SDE.

In this subsection, we briefly review the diffusion approximation giving rise to SDEs of the form (1.1), and present the associated Fokker-Planck equation (or Kolmogorov forward equation) as well as the Kolmogorov backward equation.

Denote by $\mathbb{N}_0$ the set of non-negative integers and consider a continuous-time Markov jump process Z_t^V on $\frac{\mathbb{N}_0}{V} := \left\{\frac{n}{V} : n \in \mathbb{N}_0\right\}$ with transition rates $q_V(\cdot,\cdot)$, where $V \gg 1$ is a scaling parameter. Note that the notation V has been used for the potential function defined in (1.4), but this should not cause any confusion. The main reason for using V as the parameter here is to follow the convention or tradition, as V is often interpreted as the generalized volume.

We assume for simplicity that for each $m \in \mathbb{Z}\setminus\{0\}$, there is $b_m : [0,\infty) \to [0,\infty)$ such that

$$q_V\left(\frac{n}{V}, \frac{n+m}{V}\right) = V b_m\left(\frac{n}{V}\right), \quad \forall n \in \mathbb{N}_0, \quad n+m \in \mathbb{N}_0. \tag{6.1}$$

In treating chemical reaction systems, the condition (6.1) needs to be replaced by a limiting condition:

$$\frac{1}{V} q_V\left(\frac{n}{V}, \frac{n+m}{V}\right) - b_m\left(\frac{n}{V}\right) \to 0 \quad \text{as} \quad V \to \infty$$

with certain uniformity with respect to $n \in \mathbb{N}_0$ and $n+m \in \mathbb{N}_0$. Under appropriate assumptions on b_m, $m \in \mathbb{Z}\setminus\{0\}$, the central limit theorem (see e.g. [49, 27]) ensures that as $V \to \infty$, Z_t^V converges to solutions X_t^ϵ of (1.1) with $\epsilon = \frac{1}{\sqrt{V}}$, $b = \sum_{m\neq 0} m b_m$ and $a = \sum_{m \neq 0} m^2 b_m$. More precisely, if $\lim_{V\to\infty} Z_0^V = x_0 = X_0^\epsilon$, then for any $T > 0$, $\sup_{t\in[0,T]} |Z_t^V - X_t^\epsilon| \to 0$ in probability as $V \to \infty$.

A convenient way to see that the process Z_t^V approaches a diffusion process of the form (1.1) as $V \to \infty$ is to examine the closeness between their generators. Note that the generator $\mathcal{L}_V$ of Z_t^V reads

$$\mathcal{L}_V \phi\left(\frac{n}{V}\right) = \sum_{m\neq 0} q_V\left(\frac{n}{V}, \frac{n+m}{V}\right)\left[\phi\left(\frac{n+m}{V}\right) - \phi\left(\frac{n}{V}\right)\right], \quad n \in \mathbb{N}_0.$$

We expand $\mathcal{L}_V\phi\left(\frac{n}{V}\right)$ up to the second order and use (6.1) to find

$$\mathcal{L}_V \phi\left(\frac{n}{V}\right) \approx b\left(\frac{n}{V}\right)\phi'\left(\frac{n}{V}\right) + \frac{1}{2V} a\left(\frac{n}{V}\right)\phi''\left(\frac{n}{V}\right),$$

which corresponds to the generator of the diffusion process (1.1) with $\epsilon = \frac{1}{\sqrt{V}}$, that is, the second-order differential operator $\phi \mapsto \frac{\epsilon^2}{2} a\phi'' + b\phi'$. Its formal L^2-adjoint, namely, $\phi \mapsto \frac{\epsilon^2}{2}(a\phi)'' - (b\phi)'$, is called the *Fokker-Planck operator*.

The *Kolmogorov backward equation* and *Fokker-Planck equation* (or *Kolmogorov forward equation*) associated with (1.1) are respectively given by $u_t = \frac{\epsilon^2}{2} a u_{xx} + b u_x$ and $u_t = \frac{\epsilon^2}{2}(au)_{xx} - (bu)_x$. They respectively govern the dynamics of observables and the evolution of distributions of X_t^ϵ.

6.2. Logistic diffusion processes.

Consider the following family of SDEs:

$$dx = (b_1 x - b_2 x^2)dt + \epsilon\sqrt{a_1 x + a_2 x^2}\, dW_t, \quad x \in [0,\infty), \tag{6.2}$$

where $0 < \epsilon \ll 1$ is a parameter, b_1, b_2 and a_1 are positive constants, and $a_2 \geq 0$. We roughly describe two typical situations giving rise to (6.2) via diffusion approximations (see Subsection 6.1).

Chemical reactions. Consider the following chemical reactions:

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \quad X \xrightarrow{k_2} C, \tag{6.3}$$

where k_1, k_{-1} and k_2 are reaction rates. The concentration of A molecules, denoted by x_A, is assumed to remain constant. We assume $k_1 x_A > k_2$.

Let $V \gg 1$ be the generalized volume of the system and X_t^V be the continuous-time Markov jump process counting the number of X molecules. Then, $\frac{X_t^V}{V}$ is the concentration process on $\frac{\mathbb{N}_0}{V}$, and its transition rates are given by

$$q_V\left(\frac{n}{V}, \frac{n+m}{V}\right) = \begin{cases} k_1 x_A n, & m = 1, \\ \frac{k_{-1} n(n-1)}{2V} + k_2 n, & m = -1, \\ 0, & \text{otherwise}, \end{cases}$$

whenever $n \in \mathbb{N}_0$ and $n + m \in \mathbb{N}_0$. The law of large numbers [27, 1] ensures that as the volume V grows to infinity, the re-scaled process $\frac{X_t^V}{V}$ converges to the solutions of the following mean field ODE for the concentration of X molecules:

$$\dot{x} = -\frac{k_{-1}}{2} x^2 + k_1 x_A x - k_2 x, \quad x \in [0, \infty). \tag{6.4}$$

The fluctuation of $\frac{X_t^V}{V}$ around solutions of (6.4) is captured by the central limit theorem [27, 1], leading to the diffusion approximation of $\frac{X_t^V}{V}$:

$$dx = \left(-\frac{k_{-1}}{2} x^2 + k_1 x_A x - k_2 x\right) dt + \epsilon \sqrt{\frac{k_{-1}}{2} x^2 + k_1 x_A x + k_2 x dW_t}, \quad x \in [0, \infty), \tag{6.5}$$

where $\epsilon = \frac{1}{\sqrt{V}}$ and W_t is a standard one-dimensional Wiener process.

It is not hard to check that solutions of (6.5) almost surely reach the extinction state 0 in finite time, while solutions of (6.4) with positive initial data converge exponentially fast to the unique positive equilibrium. Such a dynamical disagreement between deterministic and stochastic models is often referred to as *Keizer's paradox* [46], which is often formulated in terms of the chemical master equation satisfied by the distributions of X_t^V or $\frac{X_t^V}{V}$ (see e.g. [47, 76, 13]). Examining the QSD of (6.5) bridges the dynamics of (6.4) and (6.5) (or more generally, (6.2) and the associated unperturbed ODE $\dot{x} = b_1 x - b_2 x^2$), and successfully resolve Keizer's paradox. See Remark 6.1 for details.

Logistic BDPs. Let $\lambda > \mu > 0$. Consider a continuous-time birth-and-death process (BDP) Y_t^K on the state space $\mathbb{N}_0$ with birth rates $\lambda_n^K = \lambda n$, $n \in \mathbb{N}_0$, and death rates $\mu_n^K = n\left(\mu + \frac{n}{K}\right)$, $n \in \mathbb{N}$, where $K \gg 1$ is the scaling parameter, often called the carrying capacity. The transition rates of $\frac{Y_t^K}{K}$ is given by

$$q_K\left(\frac{n}{K}, \frac{n+m}{K}\right) = \begin{cases} \mu_n^K, & m = 1, \\ \lambda_n^K, & m = -1, \\ 0, & \text{otherwise}, \end{cases}$$

whenever $n \in \mathbb{N}_0$ and $n + m \in \mathbb{N}_0$. By the law of large numbers and central limit theorem, for sufficiently large K, the process $\frac{Y_t^K}{K}$ stays close to solutions of the following SDE:

$$dx = (\lambda x - \mu x - x^2) dt + \epsilon \sqrt{\lambda x + \mu x + x^2} dW_t, \quad x \in [0, \infty), \tag{6.6}$$

where $\epsilon = \frac{1}{\sqrt{K}}$. The SDE (6.6) is the diffusion approximation of $\frac{Y_t^K}{K}$, and is in the form of (6.2).

Going back to (6.2), we let $a(x) = a_1 x + a_2 x^2$ and $b(x) = b_1 x - b_2 x^2$. Clearly, **(H)** is satisfied. Let V be as in (1.4). Denote by X_t^ϵ the solution processes of (6.2) and by T_0^ϵ the associated extinction time. Set $x_* := \frac{b_1}{b_2}$. Note that we have used the notation V for both the generalized volume and the potential function. Its meaning should be clear in the context, and thus, no confusion shall be caused.

Theorem 6.1. *Consider* (6.2)*.*

(1) *For each ϵ,* (6.2) *admits a unique QSD μ_ϵ with a density $u_\epsilon \in C^\infty((0,\infty))$.*
(2) $u_\epsilon = \frac{R_\epsilon}{\epsilon a} e^{-\frac{2}{\epsilon^2}\int_\bullet^{x_*} \frac{b}{a} ds}$*, where $\lim_{\epsilon\to 0} R_\epsilon = a(x_*)\sqrt{-\frac{b'(x_*)}{\pi a(x_*)}}$ locally uniformly in $(0,\infty)$.*
(3) *For each $K \subset\subset (0,\infty)$, there are positive constants $\gamma = \gamma(K)$ and $\epsilon_* = \epsilon_*(K)$ such that*

$$\begin{aligned}&\sup_{\substack{\mu\in\mathcal{P}((0,\infty))\\ \mathrm{supp}(\mu)\subset K}} \left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - \left[e^{-\lambda_{\epsilon,1}t}\langle \mu, \alpha_{\epsilon,1}\rangle \mu_\epsilon + \left(1 - e^{-\lambda_{\epsilon,1}t}\langle\mu,\alpha_{\epsilon,1}\rangle\right)\delta_0\right]\right\|_{TV} \\ &\quad \le e^{\frac{\gamma}{\epsilon^2} - b_1 t}, \quad \forall t > 2 \text{ and } 0 < \epsilon < \epsilon_*,\end{aligned}$$

where $\alpha_{\epsilon,1} := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$ and $\langle\mu,\alpha_{\epsilon,1}\rangle := \int_0^\infty \alpha_{\epsilon,1} d\mu$ satisfies $\lim_{\epsilon\to 0}\langle \mu,\alpha_{\epsilon,1}\rangle = 1$.
(4) *Let w be a modulus of continuity. For each compact $K\subset(0,\infty)$, $M>0$ and sequences $\{\underline{t}_\epsilon\}_\epsilon$, $\{\overline{t}_\epsilon\}_\epsilon$ in $(0,\infty)$ satisfying $\underline{t}_\epsilon < \overline{t}_\epsilon$ for each ϵ, $\lim_{\epsilon\to 0}\underline{t}_\epsilon = \infty$ and $\lim_{\epsilon\to 0}\frac{\overline{t}_\epsilon}{\epsilon} e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds} = 0$, there holds*

$$\lim_{\epsilon\to 0}\sup_{\mathrm{supp}(\mu)\subset K}\sup_{\underline{t}_\epsilon\le t\le \overline{t}_\epsilon}\sup_{\substack{f\in w[x_*]\\ \|f\|_\infty\le M}}\left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] - \int_0^\infty f d\mu_\epsilon\right| = 0.$$

(5) *For any $\mu\in\mathcal{P}((0,\infty))$ with compact support, $\mathbb{E}_\mu^\epsilon[T_0^\epsilon] \approx_\epsilon \frac{\epsilon a_1}{b_1}\sqrt{-\frac{\pi}{a(x_*)b'(x_*)}} e^{-\frac{2}{\epsilon^2}\int_0^{x_*}\frac{b}{a}ds}$ and $\lim_{\epsilon\to 0}\mathbb{P}_\mu^\epsilon\left[\frac{T_0^\epsilon}{\mathbb{E}_\mu^\epsilon[T_0^\epsilon]} > t\right] = e^{-t}$ for all $t>0$.*
(6) $\lim_{\epsilon\to 0}\|\mu_\epsilon - \mathcal{G}_\epsilon\|_{TV} = 0$*, where $\mathcal{G}_\epsilon$ is a probability measure on $(0,\infty)$ whose density is proportional to $\exp\left\{\frac{b'(x_*)}{a(x_*)}\frac{(\bullet - x_*)^2}{\epsilon^2}\right\}$.*
(7) *For any $p\in[1,\infty)$, $\lim_{\epsilon\to 0} W_p(\mu_\epsilon,\mathcal{G}_\epsilon) = 0$, where W_p is the p-Wasserstein distance.*

Proof. (1) See [10]. (2) It follows directly from Theorem A and Corollary B (1). (3) It follows from Corollary E (1) and the fact $\lim_{\epsilon\to 0}\lambda_{\epsilon,2} = b_1$ (see [45, Theorem B]). (4) It follows from Theorem F. (5) It is a direct consequence of Corollary D (1) and Theorem D (1).

(6) Denote the density of $\mathcal{G}_\epsilon$ by $G_\epsilon(x) = \frac{1}{Z_\epsilon} e^{-\frac{V''(x_*)}{\epsilon^2}(x-x_*)^2}$, where Z_ϵ is the normalization constant. Fix $0 < \delta_0 \ll 1$ and $\kappa \in (\frac{2}{3},1)$. Set $I_\epsilon := (x_* - \epsilon^\kappa, x_* + \epsilon^\kappa)$ and $I_{\delta_0} := (x_* - \delta_0, x_* + \delta_0)$. Split

$$2\mathrm{dist}_{TV}(\mu_\epsilon,\mathcal{G}_\epsilon) = \left(\int_{(0,\infty)\setminus I_{\delta_0}} + \int_{I_{\delta_0}\setminus I_\epsilon} + \int_{I_\epsilon}\right)|u_\epsilon - G_\epsilon| dx.$$

It remains to treat the integrals.

By Corollary 4.1 and the tail of G_ϵ, there exists $\gamma_1 > 0$ (independent of ϵ) such that

$$\int_{(0,\infty)\setminus I_{\delta_0}} |u_\epsilon - G_\epsilon| dx \le e^{-\frac{\gamma_1}{\epsilon^2}}. \tag{6.7}$$

Clearly, there is $0<\eta\ll 1$ such that $V(x)-V(x_*)\geq\left[\frac{V''(x_*)}{2}-\eta\right](x-x_*)^2$ for all $x\in I_{\delta_0}$. It follows from (2) that

$$\begin{aligned}\int_{I_{\delta_0}\setminus I_\epsilon}|u_\epsilon-G_\epsilon|dx &\leq\int_{I_{\delta_0}\setminus I_\epsilon}\frac{R_\epsilon}{\epsilon a}e^{-\frac{2}{\epsilon^2}[V-V(x_*)]}dx+\int_{I_{\delta_0}\setminus I_\epsilon}G_\epsilon dx\\ &\leq C_1\int_{I_{\delta_0}\setminus I_\epsilon}\frac{1}{\epsilon}e^{-\frac{2}{\epsilon^2}\left[\frac{V''(x_*)}{2}-\eta\right](x-x_*)^2}dx+\int_{I_{\delta_0}\setminus I_\epsilon}G_\epsilon dx,\end{aligned}$$

where $C_1=1+\sup_{I_{\delta_0}}\frac{a(x_*)}{a}\sqrt{\frac{V''(x_*)}{\pi}}$. Note that there is $\gamma_2>0$ (independent of κ and ϵ) such that

$$\max\left\{\int_{I_{\delta_0}\setminus I_\epsilon}\frac{1}{\epsilon}e^{-\frac{2}{\epsilon^2}\left[\frac{V''(x_*)}{2}-\eta\right](x-x_*)^2}dx,\int_{I_{\delta_0}\setminus I_\epsilon}G_\epsilon dx\right\}\leq e^{-\frac{\gamma_2}{\epsilon^{2(1-\kappa)}}}.$$

Hence,

$$\int_{I_{\delta_0}\setminus I_\epsilon}|u_\epsilon-G_\epsilon|dx\leq 2C_1e^{-\frac{\gamma_2}{\epsilon^{2(1-\kappa)}}}. \tag{6.8}$$

Since $V\in C^3((0,\infty))$, there holds

$$V(x)=V(x_*)+\frac{V''(x_*)}{2}(x-x_*)^2+\frac{V'''(x_*)}{6}(x-x_*)^3+o(|x-x_*|^3),\quad\forall x\in I_{\delta_0}.$$

Then,

$$\int_{I_\epsilon}|u_\epsilon-G_\epsilon|dx=\int_{I_\epsilon}C_2(x,\epsilon)\frac{1}{\epsilon}e^{-\frac{V''(x_*)}{\epsilon^2}(x-x_*)^2}dx\leq\sqrt{\frac{2\pi}{V''(x_*)}}\sup_{x\in I_\epsilon}C_2(x,\epsilon),$$

where

$$C_2(x,\epsilon)=\left|\frac{R_\epsilon(x)}{a(x)}-\frac{\epsilon}{Z_\epsilon}e^{\frac{2}{\epsilon^2}\left[\frac{V'''(x_*)}{6}(x-x_*)^3+o(|x-x_*|^3)\right]}\right|e^{-\frac{2}{\epsilon^2}\left[\frac{V'''(x_*)}{6}(x-x_*)^3+o(|x-x_*|^3)\right]}.$$

Note that $\kappa>\frac{2}{3}$ gives $\sup_{x\in I_\epsilon}e^{\frac{2}{\epsilon^2}\left|\frac{V'''(x_*)}{6}(x-x_*)^3+o(|x-x_*|^3)\right|}\to 1$ as $\epsilon\to 0$. This together with (2) and the fact that $\frac{\epsilon}{Z_\epsilon}=\sqrt{-\frac{b'(x_*)}{\pi a(x_*)}}$ implies that $\lim_{\epsilon\to 0}\sup_{x\in I_\epsilon}C_2(x,\epsilon)=0$. Hence, $\lim_{\epsilon\to 0}\int_{I_\epsilon}|u_\epsilon-G_\epsilon|dx=0$, which together with (6.7) and (6.8) yields $\lim_{\epsilon\to 0}\|\mu_\epsilon-\mathcal{G}_\epsilon\|_{TV}=0$. This completes the proof.

(7) Let $p\in[1,\infty)$. By [77, Theorem 6.15], $W_p(\mu_\epsilon,\mathcal{G}_\epsilon)\leq 2^{\frac{1}{p'}}\left(\int_0^\infty|x-x_*|^p\,|u_\epsilon(x)-G_\epsilon(x)|\,dx\right)^{\frac{1}{p}}$, where p' is the dual exponent of p. It follows from (6) and the tails of u_ϵ (see Lemma 2.6) and G_ϵ that $\lim_{\epsilon\to 0}W_p(\mu_\epsilon,\mathcal{G}_\epsilon)=0$. □

Remark 6.1. *We comment on Keizer's paradox and its resolution by QSDs. Keizer's paradox, in terms of the diffusion process* (6.2) *and the associated unperturbed ODE*

$$\dot{x}=b_1x-b_2x^2,\quad x\in[0,\infty), \tag{6.9}$$

refers to the long-term dynamical disagreement of (6.2) *and* (6.9) *when they are used to model the same system. More precisely, for any* $x_0\in(0,\infty)$, *solutions* X_t^ϵ *and* $\varphi^t(x_0)$ *of* (6.2) *and* (6.9), *respectively, with* $X_0^\epsilon=x_0=\phi^0(x_0)$ *satisfy* $\lim_{t\to\infty}X_t^\epsilon=0$ *almost surely and*

$$\lim_{t\to\infty}\varphi^t(x_0)=x_*\quad\textit{exponentially fast.} \tag{6.10}$$

However, from an observable's viewpoint, (6.2) *and* (6.9) *are well-matched because the same phenomenon is observed over any reasonable periods when either of them is used. This is clearly seen from Theorem 6.1 (4) (6) and* (6.10) *(being locally uniformly in $x_0 \in (0,\infty)$), which imply that*

$$\lim_{\epsilon\to 0}\sup_{x_0\in K}\sup_{\underline{t}_\epsilon\le t\le \bar{t}_\epsilon}\sup_{\substack{f\in w[x_*]\\ \|f\|_\infty\le M}}\left|\mathbb{E}^\epsilon_{x_0}[f(X^\epsilon_t)]-f(\varphi^t(x_0))\right|=0,$$

where K, $\underline{t}_\epsilon$, $\bar{t}_\epsilon$, $w[x_*]$ and M are as in Theorem 6.1 (4). Intuitively, for the time scale $\underline{t}_\epsilon \le t \le \bar{t}_\epsilon$, X^ϵ_t is governed by the QSD μ_ϵ, which is almost Gaussian and concentrated at x_*.

Remark 6.2. *It is worth noting that the metastability in the chemical system* (6.3) *shares some similarities in phenomena with that in equilibrium statistical mechanics, meaning that there exists a state in which the process remains trapped over macroscopic time scales before moving to the state with the lowest energy. But, they differ in mechanism.*

For a closed chemical system whose mean-field model exhibits multiple stable equilibria, the concept of metastability in its stochastic counterpart bears a resemblance to that in equilibrium statistical mechanics. These systems exhibit noise-induced transitions among stable equilibria. In this scenario, stationary distributions represent equilibrium steady states in which there is no reaction flux (more precisely, reactions do not stop, but are balanced), and the equilibrium associated with their zero-noise limit (if exists) can be regarded as "the state with the lowest energy".

However, due to the fact that the reaction $A \to C$ is assumed to be irreversible, the chemical system (6.3) *can be realized in an open system in which C molecules are constantly removed from the system while A molecules are constantly supplied to the system. Noise-induced transitions occur from the sole stable equilibrium to the extinction state* 0*, but there are no transitions back. The QSDs can be seen as nonequilibrium steady states with fluxes in reactions, resulting in the eventual extinction of X molecules.*

These were more or less discussed in [76, Section 4]*, and pertain to the fundamental distinctions between open and closed chemical systems.*

6.3. **Diffusion approximation of QSDs.** We further examine logistic BDPs with the focus on the distance between QSDs of $X^K_t := \frac{Y^K_t}{K}$ and that of (6.6). This concerns the compatibility of the birth-death process X^K_t and the diffusion process (6.6) as models for the evolution of the same species, as well as the diffusion approximation of QSDs.

It is well-established (see e.g. [62, 18]) that for each $K \gg 1$, X^K_t admits a unique QSD μ^K on $\frac{\mathbb{N}}{K}$. In [17], the authors proved the following asymptotic of μ^K as $K\to\infty$.

Proposition 6.1 ([17])**.** *The following hold.*

1. *There is $C_0>0$ such that $\|\mu^K-\mathcal{G}^K\|_{TV}\le \frac{C_0}{\sqrt{K}}$ for all $K\gg 1$, where $\mathcal{G}^K=\left\{G^K\left(\frac{n}{K}\right)\right\}_{n\in\mathbb{N}}$ is a probability measure on $\frac{\mathbb{N}}{K}$ given by $G^K\left(\frac{n}{K}\right)=\frac{1}{Z^K}e^{-\frac{K}{2\lambda}\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2}$ with Z^K being the normalization constant.*
2. *For any $p\in[1,\infty)$, there exists $C_p>0$ such that $W_p(\mu^K,\mathcal{G}^K)\le\frac{C_p}{\sqrt{K}}$ for all $K\gg 1$.*

Proposition 6.1 (2) is not stated in [17]. But, it is a simple consequence of Proposition 6.1 (1), the tails of μ^K given in the proof of [17, Theorem 3.7], and the control of Wasserstein distance by weighted total variation distance (see [77, Theorem 6.15]).

We identify μ^K and $\mathcal{G}^K$ with their natural extensions to probability measures on $(0,\infty)$. In particular, they are singular with respect to the Lebesgue measure on $(0,\infty)$.

Recall that $\epsilon = \frac{1}{\sqrt{K}}$. Denote by $\mu_K := \mu_\epsilon$ the unique QSD of (6.6). As the total variation distance between μ^K and μ_K is 1, we use somewhat weaker distances.

Theorem 6.2. *The following hold.*

(1) *For any $p \in [1, \infty)$, $\lim_{K\to\infty} W_p(\mu^K, \mu_K) = 0$.*
(2) *$\lim_{K\to\infty} \mathrm{dist}_{Kol}(\mu^K, \mu_K) = 0$, where dist_{Kol} denotes the Kolmogorov metric.*

We need some elementary results. Let $\mathcal{G}_K$ be a probability measure on $(0, \infty)$ with density

$$G_K(x) = \frac{1}{Z_K} \exp\left\{-\frac{K}{2\lambda}(x - (\lambda - \mu))^2\right\}, \tag{6.11}$$

where Z_K is the normalization constant.

Lemma 6.1. *The following hold.*

(1) $\lim_{K\to\infty} \frac{Z^K}{KZ_K} = 1$.
(2) *For each $p \in [1, \infty)$, $\lim_{K\to\infty} \sum_{n\in\mathbb{N}} \left|\frac{n}{K} - (\lambda - \mu)\right|^p G^K\left(\frac{n}{K}\right) \to 0$.*

Proof. (1) Write

$$Z^K = \sum_{n\in\mathbb{N}_K} e^{-\frac{K}{2\lambda}\left(\frac{n}{K} - \frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2} + \sum_{n\in\mathbb{N}_K^c} e^{-\frac{K}{2\lambda}\left(\frac{n}{K} - \frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2} =: \mathrm{I}^K + \mathrm{II}^K,$$

where $\mathbb{N}_K = \left\{n \in \mathbb{N} : \left|\frac{n}{K} - \frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right| \le \frac{1}{K^{\frac{1}{4}}}\right\}$ and $\mathbb{N}_K^c = \mathbb{N} \setminus \mathbb{N}_K$. Since $\mathrm{I}^K \ge 1$ and

$$\mathrm{II}^K \le \sum_{\substack{n > K^{\frac{3}{4}} \\ n\in\mathbb{N}}} e^{-\frac{n}{2\lambda}K^{-\frac{1}{4}}} \le \frac{2e^{-\frac{\sqrt{K}}{2\lambda}}}{1 - e^{-\frac{1}{2\lambda}K^{-\frac{1}{4}}}} \le 8\lambda K^{\frac{1}{4}} e^{-\frac{\sqrt{K}}{2\lambda}}, \quad K \gg 1,$$

where we used the fact that $1 - e^{-x} \ge \frac{x}{2}$ for $0 < x \le \ln 2$ in the last inequality, we find

$$\mathrm{II}^K = o(\mathrm{I}^K) \quad \text{as} \quad K \to \infty \tag{6.12}$$

and $Z^K = \mathrm{I}^K + o(e^{-\frac{\sqrt{K}}{4\lambda}})$ as $K \to \infty$.

Now, we treat Z_K. Split

$$Z_K = \sum_{n\in\mathbb{N}_K} \int_{\frac{n}{K}}^{\frac{n+1}{K}} e^{-\frac{K}{2\lambda}(x-(\lambda-\mu))^2} dx + \sum_{n\in\mathbb{N}_K^c} \int_{\frac{n}{K}}^{\frac{n+1}{K}} e^{-\frac{K}{2\lambda}(x-(\lambda-\mu))^2} dx =: \mathrm{I}_K + \mathrm{II}_K.$$

Noting that for $x \in \left[\frac{n}{K}, \frac{n+1}{K}\right)$ and $n \in \mathbb{N}_K^c$, there holds

$$|x - (\lambda - \mu)| \ge \left|\frac{n}{K} - \frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right| - \left|x - (\lambda - \mu) - \left(\frac{n}{K} - \frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)\right| \ge \frac{1}{K^{\frac{1}{4}}} - \frac{2}{K} \ge \frac{1}{2K^{\frac{1}{4}}}$$

for all $K \gg 1$, we deduce

$$\begin{aligned}
\mathrm{II}_K &\le \int_{\{|x-(\lambda-\mu)|\ge\frac{1}{2}K^{-\frac{1}{4}}\}} e^{-\frac{K}{2\lambda}(x-(\lambda-\mu))^2} dx \\
&= 2\sqrt{\frac{2\lambda}{K}} \int_{\frac{K^{\frac{1}{4}}}{2\sqrt{2\lambda}}}^{\infty} e^{-y^2} dy \le 2\sqrt{\frac{2\lambda}{K}} \int_{\frac{K^{\frac{1}{4}}}{2\sqrt{2\lambda}}}^{\infty} \exp\left\{-\frac{yK^{\frac{1}{4}}}{2\sqrt{2\lambda}}\right\} dy = \frac{8\lambda}{K^{\frac{3}{4}}} e^{-\frac{1}{8\lambda}K^{\frac{1}{2}}}.
\end{aligned}$$

Then, it follows from

$$\sqrt{K} Z_K = \sqrt{K} \int_0^\infty e^{-\frac{K}{2\lambda}(x-(\lambda-\mu))^2} dx = \sqrt{\lambda} \int_{-\sqrt{\frac{K}{\lambda}}(\lambda-\mu)}^{\infty} e^{-\frac{x^2}{2}} dx \to \sqrt{2\pi\lambda} \quad \text{as} \quad K \to \infty$$

that $\lim_{K\to\infty}\sqrt{K}\mathrm{I}_K=\sqrt{2\pi\lambda}$ and

$$\mathrm{II}_K=o(\mathrm{I}_K)\quad\text{as}\quad K\to\infty. \tag{6.13}$$

Note that if $n\in\mathbb{N}_K$ and $x\in\left[\frac{n}{K},\frac{n+1}{K}\right)$, then $\left|(x-(\lambda-\mu))-\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)\right|\leq\frac{2}{K}$, and thus, for all $K\gg 1$,

$$\begin{aligned}
&\exp\left\{\frac{K}{2\lambda}\left|(x-(\lambda-\mu))^2-\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2\right|\right\}\\
&\quad\leq\exp\left\{\frac{1}{\lambda}\left|x-(\lambda-\mu)+\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right|\right\}\leq\exp\left\{\frac{1}{\lambda}\left(\frac{2}{K^{\frac14}}+\frac{2}{K}\right)\right\}\leq\exp\left\{\frac{4}{\lambda K^{\frac14}}\right\}.
\end{aligned}$$

As a consequence, we find from the mean value theorem that

$$\begin{aligned}
\mathrm{I}_K&=\frac{1}{K}\sum_{n\in\mathbb{N}_K}e^{-\frac{K}{2\lambda}(x_n^K-(\lambda-\mu))^2}\leq\frac{1}{K}e^{\frac{4}{\lambda}K^{-\frac14}}\sum_{n\in\mathbb{N}_K}e^{-\frac{K}{2\lambda}\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2}=\frac{1}{K}e^{\frac{4}{\lambda}K^{-\frac14}}\mathrm{I}^K,\\
\mathrm{I}_K&=\frac{1}{K}\sum_{n\in\mathbb{N}_K}e^{-\frac{K}{2\lambda}(x_n^K-(\lambda-\mu))^2}\geq\frac{1}{K}e^{-\frac{4}{\lambda}K^{-\frac14}}\mathrm{I}^K,
\end{aligned}$$

where $x_n^K\in\left[\frac{n}{K},\frac{n+1}{K}\right)$ for each $n\in\mathbb{N}_K$. Therefore, $\lim_{K\to\infty}\frac{I^K}{KI_K}=1$, which together with (6.12) and (6.13) leads to $\lim_{K\to\infty}\frac{Z^K}{KZ_K}=\lim_{K\to\infty}\frac{\mathrm{I}^K}{K\mathrm{I}_K}=1$.

(2) Since $Z^K\geq 1$, we derive

$$\begin{aligned}
&\sum_{n\in\mathbb{N}}\left|\frac{n}{K}-(\lambda-\mu)\right|^pG^K\left(\frac{n}{K}\right)\leq\sum_{n\in\mathbb{N}}\left|\frac{n}{K}-(\lambda-\mu)\right|^pe^{-\frac{K}{2\lambda}\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2}\\
&\quad\leq 2^p\sum_{n\in\mathbb{N}}\left(\left|\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right|^p+\left|\frac{\lfloor(\lambda-\mu)K\rfloor}{K}-(\lambda-\mu)\right|^p\right)e^{-\frac{K}{2\lambda}\left(\frac{n}{K}-\frac{\lfloor(\lambda-\mu)K\rfloor}{K}\right)^2}\\
&\quad\leq 2^p\sum_{n\in\mathbb{Z}}\left(\left(\frac{n}{K}\right)^p+\frac{1}{K^p}\right)e^{-\frac{1}{2\lambda K}n^2}.
\end{aligned}$$

Note that for each $n\in\mathbb{N}$ and $K>0$, there hold

$$\frac{1}{K}e^{-\frac{K}{2\lambda}\left(\frac{n}{K}\right)^2}<\int_{\frac{n}{K}}^{\frac{n+1}{K}}e^{-\frac{K}{2\lambda}\left(x-\frac{1}{K}\right)^2}dx,\quad\frac{1}{K}\left(\frac{n}{K}\right)^pe^{-\frac{K}{2\lambda}\left(\frac{n}{K}\right)^2}<\int_{\frac{n}{K}}^{\frac{n+1}{K}}x^pe^{-\frac{K}{2\lambda}\left(x-\frac{1}{K}\right)^2}dx,$$

It follows that

$$\begin{aligned}
\sum_{n\in\mathbb{N}}e^{-\frac{1}{2\lambda K}n^2}&<K\int_{\frac{1}{K}}^{\infty}e^{-\frac{K}{2\lambda}\left(x-\frac{1}{K}\right)^2}dx=K\int_0^\infty e^{-\frac{K}{2\lambda}x^2}dx,\\
\sum_{n\in\mathbb{N}}\left(\frac{n}{K}\right)^pe^{-\frac{1}{2\lambda K}n^2}&<K\int_{\frac{1}{K}}^{\infty}x^pe^{-\frac{K}{2\lambda}\left(x-\frac{1}{K}\right)^2}dx\\
&=K\int_0^\infty\left(x+\frac{1}{K}\right)^pe^{-\frac{K}{2\lambda}x^2}dx\leq 2^pK\int_0^\infty\left(x^p+\frac{1}{K^p}\right)e^{-\frac{K}{2\lambda}x^2}dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n\in\mathbb{N}}\left|\frac{n}{K}-(\lambda-\mu)\right|^p G^K\left(\frac{n}{K}\right) &\leq 2^p(1+2^p)K^{1-p}\int_0^\infty e^{-\frac{K}{2\lambda}x^2}dx+2^{2p}K\int_0^\infty x^p e^{-\frac{K}{2\lambda}x^2}dx\\ &= 2^p(1+2^p)\sqrt{2\lambda}K^{\frac{1}{2}-p}\int_0^\infty e^{-\frac{x^2}{2}}dx+2^{2p}(2\lambda)^{\frac{p+1}{2}}K^{\frac{1}{2}-\frac{p}{2}}\int_0^\infty x^p e^{-\frac{x^2}{2}}dx\\ &\to 0 \quad \text{as} \quad K\to\infty.\end{aligned}$$

This completes the proof. □

Now, we prove Theorem 6.2.

Proof of Theorem 6.2. Recall that $\mathcal{G}_K$ is a probability measure on $(0,\infty)$ with density G_K given in (6.11). Theorem 6.1 says that

$$\lim_{K\to\infty}\|\mu_K-\mathcal{G}_K\|_{TV}=0 \quad \text{and} \quad \lim_{K\to\infty}W_p(\mu_K,\mathcal{G}_K)=0, \quad \forall p\in[1,\infty). \tag{6.14}$$

(1) Let $p\in[1,\infty)$. Note that $W_p(\mu^K,\mu_K)\leq W_p(\mu^K,\mathcal{G}^K)+W_p(\mathcal{G}^K,\mathcal{G}_K)+W_p(\mathcal{G}_K,\mu_K)$. It follows from Proposition 6.1 (2) and (6.14) that $\lim_{K\to\infty}W_p(\mu^K,\mu_K)=0$ holds if we show

$$\lim_{K\to\infty}W_p(\mathcal{G}^K,\mathcal{G}_K)=0. \tag{6.15}$$

To show (6.15), we note that

$$\begin{aligned}W_p(\mathcal{G}^K,\mathcal{G}_K) &\leq W_p(\mathcal{G}^K,\delta_{\lambda-\mu})+W_p(\delta_{\lambda-\mu},\mathcal{G}_K)\\ &= \left(\sum_{n\in\mathbb{N}}\left|\frac{n}{K}-(\lambda-\mu)\right|^p G^K\left(\frac{n}{K}\right)\right)^{\frac{1}{p}}+\left(\int_0^\infty|x-(\lambda-\mu)|^p G_K(x)dx\right)^{\frac{1}{p}}.\end{aligned}$$

By Laplace's method, $\lim_{K\to\infty}\int_0^\infty|x-(\lambda-\mu)|^p G_K(x)dx=0$. This together with Lemma 6.1 (2) gives (6.15).

(2) Due to Proposition 6.1 (1), (6.14) and

$$\text{dist}_{Kol}(\mu^K,\mu_K)\leq\|\mu^K-\mathcal{G}^K\|_{TV}+\text{dist}_{Kol}(\mathcal{G}^K,\mathcal{G}_K)+\|\mathcal{G}_K-\mu_K\|_{TV},$$

the limit $\lim_{K\to\infty}\text{dist}_{Kol}(\mu^K,\mu_K)=0$ follows if we show

$$\lim_{K\to\infty}\text{dist}_{Kol}(\mathcal{G}^K,\mathcal{G}_K)=0. \tag{6.16}$$

Denote by $F_{\mathcal{G}^K}$ and $F_{\mathcal{G}_K}$ distribution functions of $\mathcal{G}^K$ and $\mathcal{G}_K$, respectively. Clearly,

$$F_{\mathcal{G}^K}(t)=\begin{cases}0, & t\in(0,\frac{1}{K}),\\ \displaystyle\sum_{m=1}^n G^K\left(\frac{m}{K}\right), & t=\left[\frac{n}{K},\frac{n+1}{K}\right), \quad n\in\mathbb{N},\end{cases}$$

By the definition,

$$\text{dist}_{Kol}(\mathcal{G}^K,\mathcal{G}_K)=\sup_{(0,\infty)}|F_{\mathcal{G}^K}-F_{\mathcal{G}_K}|\leq\sup\left\{\sup_{(0,\frac{1}{K})}|F_{\mathcal{G}^K}-F_{\mathcal{G}_K}|,\ \sup_{[\frac{n}{K},\frac{n+1}{K})}|F_{\mathcal{G}^K}-F_{\mathcal{G}_K}|,\ n\in\mathbb{N}\right\}.$$

Obviously, $\lim_{K\to\infty}\sup_{(0,\frac{1}{K})}|F_{\mathcal{G}^K}-F_{\mathcal{G}_K}|=0$. Recall $\mathbb{N}_K$ and $\mathbb{N}_K^c$ from the proof of Lemma 6.1. Note that

$$\sup_{n\in\mathbb{N}}\sup_{\left[\frac{n}{K},\frac{n+1}{K}\right)}|F_{\mathcal{G}^K}-F_{\mathcal{G}_K}|=\sup_{n\in\mathbb{N}}\sup_{\left[\frac{n}{K},\frac{n+1}{K}\right)}\left|\sum_{m=1}^{n}G^K\left(\frac{m}{k}\right)-\int_0^t G_K dx\right|$$
$$\leq\left(\sum_{n\in\mathbb{N}_K}+\sum_{n\in\mathbb{N}_K^c}\right)\left|G^K\left(\frac{n}{K}\right)-\int_{\frac{n}{K}}^{\frac{n+1}{K}}G_K dx\right|+\sup_{n\in\mathbb{N}}\int_{\frac{n}{K}}^{\frac{n+1}{K}}G_K dx.$$

It is easy to see that $\lim_{K\to\infty}\sup_{n\in\mathbb{N}}\int_{\frac{n}{K}}^{\frac{n+1}{K}}G_K dx=0$. The exponential tails of G^K and G_K ensure that the sum over $\mathbb{N}_K^c$ vanishes as $K\to\infty$. To treat the sum over $\mathbb{N}_K$, we set $x_K:=\frac{\lfloor(\lambda-\mu)K\rfloor}{K}$ and estimate for $n\in\mathbb{N}_K$,

$$\left|G^K\left(\frac{n}{K}\right)-\int_{\frac{n}{K}}^{\frac{n+1}{K}}G_K dx\right|\leq\frac{C_{n,K}}{Z^K}e^{-\frac{K}{2\lambda}\left(\frac{n}{K}-x_K\right)^2},\tag{6.17}$$

where $C_{n,K}=\left|1-\frac{Z^K}{Z_K}\int_{\frac{n}{K}}^{\frac{n+1}{K}}e^{-\frac{K}{2\lambda}\left[(x-(\lambda-\mu))^2-\left(\frac{n}{K}-x_K\right)^2\right]}dx\right|$. Note that

$$\sup_{n\in\mathbb{N}_K}\sup_{x\in\left(\frac{n}{K},\frac{n+1}{K}\right)}\left|-\frac{K}{2\lambda}\left[(x-(\lambda-\mu))^2-\left(\frac{n}{K}-x_K\right)^2\right]\right|$$
$$\leq\frac{K}{2\lambda}\sup_{n\in\mathbb{N}_K}\sup_{x\in\left(\frac{n}{K},\frac{n+1}{K}\right)}\left\{\left|x-\frac{n}{K}+x_K-(\lambda-\mu)\right|\cdot\left|x-\frac{n}{K}+\frac{n}{K}-x_K+x_K-(\lambda-\mu)+\frac{n}{K}-x_K\right|\right\}$$
$$\leq\frac{K}{2\lambda}\frac{2}{K}\left(\frac{2}{K}+\frac{2}{K^{\frac{1}{4}}}\right)\to 0\quad\text{as}\quad K\to\infty.$$

This together with Lemma 6.1 implies $\lim_{K\to\infty}\sup_{n\in\mathbb{N}_K}C_{n,K}=0$. Hence, we see from (6.17) that

$$\lim_{K\to\infty}\sum_{n\in\mathbb{N}_K}\left|G^K\left(\frac{n}{K}\right)-\int_{\frac{n}{K}}^{\frac{n+1}{K}}G_K(x)dx\right|\leq\lim_{K\to\infty}\sup_{n\in\mathbb{N}_K}C_{n,K}\times\lim_{K\to\infty}\sum_{n\in\mathbb{N}_K}\frac{1}{Z^K}e^{-\frac{K}{2\lambda}\left(\frac{n}{K}-x_K\right)^2}=0.$$

This proves (6.16), and completes the proof of the theorem. □

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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada

Email address: wqi2@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada

Email address: zhongwei@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada, and School of Mathematics, Jilin University, Changchun 130012, PRC

Email address: yingfei@ualberta.ca