TRANSIENT DYNAMICS OF ABSORBED SINGULAR DIFFUSIONS

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Dedicated to the memory of Professor Geneviève Raugel

Abstract. We consider a class of one-dimensional absorbed diffusion processes with small singular noises that exhibit multi-scale dynamics in the sense that typical trajectories of the solution process first quickly approach to transient states, captured by quasi-stationary distributions (QSDs), then stay with transient states for a very long period, and finally deviate from transient states and slowly relax to the absorbing state. The main purpose of the present paper is to give qualitative characterizations of such multi-scale dynamics with particular interest in the intriguing transient dynamics governed by transient states. This is achieved by the establishment of (i) noise-vanishing asymptotics of the first eigenvalue and the gap between the first and second eigenvalues of the generator that are respectively the rates for solutions to get absorbed by the absorbing state and attracted by QSDs; (ii) sophisticated estimates quantifying the distance between solutions and convex combinations of QSDs and the absorbing state. Applications to stochastic models arising in chemical reactions and population dynamics are discussed.

Contents

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1. Introduction

Transient dynamics, roughly understood as interesting and important dynamical behaviors of real or artificial systems that display only over finite time scales, arise naturally in chemical reactions (see e.g. [49, 54]), population dynamics (see e.g. [24, 25, 45]) and many other scientific areas (see e.g. [37]). In a closed chemical system, chemical oscillations could last hours and even longer, while the whole system must eventually relax to the thermal equilibrium due to the disspation of heat and energy. In population dynamics, although the eventual loss of diversity or extinction is inevitable due to limited resources, mortality, etc., species of large numbers typically coexist or persist for a very long period in comparison to human timescales. In spite of the ubiquity of transient dynamics in multi-scale systems, there have not been many mathematically rigorous studies.

The present paper is devoted to the investigation of transient dynamics of a class of multi-scale diffusion processes arising in chemical reactions, population dynamics, etc. More precisely, we consider the following family of stochastic differential equations (SDEs):

$$dx = b(x)dt + \epsilon\sqrt{a(x)}dW_t, \quad x \in [0,\infty), \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a parameter, $b : [0,\infty) \to \mathbb{R}$, $a : [0,\infty) \to [0,\infty)$ and W_t is the standard one-dimensional Wiener process on some probability space. SDEs of the form (1.1) are often derived as diffusion approximations of re-scaled Markov jump processes that model the evolution of chemical substrates or species of large numbers [19, 1]. Throughout this paper, we make the following standard assumptions on the coefficients a and b.

(H) The functions $b : [0,\infty) \to \mathbb{R}$ and $a : [0,\infty) \to [0,\infty)$ satisfy the following conditions:
(1) $b \in C^1([0,\infty))$, $b(0) = 0$, $b'(0) > 0$, and $\limsup_{x\to\infty} b(x) < 0$;
(2) $a \in C^2([0,\infty))$, $a(0) = 0$, $a'(0) > 0$, and $a > 0$ on $(0,\infty)$;
(3) $\lim_{x\to\infty} \frac{b^2(x)}{a(x)} = \infty$, and $\limsup_{x\to\infty} \frac{\max\{a(x),|a'(x)|,|a''(x)|,|b'(x)|\}}{|b(x)|} < \infty$.

Clearly, b is a generalized logistic growth rate function, and the noise coefficient $\epsilon\sqrt{a}$ vanishes and is singular at 0. This latter fact causes substantial troubles in the analysis of (1.1). The reader is directed to Section 7 for examples satisfying these assumptions. For the time being, it is beneficial to keep in mind the typical one: $dx = x(1-x)dt + \epsilon\sqrt{x}dW_t$.

We see from **(H)** that the state 0, often referred to as the extinction state, is an absorbing state of the SDE (1.1). Denote by X_t^ϵ the stochastic process generated by solutions of (1.1), and by T_0^ϵ the first time that X_t^ϵ hits 0, namely, $T_0^\epsilon = \inf\{t \geq 0 : X_t^\epsilon = 0\}$. Under **(H)**, it is a routine task to check (see e.g. [30]) that $\mathbb{P}_x^\epsilon[T_0^\epsilon < \infty] = 1$ for all $x \in [0,\infty)$, where $\mathbb{P}_\mu^\epsilon$ is the law of X_t^ϵ with initial distribution μ and $\mathbb{P}_x^\epsilon = \mathbb{P}_{\delta_x}^\epsilon$. That is, trajectories or sample paths of X_t^ϵ reach the extinction state 0 in finite time almost surely, implying the non-existence of a stationary distribution with positive concentration on

$(0,\infty)$. This is in sharp contrast to the eventual dynamics of the unperturbed mean-field ODE:

$$\dot{x} = b(x), \quad x \in (0,\infty), \tag{1.2}$$

whose solutions are attracted by the global attractor $\mathcal{A}$, which is a compact interval with its left and right endpoints being respectively the smallest positive zero and largest zero of b, and has the compact dissipativity property in the sense that for each compact $K \subset (0,\infty)$, $\sup_{x_0\in K} \operatorname{dist}(x_t,\mathcal{A}) \to 0$ as $t\to\infty$, where x_t is the solution of (1.2). On the contrary, the sample path large deviation principle (see e.g. [20, 15]) says that X^ϵ_t stays close to the solutions of (1.2) during a finite time period, which is long enough to ensure that X^ϵ_t spends considerable amount of time with the global attractor $\mathcal{A}$. Hence, X^ϵ_t exhibits multi-scale dynamics. Our goal is to characterize the multi-scale dynamics of X^ϵ_t, in which, the transient dynamics, living on the timescale that is longer than the large deviation timescale and shorter than the extinction timescale, is of particular interest and importance.

Because of the almost sure extinction in finite time of X^ϵ_t, it is natural to examine X^ϵ_t before it hits the extinction state 0 in order to understand in particular the transient dynamics of X^ϵ_t. Quasi-stationary distributions (see e.g. [42, 13]) have been widely used in literature to achieve this goal.

Definition 1.1 (Quasi-stationary distribution)**.** *A Borel probability measure μ_ϵ on $(0,\infty)$ is called a* quasi-stationary distribution (QSD) *of X^ϵ_t if*

$$\mathbb{P}^\epsilon_{\mu_\epsilon}[X^\epsilon_t \in B | t < T^\epsilon_0] = \mu_\epsilon(B), \quad \forall t \geq 0, \quad B \in \mathcal{B}((0,\infty)),$$

where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra of $(0,\infty)$.

General theory of QSDs (see e.g. [42, 13]) tells that each QSD μ_ϵ of X^ϵ_t is associated with a unique positive number λ_ϵ, often called the *extinction rate*, such that if X^ϵ_0 is distributed according to μ_ϵ, then T^ϵ_0 is exponentially distributed with rate λ_ϵ, namely, $\mathbb{P}^\epsilon_{\mu_\epsilon}[T^\epsilon_0 > t] = e^{-\lambda_\epsilon t}$ for all $t \geq 0$.

It is shown in [5] (also see Proposition 2.3) that X^ϵ_t admits a unique QSD μ_ϵ with the associated extinction rate $\lambda_{\epsilon,1}$ being the first eigenvalue of $-\mathcal{L}_\epsilon$, where $\mathcal{L}_\epsilon$ is essentially the generator of X^ϵ_t. Moreover, X^ϵ_t conditioned on non-absorption, i.e., the event $[t < T^\epsilon_0]$, converges to the QSD μ_ϵ exponentially fast with rate $\lambda_{\epsilon,2} - \lambda_{\epsilon,1}$, where $\lambda_{\epsilon,2}$ is the second eigenvalue of $-\mathcal{L}_\epsilon$. Thus, X^ϵ_t is simultaneously exponentially attracted by the extinction state 0 and the QSD μ_ϵ with rates $\lambda_{\epsilon,1}$ and $\lambda_{\epsilon,2} - \lambda_{\epsilon,1}$, respectively, suggesting the importance of comparing $\lambda_{\epsilon,1}$ with $\lambda_{\epsilon,2} - \lambda_{\epsilon,1}$. By doing so, we roughly conclude:

- (i) if $\lambda_{\epsilon,1} < \lambda_{\epsilon,2} - \lambda_{\epsilon,1}$, then X^ϵ_t first approaches to and then spend considerable amount of time with the QSD μ_ϵ before eventually relaxing to the extinction state; in this case, X_t exhibits multi-scale dynamics, and μ_ϵ is a true physical state governing the transient dynamics;
- (ii) if $\lambda_{\epsilon,1} > \lambda_{\epsilon,2} - \lambda_{\epsilon,1}$, then most trajectories of X^ϵ_t have already reached the extinction state before the conditioned process approaches to μ_ϵ, saying that X^ϵ_t exhibits simple extinction dynamics, and the QSD μ_ϵ does not play an interesting role.

The scenario described in situation (i) is expected. To obtain this piece of information, we study the asymptotics of $\lambda_{\epsilon,1}$ and $\lambda_{\epsilon,2}$ as $\epsilon \to 0$.

To put our study in a rigorous framework, we consider for each $0 < \epsilon \ll 1$ the symmetric form $\mathcal{E}_\epsilon : C^\infty_0((0,\infty)) \times C^\infty_0((0,\infty)) \to \mathbb{R}$ defined by

$$\mathcal{E}_\epsilon(\phi,\psi) = \frac{\epsilon^2}{2}\int_0^\infty a\phi'\psi' u^G_\epsilon dx, \quad \forall \phi,\psi \in C^\infty_0((0,\infty)),$$

where $u_\epsilon^G = \frac{1}{a}e^{\frac{2}{\epsilon^2}\int_0^\bullet \frac{b(s)}{a(s)}ds}$ is the non-integrable Gibbs density. More precisely, u_ϵ^G is the unique (up to constant multiplication) solution of the stationary Fokker-Planck equation $\frac{\epsilon^2}{2}(au)'' - (bu)' = 0$ in $(0,\infty)$, and satisfies $\int_0^\infty u_\epsilon^G dx = \infty$ due to its singularity near 0, that is, u_ϵ^G behaves like $\frac{1}{x}$ near $x = 0$. It is known (see e.g. [22]) that $\mathcal{E}_\epsilon$ is Markovian and closable. Its smallest closed extension, still denoted by $\mathcal{E}_\epsilon$, is a Dirichlet form. Denote by $\mathcal{L}_\epsilon$ the non-positive self-adjoint operator in the weighted space $L^2(u_\epsilon^G) := L^2((0,\infty), u_\epsilon^G(x)dx)$ associated to $\mathcal{E}_\epsilon$ such that

$$\mathcal{E}_\epsilon(\phi,\psi) = \langle -\mathcal{L}_\epsilon\phi, \psi\rangle_{L^2(u_\epsilon^G)}, \quad \phi,\psi \in D(\mathcal{L}_\epsilon), \tag{1.3}$$

where $D(\mathcal{L}_\epsilon)$ is the domain of $\mathcal{L}_\epsilon$ and has $C_0^\infty((0,\infty))$ as a dense subset. It is informative to mention

$$\mathcal{L}_\epsilon\phi = \frac{\epsilon^2}{2}a\phi'' + b\phi', \quad \forall \phi \in C_0^\infty((0,\infty)),$$

that is, $\mathcal{L}_\epsilon$ is a self-adjoint extension in $L^2(u_\epsilon^G)$ of the generator of X_t^ϵ or (1.1). In particular, $\mathcal{L}_\epsilon$ is degenerate at the extinction state 0.

We show in Proposition 2.1 that the spectrum of $-\mathcal{L}_\epsilon$ consists of simple eigenvalues $\lambda_{\epsilon,i}$, $i \in \mathbb{N} := \{1,2,3,\dots\}$ listed as follows: $0 < \lambda_{\epsilon,1} < \lambda_{\epsilon,2} < \lambda_{\epsilon,3} < \cdots \to \infty$. To study their asymptotics, in particular, that of $\lambda_{\epsilon,1}$ and $\lambda_{\epsilon,2}$, as $\epsilon \to 0$, we follow [41] to define *valleys* revealing the geometry of the potential function

$$V(x) = -2\int_0^x \frac{b(s)}{a(s)}ds, \quad x \in (0,\infty). \tag{1.4}$$

Some basic properties of V are collected in Lemma 2.1. For $I \subset (0,\infty)$, set $\gamma(I) := \sup_I V - \inf_I V$.

Definition 1.2. *An open interval $I \subset (0,\infty)$ is called a* valley *if it is one of the connected components of the sublevel set $\{x \in (0,\infty) : V(x) < \rho\}$ and satisfies $V(\partial I) = \rho$ for some $\rho \in \mathbb{R}$. We say $I \subset (0,\infty)$ a r-*valley *if it is a valley and has depth r, namely, $\gamma(I) = r$.*

As $V(0+) = 0$, it is possible that a connected component of $\{x \in (0,\infty) : V(x) < \rho\}$ for $\rho > 0$ takes the form $(0,\ell)$ with $V(\ell) = \rho > 0 = V(0+)$. The condition $V(\partial I) = \rho$ in the definition of a valley is to exclude such possibilities. Hence, $(0,\ell)$ is a valley if and only if ℓ is the smallest positive zero of V.

For $r > 0$, let $N(r)$ be the number of r-valleys. Clearly, $r \mapsto N(r)$ is a non-negative, non-increasing, left-continuous and bounded step function on $(0,\infty)$. For each $i \in \mathbb{N}$, we define

$$r_i := \inf\{r > 0 : N(r) < i\}. \tag{1.5}$$

Note that since $V(0+) = 0$, for each $i \in \mathbb{N}$, there always exists $r > 0$ such that $N(r) < i$, and hence, r_i is well-defined. Intuitively, r_i, $i \in \mathbb{N}$ are the points where $N(r)$ has jump discontinuities.

Remark 1.1. *We make some remarks about r_i, $i \in \mathbb{N}$.*

(1) *If there is $r > 0$ such that $N(r) \geq i$, then $r_i = \sup\{r > 0 : N(r) \geq i\}$. Otherwise, $r_i = 0$.*
(2) *Obviously, $r_1 \geq r_2 \geq r_3 \geq \cdots \geq 0$ and $r_1 > 0$. Moreover, r_1 has the representation:*

$$r_1 = \sup_{x\in(0,\infty)} \left[\inf_{\xi\in C_x} \sup_{t\in[0,1]} V(\xi(t)) - V(x)\right],$$

where for each $x \in (0,\infty)$,

$$C_x = \{\xi : [0,1] \to [0,\infty) : \xi \text{ is continuous and satisfies } \xi(0) = x \text{ and } \xi(1) = 0\}.$$

(3) *There is at least one r_1-valley, but not a single r-valley with $r > r_1$ exists. Note that $r_1 > r_2$ if and only if there is a unique r_1-valley, and $r_2 = 0$ if and only if b changes its sign once in $(0,\infty)$ (or equivalently, V has a single well).*

(4) *There is $i_* \in \mathbb{N}$ such that $r_{i_*} > 0 = r_{i_*+1}$. This i_* is nothing but the number of wells that V has. Hence, if $i_* \in \mathbb{N}\setminus\{1\}$, the ODE (1.2) has multiple stable equilibria separated by unstable ones, and thus, X_t^ϵ exhibits metastability (see e.g. [23, 46]).*

Our first result studies exponential asymptotics of the eigenvalues $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$ as $\epsilon\to 0$.

Theorem A. *Assume* **(H)**. *Then, $\lim_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,i} = -r_i$ for each $i\in\mathbb{N}$.*

We mention that exponential asymptotics of eigenvalues associated to non-degenerate, regular and reversible diffusion processes with small noises have been studied in [20] by large deviation methods, and in [41, 43] by variational formulas. When local minimal points of the potential function are non-degenerate, much refined results are established in [4] by a potential-theoretic approach developed in [3], and in [26, 27, 38] by a Witten complex approach. For the asymptotic of the first eigenvalue, special attention has been attracted due to its connections with the first exit time or first passage time (see e.g. [53, 21, 16, 33, 34, 17, 29, 40]). Because of the degeneracy and singularity of noises at the extinction state 0 resulting in in particular the degeneracy of $\mathcal{L}_\epsilon$ and the non-integrable singularity of u_ϵ^G at 0, known results do not directly apply in our case. Addressing difficulties caused by these unpleasant facts, we manage to adapt approaches in [29, 40, 41, 43] all based on variational formulas for eigenvalues.

Since $r_1 > 0$ as mentioned in Remark 1.1 (2), Theorem A implies that $\lambda_{\epsilon,1}$ is exponentially small in ϵ. The exponential smallness of $\lambda_{\epsilon,2}$ in ϵ follows if $r_2 > 0$. But, $r_2 = 0$ takes place in typical applications. For instance, it is the case when b is a logistic growth rate function, that is, $b(x) = b_1x - b_2x^2$ for $b_1, b_2 > 0$. In this case, Theorem A only says that the leading asymptotic of $\lambda_{\epsilon,2}$ in ϵ is non-exponential, and does not tell any further information about $\lambda_{\epsilon,2}$. In Proposition 5.2, upper bounds of the eigenvalues $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$ are established to complement Theorem A.

Our next result aims at investigating in particular the non-exponential asymptotic of $\lambda_{\epsilon,2}$ as $\epsilon\to 0$ when $r_2 = 0$. Note from Remark 1.1 (3) that $r_2 = 0$ if and only if b only changes its sign once.

Theorem B. *Assume* **(H)**. *Suppose, in addition, that*

- *there is $x^*\in(0,\infty)$ such that $b > 0$ on $(0,x^*)$, $b < 0$ on (x^*,∞) and $b'(x^*) < 0$;*
- *b is twice continuously differentiable near x^*.*

Then, $\lim_{\epsilon\to 0}\lambda_{\epsilon,i} = b'(x^*)(1-i)$ for each $i\in\mathbb{N}$.

We remark that Theorem B has been essentially proven in [14, 51, 35]. Indeed, if we consider the Schrödinger operator $-\mathcal{L}_\epsilon^S$ that is unitarily equivalent to $-\mathcal{L}_\epsilon$ (see Subsection 2.2), then the operator $-2\epsilon^2\mathcal{L}_\epsilon^S$ is in the form used for studying semi-classical limits of eigenvalues, except that there is an extra term of negligible order ϵ^4, which however blows up at the extinction state 0. Fortunately, this term can be absorbed into the main part of the operator, allowing us to apply classical results to Schrödinger operators that are comparable in the sense of quadratic forms to the original one. See Subsection 5.2 for more details.

Combining Theorem A and Theorem B, we obtain the following result.

Corollary A. *Assume* **(H)**. *Suppose, in addition, that*

- *there is $x^*\in(0,\infty)$ such that $b > 0$ on $(0,x^*)$, $b < 0$ on (x^*,∞) and $b'(x^*) < 0$;*

- *b is twice continuously differentiable near x^*.*

Then, $\lim_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,1}=-2\int_0^{x^*}\frac{b(s)}{a(s)}ds$ *and* $\lim_{\epsilon\to 0}\lambda_{\epsilon,2}=b'(x^*)(1-i)$ *for each* $i\in\mathbb{N}\setminus\{1\}$.

We discuss some implications of Theorem A and Theorem B.

- The exponential smallness of $\lambda_{\epsilon,1}$ asserts the exponentially long lifetime of X_t^ϵ. On average, it takes $\frac{1}{\lambda_{\epsilon,1}}$ long for X_t^ϵ to get absorbed by the extinction state 0.
- As long as $r_1>r_2$ (if and only if there is a unique r_1-valley), Theorem A yields that $\lambda_{\epsilon,1}<\lambda_{2,\epsilon}-\lambda_{1,\epsilon}(\approx\lambda_{\epsilon,2})$ for all small ϵ, implying the multi-scale nature of the dynamics of X_t^ϵ as mentioned earlier. When $r_2>0$, the gap $\lambda_{2,\epsilon}-\lambda_{1,\epsilon}$ is also exponentially small, saying that X_t^ϵ needs to spend exponentially long time ($\frac{1}{\lambda_{2,\epsilon}-\lambda_{1,\epsilon}}$ long on average) in order to get close to the QSD μ_ϵ. This happens because of the metastability as mentioned in Remark 1.1 (4) that trajectories of X_t^ϵ are trapped in potential wells of V for exponentially long times before transitions can be made. Nonetheless, X_t^ϵ begins to interact with μ_ϵ at a much earlier time.
- In the situation of Theorem B or Corollary A, the gap $\lambda_{\epsilon,2}-\lambda_{\epsilon,1}$ is of order $O(1)$, and hence, X_t^ϵ quickly approaches to the QSD μ_ϵ.

Our last result, establishing an upper bound on the distance between X_t^ϵ and a convex combination of the QSD μ_ϵ and the extinction state, gives a qualitative characterization of the multi-scale dynamics of X_t^ϵ. Denote by $\mathbb{E}_\mu^\epsilon$ the expectation with respect to $\mathbb{P}_\mu^\epsilon$, by $\|\cdot\|_{TV}$ the total variation distance, by $\mathcal{P}([0,\infty))$ the set of Borel probability measures on $[0,\infty)$, and by $\phi_{\epsilon,1}$ the positive eigenfunction of $-\mathcal{L}_\epsilon$ associated to $\lambda_{\epsilon,1}$ satisfying the normalization $\|\phi_{\epsilon,1}\|_{L^2(u_\epsilon^G)}=1$. We point out that $\phi_{\epsilon,1}$ also belongs to $L^1(u_\epsilon^G)$ (see Proposition 2.3). Recall that $\mathcal{A}$ is the global attractor of (1.2), and the uniqueness of r_1-valleys is equivalent to $r_1>r_2$.

Theorem C. *Assume* **(H)** *and there exists a unique r_1-valley.*

(1) *For each compact $K\subset[0,\infty)$, there are $C=C(K)>0$ and small $\epsilon_*=\epsilon_*(K)>0$ such that*

$$\sup_{\substack{\mu\in\mathcal{P}([0,\infty))\\ \mathrm{supp}(\mu)\subset K}}\left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon\in\bullet]-\left[\alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\mu_\epsilon+(1-\alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t})\delta_0\right]\right\|_{TV}$$
$$\leq\exp\left\{\frac{C}{\epsilon^2}-\lambda_{\epsilon,2}t\right\},\quad\forall t>0\text{ and }\epsilon\in(0,\epsilon_*),$$

where $\alpha_\epsilon(\mu):=\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\|\phi_{\epsilon,1}\|_{L^1(\mu)}$ satisfies

$$\sup_{\substack{\mu\in\mathcal{P}([0,\infty))\\ \mathrm{supp}(\mu)\subset K}}\alpha_\epsilon(\mu)\leq 1+e^{-\frac{r_1-r_2}{2\epsilon^2}},\quad\forall\epsilon\in(0,\epsilon_*).$$

(2) *For each $0<\delta\ll 1$ and compact sets $K_1,K_2\subset(0,\infty)$, there are $T=T(K_1,\delta)>0$, $C_1=C_1(K_2)>0$, $C_2=C_2(K_1,\delta)>0$, $C_3=C_3(K_1,\delta)>0$, $C_4=C_4(K_1,\delta)>0$ and $\epsilon_*=\epsilon_*(K_1)>0$ such that for each continuous function $f:(0,\infty)\to\mathbb{R}$ with $\mathrm{supp}(f)\subset K_2$, there holds*

$$\sup_{\substack{\mu\in\mathcal{P}([0,\infty))\\ \mathrm{supp}(\mu)\subset K_1}}\left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)]-\hat{\alpha}_\epsilon(\mu,\delta)e^{-\lambda_{\epsilon,1}t}\int_0^\infty fd\mu_\epsilon\right|$$
$$\leq\frac{C_1}{\epsilon}\exp\left\{\frac{\sup_{\mathcal{A}_\delta}V-\inf_{K_2}V}{2\epsilon^2}-\lambda_{\epsilon,2}(t-T)\right\}\|f\|_\infty$$
$$+C_2\exp\left\{-\frac{C_3}{\epsilon^2}-\frac{\lambda_{\epsilon,1}(t-T)}{C_4}\right\}\|f\|_\infty,\quad\forall t>T+2\text{ and }\epsilon\in(0,\epsilon_*),$$

where $\mathcal{A}_\delta$ is the δ-neighborhood of $\mathcal{A}$ and $\hat{\alpha}_\epsilon(\mu,\delta) := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_{\mathcal{A}_\delta} \phi_{\epsilon,1} d\mu_T^\epsilon e^{\lambda_{\epsilon,1}T}$ with $\mu_T^\epsilon := \mathbb{P}_\mu^\epsilon[X_T^\epsilon \in \bullet]$ satisfies

$$\sup_{\substack{\mu\in\mathcal{P}([0,\infty))\\ \mathrm{supp}(\mu)\subset K_1}} \sup_{0<\delta\ll 1} \hat{\alpha}_\epsilon(\mu,\delta) \leq 1 + e^{-\frac{r_1-r_2}{2\epsilon^2}}, \quad \forall \epsilon \in (0,\epsilon_*).$$

If, in addition, K_1 is contained in the basin of attraction of a local attractor $\mathcal{A}^{loc}$ of (1.2), then the same estimate holds with $\mathcal{A}_\delta$ replaced by $\mathcal{A}_\delta^{loc}$, the δ-neighborhood of $\mathcal{A}^{loc}$.

Results in Theorem C can be interpreted as follows.

- If t is such that $t \gg \frac{1}{\lambda_{\epsilon,1}}$, then $\left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - \delta_0\right\|_{TV} \ll 1$.
- If t is such that $\frac{1}{\epsilon^2\lambda_{\epsilon,2}} \ll t \ll \frac{1}{\lambda_{\epsilon,1}}$, then
$$\left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - [\alpha_\epsilon(\mu)\mu_\epsilon + (1-\alpha_\epsilon(\mu))\delta_0]\right\|_{TV} \ll 1.$$
More precise interpretation requires more detailed knowledge of the term $\alpha_\epsilon(\mu)$, which is expected to be very close to 1, and therefore, $\left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - \mu_\epsilon\right\|_{TV} \ll 1$. Since μ_ϵ tends to concentrate on the global attractor $\mathcal{A}$ and approaches to the set of invariant measures of (1.2) as $\epsilon \to 0$ (see [50, Theorem A and B]), transient dynamics of X_t^ϵ has to have a close relation to the structure of $\mathcal{A}$, that we explore in future works.

  Showing $\alpha_\epsilon(\mu) \approx 1$ amounts to proving
$$\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1} \to 1 \quad \text{locally uniformly in } (0,\infty) \quad \text{as} \quad \epsilon \to 0, \tag{1.6}$$
which turns out to be far-reaching. In Appendix A, we justify (1.6) as well as the limits of $\alpha_\epsilon(\mu)$ and $\hat{\alpha}_\epsilon(\mu,\delta)$ in the case that b changes its sign only once appealing to results established in the forthcoming work [47].
- The estimate in (2) for the dynamics of compactly supported observables can be interpreted in the similar way, can be made much more concise if the constants are allowed to depend on all of K_1, K_2 and δ, and has an advantage over that in (1) if $\sup_{\mathcal{A}_\delta} V < \inf_{K_2} V$, or $\sup_{\mathcal{A}_\delta^{loc}} V < \inf_{K_2} V$. In which case, the estimate in (2) begins to tell useful dynamical information of X_t^ϵ when $t \gg 1$ (independent of $0 < \epsilon \ll 1$). In contrast, the estimate in (1) does not tell anything until $t \gg \frac{1}{\epsilon^2\lambda_{\epsilon,2}}$.

  We point out that when the global attractor $\mathcal{A}$ is a singleton set $\{x^*\}$, the potential function V has a unique local minimal point at x^*, and hence, the condition $\sup_{\mathcal{A}_\delta} V < \inf_{K_2} V$ is true as long as $\mathcal{A}_\delta \cap K_2 = \emptyset$. More generally, if the local attractor $\mathcal{A}^{loc}$ is contained in $E := \{x \in (0,\infty) : V(x) = \min V\}$, then for each compact $K_2 \subset (0,\infty)$ satisfying $K_2 \cap E = \emptyset$, there is $\delta_* = \delta_*(K_2) > 0$ such that $\sup_{\mathcal{A}_\delta^{loc}} V < \inf_{K_2} V$ for all $\delta \in (0,\delta_*)$.

Estimates as in Theorem C (1) have been established in [11, 12] for absorbed birth-and-death processes whose mean-field ODEs, restricted to the interior of the phase space, have simple dynamics in the sense that the global attractor is a non-degenerate fixed point. If we further explore certain uniform-in-ϵ "coming down from infinity" of X_t^ϵ as did in [11, 12], it is possible to release Theorem C (1) from the dependence on the compact set K. Since no additional dynamical information can be obtained from doing so, we do not pursue here.

In Theorem C (2), $\mathrm{supp}(\mu)$ being contained in the compact set K_1 plays an important role. In fact, in the proof, we first use the compact dissipativity property of the dynamical system generated by (1.2) to find T, which is the smallest time such that $\bigcup_{x_0\in K_1}\bigcup_{t>T} x_t \subset \mathcal{A}_{\frac{\delta}{2}}$ (resp. $\mathcal{A}_{\frac{\delta}{2}}^{loc}$), where

x_t is the solution of (1.2), and then apply the sample path large deviation principle to conclude that the distribution of X_T^ϵ mainly concentrates in $\mathcal{A}_\delta$ (resp. $\mathcal{A}_\delta^{loc}$). From where, better estimates can be derived.

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results for later use. In Section 3, we prove Theorem A in the case $i=1$. In Section 4, we prove Theorem A in the case $i\in\mathbb{N}\setminus\{1\}$. Section 5 and Section 6 are respectively devoted to the proof of Theorem B and Theorem C. In Section 7, we discuss some applications. We justify (1.6) in Appendix A.

2. Preliminaries

This is a service section. We collect preliminary results for later purposes.

2.1. **The potential function.** Basic properties of the potential function V defined in (1.4) are summarized in the next lemma.

Lemma 2.1. *The following hold.*

- $V\in C^2((0,\infty))\cap C((0,\infty))$, $V(0+):=\lim_{x\to 0^+}V(x)=0$ *and* $V(\infty):=\lim_{x\to\infty}V(x)=\infty$.
- $V(x)<0$ *for all* $0<x\ll 1$. *In particular, the minimum of* V *is negative and attainable.*

If, in addition, that zeros of b *in* $(0,\infty)$ *are isolated, and there is a zero* $x^*\in(0,\infty)$ *of* b *such that* $b\geq 0$ *on* $(0,x^*)$, $b\leq 0$ *on* (x^*,∞) *and* $b'(x^*)<0$, *then* V *has a unique local minimal point at* x^*, *and satisfies* $V''(x^*)>0$.

2.2. **Semi-classical Schrödinger operators.** Consider the change of variable

$$y=\xi(x)=\int_0^x\frac{1}{\sqrt{a(z)}}dz,\quad x\in(0,\infty).$$

Assumptions on a ensure that $\xi'>0$ on $(0,\infty)$, $\lim_{x\to 0^+}\xi(x)=0$ and $y_\infty:=\lim_{x\to\infty}\xi(x)\in(0,\infty]$. In particular, $\xi:(0,\infty)\to(0,y_\infty)$ is invertible.

Let $v_\epsilon^G(y):=\frac{u_\epsilon^G(x)}{\xi'(x)}$ and define

$$\mathcal{L}_\epsilon^Y=\frac{\epsilon^2}{2}\frac{d^2}{dy^2}-q_\epsilon(y)\frac{d}{dy}:L^2(v_\epsilon^G)\to L^2(v_\epsilon^G),$$

where $q_\epsilon=-(\mathcal{L}_\epsilon\xi)\circ\xi^{-1}$ and $L^2(v_\epsilon^G):=L^2((0,y_\infty),v_\epsilon^G(y)dy)$. It is straightforward to check that $U_\epsilon\mathcal{L}_\epsilon=\mathcal{L}_\epsilon^YU_\epsilon$, where $U_\epsilon:L^2(u_\epsilon^G)\to L^2(v_\epsilon^G)$, $f\mapsto f\circ\xi^{-1}$ is a unitary transform.

Consider the following semi-classical Schrödinger operator

$$\mathcal{L}_\epsilon^S=\frac{\epsilon^2}{2}\frac{d^2}{dy^2}-\frac{1}{2}\left[\frac{q_\epsilon^2(y)}{\epsilon^2}-q_\epsilon'(y)\right]:L^2((0,y_\infty))\to L^2((0,y_\infty)).\tag{2.1}$$

It is easy to verify that $\tilde{U}_\epsilon\mathcal{L}_\epsilon^Y=\mathcal{L}_\epsilon^S\tilde{U}_\epsilon$, where $\tilde{U}_\epsilon:L^2(v_\epsilon^G)\to L^2((0,y_\infty))$, $f\mapsto f\sqrt{v_\epsilon^G}$ is a unitary transform. We include the following commutative diagram for readers' convenience:

$$\begin{array}{ccccc}
L^2(u_\epsilon^G) & \xrightarrow{U_\epsilon} & L^2(v_\epsilon^G) & \xrightarrow{\tilde{U}_\epsilon} & L^2((0,y_\infty))\\
\downarrow{\scriptstyle\mathcal{L}_\epsilon} & & \downarrow{\scriptstyle\mathcal{L}_\epsilon^Y} & & \downarrow{\scriptstyle\mathcal{L}_\epsilon^S}\\
L^2(u_\epsilon^G) & \xrightarrow{U_\epsilon} & L^2(v_\epsilon^G) & \xrightarrow{\tilde{U}_\epsilon} & L^2((0,y_\infty))
\end{array}$$

Denote by V_ϵ the potential of the Schrödinger operator $-\mathcal{L}_\epsilon^S$, namely, $V_\epsilon=\frac{1}{2}\left(\frac{q_\epsilon^2}{\epsilon^2}-q_\epsilon'\right)$.

Lemma 2.2. *The following hold.*

(1) $2V_\epsilon(y) = \epsilon^2 \left\{ \frac{3}{16} \frac{[a'(x)]^2}{a(x)} - \frac{a''(x)}{4} \right\} + b'(x) - \frac{a'(x)b(x)}{a(x)} + \frac{1}{\epsilon^2} \frac{b^2(x)}{a(x)}$.

(2) *There exists* $y_1 \in (0, y_\infty)$ *such that*

$$V_\epsilon(y) \gtrsim \frac{\epsilon^2}{\xi^{-1}(y)}, \quad \forall y \in (0, y_1],\ 0 < \epsilon \ll 1 \quad \textit{and} \quad \inf_{0<\epsilon\ll 1} \inf_{(0,y_1]} V_\epsilon > 0.$$

(3) *There exists* $y_2 \in (0, y_\infty)$ *such that*

$$V_\epsilon(y) \gtrsim \frac{1}{\epsilon^2} \frac{b^2(\xi^{-1}(y))}{a(\xi^{-1}(y))}, \quad \forall y \in [y_2, y_\infty),\ 0 < \epsilon \ll 1.$$

(4) *The family* $\{V_\epsilon\}_{0<\epsilon\ll 1}$ *is uniformly lower bounded, that is,* $\inf_{0<\epsilon\ll 1} \min_{(0,y_\infty)} V_\epsilon > -\infty$.

Proof. (1) Note that $q_\epsilon(y) = -\mathcal{L}_\epsilon \xi(x)$ and $q'_\epsilon(y) = -\frac{(\mathcal{L}_\epsilon \xi)'(x)}{\xi'(x)}$. Straightforward calculations give

$$\begin{aligned} \mathcal{L}_\epsilon \xi(x) &= -\frac{\epsilon^2}{4} \frac{a'(x)}{\sqrt{a(x)}} + \frac{b(x)}{\sqrt{a(x)}}, \\ \frac{(\mathcal{L}_\epsilon \xi)^2(x)}{\epsilon^2} &= \frac{\epsilon^2}{16} \frac{[a'(x)]^2}{a(x)} - \frac{1}{2} \frac{a'(x)b(x)}{a(x)} + \frac{1}{\epsilon^2} \frac{b^2(x)}{a(x)}, \\ \frac{(\mathcal{L}_\epsilon \xi)'(x)}{\xi'(x)} &= -\frac{\epsilon^2}{4} a''(x) + \frac{\epsilon^2}{8} \frac{[a'(x)]^2}{a(x)} + b'(x) - \frac{1}{2} \frac{a'(x)b(x)}{a(x)}. \end{aligned}$$

The formula for V_ϵ follows.

(2) Let $x_1 \in (0,1)$ and $0 < \delta \ll 1$ be such that the following hold for $x \in (0, x_1]$:

$$\begin{aligned} &0 \le a(x) \le (a'(0) + \delta)\, x, \quad a'(x) \in [a'(0) - \delta, a'(0) + \delta], \quad a''(x) \le a''(0) + \delta, \\ &b(x) \in [(b'(0) - \delta)\, x, (b'(0) + \delta)\, x], \quad b'(x) \ge b'(0) - \delta. \end{aligned}$$

Then, for each $x \in (0, x_1]$ and $0 < \epsilon \ll 1$,

$$\begin{aligned} 2V_\epsilon(\xi(x)) &\ge \epsilon^2 \left\{ \frac{3}{16} \frac{(a'(0) - \delta)^2}{(a'(0) + \delta)\, x} - \frac{a''(0) + \delta}{4} \right\} + b'(0) - \delta \\ &\qquad - \frac{(a'(0) + \delta)\,(b'(0) + \delta)\, x}{(a'(0) + \delta)\, x} + \frac{1}{\epsilon^2} \frac{(b'(0) - \delta)^2 x^2}{(a'(0) + \delta)\, x} \\ &= \epsilon^2 \left(\frac{c_1}{x} - \frac{a''(0) + \delta}{4} \right) - 2\delta + \frac{c_2 x}{\epsilon^2} \ge \frac{\epsilon^2 c_1}{x} - 3\delta + \frac{c_2 x}{\epsilon^2}, \end{aligned} \tag{2.2}$$

where $c_1 = \frac{3}{16} \frac{(a'(0) - \delta)^2}{a'(0) + \delta}$ and $c_2 = \frac{(b'(0) - \delta)^2}{a'(0) + \delta}$. The estimate (2.2) gives $V_\epsilon(\xi(x)) \gtrsim \frac{\epsilon^2}{x}$ for all $x \in (0, x_1]$ and $0 < \epsilon \ll 1$. Moreover,

$$\inf_{0<\epsilon\ll 1} \inf_{(0,x_1]} V_\epsilon \circ \xi \ge \frac{1}{2} \inf_{0<\epsilon\ll 1} \min_{x\in(0,x_1]} \left(\frac{\epsilon^2 c_1}{x} - 3\delta + \frac{c_2 x}{\epsilon^2} \right) = \sqrt{c_1 c_2} - \frac{3\delta}{2} > 0.$$

The results follow by setting $y_1 = \xi(x_1)$.

(3) It is a simple consequence of **(H)**(3).

(4) Let y_1 and y_2 be as in (2) and (3), respectively. We may assume, without loss of generality, that $y_1 < y_2$. Then,

$$\inf_{0<\epsilon\ll 1} \min_{[y_1,y_2]} V_\epsilon \ge \frac{1}{2} \inf_{0<\epsilon\ll 1} \min_{[\xi^{-1}(y_1), \xi^{-1}(y_2)]} \left(-\frac{\epsilon^2 a''}{4} + b' - \frac{a'b}{a} \right) > -\infty.$$

This together with (2) and (3) yields the result. □

Remark 2.1. As $\xi^{-1}(y)\to 0$ as $y\to 0^+$, Lemma 2.2 (2) implies that for each $0<\epsilon\ll 1$, there holds $V_\epsilon(y)\to\infty$ as $y\to 0^+$. But, it is an unpleasant fact that this blowup is not uniform in $0<\epsilon\ll 1$. In fact, for $x\approx 0$, there holds

$$\begin{aligned}2V_\epsilon(\xi(x)) &\approx \epsilon^2\left\{\frac{3}{16}\frac{(a'(0))^2}{a'(0)x}-\frac{a''(0)}{4}\right\}+b'(0)-\frac{(a'(0)+a''(0)x)\,b'(0)x}{a'(0)x}+\frac{1}{\epsilon^2}\frac{(b'(0))^2\,x^2}{a'(0)x}\\ &=\epsilon^2\left[\frac{3}{16}\frac{a'(0)}{x}-\frac{a''(0)}{4}\right]-\frac{a''(0)b'(0)}{a'(0)}x+\frac{(b'(0))^2}{a'(0)}\frac{x}{\epsilon^2},\end{aligned}$$

leading to

$$2V_\epsilon(\xi(\epsilon^2))\approx\frac{3a'(0)}{16}-\frac{\epsilon^2a''(0)}{4}-\frac{\epsilon^2a''(0)b'(0)}{a'(0)}+\frac{(b'(0))^2}{a'(0)}\approx\frac{3a'(0)}{16}+\frac{(b'(0))^2}{a'(0)}.$$

2.3. **Spectrum and semigroup.** We present the following result on the spectrum of $-\mathcal{L}_\epsilon$ and the semigroup generated by $\mathcal{L}_\epsilon$.

Proposition 2.1. *For each $0<\epsilon\ll 1$, the following hold.*

(1) *$-\mathcal{L}_\epsilon$ has purely discrete spectrum contained in $(0,\infty)$ and listed as follows:*

$$\lambda_{\epsilon,1}<\lambda_{\epsilon,2}<\lambda_{\epsilon,3}<\cdots\to\infty.$$

(2) *Each $\lambda_{\epsilon,i}$ is associated with a unique eigenfunction $\phi_{\epsilon,i}\in L^2(u^G_\epsilon)\cap L^1(u^G_\epsilon)\cap C^2((0,\infty))$ subject to the normalization $\|\phi_{\epsilon,i}\|_{L^2(u^G_\epsilon)}=1$. Moreover, $\phi_{\epsilon,1}$ is positive on $(0,\infty)$.*

(3) *The set $\{\phi_{\epsilon,i},i\in\mathbb{N}\}$ is an orthonormal basis of $L^2(u^G_\epsilon)$.*

(4) *$\mathcal{L}_\epsilon$ generates a positive analytic semigroup $(P^\epsilon_t)_{t\geq 0}$ of contractions on $L^2(u^G_\epsilon)$ satisfying*

$$P^\epsilon_tf=\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}],\quad\forall f\in L^2(u^G_\epsilon)\cap C_b([0,\infty)),\quad t\geq 0.$$

(5) *For each $k\in\mathbb{N}$,*

$$P^\epsilon_tf=\sum_{i=1}^{k-1}e^{-\lambda_{\epsilon,i}t}\langle f,\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\phi_{\epsilon,i}+P^\epsilon_tQ^\epsilon_kf,\quad\forall t\geq 0\text{ and }f\in L^2(u^G_\epsilon),\tag{2.3}$$

where Q^ϵ_k is the spectral projection of $\mathcal{L}_\epsilon$ corresponding to the eigenvalues $\{\lambda_{\epsilon,j}\}_{j\geq k}$. Moreover,

$$\|P^\epsilon_tQ^\epsilon_kf\|_{L^2(u^G_\epsilon)}\leq e^{-\lambda_{\epsilon,k}t}\|f\|_{L^2(u^G_\epsilon)},\quad\forall f\in L^2(u^G_\epsilon).$$

(6) *For each $f\in C_b([0,\infty))$, (2.3) holds pointwise.*

Proof. Fix $0<\epsilon\ll 1$. By Lemma 2.2, V_ϵ is lower bounded and satisfies $V_\epsilon(y)\to\infty$ as $y\to 0^+$ and $y\to y^-_\infty$. As a result (see e.g. [2]), the spectrum of $-\mathcal{L}^S_\epsilon$ is purely discrete and consists of simple eigenvalues. The associated normalized eigenfunctions $\{\Phi_{\epsilon,i}\}_{i\in\mathbb{N}}$ are twice continuously differentiable and form an orthonormal basis of $L^2((0,y_\infty))$. The eigenfunction $\Phi_{\epsilon,1}$ associated to $\lambda_{\epsilon,1}$ can be chosen to be positive on $(0,y_\infty)$. Moreover, $\{\Phi_{\epsilon,i}\}_{i\in\mathbb{N}}\subset L^2((0,y_\infty),(V_\epsilon+M)dy)$ for some $M>0$ satisfying $\inf V_\epsilon+M>0$.

As $\mathcal{L}_\epsilon$ and $\mathcal{L}^S_\epsilon$ are unitarily equivalent, all conclusions in (1)-(3) except $\lambda_{\epsilon,1}>0$ and $\phi_{\epsilon,i}\in L^1(u^G_\epsilon)$ follow, where $\phi_{\epsilon,i}:=U^{-1}_\epsilon\tilde{U}^{-1}_\epsilon\Phi_{\epsilon,i}$. Obviously, $\lambda_{\epsilon,1}\geq 0$. Suppose $\lambda_{\epsilon,1}=0$ for contradiction. Then, $-\mathcal{L}_\epsilon\phi_{\epsilon,1}=0$. It follows from (1.3) that $\frac{\epsilon^2}{2}\int_0^\infty a|\phi'_{\epsilon,1}|^2u^G_\epsilon dx=0$, which implies that $\phi_{\epsilon,1}$ is a positive constant function on $(0,\infty)$. As u^G_ϵ is non-integrable on $(0,\infty)$, we find $\int_0^\infty\phi^2_{\epsilon,1}u^G_\epsilon dx=\infty$, which is contradictory to $\phi_{\epsilon,1}\in L^2(u^G_\epsilon)$. Hence, $\lambda_{\epsilon,1}>0$. The decay of u^G_ϵ near ∞ ensures $\int_1^\infty\phi_{\epsilon,i}u^G_\epsilon dx<\infty$, and the combination of $\Phi_{\epsilon,i} \in L^2((0,y_\infty),(V_\epsilon+M)dy)$ and the behavior of V_ϵ near 0 guarantee $\int_0^1 \phi_{\epsilon,i} u^G_\epsilon dx < \infty$ (see [5, Proposition 4.3]). Thus, $\phi_{\epsilon,i} \in L^1(u^G_\epsilon)$.

Due to the self-adjointness of $\mathcal{L}_\epsilon$,

$$\|(\lambda-\mathcal{L}_\epsilon)^{-1}\|_{L^2(u^G_\epsilon)\to L^2(u^G_\epsilon)} = \frac{1}{\mathrm{dist}(\lambda,\sigma(\mathcal{L}_\epsilon))} \le \frac{1}{|\lambda|}, \quad \forall \lambda\in\mathbb{C} \text{ with } \Re\lambda>0.$$

Therefore (see e.g. [48, Theorem 2.5.2]), $\mathcal{L}_\epsilon$ generates an analytic semigroup $(P^\epsilon_t)_{t\ge 0}$ of contractions on $L^2(u^G_\epsilon)$. The positivity of $(P^\epsilon_t)_{t\ge0}$ follows from the maximum principle. The theory of symmetric Dirichlet forms (see e.g. [5]) gives the following stochastic representation of P^ϵ_t:

$$P^\epsilon_t f(x) = \mathbb{E}^\epsilon_x[f(X^\epsilon_t)\mathbb{1}_{t<T^\epsilon_0}], \quad \forall x\in(0,\infty) \text{ and } f\in C_b([0,\infty))\cap L^2(u^G_\epsilon).$$

This proves (4). The statements in (5) follow from general theory concerning the asymptotic behaviors of analytic semigroups (see e.g. [18, Theorem V.3.1]) and the simplicity of eigenvalues $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$.

Since $\{\phi_{\epsilon,i}\}_{i\in\mathbb{N}} \subset L^1(u^G_\epsilon)$ and $Q^\epsilon_k = id - \sum_{i=1}^{k-1}\langle\bullet,\phi_{\epsilon,i}\rangle_{L^2(u^G_\epsilon)}\phi_{\epsilon,i}$, each term on the right-hand side of (2.3) is well-defined. Standard approximations then ensure that (2.3) holds in the pointwise sense for each $f\in C_b([0,\infty))$. This proves (6), and finishes the proof of the proposition. □

As a result of Proposition 2.1, the following variational formulas for $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$ hold:

$$\lambda_{\epsilon,1} = \inf_{\phi\in C^\infty_0((0,\infty))\setminus\{0\}} \frac{\langle-\mathcal{L}_\epsilon\phi,\phi\rangle_{L^2(u^G_\epsilon)}}{\|\phi\|^2_{L^2(u^G_\epsilon)}} = \inf_{\phi\in C^\infty_0((0,\infty))\setminus\{0\}} \frac{\frac{\epsilon^2}{2}\displaystyle\int_0^\infty a|\phi'|^2 u^G_\epsilon dx}{\displaystyle\int_0^\infty \phi^2 u^G_\epsilon dx}, \tag{2.4}$$

$$\lambda_{\epsilon,i} = \inf_{\substack{\phi\in C^\infty_0((0,\infty))\setminus\{0\}\\ \phi\perp\phi_{\epsilon,j}, j\in\{1,\dots,i-1\}}} \frac{\langle-\mathcal{L}_\epsilon\phi,\phi\rangle_{L^2(u^G_\epsilon)}}{\|\phi\|^2_{L^2(u^G_\epsilon)}}, \quad i\in\mathbb{N}\setminus\{1\}, \tag{2.5}$$

where the orthogonality $\perp$ is taken in $L^2(u^G_\epsilon)$.

2.4. **Uniform estimate of the first eigenfunctions.** We prove the following result asserting that the normalized eigenfunction $\phi_{\epsilon,1}$ given in Proposition 2.1, as an element of $L^2(u^G_\epsilon)$, has uniform-in-ϵ positive concentration away from 0 and ∞.

Proposition 2.2. *There are $\kappa\in(0,1)$ and $K\gg1$ such that $\int_{(0,\infty)\setminus(\frac{1}{K},K)}\phi^2_{\epsilon,1}u^G_\epsilon dx \le \kappa$ for all $0<\epsilon\ll1$.*

Set $\Phi_{\epsilon,1} := \tilde{U}_\epsilon U_\epsilon \phi_{\epsilon,1}$, where U_ϵ and $\tilde{U}_\epsilon$ are the unitary transforms defined in Subsection 2.2. The next result is a simple consequence of the definitions and Proposition 2.1.

Lemma 2.3. *The following hold:*

1. $\Phi_{\epsilon,1}\in L^2((0,y_\infty))\cap C^2((0,y_\infty))$, $\|\Phi_{\epsilon,1}\|_{L^2((0,y_\infty))}=1$;
2. $-\mathcal{L}^S_\epsilon\Phi_{\epsilon,1} = \lambda_{\epsilon,1}\Phi_{\epsilon,1}$;
3. *for each $I\subset(0,\infty)$, there holds $\int_I \phi^2_{\epsilon,1}u^G_\epsilon dx = \int_{\xi(I)}\Phi^2_{\epsilon,1}dy$.*

Lemma 2.4. *There holds $\lim_{\epsilon\to0}\lambda_{\epsilon,1}=0$.*

Proof. Clearly, $E := \{x\in(0,\infty): V(x)=\min V\}$ is compact in $(0,\infty)$. Let I be an open interval satisfying $E\subset I\subset\subset(0,\infty)$. Let $\phi\in C^\infty_0((0,\infty))$ satisfy $\phi(x)=x$ for $x\in I$.

As $\min V < 0$ and $\inf_{(0,\infty)\setminus I} V > \min_I V$, we find $\int_I e^{-\frac{V(x)}{\epsilon^2}}dx \to \infty$ and $\frac{\int_{(0,\infty)\setminus I} e^{-\frac{V(x)}{\epsilon^2}}dx}{\int_I e^{-\frac{V(x)}{\epsilon^2}}dx} \to 0$ as $\epsilon \to 0$. It follows from (2.4) that

$$\begin{aligned}\lambda_{\epsilon,1} &\le \frac{\epsilon^2}{2}\frac{\displaystyle\int_I a\,|\phi'|^2\,u_\epsilon^G dx + \int_{(0,\infty)\setminus I} a\,|\phi'|^2\,u_\epsilon^G dx}{\displaystyle\int_I \phi^2 u_\epsilon^G dx}\\ &\le \frac{\epsilon^2}{2}\frac{\displaystyle\int_I e^{-\frac{V(x)}{\epsilon^2}}dx + \left(\sup_{\mathrm{supp}(\phi)}|\phi'|^2\right)\int_{(0,\infty)\setminus I} e^{-\frac{V(x)}{\epsilon^2}}dx}{\displaystyle\left(\inf_{x\in I}\frac{x^2}{a(x)}\right)\int_I e^{-\frac{V(x)}{\epsilon^2}}dx} \to 0 \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

This proves the lemma. $\square$

Proof of Proposition 2.2. Let y_1 and y_2 be as in Lemma 2.2 (2) and (3), respectively. We may assume, without loss of generality, that $y_1 < y_2$. Then,

$$V_{\inf} := \inf_{0<\epsilon\ll 1}\min V_\epsilon > -\infty \quad \text{and} \quad V_0 := \inf_{0<\epsilon\ll 1}\inf_{(0,y_1]\cup[y_2,y_\infty)} V_\epsilon > 0.$$

Denote by $D(\mathcal{L}_\epsilon^S)$ the domain of $\mathcal{L}_\epsilon^S$. By (1.3) and the unitary equivalence between $\mathcal{L}_\epsilon$ and $\mathcal{L}_\epsilon^S$,

$$\frac{\epsilon^2}{2}\int_0^{y_\infty}\phi'\psi' dy + \int_0^{y_\infty} V_\epsilon \phi\psi dy = -\langle \mathcal{L}_\epsilon^S\phi,\psi\rangle_{L^2((0,y_\infty))}, \quad \forall \phi,\psi \in D(\mathcal{L}_\epsilon^S),$$

which together with Lemma 2.3 (1)(2) ensures $\frac{\epsilon^2}{2}\int_0^{y_\infty}|\Phi_{\epsilon,1}'|^2 dy + \int_0^{y_\infty} V_\epsilon \Phi_{\epsilon,1}^2 dy = \lambda_{\epsilon,1}$, leading to $\int_{(0,y_1]\cup[y_2,y_\infty)} V_\epsilon\Phi_{\epsilon,1}^2 dy \le \lambda_{\epsilon,1} - \int_{y_1}^{y_2} V_\epsilon \Phi_{\epsilon,1}^2 dy$. Therefore,

$$V_0\int_{(0,y_1]\cup[y_2,y_\infty)}\Phi_{\epsilon,1}^2 dy \le \lambda_{\epsilon,1} - \min\{0,V_{\inf}\}\int_{y_1}^{y_2}\Phi_{\epsilon,1}^2 dy.$$

Since $\int_{(0,y_1]\cup[y_2,y_\infty)}\Phi_{\epsilon,1}^2 dy + \int_{y_1}^{y_2}\Phi_{\epsilon,1}^2 dy = 1$, we deduce $\int_{(0,y_1]\cup[y_2,y_\infty)}\Phi_{\epsilon,1}^2 dy \le \kappa_\epsilon := \frac{\lambda_{\epsilon,1}-\min\{0,V_{\inf}\}}{V_0-\min\{0,V_{\inf}\}}$, which together with Lemma 2.3 (3) yields $\int_{(0,\xi^{-1}(y_1)]\cup[\xi^{-1}(y_2),\infty)}\phi_{\epsilon,1}^2 u_\epsilon^G dx \le \kappa_\epsilon$.

Lemma 2.4 guarantees $\kappa := \sup_{0<\epsilon\ll 1}\kappa_\epsilon \in (0,1)$. Setting $K := \max\left\{\frac{1}{\xi^{-1}(y_1)}, \xi^{-1}(y_2)\right\}$ gives $\int_{(0,\infty)\setminus(\frac{1}{K},K)}\phi_{\epsilon,1}^2 u_\epsilon^G dx \le \kappa$ for all $0<\epsilon\ll 1$. This completes the proof. $\square$

2.5. **QSD: existence, uniqueness and convergence.** Recall that X_t^ϵ is the stochastic process generated by solutions of (1.1), and T_0^ϵ is the first time that X_t^ϵ hits 0, namely, $T_0^\epsilon = \inf\{t \ge 0 : X_t^\epsilon = 0\}$. The next result is proven in [5]. We refer the reader to [6, 39, 44, 8, 28, 9, 10] and references therein for further developments of QSDs for singular diffusion processes.

Proposition 2.3 ([5]). *For each $0<\epsilon\ll 1$, the following hold.*

(1) *X_t^ϵ admits a unique QSD μ_ϵ with density $u_\epsilon = \frac{\phi_{\epsilon,1}u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}}$. The extinction rate associated to μ_ϵ is given by $\lambda_{\epsilon,1}$.*

(2) *For each Borel probability measure μ on $(0,\infty)$ with compact support,*

$$\lim_{t\to\infty} e^{(\lambda_{\epsilon,2}-\lambda_{\epsilon,1})t}\left(\mathbb{P}^{\epsilon}_{\mu}\left[X^{\epsilon}_t\in B|t<T^{\epsilon}_0\right]-\mu_\epsilon(B)\right)$$

$$=\frac{\displaystyle\int_0^\infty \phi_{\epsilon,2}d\mu}{\displaystyle\int_0^\infty \phi_{\epsilon,1}d\mu}\left(\frac{\langle \mathbb{1}_B,\phi_{\epsilon,2}\rangle_{L^2(u^G_\epsilon)}}{\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}}-\frac{\langle \mathbb{1}_B,\phi_{\epsilon,1}\rangle_{L^2(u^G_\epsilon)}\langle \mathbb{1},\phi_{\epsilon,2}\rangle_{L^2(u^G_\epsilon)}}{\|\phi_{\epsilon,1}\|^2_{L^1(u^G_\epsilon)}}\right),\quad \forall B\in\mathcal{B}((0,\infty)),$$

where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra of $(0,\infty)$.

Proof. Given Proposition 2.1, it is easy to see that μ_ϵ is a QSD with extinction rate $\lambda_{\epsilon,1}$. The uniqueness and convergence follow respectively from [5, Theorem 7.3] and [5, Proposition 5.5]. □

3. **Exponential asymptotic of the first eigenvalue**

In this section, we study the asymptotic of the first eigenvalue $\lambda_{\epsilon,1}$ as $\epsilon\to 0$ and prove Theorem A in the case $i=1$, which is restated in the following theorem.

Theorem 3.1. *Assume* **(H)**. *Then,* $\lim_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,1}=-r_1$.

We first establish the upper bound.

Proof of Theorem 3.1: Upper Bound. By Lemma 2.1 and Remark 1.1 (2), there is $x_*\in(0,\infty)$ such that $r_1=\inf_{\xi\in C_{x_*}}\sup_{t\in[0,1]}\left[V(\xi(t))-V(x_*)\right]$. Then, x_* must be contained in some r_1-valley I, and hence,

$$V(x_*)=\inf_I V\quad\text{and}\quad V(\partial I)-\inf_I V=r_1. \tag{3.1}$$

For $0<\delta\ll 1$, let $0<\eta\ll 1$ be such that $|V(x)-V(y)|\leq\delta$ whenever $x,y\in\overline{I}$ and $|x-y|\leq 3\eta$. Let $\psi\in C_0^\infty((0,\infty))$ satisfy $\psi=1$ on $[\inf I+3\eta,\sup I-3\eta]$ and $\psi=0$ on $(0,\infty)\setminus[\inf I+\eta,\sup I-\eta]$. According to Laplace's method for integrals or the large deviation principle, we find

$$\limsup_{\epsilon\to 0}\epsilon^2\ln\int_0^\infty|\psi'|^2e^{-\frac{V}{\epsilon^2}}dx\leq\limsup_{\epsilon\to 0}\epsilon^2\ln\left[\max_{J_1}|\psi'|^2\int_{J_1}e^{-\frac{V}{\epsilon^2}}dx\right]\leq-\min_{J_1}V+\delta\leq -V(\partial I)+2\delta,$$

where $J_1=[\inf I+\eta,\inf I+3\eta]\cup[\sup I-3\eta,\sup I-\eta]$. Moreover,

$$\liminf_{\epsilon\to 0}\epsilon^2\ln\int_0^\infty\frac{\psi^2}{a}e^{-\frac{V}{\epsilon^2}}dx\geq\liminf_{\epsilon\to 0}\epsilon^2\ln\left[\frac{1}{\max_{J_2}a}\int_{J_2}e^{-\frac{V}{\epsilon^2}}dx\right]\geq-\min_{J_2}V-\delta=-\inf_I V-\delta,$$

where $J_2=[\inf I+3\eta,\sup I-3\eta]$. It follows from (2.4) and (3.1) that

$$\limsup_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,1}\leq -V(\partial I)+\delta+\inf_I V=-r_1+3\delta.$$

Since $0<\delta\ll 1$ is arbitrary, we conclude the upper bound $\limsup_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,1}\leq -r_1$. □

We proceed to prove the lower bound. In the next lemma, we construct special minimizing sequences that help to overcome the difficulties caused by the non-integrability of u^G_ϵ near 0.

Lemma 3.1. *There are sequences $\{\psi_{\epsilon,n}\}_{n\in\mathbb{N},0<\epsilon\ll 1}\subset C_0^\infty((0,\infty))$ satisfying the following properties:*

- $\|\psi_{\epsilon,n}\|_{L^2(u^G_\epsilon)}=1$ *for each* $n\in\mathbb{N}$ *and* $0<\epsilon\ll 1$;
- $\lambda_{\epsilon,1}=\lim_{n\to\infty}\frac{\epsilon^2}{2}\int_0^\infty a\left|\psi'_{\epsilon,n}\right|^2u^G_\epsilon dx$ *for each* $0<\epsilon\ll 1$;
- $\sup_{0<\epsilon\ll 1}\|\psi_{\epsilon,n}-\phi_{\epsilon,1}\|_{L^2(u^G_\epsilon)}\to 0$ *as* $n\to\infty$.

In particular, there are $\kappa\in(0,1)$ and $K\gg1$ such that $\int_{(0,\infty)\setminus(\frac{1}{K},K)}\psi_{\epsilon,n}^2u_\epsilon^G dx\leq\kappa$ for all $0<\epsilon\ll1$ and $n\gg1$.

Proof. For each $0<\epsilon\ll1$, let $\{\tilde{\psi}_{\epsilon,n}\}_{n\in\mathbb{N}}\subset C_0^\infty((0,\infty))$ satisfy $\|\tilde{\psi}_{\epsilon,n}\|_{L^2(u_\epsilon^G)}=1$ for each $n\in\mathbb{N}$ and $\tilde{\psi}_{\epsilon,n}\to\phi_{\epsilon,1}$ in $D(\mathcal{L}_\epsilon)$ as $n\to\infty$, where $D(\mathcal{L}_\epsilon)$ is the domain of $\mathcal{L}_\epsilon$. The expected sequences $\{\psi_{\epsilon,n}\}_{n\in\mathbb{N},0<\epsilon\ll1}$ can be obtained by extracting subsequences from $\{\tilde{\psi}_{\epsilon,n}\}_{n\in\mathbb{N},0<\epsilon\ll1}$. Indeed, for each $m\in\mathbb{N}$, let $n(\epsilon,m)\gg1$ be such that $\|\tilde{\psi}_{\epsilon,n(\epsilon,m)}-\phi_{\epsilon,1}\|_{D(\mathcal{L}_\epsilon)}\leq\frac{1}{m}$. We then only need to define $\psi_{\epsilon,m}:=\tilde{\psi}_{\epsilon,n(\epsilon,m)}$ for each $m\in\mathbb{N}$ and $0<\epsilon\ll1$. The "In particular" part is a simple consequence of Proposition 2.2 and the construction. □

Proof of Theorem 3.1: Lower Bound. As $V\in C([0,\infty))$, it can be extended to be a function in $C(\mathbb{R})$, still denoted by V (note we do not need any specific property of V on $(-\infty,0)$). Let $\{\psi_{\epsilon,n}\}_{n\in\mathbb{N},0<\epsilon\ll1}$, κ and K be as in Lemma 3.1. By zero extension on $(-\infty,0]$, we consider $\{\psi_{\epsilon,n}\}_{n\in\mathbb{N},0<\epsilon\ll1}$ as functions on $\mathbb{R}$.

Fix $0<\delta\ll1$. Let $\eta:\mathbb{R}\to(0,\infty)$ be continuously differentiable and satisfies $|V(y)-V(z)|\leq\delta$ whenever $x,y,z\in\mathbb{R}$ satisfy $|y-x|\leq\eta(x)$ and $|z-x|\leq\eta(x)$.

Let $\{x_i\}_{i=1}^N\subset(\frac{1}{K},K)$ be such that $I_i:=(x_i-\eta(x_i),x_i+\eta(x_i))\subset\left(\frac{1}{K+1},K+1\right)$ for all $i\in\{1,\dots,N\}$ and $\left[\frac{1}{K},K\right]\subset\bigcup_{i=1}^N I_i$. By Lemma 3.1, for $0<\epsilon\ll1$ and $n\gg1$,

$$1=\int_{(0,\infty)\setminus(\frac{1}{K},K)}\frac{\psi_{\epsilon,n}^2}{a}e^{-\frac{V}{\epsilon^2}}dx+\int_{\frac{1}{K}}^K\frac{\psi_{\epsilon,n}^2}{a}e^{-\frac{V}{\epsilon^2}}dx\leq\kappa+\frac{1}{\min_{[\frac{1}{K},K]}a}\sum_{i=1}^N e^{\frac{-V(x_i)+\delta}{\epsilon^2}}\int_{I_i}\psi_{\epsilon,n}^2dx. \tag{3.2}$$

Fix $i\in\{1,\dots,N\}$. Let $\xi_i\in C_{x_i}$ be C^1. For $x\in I_i$, we define

$$\ell_{i,x}(t)=\xi_i(t)+\frac{(1-t)(x-x_i)+t\frac{\eta(0)}{2\eta(x_i)}(x-x_i-\eta(x_i))}{(1-t)\eta(x_i)+t\eta(0)}\eta(\xi_i(t)),\quad t\in[0,1],$$

It is straightforward to check that the family $\{\ell_{i,x}\}_{x\in I_i}$ satisfies the following properties:

- $\ell_{i,x}$ is C^1 for each $x\in I_i$, and $\sup_{x\in I_i}\sup_{t\in[0,1]}|\ell_{i,x}'(t)|$ is finite;
- $\ell_{i,x}(0)=x$ and $\ell_{i,x}(1)=\frac{\eta(0)}{2\eta(x_i)}(x-x_i-\eta(x_i))\in(-\eta(0),0)$ for all $x\in I_i$;
- $|\ell_{i,x}(t)-\xi_i(t)|\leq\eta(\xi_i(t))$ for all $t\in[0,1]$ and $x\in I_i$; thus, $|V(\ell_{i,x}(t))-V(\xi_i(t))|\leq\delta$ for all $t\in[0,1]$ and $x\in I_i$;
- for each $t\in[0,1]$, there holds $\frac{d}{dx}\ell_{i,x}(t)=\frac{1-t+t\frac{\eta(0)}{2\eta(x_i)}}{(1-t)\eta(x_i)+t\eta(0)}\eta(\xi_i(t))$ for $x\in I_i$.

As $\ell_{i,x}(0)=x$ and $\psi_{\epsilon,n}(\ell_{i,x}(1))=0$, we integrate $\frac{d}{dt}\psi_{\epsilon,n}(\ell_{i,x}(t))=\psi_{\epsilon,n}'(\ell_{i,x}(t))\ell_{i,x}'(t)$ over $[0,1]$ to find $-\psi_{\epsilon,n}(x)=\int_0^1\psi_{\epsilon,n}'(\ell_{i,x}(t))\ell_{i,x}'(t)dt$ for $x\in I_i$. It follows that

$$\begin{aligned}\int_{I_i}\psi_{\epsilon,n}^2dx&=\int_{I_i}\left[\int_0^1\psi_{\epsilon,n}'(\ell_{i,x}(t))\ell_{i,x}'(t)dt\right]^2dx\\&\leq\sup_{x\in I_i}\max_{t\in[0,1]}|\ell_{i,x}'(t)|^2\int_0^1\int_{I_i}\left|\psi_{\epsilon,n}'(\ell_{i,x}(t))\right|^2dxdt\\&\leq\sup_{x\in I_i}\max_{t\in[0,1]}|\ell_{i,x}'(t)|^2\int_0^1e^{\frac{V(\xi_i(t))+\delta}{\epsilon^2}}\int_{I_i}\left|\psi_{\epsilon,n}'(\ell_{i,x}(t))\right|^2e^{-\frac{V(\ell_{i,x}(t))}{\epsilon^2}}dxdt\\&\leq\sup_{x\in I_i}\max_{t\in[0,1]}|\ell_{i,x}'(t)|^2\exp\left\{\frac{\max_{t\in[0,1]}V(\xi_i(t))+\delta}{\epsilon^2}\right\}\int_0^1\int_{I_i}\left|\psi_{\epsilon,n}'(\ell_{i,x}(t))\right|^2e^{-\frac{V(\ell_{i,x}(t))}{\epsilon^2}}dxdt.\end{aligned}$$

Note that for each $t \in [0,1]$, the function $x \mapsto \frac{d}{dx}\ell_{i,x}(t)$ is a constant function on I_i, leading to

$$\begin{aligned}\int_{I_i} \left|\psi'_{\epsilon,n}(\ell_{i,x}(t))\right|^2 e^{-\frac{V(\ell_{i,x}(t))}{\epsilon^2}} dx &= \frac{1}{\frac{d}{dx}\ell_{i,x}(t)} \int_{I_i} \left|\psi'_{\epsilon,n}(\ell_{i,x}(t))\right|^2 e^{-\frac{V(\ell_{i,x}(t))}{\epsilon^2}} \frac{d}{dx}\ell_{i,x}(t)dx \\ &\leq \frac{1}{\frac{d}{dx}\ell_{i,x}(t)} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx.\end{aligned}$$

Setting $D_i := \sup_{x\in I_i} \max_{t\in[0,1]} |\ell'_{i,x}(t)|^2 \int_0^1 \frac{1}{\frac{d}{dx}\ell_{i,x}(t)} dt$, we arrive at

$$\int_{I_i} \psi^2_{\epsilon,n} dx \leq D_i \exp\left\{\frac{\max_{t\in[0,1]} V(\xi_i(t)) + \delta}{\epsilon^2}\right\} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx,$$

which together with (3.2) yields

$$1-\kappa \leq \frac{\sum_{i=1}^N D_i \exp\left\{\frac{\max_{t\in[0,1]} V(\xi_i(t)) - V(x_i) + 2\delta}{\epsilon^2}\right\}}{\min_{[\frac{1}{K},K]} a} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx.$$

Since $\xi_i \in C_{x_i}$ being C^1 is arbitrary and the set of such curves are dense in C_{x_i}, we find

$$\begin{aligned}1-\kappa &\leq \frac{\sum_{i=1}^N D_i \exp\left\{\frac{\inf_{\xi\in C_{x_i}} \max_{t\in[0,1]} V(\xi(t)) - V(x_i) + 2\delta}{\epsilon^2}\right\}}{\min_{[\frac{1}{K},K]} a} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx \\ &\leq \frac{\sum_{i=1}^N D_i}{\min_{[\frac{1}{K},K]} a} \exp\left\{\frac{\sup_{x\in(0,\infty)} \left[\inf_{\xi\in C_x} \max_{t\in[0,1]} V(\xi(t)) - V(x)\right] + 2\delta}{\epsilon^2}\right\} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx \\ &= \frac{\sum_{i=1}^N D_i}{\min_{[\frac{1}{K},K]} a} e^{\frac{r_1+2\delta}{\epsilon^2}} \int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx.\end{aligned}$$

Thus,

$$\int_0^\infty \left|\psi'_{\epsilon,n}\right|^2 e^{-\frac{V}{\epsilon^2}} dx \geq \frac{\min_{[\frac{1}{K},K]} a}{\sum_{i=1}^N D_i} e^{-\frac{r_1+2\delta}{\epsilon^2}} (1-\kappa), \quad \forall 0<\epsilon \ll 1 \quad \text{and} \quad n \gg 1.$$

Note that the term $\frac{\min_{[\frac{1}{K},K]} a}{\sum_{i=1}^N D_i}$ is independent of the functions $\{\psi_{\epsilon,n}\}_{\epsilon,n}$. Letting $n\to\infty$ yields

$$\lambda_{\epsilon,1} \geq \frac{\epsilon^2}{2} \frac{\min_{[\frac{1}{K},K]} a}{\sum_{i=1}^N D_i} e^{-\frac{r_1+2\delta}{\epsilon^2}} (1-\kappa), \quad \forall 0<\epsilon \ll 1,$$

which gives $\liminf_{\epsilon\to 0} \epsilon^2 \ln \lambda_{\epsilon,1} \geq -r_1 - 2\delta$. Since $0<\delta\ll 1$ is arbitrary, we arrive at the lower bound $\liminf_{\epsilon\to 0} \epsilon^2 \ln \lambda_{\epsilon,1} \geq -r_1$. □

4. Exponential asymptotics of other eigenvalues

In this section, we prove Theorem A in the case $i \in \mathbb{N}\setminus\{1\}$, which is restated as follows.

Theorem 4.1. *Assume* **(H)**. *Then,* $\lim_{\epsilon\to 0} \epsilon^2 \ln \lambda_{\epsilon,i} = -r_i$ *for each* $i \in \mathbb{N}\setminus\{1\}$.

The upper and lower bounds are respectively established in Subsection 4.1 and Subsection 4.3. In Subsection 4.2, we state results on asymptotics of the first eigenvalues of some auxiliary operators.

4.1. **Upper bound.** For an open interval $I \subset\subset (0,\infty)$, let $\mathcal{L}_\epsilon^I$ be the generator of the diffusion process X_t^ϵ or (1.1) restricted to I and killed on ∂I, which is non-degenerate, regular and reversible. Moreover, $-\mathcal{L}_\epsilon^I$, considered as an operator in $L^2(I, u_\epsilon^G dx)$, has purely discrete spectrum consisting of simple eigenvalues. Let $\lambda_\epsilon^I > 0$ be the first eigenvalue of $-\mathcal{L}_\epsilon^I$. Then,

$$\lambda_\epsilon^I = \inf_{\phi\in C_0^\infty(I)} \frac{\frac{\epsilon^2}{2}\displaystyle\int_I a\,|\phi'|^2\,u_\epsilon^G dx}{\displaystyle\int_I \phi^2 u_\epsilon^G dx} = \inf_{\phi\in C_0^\infty(I)} \frac{\frac{\epsilon^2}{2}\displaystyle\int_I |\phi'|^2\,e^{-\frac{V}{\epsilon^2}}dx}{\displaystyle\int_I \frac{\phi^2}{a} e^{-\frac{V}{\epsilon^2}}dx}. \tag{4.1}$$

For $x \in I$ and $y \in \partial I$, let

$$C_{x,y}^I := \left\{\xi : [0,1] \to \overline{I} : \xi \text{ is continuous and satisfies } \xi(0) = x \text{ and } \xi(1) = y\right\}.$$

Define

$$r^I := \sup_{x\in I}\left[\inf_{y\in\partial I}\inf_{\xi\in C_{x,y}^I}\sup_{t\in[0,1]} V(\xi(t)) - V(x)\right].$$

Proposition 4.1 ([41])**.** *There holds* $\lim_{\epsilon\to 0}\epsilon^2 \ln \lambda_\epsilon^I = -r^I$.

Recall from Section 1 that for $r > 0$, $N(r)$ is the number of r-valleys.

Proof of Theorem 4.1: Upper Bound. Fix $i_* \geq 2$. We show

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,i_*} \leq -r_{i_*}. \tag{4.2}$$

We distinguish the following two cases:

Case 1. There is $r > 0$ such that $N(r) \geq i_*$.
Case 2. $N(r) < i_*$ for all $r > 0$.

Case 1. In this case, $\sup\{r > 0 : N(r) \geq i_*\}$ is well-defined, and hence, by Remark 1.1 (1),

$$r_{i_*} = \sup\{r > 0 : N(r) \geq i_*\}.$$

Let $0 < \delta_* \ll 1$. For $r \in (r_{i_*} - \delta_*, r_{i_*})$, let I_k, $k = 1,\dots,N(r)$ be the r-valleys. They are ordered in the nature way: $\sup I_k \leq \inf I_{k+1}$ for all $k \in \{1,\dots,N(r)-1\}$.

We claim that

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,N(r)} \leq -r, \quad \forall r \in (r_{i_*} - \delta_*, r_{i_*}). \tag{4.3}$$

Note that $N(r) \geq i_*$ for all $r \in (r_{i_*} - \delta_*, r_{i_*})$, implying $\lambda_{\epsilon,N(r)} \geq \lambda_{\epsilon,i_*}$ for all $r \in (r_{i_*} - \delta_*, r_{i_*})$. Thus, if the claim (4.3) is true, then $\limsup_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,i_*} \leq -r$ for all $r \in (r_{i_*} - \delta_*, r_{i_*})$. Letting $r \to r_{i_*}^-$ yields (4.2).

It suffices to show (4.3). Making δ_* smaller if necessary, we claim that

$$I_k \subset\subset (0,\infty), \quad \forall k \in \{1,\dots,N(r)\}, \quad r \in (r_{i_*} - \delta_*, r_{i_*}). \tag{4.4}$$

Indeed, the only possible exception takes place if $\inf I_1 = 0$ for some $r \in (r_{i_*} - \delta_*, r_{i_*})$. If this is the case, then $I_1 = (0,\beta)$, where $\beta = \min\{x \in (0,\infty) : V(x) = 0\}$. Thus, $\gamma(I_1)$ depends only on V. Note that there always holds $\gamma(I_1) \notin (r_{i_*} - \delta_*, r_{i_*})$ if δ_* is sufficiently small, which contradicts the fact $\gamma(I_1) = r \in (r_{i_*} - \delta_*, r_{i_*})$.

Fix $0<\delta\ll 1$. For $k\in\{1,\dots,N(r)\}$ and $0<\epsilon\ll 1$, let $\psi_{\epsilon,k}\in C_0^\infty(I_k)$ be such that

$$\int_{I_k}\psi_{\epsilon,k}^2 u_\epsilon^G dx=1\quad\text{and}\quad \frac{\frac{\epsilon^2}{2}\int_{I_k}\left|\psi_{\epsilon,k}'\right|^2 e^{-\frac{V}{\epsilon^2}}dx}{\int_{I_k}\frac{\psi_{\epsilon,k}^2}{a}e^{-\frac{V}{\epsilon^2}}dx}\le \lambda_\epsilon^{I_k}e^{\frac{\delta}{\epsilon^2}},$$

where $\lambda_\epsilon^{I_k}$ is given in (4.1). Moreover, let $\alpha_{\epsilon,k}$, $k\in\{1,\dots,N(r)\}$ and $0<\epsilon\ll 1$ be such that

$$\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2=1\quad\text{and}\quad \sum_{k=1}^{N(r)}\alpha_{\epsilon,k}\int_{I_k}\psi_{\epsilon,k}\phi_{\epsilon,\ell}u_\epsilon^G dx=0,\quad \forall \ell\in\{1,\dots,N(r)-1\},$$

where we recall from Proposition 2.1 that $\phi_{\epsilon,\ell}$ is the normalized eigenfunction of $\mathcal{L}_\epsilon$ associated to the eigenvalue $\lambda_{\epsilon,\ell}$.

Set $\psi_\epsilon=\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}\psi_{\epsilon,k}$. Since I_k, $k\in\{1,\dots,N(r)\}$ are pairwise disjoint, we find:

$$\int_0^\infty \psi_\epsilon^2 u_\epsilon^G dx=\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2\int_{I_k}\psi_{\epsilon,k}^2u_\epsilon^G dx=\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2=1.$$

Moreover,

$$\int_0^\infty \psi_\epsilon\phi_{\epsilon,\ell}u_\epsilon^G dx=\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}\int_{I_k}\psi_{\epsilon,k}\phi_{\epsilon,\ell}u_\epsilon^G dx=0,\quad \forall \ell\in\{1,\dots,N(r)-1\}.$$

In addition, the pairwise disjointness of I_k, $k\in\{1,\dots,N(r)\}$ implies that

$$\begin{aligned}\frac{\epsilon^2}{2}\int_0^\infty\left|\psi_\epsilon'\right|^2e^{-\frac{V}{\epsilon^2}}dx&=\frac{\epsilon^2}{2}\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2\int_{I_k}\left|\psi_{\epsilon,k}'\right|^2e^{-\frac{V}{\epsilon^2}}dx\\ &\le\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2\lambda_\epsilon^{I_k}\int_{I_k}\frac{\psi_{\epsilon,k}^2}{a}e^{-\frac{V}{\epsilon^2}}dx\\ &\le\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\left(\max_{k\in\{1,\dots,N(r)\}}\lambda_\epsilon^{I_k}\right)\sum_{k=1}^{N(r)}\alpha_{\epsilon,k}^2\int_{I_k}\frac{\psi_{\epsilon,k}^2}{a}e^{-\frac{V}{\epsilon^2}}dx\\ &=\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\left(\max_{k\in\{1,\dots,N(r)\}}\lambda_\epsilon^{I_k}\right)\int_{I_k}\frac{\psi_\epsilon^2}{a}e^{-\frac{V}{\epsilon^2}}dx.\end{aligned}$$

The variational formula for $\lambda_{\epsilon,N(r)}$ given in (2.5) then yields $\lambda_{\epsilon,N(r)}\le\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\left(\max_{k\in\{1,\dots,N(r)\}}\lambda_\epsilon^{I_k}\right)$. As $\ln\left(\max_{k\in\{1,\dots,N(r)\}}\lambda_\epsilon^{I_k}\right)=\max_{k\in\{1,\dots,N(r)\}}\ln\lambda_\epsilon^{I_k}$, we deduce from Proposition 4.1 that

$$\limsup_{\epsilon\to0}\epsilon^2\ln\lambda_{\epsilon,N(r)}\le\delta-\min_{k\in\{1,\dots,N(r)\}}r^{I_k}.$$

Since $0<\delta\ll1$ is arbitrary, we conclude $\limsup_{\epsilon\to0}\epsilon^2\ln\lambda_{\epsilon,N(r)}\le-\min_{k\in\{1,\dots,N(r)\}}r^{I_k}$. As each I_k is a r-valley, the definition of r^{I_k} ensures that $r^{I_k}=r$. This proves (4.3), and thus, finishes the proof in this case.

Case 2. In this case, $r_{i_*}=0$ (see Remark 1.1 (1)) so that (4.2) reads

$$\limsup_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,i_*}\leq 0. \tag{4.5}$$

Let $I_k\subset\subset(0,\infty)$, $k\in\{1,\dots,i_*\}$ be pairwise disjoint, and on at least one of them, V is monotone. As a result, at least one of r^{I_k}, $k\in\{1,\dots,i_*\}$ is zero, that is, $\min_{k\in\{1,\dots,i_*\}}r^{I_k}=0$.

Fix $0<\delta\ll 1$. For $k\in\{1,\dots,i_*\}$ and $0<\epsilon\ll 1$, let $\psi_{\epsilon,k}\in C_0^\infty(I_k)$ be such that

$$\int_{I_k}\psi_{\epsilon,k}^2u_\epsilon^G dx=1\quad\text{and}\quad\frac{\frac{\epsilon^2}{2}\int_{I_k}\left|\psi_{\epsilon,k}'\right|^2e^{-\frac{V}{\epsilon^2}}dx}{\int_{I_k}\frac{\psi_{\epsilon,k}^2}{a}e^{-\frac{V}{\epsilon^2}}dx}\leq\lambda_\epsilon^{I_k}e^{\frac{\delta}{\epsilon^2}}.$$

Moreover, let $\alpha_{\epsilon,k}$, $k\in\{1,\dots,i_*\}$ and $0<\epsilon\ll 1$ be such that

$$\sum_{k=1}^{i_*}\alpha_{\epsilon,k}^2=1\quad\text{and}\quad\sum_{k=1}^{i_*}\alpha_{\epsilon,k}\int_{I_k}\psi_{\epsilon,k}\phi_{\epsilon,\ell}u_\epsilon^G dx=0.\quad\forall\ell\in\{1,\dots,i_*-1\}.$$

Set $\psi_\epsilon=\sum_{k=1}^{i_*}\alpha_{\epsilon,k}\psi_{\epsilon,k}$. Arguing as in **Case 1**, we find

$$\begin{aligned}&\int_0^\infty\psi_\epsilon^2u_\epsilon^G dx=1,\quad\int_0^\infty\psi_\epsilon\phi_{\epsilon,\ell}u_\epsilon^G dx=0,\quad\forall\ell\in\{1,\dots,i_*-1\},\\&\frac{\epsilon^2}{2}\int_0^\infty\left|\psi_\epsilon'\right|^2e^{-\frac{V}{\epsilon^2}}dx\leq\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\left(\max_{k\in\{1,\dots,i_*\}}\lambda_\epsilon^{I_k}\right)\int_{I_k}\frac{\psi_\epsilon^2}{a}e^{-\frac{V}{\epsilon^2}}dx.\end{aligned}$$

Hence, $\lambda_{\epsilon,i_*}\leq\frac{\epsilon^2}{2}e^{\frac{\delta}{\epsilon^2}}\left(\max_{k\in\{1,\dots,i_*\}}\lambda_\epsilon^{I_k}\right)$, implying $\limsup_{\epsilon\to0}\epsilon^2\ln\lambda_{\epsilon,i_*}\leq\delta-\min_{k\in\{1,\dots,i_*\}}r^{I_k}=\delta$. As $0<\delta\ll1$ is arbitrary, the upper bound (4.5) follows. □

4.2. **Auxiliary eigenvalue asymptotics.** In this subsection, we consider the first eigenvalue asymptotics of three types of operators who are generators of the diffusion process X_t^ϵ or (1.1) restricted to different types of intervals with reflection on boundaries. More precisely, let $I\subset(0,\infty)$ be a proper open interval. There are only three types:

$$\begin{aligned}&\text{Type 1}:\quad I=(0,b)\quad\text{for } 0<b<\infty,\\&\text{Type 2}:\quad I=(a,b)\quad\text{for } 0<a<b<\infty,\\&\text{Type 3}:\quad I=(a,\infty)\quad\text{for } 0<a<\infty.\end{aligned}$$

Let $\mathcal{L}_\epsilon^{I,R}$ be the generator of the diffusion process X_t^ϵ or (1.1) restricted to I and reflected on the right boundary (Type 1), both boundaries (Type 2) and the left boundary (Type 3). The superscript "R" stands for reflection. This operator extends to a self-adjoint operator in $L^2(I,u_\epsilon^G dx)$, still denoted by $\mathcal{L}_\epsilon^{I,R}$. The rigorous formulation can be done using Dirichlet forms as done in Section 1 for the operator $\mathcal{L}_\epsilon$. Proposition 2.1 has a version for $-\mathcal{L}_\epsilon^{I,R}$.

Note that for a Type 2 or Type 3 interval I, u_ϵ^G is integrable on I, and hence, $u_\epsilon^{Gibbs}:=\frac{u_\epsilon^G}{\int_I u_\epsilon^G dx}$ is the Gibbs density, namely, the density of the unique stationary distribution of the restricted diffusion process. In particular, the first eigenvalue of $-\mathcal{L}_\epsilon^{I,R}$ is zero.

Let

$$\lambda_\epsilon^{I,R}=\begin{cases}\text{the first eigenvalue of } -\mathcal{L}_\epsilon^{I,R}, & \text{if } I \text{ is of Type 1},\\ \text{the first non-zero eigenvalue of } -\mathcal{L}_\epsilon^{I,R}, & \text{if } I \text{ is of Type 2 or Type 3}.\end{cases}$$

Recall that

- if I is of Type 1, then

$$\lambda_\epsilon^{I,R} = \inf_{\substack{\phi\in C^\infty(I)\setminus\{0\}\\ \inf\operatorname{supp}(\phi)>0}} \frac{\frac{\epsilon^2}{2}\int_I a\left|\phi'\right|^2 u_\epsilon^G dx}{\int_I \phi^2 u_\epsilon^G dx}, \tag{4.6}$$

- if I is of Type 2, then

$$\lambda_\epsilon^{I,R} = \inf_{\phi\in C^\infty(I)\setminus\{0\}} \frac{\frac{\epsilon^2}{2}\int_I a\left|\phi'\right|^2 u_\epsilon^G dx}{\int_I \left(\phi - \int_I \phi u_\epsilon^{Gibbs} dx\right)^2 u_\epsilon^G dx}, \tag{4.7}$$

- if I is of Type 3, then

$$\lambda_\epsilon^{I,R} = \inf_{\substack{\phi\in C^\infty(I)\setminus\{0\}\\ \sup\operatorname{supp}(\phi)<\infty}} \frac{\frac{\epsilon^2}{2}\int_I a\left|\phi'\right|^2 u_\epsilon^G dx}{\int_I \left(\phi - \int_I \phi u_\epsilon^{Gibbs} dx\right)^2 u_\epsilon^G dx}. \tag{4.8}$$

We define the following quantity:

$$r^{I,R} = \begin{cases} \sup_{x\in I}\left[\inf_{\xi\in C_x^{I,R}}\sup_{t\in[0,1]} V(\xi(t)) - V(x)\right], & \text{if } I \text{ is of Type 1,}\\ \sup_{x,y\in I}\left[\inf_{\xi\in C_{x,y}^{I,R}}\sup_{t\in[0,1]} V(\xi(t)) - V(x) - V(y)\right] + \inf_I V, & \text{if } I \text{ is of Type 2 or Type 3,}\end{cases} \tag{4.9}$$

where for $x, y \in I$,

$$C_x^{I,R} = \left\{\xi : [0,1] \to \overline{I} : \xi \text{ is continuous and satisfies } \xi(0) = x \text{ and } \xi(1) = 0\right\},$$
$$C_{x,y}^{I,R} = \left\{\xi : [0,1] \to \overline{I} : \xi \text{ is continuous and satisfies } \xi(0) = x \text{ and } \xi(1) = y\right\}.$$

Proposition 4.2. *There holds* $\lim_{\epsilon\to 0}\epsilon^2 \ln \lambda_\epsilon^{I,R} = -r^{I,R}$.

Proof. If I is of Type 2, the result is proven in [29, 40]. As $V(\infty) = \infty$, the role played by a reflection boundary at a finite right endpoint is similar to that of ∞. In consideration of this, when I is of Type 1, the proof is similar to that of Theorem A in the case $i = 1$. If I is of Type 3, the proof is similar to that when I is of Type 2. □

4.3. **Lower bound.** We prove the lower bound in this subsection.

Proof of Theorem 4.1: Lower Bound. Fix $i_* \geq 2$. We prove

$$\liminf_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,i_*} \geq -r_{i_*}. \tag{4.10}$$

Let $0 < \delta_* \ll 1$ be fixed. For $r \in (r_{i_*}, r_{i_*} + \delta_*)$, let $I_k(r)$, $k \in \{1, \dots, N(r)\}$ be the r-valleys. They are ordered in the natural way: $\sup I_k(r) \leq \inf I_{k+1}(r)$ for all $k \in \{1, \dots, N(r) - 1\}$.

We claim that

$$\liminf_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,N(r)+1} \geq -r, \quad \forall r \in (r_{i_*}, r_{i_*} + \delta_*). \tag{4.11}$$

Note that making δ_* smaller if necessary, there exists $n_0 \in \mathbb{N}$ such that

$$N(r) + n_0 = i_*, \quad \forall r \in (r_{i_*}, r_{i_*} + \delta_*). \tag{4.12}$$

Thus, if the claim (4.11) is true, then

$$\limsup_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,i_*}=\limsup_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,N(r)+n_0}\geq\liminf_{\epsilon\to 0}\epsilon^2\ln\lambda_{\epsilon,N(r)+1}\geq -r,\quad\forall r\in(r_{i_*},r_{i_*}+\delta_*).$$

Letting $r\to r_{i_*}^+$ yields (4.10). Hence, it suffices to prove (4.11), which is done in three steps.

Step 1. Note that making δ_* smaller if necessary, there hold for each $r\in(r_{i_*},r_{i_*}+\delta_*)$,

$$I_k(r)\subset\subset(0,\infty),\quad\forall k\in\{1,\dots,N(r)\},$$
$$\sup I_k(r)<\inf I_{k+1}(r),\quad\forall k=1,\dots,N(r)-1.$$

The first conclusion follows from arguments leading to (4.4). The second one follows from (4.12) saying that the number of r-valleys remains constant for $r\in(r_{i_*},r_{i_*}+\delta_*)$. Otherwise, two or more of them merge as r increases so that the number of r-valleys drops.

The purpose of this step is to find for each $r\in(r_{i_*},r_{i_*}+\delta_*)$ some special points in the following disjoint intervals:

$$(0,\inf I_1(r)),\quad(\sup I_k(r),\inf I_{k+1}(r)),\quad\forall k\in\{1,\dots,N(r)-1\}$$

in order to re-partition $(0,\infty)$.

We first treat the intervals $(\sup I_k(r),\inf I_{k+1}(r))$ for $k\in\{1,\dots,N(r)-1\}$ and $r\in(r_{i_*},r_{i_*}+\delta_*)$. Note that

$$\sup_{(\sup I_k(r),\inf I_{k+1}(r))}V>\max\left\{V(\sup I_k(r)),V(\inf I_{k+1}(r))\right\},\ \forall k\in\{1,\dots,N(r)-1\},\ r\in(r_{i_*},r_{i_*}+\delta_*).\tag{4.13}$$

In fact, if (4.13) fails for some $r_0\in(r_{i_*},r_{i_*}+\delta_*)$ and $k_0\in\{1,\dots,N(r_0)-1\}$, then for all $r\in(r_0,r_{i_*}+\delta_*)$, $I_{k_0}(r)$ and $I_{k_0+1}(r)$ merge, and therefore, $N(r)<N(r_0)$ for all $r\in(r_0,r_{i_*}+\delta_*)$, which contradicts (4.12).

According to (4.13), for each $r\in(r_{i_*},r_{i_*}+\delta_*)$ and $k\in\{1,\dots,N(r)-1\}$ there exists $x_k(r)\in(\sup I_k(r),\inf I_{k+1}(r))$ such that

$$V(x_k(r))=\sup_{(\sup I_k(r),\inf I_{k+1}(r))}V.$$

Now, we treat the intervals $(0,\inf I_1(r))$ for $r\in(r_{i_*},r_{i_*}+\delta_*)$. Note that

$$\sup_{(0,\inf I_1(r))}V>V(\inf I_1(r)),\quad\forall r\in(r_{i_*},r_{i_*}+\delta_*).\tag{4.14}$$

Indeed, if (4.14) fails for some $r_0\in(r_{i_*},r_{i_*}+\delta_*)$, then $I_1(r)$ is no longer a r-valley for all $r\in(r_0,r_{i_*}+\delta_*)$, leading to a contradiction.

For $r\in(r_{i_*},r_{i_*}+\delta_*)$, set $E(r)=\left\{x\in(0,\inf I_1(r)):V(x)=\sup_{(0,\inf I_1(r))}V\right\}$. We claim that there holds either

$$E(r)\neq\emptyset,\quad\forall r\in(r_{i_*},r_{i_*}+\delta_*),\tag{4.15}$$

or

$$E(r)=\emptyset,\quad\forall r\in(r_{i_*},r_{i_*}+\delta_*).\tag{4.16}$$

Suppose on the contrary that there are $r_1,r_2\in(r_{i_*},r_{i_*}+\delta_*)$ such that $E(r_1)\neq\emptyset$ and $E(r_2)=\emptyset$. If $r_1>r_2$, then $I_1(r_1)\supset I_1(r_2)$. It follows that $E(r_2)=E(r_1)$, leading to a contradiction. If $r_1<r_2$, then $I_1(r_1)\subset I_1(r_2)$. By (4.14) and the assumptions, there holds

$$V(\inf I_1(r_2))<V(0+)\leq\sup_{(0,\inf I_1(r_1))}V,$$

which implies $\inf I_1(r_2) > \max E(r_1)$. It follows that $E(r_2) = E(r_1)$, leading to a contradiction. Hence, either (4.15) or (4.16) holds.

If (4.15) is true, we fix any

$$x_0(r) \in E(r), \quad \forall r \in (r_{i_*}, r_{i_*} + \delta_*).$$

If (4.16) is the case, we see from (4.14) that $V(0+) > V(x)$ for all $x \in (0, \inf I_1(r))$ and $r \in (r_{i_*}, r_{i_*} + \delta_*)$. For each $r \in (r_{i_*}, r_{i_*} + \delta_*)$, let

$$x_0(r) := \begin{cases} \min\left\{x \in (0, \inf I_1(r)) : V(0+) - V(x) = r\right\}, & \text{if} \quad V(0+) > \inf\limits_{(0,\inf I_1(r))} V + r, \\ \inf I_1(r), & \text{if} \quad V(0+) \le \inf\limits_{(0,\inf I_1(r))} V + r. \end{cases}$$

Hence, we have defined the points: $x_k(r)$, $k \in \{0, 1, \dots, N(r) - 1\}$, $r \in (r_{i_*}, r_{i_*} + \delta_*)$.

Step 2. We re-partition $(0, \infty)$ by defining the following intervals: for each $r \in (r_{i_*}, r_{i_*} + \delta_*)$,

$$\begin{aligned} \tilde{I}_0(r) &= (0, x_0(r)), \\ \tilde{I}_k(r) &= (x_{k-1}(r), x_k(r)), \quad k \in \{1, \dots, N(r) - 1\}, \\ \tilde{I}_{N(r)}(r) &= (x_{N(r)-1}(r), \infty). \end{aligned}$$

Note that $\tilde{I}_0(r)$ is of Type 1, $\tilde{I}_k(r)$ is of Type 2 for each $k \in \{1, \dots, N(r) - 1\}$, and $\tilde{I}_{N(r)}(r)$ is of Type 3. Recall from (4.9) the definition of $r^{\tilde{I}_k(r),R}$. We claim that

$$r^{\tilde{I}_k(r),R} \le r, \quad \forall k \in \{0, 1, \dots, N(r)\}, \quad r \in (r_{i_*}, r_{i_*} + \delta_*). \tag{4.17}$$

The rest of this step is devoted to the proof of (4.17). Note that $I_k(r)$ is the only r-valley contained in $\tilde{I}_k(r)$. Otherwise, the number of r-valleys changes as r varies.

By the definition of $x_0(r)$, it is not hard to see that $V(0+) - \inf_{(0,x_0(r))} V \le r$. This is trivial if (4.16) is the case. If (4.15) is the case, there must hold $V(0+) - \inf_{(0,x_0(r))} V < r$ as there exists no r-valley contained in $(0, x_0(r))$. The claim (4.17) in the case $k = 0$ follows.

Let $r \in (r_{i_*}, r_{i_*} + \delta_*)$ and $k \in \{1, \dots, N(r)\}$. Note that

$$r^{\tilde{I}_k(r),R} = \sup_{\substack{x,y \in \tilde{I}_k(r) \\ x<y}} \left[\inf_{\xi \in C_{x,y}^{\tilde{I}_k(r),R}} \sup_{t \in [0,1]} V(\xi(t)) - V(x) - V(y) \right] + \inf_{\tilde{I}_k(r)} V.$$

Let $E_k(r)$ be the set of local maximal points of $V|_{\tilde{I}_k(r)}$ with connected components attached to $\partial \tilde{I}_k(r)$ (if there is any) removed. If $E_k(r) = \emptyset$, then it is easy to check that $r_1^{\tilde{I}_k(r),R} = 0$. Suppose $E_k(r) \ne \emptyset$. Then, for each $x, y \in \tilde{I}_k(r)$ with $x < y$, there holds

$$\inf_{\xi \in C_{x,y}^{\tilde{I}_k(r),R}} \sup_{t \in [0,1]} V(\xi(t)) = \begin{cases} \max\left\{V(x), V(y)\right\}, & \text{if} \quad (x,y) \cap E_k(r) = \emptyset, \\ \max\limits_{\{x,y\} \cup ((x,y) \cap E_k(r))} V, & \text{if} \quad (x,y) \cap E_k(r) \ne \emptyset, \end{cases}$$

leading to

$$\begin{aligned} &\inf_{\xi \in C_{x,y}^{\tilde{I}_k(r),R}} \sup_{t \in [0,1]} V(\xi(t)) - V(x) - V(y) \\ &\quad = \begin{cases} -\min\left\{V(x), V(y)\right\}, & \text{if} \quad (x,y) \cap E_k(r) = \emptyset, \\ \max\limits_{\{x,y\} \cup ((x,y) \cap E_k(r))} V - V(x) - V(y), & \text{if} \quad (x,y) \cap E_k(r) \ne \emptyset. \end{cases} \end{aligned}$$

Thus,

$$\sup_{\substack{x,y\in\tilde{I}_k(r)\\(x,y)\cap E_k(r)=\emptyset}}\left[\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y)\right]=-\inf_{\tilde{I}_k(r)}V.$$

We claim that

$$\tilde{r}^{\tilde{I}_k(r),R}:=\sup_{\substack{x,y\in\tilde{I}_k(r)\\(x,y)\cap E_k(r)\neq\emptyset}}\left[\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y)\right]\leq r-\inf_{\tilde{I}_k(r)}V. \tag{4.18}$$

Let $[\alpha_1,\beta_1],\dots,[\alpha_{\tilde{N}},\beta_{\tilde{N}}]$ for some $\tilde{N}=\tilde{N}(k,r)\in\mathbb{N}$ be the connected components of $E_k(r)$. They are ordered in the natural way: $\beta_\ell<\alpha_{\ell+1}$ for all $\ell\in\{1,\dots,\tilde{N}-1\}$. Set $\beta_0:=\inf\tilde{I}_k(r)$ and $\alpha_{\tilde{N}+1}:=\sup\tilde{I}_k(r)$. Clearly, $\beta_0<\alpha_1$ and $\beta_{\tilde{N}}<\alpha_{\tilde{N}+1}$.

Note that if $x_0\in[\alpha_{\ell_0},\beta_{\ell_0}]$ for some $\ell_0\in\{1,\dots,\tilde{N}\}$ and $y\in(x_0,\sup\tilde{I}_k(r))$, then

$$\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x_0,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x_0)-V(y)\leq\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y)$$

for all $x<\alpha_{\ell_0}$ with $\alpha_{\ell_0}-x\ll1$. Similarly, if $y_0\in[\alpha_{\ell_0},\beta_{\ell_0}]$ for some $\ell_0\in\{1,\dots,\tilde{N}\}$ and $x\in(\inf\tilde{I}_k(r),y_0)$, then

$$\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y_0}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y_0)\leq\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y)$$

for all $y>\beta_{\ell_0}$ with $y-\beta_{\ell_0}\ll1$. Therefore,

$$\begin{aligned}\tilde{r}^{\tilde{I}_k(r),R}&=\sup_{\substack{\ell_1,\ell_2\in\{1,\dots,\tilde{N}\}\\\ell_1\leq\ell_2}}\sup_{\substack{x\in(\beta_{\ell_1-1},\alpha_{\ell_1})\\y\in(\beta_{\ell_2},\alpha_{\ell_2+1})}}\left[\inf_{\xi\in C^{\tilde{I}_k(r),R}_{x,y}}\sup_{t\in[0,1]}V(\xi(t))-V(x)-V(y)\right]\\&=\sup_{\substack{\ell_1,\ell_2\in\{1,\dots,\tilde{N}\}\\\ell_1\leq\ell_2}}\sup_{\substack{x\in(\beta_{\ell_1-1},\alpha_{\ell_1})\\y\in(\beta_{\ell_2},\alpha_{\ell_2+1})}}\left[\max_{\{x,y\}\cup(\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell])}V-V(x)-V(y)\right]\\&=\sup_{\substack{\ell_1,\ell_2\in\{1,\dots,\tilde{N}\}\\\ell_1\leq\ell_2}}\left[\max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_1-1},\alpha_{\ell_1})}V-\min_{(\beta_{\ell_2},\alpha_{\ell_2+1})}V\right],\end{aligned} \tag{4.19}$$

where the minimals are attained due to the definition of $E_k(r)$.

We show that for each $\ell_1,\ell_2\in\{1,\dots,\tilde{N}\}$ with $\ell_1\leq\ell_2$ there holds

$$\min\left\{\max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_1-1},\alpha_{\ell_1})}V,\ \max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_2},\alpha_{\ell_2+1})}V\right\}\leq r. \tag{4.20}$$

To see this, we recall that $I_k(r)$ is the only r-valley contained in $\tilde{I}_k(r)$, and the following hold:

$$\begin{aligned}&V(\inf\tilde{I}_k(r))=\sup_{(\inf\tilde{I}_k(r),\sup I_k(r))}V,\quad V(\sup\tilde{I}_k(r))=\sup_{(\inf I_k(r),\sup\tilde{I}_k(r))}V,\\&\max\left\{V(\inf\tilde{I}_k(r)),V(\sup\tilde{I}_k(r))\right\}=\sup_{\tilde{I}_k(r)}V>\sup_{I_k(r)}V.\end{aligned} \tag{4.21}$$

Suppose for contradiction that (4.20) fails for some $\ell_1,\ell_2\in\{1,\dots,\tilde N\}$ with $\ell_1\le\ell_2$, that is,

$$\min\left\{\max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_1-1},\alpha_{\ell_1})}V,\ \max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_2},\alpha_{\ell_2+1})}V\right\}>r.$$

Let $x_*\in\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]$ be such that $V(x_*)=\max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V$. Then, the third identity in (4.21) ensures that there must hold

$$\text{either}\quad I_k(r)\subset(\inf\tilde I_k(r),x_*)\quad\text{or}\quad I_k(r)\subset(x_*,\sup\tilde I_k(r)).\tag{4.22}$$

If the first inclusion in (4.22) is the case, we deduce from the second identity in (4.21) that

$$V(\sup\tilde I_k(r))=\sup_{(\inf I_k(r),\sup\tilde I_k(r))}V\ge\sup_{(x_*,\sup\tilde I_k(r))}V\ge V(x_*)>\min_{(\beta_{\ell_2},\alpha_{\ell_2+1})}V+r.$$

As a result, there exists a r-valley contained in $(x_*,\sup\tilde I_k(r))$, which contradicts the fact that $I_k(r)$ is the only r-valley contained in $\tilde I_k(r)$. Similarly, a contradiction can be deduced if the second inclusion in (4.22) is the case. Hence, (4.20) is true.

Obviously, (4.20) implies that for each $\ell_1,\ell_2\in\{1,\dots,\tilde N\}$ with $\ell_1\le\ell_2$ there holds

$$\max_{\cup_{\ell=\ell_1}^{\ell_2}[\alpha_\ell,\beta_\ell]}V-\min_{(\beta_{\ell_1-1},\alpha_{\ell_1})}V-\min_{(\beta_{\ell_2},\alpha_{\ell_2+1})}V\le r-\inf_{\tilde I_k(r)}V.$$

This together with (4.19) implies (4.18). The claim (4.17) follows.

Step 3. By the min-max principle, there holds for each $0<\epsilon\ll1$ and $r\in(r_{i_*},r_{i_*}+\delta_*)$

$$\lambda_{\epsilon,N(r)+1}\ge\inf\left\{\frac{\frac{\epsilon^2}{2}\int_0^\infty a\,|\phi'|^2\,u_\epsilon^G dx}{\int_0^\infty\phi^2u_\epsilon^Gdx}:\phi\in C_0^\infty((0,\infty)),\ \int_{\tilde I_k(r)}\phi u_\epsilon^Gdx=0,\ \forall k\in\{1,\dots,N(r)\}\right\}.$$

Fix $0<\epsilon\ll1$ and $r\in(r_{i_*},r_{i_*}+\delta_*)$. Let $\phi\in C_0^\infty((0,\infty))$ satisfy $\int_{\tilde I_k(r)}\phi u_\epsilon^Gdx=0$ for all $k=1,\dots,N(r)$. Then, $\int_0^\infty\phi^2u_\epsilon^Gdx=\int_{\tilde I_0(r)}\phi^2u_\epsilon^Gdx+\sum_{k=1}^{N(r)}\int_{\tilde I_k(r)}\phi^2u_\epsilon^Gdx$. According to (4.6),

$$\int_{\tilde I_0(r)}\phi^2u_\epsilon^Gdx\le\frac{1}{\lambda_\epsilon^{\tilde I_0(r),R}}\frac{\epsilon^2}{2}\int_{\tilde I_0(r)}a(\phi')^2u_\epsilon^Gdx.$$

According to (4.7), (4.8) and the choice of ϕ, we find

$$\sum_{k=1}^{N(r)}\int_{\tilde I_k(r)}\phi^2u_\epsilon^Gdx\le\sum_{k=1}^{N(r)}\frac{1}{\lambda_\epsilon^{\tilde I_k(r),R}}\frac{\epsilon^2}{2}\int_{I_k(r)}a(\phi')^2u_\epsilon^Gdx.$$

It follows that

$$\frac{\frac{\epsilon^2}{2}\int_0^\infty a\,|\phi'|^2\,u_\epsilon^Gdx}{\int_0^\infty\phi^2u_\epsilon^Gdx}\ge\min_{k\in\{0,1,\dots,N(r)\}}\lambda_\epsilon^{\tilde I_k(r),R},$$

implying $\lambda_{\epsilon,N(r)+1}\ge\min_{k\in\{0,1,\dots,N(r)\}}\lambda_\epsilon^{\tilde I_k(r),R}$ for all $0<\epsilon\ll1$, which together with (4.17) yields

$$\liminf_{\epsilon\to0}\epsilon^2\ln\lambda_{\epsilon,N(r)+1}\ge-\max_{k\in\{0,1,\dots,N(r)\}}r^{\tilde I_k(r),R}\ge-r.$$

This proves (4.11), and hence, finishes the proof. □

5. Non-exponential asymptotics of eigenvalues

In Subsection 5.1, we state an auxiliary result on the asymptotics of eigenvalues of a family of Schrödinger operators. This is used in Subsection 5.2 to prove Theorem B. In Subsection 5.3, we establish auxiliary upper bounds of the eigenvalues $\lambda_{\epsilon,i}$, $i\in\mathbb{N}$.

5.1. **Asymptotics of Schrödinger eigenvalues.** Recall that y_∞ is defined in Subsection 2.2. Consider the following family of one-dimensional Schrödinger operators

$$H_h=-h^2\frac{d^2}{dy^2}+f(y)+hg(y):L^2((0,y_\infty))\to L^2((0,y_\infty)),\tag{5.1}$$

where $0<h\ll 1$, $f:(0,y_\infty)\to[0,\infty)$ and $g:(0,y_\infty)\to\mathbb{R}$ are continuous functions and satisfy:

- $\lim_{y\to y_\infty^-}f(y)=\infty$; f admits a unique zero y^*, and is twice continuously differentiable near y^* with $f''(y^*)>0$;
- g is lower bounded, and satisfies $g\le C_1f+C_2$ on $(0,y_\infty)$ for some $C_1,C_2>0$.

For each $0<h\ll 1$, the spectrum of H_h, as a self-adjoint realization with core $C_0^\infty((0,\infty))$, is purely discrete and is denoted by: $E_{h,1}<E_{h,2}<E_{h,3}\cdots\to\infty$. The following result concerning the asymptotics of $E_{h,i}$ as $h\to 0$ is well-known (see e.g. [14, 51, 35]).

Proposition 5.1. *For each $i\in\mathbb{N}$, there holds $\lim_{h\to 0^+}\frac{E_{h,i}}{h}=(2i-1)\sqrt{\frac{f''(y^*)}{2}}+g(y^*)$.*

5.2. **Proof of Theorem B.** Let the assumptions in Theorem B be satisfied. Recall $\mathcal{L}_\epsilon^S$ from (2.1) and consider the Schrödinger operator

$$H_\epsilon:=-2\epsilon^2\mathcal{L}_\epsilon^S=-\epsilon^4\frac{d^2}{dy^2}+\left[\frac{b^2}{a}+\epsilon^2\left(b'-\frac{a'b}{a}\right)+\epsilon^4\left(\frac{3}{16}\frac{(a')^2}{a}-\frac{a''}{4}\right)\right](\xi^{-1}(y)).\tag{5.2}$$

Denote by $E_{\epsilon,1}<E_{\epsilon,2}<E_{\epsilon,3}<\cdots$ the eigenvalues of H_ϵ, and recall that the eigenvalues of $-\mathcal{L}_\epsilon^\epsilon$ are $\lambda_{\epsilon,1}<\lambda_{\epsilon,2}<\lambda_{\epsilon,3}<\cdots$. Obviously,

$$E_{\epsilon,i}=2\epsilon^2\lambda_{\epsilon,i},\quad \forall i\in\mathbb{N}\quad\text{and}\quad 0<\epsilon\ll 1.\tag{5.3}$$

Note that with $\epsilon^2=h$, the operator H_ϵ is almost in the form of H_h given in (5.1). A careful analysis of the extra term $\epsilon^4\left(\frac{3}{16}\frac{(a')^2}{a}-\frac{a''}{4}\right)\circ\xi^{-1}$, which is unbounded near 0, allows us to apply Proposition 5.1 to prove the following result.

Lemma 5.1. *For each $i\in\mathbb{N}$, there holds $\lim_{\epsilon\to 0}\frac{E_{\epsilon,i}}{\epsilon^2}=2b'(x^*)(1-i)$.*

Proof. In order to apply Proposition 5.1, we construct operators that are in the form of (5.1) and comparable in the sense of quadratic forms to H_ϵ .

Let $c:(0,\infty)\to[0,\infty)$ be continuous and satisfy the following conditions:

- $c(x)=\frac{1}{x}$ for $x\in(0,\frac{x^*}{2})$, $c(x)=\frac{b^2(x)}{a(x)}$ for $x\gg x^*$;
- c is twice continuously differentiable near x^*, and satisfies $c(x^*)=c'(x^*)=c''(x^*)=0$.

By **(H)**, for each $0<\delta\ll 1$, there exists $0<\epsilon_\delta\ll 1$ such that $\epsilon^4\left(\frac{3}{16}\frac{(a')^2}{a}-\frac{a''}{4}\right)\le c+\epsilon^2\delta$ on $(0,\infty)$ for all $\epsilon\in(0,\epsilon_\delta)$, and thus, in the sense of quadratic forms,

$$H_\epsilon\le\overline{H}_\epsilon^\delta:=-\epsilon^4\frac{d^2}{dy^2}+\overline{f}(y)+\epsilon^2\overline{g}^\delta(y),\tag{5.4}$$

where $\overline{f}=\left(\frac{b^2}{a}+c\right)\circ\xi^{-1}$ and $\overline{g}^\delta=\left(b'-\frac{a'b}{a}+\delta\right)\circ\xi^{-1}$.

Obviously, $\overline{f}$ is continuously differentiable, satisfies $\overline{f}(y) \to \infty$ as $y \to y_\infty^-$, has $y^* := \xi(x^*)$ as the unique zero, and is twice continuously differentiable near y^* with $\overline{f}''(y^*) = 2(b'(x^*))^2$. By **(H)** and the construction of the function c, it is easy to see that $\overline{g}^\delta$ is lower bounded, and there are $C_1, C_2 > 0$ such that $\overline{g}^\delta \le C_1 \overline{f} + C_2$ for all $0 < \delta \ll 1$.

Denote by $\overline{E}^\delta_{\epsilon,1} \le \overline{E}^\delta_{\epsilon,2} \le \cdots$ the eigenvalues of $\overline{H}^\delta_\epsilon$. We apply Proposition 5.1 to conclude that

$$\lim_{\epsilon\to 0} \frac{\overline{E}^\delta_{\epsilon,i}}{\epsilon^2} = (2i-1)\sqrt{\frac{\overline{f}''(y^*)}{2}} + \overline{g}^\delta(y^*) = 2b'(x^*)(1-i) + \delta, \quad \forall 0 < \delta \ll 1 \text{ and } i \in \mathbb{N}.$$

It follows from (5.4) that $E_{\epsilon,i} \le \overline{E}^\delta_{\epsilon,i}$, leading to $\limsup_{\epsilon\to 0} \frac{E_{\epsilon,i}}{\epsilon^2} \le 2b'(x^*)(1-i)$ for all $i \in \mathbb{N}$.

It remains to establish the lower bound. Fix $x_0 \gg x^*$ and let $\eta : (0,\infty) \to [0, \frac{1}{2}]$ be smooth and satisfy $\eta(x) = 0$ for $x \le x_0$ and $\eta(x) = \frac{1}{2}$ for $x \ge 2x_0$. Then, for each $0 < \delta \ll 1$, there exists $0 < \epsilon_\delta \ll 1$ such that $\eta \frac{b^2}{a} + \epsilon^4 \left(\frac{3}{16}\frac{(a')^2}{a} - \frac{a''}{4} \right) \ge -\epsilon^2 \delta$ on $(0,\infty)$ for all $\epsilon \in (0, \epsilon_\delta)$, which gives

$$H_\epsilon \ge \underline{H}^\delta_\epsilon := -\epsilon^4 \frac{d^2}{dy^2} + \underline{f}(y) + \epsilon^2 \underline{g}^\delta(y), \tag{5.5}$$

where $\underline{f} = \left((1-\eta)\frac{b^2}{a} \right) \circ \xi^{-1}$ and $\underline{g}^\delta = \left(b' - \frac{a'b}{a} - \delta \right) \circ \xi^{-1}$.

Denote by $\underline{E}^\delta_{\epsilon,1} \le \underline{E}^\delta_{\epsilon,2} \le \cdots$ the eigenvalues of $\underline{H}^\delta_\epsilon$. We apply Proposition 5.1 to conclude that

$$\lim_{\epsilon\to 0} \frac{\underline{E}^\delta_{\epsilon,i}}{\epsilon^2} = (2i-1)\sqrt{\frac{\underline{f}''(y^*)}{2}} + \underline{g}^\delta(y^*) = 2b'(x^*)(1-i) - \delta, \quad \forall 0 < \delta \ll 1 \text{ and } i \in \mathbb{N}.$$

It follows from (5.5) that $\liminf_{\epsilon\to 0} \frac{E_{\epsilon,i}}{\epsilon^2} \ge 2b'(x^*)(1-i)$ for all $i \in \mathbb{N}$. This completes the proof. □

Theorem B now is a simple consequence of Lemma 5.1.

Proof of Theorem B. By Lemma 5.1 and (5.3), $\lim_{\epsilon\to 0} \lambda_{\epsilon,i} = b'(x^*)(1-i)$ for all $i \in \mathbb{N}$. □

5.3. **Upper bounds of eigenvalues.** In this subsection, we use Proposition 5.1 to establish non-exponential upper bounds of the eigenvalues $\lambda_{\epsilon,i}$, $i \in \mathbb{N}$ that complement Theorem A, which provides almost no information about $\lambda_{\epsilon,i}$ when $r_i = 0$.

Proposition 5.2. *Assume* **(H)**. *There are $\Lambda_1, \Lambda_2 > 0$ such that $\lambda_{\epsilon,i} \le \Lambda_1 i + \Lambda_2$ for all $i \in \mathbb{N}$ and $0 < \epsilon \ll 1$.*

Proof. Recall that $\lambda_{\epsilon,i}$, $i \in \mathbb{N}$ are also the eigenvalues of the Schrödinger operator $-\mathcal{L}^S_\epsilon$. The operator H_ϵ is given in (5.2). Its eigenvalues $E_{\epsilon,i}$, $i \in \mathbb{N}$ satisfy $E_{\epsilon,i} = 2\epsilon^2 \lambda_{\epsilon,i}$ for all $i \in \mathbb{N}$.

Arguing as in the proof of Lemma 5.1, it is not hard to find smooth functions $\tilde{f} : (0, y_\infty) \to [0,\infty)$ and $\tilde{g} : (0, y_\infty) \to \mathbb{R}$ satisfying

- $\tilde{f}(y) \to \infty$ as $y \to 0^+$ and $y \to y_\infty^-$; $\tilde{f}$ admits a unique zero at $\tilde{y}_*$ and satisfies $\tilde{f}''(\tilde{y}_*) > 0$,
- $\tilde{g}$ is lower bounded and there are constants $C_1, C_2 > 0$ such that $\tilde{g} \le C_1 \tilde{f} + C_2$,

such that $H_\epsilon \le \tilde{H}_\epsilon := -\epsilon^4 \frac{d^2}{dy^2} + \tilde{f}(y) + \epsilon^2 \tilde{g}(y)$ in the sense of quadratic forms. Denote by $\tilde{E}_{\epsilon,1} < \tilde{E}_{\epsilon,2} < \tilde{E}_{\epsilon,3} \cdots \to \infty$ the eigenvalues of $\tilde{H}_\epsilon$. We apply Proposition 5.1 with $h = \epsilon^2$ to $\tilde{H}_\epsilon$ to conclude

$$\lim_{\epsilon\to 0} \frac{\tilde{E}_{\epsilon,i}}{\epsilon^2} = (2i-1)\sqrt{\frac{\tilde{f}''(\tilde{y}_*)}{2}} + \tilde{g}(\tilde{y}_*), \quad \forall i \in \mathbb{N}.$$

It follows that $\limsup_{\epsilon\to 0} \lambda_{\epsilon,i} \le \frac{2i-1}{2}\sqrt{\frac{\tilde{f}''(\tilde{y}_*)}{2}} + \frac{\tilde{g}(\tilde{y}_*)}{2}$ for all $i \in \mathbb{N}$. This completes the proof. □

6. Qualitative characterizations of multi-scale dynamics

In this section, we study qualitative characterization of the multi-scale dynamics of X_t^ϵ, and prove Theorem C.

We need the following result. Recall from Proposition 2.1 that $(P_t^\epsilon)_{t\geq 0}$ is the positive analytic semigroup generated by $\mathcal{L}_\epsilon$, and Q_k^ϵ is the spectral projection of $\mathcal{L}_\epsilon$ corresponding to $\{\lambda_{\epsilon,j}\}_{j\geq k}$.

Lemma 6.1. *For each $k\in\mathbb{N}$, there hold the following statements.*

(1) *There exists $C>0$ such that for each $0<\epsilon\ll 1$,*

$$|P_t^\epsilon Q_k^\epsilon f|\leq \frac{C}{\epsilon}a^{\frac{1}{4}}e^{\frac{V}{2\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_{L^2(u_\epsilon^G)}\quad in\quad (0,\infty),\quad \forall f\in L^2(u_\epsilon^G)\ and\ t>1.$$

(2) *There exist $C>0$ such that for each $0<\epsilon\ll 1$,*

$$|P_t^\epsilon Q_k^\epsilon f|\leq a^{\frac{1}{4}}e^{\frac{V+C}{2\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_\infty\quad in\quad (0,\infty),\quad \forall f\in C_b([0,\infty))\ and\ t>2.$$

Proof. Fix $k\in\mathbb{N}$ and let $0<\epsilon\ll 1$. For $t\geq 0$, we define $\tilde{P}_t^\epsilon:=\tilde{U}_\epsilon U_\epsilon P_t^\epsilon U_\epsilon^{-1}\tilde{U}_\epsilon^{-1}$, where U_ϵ and $\tilde{U}_\epsilon$ are unitary transforms specified in Subsection 2.2. Then, $(\tilde{P}_t^\epsilon)_{t\geq 0}$ is an analytic semigroup of contractions on $L^2((0,y_\infty))$ with generator $\mathcal{L}_\epsilon^S$. The spectrum of $\mathcal{L}_\epsilon^S$ is the same as that of $\mathcal{L}_\epsilon$ consisting of simple eigenvalues $\{-\lambda_{\epsilon,k}\}_{k\in\mathbb{N}}$.

(1) Set $M:=1+|\inf_{0<\epsilon\ll 1}\min V_\epsilon|$, which is finite thanks to Lemma 2.2 (4). We claim that for each $p\in(2,\infty]$, there is $\tilde{C}_1=\tilde{C}_1(p)>0$ such that

$$\|\tilde{P}_t^\epsilon\tilde{f}\|_{L^p}\leq\frac{\tilde{C}_1}{t\epsilon}e^{Mt}\|\tilde{f}\|_{L^2},\quad \forall\tilde{f}\in L^2((0,y_\infty))\text{ and }t>0.\tag{6.1}$$

Since $\|(\lambda-\mathcal{L}_\epsilon^S)^{-1}\|_{L^2\to L^2}=\frac{1}{\mathrm{dist}(\lambda,\sigma(\mathcal{L}_\epsilon^S))}$ for all $\lambda\in\rho(\mathcal{L}_\epsilon^S)$ and $\sigma(\mathcal{L}_\epsilon^S)\subset(-\infty,0)$, we find

$$\|(\lambda-(\mathcal{L}_\epsilon^S-M))^{-1}\|_{L^2\to L^2}\leq\frac{1}{|\lambda|},\quad\forall\lambda\in\mathbb{C}\text{ with }\Re\lambda>0.\tag{6.2}$$

As $\mathcal{L}_\epsilon^S-M$ generates the analytic semigroup $(e^{-Mt}\tilde{P}_t^\epsilon)_{t\geq 0}$ of contractions on $L^2((0,y_\infty))$, and the right-hand side of (6.2) is independent of ϵ, we apply [48, Theorem 2.5.2] to find $C_1>0$ such that

$$\|(\mathcal{L}_\epsilon^S-M)e^{-Mt}\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}\leq\frac{C_1}{t}\|\tilde{f}\|_{L^2},\quad\forall\tilde{f}\in L^2((0,y_\infty))\text{ and }t>0.\tag{6.3}$$

Let $D(\mathcal{L}_\epsilon^S)$ be the domain of $\mathcal{L}_\epsilon^S$. Note that

$$\langle-(\mathcal{L}_\epsilon^S-M)u,u\rangle_{L^2}=\frac{\epsilon^2}{2}\int_0^{y_\infty}|u'|dy+\int_0^{y_\infty}(V_\epsilon+M)|u|^2dy,\quad\forall u\in D(\mathcal{L}_\epsilon^S),$$

which together with (6.3) and the fact $\inf_{0<\epsilon\ll 1}(\min V_\epsilon+M)\geq 1$ leads to

$$\begin{aligned}&\frac{\epsilon^2}{2}\int_0^{y_\infty}|\partial_y\tilde{P}_t^\epsilon\tilde{f}|^2dy+\int_0^{y_\infty}(V_\epsilon+M)|\tilde{P}_t^\epsilon\tilde{f}|^2dy\\&\quad\leq\|(\mathcal{L}_\epsilon^S-M)\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}\|\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}\\&\quad\leq\frac{C_1e^{Mt}}{t}\|\tilde{f}\|_{L^2}\|\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}\leq\frac{C_1e^{Mt}}{t}\|\tilde{f}\|_{L^2}\left(\frac{\epsilon^2}{2}\int_0^{y_\infty}|\partial_y\tilde{P}_t^\epsilon\tilde{f}|^2dy+\int_0^{y_\infty}(V_\epsilon+M)|\tilde{P}_t^\epsilon\tilde{f}|^2dy\right)^{\frac{1}{2}},\end{aligned}$$

yielding

$$\frac{\epsilon^2}{2}\int_0^{y_\infty}|\partial_y\tilde{P}_t^\epsilon\tilde{f}|^2dy+\int_0^{y_\infty}(V_\epsilon+M)|\tilde{P}_t^\epsilon\tilde{f}|^2dy\leq\frac{C_1^2e^{2Mt}}{t^2}\|\tilde{f}\|_{L^2}^2,\quad\forall\tilde{f}\in L^2((0,y_\infty)),\ t>0.$$

By Sobolev's embedding theorem, we find for each $p\in(2,\infty]$ the existence of $C_2(p)>0$ such that

$$\|\tilde{P}_t^\epsilon\tilde{f}\|_{L^p}\leq C_2(p)\left(\|\partial_y\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}+\|\tilde{P}_t^\epsilon\tilde{f}\|_{L^2}\right)\leq\frac{2C_1C_2(p)}{t\epsilon}e^{Mt}\|\tilde{f}\|_{L^2},\quad\forall\tilde{f}\in L^2((0,y_\infty)),\ t>0.$$

This gives (6.1) with $\tilde{C}_1:=2C_1C_2(p)$.

Now, let $f\in L^2(u_\epsilon^G)$ and set $\tilde{f}_\epsilon:=\tilde{U}_\epsilon U_\epsilon f$. Note that $\|\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^2}\leq e^{-\lambda_{\epsilon,k}t}\|\tilde{f}_\epsilon\|_{L^2}$ for all $t>0$ and $0<\epsilon\ll1$, where $\tilde{Q}_k^\epsilon:=\tilde{U}_\epsilon U_\epsilon Q_k^\epsilon U_\epsilon^{-1}\tilde{U}_\epsilon^{-1}$ is the spectral projection of $\mathcal{L}_\epsilon^S$ corresponding to $\{-\lambda_{\epsilon,j}\}_{j\geq k}$. Applying (6.1) with $p=\infty$, we deduce

$$\|\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^\infty}\leq\frac{\tilde{C}_1e^M}{\epsilon}\|\tilde{P}_{t-1}^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^2}\leq\frac{\tilde{C}_1e^M}{\epsilon}e^{-\lambda_{\epsilon,k}(t-1)}\|\tilde{f}_\epsilon\|_{L^2},\quad\forall t>1,\tag{6.4}$$

leading to

$$\begin{aligned}|P_t^\epsilon Q_k^\epsilon f|&=|U_\epsilon^{-1}\tilde{U}_\epsilon^{-1}\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon|\\&=|(\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon)\circ\xi|\frac{\sqrt{\xi'}}{\sqrt{u_\epsilon^G}}\leq\|\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^\infty}a^{\frac14}e^{\frac{V}{2\epsilon^2}}\leq\frac{\tilde{C}_1e^{M+\lambda_{\epsilon,1}}}{\epsilon}a^{\frac14}e^{\frac{V}{2\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_{L^2(u_\epsilon^G)},\quad\forall t>1.\end{aligned}$$

Since $\lambda_{\epsilon,1}\to0$ as $\epsilon\to0$ by Theorem A, the conclusion of (1) follows.

(2) Fix $f\in C_b([0,\infty)$ and set $\tilde{f}_\epsilon:=\tilde{U}_\epsilon U_\epsilon f$. We claim that there exist $\tilde{C}_2,\tilde{C}_3>0$ such that

$$\|\tilde{P}_1^\epsilon\tilde{f}_\epsilon\|_{L^2}\leq\frac{\tilde{C}_2}{\epsilon}e^{\frac{\tilde{C}_3}{\epsilon^2}}\|f\|_\infty.\tag{6.5}$$

To do so, we fix $p>2$. By (6.1), $\tilde{P}_1^\epsilon:L^2((0,y_\infty))\to L^p((0,y_\infty))$ is linear and bounded and satisfies $\|\tilde{P}_1^\epsilon\|_{L^2\to L^p}\leq\frac{\tilde{C}_1}{\epsilon}e^M$. This together with the symmetry of $\tilde{P}_1^\epsilon:L^2((0,y_\infty))\to L^2((0,y_\infty))$ yields

$$\|\tilde{P}_1^\epsilon\tilde{f}\|_{L^2}\leq\frac{\tilde{C}_1}{\epsilon}e^M\|\tilde{f}\|_{L^{p'}},\quad\forall\tilde{f}\in L^2((0,y_\infty))\cap L^{p'}((0,y_\infty)),\tag{6.6}$$

where $p'=\frac{p}{p-1}\in(1,2)$ is the dual exponent of p. As a consequence, $\tilde{P}_1^\epsilon$ uniquely extends to be a bounded linear operator from $L^{p'}((0,y_\infty))$ to $L^2((0,y_\infty))$, and satisfies $\|\tilde{P}_1^\epsilon\|_{L^{p'}\to L^2}\leq\frac{\tilde{C}_1}{\epsilon}e^M$.

Straightforward calculations give

$$\begin{aligned}\int_0^{y_\infty}|\tilde{f}_\epsilon|^{p'}dy&=\int_0^{y_\infty}|f\circ\xi^{-1}|^{p'}\left|\frac{u_\epsilon^G\circ\xi^{-1}}{\xi'\circ\xi^{-1}}\right|^{\frac{p'}{2}}dy\\&=\int_0^\infty|f|^{p'}|u_\epsilon^G|^{\frac{p'}{2}}|\xi'|^{1-\frac{p'}{2}}dx=\int_0^\infty|f|^{p'}a^{-\frac12-\frac{p'}{4}}e^{-\frac{p'V}{2\epsilon^2}}dx.\end{aligned}$$

Since $\int_0^\infty a^{-\frac12-\frac{p'}{4}}e^{-\frac{p'V}{2\epsilon^2}}dx\leq C_3^{p'}e^{\frac{C_4p'}{\epsilon^2}}$ for some $C_3,C_4>0$, we find $\|\tilde{f}_\epsilon\|_{L^{p'}}\leq C_3e^{\frac{C_4}{\epsilon^2}}\|f\|_\infty$, which together with (6.6) gives $\|\tilde{P}_1^\epsilon\tilde{f}_\epsilon\|_{L^2}\leq\frac{\tilde{C}_1C_3}{\epsilon}e^Me^{\frac{C_4}{\epsilon^2}}\|f\|_\infty$. The claim then follows readily by setting $\tilde{C}_2:=\tilde{C}_1C_3e^M$ and $\tilde{C}_3:=C_4$.

For $t>2$, we deduce from (6.4) and (6.5) that

$$\begin{aligned}\|\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^\infty}&\leq\frac{\tilde{C}_1e^M}{\epsilon}\|\tilde{P}_{t-1}^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^2}\\&\leq\frac{\tilde{C}_1e^M}{\epsilon}e^{-\lambda_{\epsilon,k}(t-2)}\|\tilde{P}_1^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon\|_{L^2}\leq\frac{\tilde{C}_1\tilde{C}_2e^{M+2\lambda_{\epsilon,k}}}{\epsilon^2}e^{-\lambda_{\epsilon,k}t}e^{\frac{\tilde{C}_3}{\epsilon^2}}\|f\|_\infty,\end{aligned}$$

and hence,

$$|P_t^\epsilon Q_k^\epsilon f| = |U_\epsilon^{-1}\tilde{U}_\epsilon^{-1}\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon| = |(\tilde{P}_t^\epsilon\tilde{Q}_k^\epsilon\tilde{f}_\epsilon)\circ\xi|\frac{\sqrt{\xi'}}{\sqrt{u_\epsilon^G}} \leq \frac{\tilde{C}_1\tilde{C}_2 e^{M+2\lambda_{\epsilon,k}}}{\epsilon^2}e^{-\lambda_{\epsilon,k}t}e^{\frac{\tilde{C}_3}{\epsilon^2}}\|f\|_\infty \times a^{\frac{1}{4}}e^{\frac{V}{2\epsilon^2}}.$$

By Proposition 5.2, it is easy to see the existence of $\tilde{C}_4 > 0$ (depending on k) such that $|P_t^\epsilon Q_k^\epsilon f| \leq a^{\frac{1}{4}}e^{\frac{V+\tilde{C}_4}{2\epsilon^2}}e^{-\lambda_{\epsilon,k}t}\|f\|_\infty$ for all $t > 2$ and $0 < \epsilon \ll 1$. This completes the proof. □

Proof of Theorem C. (1) Let $K \subset [0,\infty)$ be a fixed compact set, and $\mu \in \mathcal{P}([0,\infty))$ satisfy $\mathrm{supp}(\mu) \subset K$. Fix $f \in C_b([0,\infty))$ and $0 < \epsilon \ll 1$. Then, Proposition 2.1 (5) gives

$$\begin{aligned}\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)\mathbb{1}_{t<T_0^\epsilon}] &= \langle f,\phi_{\epsilon,1}\rangle_{L^2(u_\epsilon^G)}\|\phi_{\epsilon,1}\|_{L^1(\mu)}e^{-\lambda_{\epsilon,1}t} + \int_0^\infty P_t^\epsilon Q_2^\epsilon f d\mu \\ &= \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon + \int_0^\infty P_t^\epsilon Q_2^\epsilon f d\mu, \quad \forall t > 0,\end{aligned}$$

where $\alpha_\epsilon(\mu) := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\|\phi_{\epsilon,1}\|_{L^1(\mu)}$ and μ_ϵ is the unique QSD of X_t^ϵ (see Proposition 2.3). In particular, setting $f = \mathbb{1}$ yields

$$\mathbb{P}_\mu^\epsilon[t < T_0^\epsilon] = \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t} + \int_0^\infty P_t^\epsilon Q_2^\epsilon \mathbb{1} d\mu, \quad \forall t > 0. \tag{6.7}$$

It follows that

$$\begin{aligned}\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] &= \mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)\mathbb{1}_{t<T_0^\epsilon}] + \mathbb{E}_\mu^\epsilon[f(0)\mathbb{1}_{t\geq T_0^\epsilon}] \\ &= \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon + \int_0^\infty P_t^\epsilon Q_2^\epsilon f d\mu + f(0)\left(1 - \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t} - \int_0^\infty P_t^\epsilon Q_2^\epsilon \mathbb{1} d\mu\right) \\ &= \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon + \left(1 - \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\right)f(0) \\ &\quad + \int_0^\infty P_t^\epsilon Q_2^\epsilon f d\mu - f(0)\int_0^\infty P_t^\epsilon Q_2^\epsilon \mathbb{1} d\mu, \quad \forall t > 0.\end{aligned}$$

Lemma 6.1 (2) (with $k = 2$) yields the existence of $C_1 > 0$ such that

$$\begin{aligned}&\left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] - \left[\alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon + \left(1 - \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\right)f(0)\right]\right| \\ &\quad \leq \int_0^\infty \left(|P_t^\epsilon Q_2^\epsilon f| + |P_t^\epsilon Q_2^\epsilon \mathbb{1}||f(0)|\right)d\mu \leq 2\int_0^\infty a^{\frac{1}{4}}e^{\frac{V+C_1}{2\epsilon^2}}d\mu e^{-\lambda_{\epsilon,2}t}\|f\|_\infty, \quad \forall t > 2.\end{aligned}$$

Clearly, there is $C_2 = C_2(K) > 0$ such that $2\int_0^\infty a^{\frac{1}{4}}e^{\frac{V+C_1}{2\epsilon^2}}d\mu \leq e^{\frac{C_2}{\epsilon^2}}$, leading to

$$\left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - \left[\alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t}\mu_\epsilon + (1 - \alpha_\epsilon(\mu)e^{-\lambda_{\epsilon,1}t})\delta_0\right]\right\|_{TV} \leq \exp\left\{\frac{C_2}{\epsilon^2} - \lambda_{\epsilon,2}t\right\}, \quad \forall t > 2.$$

It remains to estimate $\alpha_\epsilon(\mu)$. Note from (6.7) that

$$\begin{aligned}\alpha_\epsilon(\mu) &= e^{\lambda_{\epsilon,1}t}\left(\mathbb{P}_\mu^\epsilon[t < T_0^\epsilon] - \int_0^\infty P_t^\epsilon Q_2^\epsilon \mathbb{1} d\mu\right) \\ &\leq e^{\lambda_{\epsilon,1}t}\left(1 + \int_0^\infty a^{\frac{1}{4}}e^{\frac{V+C_1}{2\epsilon^2}}d\mu e^{-\lambda_{\epsilon,2}t}\right) \leq e^{\lambda_{\epsilon,1}t}\left(1 + e^{\frac{C_2}{\epsilon^2} - \lambda_{\epsilon,2}t}\right), \quad \forall t > 0.\end{aligned}$$

Since the function $t \mapsto e^{\lambda_{\epsilon,1}t}\left(1+e^{\frac{C_2}{\epsilon^2}-\lambda_{\epsilon,2}t}\right)$ is minimized at $t=\left(\frac{C_2}{\epsilon^2}-\ln\frac{\lambda_{\epsilon,1}}{\lambda_{\epsilon,2}-\lambda_{\epsilon,1}}\right)\frac{1}{\lambda_{\epsilon,2}}$, we find

$$\alpha_\epsilon(\mu) \le \exp\left\{\left(\frac{C_2}{\epsilon^2}-\ln\frac{\lambda_{\epsilon,1}}{\lambda_{\epsilon,2}-\lambda_{\epsilon,1}}\right)\frac{\lambda_{\epsilon,1}}{\lambda_{\epsilon,2}}\right\}\left(1+\frac{\lambda_{\epsilon,1}}{\lambda_{\epsilon,2}-\lambda_{\epsilon,1}}\right).$$

By the asymptotics of $\lambda_{\epsilon,1}$ and $\lambda_{\epsilon,2}$ (see Theorem A), we find the existence of some small $\epsilon_*=\epsilon_*(C_2)=\epsilon_*(K)>0$ such that $\alpha_\epsilon(\mu)\le 1+e^{-\frac{r_1-r_2}{2\epsilon^2}}$ for all $\epsilon\in(0,\epsilon_*)$. This proves (1).

(2) We prove the result in the global attractor case; the local attractor case can be treated in the same manner. Let $0<\delta\ll 1$ and K_1, K_2 be compact sets in $(0,\infty)$. Let $\mu\in\mathcal{P}([0,\infty))$ satisfy $\operatorname{supp}(\mu)\subset K_1$, and f be a continuous function on $(0,\infty)$ with $\operatorname{supp}(f)\subset K_2$.

Recall that $\mathcal{A}$ is the global attractor of (1.2). Let $T=T(K_1,\delta)>0$ be the smallest time such that $\sup_{x_0\in K}\operatorname{dist}(x_t,\mathcal{A})<\frac{\delta}{2}$ for all $t>T$, where x_t denotes the solution of (1.2). By the sample path large deviation principle, there are $D_1=D_1(K_1,\delta)>0$ and $D_2=D_2(K_1,\delta)>0$ such that

$$\sup_{x_0\in K_1}\mathbb{P}^\epsilon_{x_0}\left[\sup_{t\in[0,T]}|X^\epsilon_t-x_t|>\frac{\delta}{2}\right]\le D_1e^{-\frac{D_2}{\epsilon^2}},\quad \forall 0<\epsilon\ll 1,$$

resulting in

$$\mathbb{P}^\epsilon_\mu[X^\epsilon_T\in\mathcal{A}^c_\delta]\le\mathbb{P}^\epsilon_\mu\left[|X^\epsilon_T-x_T|>\frac{\delta}{2}\right]\le D_1e^{-\frac{D_2}{\epsilon^2}},\quad \forall 0<\epsilon\ll 1, \tag{6.8}$$

where $\mathcal{A}_\delta:=\{x>0:\operatorname{dist}(x,\mathcal{A})<\delta\}$ and $\mathcal{A}^c_\delta:=(0,\infty)\setminus\mathcal{A}_\delta$.

Setting $\mu^\epsilon_T:=\mathbb{P}^\epsilon_\mu[X^\epsilon_T\in\bullet]$, we use the Markov property of X^ϵ_t to write

$$\begin{aligned}\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)]&=\mathbb{E}^\epsilon_\mu\left[\mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)|X^\epsilon_T]\right]\\&=\int_{\mathcal{A}_\delta}\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_{t-T})]d\mu^\epsilon_T+\int_{\mathcal{A}^c_\delta}\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_{t-T})]d\mu^\epsilon_T=:E^\epsilon_1+E^\epsilon_2,\quad\forall t\ge T.\end{aligned}\tag{6.9}$$

We estimate E^ϵ_1. Since Proposition 2.1 gives

$$\mathbb{E}^\epsilon_\bullet[f(X^\epsilon_{t-T})]-\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1}e^{-\lambda_{\epsilon,1}(t-T)}\int_0^\infty fd\mu_\epsilon=P^\epsilon_{t-T}Q^\epsilon_2f,\quad\forall t>T,$$

we apply Lemma 6.1 (1) (with $k=2$) to find $D_3>0$ such that

$$\begin{aligned}&\left|E^\epsilon_1-\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\int_{\mathcal{A}_\delta}\phi_{\epsilon,1}d\mu^\epsilon_Te^{-\lambda_{\epsilon,1}(t-T)}\int_0^\infty fd\mu_\epsilon\right|\\&\quad\le\frac{D_3}{\epsilon}e^{-\lambda_{\epsilon,2}(t-T)}\left(\int_{\mathcal{A}_\delta}a^{\frac14}e^{\frac{V}{2\epsilon^2}}d\mu^\epsilon_T\right)\left(\int_0^\infty|f|^2u^G_\epsilon dx\right)^{\frac12}\\&\quad=\frac{D_3}{\epsilon}e^{-\lambda_{\epsilon,2}(t-T)}\left(\int_{\mathcal{A}_\delta}a^{\frac14}e^{\frac{V}{2\epsilon^2}}d\mu^\epsilon_T\right)\left(\int_0^\infty\frac{|f|^2}{a}e^{-\frac{V}{\epsilon^2}}dx\right)^{\frac12}\\&\quad\le\frac{D_3}{\epsilon}\left(\sup_{\mathcal{A}_\delta}a^{\frac14}\right)\left(\int_{K_2}\frac1a dx\right)^{\frac12}\exp\left\{\frac{\sup_{\mathcal{A}_\delta}V}{2\epsilon^2}-\frac{\inf_{K_2}V}{2\epsilon^2}-\lambda_{\epsilon,2}(t-T)\right\}\|f\|_\infty\\&\quad\le\frac{D_4}{\epsilon}\exp\left\{\frac{\sup_{\mathcal{A}_\delta}V}{2\epsilon^2}-\frac{\inf_{K_2}V}{2\epsilon^2}-\lambda_{\epsilon,2}(t-T)\right\}\|f\|_\infty,\quad\forall t>T+1,\end{aligned}\tag{6.10}$$

where $D_4=D_4(K_2):=D_3\left(\sup_{\mathcal{A}_1}a^{\frac14}\right)\left(\int_{K_2}\frac1a dx\right)^{\frac12}$.

To estimate E_2^ϵ, we see from (6.8) that

$$E_2^\epsilon \leq \mathbb{P}_\mu^\epsilon[X_T^\epsilon \in \mathcal{A}_\delta^c]\|f\|_\infty \leq D_1 e^{-\frac{D_2}{\epsilon^2}}\|f\|_\infty, \quad \forall t > T.$$

Since Lemma 6.1 (2) (with $k = 1$) gives $D_5 > 0$ such that

$$E_2^\epsilon \leq \mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] \leq \int_0^\infty a^{\frac{1}{4}} e^{\frac{V+D_5}{\epsilon^2}} d\mu e^{-\lambda_{\epsilon,1}(t-T)}\|f\|_\infty, \quad \forall t > T+2,$$

there exists $D_6 = D_6(K_1) > 0$ such that

$$E_2^\epsilon \leq \exp\left\{\frac{D_6}{\epsilon^2} - \lambda_{\epsilon,1}(t-T)\right\}\|f\|_\infty, \quad \forall t > T+2.$$

Clearly, there exist $p = p(K_1,\delta) > 1$ and $D_7 = D_7(K_1,\delta) > 0$ such that $-\frac{D_2}{p} + \frac{D_6}{p'} < -D_7$, where $p' = \frac{p}{p-1}$ is the dual exponent of p. Then,

$$\begin{aligned} E_2^\epsilon = (E_2^\epsilon)^{\frac{1}{p}}(E_2^\epsilon)^{\frac{1}{p'}} &\leq D_1^{\frac{1}{p}} \exp\left\{-\frac{D_2}{p\epsilon^2} + \frac{D_6}{p'\epsilon^2} - \frac{\lambda_{\epsilon,1}(t-T)}{p'}\right\}\|f\|_\infty \\ &\leq D_1^{\frac{1}{p}} \exp\left\{-\frac{D_7}{\epsilon^2} - \frac{\lambda_{\epsilon,1}(t-T)}{p'}\right\}\|f\|_\infty, \quad \forall t > T+2. \end{aligned} \tag{6.11}$$

Setting $\hat{\alpha}_\epsilon(\mu,\delta) := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_{\mathcal{A}_\delta} \phi_{\epsilon,1} d\mu_T^\epsilon e^{\lambda_{\epsilon,1}T}$, we find from (6.9), (6.10) and (6.11) that

$$\begin{aligned} \left|\mathbb{E}_\mu^\epsilon[f(X_t^\epsilon)] - \hat{\alpha}_\epsilon(\mu,\delta)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon\right| &\leq \left|E_1^\epsilon - \hat{\alpha}_\epsilon(\mu,\delta)e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon\right| + E_2^\epsilon \\ &\leq \frac{D_4}{\epsilon}\exp\left\{\frac{\sup_{\mathcal{A}_\delta} V}{2\epsilon^2} - \frac{\inf_{K_2} V}{2\epsilon^2} - \lambda_{\epsilon,2}(t-T)\right\}\|f\|_\infty \\ &\quad + D_1^{\frac{1}{p}}\exp\left\{-\frac{D_7}{\epsilon^2} - \frac{\lambda_{\epsilon,1}(t-T)}{p'}\right\}\|f\|_\infty, \quad \forall t > T+2. \end{aligned}$$

It remains to estimate $\hat{\alpha}(\mu,\delta)$. Note that

$$\begin{aligned} \hat{\alpha}_\epsilon(\mu,\delta) &\leq \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_0^\infty \phi_{\epsilon,1} d\mu_T^\epsilon e^{\lambda_{\epsilon,1}T} \\ &= \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_0^\infty P_T^\epsilon \phi_{\epsilon,1} d\mu e^{\lambda_{\epsilon,1}T} = \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)} \int_0^\infty \phi_{\epsilon,1} d\mu = \alpha_\epsilon(\mu). \end{aligned} \tag{6.12}$$

The upper bound of $\hat{\alpha}(\mu,\delta)$ follows from that of $\alpha_\epsilon(\mu)$ given in (1). This completes the proof. □

7. Applications

In this section, we discuss some applications of Corollary A and Theorem C. Two typical situations where SDEs of the form (1.1) arise are described as follows.

Chemical reactions. Consider the chemical reactions:

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \quad X \xrightarrow{k_2} C,$$

where the positive numbers k_1, k_{-1} and k_2 are reaction rates. The concentration of the substance A, denoted by x_A, is assumed to remain constant. We assume $k_1 x_A > k_2$.

Denote by $V \gg 1$ the volume of the system and X_t^V be the continuous-time Markov jump process counting the number of the substance X. The law of large numbers [19, 1] ensures that as the volume

V grows to infinity, the re-scaled process $\frac{X_t^V}{V}$ converges to solutions of the following classical mean-field ODE for the concentration of X:

$$\dot{x} = -k_{-1}x^2 + k_1 x_A x - k_2 x, \quad x \in [0,\infty). \tag{7.1}$$

Obviously, (7.1) is a standard logistic equation, and its unique positive equilibrium is given by $x_e := \frac{k_1 x_A - k_2}{k_{-1}}$. The fluctuation of $\frac{X_t^V}{V}$ around solutions of (7.1) is captured by the central limit theorem [19, 1], leading to the following *diffusion approximation* of $\frac{X_t^V}{V}$:

$$dx = (-k_{-1}x^2 + k_1 x_A x - k_2 x)dt + \epsilon\sqrt{k_{-1}x^2 + k_1 x_A x + k_2 x}dW_t, \quad x \in [0,\infty), \tag{7.2}$$

where $\epsilon = \frac{1}{\sqrt{V}}$ and W_t is a standard one-dimensional Wiener process.

It is a routine task to check (see e.g. [30]) that solutions of (7.2) hit the extinction state 0 in finite time almost surely, while positive solutions of (7.1) are attracted by the equilibrium x_e. Such a dynamical disagreement between a mean-field model and its stochastic counterpart is often referred to as *Keizer's paradox* [31]. This is originally formulated in terms of the master equation for distributions of $\frac{X_t^V}{V}$ (see e.g. [32, 52, 7]). In the work [7], the authors use a slow manifold reduction method to estimate the first eigenvalue and the gap between the first and second eigenvalues of the generator of $\frac{X_t^V}{V}$ that respectively quantify the first passage time to the extinction state and QSD.

Logistic BDPs. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda > \mu > 0$. Consider a continuous-time birth-and-death process (BDP) Z_t^K on the state space $\mathbb{N}_0$ with birth rates $\lambda_n^K = \lambda n$, $n \in \mathbb{N}_0$ and death rates $\mu_n^K = n\left(\mu + \frac{n}{K}\right)$, $n \in \mathbb{N}$, where $K \gg 1$ is the carrying capacity. By the central limit theorem (see e.g. [36, 19]), for sufficiently large K, the process $\frac{Z_t^K}{K}$ stays close to solutions of the following SDE:

$$dx = (\lambda x - \mu x - x^2)dt + \epsilon\sqrt{\lambda x + \mu x + x^2}dW_t, \quad x \in [0,\infty), \tag{7.3}$$

where $\epsilon = \frac{1}{\sqrt{K}}$. The SDE (7.3) is the diffusion approximation of $\frac{Z_t^K}{K}$.

Both situations give rise to SDEs of the form :

$$dx = (b_1 x - b_2 x^2)dt + \epsilon\sqrt{a_1 x + a_2 x^2}dW_t, \quad x \in [0,\infty), \tag{7.4}$$

where $0 < \epsilon \ll 1$, b_1, b_2 and a_1 are positive constants, and $a_2 \geq 0$. It is straightforward to check that assumptions in Corollary A and Proposition A.1 are satisfied. Applying Corollary A, Theorem C and Proposition A.1 to (7.4), we find the following result.

Theorem 7.1. *The following statements hold.*

(1) *There hold*

$$-\lim_{\epsilon\to 0}\epsilon^2 \ln \lambda_{\epsilon,1} = r_1 := \begin{cases} 2\left[\frac{a_1 b_2 + a_2 b_1}{a_2^2}\ln\left(1 + \frac{a_2 b_1}{a_1 b_2}\right) - \frac{b_1}{a_2}\right], & \text{if} \quad a_2 > 0, \\ \frac{b_1^2}{a_1 b_2}, & \text{if} \quad a_2 = 0 \end{cases}$$

and $\lim_{\epsilon\to 0}\lambda_{\epsilon,2} = b_1$.

(2) *For each compact* $K \subset [0,\infty)$*, there are* $C = C(K) > 0$ *and small* $\epsilon_* = \epsilon_*(K) > 0$ *such that*

$$\begin{aligned} &\sup_{\substack{\mu\in\mathcal{P}([0,\infty)) \\ \operatorname{supp}(\mu)\subset K}} \left\|\mathbb{P}_\mu^\epsilon[X_t^\epsilon \in \bullet] - \left[e^{-\lambda_{\epsilon,1}t}\mu_\epsilon + (1 - e^{-\lambda_{\epsilon,1}t})\delta_0\right]\right\|_{TV} \\ &\qquad \leq \exp\left\{\frac{C}{\epsilon^2} - \frac{b_1}{2}t\right\} + \gamma(\epsilon)e^{-\lambda_{\epsilon,1}t}, \quad \forall t > 0 \text{ and } \epsilon \in (0,\epsilon_*), \end{aligned}$$

where $\gamma = \gamma_K : (0,1) \to (0,1)$ *satisfies* $\gamma(\epsilon) \to 0$ *as* $\epsilon \to 0$.

(3) *For each* $0 < \delta \ll 1$ *and compact sets* $K_1, K_2 \subset (0,\infty)$ *with* $(\frac{b_1}{b_2} - \delta, \frac{b_1}{b_2} + \delta) \cap K_2 = \emptyset$, *there are* $T = T(K_1,\delta) > 0$, $C = C(K_1,K_2,\delta) > 0$ *and* $\epsilon_* = \epsilon_*(K_1,K_2,\delta) > 0$ *such that for each continuous function* $f : (0,\infty) \to \mathbb{R}$ *with* $\mathrm{supp}(f) \subset K_2$, *there holds*

$$\sup_{\substack{\mu\in\mathcal{P}([0,\infty))\\ \mathrm{supp}(\mu)\subset K_1}} \left| \mathbb{E}^\epsilon_\mu[f(X^\epsilon_t)] - e^{-\lambda_{\epsilon,1}t}\int_0^\infty f d\mu_\epsilon \right| \le \Gamma(\epsilon) e^{-C\lambda_{\epsilon,1}t}\|f\|_\infty, \quad \forall t > T \text{ and } \epsilon \in (0,\epsilon_*),$$

where $\Gamma = \Gamma_{K_1,K_2,\delta} : (0,1) \to (0,1)$ *satisfies* $\Gamma(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Note that $\frac{a_1b_2+a_2b_1}{a_2^2}\ln\left(1+\frac{a_2b_1}{a_1b_2}\right) = \frac{a_1b_2}{a_2^2}\left(1+\frac{a_2b_1}{a_1b_2}\right)\ln\left(1+\frac{a_2b_1}{a_1b_2}\right) > \frac{a_1b_2}{a_2^2}\frac{a_2b_1}{a_1b_2} = \frac{b_1}{a_2}$.

Appendix A. Asymptotic of the first eigenfunctions

We justify (1.6) when b changes its sign only once appealing to results established in the forthcoming work [47]. The formal statement is as follows. Recall $\alpha_\epsilon(\mu)$ and $\hat\alpha_\epsilon(\mu,\delta)$ from Theorem C.

Proposition A.1. *Assume* **(H)**. *Suppose, in addition, that*

- $b \in C^2((0,\infty))$, *b changes its sign only at* $x^* \in (0,\infty)$ *and* $b'(x^*) < 0$;
- $a \in C^3((0,\infty))$ *and* $\int_1^\infty \frac{1}{\sqrt{a(s)}}ds = \infty$.

Then, $\lim_{\epsilon\to 0}\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}\phi_{\epsilon,1} = 1$ *locally uniformly in* $(0,\infty)$. *In particular,*

(1) *for any compact* $K \subset [0,\infty)$,

$$\lim_{\epsilon\to 0}\alpha_\epsilon(\mu) = 1 \quad \text{uniformly in } \mu \in \mathcal{P}([0,\infty)) \text{ with } \mathrm{supp}(\mu) \subset K;$$

(2) *for any compact* $K \subset (0,\infty)$ *and* $0 < \delta_1 < \delta_2 \ll 1$,

$$\lim_{\epsilon\to 0}\hat\alpha_\epsilon(\mu,\delta) = 1 \quad \text{uniformly in } \mu \in \mathcal{P}([0,\infty)) \text{ with } \mathrm{supp}(\mu) \subset K \text{ and } \delta \in [\delta_1,\delta_2].$$

We remark that the condition $\int_1^\infty \frac{1}{\sqrt{a(s)}}ds = \infty$ is only to make $y_\infty = \lim_{x\to\infty}\xi(x) = \infty$, where ξ is defined in Subsection 2.2, so that the Schrödinger operator $\mathcal{L}^S_\epsilon$ is on $(0,\infty)$. Then, we can use decaying properties of its eigenfunctions near ∞.

Recall that $u_\epsilon = \frac{\phi_{\epsilon,1}u^G_\epsilon}{\|\phi_{\epsilon,1}\|_{L^1(u^G_\epsilon)}}$ is the density of the unique QSD μ_ϵ of X^ϵ_t. We need the following results from [47], whose proof indeed benefits from decaying properties of $\tilde U_\epsilon U_\epsilon\phi_{\epsilon,1}$, an eigenfunction of $\mathcal{L}^S_\epsilon$ associated to $\lambda_{\epsilon,1}$, near ∞, where $\tilde U_\epsilon$ and U_ϵ are unitary transforms defined in Subsection 2.2.

Lemma A.1 ([47]). *Suppose the conditions in Proposition A.1 hold.*

(1) *There holds* $u_\epsilon = \frac{R_\epsilon}{\epsilon a}e^{-\frac{2}{\epsilon^2}\int_\bullet^{x_*}\frac{b(s)}{a(s)}ds}$, *where*

$$\lim_{\epsilon\to 0}R_\epsilon(x) = R_0 := \sqrt{-\frac{a(x_*)b'(x_*)}{\pi}} \text{ locally uniformly in } x \in (0,\infty).$$

(2) *For each* $0 < \delta \ll 1$, *there are* $0 < x_\delta \ll 1$ *and* $0 < \epsilon_\delta \ll 1$ *such that*

$$e^{-\frac{2}{\epsilon^2}\left(\int_0^{x^*}\frac{b(s)}{a(s)}ds+\delta\right)} \le u_\epsilon(x) \le e^{-\frac{2}{\epsilon^2}\left(\int_0^{x^*}\frac{b(s)}{a(s)}ds-\delta\right)}, \quad \forall x \in (0,x_\delta), \quad \epsilon \in (0,\epsilon_\delta).$$

(3) *There exist* $L \gg 1$, $C > 0$, $r_* > 0$ *and* $0 < \epsilon_* \ll 1$ *such that*

$$u_\epsilon(x) \le \frac{C}{[a(x)]^{\frac{3}{4}}}e^{-\frac{r_*}{\epsilon^2}(\xi(x)-\xi(L))}e^{\frac{1}{\epsilon^2}\int_L^x\frac{b(s)}{a(s)}ds}, \quad \forall x \in [L,\infty), \quad \epsilon \in (0,\epsilon_*).$$

Note that Lemma A.1 (1) justifies the WKB approximation of u_ϵ up to the second order.

We prove Proposition A.1.

Proof of Proposition A.1. Lemma A.1 (1) gives $\frac{\phi_{\epsilon,1}u_\epsilon^G}{\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}} = \frac{R_\epsilon}{\epsilon a}e^{-\frac{2}{\epsilon^2}\int_\bullet^{x_*}\frac{b(s)}{a(s)}ds}$, which together with the expression for u_ϵ^G leads to

$$\phi_{\epsilon,1} = \frac{1}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x^*}\frac{b(s)}{a(s)}ds}\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}R_\epsilon. \tag{A.1}$$

It follows that

$$1 = \int_0^\infty \phi_{\epsilon,1}^2 u_\epsilon^G dx = \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\int_0^\infty u_\epsilon\phi_{\epsilon,1}dx = \frac{1}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x^*}\frac{b(s)}{a(s)}ds}\|\phi_{\epsilon,1}\|^2_{L^1(u_\epsilon^G)}\int_0^\infty u_\epsilon R_\epsilon dx. \tag{A.2}$$

We claim $\lim_{\epsilon\to0}\int_0^\infty u_\epsilon R_\epsilon dx = R_0$. Indeed, $u_\epsilon R_\epsilon = \epsilon a u_\epsilon^2 e^{\frac{2}{\epsilon^2}\int_\bullet^{x_*}\frac{b(s)}{a(s)}ds}$ thanks to Lemma A.1 (1). Let $K\gg1$. It is easy to see from Lemma A.1 (2) that $\lim_{\epsilon\to0}\int_0^{\frac{1}{K}}u_\epsilon R_\epsilon dx = 0$. By Lemma A.1 (3),

$$\begin{aligned}\int_K^\infty u_\epsilon R_\epsilon dx &\le \epsilon C^2 e^{\frac{2\gamma_*}{\epsilon^2}\xi(L)}e^{\frac{2}{\epsilon^2}\int_L^{x_*}\frac{b(s)}{a(s)}ds}\int_K^\infty\frac{1}{\sqrt{a(x)}}e^{-\frac{2\gamma_*}{\epsilon^2}\xi(x)}dx\\ &= \epsilon C^2 e^{\frac{2\gamma_*}{\epsilon^2}\xi(L)}e^{\frac{2}{\epsilon^2}\int_L^{x_*}\frac{b(s)}{a(s)}ds}\int_{\xi(K)}^\infty e^{-\frac{2\gamma_*}{\epsilon^2}y}dy\to0 \quad\text{as}\quad \epsilon\to0.\end{aligned}$$

Hence, $\lim_{\epsilon\to0}\int_0^\infty u_\epsilon R_\epsilon dx = \lim_{\epsilon\to0}\int_{\frac{1}{K}}^K u_\epsilon R_\epsilon dx$. Since $\int_0^\infty u_\epsilon dx = 1$ and Lemma A.1 (2)(3) gives $\lim_{\epsilon\to0}\int_{\frac{1}{K}}^K u_\epsilon dx = 1$, we conclude from Lemma A.1 (1) that $\lim_{\epsilon\to0}\int_{\frac{1}{K}}^K u_\epsilon R_\epsilon dx = R_0$. This proves the claim.

Letting $\epsilon\to0$ in (A.2), the claim implies $\lim_{\epsilon\to0}\frac{1}{\epsilon}e^{-\frac{2}{\epsilon^2}\int_0^{x^*}\frac{b(s)}{a(s)}ds}\|\phi_{\epsilon,1}\|^2_{L^1(u_\epsilon^G)} = \frac{1}{R_0}$, which together with (A.1) and Lemma A.1 (1) yields the local uniform convergence of $\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}$ to 1 as $\epsilon\to0$.

(1) follows readily. To prove (2), we recall that $\hat{\alpha}_\epsilon(\mu,\delta) := \|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\int_{\mathcal{A}_\delta}\phi_{\epsilon,1}d\mu_T^\epsilon e^{\lambda_{\epsilon,1}T}$, where $T = T(K,\delta)$ is the smallest time such that $\sup_{x_0\in K}\operatorname{dist}(x_t,\mathcal{A}) < \frac{\delta}{2}$ for all $t>T$, and $\mu_T^\epsilon = \mathbb{P}_\mu^\epsilon[X_T^\epsilon\in\bullet]$.

Let $\eta_\delta:(0,\infty)\to[0,1]$ be a smooth function satisfying $\eta_\delta = 1$ on $\mathcal{A}_{\frac{3\delta}{4}}$ and $\eta_\delta = 0$ on $\mathcal{A}_\delta^c$. Then,

$$\begin{aligned}\hat{\alpha}_\epsilon(\mu,\delta) &\ge e^{\lambda_{\epsilon,1}T}\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\int_0^\infty\eta_\delta\phi_{\epsilon,1}d\mu_T^\epsilon\\ &= e^{\lambda_{\epsilon,1}T}\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\int_0^\infty P_T^\epsilon(\eta_\delta\phi_{\epsilon,1})d\mu = e^{\lambda_{\epsilon,1}T}\int_0^\infty\mathbb{E}_\bullet^\epsilon\left[(\eta_\delta\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1})(X_T^\epsilon)\mathbb{1}_{T<T_0^\epsilon}\right]d\mu,\end{aligned}$$

where we used Proposition 2.1 (4) in the second equality. Note that

$$\begin{aligned}&\mathbb{E}_\bullet^\epsilon\left[(\eta_\delta\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1})(X_T^\epsilon)\mathbb{1}_{T<T_0^\epsilon}\right]\\ &\quad\ge\int_{[X_T^\epsilon\in\mathcal{A}_{\frac{3\delta}{4}}]}(\eta_\delta\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1})(X_T^\epsilon)\mathbb{1}_{T<T_0^\epsilon}d\mathbb{P}_\bullet^\epsilon\\ &\quad\ge\left(\inf_{\mathcal{A}_{\frac{3\delta}{4}}}\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}\right)\mathbb{P}_\bullet^\epsilon\left[X_T^\epsilon\in\mathcal{A}_{\frac{3\delta}{4}},T<T_0^\epsilon\right]\\ &\quad\ge\left(\inf_{\mathcal{A}_{\frac{3\delta}{4}}}\|\phi_{\epsilon,1}\|_{L^1(u_\epsilon^G)}\phi_{\epsilon,1}\right)\left\{\mathbb{P}_\bullet^\epsilon\left[X_T^\epsilon\in\mathcal{A}_{\frac{3\delta}{4}}\right]+\mathbb{P}_\bullet^\epsilon\left[T<T_0^\epsilon\right]-1\right\}.\end{aligned}$$

Since the limits $\lim_{\epsilon\to 0} e^{\lambda_{\epsilon,1}T} = 1$, $\lim_{\epsilon\to 0} \mathbb{P}_x^\epsilon \left[X_T^\epsilon \in \mathcal{A}_{\frac{3\delta}{4}}\right] = 1$ and $\lim_{\epsilon\to 0} \mathbb{P}_x^\epsilon \left[T < T_0^\epsilon\right] = 1$ are uniform in $x \in K$ and $\delta \in [\delta_1, \delta_2]$, we conclude the lower bound:

$$\liminf_{\epsilon\to 0} \hat{\alpha}_\epsilon(\mu, \delta) \geq 1 \quad \text{uniformly in } \mu \in \mathcal{P}([0,\infty)) \text{ with } \operatorname{supp}(\mu) \subset K \text{ and } \delta \in [\delta_1, \delta_2].$$

The upper bound follows from $\hat{\alpha}_\epsilon(\mu, \delta) \leq \alpha_\epsilon(\mu)$ due to (6.12), and the limit proven in (1) for $\alpha_\epsilon(\mu)$. This completes the proof. □

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