CONVERGENCE TO PERIODIC PROBABILITY SOLUTIONS IN FOKKER-PLANCK EQUATIONS

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Abstract. The present paper is devoted to the study of convergence of solutions of a Fokker-Planck equation (FPE) associated to a periodic stochastic differential equation with less regular coefficients, under various Lyapunov conditions. In the case of non-degenerate noises, we prove two types of convergence of solutions to the unique periodic probability solution, namely, convergence in mean and exponential convergence. In the case of degenerate noises, we show the convergence of solutions in mean to the set of periodic probability solutions. New results on the uniqueness of periodic probability solutions and global probability solutions of the FPE are also obtained. As applications, we study the long-time behaviors of the FPEs associated to stochastic damping Hamiltonian systems and stochastic slow-fast systems, and of weak solutions of periodic stochastic differential equations with less regular coefficients.

Contents

2000 *Mathematics Subject Classification.* Primary 35Q84; Secondary 35J25, 37B25, 60J60.

Key words and phrases. Fokker-Planck equation, periodic probability solution, uniqueness, convergence.

The first author was partially supported by NSFC grant 11571344. The second author was supported by NSFC grant 11571344 and a scholarship from the China Scholarship Council. The third author was supported by a start-up grant from the University of Alberta, NSERC RGPIN-2018- 04371 and NSERC DGECR-2018-00353. The fourth author was partially supported by NSERC discovery grant 1257749, PIMS CRG grant, a faculty development grant from the University of Alberta, and a Scholarship from Jilin University.

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1. **Introduction**

Consider ordinary differential equations (ODEs) of the form

$$\dot{x} = V(x,t), \quad x \in \mathcal{U}, \tag{1.1}$$

where t is the time variable, $\dot{x}$ stands for the time derivative of $x = x(t)$, $\mathcal{U} \subset \mathbb{R}^d$ is an open connected domain and $V = (V^i) : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^d$ is a time T-periodic vector field, called the *drift field*, for some $T > 0$. The periodic time dependence in (1.1) is frequently used in applications, for instance in biology, ecology, physics, and engineering, to model time recurrence and seasonal variations in the vector field. As real world problems are often subject to noise perturbations from either surrounding environments or intrinsic uncertainties [20], more realistic models should often take the fluctuations or noises into consideration. This motivates us to consider noise perturbations to the ODE (1.1) that result in the following stochastic differential equation (SDE):

$$\mathrm{d}x = V(x,t)\mathrm{d}t + G(x,t)\mathrm{d}W_t, \quad x \in \mathcal{U}, \tag{1.2}$$

where $G : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^{d\times m}$ is a time T-periodic *noise coefficient matrix* with $m \geq d$, and $W = (W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process. The SDE (1.2) is naturally connected to the following Fokker-Planck equation (FPE):

$$\partial_t u = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad x \in \mathcal{U}, \tag{1.3}$$

where $A := (a^{ij}) = \frac{1}{2}GG^\top$ is the *diffusion matrix*, $\partial_i = \partial_{x_i}$, $\partial^2_{ij} = \partial^2_{x_i x_j}$ and the summation convention is used in the right hand side of (1.3). Not only does the FPE (1.3) govern the distributions of the solutions of (1.2), but also it has been directly used to model the evolution of the distributions for many stochastic processes [33].

Two fundamental problems concerning the long-time dynamics of the SDE (1.2) and the FPE (1.3) are the existence and uniqueness of *steady states*, and the convergence of their solutions to the steady states. These problems have been extensively studied when $V(x,t) = V(x)$ and $G(x,t) = G(x)$ are autonomous in both regular or less regular cases, in which steady states are often defined to be the *stationary measures*, or *stationary distributions*. We refer the reader to [9, 5, 6, 7, 18, 19] and references therein for the existence and uniqueness of stationary measures, and [25, 26, 32, 29, 3, 23, 9, 5, 6, 7] and references therein for the convergence of solutions of (1.2) and (1.3) to stationary measures. Many different approaches have been taken and developed to study these problems. For instance, ergodic properties of Markov processes and stochastic analytical techniques are adopted in [3, 25, 26, 32, 29], theories of Dirichlet forms and semigroups are used in [5, 6, 9], and PDE techniques are developed in [5, 6, 7, 9, 18, 19, 23]. We emphasize that (1.2) and (1.3) with less regular coefficients arise naturally in applications, for instance in modeling complex fluid flows [31], and their study gives rise to challenging mathematical problems.

When V and G are T-periodic in t and admit at least Lipschitz regularity in x, steady states of (1.2) and (1.3) are characterized by the periodic analogs of stationary measures, called *periodic solutions*, that appeared in literature under different names and definitions. The investigation of these fundamental problems for (1.2) and (1.3) with locally Lipschitz coefficients has attracted much attention especially in recent years. In [26], Khasminskii defined periodic solutions for the SDE

(1.2) in the sense of periodic Markov processes and proved the existence under periodic Lyapunov conditions. In [11], Chen-Han-Li-Yang studied the existence of classical periodic solutions of the FPE (1.3) assuming the existence of an uncommon Lyapunov function. The existence of periodic solutions of semi-linear SDEs has been established in [30, 21, 12, 10] and references therein. Zhao-Zheng [35] and Feng-Zhao-Zhou [15] studied the existence of the so-called random periodic solutions of (1.2) in the framework of random dynamical systems. As for the convergence, Feng-Zhao-Zhong investigated in [16] the ergodic property of (1.2) that generalizes the classical ergodic theory of (1.2) in the autonomous case.

For (1.2) and (1.3) with less regular coefficients, the authors of the present paper adopted PDE techniques in [22] to show the existence of periodic probability solutions (see Definition 1.1) of (1.3) under a Lyapunov condition. The uniqueness of periodic probability solutions and the convergence of solutions of (1.3) remained open.

The main purpose of the present paper is to investigate the uniqueness of periodic probability solutions of (1.3) as well as the convergence of the solutions of (1.2) and (1.3) when V and G are less regular. Our study of the convergence issue also gives an alternative approach for the existence of periodic probability solutions of (1.3) but under stronger conditions than those required in [22]. We recall from [22] the definition of periodic probability solutions of (1.3). Denote by

$$\mathcal{L} := \partial_t + a^{ij}\partial_{ij}^2 + V^i\partial_i$$

the parabolic operator associated to the dual equation of (1.3).

Definition 1.1 (Periodic probability solution)**.** A Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is called a *periodic probability solution* of (1.3) if there exists a family of Borel probability measures $(\mu_t)_{t\in\mathbb{R}}$ on $\mathcal{U}$ satisfying

$$\mu_t = \mu_{t+T}, \quad \forall t\in\mathbb{R},$$
$$a^{ij}, V^i \in L^1_{loc}(\mathcal{U}\times\mathbb{R}, \mathrm{d}\mu_t\mathrm{d}t), \quad \forall i,\, j\in\{1,\dots,d\}$$

and

$$\int_{\mathbb{R}}\int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_t\mathrm{d}t = 0, \quad \forall \phi\in C_0^{2,1}(\mathcal{U}\times\mathbb{R}),$$

such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$, writing $\mu = (\mu_t)_{t\in\mathbb{R}}$ in short.

To proceed, dissipative conditions in terms of Lyapunov type of functions are needed. For a nonnegative function $U\in C_T(\mathcal{U}\times\mathbb{R})$, we define for each $\rho>0$, the *ρ-sublevel set*

$$\Omega_\rho = \{(x,t)\in\mathcal{U}\times\mathbb{R} : U(x,t)<\rho\},$$

and its *t-sections*

$$\Omega_\rho^t = \{x\in\mathcal{U} : U(x,t)<\rho\}, \quad \forall t\in\mathbb{R}.$$

From now on, we begin to use some function spaces, which, except the usual ones, are collected in Table 1 at the end of this section. We define four Lyapunov type of functions as follows.

Definition 1.2. A function $U\in C_T(\mathcal{U}\times\mathbb{R})$ is called an *unbounded compact function* if $U\geq 0$ and there is a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open sets in $\mathcal{U}$ satisfying

$$\mathcal{U}_n\subset\mathcal{U}_{n+1}\subset\subset\mathcal{U}, \quad \forall n\in\mathbb{N} \quad \text{and} \quad \mathcal{U} = \bigcup_{n=1}^{\infty}\mathcal{U}_n$$

such that

$$\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U \to \infty \quad \text{as} \quad n \to \infty. \tag{1.4}$$

An unbounded compact function $U \in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ is called

(1) a *Lyapunov-like function* (with respect to $\mathcal{L}$) if there exist positive constants ρ_m, C_1 and C_2 such that

$$\mathcal{L}U \le C_1 U + C_2 \quad \text{in} \quad (\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}; \tag{1.5}$$

(2) a *Lyapunov function* (with respect to $\mathcal{L}$) if there exist positive constants ρ_m and γ such that

$$\mathcal{L}U \le -\gamma \quad \text{in} \quad (\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}; \tag{1.6}$$

(3) a *strong Lyapunov function* (with respect to $\mathcal{L}$) if

$$\lim_{n\to\infty}\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} \mathcal{L}U = -\infty; \tag{1.7}$$

(4) an *exponentially strong Lyapunov function* (with respect to $\mathcal{L}$) if there exist positive constants ρ_m, C_1 and C_2 such that

$$\mathcal{L}U \le -C_1 U + C_2 \quad \text{in} \quad (\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}. \tag{1.8}$$

As the definitions of these Lyapunov type of functions are based on *unbounded* compact functions, they are necessarily unbounded. The word "unbounded" often appears in front of these functions in the literature just to highlight the unboundedness of them. In this paper, we choose to suppress the word "unbounded" in front of these functions for the sake of simplicity. Whenever no confusion is caused, we also suppress the phrase "with respect to $\mathcal{L}$".

To study the uniqueness of periodic probability solutions of (1.3), we make the following assumption.

(H1) For fixed $p > d+2$, $a^{ij} \in L^\infty(\mathbb{R}; W^{1,p}_{loc}(\mathcal{U}))$ and $V^i \in L^p_{loc}(\mathcal{U}\times\mathbb{R})$ for each $i,j = 1,\dots,d$. The diffusion matrix $A = (a^{ij})$ is *locally uniformly positive definite*, that is, for every open set $\mathcal{V} \subset\subset \mathcal{U}$, there exist positive constants $\lambda_\mathcal{V}$ and $\Lambda_\mathcal{V}$ such that

$$\lambda_\mathcal{V}|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \Lambda_\mathcal{V}|\xi|^2, \quad \forall (x,t) \in \mathcal{V}\times\mathbb{R} \text{ and } \xi \in \mathbb{R}^d. \tag{1.9}$$

Our result on the uniqueness states as follows.

Theorem A (Uniqueness)**.** *Assume* **(H1)**. *If $\mathcal{L}$ admits a Lyapunov-like function, then there exists at most one periodic probability solution of* (1.3).

We remark that the existence of periodic probability solutions of (1.3) is established in [22, Theorem A] under **(H1)** and a Lyapunov function. This together with Theorem A gives the following corollary.

Corollary A. *Assume* **(H1)**. *If $\mathcal{L}$ admits a Lyapunov function, then there exists a unique periodic probability solution of* (1.3).

Let $\mathcal{M}_p(\mathcal{U})$ be the space of all Borel probability measures on $\mathcal{U}$. We recall the definition of global probability solutions, whose existence and uniqueness are investigated in Subsection 2.2.

Definition 1.3. Let $\mathcal{I} \subset \mathbb{R}$ be an open interval and $s \in \mathbb{R}$.

(1) A Borel measure μ on $\mathcal{U}\times\mathcal{I}$ is called a *measure solution* of (1.3) (in $\mathcal{U}\times\mathcal{I}$) if there exists a family of Borel measures $(\mu_t)_{t\in\mathcal{I}}$ on $\mathcal{U}$ satisfying

$$a^{ij}, V^i \in L^1_{loc}(\mathcal{U}\times\mathcal{I}, \mathrm{d}\mu_t\mathrm{d}t), \quad \forall i,j \in \{1,\dots,d\}$$

and

$$\iint_{\mathcal{U}\times\mathcal{I}} \mathcal{L}\phi \mathrm{d}\mu_t \mathrm{d}t = 0, \quad \forall \phi \in C_0^{2,1}(\mathcal{U}\times\mathcal{I}), \tag{1.10}$$

such that $\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t$. In this case, we write $\mu = (\mu_t)_{t\in\mathcal{I}}$.

If, in addition, $\mathcal{I} = (s,\infty)$ and $\mu_t(\mathcal{U}) \le 1$ (resp. $\mu_t(\mathcal{U}) = 1$) for a.e. $t \in \mathcal{I}$, then μ is called a *global sub-probability solution* (resp. *global probability solution*) of (1.3).

(2) Let $\mathcal{I} = (s, t_0)$ for some $t_0 \in (s,\infty]$. A measure solution $\mu = (\mu_t)_{t\in\mathcal{I}}$ of (1.3) is said to satisfy the initial condition

$$\mu_s = \nu \in \mathcal{M}_p(\mathcal{U}) \tag{1.11}$$

if for each $\phi \in C_c^\infty(\mathcal{U})$, there is a set $J_\phi \subset \mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi| = 0$ such that

$$\lim_{J_\phi \ni t\to s} \int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\nu. \tag{1.12}$$

In this case, $\mu = (\mu_t)_{t\in\mathcal{I}}$ is simply called a *measure solution* of the Cauchy problem (1.3)-(1.11).

If, in addition, $\mathcal{I} = (s,\infty)$ and $\mu = (\mu_t)_{t\in(s,\infty)}$ is a global sub-probability solution (resp. global probability solution) of (1.3), then μ is called a *global sub-probability solution* (resp. *global probability solution*) of the Cauchy problem (1.3)-(1.11).

We prove three results on the convergence of global probability solutions of the Cauchy problem (1.3)-(1.11) to periodic probability solutions. To state the first one, we make the following assumptions on A and V.

(H2) a^{ij}, $V^i \in C(\mathcal{U}\times\mathbb{R})$ for each $i, j = 1, \dots, d$.

Theorem B (Convergence in Mean)**.** *Assume* **(H2)** *and that $\mathcal{L}$ admits a strong Lyapunov function U. Let $\mu = (\mu_t)_{t\in(s,\infty)}$ be a global probability solution of the Cauchy problem* (1.3)-(1.11) *with $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$. Then for any sequence of positive integers $\{n_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} n_j = \infty$, there exists a subsequence, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, and a periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ of* (1.3) *such that*

(1) *for each bounded $\phi \in C_T(\mathcal{U}\times\mathbb{R})$, there holds*

$$\lim_{j\to\infty} \frac{1}{n_j T} \int_t^{t+n_j T} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau = \frac{1}{T}\int_0^T \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau, \quad \forall t \ge s, \tag{1.13}$$

(2) *for each $\psi \in C_c^2(\mathcal{U})$, there holds*

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t+kT} = \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t, \quad \textit{for a.e. } t > s. \tag{1.14}$$

In particular, if (1.3) *admits a unique periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$, then* (1.13) *and* (1.14) *hold for the whole sequence $\mathbb{N}$.*

Under the conditions of Theorem B, the diffusion matrix A is allowed to be degenerate in $\mathcal{U}$, in which case the FPE (1.3) can admit multiple periodic probability solutions. This is why the main part in the statement of Theorem B only asserts the average attractiveness of global probability solutions of the Cauchy problem (1.3)-(1.11) by the set of periodic probability solutions of (1.3). If we assume, in addition, that A is locally uniformly positive definite as in **(H1)**, then Theorem A guarantees the validity of the "In particular" part in the statement of Theorem B. The same results can be established under slightly weaker conditions on the coefficients. As the proof is almost the same, we state the results in the next corollary, which is our second result on the convergence.

Corollary B. *Assume* **(H1)** *and that $\mathcal{L}$ admits a strong Lyapunov function U. Then for any global probability solution $\mu=(\mu_t)_{t\in(s,\infty)}$ of the Cauchy problem (1.3)-(1.11) with $\int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu<\infty$, there holds for any $\psi\in C_b(\mathcal{U})$*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\mathcal{U}}\psi\mathrm{d}\mu_{t+kT}=\int_{\mathcal{U}}\psi\mathrm{d}\tilde{\mu}_t,\quad \forall t\in(s,s+T], \tag{1.15}$$

where $\tilde{\mu}=(\tilde{\mu}_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (1.3).

Compared to (1.14), the convergence (1.15) holds for a larger class of test functions. This is because the assumption **(H1)** (more precisely, the locally uniform positive definiteness of A in **(H1)**) ensures the existence of the continuous density of a global probability solution $\mu=(\mu_t)_{t\in(s,\infty)}$, which guarantees the continuity of the function $t\mapsto\int_{\mathcal{U}}\phi\mathrm{d}\mu_t$ on (s,∞) for each $\phi\in C_b(\mathcal{U})$, while under the conditions in the statement of Theorem B, the continuity of the function $t\mapsto\int_{\mathcal{U}}\phi\mathrm{d}\mu_t$ on (s,∞) is only obtained when $\phi\in C_c^2(\mathcal{U})$.

Conclusions in Theorem B and Corollary B can be regarded as weak forms of Birkhoff's ergodic theorem. Moreover, their proofs do not require the standard semi-flow property that plays essential roles in the proof of the classical ergodic theorem for measure-preserving dynamical systems and Markov processes. Indeed, under the assumption **(H1)** or **(H2)**, the uniqueness of solutions of the Cauchy problem (1.3)-(1.11) is unknown. Even if we assume the uniqueness, they are only known to generate a semi-flow under the weak*-topology. Such weak ergodic theorems without semi-flow property can potentially serve as theoretical foundations for the evolution of practical systems that are often too complicated to admit the standard semi-flow property or get it tested.

Our third convergence result concerns the exponential convergence of global probability solutions of the Cauchy problem (1.3)-(1.11) to periodic probability solutions under exponentially strong Lyapunov functions. This requires Lipschitz conditions on $A=(a^{ij})$ as follows.

- **(H3)** For each $i,j=1,\dots,d$, the entry a^{ij} is *locally Lipschitz* in x, that is, for each open set $\mathcal{V}\subset\subset\mathcal{U}$, there is a $L_{\mathcal{V}}>0$ such that

$$\left|a^{ij}(x_1,t)-a^{ij}(x_2,t)\right|\le L_{\mathcal{V}}|x_1-x_2|,\quad \forall x_1,x_2\in\overline{\mathcal{V}}\text{ and a.e. }t\in\mathbb{R}. \tag{1.16}$$

Theorem C (Exponential convergence)**.** *Assume* **(H1)** *and* **(H3)**. *Suppose $\mathcal{L}$ admits an exponentially strong Lyapunov function U. Then, there exist positive constants C_1 and C_2 such that for any global probability solution $\mu=(\mu_t)_{t\in(s,\infty)}$ of the Cauchy problem (1.3)-(1.11) with $\int_{\mathcal{U}}U(\cdot,s)\mathrm{d}\nu<\infty$, there holds*

$$\|\mu_t-\tilde{\mu}_t\|_{TV}\le C_1e^{-C_2(t-s)},\quad \forall t>s,$$

where $\tilde{\mu}=(\tilde{\mu}_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (1.3) and $\|\cdot\|_{TV}$ denotes the total variation norm.

An important piece in the proof of Theorem C is the construction of the transition probability densities $p(s,x,t,y)$ for $s<t$ and $x,y\in\mathcal{U}$ associated to the global probability solutions of the Cauchy problem (1.3)-(1.11) (see Subsection 5.1). This benefits from the assumption **(H3)**, which together with the assumption **(H1)**, ensures the existence, regularity and uniqueness of global probability solutions of the Cauchy problem (1.3)-(1.11) (see Theorem 2.3). Consequently, the transition probability

TABLE 1. Notations

$\mathcal{I} \subset \mathbb{R}$	An interval
$C_c(\mathcal{U})$	The space of compactly supported continuous functions on $\mathcal{U}$
$C_b(\mathcal{U})$	The space of bounded continuous functions on $\mathcal{U}$
$C_c^2(\mathcal{U})/C_c^\infty(\mathcal{U})$	$C_c(\mathcal{U}) \cap C^2(\mathcal{U})/C_c(\mathcal{U}) \cap C^\infty(\mathcal{U})$
$C_0(\mathcal{U} \times \mathcal{I})$	The space of compactly supported continuous functions on $\mathcal{U} \times \mathcal{I}$
$C_c(\mathcal{U} \times \mathcal{I})$	The space of continuous functions $u : \mathcal{U} \times \mathcal{I} \to \mathbb{R}$ such that $u(t, \cdot) \in C_c(\mathcal{U})$ for each $t \in \mathcal{I}$
$C_T(\mathcal{U} \times \mathbb{R})$	The space of T-periodic and continuous functions on $\mathcal{U} \times \mathbb{R}$
$C^{2,1}(\mathcal{U} \times \mathcal{I})$	The space of continuous functions that is twice continuously differentiable in x and continuously differentiable in t
$C_c^{2,1}(\mathcal{U} \times \mathcal{I})$	$C^{2,1}(\mathcal{U} \times \mathcal{I}) \cap C_c(\mathcal{U} \times \mathcal{I})$
$C_T^{2,1}(\mathcal{U} \times \mathbb{R})$	$C^{2,1}(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$
$C_0^{2,1}(\mathcal{U} \times \mathcal{I})$	The space of functions in $C^{2,1}(\mathcal{U} \times \mathcal{I})$ with compact support in $\mathcal{U} \times \mathcal{I}$
$L^\infty(\mathbb{R}; W_{loc}^{1,p}(\mathcal{U}))$	The space of measurable functions $u : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ such that $u(t, \cdot) \in W_{loc}^{1,p}(\mathcal{U})$ for a.e. $t \in \mathbb{R}$ and for each subdomain $\Omega \subset\subset \mathcal{U}$, the function $t \mapsto \|u(t, \cdot)\|_{W^{1,p}(\Omega)}$ is essentially bounded

densities can be defined and shown to satisfy expected properties resulting in the applicability of classical arguments leading to the exponential convergence.

It is worthwhile to point out that our approaches for the convergence are different from those in [23] for the autonomous case that are based on the sophisticated theory of *generalized Markov semigroups* associated to stationary measures developed in [9]. In fact, it is unclear whether there is an analogous theory of generalized Markov semigroups associated to periodic probability solutions. Any progress along this direction would be helpful for improving the convergence results in Theorem B and Corollary B.

In this paper, we also consider three applications of Theorem B, Corollary B and Theorem C as follows. (i) For a class of stochastic damping Hamiltonian systems, strong Lyapunov functions are constructed to ensure the convergence of global probability solutions of the associated FPEs as stated in Theorem B. (ii) For a class of stochastic slow-fast systems with very strong dissipative properties along the fast directions and non-degenerate noises only along the slow directions, we show the existence and uniqueness of periodic probability solutions as well as the convergence of global probability solutions of the associated FPEs under Lyapunv conditions along the slow directions. (iii) For a SDE with less regular coefficients, we show that the distributions of their globally defined weak solutions are global probability solutions of the associated FPE, and hence, under appropriate Lyapunov conditions, the convergence of globally defined weak solutions are established as simple consequences of our convergence results. The details of these applications are given in Section 6.

The rest of the paper is organized as follows. In Section 2, we recall some basic facts including in particular equivalent formalisms of global probability solutions of the Cauchy problem (1.3)-(1.11) and the regularity theory of measure solutions of (1.3), and prove the global well-posedness of the Cauchy problem (1.3)-(1.11). In Section 3, we study the uniqueness of periodic probability solutions of (1.3) with non-degenerate noises and prove Theorem A. We study the convergence of global probability solutions of the Cauchy problem (1.3)-(1.11) in Section 4 and Section 5. In particular, Theorem B and

Theorem C are respectively proven in Section 4 and Section 5. The proof of Corollary B is sketched out at the end of Subsection 4.2. Some applications of our convergence results, namely, Theorem B, Corollary B and Theorem C, are presented in Section 6. In Appendix A, the proof of a technical inequality is given.

2. Preliminaries

In Subsection 2.1, we present some equivalent formalisms of measure solutions, given in Definition 1.3, of (1.3) or the Cauchy problem (1.3)-(1.11), and recall the regularity theory. In Subsection 2.2, we present some results on the global well-posedness of the Cauchy problem (1.3)-(1.11).

2.1. Measure solutions and regularity.

Arguing as in [6, Proposition 6.1.2] and [8, Lemma 1.1], the following equivalent formalisms hold for (1.10) or (1.10)-(1.12) in $\mathcal{U}\times\mathcal{I}$, where $\mathcal{I}\subset\mathbb{R}$ is an open interval.

Lemma 2.1. *Let $(\mu_t)_{t\in\mathcal{I}}$ be a family of Borel measures such that a^{ij}, $V^i\in L^1_{loc}(\mathcal{U}\times\mathcal{I},\mathrm{d}\mu_t\mathrm{d}t)$ for all $i,j=1,\dots,d$.*

(1) *The following statements are equivalent to (1.10).*

(a) *For each $\phi\in C^2_c(\mathcal{U})$, there is a set $J_\phi\subset\mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi|=0$ such that*

$$\int_{\mathcal{U}}\phi\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi\mathrm{d}\mu_r+\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad\forall r,\,t\in J_\phi \text{ with } r<t.$$

(b) *For each $\phi\in C^{2,1}_c(\mathcal{U}\times\mathcal{I})$, there is a set $J_\phi\subset\mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi|=0$ such that*

$$\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi(\cdot,r)\mathrm{d}\mu_r+\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad\forall r,\,t\in J_\phi \text{ with } r<t.$$

(2) *Let $\mathcal{I}=(s,t_0)$ for some $-\infty<s<t_0\le\infty$ The following statements are equivalent to (1.10)-(1.12).*

(a) *For each $\phi\in C^2_c(\mathcal{U})$, there is a set $J_\phi\subset\mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi|=0$ such that*

$$\int_{\mathcal{U}}\phi\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi\mathrm{d}\nu+\lim_{J_\phi\ni r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad\forall t\in J_\phi,\tag{2.1}$$

(b) *For each $\phi\in C^{2,1}_c(\mathcal{U}\times[s,t_0))$, there is a set $J_\phi\subset\mathcal{I}$ satisfying $|\mathcal{I}\setminus J_\phi|=0$ such that*

$$\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi(\cdot,s)\mathrm{d}\nu+\lim_{J_\phi\ni r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad\forall t\in J_\phi.$$

Lemma 2.2. *Let $(\mu_t)_{t\in\mathcal{I}}$ be as in Lemma 2.1. If the function $t\mapsto\int_{\mathcal{U}}\phi\mathrm{d}\mu_t$ is continuous on $\mathcal{I}$ for any $\phi\in C^2_c(\mathcal{U})$, then J_ϕ can be taken to be $\mathcal{I}$ in each case of Lemma 2.1.*

Proof. We only show the case in Corollary (2)(a); the other cases can be proven in the same manner.

For fixed $\phi\in C^2_c(\mathcal{U})$, it is clear that the function $(r,t)\mapsto\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ is continuous on $\{(r,t)\in\mathcal{I}^2:r<t\}$. Fix $t_*\in\mathcal{I}\setminus J_\phi$. There is a sequence $\{t_n\}_{n\in\mathbb{N}}\subset J_\phi$ such that $t_n\to t_*$ as $n\to\infty$. Setting $t=t_n$ in (2.1) and letting $n\to\infty$, we see that (2.1) holds for $t=t_*$.

It remains to show for each fixed $t\in\mathcal{I}$, there holds the limit

$$\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau=\int_{\mathcal{U}}\phi\mathrm{d}\mu_t-\int_{\mathcal{U}}\phi\mathrm{d}\nu=:A_t.$$

Clearly, the above limit is the case if r takes values in J_ϕ. If the above limit is not the case, then there exists an $\epsilon_0>0$ and a sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}\setminus J_\phi$ satisfying $r_n\to s$ as $n\to\infty$ such that

$$\left|\int_{r_n}^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau-A_t\right|>\epsilon_0,\quad\forall n\gg1.$$

By the continuity of $r\mapsto\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ and the density of J_ϕ in $\mathcal{I}$, we find a sequence $\{\tilde{r}_n\}_{n\in\mathbb{N}}\subset J_\phi$ satisfying $\tilde{r}_n\to s$ as $n\to\infty$ such that

$$\left|\int_{\tilde{r}_n}^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau-A_t\right|>\frac{\epsilon_0}{2},\quad\forall n\gg1,$$

which leads to a contradiction. This proves (2)(a). □

Now, we recall the regularity theory of measure solutions of (1.3) in $\mathcal{U}\times\mathcal{I}$. Recall $p>d+2$. Let $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$ be the space of measurable functions u on $\mathcal{U}\times\mathcal{I}$ such that $u(\cdot,t)\in W_0^{1,p}(\mathcal{U})$ for a.e. $t\in\mathcal{I}$ and the function $t\mapsto\|u(t,\cdot)\|_{W_0^{1,p}(\mathcal{U})}$ lies in $L^p(\mathcal{I})$. Let $\mathbb{H}^{-1,p'}(\mathcal{U}\times\mathcal{I})$ be the dual space of $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$, where $p'>1$ is such that $\frac{1}{p}+\frac{1}{p'}=1$.

Let $\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$ be the space of measurable functions u on $\mathcal{U}\times\mathcal{I}$ such that $\eta u\in\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathcal{I})$ and $\partial_t(\eta u)\in\mathbb{H}^{-1,p}(\mathcal{U}\times\mathcal{I})$ for each $\eta\in C_0^\infty(\mathbb{R}^{d+1})$. By [6, Theorem 6.2.2], there exist $\alpha>\frac{1}{p}$ and $\gamma>0$, depending only on d and p, such that $\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$ is continuously embedded into $C^{\alpha-\frac{1}{p}}(\mathcal{I},C^\gamma(\mathcal{U}))$. Here, $C^\alpha(\mathcal{I},C^\gamma(\mathcal{U}))$ denotes the space of all continuous functions $u:\mathcal{U}\times\mathcal{I}\to\mathbb{R}$ such that $u(t,\cdot)\in C^\gamma(\mathcal{U})$ for all $t\in\mathcal{I}$ and for each subdomain $\Omega\subset\subset\mathcal{U}$, the function $t\mapsto|u(t,\cdot)|_{C^\gamma(\overline{\Omega})}$ lies in $C^\alpha(\mathcal{I})$.

Theorem 2.1 ([4, 6]). *Assume* **(H1)**. *Let $\mu=(\mu_t)_{t\in\mathcal{I}}$ be a measure solution of* (1.3). *Then, μ admits a positive density $u\in\mathcal{H}_{loc}^{1,p}(\mathcal{U}\times\mathcal{I})$. Moreover, for closed intervals $[s_1,t_1]\subset\subset[s_2,t_2]\subset\mathcal{I}$ and open subsets $\mathcal{W}\subset\subset\mathcal{W}_1\subset\subset\mathcal{U}$, there exist $\alpha>\frac{1}{p}$, $\gamma>0$ and $N>0$, independent of μ or u, such that*

$$\|u\|_{C^{\alpha-\frac{1}{p}}((s_1,t_1),C^\gamma(\mathcal{W}))}\le N\int_{s_2}^{t_2}\mu_\tau(\mathcal{W}_1)\mathrm{d}\tau.\tag{2.2}$$

2.2. **Global well-posedness.**

The following result on the existence of global probability solutions of the Cauchy problem (1.3)-(1.11) is taken from [28].

Theorem 2.2 ([28]). *Assume* **(H1)** *or* **(H2)**. *Suppose $\mathcal{L}$ admits a Lyapunov-like function. Then, the Cauchy problem* (1.3)-(1.11) *admits a global probability solution $\mu=(\mu_t)_{t\in(s,\infty)}$. Moreover, under* **(H1)**, *μ admits a density in $C^\alpha((s,\infty),C^\gamma(\mathcal{U}))$ for some $\alpha>\frac{1}{p}$ and $\gamma>0$.*

We prove a uniqueness result.

Theorem 2.3. *Assume* **(H1)** *and* **(H3)**. *Suppose $\mathcal{L}$ admit a Lyapunov-like function. Then, the Cauchy problem* (1.3)-(1.11) *admits a unique (in the class of global sub-probability solutions) global probability solution.*

Proof. Let $\mu^1=(\mu_t^1)_{t\in(s,\infty)}$ and $\mu^2=(\mu_t^2)_{t\in(s,\infty)}$ be respectively a global probability solution and a global sub-probability solution of the Cauchy problem (1.3)-(1.11). Applying Theorem 2.1, we may assume that for each $i=1,2$, μ^i admits a positive density $\rho_i\in C(\mathcal{U}\times(s,\infty))$. We show that $\rho_1=\rho_2$. Setting $w:=\frac{\rho_2}{\rho_1}$, it is equivalent to prove $w\equiv1$ on $\mathcal{U}\times(s,\infty)$.

Define $f_\lambda(t) := e^{\lambda(1-t)} - e^\lambda$ for $t \geq 0$, where $\lambda > 0$ is a parameter. Following the main ideas of [34, Lemma 2.2], we deduce that for any non-negative function $\phi \in C_c^{2,1}(\mathcal{U} \times \mathbb{R})$ there holds

$$\int_{\mathcal{U}} f_\lambda(w)\phi \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \phi \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\mathcal{L}\phi \mathrm{d}\mu_\tau^1 \mathrm{d}\tau, \quad \forall t > s. \tag{2.3}$$

As the proof of (2.3) is relatively independent and long, we include it in Appendix A for the sake of readability and completeness.

Let U be the Lyapunov-like function as in (1.5). Fix $\rho_0 > \rho_m$. We introduce a smooth and non-decreasing function θ satisfying

$$\theta(t) = \begin{cases} 0, & t \in [0, \rho_m], \\ t, & t \in [\rho_0, \infty). \end{cases} \tag{2.4}$$

By the definition of θ, it is easy to find a $\bar{C} > 0$ such that $t\theta'(t) \leq \bar{C}\theta(t)$ for $t \geq 0$. Since $\theta'' \not\equiv 0$ on $[\rho_m, \rho_0]$ and $\theta'' = 0$ otherwise, we find from (1.5) and (1.9) that there exist $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\begin{aligned} \mathcal{L}\theta(U) &= \theta'(U)\mathcal{L}U + \theta''(U)a^{ij}\partial_i U \partial_j U \\ &\leq \theta'(U)(C_1 U + C_2) + \Lambda_{\Omega_{\rho_0}} |\theta''|_\infty \max_{\Omega_{\rho_0}} |\nabla U|^2 \\ &\leq \tilde{C}_1 \theta(U) + \tilde{C}_2 \quad \text{in} \quad \mathcal{U} \times \mathbb{R}. \end{aligned}$$

Hence, we find a new Lyapunov-like function $\tilde{U} := \theta(U)$ whose Lyapunov condition holds on the whole space. This allows us to proceed as in [28, Theorem 3.5].

Let $\zeta \in C_c^\infty([0,\infty))$ satisfy

$$\zeta(0) = 1, \quad \zeta = 0 \text{ on } [1, \infty), \quad \zeta' \leq 0 \quad \text{and} \quad \zeta'' \geq 0.$$

It is clear that $\zeta(\frac{\tilde{U}}{N}) \in C_c^{2,1}(\mathcal{U} \times (s,\infty))$ for $N \gg 1$. Setting $\phi = \zeta(\frac{\tilde{U}}{N})$ in (2.3), we arrive at

$$\begin{aligned} \int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 &\leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\mathcal{L}\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \\ &= f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N}\zeta'\left(\frac{\tilde{U}}{N}\right) \mathcal{L}\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau \\ &\quad + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N^2}\zeta''\left(\frac{\tilde{U}}{N}\right) a^{ij}\partial_i\tilde{U}\partial_j\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau, \quad \forall t > s. \end{aligned} \tag{2.5}$$

Since $f_\lambda \leq 0$, $\zeta'' \geq 0$ and (a^{ij}) is positive definite, the last term in (2.5) is non-positive. Thus,

$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \int_s^t \int_{\mathcal{U}} f_\lambda(w)\frac{1}{N}\zeta'\left(\frac{\tilde{U}}{N}\right) \mathcal{L}\tilde{U} \mathrm{d}\mu_\tau^1 \mathrm{d}\tau.$$

Since $\mathcal{L}\tilde{U} \leq \tilde{C}_1\tilde{U} + \tilde{C}_2$, $\zeta' \leq 0$ and $f_\lambda \leq 0$, we find

$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \frac{1}{N}\int_s^t \int_{\mathcal{U}} f_\lambda(w)\zeta'\left(\frac{\tilde{U}}{N}\right) \left[\tilde{C}_1\tilde{U} + \tilde{C}_2\right] \mathrm{d}\mu_\tau^1 \mathrm{d}\tau.$$

It follows from $\zeta'(t) = 0$ for $t \geq 1$ and $|f_\lambda| \leq e^\lambda$ that

$$\int_{\mathcal{U}} f_\lambda(w)\zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}} \zeta\left(\frac{\tilde{U}}{N}\right) \mathrm{d}\nu + \frac{C_3}{N}\int_s^t \int_{\{0 \leq \tilde{U} \leq N\}} \left[\tilde{C}_1\tilde{U} + \tilde{C}_2\right] \mathrm{d}\mu_\tau^1 \mathrm{d}\tau, \tag{2.6}$$

where $C_3 = e^{\lambda}|\zeta'|_\infty$.

Applying the dominated convergence theorem, we find

$$\lim_{N\to+\infty}\frac{1}{N}\int_s^t\int_{\{0\le\tilde U\le N\}}\left[\tilde{C_1}\tilde U+\tilde{C_2}\right]\mathrm{d}\mu^1_\tau\mathrm{d}\tau=0.$$

Since $\lim_{N\to\infty}\zeta(\frac{t}{N})=1$ for each $t\in[0,\infty)$, we pass to the limit $N\to\infty$ in (2.6) to find from the dominated convergence theorem that $\int_{\mathcal U} f_\lambda(w)\mathrm{d}\mu^1_t\le f_\lambda(1)$ for all $t>s$, namely,

$$\int_{\mathcal U}\left[e^{\lambda(1-w)}-e^{\lambda}\right]\mathrm{d}\mu^1_t\le(1-e^{\lambda}),\quad \forall t>s.$$

Since μ^1 is a global probability solution of the Cauchy problem (1.3)-(1.11) so that $\mu^1_t(\mathcal U)=1$ for all $t>s$, we deduce

$$\int_{\mathcal U} e^{\lambda(1-w)}\mathrm{d}\mu^1_t\le 1,\quad \forall t>s. \tag{2.7}$$

For fixed $t\in\mathbb R$, if there exists an $a\in(0,1)$ such that

$$\mu^1_t\left(\{x\in\mathcal U: 0<w(x,t)<a\}\right)>0,$$

then

$$\int_{\{x\in\mathcal U:0<w(x,t)<a\}} e^{\lambda(1-a)}\mathrm{d}\mu^1_t\le\int_{\mathcal U} e^{\lambda(1-w)}\mathrm{d}\mu^1_t\le 1.$$

Letting $\lambda\to\infty$, the left hand side in the above inequality approaches ∞, giving rise to a contradiction. Thus $w(x,t)\ge 1$ for μ^1_t-a.e. $x\in\mathcal U$. As μ^1 has a pointwise positive density ρ_1 and w on continuous in $\mathcal U\times(s,\infty)$, then $w(x,t)\ge1$ for all $(x,t)\in\mathcal U\times(s,\infty)$. If $w\not\equiv 1$, we integrate the equality $w\rho_1=\rho_2$ to find

$$T<\int_0^T\int_{\mathcal U} w(x,t)\rho_1(x,t)dxdt=\int_0^T\int_{\mathcal U}\rho_2(x,t)dxdt\le T,$$

which leads to a contradiction. □

3. Proof of Theorem A

Throughout this section, we assume **(H1)**. Let μ^1 and μ^2 be two periodic probability solutions of (1.3). By Theorem 2.1, for each $i=1,2$, μ^i admits a positive and T-periodic density $\rho_i\in\mathcal H^{1,p}_{loc}(\mathcal U\times\mathbb R)\cap C^{\alpha-\frac1p}(\mathbb R, C^{\gamma}(\mathcal U))$ for some $\alpha>\frac1p$ and $\gamma>0$. To show $\mu_1=\mu_2$, it suffices to prove that $w:=\frac{\rho_2}{\rho_1}\equiv1$.

Define $f(t):=e^{1-t}-e$ for $t\in[0,\infty)$ and

$$\eta(x)=\begin{cases}c_d e^{-\frac{1}{1-|x|^2}}, & \text{if}\quad |x|\le1,\\ 0, & \text{if}\quad |x|>1,\end{cases}$$

where $c_d>0$ is such that $\int_{\mathbb R^d}\eta dx=1$. It is well-known that $\eta\in C^\infty_c(\mathbb R^d)$. Let $\eta_\epsilon(x):=\frac{1}{\epsilon^d}\eta(\frac{x}{\epsilon})$ for $x\in\mathbb R^d$ and $0<\epsilon\ll1$. For a T-periodic measurable function g on $\mathcal U\times\mathbb R$, we define

$$g_\epsilon(x,t):=\int_{\{y\in\mathbb R^d:x-y\in\mathcal U\}} g(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad (x,t)\in\mathcal U\times\mathbb R. \tag{3.1}$$

In particular, for each $i=1,2$,

$$\rho_{i,\epsilon}(x,t)=\int_{\{y\in\mathbb R^d:x-y\in\mathcal U\}}\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad (x,t)\in\mathcal U\times\mathbb R. \tag{3.2}$$

Set

$$w_\epsilon := \frac{\rho_{2,\epsilon}}{\rho_{1,\epsilon}}, \qquad W_\epsilon := (V\rho_2)_\epsilon - (V\rho_1)_\epsilon w_\epsilon, \quad \text{and}$$
$$R^k_{\rho_i,\epsilon} := \partial_l(a^{kl}\rho_i)_\epsilon - \partial_l(a^{kl}\rho_{i,\epsilon}), \qquad \forall i=1,2 \text{ and } k=1,\dots,d.$$

Lemma 3.1. *Both W_ϵ and $R^k_{\rho_i,\epsilon}$ (for each $i=1,2$ and $k=1,\dots,d$) converge to 0 in $L^p_{loc}(\mathcal{U}\times\mathbb{R})$ as $\epsilon\to 0$.*

Proof. Let $\mathcal{K}\subset\subset\mathcal{U}$ and $t>s$. We see from the formula (3.1) that there is an $\epsilon_{\mathcal{K}}>0$ such that for each T-periodic measurable function g on $\mathcal{U}\times\mathbb{R}$, there holds

$$g_\epsilon(x,t)=\int_{\mathbb{R}^d} g(x-y,t)\eta_\epsilon(y)\mathrm{d}y, \quad (x,t)\in\mathcal{K}\times\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_{\mathcal{K}}).$$

Then, it follows from definitions of W_ϵ and w that for each $\epsilon\in(0,\epsilon_{\mathcal{K}})$, there holds

$$\begin{aligned} W_\epsilon(x,t) &= \int_{\mathbb{R}^d} V(x-y,t)\rho_2(x-y,t)\eta_\epsilon(y)\mathrm{d}y - \int_{\mathbb{R}^d} V(x-y,t)\rho_1(x-y,t)\eta_\epsilon(y)\mathrm{d}y w_\epsilon(x,t)\\ &= \int_{\mathbb{R}^d} V(x-y,t)\rho_1(x-y,t)\eta_\epsilon(y)\left[w(x-y,t)-w_\epsilon(x,t)\right]\mathrm{d}y, \quad \forall (x,t)\in\mathcal{K}\times\mathbb{R}. \end{aligned}$$

Since $w\in C_T(\mathcal{U}\times\mathbb{R})$, we see that for any $0<\delta\ll 1$, there is an $\epsilon_0=\epsilon_0(\delta)\in(0,\epsilon_{\mathcal{K}})$ such that $\mathcal{K}_{\epsilon_0}:=\{x\in\mathcal{U}:\mathrm{dist}(x,\mathcal{K})<\epsilon_0\}\subset\subset\mathcal{U}$,

$$\sup_{|y|\le\epsilon_0}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}|w(x-y,t)-w(x,t)|<\frac{\delta}{2}, \quad \text{and}$$
$$\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}|w(x,t)-w_\epsilon(x,t)|<\frac{\delta}{2}, \quad \forall\epsilon\in(0,\epsilon_0).$$

It follows that

$$\sup_{|y|\le\epsilon_0}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}|w(x-y,t)-w_\epsilon(x,t)|<\delta, \quad \forall\epsilon\in(0,\epsilon_0).$$

This together with Hölder's inequality yields

$$\begin{aligned} \int_s^t\int_{\mathcal{K}}|W_\epsilon|^p\mathrm{d}x\mathrm{d}\tau &\le \delta^p\int_s^t\int_{\mathcal{K}}\left|\int_{\mathbb{R}^d}V(x-y,\tau)\rho_1(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right|^p\mathrm{d}x\mathrm{d}\tau\\ &\le \delta^p\int_s^t\int_{\mathcal{K}}\left[\left(\int_{\mathbb{R}^d}|V(x-y,\tau)|^p\rho_1^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right)\left(\int_{\mathbb{R}^d}1^{p'}\eta_\epsilon(y)\mathrm{d}y\right)^{\frac{p}{p'}}\right]\mathrm{d}x\mathrm{d}\tau\\ &= \delta^p\int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d}|V(x-y,\tau)|^p\rho_1^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau, \quad \forall\epsilon\in(0,\epsilon_0). \end{aligned}$$

A simple change of variable gives

$$\begin{aligned} \int_s^t\int_{\mathcal{K}}|W_\epsilon|^p\mathrm{d}x\mathrm{d}\tau &\le \delta^p\int_s^t\int_{\mathcal{K}}\int_{\mathcal{K}_{\epsilon_0}}|V(z,\tau)|^p\rho_1^p(z,\tau)\eta_\epsilon(x-z)\mathrm{d}z\mathrm{d}x\mathrm{d}\tau\\ &\le \delta^p\int_s^t\int_{\mathcal{K}_{\epsilon_0}}|V(z,\tau)|^p\rho_1^p(z,\tau)\left(\int_{\mathbb{R}^d}\eta_\epsilon(x-z)\mathrm{d}x\right)\mathrm{d}z\mathrm{d}\tau\\ &\le \delta^p\left(\sup_{\mathcal{K}_{\epsilon_0}\times\mathbb{R}}\rho_1^p\right)\int_s^t\int_{\mathcal{K}_{\epsilon_0}}|V|^p\mathrm{d}z\mathrm{d}\tau, \quad \forall\epsilon\in(0,\epsilon_0), \end{aligned}$$

where we used Fubini's theorem in the second inequality. Hence, $\lim_{\epsilon\to 0}\|W_\epsilon\|_{L^p(\mathcal{K}\times[s,t])}=0$.

Now, we deal with $R^k_{\rho_i,\epsilon}$. Note that for $\epsilon\in(0,\epsilon_{\mathcal{K}})$,

$$\begin{aligned}R^k_{\rho_i,\epsilon}(x,t)&=\partial_l\int_{\mathbb{R}^d}a^{kl}(x-y,t)\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y-\partial_l\int_{\mathbb{R}^d}a^{kl}(x,t)\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y\\&=\int_{\mathbb{R}^d}\left[\partial_l a^{kl}(x-y,t)-\partial_l a^{kl}(x,t)\right]\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y\\&\quad-\int_{\mathbb{R}^d}\left(a^{kl}(x-y,t)-a^{kl}(x,t)\right)\partial_l\rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y,\quad (x,t)\in\mathcal{K}\times\mathbb{R}.\end{aligned}$$

Since $a^{kl}\in L^\infty(\mathbb{R},W^{1,p}_{loc}(\mathcal{U}))$ is T-periodic for each $k,l=1,\dots,d$, we find

$$\sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\to 0\quad\text{as}\quad\epsilon\to0,\qquad\text{and}\tag{3.3}$$

$$\sup_{|y|\le\epsilon}\sup_{(x,t)\in\mathcal{K}\times\mathbb{R}}\left|a^{kl}(x-y,t)-a^{kl}(x,t)\right|\le\epsilon^{1-\frac{d}{p}}\operatorname*{ess\,sup}_{t\in\mathbb{R}}\|a^{kl}(\cdot,t)\|_{W^{1,p}(\mathcal{K}_\epsilon\times\mathbb{R})}\to0\quad\text{as}\quad\epsilon\to0,\tag{3.4}$$

where we used the Sobolev embedding theorem.

Applying Hölder's inequality and (3.3), we find

$$\begin{aligned}&\int_s^t\int_{\mathcal{K}}\left|\int_{\mathbb{R}^d}\left(\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right)\rho_i(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right|^p\mathrm{d}x\mathrm{d}\tau\\&\le\int_s^t\int_{\mathcal{K}}\left[\left(\int_{\mathbb{R}^d}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\rho_i^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right)\left(\int_{\mathbb{R}^d}1^{p'}\eta_\epsilon(y)\mathrm{d}y\right)^{\frac{p}{p'}}\right]\mathrm{d}x\mathrm{d}\tau\\&=\int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\rho_i^p(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\\&\le\left(\sup_{\mathcal{K}_\epsilon\times\mathbb{R}}\rho_i^p\right)\times\sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\times\int_{\mathbb{R}^d}\eta_\epsilon(y)\mathrm{d}y\\&=\left(\sup_{\mathcal{K}_\epsilon\times\mathbb{R}}\rho_i^p\right)\times\sup_{|y|\le\epsilon}\int_s^t\int_{\mathcal{K}}\left|\partial_l a^{kl}(x-y,\tau)-\partial_l a^{kl}(x,\tau)\right|^p\mathrm{d}x\mathrm{d}\tau\to0\quad\text{as}\quad\epsilon\to0.\end{aligned}\tag{3.5}$$

Applying Hölder's inequality, a change of variable and (3.4), we find

$$\begin{aligned}&\int_s^t\int_{\mathcal{K}}\left|\int_{\mathbb{R}^d}\left(a^{kl}(x-y,\tau)-a^{kl}(x,\tau)\right)\partial_l\rho_i(x-y,\tau)\eta_\epsilon(y)\mathrm{d}y\right|^p\mathrm{d}x\mathrm{d}\tau\\&\le\int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d}\left|a^{kl}(x-y,\tau)-a^{kl}(x,\tau)\right|^p|\partial_l\rho_i(x-y,\tau)|^p\eta_\epsilon(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\\&\le\sup_{|y|\le\epsilon}\sup_{(x,\tau)\in\mathcal{K}\times\mathbb{R}}\left|a^{kl}(x-y,\tau)-a^{kl}(x,\tau)\right|^p\times\int_s^t\int_{\mathcal{K}}\int_{\mathbb{R}^d}|\partial_l\rho_i(x-y,t)|^p\eta_\epsilon(y)\mathrm{d}x\mathrm{d}\tau\\&\le\sup_{|y|\le\epsilon}\sup_{(x,\tau)\in\mathcal{K}\times\mathbb{R}}\left|a^{kl}(x-y,\tau)-a^{kl}(x,\tau)\right|^p\times\int_s^t\int_{\mathcal{K}_\epsilon}|\partial_l\rho_i(z,t)|^p\left(\int_{\mathbb{R}^d}\eta_\epsilon(x-z)\mathrm{d}x\right)\mathrm{d}z\mathrm{d}\tau\\&\le\epsilon^{p-d}\operatorname*{ess\,sup}_{t\in\mathbb{R}}\|a^{kl}(\cdot,t)\|^p_{W^{1,p}(\mathcal{K}_\epsilon\times\mathbb{R})}\times\int_s^t\int_{\mathcal{K}_\epsilon}|\partial_l\rho_i|^p\mathrm{d}x\mathrm{d}\tau\\&\to0\quad\text{as}\quad\epsilon\to0,\end{aligned}\tag{3.6}$$

where the L^p-integrability of $\partial_l\rho_i$ (for each $l=1,\dots,d$ and $i=1,2$) on $\mathcal{K}_\epsilon\times[s,t]$ follows from $\rho_i\in\mathcal{H}^{1,p}_{loc}(\mathcal{U}\times\mathbb{R})$ for $i=1,2$.

It follows from (3.5) and (3.6) that $\lim_{\epsilon\to 0}\|R^k_{\rho_i,\epsilon}\|_{L^p(\mathcal{K}\times[s,t])}=0$. This completes the proof. □

Next, we show an inequality for w. Set

$$C^{2,1}_{c,T}(\mathcal{U}\times\mathbb{R}):=C^{2,1}_c(\mathcal{U}\times\mathbb{R})\cap C^{2,1}_T(\mathcal{U}\times\mathbb{R}).$$

Lemma 3.2. *There is a $C>0$ such that for each non-negative function $\phi\in C^{2,1}_{c,T}(\mathcal{U}\times\mathbb{R})$, there holds*

$$\int_t^{t+T}\int_{\mathcal{U}}\phi f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu^1_s\mathrm{d}s\le C\int_t^{t+T}\int_{\mathcal{U}}f(w)\mathcal{L}\phi\mathrm{d}\mu^1_s\mathrm{d}s,\quad\forall t\in\mathbb{R}.$$

Proof. As the proof follows from similar arguments leading to (2.3), we only point out the differences. Fix a non-negative function $\phi\in C^{2,1}_{c,T}(\mathcal{U}\times\mathbb{R})$. We see that the inequality (A.7) holds with t_1, t_2, ψ and f_λ replaced by t, $t+T$, ϕ and f, respectively. That is,

$$\begin{aligned}\int_t^{t+T}\int_{\mathcal{U}}\partial_t(f(w_\epsilon)\rho_{1,\epsilon})\phi\mathrm{d}x\mathrm{d}\tau&+\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &\le\int_t^{t+T}\int_{\mathcal{U}}[f(w_\epsilon)\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi+f(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\phi]\mathrm{d}x\mathrm{d}\tau\\ &\quad+3\delta\int_t^{t+T}\int_{\mathcal{U}}\phi f''(w_\epsilon)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\delta),\quad\forall t\in\mathbb{R},\end{aligned}\tag{3.7}$$

where $\delta>0$ is to be determined and

$$\begin{aligned}\Omega(\epsilon,\delta)=&\lambda e^\lambda\sup_{\mathcal{U}\times[t,t+T]}|\nabla\phi|\int_t^{t+T}\int_{\mathrm{supp}(\phi(\cdot,\tau))}|W_\epsilon|e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &+\frac{\lambda^2}{4\delta}e^\lambda\sup_{\mathcal{U}\times[t,t+T]}|\phi|\int_t^{t+T}\int_{\mathrm{supp}(\phi(\cdot,\tau))}\frac{|W_\epsilon|^2}{\rho_{1,\epsilon}}e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &+\lambda e^\lambda\sup_{\mathcal{U}\times[t,t+T]}|\nabla\phi|\int_t^{t+T}\int_{\mathrm{supp}(\phi(\cdot,\tau))}[|R_{\rho_2,\epsilon}|+|R_{\rho_1,\epsilon}|(1+\lambda w_\epsilon)]\,e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &+\frac{\lambda^2}{4\delta}e^\lambda\sup_{\mathcal{U}\times[t,t+T]}|\phi|\int_t^{t+T}\int_{\mathrm{supp}(\phi(\cdot,\tau))}\frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}}e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &+\frac{\lambda^2}{4\delta}e^\lambda\sup_{\mathcal{U}\times[t,t+T]}|\phi|\int_t^{t+T}\int_{\mathrm{supp}(\phi(\cdot,\tau))}\frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}}|w_\epsilon|^2e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

It follows from Lemma 3.1 that $\lim_{\epsilon\to 0}\Omega(\epsilon,\delta)=0$ for any $\delta>0$.

From the T-periodicity of $w,\rho_{1,\epsilon}$ and w_ϵ, and the Newton-Leibniz formula, we obtain

$$\int_t^{t+T}\int_{\mathbb{R}^d}\partial_t(f(w_\epsilon)\rho_{1,\epsilon})\phi\mathrm{d}x\mathrm{d}\tau=-\int_t^{t+T}\int_{\mathbb{R}^d}f(w_\epsilon)\rho_{1,\epsilon}\partial_t\phi\mathrm{d}x\mathrm{d}\tau.$$

It follows from (3.7) that

$$\begin{aligned}&\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &\quad\le\int_t^{t+T}\int_{\mathcal{U}}[f(w_\epsilon)\rho_{1,\epsilon}(\partial_t\phi+a^{kl}\partial_{kl}\phi)+f(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\phi]\mathrm{d}x\mathrm{d}\tau\\ &\quad\quad+3\delta\int_t^{t+T}\int_{\mathcal{U}}\phi f''(w)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\delta),\quad\forall t\in\mathbb{R},\end{aligned}\tag{3.8}$$

Since ϕ, when restricted on $\mathcal{U}\times[t,t+T]$, is compactly supported, and (a^{ij}) is locally uniform positive definite, there exists $\lambda>0$ such that

$$\left(a^{kl}\partial_k w_\epsilon \partial_l w_\epsilon\right)(x,\tau)\geq \lambda|\nabla w_\epsilon(x,\tau)|^2,\quad \forall (x,\tau)\in \mathrm{supp}(\phi)\cap(\mathcal{U}\times[t,t+T]),$$

which together with the positiveness of f'' and ϕ gives

$$\frac{\lambda}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\leq \frac{1}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau.$$

Setting $\delta=\frac{\lambda}{6}$ in (3.8), we use the above inequality to find

$$\begin{aligned}&\frac{1}{2}\int_t^{t+T}\int_{\mathbb{R}^d}\phi f''(w)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq\int_t^{t+T}\int_{\mathbb{R}^d}[\rho_{1,\epsilon}f(w_\epsilon)(\partial_t\phi+a^{kl}\partial_{kl}\phi)+f(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\phi]\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\frac{\lambda}{6}),\quad\forall t\in\mathbb{R}.\end{aligned}$$

The result follows from letting $\epsilon\to 0$. □

Now, we prove Theorem A.

Proof of Theorem A. Let $\zeta\in C^\infty([0,+\infty))$ be a non-negative function satisfying

$$\zeta(t)=\begin{cases}1, & t=0,\\ 0, & t\in[1,\infty),\end{cases}\qquad \zeta'\leq 0\quad\text{and}\quad \zeta''\geq 0.$$

Let θ be defined as in (2.4). Clearly, $\tilde{U}:=\theta(U)$ satisfies $\mathcal{L}\tilde{U}\leq\tilde{C}_1\tilde{U}+\tilde{C}_2$ for some $\tilde{C}_1,\tilde{C}_2>0$. It is easy to see that $\zeta(\frac{\tilde{U}}{N})\in C^{2,1}_{c,T}(\mathcal{U}\times\mathbb{R})$ for $N\gg 1$. Applying Lemma 3.2 with $\phi:=\zeta(\frac{\tilde{U}}{N})$, we find

$$\begin{aligned}&\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde{U}}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\\ &\quad\leq C\int_t^{t+T}\int_{\mathcal{U}}f(w)\mathcal{L}\zeta\left(\frac{\tilde{U}}{N}\right)\mathrm{d}\mu_s^1\mathrm{d}s\\ &\quad= C\int_t^{t+T}\int_{\mathcal{U}}f(w)\left[\zeta''\left(\frac{\tilde{U}}{N}\right)\frac{1}{N^2}a^{ij}\partial_i\tilde{U}\partial_j\tilde{U}+\zeta'\left(\frac{\tilde{U}}{N}\right)\frac{1}{N}\mathcal{L}\tilde{U}\right]\mathrm{d}\mu_s^1\mathrm{d}s,\quad\forall t\in\mathbb{R}.\end{aligned}$$

Since (a^{ij}) is positive definite, $f<0$ and $\zeta''\geq 0$, the term $\int_t^{t+T}\int_{\mathcal{U}}f(w)\zeta''\left(\frac{\tilde{U}}{N}\right)\frac{1}{N^2}a^{ij}\partial_i\tilde{U}\partial_j\tilde{U}\mathrm{d}\mu_s^1\mathrm{d}s$ is non-positive. As a result,

$$\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde{U}}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\leq\frac{C}{N}\int_t^{t+T}\int_{\mathcal{U}}f(w)\zeta'\left(\frac{\tilde{U}}{N}\right)\mathcal{L}\tilde{U}\mathrm{d}\mu_s^1\mathrm{d}s.$$

Using $\mathcal{L}\tilde{U}\leq\tilde{C}\tilde{U}+\tilde{C}_2$ and $|f|\leq e$, we find

$$\begin{aligned}0&\leq\int_t^{t+T}\int_{\mathcal{U}}\zeta\left(\frac{\tilde{U}}{N}\right)f''(w)a^{ij}\partial_i w\partial_j w\mathrm{d}\mu_s^1\mathrm{d}s\\ &\leq\frac{Ce}{N}|\zeta'|_\infty\int_t^{t+T}\int_{\{(x,s):\tilde{U}(x,s))\leq N\}}\left(\tilde{C}_1\tilde{U}+\tilde{C}_2\right)\mathrm{d}\mu_s^1\mathrm{d}s,\quad\forall t\in\mathbb{R}.\end{aligned}$$

Letting $N \to \infty$ in the above inequality, we conclude from $f''(t) = e^{1-t}$ and the dominated convergence theorem that

$$\int_t^{t+T} \int_{\mathcal{U}} e^{1-w} a^{ij} \partial_i w \partial_j w \rho_1 \mathrm{d}x\mathrm{d}t = 0, \quad \forall t \in \mathbb{R}.$$

Since (a^{ij}) is locally uniformly positive definite, we conclude that $\nabla w = 0$ a.e. on $\mathcal{U} \times \mathbb{R}$, which together with the continuity of w implies that $w(\cdot, t) \equiv$ const for $t \in \mathbb{R}$. As $w = \frac{\rho_2}{\rho_1}$ and both $\rho_1(\cdot, t)$ and $\rho_2(\cdot, t)$ are continuous probability densities for each $t \in \mathbb{R}$, there must hold $w \equiv 1$. This completes the proof. □

4. **Proof of Theorem B**

In Subsection 4.1, we establish an estimate for global probability solutions of the Cauchy problem (1.3)-(1.11). It is then applied to prove Theorem B in Subsection 4.2.

4.1. **An estimate.**

We first prove the following result on the time regularity of global sub-probability solutions of (1.3). It is indeed a variation of a classical result (see [2, Lemma 8.1.2]). However, there is no global integrability of a^{ij} and V^i in our case.

Definition 4.1. Let $\mathcal{I}$ be an interval and $\mu = (\mu_t)_{t \in \mathcal{I}}$ be a family of Borel measures on $\mathcal{U}$. A *continuous modification* of μ is a family of Borel measures $\tilde{\mu} = (\tilde{\mu}_t)_{t \in \mathcal{I}}$ on $\mathcal{U}$ satisfying the property:

$$\forall \phi \in C_c^{2,1}(\mathcal{U} \times \mathcal{I}), \text{ the function } t \mapsto \int_{\mathcal{U}} \phi(\cdot, t) \mathrm{d}\tilde{\mu}_t \text{ is continuous on } \mathcal{I},$$

such that $\mu_t = \tilde{\mu}_t$ for a.e. $t \in \mathcal{I}$.

Lemma 4.1. *Let $\mathcal{I}$ be an interval and $\mu = (\mu_t)_{t \in \mathcal{I}}$ be a family of Borel measures on $\mathcal{U}$. Then there exists at most one continuous modification of μ.*

Proof. Suppose both $\tilde{\mu}^1 = (\tilde{\mu}_t^1)_{t \in \mathcal{I}}$ and $\tilde{\mu}^2 = (\tilde{\mu}_t^2)_{t \in \mathcal{I}}$ are continuous modifications of $\mu = (\mu_t)_{t \in \mathcal{I}}$. By Definition 4.1, $\tilde{\mu}_t^1 = \tilde{\mu}_t^2$ for a.e. $t \in \mathbb{R}$ and for each $\phi \in C_c^2(\mathcal{U})$, the functions $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^1$ and $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^2$ are continuous on $\mathcal{I}$. It follows that for each $\phi \in C_c^2(\mathcal{U})$, there holds

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^1 = \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t^2, \quad \forall t \in \mathcal{I}.$$

This implies that $\tilde{\mu}_t^1 = \tilde{\mu}_t^2$ for all $t \in \mathcal{I}$. □

Lemma 4.2. *Assume* **(H2)**. *Let $\mu = (\mu_t)_{t \in (s,\infty)}$ be a global sub-probability solution of* (1.3). *Then μ admits a unique continuous modification.*

Proof. By Lemma 4.1, we only need to show the existence. We first show that there exists a family of sub-probability measures $(\tilde{\mu}_t)_{t \in (s,\infty)}$ on $\mathcal{U}$ satisfying the property:

$$\forall \phi \in C_c^2(\mathcal{U}), \text{ the function } t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t \text{ is continuous on } (s, \infty),$$

such that $\tilde{\mu}_t = \mu_t$ for a.e. $t \in (s, \infty)$.

As $\mu = (\mu_t)_{t \in (s,\infty)}$ is a sub-probability solution of (1.3), we see from Lemma 2.1 (1)(a) that for any $\phi \in C_c^2(\mathcal{U})$, there exists a set $J_\phi \subset (s, \infty)$ satisfying $|(s, \infty) \setminus J_\phi| = 0$ such that

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall r, t \in J_\phi \text{ with } r < t. \tag{4.1}$$

For each $\phi \in C_c^2(\mathcal{U})$, we define a function f_ϕ on J_ϕ by setting

$$f_\phi(t) := \int_{\mathcal{U}} \phi \mathrm{d}\mu_t, \quad t \in J_\phi.$$

Since a^{ij} and V^i are locally bounded and T-periodic for each $i,j = 1,\dots,d$, and ϕ is compactly supported in $\mathcal{U}$, the boundedness of $\mathcal{L}\phi$ follows. As a result, we have

$$\begin{aligned}|f_\phi(t) - f_\phi(r)| &= \left|\int_{\mathcal{U}} \phi \mathrm{d}\mu_t - \int_{\mathcal{U}} \phi \mathrm{d}\mu_r\right| \\ &= \left|\int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau\right| \\ &\le \max_{\mathcal{U}\times[r,t]} |\mathcal{L}\phi| \times (t-r), \quad \forall r,t \in J_\phi \text{ with } r<t.\end{aligned}$$

It follows that there exists a locally Lipschitz continuous function $\tilde{f}_\phi$ on (s,∞) such that $\tilde{f}_\phi(t) = f_\phi(t)$ for $t \in J_\phi$. Obviously, $\tilde{f}_\phi \ge 0$ if $\phi \ge 0$ and $|\tilde{f}_\phi(t)| \le |\phi|_\infty$ for $\phi \in C_c^2(\mathcal{U})$ and $t \in (s,\infty)$.

For each $t \in [s,\infty)$, we define a functional as follows:

$$K_t : C_c^2(\mathcal{U}) \to \mathbb{R}, \quad \phi \mapsto \tilde{f}_\phi(t).$$

Obviously, K_t is linear, positive and $|K_t\phi| \le |\phi|_\infty$. As $C_c^2(\mathcal{U})$ is dense in $C_c(\mathcal{U})$ under the topology of uniform convergence on $\mathcal{U}$, K_t has a unique linear continuous extension $\overline{K}_t$ onto $C_c(\mathcal{U})$ satisfying $\overline{K}_t\phi = K_t\phi$ for all $\phi \in C_c^2(\mathcal{U})$. We see that $\overline{K}_t$ is positive. In fact, for any non-negative function $\phi \in C_c(\mathcal{U})$, there exists a sequence of non-negative functions $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^2(\mathcal{U})$ that converges uniformly to ϕ on $\mathcal{U}$. Therefore,

$$\overline{K}_t\phi = \lim_{n\to\infty} K_t\phi_n = \lim_{n\to\infty} \tilde{f}_{\phi_n}(t) \ge 0.$$

Applying the Riesz representation theorem, we find a Borel measure $\tilde{\mu}_t$ on $\mathcal{U}$ such that

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \overline{K}_t\phi, \quad \forall \phi \in C_c(\mathcal{U}).$$

As a consequence, we obtain a family of Borel measures $(\tilde{\mu}_t)_{t\in(s,\infty)}$ on $\mathcal{U}$ satisfying

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \tilde{f}_\phi(t), \quad \forall t \in (s,\infty) \text{ and } \phi \in C_c^2(\mathcal{U}).$$

In particular, the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t$ is continuous on (s,∞) for any $\phi \in C_c^2(\mathcal{U})$.

Let $\mathcal{D}$ be a countable basis of $C_c^2(\mathcal{U})$ under the topology of uniform convergence on $\mathcal{U}$ and set $J := \cap_{\phi\in\mathcal{D}} J_\phi$. Cleary, $|(s,\infty) \setminus J| = 0$ and

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \tilde{f}_\phi(t) = f_\phi(t) = \int_{\mathcal{U}} \phi \mathrm{d}\mu_t, \quad \forall \phi \in \mathcal{D} \text{ and } t \in J. \tag{4.2}$$

As $C_c^2(\mathcal{U})$ is dense in $C_c(\mathcal{U})$ and $\mathcal{D}$ is dense in $C_c^2(\mathcal{U})$, (4.2) holds for all $\phi \in C_c(\mathcal{U})$ and $t \in J$. Hence, $\tilde{\mu}_t = \mu_t$ and $\tilde{\mu}_t(\mathcal{U}) = \mu_t(\mathcal{U}) \le 1$ for all $t \in J$. From the continuity of the function $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t$ on (s,∞) for each $\phi \in C_c^2(\mathcal{U})$, we conclude that $\tilde{\mu}_t(\mathcal{U}) \le 1$ for all $t \in (s,\infty)$.

It remains to show that for each $\phi \in C_c^{2,1}(\mathcal{U}\times(s,\infty))$, the function $t \mapsto \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t$ is continuous on (s,∞). For fixed $t \in (s,\infty)$ and $\phi \in C_c^{2,1}(\mathcal{U}\times(s,\infty))$, the local Lipschitz continuity of $\tilde{f}_{\phi(\cdot,t)}$ on (s,∞) implies that

$$\left|\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_r - \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right| = \left|\tilde{f}_{\phi(\cdot,t)}(r) - \tilde{f}_{\phi(\cdot,t)}(t)\right| \to 0 \quad \text{as} \quad r \to t.$$

It follows that

$$\left|\int_{\mathcal{U}}\phi(\cdot,r)\mathrm{d}\tilde{\mu}_r-\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right|\le\int_{\mathcal{U}}|\phi(\cdot,r)-\phi(\cdot,t)|\mathrm{d}\tilde{\mu}_r+\left|\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_r-\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right|$$
$$\le\max_{x\in\mathcal{U}}|\phi(x,r)-\phi(x,t)|+\left|\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_r-\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_t\right|$$
$$\to 0\quad\text{as}\quad r\to t.$$

This proves the required continuity, and hence, completes the proof. □

A similar result can be proven for global sub-probability solutions of the Cauchy problem (1.3)-(1.11).

Lemma 4.3. *Assume* **(H2)**. *Let $\mu=(\mu_t)_{t\in(s,\infty)}$ be a global sub-probability solution of the Cauchy problem* (1.3)-(1.11). *Then μ admits a continuous modification $\tilde{\mu}=(\tilde{\mu}_t)_{t\in\mathbb{R}}$ satisfying:*

$$\lim_{t\to s^+}\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\tilde{\mu}_t=\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\nu.$$

Proof. The proof follows from arguments as in the proof of Lemma 4.1. The differences are that we use Lemma 2.1 (2)(a) instead of Lemma 2.1 (1)(a) and define $\tilde{f}_\phi$ on $[s,\infty)$ for $\phi\in C_c^2(\mathcal{U})$ with $\mu_s=\nu$. □

Remark 4.1. *If μ is a global sub-probability solution of* (1.3) *or a global sub-probability solution of the Cauchy problem* (1.3)-(1.11)*, so is its continuous modification $\tilde{\mu}$. Moreover, Lemma 2.2 applies in particular to $\tilde{\mu}$. This would allow us to get rid of J_ϕ in many situations in the sequel.*

The expected estimate is stated in the next result.

Proposition 4.1. *Assume* **(H2)** *and that $\mathcal{L}$ admits a strong Lyapunov function U. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathcal{U}$ as in Definition 1.2 and $\mu=(\mu_t)_{t\in(s,\infty)}$ be a global sub-probability solution of the Cauchy problem* (1.3)-(1.11) *with $\int_{\mathcal{U}}U(\cdot,s)\mathrm{d}\nu<\infty$. Let $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(s,\infty)}$ be the continuous modification of $\mu=(\mu_t)_{t\in(s,\infty)}$ given in Lemma 4.3. Then,*

$$\tilde{\mu}_t(\mathcal{U})=1,\quad\forall t>s,$$

and there exists some $C>0$, independent of s, ν and μ, such that

$$C_n\int_s^t\tilde{\mu}_\tau(\mathcal{U}\setminus\mathcal{U}_n)\mathrm{d}\tau+D_n\tilde{\mu}_t(\mathcal{U}\setminus\mathcal{U}_n)\le\int_{\mathcal{U}}U(\cdot,s)\mathrm{d}\nu+C(t-s),\quad\forall t>s\text{ and }n\in\mathbb{N},\tag{4.3}$$

where $C_n:=-\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\mathcal{L}U>0$ and $D_n:=\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}U$.

Proof. For notational simplicity, we write $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(s,\infty)}$ as $\mu=(\mu_t)_{t\in(s,\infty)}$ throughout the proof.

We see from Lemma 2.1 (2)(b) and Lemma 2.2 that for each $\phi\in C_c^{2,1}(\mathcal{U}\times[s,\infty))$ there holds

$$\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi(\cdot,s)\mathrm{d}\mu_s+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad\forall t>s.\tag{4.4}$$

Since U is a strong Lyapunov function, there is a $\rho_m>0$ such that $\mathcal{L}U\le 0$ on $(\mathcal{U}\times\mathbb{R})\setminus\overline{\Omega}_{\rho_m}$. Fix $\rho_0>\rho_m$ and let $\{\zeta_\rho\}_{\rho>\rho_0}$ be a family of smooth and non-decreasing functions on $\mathbb{R}$ satisfying

$$\zeta_\rho(t)=\begin{cases}0, & t\in[0,\rho_m],\\ t, & t\in[\rho_0,\rho],\\ \rho+1, & t\in[\rho+2,\infty),\end{cases}\qquad\zeta_\rho(t)\le t,\ t\in[\rho_m,\rho_0]\quad\text{and}\quad\zeta_\rho''(t)\le 0,\ t\in[\rho,\rho+2].$$

In addition, we let the functions $\{\zeta_\rho\}_{\rho\geq\rho_0}$ coincide on $[0,\rho_0]$.

Obviously, $\zeta_\rho(U)-(\rho+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$. Setting $\phi=\zeta_\rho(U)-(\rho+1)$ in (4.4), we find

$$\begin{aligned}\int_{\mathcal{U}}(\zeta_\rho(U)-(\rho+1))\mathrm{d}\mu_t&=\int_{\mathcal{U}}(\zeta_\rho(U)-(\rho+1))\mathrm{d}\nu+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}(\zeta_\rho(U)-(\rho+1))\mathrm{d}\mu_\tau\mathrm{d}\tau\\&=\int_{\mathcal{U}}(\zeta_\rho(U)-(\rho+1))\mathrm{d}\nu+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\mathcal{L}(\zeta_\rho(U))\mathrm{d}\mu_\tau\mathrm{d}\tau.\end{aligned}$$

It follows from

$$\mathcal{L}(\zeta_\rho(U))=\zeta_\rho'(U)\mathcal{L}U+\zeta_\rho''(U)a^{ij}\partial_iU\partial_jU,$$

that

$$\begin{aligned}\int_{\mathcal{U}}\zeta_\rho(U)\mathrm{d}\mu_t=&\int_{\mathcal{U}}\zeta_\rho(U)\mathrm{d}\nu+(\rho+1)\times[\mu_t(\mathcal{U})-\nu(\mathcal{U})]\\&+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\zeta_\rho'(U)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau\\&+\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\zeta_\rho''(U)a^{ij}\partial_iU\partial_jU\mathrm{d}\mu_\tau\mathrm{d}\tau.\end{aligned}\tag{4.5}$$

Due to (1.4), there exists an $n_0\in\mathbb{N}$ such that $\Omega_{\rho_0}\subset\subset\mathcal{U}_n\times\mathbb{R}$ for all $n>n_0$. Since $\zeta_\rho'=0$ on $[0,\rho_m]$, $\zeta_\rho'=1$ on $[\rho_0,\rho]$ and $\zeta_\rho'\geq0$ otherwise, we see from $\mathcal{L}U\leq0$ in $(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m}$ that

$$\zeta_\rho'(U)\mathcal{L}U\leq\begin{cases}\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\mathcal{L}U, & \text{in}\quad\Omega_\rho\setminus(\mathcal{U}_n\times\mathbb{R}),\\0, & \text{otherwise.}\end{cases}$$

Thus,

$$\begin{aligned}\lim_{r\to s}\int_r^t\int_{\mathcal{U}}\zeta_\rho'(U)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau&\leq\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\mathcal{L}U\times\lim_{r\to s}\int_r^t\mu_\tau(\Omega_\rho^\tau\setminus\mathcal{U}_n)\mathrm{d}\tau\\&=-C_n\int_s^t\mu_\tau(\Omega_\rho^\tau\setminus\mathcal{U}_n)\mathrm{d}\tau,\quad n>n_0,\end{aligned}\tag{4.6}$$

where $C_n:=-\sup_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\mathcal{L}U>0$ and the monotone convergence theorem is used in the above equality.

As $\zeta_\rho''\neq0$ on $[\rho_m,\rho_0]$, $\zeta''\leq0$ on $[\rho,\rho+2]$ and $\zeta''=0$ otherwise, we find from the non-negative definiteness of (a^{ij}) that

$$\zeta_\rho''(U)a^{ij}\partial_iU\partial_jU\leq\begin{cases}C_*\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU, & \text{in}\quad\Omega_{\rho_0}\setminus\Omega_{\rho_m},\\0, & \text{otherwise,}\end{cases}$$

where $C_*:=\max_{t\in[\rho_m,\rho_0]}\zeta_\rho''(t)$ is independent of ρ due to the construction of $\{\zeta_\rho\}_{\rho>\rho_0}$. Hence,

$$\int_r^t\int_{\mathcal{U}}\zeta_\rho''(U)a^{ij}\partial_iU\partial_jU\mathrm{d}\mu_\tau\mathrm{d}\tau\leq C_*\left(\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU\right)\times(t-r)=C(t-r),\tag{4.7}$$

where $C=C_*\left(\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU\right)$.

Substituting (4.6) and (4.7) into (4.5) yields

$$\begin{aligned}\int_{\mathcal{U}} \zeta_\rho(U)\mathrm{d}\mu_t \leq \int_{\mathcal{U}} \zeta_\rho(U)\mathrm{d}\nu + (\rho+1)\times[\mu_t(\mathcal{U})-\nu(\mathcal{U})] \\ - C_n\int_s^t \mu_\tau(\Omega_\rho^\tau\setminus\mathcal{U}_n)\mathrm{d}\tau + C(t-s), \quad \forall t>s.\end{aligned}\tag{4.8}$$

As $\zeta_\rho \geq 0$ and $\zeta_\rho(t)=t$ for $t\in[\rho_0,\rho]$, we derive from $\Omega_{\rho_0}\subset\mathcal{U}_n\times\mathbb{R}$ that

$$\int_{\mathcal{U}} \zeta_\rho(U)\mathrm{d}\mu_t \geq \int_{\Omega_\rho^t\setminus\Omega_{\rho_0}^t} U\mathrm{d}\mu_t \geq \int_{\Omega_\rho^t\setminus\mathcal{U}_n} U\mathrm{d}\mu_t \geq D_n\mu_t(\Omega_\rho^t\setminus\mathcal{U}_n), \quad \forall n>n_0,\tag{4.9}$$

where $D_n := \inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U$. As $\zeta_\rho(t)\leq t$ for $t\geq 0$ and $\rho>\rho_0$, we find from (4.8) and (4.9) that

$$\begin{aligned}D_n\mu_t(\Omega_\rho^t\setminus\mathcal{U}_n) \leq \int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + (\rho+1)\times[\mu_t(\mathcal{U})-\nu(\mathcal{U})] \\ - C_n\int_s^t \mu_\tau(\Omega_\rho^\tau\setminus\mathcal{U}_n)\mathrm{d}\tau + C(t-s), \quad \forall t>s.\end{aligned}\tag{4.10}$$

Note the ν-integrability of $U(\cdot,s)$ ensures the non-triviality of the above inequalities. If $\mu_t(\mathcal{U}) < \nu(\mathcal{U}) = 1$ for some $t>s$, we deduce from (4.10) that

$$0 \leq \int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + (\rho+1)\times(\mu_t(\mathcal{U})-\nu(\mathcal{U})) + C(t-s) \to -\infty \quad \text{as} \quad \rho\to\infty,$$

which leads to a contradiction. Therefore, $\mu_t(\mathcal{U})=\nu(\mathcal{U})=1$ for all $t>s$. Consequently, letting $\rho\to\infty$ in (4.10) leads to

$$C_n\int_s^t \mu_\tau(\mathcal{U}\setminus\mathcal{U}_n)\mathrm{d}\tau + D_n\mu_t(\mathcal{U}\setminus\mathcal{U}_n) \leq \int_{\mathcal{U}} U(\cdot,s)\mathrm{d}\nu + C(t-s), \quad \forall t>s.$$

This completes the proof. □

4.2. **Proof of Theorem B.** We recall the definition of the weak*-topology for Borel measures on $\mathcal{U}\times\mathbb{R}$.

Definition 4.2. A sequence of σ-finite Borel measures $\{\mu^n, n\in\mathbb{N}\}$ on $\mathcal{U}\times\mathbb{R}$ is said to converge to a σ-finite Borel measure μ on $\mathcal{U}\times\mathbb{R}$ under the *weak*-topology* as $n\to\infty$ if

$$\lim_{n\to\infty}\iint_{\mathcal{U}\times\mathbb{R}} \phi\mathrm{d}\mu^n = \iint_{\mathcal{U}\times\mathbb{R}} \phi\mathrm{d}\mu, \quad \forall\phi\in C_0(\mathcal{U}\times\mathbb{R}).$$

Set

$$C_{c,T}(\mathcal{U}\times\mathbb{R}) := C_c(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R}).$$

Proof of Theorem B. For clarity, we assume $s=0$. Applying Lemma 4.3, we may replace $\mu = (\mu_t)_{t\in(0,\infty)}$ by its continuous modification, still denoted by $\mu = (\mu_t)_{t\in(0,\infty)}$. Since $U(\cdot,0)$ is ν-integrable, Proposition 4.1 yields the existence of some $C>0$ such that

$$C_n\int_0^t \mu_\tau(\mathcal{U}\setminus\mathcal{U}_n)\mathrm{d}\tau + D_n\mu_t(\mathcal{U}\setminus\mathcal{U}_n) \leq \int_{\mathcal{U}} U(\cdot,0)\mathrm{d}\nu + Ct, \quad t>0,\tag{4.11}$$

where $\mathcal{U}_n$, C_n and D_n are as in the statement of Proposition 4.1.

For each $n\in\mathbb{N}$, we define

$$\mu_t^n := \frac{1}{n}\sum_{k=0}^{n-1}\mu_{t+kT} \quad \text{for } t>0 \quad \text{and} \quad \mathrm{d}\mu^n := \mathrm{d}\mu_t^n\mathrm{d}t \quad \text{on} \quad \mathcal{U}\times(0,\infty).$$

Then, for any bounded $\phi \in C_T(\mathcal{U}\times\mathbb{R})$, there holds

$$\begin{aligned}\int_t^{t+T}\int_{\mathcal{U}}\phi \mathrm{d}\mu_\tau^n \mathrm{d}\tau &= \frac{1}{n}\sum_{k=0}^{n-1}\int_t^{t+T}\int_{\mathcal{U}}\phi \mathrm{d}\mu_{\tau+kT}\mathrm{d}\tau\\ &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{t+kT}^{t+(k+1)T}\int_{\mathcal{U}}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau\\ &= \frac{1}{n}\int_t^{t+nT}\int_{\mathcal{U}}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall t>0.\end{aligned} \tag{4.12}$$

Let $\{n_j\}_{j\in\mathbb{N}}\subset\mathbb{N}$ be fixed. The proof is finished in seven steps. In **Step 1-Step 5**, we construct the limiting periodic probability solution of (1.3). The convergence results are proven in **Step 6** and **Step 7**.

Step 1. We show the existence of a subsequence of $\{n_j\}_{j\in\mathbb{N}}$, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, such that μ^{n_j} converges under the weak*-topology to some Borel measure $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$, and for each $t>0$, there holds

$$\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}\phi \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau = \iint_{\mathcal{U}\times[t,t+T]}\phi \mathrm{d}\tilde{\mu}, \quad \forall \text{ bounded } \phi\in C(\mathcal{U}\times[t,t+T]). \tag{4.13}$$

For any compact set $K\subset\mathcal{U}\times(0,\infty)$, there holds $\sup_{j\in\mathbb{N}}\mu^{n_j}(K)<\infty$. Applying [14, Corollary A2.6.V], we conclude the existence of a subsequence of $\{n_j\}_{j\in\mathbb{N}}$, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, such that μ^{n_j} converges under the weak*-topology to some Borel measure $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$.

To show (4.13), we may apply [13, Theorem 4.4.2] that says in particular it is equivalent to show

(i) for each $f\in C_0(\mathcal{U}\times[t,t+T])$, there holds

$$\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}} f \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau = \iint_{\mathcal{U}\times[t,t+T]} f \mathrm{d}\tilde{\mu};$$

(ii) $\tilde{\mu}(\mathcal{U}\times[t,t+T]) = T$.

We prove (i) and (ii) in the rest of **Step 1**.

(i) Note that for any $f\in C_0(\mathcal{U}\times[t,t+T])$, there is an $\epsilon_0\in(0,1)$ and a family of functions $\{f_\epsilon\}_{\epsilon\in(0,\epsilon_0)}\subset C_0(\mathcal{U}\times(0,\infty))$ satisfying

- $|f_\epsilon|_\infty\le|f|_\infty$ for all $\epsilon\in(0,\epsilon_0)$,
- for each $\epsilon\in(0,\epsilon_0)$,

$$f_\epsilon = \begin{cases} f & \text{on} \quad \mathcal{U}\times[t,t+T],\\ 0 & \text{on} \quad \mathcal{U}\times(0,t-\epsilon]\cup[t+\epsilon+T,\infty).\end{cases}$$

Clearly,

$$\lim_{\epsilon\to 0} f_\epsilon(x,\tau) = f(x,\tau)\mathbb{1}_{[t,t+T]}(\tau), \quad (x,\tau)\in\mathcal{U}\times(0,\infty).$$

As μ^{n_j} converges to $\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$ as $j\to\infty$ under the weak*-topology, there holds

$$\lim_{j\to\infty}\int_0^\infty\int_{\mathcal{U}} f_\epsilon \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau = \iint_{\mathcal{U}\times(0,\infty)} f_\epsilon \mathrm{d}\tilde{\mu}, \quad \forall\epsilon\in(0,\epsilon_0). \tag{4.14}$$

It follows from the construction of $\{f_\epsilon\}_{\epsilon\in(0,\epsilon_0)}$ that

$$\left|\int_0^\infty\int_{\mathcal{U}} f_\epsilon \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau - \int_t^{t+T}\int_{\mathcal{U}} f \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\right| \le 2\epsilon|f|_\infty, \quad \forall\epsilon\in(0,\epsilon_0),$$

which is equivalent to

$$\int_0^\infty \int_{\mathcal{U}} f_\epsilon \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau - 2\epsilon|f|_\infty \le \int_t^{t+T} \int_{\mathcal{U}} f \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau \le \int_0^\infty \int_{\mathcal{U}} f_\epsilon \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau + 2\epsilon|f|_\infty, \quad \forall \epsilon \in (0, \epsilon_0).$$

Letting $j \to \infty$ in the above inequality, we find from (4.14) that

$$\begin{aligned} \iint_{\mathcal{U}\times(0,\infty)} f_\epsilon \mathrm{d}\tilde{\mu} - 2\epsilon|f|_\infty &\le \liminf_{j\to\infty} \int_t^{t+T} \int_{\mathcal{U}} f \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau \\ &\le \limsup_{j\to\infty} \int_t^{t+T} \int_{\mathcal{U}} f \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau \\ &\le \iint_{\mathcal{U}\times(0,\infty)} f_\epsilon \mathrm{d}\tilde{\mu} + 2\epsilon|f|_\infty. \end{aligned} \tag{4.15}$$

Since

$$\lim_{\epsilon\to 0} \iint_{\mathcal{U}\times(0,\infty)} f_\epsilon \mathrm{d}\tilde{\mu} = \iint_{\mathcal{U}\times[t,t+T]} f \mathrm{d}\tilde{\mu}$$

thanks to the dominated convergence theorem, passing to the limit $\epsilon \to 0$ in (4.15) yields (i).

(ii) It follows from the definition of $\{\mu^{n_j}\}_{j\in\mathbb{N}}$ that

$$\begin{aligned} \mu^{n_j}((\mathcal{U}\setminus\mathcal{U}_m)\times[t,t+T]) &= \frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_t^{t+T} \mu_{\tau+kT}(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau \\ &= \frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{t+kT}^{t+(k+1)T} \mu_\tau(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau \\ &= \frac{1}{n_j}\int_t^{t+n_jT} \mu_\tau(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau, \quad \forall m \in \mathbb{N} \text{ and } j \in \mathbb{N}. \end{aligned}$$

By (4.11),

$$\frac{1}{n_j}\int_0^{t+n_jT} \mu_\tau(\mathcal{U}\setminus\mathcal{U}_m)\mathrm{d}\tau \le \frac{1}{n_jC_m}\left(\int_{\mathcal{U}} U(\cdot,0)\mathrm{d}\nu + C\times(t+n_jT)\right), \quad \forall m \in \mathbb{N} \text{ and } j \in \mathbb{N},$$

where we recall that $C_m = -\sup_{(\mathcal{U}\setminus\mathcal{U}_m)\times\mathbb{R}} \mathcal{L}U \to \infty$. As a result, for any $0 < \epsilon \ll 1$, there exists an $m_0 = m_0(\epsilon) \in \mathbb{N}$ such that

$$\mu^{n_j}((\mathcal{U}\setminus\mathcal{U}_m)\times[t,t+T]) \le \epsilon, \quad \forall m \ge m_0 \text{ and } j \in \mathbb{N}.$$

Equivalently,

$$\mu^{n_j}(\mathcal{U}_m\times[t,t+T]) \ge T-\epsilon, \quad \forall m \ge m_0 \text{ and } j \in \mathbb{N}.$$

This means, $\{\mu^{n_j}\}$, when restricted on $\mathcal{U}\times[t,t+T]$, is tight. Hence, we apply the Portmanteau theorem to find that

$$\tilde{\mu}(\overline{\mathcal{U}}_m\times[t,t+T]) \ge \limsup_{j\to\infty} \mu^{n_j}(\overline{\mathcal{U}}_m\times[t,t+T]) \ge T-\epsilon, \quad \forall m \ge m_0.$$

Letting $\epsilon \to 0$, we conclude that $\tilde{\mu}(\mathcal{U}\times[t,t+T]) \ge T$.

By (i), we deduce that

$$\tilde{\mu}(\mathcal{U}_m\times[t,t+T]) \le \liminf_{j\to\infty} \mu^{n_j}(\mathcal{U}_m\times[t,t+T]) \le T,$$

which implies $\tilde{\mu}(\mathcal{U}\times[t,t+T]) \le T$. Hence, $\tilde{\mu}(\mathcal{U}\times[t,t+T]) = T$, and (ii) follows.

Step 2. We show that the measure $\tilde{\mu}$ obtained in **Step 1** admits t-sections. More precisely, we show the existence of a family of Borel measures $\{\tilde{\mu}_t\}_{t\in(0,\infty)}$ on $\mathcal{U}$ satisfying

$$\tilde{\mu}_t = \tilde{\mu}_{t+T} \quad \text{and} \quad \tilde{\mu}_t(\mathcal{U}) = 1 \quad \text{for a.e. } t > 0,$$

such that $\tilde{\mu} = (\tilde{\mu}_t)_{t\in(0,\infty)}$.

Let $\eta \in C_c(0,\infty)$ and $|\mathrm{supp}(\eta)| \leq T$. Setting $\phi = \eta$ in (4.13) gives

$$\iint_{\mathcal{U}\times(0,\infty)} \eta \mathrm{d}\tilde{\mu} = \lim_{j\to\infty} \int_0^\infty \int_{\mathcal{U}} \eta \mathrm{d}\mu_t^{n_j} \mathrm{d}t = \int_0^\infty \eta \mathrm{d}t. \tag{4.16}$$

Arguing as in the proof of [22, Lemma 4.2], we derive from (4.16) the existence of a family of Borel measures $\{\tilde{\mu}_t\}_{t\in(0,\infty)}$ satisfying $\tilde{\mu}_t(\mathcal{U}) = 1$ for a.e. $t > 0$ such that $\tilde{\mu} = (\tilde{\mu}_t)_{t\in(0,\infty)}$.

It remains to show $\tilde{\mu}_t = \tilde{\mu}_{t+T}$ for a.e. $t > 0$. It follows from (4.12) that for any $\phi \in C_{c,T}(\mathcal{U}\times\mathbb{R})$ and $t_1, t_2 \in (0,\infty)$ with $t_1 < t_2$, there holds

$$\begin{aligned}
\int_{t_1}^{t_1+T} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau &= \frac{1}{n_j} \int_{t_1}^{t_1+n_jT} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \\
&= \frac{1}{n_j}\left(\int_{t_1}^{t_2} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau - \int_{t_1+n_jT}^{t_2+n_jT} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau + \int_{t_2}^{t_2+n_jT} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \right) \\
&= \frac{1}{n_j}\left(\int_{t_1}^{t_2} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau - \int_{t_1+n_jT}^{t_2+n_jT} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \right) + \int_{t_2}^{t_2+T} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau.
\end{aligned}$$

Letting $j \to \infty$ in the above equality, we find from (4.13) that

$$\iint_{\mathcal{U}\times[t_1,t_1+T]} \phi \mathrm{d}\tilde{\mu} = \iint_{\mathcal{U}\times[t_2,t_2+T]} \phi \mathrm{d}\tilde{\mu}, \quad \forall t_1, t_2 > 0 \text{ with } t_1 < t_2.$$

We then argue as in the proof of [22, Lemma 4.1] to find $\tilde{\mu}_t = \tilde{\mu}_{t+T}$ for a.e. $t > 0$.

Step 3. We show that $\tilde{\mu} = (\tilde{\mu}_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3).

We claim that for each $t > 0$, there holds

$$\int_t^{t+T} \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau = 0, \quad \forall \phi \in C_0^{2,1}(\mathcal{U}\times(t,t+T)) \tag{4.17}$$

Fix $t \in \mathbb{R}$. For any $\phi \in C_0^{2,1}(\mathcal{U}\times(t,t+T))$ and $k \in \mathbb{N}\cup\{0\}$, we define

$$\phi_k(x,\tau) := \begin{cases} \phi(x,\tau-kT), & (x,\tau) \in \mathcal{U}\times(t+kT, t+(k+1)T), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\phi_k \in C_0^{2,1}(\mathcal{U}\times(0,\infty))$ for each $k \in \mathbb{N}\cup\{0\}$. As $\mu = (\mu_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3), there holds

$$\int_{t+kT}^{t+(k+1)T} \int_{\mathcal{U}} \mathcal{L}\phi_k \mathrm{d}\mu_\tau \mathrm{d}\tau = \int_0^\infty \int_{\mathcal{U}} \mathcal{L}\phi_k \mathrm{d}\mu_\tau \mathrm{d}\tau = 0.$$

This together with the T-periodicity of (a^{ij}) and (V^i) gives

$$\int_t^{t+T} \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_{\tau+kT} \mathrm{d}\tau = \int_{t+kT}^{t+(k+1)T} \int_{\mathcal{U}} \mathcal{L}\phi_k \mathrm{d}\mu_\tau \mathrm{d}\tau = 0,$$

which yields

$$\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_\tau^n \mathrm{d}\tau=\frac{1}{n}\sum_{k=0}^{n-1}\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_{\tau+kT}\mathrm{d}\tau=0,\quad \forall n\in\mathbb{N}. \tag{4.18}$$

As clearly $\mathcal{L}\phi\in C_0(\mathcal{U}\times(t,t+T))$, we deduce from (4.18) and (4.13) that

$$\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau=\lim_{j\to\infty}\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau=0.$$

This proves (4.17).

From (4.17) and Lemma 2.1 (1)(a), we find that for each $t>0$ and $\phi\in C_c^2(\mathcal{U})$, there exists a subset $J_\phi^t\subset(t,t+T)$ satisfying $|(t,t+T)\setminus J_\phi^t|=0$ such that

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau,\quad \forall t_1,t_2\in J_\phi^t \text{ with } t_1<t_2.$$

As t is arbitrary in $(0,\infty)$, we see the existence of a set $J_\phi\subset(0,\infty)$ with $|(0,\infty)\setminus J_\phi|=0$ such that

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau,\quad \forall t_1,t_2\in J_\phi \text{ with } t_1<t_2.$$

That is, $\tilde{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times(0,\infty)$. As $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$ by **Step 2**, $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3).

Step 4. We show that $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$ admits a continuous modification, still denoted by $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, such that $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ and $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$.

By Lemma 4.2, there is a modification of $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, still denoted by $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, satisfying the property:

$$\forall\phi\in C_c^2(\mathcal{U}),\text{ the function } t\mapsto\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_t \text{ is continuous on } (0,\infty).$$

This together with the fact that $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for a.e. $t>0$ (from **Step 2**) yields that

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_t=\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t+T},\quad \forall\phi\in C_c^2(\mathcal{U})\quad\text{and}\quad t>0.$$

Hence, $\tilde{\mu}_t=\tilde{\mu}_{t+T}$ for all $t>0$.

It remains to show $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$. Note that up to now we only know $\tilde{\mu}_t(\mathcal{U})\le 1$ for all $t>0$ and $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$. Fix $t_0>0$. Since U is a strong Lyapunov function, we can follow the arguments as in the proof of [28, Proposition 2.8] (see the proof of [22, Theorem A] for more details) to find a non-negative function $\tilde{U}\in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ satisfying the following properties:

(1) $\int_{\mathcal{U}}\tilde{U}(\cdot,t_0)\mathrm{d}\tilde{\mu}_{t_0}<\infty$,
(2) $\lim_{n\to\infty}\inf_{x\in\mathcal{U}\setminus\mathcal{U}_n}\tilde{U}(x,t)=\infty$ for all $t\in\mathbb{R}$,
(3) there is a $\tilde{\rho}_m>0$ such that $\mathcal{L}\tilde{U}\le 0$ on $(\mathcal{U}\times\mathbb{R})\setminus\tilde{\Omega}_{\rho_m}$, where $\tilde{\Omega}_{\rho_m}:=\{(x,t)\in\mathcal{U}\times\mathbb{R}:\tilde{U}(x,t)<\tilde{\rho}_m\}$.

We see from Lemma 2.1 (1)(b) and Lemma 2.2 that for each $\phi\in C_c^{2,1}(\mathcal{U}\times(0,\infty))$, there holds

$$\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_2}-\int_{\mathcal{U}}\phi \mathrm{d}\tilde{\mu}_{t_1}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau,\quad \forall 0<t_1<t_2. \tag{4.19}$$

Note $\zeta_\rho(\tilde{U})-(\rho+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$ thanks to property (2), where $\{\zeta_\rho\}$ is defined as in the proof of Proposition 4.1. Setting $\phi=\zeta_\rho(\tilde{U})-(\rho+1)$ in (4.19), we find

$$\begin{aligned}\int_{\mathcal{U}}(\zeta_\rho(\tilde{U})-(\rho+1))\mathrm{d}\tilde{\mu}_t&=\int_{\mathcal{U}}(\zeta_\rho(\tilde{U})-(\rho+1))\mathrm{d}\tilde{\mu}_{t_0}\\&\quad+\int_{t_0}^{t}\int_{\mathcal{U}}\left[\zeta_\rho'(\tilde{U})\mathcal{L}\tilde{U}+\zeta_\rho''(\tilde{U})a^{ij}\partial_i\tilde{U}\partial_j\tilde{U}\right]\mathrm{d}\tilde{\mu}_\tau\mathrm{d}\tau,\quad\forall t>t_0.\end{aligned}$$

Arguing as in the proof of Proposition 4.1 yields the existence of some $C>0$ such that

$$0\le\int_{\mathcal{U}}\tilde{U}(\cdot,t_0)\mathrm{d}\tilde{\mu}_{t_0}+(\rho+1)\times[\tilde{\mu}_t(\mathcal{U})-\tilde{\mu}_{t_0}(\mathcal{U})]+C(t-t_0),\quad\forall t>t_0.$$

Since $\tilde{U}(\cdot,t_0)$ is $\tilde{\mu}_{t_0}$-integrable, if $\tilde{\mu}_t(\mathcal{U})<\tilde{\mu}_{t_0}(\mathcal{U})$ for some $t>t_0$, a contradiction is readily derived by letting $\rho\to\infty$ in the above inequality. As a result, $\tilde{\mu}_t(\mathcal{U})\ge\tilde{\mu}_{t_0}(\mathcal{U})$ for all $t>t_0$. Since $t_0>0$ is arbitrary and $\tilde{\mu}_t(\mathcal{U})=1$ for a.e. $t>0$, we conclude that $\tilde{\mu}_t(\mathcal{U})=1$ for all $t>0$.

Step 5. We extend $\tilde{\mu}=(\tilde{\mu}_t)_{t\in(0,\infty)}$, the continuous modification obtained in **Step 4**, to obtain a periodic probability solution $\hat{\mu}=(\hat{\mu}_t)_{t\in\mathbb{R}}$ of (1.3).

To do so, we define

$$\hat{\mu}_t=\begin{cases}\tilde{\mu}_t, & t>0,\\ \tilde{\mu}_{t+kT}, & t\in(-kT,-(k-1)T]\text{ and }k\in\mathbb{N}.\end{cases}$$

Obviously, $\hat{\mu}_t(\mathcal{U})=1$ and $\hat{\mu}_t=\hat{\mu}_{t+T}$ for all $t\in\mathbb{R}$. Thus, $\hat{\mu}:=(\hat{\mu}_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (1.3) if we can show $\hat{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times\mathbb{R}$.

As $\tilde{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times(0,\infty)$, the definition of $\hat{\mu}$ implies that for any $\phi\in C_c^2(\mathcal{U})$ and $k\in\mathbb{N}$, there holds

$$\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_{t_1}-\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_{t_2}=\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau,\quad\forall t_1,t_2\in(-kT,-(k-1)T]\text{ with }t_1<t_2.\tag{4.20}$$

As $k\in\mathbb{N}$ is arbitrary and $\hat{\mu}=\tilde{\mu}$ on $\mathcal{U}\times(0,\infty)$, we see that (4.20) holds for all $t_1,t_2\in\mathbb{R}$ with $t_1<t_2$. That is, $\hat{\mu}$ is a measure solution of (1.3) in $\mathcal{U}\times\mathbb{R}$.

Step 6. We show that for any bounded $\phi\in C_T(\mathcal{U}\times\mathbb{R})$, there holds

$$\lim_{j\to\infty}\frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau=\frac{1}{T}\int_0^T\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau,\quad\forall t\ge0.\tag{4.21}$$

It follows from (4.12), (4.13) and the definition of $\hat{\mu}=(\hat{\mu}_t)_{t\in\mathbb{R}}$ that for each bounded $\phi\in C_T(\mathcal{U}\times\mathbb{R})$,

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{n_jT}\int_t^{t+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau&=\lim_{j\to\infty}\frac{1}{n_jT}\left(\int_t^{t+\epsilon}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau-\int_{t+n_jT}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\right)\\&\quad+\lim_{j\to\infty}\frac{1}{n_jT}\int_{t+\epsilon}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\\&=\lim_{j\to\infty}\frac{1}{n_jT}\int_{t+\epsilon}^{t+\epsilon+n_jT}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau\\&=\lim_{j\to\infty}\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau^{n_j}\mathrm{d}\tau\\&=\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi\mathrm{d}\hat{\mu}_\tau\mathrm{d}\tau,\quad\forall t\ge0\text{ and }\epsilon>0.\end{aligned}$$

The T-periodicity of ϕ and $\hat{\mu} = (\hat{\mu}_t)_{t\in\mathbb{R}}$ then ensures that

$$\frac{1}{T}\int_{t+\epsilon}^{t+\epsilon+T}\int_{\mathcal{U}}\phi \mathrm{d}\hat{\mu}_\tau \mathrm{d}\tau = \frac{1}{T}\int_0^T\int_{\mathcal{U}}\phi \mathrm{d}\hat{\mu}_\tau \mathrm{d}\tau, \quad \forall t \geq 0 \text{ and } \epsilon > 0.$$

Hence, (4.21) follows.

Step 7. We show that for any $\psi \in C_c^2(\mathcal{U})$, there holds

$$\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t+kT} = \int_{\mathcal{U}}\psi \mathrm{d}\hat{\mu}_t, \quad \forall t > 0. \tag{4.22}$$

Fix $\psi \in C_c^2(\mathcal{U})$. Clearly, $\mathcal{L}\psi$ is bounded on $\mathcal{U}\times\mathbb{R}$. Since $\mu = (\mu_t)_{t\in(0,\infty)}$ is a global probability solution of (1.3), Lemma 2.1 (1)(a) and Lemma 2.2 imply that

$$\begin{aligned}\left|\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_1} - \int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_2}\right| &\leq \int_{t_1}^{t_2}\int_{\mathcal{U}}|\mathcal{L}\psi| \mathrm{d}\mu_\tau \mathrm{d}\tau \\ &\leq \max_{\mathrm{supp}(\psi)\times\mathbb{R}}|\mathcal{L}\psi| \times (t_2 - t_1), \quad \forall t_1, t_2 > 0 \text{ with } t_1 < t_2.\end{aligned} \tag{4.23}$$

Fix $t_0 \in (0, T)$ and let $\eta \in C_c^\infty(\mathbb{R})$ be non-negative and satisfy $\mathrm{supp}(\eta) \subset [-1, 1]$ and $\int_{\mathbb{R}}\eta \mathrm{d}t = 1$. Define $\eta_\epsilon(t) := \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})$ for $t \in \mathbb{R}$ and $0 < \epsilon \ll 1$. Clearly, $\int_{\mathbb{R}}\eta_\epsilon \mathrm{d}t = 1$ for $0 < \epsilon \ll 1$. It follows that

$$\begin{aligned}&\left|\int_t^{t+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon(\tau - (t+t_0))\mathrm{d}\tau - \int_{\mathcal{U}}\psi \mathrm{d}\mu_{t+t_0}\right| \\ &\quad\leq \int_t^{t+T}\left|\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau - \int_{\mathcal{U}}\psi \mathrm{d}\mu_{t+t_0}\right|\eta_\epsilon(\tau - (t+t_0))\mathrm{d}\tau \\ &\quad\leq \epsilon \times \max_{\mathrm{supp}(\psi)\times\mathbb{R}}|\mathcal{L}\psi|, \quad \forall t > 0 \text{ and } 0 < \epsilon \ll 1,\end{aligned}$$

where we used (4.23). Equivalently,

$$\begin{aligned}&\int_t^{t+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon(\tau - (t+t_0))\mathrm{d}\tau - C\epsilon \\ &\quad\leq \int_{\mathcal{U}}\psi \mathrm{d}\mu_{t+t_0} \\ &\quad\leq \int_t^{t+T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon(\tau - (t+t_0))\mathrm{d}\tau + C\epsilon, \quad \forall t > 0 \text{ and } 0 < \epsilon \ll 1,\end{aligned} \tag{4.24}$$

where $C = \max_{\mathrm{supp}(\psi)\times\mathbb{R}}|\mathcal{L}\psi|$.

Fix $t_1 > 0$. For each fixed n_j, setting $t = t_1 + kT$ for $k = 0, \ldots, n_j - 1$ in (4.24) and then summarizing the resulting inequalities, we arrive at

$$\begin{aligned}&\sum_{k=0}^{n_j-1}\int_{t_1+kT}^{t_1+(k+1)T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon\left(\tau - (t_0 + t_1 + kT)\right)\mathrm{d}\tau - n_jC\epsilon \\ &\quad\leq \sum_{k=0}^{n_j-1}\int_{\mathcal{U}}\psi \mathrm{d}\mu_{t_0+t_1+kT} \\ &\quad\leq \sum_{k=0}^{n_j-1}\int_{t_1+kT}^{t_1+(k+1)T}\left(\int_{\mathcal{U}}\psi \mathrm{d}\mu_\tau\right)\eta_\epsilon(\tau - (t_0 + t_1 + kT))\mathrm{d}\tau + n_jC\epsilon, \quad 0 < \epsilon \ll 1.\end{aligned} \tag{4.25}$$

For each $\epsilon > 0$, we define a function $\tilde{\eta}_\epsilon$ on $\mathbb{R}$ by setting

$$\tilde{\eta}_\epsilon(t) = \eta_\epsilon(t - (t_0 + t_1 + kT)), \quad t \in [t_1 + kT, t_1 + (k+1)T) \text{ and } k \in \mathbb{Z}.$$

Obviously, $\tilde{\eta}_\epsilon$ is T-periodic and $\tilde{\eta}_\epsilon \in C_c^\infty(\mathbb{R})$ for each $0 < \epsilon \ll 1$. Setting $\phi(x,t) := \psi(x)\tilde{\eta}_\epsilon(t)$ for $(x,t) \in \mathcal{U} \times \mathbb{R}$ in (4.21) gives

$$\lim_{j\to\infty} \frac{1}{n_jT} \int_{t_1}^{t_1+n_jT} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau = \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau. \tag{4.26}$$

As

$$\begin{aligned} &\lim_{j\to\infty} \frac{1}{n_jT} \int_{t_1}^{t_1+n_jT} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau \\ &\quad= \lim_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{t_1+kT}^{t_1+(k+1)T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau \\ &\quad= \lim_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{t_1+kT}^{t_1+(k+1)T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau \right) \eta_\epsilon(\tau - (t_0 + t_1 + kT))\mathrm{d}\tau, \end{aligned}$$

we obtain

$$\lim_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{t_1+kT}^{t_1+(k+1)T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\mu_\tau \right) \eta_\epsilon(\tau - (t_0 + t_1 + kT))\mathrm{d}\tau = \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau.$$

Thus, dividing (4.25) by n_jT and then letting $j \to \infty$, we derive from the above limit that

$$\begin{aligned} \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau - \frac{1}{T}C\epsilon &\le \liminf_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_0+t_1+kT} \\ &\le \limsup_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_0+t_1+kT} \\ &\le \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau + \frac{1}{T}C\epsilon, \quad 0 < \epsilon \ll 1. \end{aligned} \tag{4.27}$$

Since the continuity of the function $t \mapsto \int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_t$ on $\mathbb{R}$ implies that

$$\lim_{\epsilon\to0} \frac{1}{T} \int_{t_1}^{t_1+T} \left(\int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_\tau \right) \tilde{\eta}_\epsilon(\tau)\mathrm{d}\tau = \frac{1}{T} \int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_{t_0+t_1},$$

we pass to the limit $\epsilon \to 0$ in (4.27) to find

$$\lim_{j\to\infty} \frac{1}{n_jT} \sum_{k=0}^{n_j-1} \int_{\mathcal{U}} \psi \mathrm{d}\mu_{t_0+t_1+kT} = \frac{1}{T} \int_{\mathcal{U}} \psi \mathrm{d}\hat{\mu}_{t_0+t_1}, \quad \forall t_0 \in (0,T) \text{ and } t_1 > 0.$$

This proves (4.22).

If the periodic probability solution of (1.3) is unique, then it is clear that (4.21) and (4.22) hold for the whole sequence $\{\mu^n\}_{n\in\mathbb{N}}$. This completes the proof. □

We sketch the proof of Corollary B.

Proof of Corollary B. The poof is almost the same as that of Theorem B. The main difference lies in **Step 7**. More precisely, in this situation, we apply Theorem A to find that $\tilde{\mu} = (\tilde{\mu})_{t\in\mathbb{R}}$ is the unique periodic probability solution of (1.3) and admits a continuous density. Therefore, the function $t \mapsto \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t$ is continuous on $\mathbb{R}$ for any $\psi \in C_b(\mathcal{U})$. Hence, the result follows from arguments as in **Step 7**. □

5. Proof of Theorem C

In Subsection 5.1, we construct and study the transition probability densities associated to the global probability solutions of the Cauchy problem (1.3)-(1.11). The proof of Theorem C is given in Subsection 5.2.

Throughout this section, we assume **(H1)** and **(H3)**, and that $\mathcal{L}$ admits an exponentially strong Lyapunov function U. Hence, Theorem 2.3 and Theorem A hold. Moreover, we denote $M_b(\mathcal{U})$ as the collection of all bounded measurable functions on $\mathcal{U}$ and write

$$\langle \mu, \phi \rangle := \int_{\mathcal{U}} \phi \mathrm{d}\mu, \quad \mu \in \mathcal{M}_p(\mathcal{U}) \text{ and } \phi \in M_b(\mathcal{U}),$$

where we recall that $\mathcal{M}_p(\mathcal{U})$ is the set of all Borel probability measures on $\mathcal{U}$.

5.1. Transition probability densities.

For fixed $s \in \mathbb{R}$ and $x \in \mathcal{U}$, let $\mu^{s,x}$ be the unique global probability solution of the Cauchy problem (1.3)-(1.11) with $\nu = \delta_x$ given in Theorem 2.3. Following Theorem 2.1, $\mu^{s,x}$ admits a Hölder continuous density $(y,t) \mapsto p(s,x,t,y)$ on $\mathcal{U} \times (s,\infty)$. We prove some properties of $p(s,x,t,y)$ in the rest of this subsection.

Lemma 5.1. *The following hold.*

1. *For $s,t \in \mathbb{R}$ with $s < t$ and $y \in \mathcal{U}$, the function $x \mapsto p(s,x,t,y)$ is continuous on $\mathcal{U}$.*
2. *For $s \in \mathbb{R}$, the function $(x,t,y) \mapsto p(s,x,t,y)$ is measurable on $\mathcal{U} \times (s,\infty) \times \mathcal{U}$.*

Proof. (1) Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ converge to some $x_* \in \mathcal{U}$ as $n \to \infty$. We show

$$\lim_{n\to\infty} p(s,x_n,t,y) = p(s,x_*,t,y), \quad \forall (y,t) \in \mathcal{U} \times (s,\infty). \tag{5.1}$$

For convenience, we define

$$u_n(y,t) := p(s,x_n,t,y), \quad (y,t) \in \mathcal{U} \times (s,\infty),$$
$$\mathrm{d}\mu^n = \mathrm{d}\mu_t^n \mathrm{d}t := u_n(y,t)\mathrm{d}y\mathrm{d}t.$$

Note that μ^n is nothing but μ^{s,x_n}.

By Theorem 2.1, we see that for any $\mathcal{V} \subset\subset \mathcal{U}$ and $t_1, t_2 \in (s,\infty)$ with $t_1 < t_2$, there exists a $C > 0$, independent of n, such that

$$|u_n|_{C^{\alpha-\frac{1}{p}}\left([t_1,t_2],C^\gamma(\overline{\mathcal{V}})\right)} \leq C, \quad \forall n \in \mathbb{N}.$$

Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is pre-compact under the topology of locally uniform convergence on $\mathcal{U}\times(s,\infty)$ thanks to the Arzelà-Ascoli theorem and the standard diagonal argument. In particular, any subsequence of $\{u_n\}_{n\in\mathbb{N}}$ has a further subsequence that is locally uniformly convergent on $\mathcal{U} \times (s,\infty)$.

Let us fix a subsequence $\{u_{n_j}\}$ that converges locally uniformly to some non-negative continuous function u on $\mathcal{U} \times (s,\infty)$. We show that the Borel measure μ defined by $\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t := u(y,t)\mathrm{d}y\mathrm{d}t$ coincides with μ^{s,x_*}. That is, for any $\phi \in C_c^2(\mathcal{U})$, there holds

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \phi(x_*) + \lim_{r\to s} \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall t > s. \tag{5.2}$$

As μ^{n_j} is the global probability solution of the Cauchy problem (1.3)-(1.11) with $\nu = \delta_{x_{n_j}}$, we apply Lemma 2.1 (1)(a) and Lemma 2.2 to find for any $\phi \in C_c^2(\mathcal{U})$ and $t > r > s$,

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{n_j} = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r^{n_j} + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau,$$

which is rewritten as

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{n_j} - \phi(x_{n_j}) - \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r^{n_j} - \phi(x_{n_j}).$$

Following the proof of [28, Theorem 2.3], we see that for fixed $t_0 > s$ and $\phi \in C_c^2(\mathcal{U})$ there exist $C_1 > 0$ and $\alpha > 0$, independent of n, such that

$$\left| \int_{\mathcal{U}} \phi \mathrm{d}\mu_t^n - \phi(x_n) \right| \leq C_1 |t - s|^\alpha, \quad \forall t \in (s, t_0) \text{ and } n \in \mathbb{N}.$$

Hence, for any $t > s$, there holds

$$\begin{aligned} \left| \int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{n_j} - \phi(x_{n_j}) - \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau^{n_j} \mathrm{d}\tau \right| &\leq \left| \int_{\mathcal{U}} \phi \mathrm{d}\mu_r^{n_j} - \phi(x_{n_j}) \right| \\ &\leq C_1 |r - s|^\alpha, \quad \forall s < r < \min\{t, t_0\} \text{ and } j \in \mathbb{N}. \end{aligned}$$

Letting $j \to \infty$, we find

$$\left| \int_{\mathcal{U}} \phi \mathrm{d}\mu_t - \phi(x_*) - \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau \right| \leq C |r - s|^\alpha, \quad \forall s < r < \min\{t, t_0\}.$$

Letting $r \to s$ in the above inequality yields (5.2).

Since the above result holds for any locally uniformly convergent subsequence of $\{u_n\}_{n \in \mathbb{N}}$, the sequence u_n converges locally uniformly to $p(s, x_*, \cdot, \cdot)$ as $n \to \infty$. In particular, (5.14) follows.

(2) In addition to (1), we know that for each $s \in \mathbb{R}$ and $x \in \mathcal{U}$, the function $(y, t) \mapsto p(s, x, t, y)$ is continuous on $\mathcal{U} \times (s, \infty)$. Hence, the function $(x, t, y) \mapsto p(s, x, t, y)$ is a Carathéodory function on $\mathcal{U} \times (s, \infty) \times \mathcal{U}$ and its measurability follows from [1, Lemma 4.51]. □

Lemma 5.2. *Let $\mu = (\mu_t)_{t \in (s, \infty)}$ be the unique global probability solution of the Cauchy problem (1.3)-(1.11). Then, there holds*

$$\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t = \int_{\mathcal{U}} p(s, x, t, y) \mathrm{d}\nu(x) \mathrm{d}y \mathrm{d}t.$$

In particular, for any $\phi \in M_b(\mathcal{U})$, there holds

$$\langle \mu_t, \phi \rangle = \langle \nu, \langle \mu_t^{s, \bullet}, \phi \rangle \rangle = \int_{\mathcal{U}} \langle \mu_t^{s,x}, \phi \rangle \mathrm{d}\nu(x).$$

Proof. Define

$$\mathrm{d}\tilde{\mu} = \mathrm{d}\tilde{\mu}_t \mathrm{d}t := \int_{\mathcal{U}} p(s, x, t, y) \mathrm{d}\nu(x) \mathrm{d}y \mathrm{d}t \quad \text{on} \quad \mathcal{U} \times (s, \infty).$$

By the definition of $\mu^{s,x}$, there hold for any $\phi \in C_c^2(\mathcal{U})$

$$\lim_{t \to s} \int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} = \phi(x), \quad \text{and} \tag{5.3}$$

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} = \int_{\mathcal{U}} \phi \mathrm{d}\mu_r^{s,x} + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau^{s,x} \mathrm{d}\tau, \quad \forall s < r < t. \tag{5.4}$$

It follows from Lemma 5.1 that each term in (5.4) is measurable with respect to x. Integrating (5.4) with respect to ν and applying Fubini's theorem, we find

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_r + \int_r^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau, \quad \forall s < r < t. \tag{5.5}$$

That is, $\tilde{\mu}$ is a global probability solution of (1.3) in $\mathcal{U} \times (s, \infty)$.

For $\phi \in C_c^2(\mathcal{U})$, we deduce from $\left|\int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x}\right| \leq |\phi|_\infty$, (5.3) and the dominated convergence theorem that

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}} \int_{\mathcal{U}} \phi \mathrm{d}\mu_t^{s,x} \mathrm{d}\nu(x) \to \int_{\mathcal{U}} \phi \mathrm{d}\nu \quad \text{as} \quad t \to s.$$

Hence, $\tilde{\mu}$ is a global probability solution of the Cauchy problem (1.3)-(1.11). The uniqueness result in Theorem 2.3 ensures that $\tilde{\mu} = \mu$.

The "In particular" part follows readily. □

Corollary 5.1. *There holds*

$$p(s, x, t_2, y) = \int_{\mathcal{U}} p(s, x, t_1, z) p(t_1, z, t_2, y) \mathrm{d}z$$

for all $x, y \in \mathcal{U}$ and $t_2 > t_1 > s$.

Proof. Fix $s \in \mathbb{R}$ and $x \in \mathcal{U}$. Lemma 5.2 ensures that the measure

$$\begin{aligned} \mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t &:= \left(\int_{\mathcal{U}} p(t_1, z, t, y) \mathrm{d}\mu_{t_1}^{s,x}(z) \right) \mathrm{d}y \mathrm{d}t \\ &= \left(\int_{\mathcal{U}} p(t_1, z, t, y) p(s, x, t_1, z) \mathrm{d}z \right) \mathrm{d}y \mathrm{d}t \quad \text{on} \quad \mathcal{U} \times (t_1, \infty) \end{aligned}$$

is the unique global probability solution of the Cauchy problem (1.3)-(1.11) (with $s = t_1$ and $\nu = \mu_{t_1}^{s,x}$). So is the measure $\mu^{s,x}$ restricted on $\mathcal{U} \times (t_1, \infty)$. Theorem 2.3 then yields

$$\mu^{s,x} = \mu \quad \text{on} \quad \mathcal{U} \times (t_1, \infty).$$

Hence, they have the same densities, that is,

$$p(s, x, t, y) = \int_{\mathcal{U}} p(s, x, t_1, z) p(t_1, z, t, y) \mathrm{d}z, \quad \forall t > t_1 \text{ and } y \in \mathcal{U}.$$

The corollary follows. □

5.2. **Proof of Theorem C.** We prove two lemmas before proving Theorem C. Recall that U is an exponentially strong Lyapunov function.

The first lemma gives evolutionary estimates of a global probability solution of the Cauchy problem (1.3)-(1.11) against U.

Lemma 5.3. *There are positive constants C_1 and C_2 such that for any global probability solution $\mu = (\mu_t)_{t \in (s,\infty)}$ of the Cauchy problem (1.3)-(1.11) with $\int_{\mathcal{U}} U(\cdot, s) \mathrm{d}\nu < \infty$, there holds*

$$\int_{\mathcal{U}} U(\cdot, t) \mathrm{d}\mu_t \leq e^{-C_1(t-s)} \int_{\mathcal{U}} U(\cdot, s) \mathrm{d}\nu + C_2, \quad \forall t > s.$$

Proof. For notational simplicity, the integrals of the forms $\int_{\mathcal{U}} g(\cdot, t) \mathrm{d}\mu_t$ and $\int_{\mathcal{U}} g(\cdot, s) \mathrm{d}\nu$ are respectively written as $\int_{\mathcal{U}} g \mathrm{d}\mu_t$ and $\int_{\mathcal{U}} g \mathrm{d}\nu$ in the rest of the proof.

By Theorem 2.1, μ admits a density $u \in C(\mathcal{U}\times(s,\infty))$, namely, $\mathrm{d}\mu=\mathrm{d}\mu_t\mathrm{d}t=u(x,t)\mathrm{d}x\mathrm{d}t$. By Lemma 2.1 (1)(b) and Lemma 2.2, there holds for each $\phi\in C_c^{2,1}(\mathcal{U}\times(s,\infty))$

$$\int_{\mathcal{U}}\phi\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi\mathrm{d}\mu_r+\int_r^t\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,\quad \forall t>r>s,$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\phi\mathrm{d}\mu_t=\int_{\mathcal{U}}\mathcal{L}\phi\mathrm{d}\mu_t,\quad \forall t>s. \tag{5.6}$$

As U is an exponentially strong Lyapunov function, there are positive constants C_1, C_2 and ρ_m such that

$$\mathcal{L}U\leq -C_1U+C_2<0\quad\text{in}\quad(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m}.$$

Fix $\rho_0>\rho_m$ and set $N_0=[\rho_0]+1$, where $[\rho_0]$ is the integer part of ρ_0. Let $\{\zeta_N\}_{N\geq N_0}$ be a sequence of smooth and non-decreasing functions on $\mathbb{R}$ satisfying

$$\zeta_N(t)=\begin{cases}0, & t\in[0,\rho_m],\\ t, & t\in[\rho_0,N],\\ N+1, & t\in[N+2,\infty),\end{cases}\qquad \zeta_N(t)\leq t,\ t\in[\rho_m,\rho_0]\ \text{ and }\ \zeta_N''(t)\leq 0,\ t\in[N,N+2].$$

In addition, let the functions $\{\zeta_N\}_{N\geq N_0}$ coincide on $[0,\rho_0]$.

We claim that there exists $\tilde{C}_1>0$ such that

$$\begin{aligned}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t\leq e^{-C_1(t-s)}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\nu+\frac{\tilde{C}_1}{C_1}&\\ +C_1(N+1)\int_s^t\mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau,&\quad\forall t>s\text{ and }N\geq N_0.\end{aligned}\tag{5.7}$$

Note that $\zeta_N(U)-(N+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$. Applying Lemma 2.1 (2)(b) and Lemma 2.2 with $\phi=\zeta_N(U)-(N+1)$, we find

$$\lim_{r\to s}\int_{\mathcal{U}}[\zeta_N(U)-(N+1)]\mathrm{d}\mu_r=\int_{\mathcal{U}}[\zeta_N(U)-(N+1)]\mathrm{d}\nu.$$

It follows from $\mu_r(\mathcal{U})=\nu(\mathcal{U})=1$ for all $r>s$ that

$$\lim_{r\to s}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_r=\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\nu. \tag{5.8}$$

Setting $\phi=\zeta_N(U)-(N+1)$ in (5.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\left[\zeta_N(U)-(N+1)\right]\mathrm{d}\mu_t=\int_{\mathcal{U}}\mathcal{L}\left(\zeta_N(U)-(N+1)\right)\mathrm{d}\mu_t.$$

Since $\mu_t(\mathcal{U})=1$ for all $t>0$ and

$$\mathcal{L}(\zeta_N(U)-(N+1))=\zeta_N'(U)\mathcal{L}U+\zeta_N''(U)a^{ij}\partial_iU\partial_jU,$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\zeta_N(U)\mathrm{d}\mu_t=\int_{\mathcal{U}}\zeta_N'(U)\mathcal{L}U\mathrm{d}\mu_t+\int_{\mathcal{U}}\zeta_N''(U)a^{ij}\partial_iU\partial_jU\mathrm{d}\mu_t,\quad\forall t>s. \tag{5.9}$$

Since $\zeta_N'=0$ on $[0,\rho_m]$, $\zeta_N'(t)=1$ on $[\rho_0,N]$ and $\zeta_N'\geq 0$ otherwise, we deduce that

$$\zeta_N'(U)\mathcal{L}U\leq\begin{cases}-C_1U+C_2, & \text{on}\quad\Omega_N\setminus\Omega_{\rho_0},\\ 0, & \text{otherwise},\end{cases}$$

which implies that

$$\int_{\mathcal{U}} \zeta_N'(U)\mathcal{L}U \mathrm{d}\mu_t \le \int_{\Omega_N^t\setminus\Omega_{\rho_0}^t} (-C_1U + C_2)\mathrm{d}\mu_t. \tag{5.10}$$

Since $\zeta_N'' \neq 0$ on $[\rho_m, \rho_0]$, $\zeta'' \le 0$ on $[N, N+2]$ and $\zeta_N'' = 0$ otherwise, we see from the non-negative definiteness of (a^{ij}) that

$$\zeta_N'' a^{ij}\partial_i U \partial_j U \le \begin{cases} C_* \max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U, & \text{on} \quad \Omega_{\rho_0}\setminus\Omega_{\rho_m}, \\ 0, & \text{otherwise}, \end{cases}$$

where $C_* := \max_{t\in[\rho_m,\rho_0]} \zeta_N''(t)$, which is independent of N due to the coincidence of $\{\zeta_N\}_{N\ge N_0}$ on $[0, \rho_0]$. Thus,

$$\begin{aligned} \int_{\mathcal{U}} \zeta_N''(U) a^{ij}\partial_i U\partial_j U \mathrm{d}\mu_t &\le \int_{\Omega_{\rho_0}^t\setminus\Omega_{\rho_m}^t} \zeta_N''(U) a^{ij}\partial_i U\partial_j U\mathrm{d}\mu_t \\ &\le C_* \max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U. \end{aligned} \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.9) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t + C_1\int_{\Omega_N^t\setminus\Omega_{\rho_0}^t} U\mathrm{d}\mu_t \le C_2 + C_*\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U, \tag{5.12}$$

As $\zeta_N \le \rho_0$ on $[0, \rho_0]$, $\zeta_N(t) = t$ on $[\rho_0, N]$ and $\zeta_N \le N+1$ on $[N, \infty)$, we see that

$$\begin{aligned} \int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t &= \left(\int_{\Omega_{\rho_0}^t} + \int_{\Omega_N^t\setminus\Omega_{\rho_0}^t} + \int_{\mathcal{U}\setminus\Omega_N^t}\right)\zeta_N(U)\mathrm{d}\mu_t \\ &\le \rho_0 + \int_{\Omega_N^t\setminus\Omega_{\rho_0}^t} U\mathrm{d}\mu_t + (N+1)\mu_t(\mathcal{U}\setminus\Omega_N^t). \end{aligned} \tag{5.13}$$

Setting $\tilde{C}_1 := C_1\rho_0 + C_2 + C_*\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U$, we find from (5.12) and (5.13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t + C_1\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t \le \tilde{C}_1 + C_1(N+1)\mu_t(\mathcal{U}\setminus\Omega_N^t), \quad \forall t > s.$$

Applying Gronwall's inequality yields

$$\begin{aligned} \int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t \le e^{-C_1(t-r)}\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_r + \frac{\tilde{C}_1}{C_1} \\ + C_1(N+1)\int_r^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau, \quad \forall t > r > s. \end{aligned}$$

Letting $r \to s$ in the above inequality, we conclude (5.7) from (5.8) and the monotone convergence theorem.

Note that if there holds

$$(N+1)\int_s^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau \to 0 \quad \text{as} \quad N\to\infty, \tag{5.14}$$

then we can pass to the limit $N\to\infty$ in (5.7) to find from $\zeta_N(t) \le t$ for $N \ge N_0$ and $t \ge 0$ that

$$\int_{\mathcal{U}\setminus\Omega_{\rho_0}^t} U\mathrm{d}\mu_t \le e^{-C_1(t-s)}\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu + \frac{\tilde{C}_1}{C_1}, \quad \forall t > s,$$

which readily leads to the lemma.

To finish the proof, we show (5.14). Fix $t > s$. We define

$$f(x,\tau) := \begin{cases} [1+U(x,\tau)]u(x,\tau), & \forall (x,\tau) \in \bigcup_{\tau\in[s,t]} \left((\mathcal{U}\setminus\Omega_{\rho_0})\times\{\tau\}\right), \\ 0, & \text{otherwise,} \end{cases}$$

and for $N \gg 1$

$$f_N(x,\tau) := \begin{cases} (N+1)u(x,\tau)e^{-C_1(t-\tau)}, & \forall (x,\tau) \in \bigcup_{\tau\in[s,t]} \left((\mathcal{U}\setminus\Omega_{\rho_N})\times\{\tau\}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $f_N \le f$ for $N \gg 1$ and $f_N \to 0$ as $N \to \infty$. As

$$\int_s^t \int_{\mathcal{U}\setminus\Omega_{\rho_0}^\tau} f(x,\tau)\mathrm{d}x\mathrm{d}\tau = \int_s^t \int_{\mathcal{U}\setminus\Omega_{\rho_0}^\tau} U\mathrm{d}\mu_\tau\mathrm{d}\tau, \quad \text{and}$$

$$\int_s^t \int_{\mathcal{U}\setminus\Omega_{\rho_0}^\tau} f_N(x,\tau)\mathrm{d}x\mathrm{d}\tau = (N+1)\int_s^t \mu_\tau(\mathcal{U}\setminus\Omega_N^\tau)e^{-C_1(t-\tau)}\mathrm{d}\tau,$$

the limit (5.14) follows from the dominated convergence theorem if there holds

$$\int_s^t \int_{\mathcal{U}\setminus\Omega_{\rho_0}^\tau} U\mathrm{d}\mu_\tau\mathrm{d}\tau < \infty. \tag{5.15}$$

It remains to show (5.15). Set $\tilde{C}_2 := C_2 + C_* \max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U$. For any fixed $r \in (s,t)$, integrating (5.12) over $[r,t]$ gives

$$\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t + C_1\int_r^t \int_{\Omega_N^\tau\setminus\Omega_{\rho_0}^\tau} U\mathrm{d}\mu_\tau\mathrm{d}\tau \le \int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_r + \tilde{C}_2(t-r).$$

Letting $r \to s$ in the above inequality, we deduce from (5.8) that

$$\int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\mu_t + C_1\int_s^t \int_{\Omega_N^\tau\setminus\Omega_{\rho_0}^\tau} U\mathrm{d}\mu_\tau\mathrm{d}\tau \le \int_{\mathcal{U}} \zeta_N(U)\mathrm{d}\nu + \tilde{C}_2(t-s).$$

Since $\zeta_N(t) \le t$ for all $t > 0$ and $N \ge N_0$, and $U(\cdot,s)$ is ν-integrable, passing to the limit $N \to \infty$ in the above inequality yields

$$\int_s^t \int_{\mathcal{U}\setminus\Omega_{\rho_0}^\tau} U\mathrm{d}\mu_\tau\mathrm{d}\tau \le \int_{\mathcal{U}} U\mathrm{d}\nu + \tilde{C}_2(t-s) < \infty.$$

Hence, (5.15) holds. This completes the proof. □

The second lemma is a version of the minorization condition of the measures $\{\mu^{s,x}\}$, where $\mu^{s,x} = (\mu_t^{s,x})_{t\in(0,\infty)}$ is defined at the beginning of Subsection 5.1.

Lemma 5.4. *Let $s,t \in \mathbb{R}$ with $s < t$. For each $R > 0$, there is a constant $\alpha > 0$, such that*

$$\|\mu_t^{s,x_1} - \mu_t^{s,x_2}\|_{TV} \le 2(1-\alpha) \tag{5.16}$$

for all $x_1, x_2 \in \mathcal{U}$ satisfying $U(x_1,s) + U(x_2,s) \le R$, where $\|\cdot\|_{TV}$ denotes the total variation norm.

Proof. Fix $s,t \in \mathbb{R}$ with $s < t$ and $R > 0$. Note that

$$\{(x_1,x_2) \in \mathcal{U}\times\mathcal{U} : U(x_1,s) + U(x_2,s) \le R\} \subset \overline{\Omega}_R^s \times \overline{\Omega}_R^s,$$

where we recall that $\Omega_\rho^\tau := \{x \in \mathcal{U} : U(x,\tau) < \rho\}$ for $\tau \in \mathbb{R}$ and $\overline{\Omega}_R^\tau$ denotes the closure of Ω_R^τ.

We first claim that there exist positive constants ρ_1 and M such that

$$\inf_{y\in\Omega^t_{\rho_1}} p(s,x,t,y)\geq M,\quad \forall x\in\overline{\Omega}^s_R. \tag{5.17}$$

For $x\in\overline{\Omega}^s_R$, we denote $\mu=(\mu_t)_{t\in(s,\infty)}:=\mu^{s,x}$ and $u(y,t):=p(s,x,t,y)$ for $y\in\mathcal{U}$ for notational simplicity. Applying Lemma 5.3, we find

$$\int_{\mathcal{U}} U(y,\tau)u(y,\tau)\mathrm{d}y=\int_{\mathcal{U}} U(\cdot,\tau)\mathrm{d}\mu_\tau\leq e^{-C_1(\tau-s)}U(x,s)+C_2,\quad \forall \tau>s. \tag{5.18}$$

Set $\Delta:=\frac{t-s}{4}$. Integrating (5.18) with respect to τ over $[s+\Delta,s+2\Delta]$ gives

$$\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}} U(y,\tau)u(t,\tau)\mathrm{d}y\mathrm{d}\tau\leq \frac{U(x,s)}{C_1}\left(e^{-C_1\Delta}-e^{-2C_1\Delta}\right)+C_2\Delta.$$

Setting

$$C_3:=\frac{1}{C_1}\left(e^{-C_1\Delta}-e^{-2C_1\Delta}\right)\max_{x\in\overline{\Omega}^s_R}U(x,s)+C_2\Delta,$$

we arrive at

$$\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_\rho} U(y,\tau)u(t,\tau)\mathrm{d}y\mathrm{d}\tau\leq C_3,\quad \forall\rho\geq\min_{\mathcal{U}\times\mathbb{R}}U. \tag{5.19}$$

As U satisfies (1.4), there holds

$$\lim_{\rho\to\infty}\inf_{(\mathcal{U}\times\mathbb{R})\setminus\Omega_\rho}U=\infty.$$

This together with (5.19) yields the existence of some $\rho_1>\min_{\mathcal{U}\times\mathbb{R}}U$ such that

$$\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_1}} u(y,\tau)\mathrm{d}y\mathrm{d}\tau\leq\frac{\Delta}{2},$$

which implies that

$$\frac{\Delta}{2}\leq\Delta-\int_{s+\Delta}^{s+2\Delta}\int_{\mathcal{U}\setminus\Omega^\tau_{\rho_1}} u(y,\tau)\mathrm{d}y\mathrm{d}\tau=\int_{s+\Delta}^{s+2\Delta}\int_{\Omega^\tau_{\rho_1}} u(y,\tau)\mathrm{d}y\mathrm{d}\tau\leq|Q^1_{\rho_1}|\sup_{Q^1_{\rho_1}}u,$$

where $Q^1_{\rho_1}:=\cup_{\tau\in[s+\Delta,s+2\Delta]}\left(\Omega^\tau_{\rho_1}\times\{\tau\}\right)$. Applying Harnack's inequality (e.g., [27, Theorem 10.1]) to u, there exists a $C>0$, independent of u, such that

$$\frac{\Delta}{2|Q^1_{\rho_1}|}\leq\sup_{Q^1_{\rho_1}}u\leq C\inf_{Q^2_{\rho_1}}u\leq C\inf_{y\in\Omega^t_{\rho_1}}u(y,t), \tag{5.20}$$

where $Q^2_{\rho_1}:=\cup_{\tau\in[s+3\Delta,t]}\left(\Omega^\tau_{\rho_1}\times\{\tau\}\right)$. Setting $M:=\frac{\Delta}{2C|Q^1_{\rho_1}|}$, we find (5.17) follows from (5.20).

Now, we prove the lemma. For $x_1,x_2\in\overline{\Omega}^s_R$, we denote $\mu^i=(\mu^i_t)_{t\in(0,\infty)}:=\mu^{s,x_i}$ and $u_i(y,t):=p(s,x_i,t,y)$ for $y\in\mathcal{U}$ and $i=1,2$. We find from (5.17) that for $i,j=1,2$ with $i\neq j$ there holds

$$\begin{aligned}\int_{\Omega^t_{\rho_1}}(u_i(y,t)-u_j(y,t))^+\mathrm{d}y&\leq\int_{\Omega^t_{\rho_1}}(u_i(y,t)-M)\mathbb{1}_{\{u_i\geq u_j\}}(y,t)\mathrm{d}y\\&\leq\mu^i_t(\Omega^t_{\rho_1})-M\int_{\Omega^t_{\rho_1}}\mathbb{1}_{\{u_i\geq u_j\}}(y,t)\mathrm{d}y.\end{aligned}$$

As a result, there holds for $i,\ j=1,2$ with $i\neq j$,

$$\begin{aligned}\int_{\mathcal{U}}(u_i(y,t)-u_j(y,t))^+\mathrm{d}y&=\int_{\mathcal{U}\setminus\Omega^t_{\rho_1}}(u_i(y,t)-u_j(y,t))^+\mathrm{d}y+\int_{\Omega^t_{\rho_1}}(u_i(y,t)-u_j(y,t))^+\mathrm{d}y\\&\leq \mu^i_t(\mathcal{U}\setminus\Omega^t_{\rho_1})+\mu^i_t(\Omega^t_{\rho_1})-M\int_{\Omega^t_{\rho_1}}\mathbb{1}_{\{u_i\geq u_j\}}(y,t)\mathrm{d}y,\\&=1-M\int_{\Omega^t_{\rho_1}}\mathbb{1}_{\{u_i\geq u_j\}}(y,t)\mathrm{d}y.\end{aligned}$$

As $|u_1-u_2|=(u_1-u_2)^++(u_2-u_1)^+$, we arrive at

$$\int_{\mathcal{U}}|u_1(y,t)-u_2(y,t)|\,\mathrm{d}y\leq 2-M|\Omega^t_{\rho_1}|.$$

It follows that

$$\begin{aligned}\|\mu^1_t-\mu^2_t\|_{TV}&=\sup_{A\in\mathcal{B}(\mathbb{R}^d)}|\mu^1_t(A)-\mu^2_t(A)|\\&\leq\sup_{A\in\mathcal{B}(\mathbb{R}^d)}\int_A|u_1(x,t)-u_2(x,t)|\mathrm{d}x\\&\leq 2\left(1-\frac{1}{2}M|\Omega^t_{\rho_1}|\right).\end{aligned}$$

As $0<M|\Omega^t_{\rho_1}|\leq\int_{\Omega^t_{\rho_1}}u_1(y,t)\mathrm{d}x\leq 1$, we find $\alpha:=\frac{1}{2}M|\Omega^t_{\rho_1}|\in(0,1)$. This completes the proof. □

We are ready to prove Theorem C.

Proof of Theorem C. We assume, without loss of generality, that $s=0$. For each $x\in\mathcal{U}$, we denote $\mu^x=(\mu^x_t)_{t\in(0,\infty)}:=\mu^{0,x}$, where we recall $\mu^{0,x}$ is the unique global probability solution of the Cauchy problem (1.3)-(1.11) with $s=0$ and $\nu=\delta_x$. Then $\mathrm{d}\mu^x:=\mathrm{d}\mu^x_t\mathrm{d}t=p(0,x,t,y)\mathrm{d}y\mathrm{d}t$. The proof is divided into four steps.

Step 1. We show that there exists a unique measure $\mu_*\in\mathcal{M}_p(\mathcal{U})$ and positive constants C and $\varrho\in(0,1)$ such that

$$\|\mu^x_{nT}-\mu_*\|_{TV}\leq C\varrho^n(1+U(x,0)),\quad\forall x\in\mathcal{U}\text{ and }n\in\mathbb{N}.\tag{5.21}$$

Let $\mathcal{P}:M_b(\mathcal{U})\to M_b(\mathcal{U})$ be defined by

$$\mathcal{P}\phi(x):=\langle\mu^x_T,\phi\rangle=\int_{\mathcal{U}}p(0,x,T,y)\phi(y)\mathrm{d}y,\quad\forall x\in\mathcal{U}\text{ and }\phi\in M_b(\mathcal{U}).$$

By Lemma 5.1, $\mathcal{P}$ is well-defined. In particular, $\mathcal{P}\mathbf{1}=\mathbf{1}$, where $\mathbf{1}\equiv 1$. Let $\mathcal{P}^*:\mathcal{M}_p(\mathcal{U})\to\mathcal{M}_p(\mathcal{U})$ be the adjoint operator of $\mathcal{P}$ defined by

$$\langle\mathcal{P}^*\mu,\phi\rangle=\langle\mu,\mathcal{P}\phi\rangle,\quad\forall\mu\in\mathcal{M}_p(\mathcal{U})\text{ and }\phi\in M_b(\mathcal{U}).$$

Since a^{ij} and V^i are T-periodic for $i,j=1,\dots,d$, we find from Theorem 2.3 that

$$p(nT,x,(n+1)T,y)=p(0,x,T,y),\quad\forall n\in\mathbb{N}\text{ and }x,\ y\in\mathcal{U},$$

which implies that

$$\mathcal{P}\phi(x)=\int_{\mathcal{U}}p(nT,x,(n+1)T,y)\phi(y)\mathrm{d}y,\quad x\in\mathcal{U}.$$

It follows from Corollary 5.1 that

$$\mathcal{P}^n\phi(x) = \langle \mu^x_{nT}, \phi\rangle, \quad x \in \mathcal{U}. \tag{5.22}$$

Define a wighted supremum norm:

$$\|\phi\|_* := \operatorname*{ess\,sup}_{x\in\mathcal{U}} \left|\frac{\phi(x)}{1+U(x,0)}\right|, \quad \forall \phi \in M_b(\mathcal{U}).$$

Thanks to Lemma 5.3 and Lemma 5.4, we apply Harris's Theorem (see e.g. [17, Theorem 3.6]) to find that $\mathcal{P}$ admits a unique invariant measure μ_* of $\mathcal{P}$, namely, $\mathcal{P}^*\mu_* = \mu_*$, and there exist constants $C > 0$ and $\varrho \in (0,1)$ such that

$$\|\langle \mu^\bullet_{nT}, \phi\rangle - \langle \mu_*, \phi\rangle\|_* = \|\mathcal{P}^n\phi - \langle \mu_*, \phi\rangle\|_* \le C\varrho^n \|\phi - \langle \mu_*, \phi\rangle\|_* \le 2C\varrho^n\|\phi\|_\infty$$

holds for all $\phi \in M_b(\mathcal{U})$, where we used (5.22) in the equality. Consequently,

$$\begin{aligned}
\|\mu^x_{nT} - \mu_*\|_{TV} &= \sup_{|\phi|\le 1} |\langle \mu^x_{nT}, \phi\rangle - \langle \mu_*, \phi\rangle| \\
&= \sup_{|\phi|\le 1} \frac{|\langle \mu^x_{nT}, \phi\rangle - \langle \mu_*, \phi\rangle|}{1+U(x,0)} [1+U(x,0)] \\
&\le \sup_{|\phi|\le 1} \|\langle \mu^\bullet_{nT}, \phi\rangle - \langle \mu_*, \phi\rangle\|_* [1+U(x,0)] \\
&\le 2C\varrho^n [1+U(x,0)].
\end{aligned}$$

This proves (5.21).

Step 2. We show that (1.3) admits a unique periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$. Moreover, there holds $\tilde{\mu}_{nT} = \mu_*$ for all $n \in \mathbb{N}$.

Denote $\mathrm{d}\tilde{\mu} := \mathrm{d}\tilde{\mu}_t\mathrm{d}t$ as the unique global probability solution of the Cauchy problem (1.3)-(1.11) with $s = 0$ and $\nu = \mu_*$. It follows from Lemma 5.2 that

$$\langle \tilde{\mu}_t, \phi\rangle = \langle \mu_*, \langle \mu^\bullet_t, \phi\rangle\rangle, \quad \forall t > 0 \text{ and } \phi \in M_b(\mathcal{U}),$$

which together with (5.22) and the fact that μ_* is invariant under $\mathcal{P}$ implies that

$$\langle \tilde{\mu}_{nT}, \phi\rangle = \langle \mu_*, \langle \mu^\bullet_{nT}, \phi\rangle\rangle = \langle \mu_*, \mathcal{P}^n\phi\rangle = \langle \mathcal{P}^{*n}\mu_*, \phi\rangle = \langle \mu_*, \phi\rangle, \quad \forall n \in \mathbb{N} \text{ and } \phi \in M_b(\mathcal{U}).$$

That is, $\tilde{\mu}_{nT} = \mu_*$ for all $n \in \mathbb{N}$. Therefore, $\tilde{\mu}_{t+T} = \tilde{\mu}_t$ for all $t > 0$. We then extend $\tilde{\mu}$ to $\mathcal{U} \times (-\infty, 0)$ by defining

$$\tilde{\mu}_t := \tilde{\mu}_{t+nT}, \quad t \in (-nT, (n-1)T] \text{ and } n \in \mathbb{N}.$$

It is not hard to check that $\tilde{\mu} := (\tilde{\mu}_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (1.3).

The uniqueness follows from Theorem A.

Step 3. We prove that there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\|\mu^x_t - \tilde{\mu}_t\|_{TV} \le \tilde{C}_1 e^{-\tilde{C}_2 t} [1+U(x,0)], \quad \forall x \in \mathcal{U} \text{ and } t > 0. \tag{5.23}$$

For $t > 0$, there are unique $n_t \in \mathbb{N}_0$ and $0 \le r_t < T$ such that $t = n_tT + r_t$. For $\phi \in M_b(\mathcal{U})$, we denote

$$\phi_{r_t}(x) := \langle \mu^x_{r_t}, \phi\rangle, \quad x \in \mathcal{U}.$$

Clearly, $\|\phi_{r_t}\| \le \|\phi\|_\infty$ if ϕ is bounded. As $p(0,x,r,y) = p(nT, x, r+nT, y)$ for all $n \in \mathbb{N}$, $x, y \in \mathcal{U}$ and $r > 0$, we find

$$\phi_{r_t}(x) := \int_{\mathcal{U}} p(n_tT, x, t, y)\phi(y)\mathrm{d}y, \quad x \in \mathcal{U}.$$

It follows from Lemma 5.2 and Corollary 5.1 that for each $\phi \in M_b(\mathcal{U})$ there hold

$$\langle \tilde{\mu}_t, \phi \rangle = \langle \tilde{\mu}_{r_t}, \phi \rangle = \left\langle \mu_*, \langle \mu^{\bullet}_{r_t}, \phi \rangle \right\rangle = \langle \mu_*, \phi_{r_t} \rangle$$

and

$$\langle \mu^x_t, \phi \rangle = \int_{\mathcal{U}} p(0,x,nT,z) \left[\int_{\mathcal{U}} p(nT,z,t,y)\phi(y)\mathrm{d}y \right] \mathrm{d}z = \mathcal{P}^n \phi_{r_t}(x) = \langle \mu^x_{nT}, \phi_{r_t} \rangle, \quad \forall x \in \mathcal{U}.$$

Consequently, we derive from (5.21) that

$$\begin{aligned} \|\mu^x_t - \tilde{\mu}_t\|_{TV} &= \sup_{|\phi|\le 1} |\langle \mu^x_t, \phi \rangle - \langle \tilde{\mu}_t, \phi \rangle| \\ &= \sup_{|\phi|\le 1} |\langle \mu^x_{nT}, \phi_{r_t} \rangle - \langle \mu_*, \phi_{r_t} \rangle| \\ &\le C\varrho^n \left[1 + U(x,0)\right] \\ &\le \tilde{C}_1 e^{-\tilde{C}_2 t} \left[1 + U(x,0)\right], \end{aligned}$$

where $\tilde{C}_1 = C\varrho^{-1}$ and $\tilde{C}_2 = \frac{1}{T}\ln\varrho$.

Step 4. We show that

$$\|\mu_t - \tilde{\mu}_t\|_{TV} \le \tilde{C}_1 e^{-\tilde{C}_2 t} \int_{\mathcal{U}} \left[1 + U(\cdot,0)\right] \mathrm{d}\nu, \quad \forall t > 0.$$

We apply Lemma 5.2 to find

$$\langle \mu_t, \phi \rangle = \langle \nu, \langle \mu^{\bullet}_t, \phi \rangle \rangle = \int_{\mathcal{U}} \langle \mu^x_t, \phi \rangle \mathrm{d}\nu(x), \quad \forall \phi \in M_b(\mathcal{U}).$$

It follows from (5.23) that

$$\begin{aligned} \|\mu_t - \tilde{\mu}_t\|_{TV} &= \sup_{|\phi|\le 1} |\langle \mu_t, \phi \rangle - \langle \tilde{\mu}_t, \phi \rangle| \\ &= \sup_{|\phi|\le 1} \left| \int_{\mathcal{U}} \left[\langle \mu^x_t, \phi \rangle - \langle \tilde{\mu}_t, \phi \rangle\right] \mathrm{d}\nu(x) \right| \\ &\le \int_{\mathcal{U}} \sup_{|\phi|\le 1} |\langle \mu^x_t, \phi \rangle - \langle \tilde{\mu}_t, \phi \rangle| \, \mathrm{d}\nu(x) \\ &\le \tilde{C}_1 e^{-\tilde{C}_2 t} \int_{\mathcal{U}} \left[1 + U(\cdot,0)\right] \mathrm{d}\nu. \end{aligned}$$

This completes the proof. □

6. Applications

In this section, we discuss some applications of Theorem B, Corollary B and Theorem C. Applications to stochastic damping Hamiltonian systems and stochastic slow-fast systems are discussed respectively in Subsection 6.1 and Subsection 6.2. In Subsection 6.3, we investigate the convergence of weak solutions of a SDE with less regular coefficients.

6.1. **Stochastic damping Hamiltonian systems.** Consider the following stochastic damping Hamiltonian system:

$$\begin{cases} \dot{x} = y, \\ \mathrm{d}y = -\left[b(x,y)y + \nabla V(x,t)\right]\mathrm{d}t + F(x,y,t)\mathrm{d}t + \sigma(x,y,t)\mathrm{d}W_t, \end{cases} \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{6.1}$$

where the damping $b = (b^{ij}) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ is continuous, the potential $V : \mathbb{R}^d \times \mathbb{R} \mapsto (0, \infty)$ is twice continuously differentiable in x and continuously differentiable in t, the external force $F : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d$ is continuous, the noise coefficient matrix $\sigma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^{d\times m}$ belongs to $C(\mathbb{R}, W^{1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^d))$, where $p > d+2$ and $m \geq d$ are fixed, and $(W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process. We assume V, F and σ are all T-periodic in t for some $T > 0$.

The FPE associated to (6.1) reads

$$\partial_t u = \partial^2_{y_i y_j}(a^{ij}u) - \partial_{x_i}(y_i u) + \partial_{y_i}\left((b^{ij}y_j + \partial_{x_i}V)u\right) - \partial_{y_i}(F^i u), \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \tag{6.2}$$

where $(a^{ij}) := \frac{\sigma\sigma^\top}{2}$ is the diffusion matrix. Denote

$$\mathcal{L}_H := \partial_t + a^{ij}\partial^2_{y_i y_j} + y^i\partial_{x_i} - (b^{ij}y_j + \partial_{x_i}V)\partial_{y_i} + F^i\partial_{y_i}.$$

We make the following additional assumptions on the coefficients.

- **(A1)** There is $b_0 > 0$ such that $b^{ij}y_i y_j \geq b_0|y|^2$ for all $y \in \mathbb{R}^d$.
- **(A2)** The functions F and σ, and $\partial_t V$ are uniformly bounded on $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$ and $\mathbb{R}^d\times\mathbb{R}$, respectively.
- **(A3)** There exists a lower bounded function $\Phi \in C^2(\mathbb{R}^d)$ such that

$$\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} \sum_{i,j=1}^d \left| -b^{ji}(x,y)\frac{x_j}{|x|} + \partial_{x_i}\Phi(x) \right| < \infty.$$

- **(A4)** $\nabla_x V \cdot \frac{x}{|x|} \to \infty$ as $|x| \to \infty$.

Note that **(A1)** says that the system (6.1) is damped. When $b(x,y)$ is bounded, the function Φ in **(A3)** can be taken to be 0.

Following the arguments as in the proof of [22, Theorem 5.1], we can construct a strong Lyapunov function with respect to $\mathcal{L}_H$. Hence, Theorem B is applied to give the following result.

Theorem 6.1. *Assume* **(A1)**-**(A4)**. *Let $\mu = (\mu_t)_{t\in(s,\infty)}$ be a global probability solution of the Cauchy problem associated to (6.2) with initial condition $\mu_s = \nu$, where $\nu \in \mathcal{M}_p(\mathbb{R}^d \times \mathbb{R}^d)$ is compactly supported. Then for any sequence of positive integers $\{n_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} n_j = \infty$, there exists a subsequence, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, and a periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ of (6.2) such that*

(1) *for each bounded $\phi \in C_T(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$, there holds*

$$\lim_{j\to\infty} \frac{1}{n_j T}\int_t^{t+n_jT}\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi\,\mathrm{d}\mu_\tau\mathrm{d}\tau = \frac{1}{T}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi\,\mathrm{d}\tilde{\mu}_\tau\mathrm{d}\tau, \quad \forall t \geq s,$$

(2) *for each $\psi \in C^2_c(\mathbb{R}^d \times \mathbb{R}^d)$, there holds*

$$\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi\,\mathrm{d}\mu_{t+kT} = \int_{\mathbb{R}^d\times\mathbb{R}^d}\psi\,\mathrm{d}\tilde{\mu}_t, \quad \text{for a.e. } t \in (s, s+T].$$

We point out that the uniqueness of periodic probability solutions of (6.2) (with non-smooth coefficients) remains an interesting open question.

6.2. **Stochastic slow-fast systems.** Consider the following SDE

$$\begin{cases}\epsilon\dot{x} = f(x,y,t),\\ \mathrm{d}y = g(x,y,t)\mathrm{d}t + \sigma(x,y,t)\mathrm{d}W_t,\end{cases} \qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n, \tag{6.3}$$

where $0<\epsilon\ll 1$, $f=(f^k):\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}^m$, $g=(g^i):\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}^n$, $\sigma=(\sigma^{ij}):\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}^{n\times\ell}$ is the *noise coefficient matrix* with $\ell\geq n$, and $W=(W_t)_{t\in\mathbb{R}}$ is a standard ℓ-dimensional Wiener process. We assume f, g and σ are T-periodic in t for some $T>0$.

As here we are only interested in the dynamics of (6.3) for each fixed $0<\epsilon\ll 1$, we set $\epsilon=1$ in (6.3) and consider the following system for clarity.

$$\begin{cases}\dot{x} = f(x,y,t),\\ \mathrm{d}y = g(x,y,t)\mathrm{d}t + \sigma(x,y,t)\mathrm{d}W_t,\end{cases} \qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n.$$

The associated FPE reads

$$\partial_t u = \partial^2_{y_iy_j}(a^{ij}u) - \partial_{x_k}(f^k u) - \partial_{y_j}(g^i u), \quad (x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}, \tag{6.4}$$

where $A:=(a^{ij})=\frac{1}{2}\sigma\sigma^\top$. Denote

$$\mathcal{L}_{SF} := \partial_t + a^{ij}\partial_{y_iy_j} + f^k\partial_{x_k} + g^i\partial_{y_i}.$$

We make the following assumptions on the coefficients.

(B1) Let $p>m+n+2$. $A(x,y,t)$ is positive definite for each $(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$, and $a^{ij}\in C_T(\mathbb{R},W^{1,p}_{loc}(\mathbb{R}^m\times\mathbb{R}^n))$ and $g^i\in C_T(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for each $i,j=1,\dots,n$. Moreover, for each $a>0$, there holds

$$\sup_{\{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|\leq a\}}|A|<\infty,$$

(B2) There exists some positive T-periodic function $U\in C^{1,1}(\mathbb{R}^m\times\mathbb{R})$ satisfying

$$\lim_{|x|\to\infty}U(x,t)=\infty, \quad \forall t\in\mathbb{R}$$

such that

$$\sup_{|y|\leq a}\sup_{t\in\mathbb{R}}\mathcal{L}_{SF}U\to-\infty \quad \text{as} \quad |x|\to\infty$$

for each $a>0$, and

$$\mathcal{L}_{SF}U=0 \quad \text{on} \quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|x|=0\},$$
$$\mathcal{L}_{SF}U<0 \quad \text{on} \quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|x|\neq 0\}.$$

To proceed, we need some dissipative conditions along the slow direction, namely, the y-direction.

Definition 6.1. Let $V\in C^{2,1}_T(\mathbb{R}^n\times\mathbb{R})$ be non-negative and satisfy

$$\lim_{|y|\to\infty}\inf_{t\in\mathbb{R}}V(y,t)=\infty.$$

It is called

(1) a *semi-Lyapunov function* (with respect to $\mathcal{L}_{SF}$) if there exist positive constants γ and a such that

$$\mathcal{L}_{SF}V\leq-\gamma \quad \text{on} \quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|>a\}, \tag{6.5}$$

(2) a *strong semi-Lyapunov function* (with respect to $\mathcal{L}_{SF}$) if

$$\lim_{|y|\to\infty}\sup_{(x,t)\in\mathbb{R}^m\times\mathbb{R}}\mathcal{L}_{SF}V(x,y,t)=-\infty.$$

Theorem 6.2. *Assume* **(B1)** *and* **(B2)***. If $\mathcal{L}_{SF}$ admits a semi-Lyapunov function, then there exists a unique periodic probability solution $\mu=(\mu_t)_{t\in\mathbb{R}}$ of (6.4). Moreover,* $\mathrm{supp}(\mu)=\{0_m\}\times\mathbb{R}^n\times\mathbb{R}$*, where 0_m denotes the origin in $\mathbb{R}^m$.*

Proof. We write $\mathcal{L}_{SF}$ as $\mathcal{L}$ for notational simplicity. The proof is divided into three steps.

Step 1. We show that (6.4) admits a periodic probability solution $\mu=(\mu_t)_{t\in\mathbb{R}}$.

Let V be a semi-Lyapunov function with respect to $\mathcal{L}$ and $\gamma,a>0$ be constants such that (6.5) holds. Define

$$W(x,y,t):=U(x,t)+V(y,t),\quad (x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}.$$

Obviously, W is non-negative and T-periodic, and satisfies

$$\inf_{t\in\mathbb{R}}W(x,y,t)\to\infty\quad\text{as}\quad |x|+|y|\to\infty,$$

and

$$\mathcal{L}W=\mathcal{L}U+\mathcal{L}V\le-\gamma\quad\text{on}\quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|>a\}.$$

Moreover, it follows from **(B1)** that $\mathcal{L}V$ is bounded on $\{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|\le a\}$ and from **(B2)** that

$$\lim_{|x|\to\infty}\sup_{|y|\le a}\sup_{t\in\mathbb{R}}\mathcal{L}U=-\infty.$$

Hence, there is a constant $b>0$ such that

$$\mathcal{L}W=\mathcal{L}U+\mathcal{L}V\le-\gamma\quad\text{on}\quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|\le a,\ |x|>b\}.$$

As a result, we find

$$\mathcal{L}W\le-\gamma\quad\text{on}\quad \{(x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}:|y|>a\text{ or }|x|>b\}.$$

That is, W is a Lyapunov function. Hence, we apply [22, Theorem B] to find a periodic probability solution $\mu=(\mu_t)_{t\in\mathbb{R}}$ of (6.4).

By virtue of Lemma 4.2, we may, assume without loss of generality, that for any $\phi\in C_c^{2,1}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$, the function $t\mapsto\iint_{\mathbb{R}^m\times\mathbb{R}^n}\phi(\cdot,t)\mathrm{d}\mu_t$ is continuous on $\mathbb{R}$.

Step 2. We show that μ is supported on $\{0_m\}\times\mathbb{R}^n\times\mathbb{R}$. By Lemma 2.1 (1)(b) and Lemma 2.2, there holds for any T-periodic $\phi\in C_c^{2,1}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$

$$\begin{aligned}\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau&=\iint_{\mathbb{R}^m\times\mathbb{R}^n}\phi(x,y,t+T)\mathrm{d}\mu_{t+T}-\iint_{\mathbb{R}^m\times\mathbb{R}^n}\phi(x,y,t)\mathrm{d}\mu_t\\&=0,\quad\forall t\in\mathbb{R}.\end{aligned}\tag{6.6}$$

For each $\alpha>1$, define

$$W_\alpha(x,y,t):=\alpha U(x,t)+V(y,t),\quad (x,y,t)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}.$$

Obviously, $W_\alpha(x,y,t)\to\infty$ as $|x|+|y|\to\infty$, for each $\alpha>1$.

Let $\{\zeta_\rho\}_{\rho>0}$ be a family of smooth and non-decreasing functions on $\mathbb{R}$ satisfying

$$\zeta_\rho(t)=\begin{cases}t, & t\in[0,\rho],\\ \rho+1, & t\in[\rho+2,\infty),\end{cases}\qquad\text{and}\qquad \zeta_\rho''\le0\text{ on }[\rho,\rho+2].$$

Clearly, for each $\rho>0$, the function $\zeta_\rho(W_\alpha)-(\rho+1)$ is T-periodic and belongs to $C_c^{2,1}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$. Setting $\phi:=\zeta_\rho(W_\alpha)-(\rho+1)$ in (6.6), we find from

$$\begin{aligned}\mathcal{L}\zeta_\rho(W_\alpha)&=\zeta_\rho'(W_\alpha)\mathcal{L}W_\alpha+\zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}W_\alpha\partial_{y_j}W_\alpha\\&=\zeta_\rho'(W_\alpha)(\alpha\mathcal{L}U+\mathcal{L}V)+\zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V\end{aligned}$$

that

$$\begin{aligned}0&=\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\mathcal{L}\zeta_\rho(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau\\&=\alpha\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\zeta_\rho'(W_\alpha)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau+\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\zeta_\rho'(W_\alpha)\mathcal{L}V\mathrm{d}\mu_\tau\mathrm{d}\tau\\&\quad+\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V\mathrm{d}\mu_\tau\mathrm{d}\tau.\end{aligned}\tag{6.7}$$

As $\zeta_\rho'\geq 0$ on $[0,\infty)$, we see from (6.5) that

$$\zeta_\rho'(W_\alpha)\mathcal{L}V\leq\begin{cases}\max_{\Omega\times\mathbb{R}}|\mathcal{L}U|\zeta_\rho'(W_\alpha), & (x,y,t)\in\Omega\times\mathbb{R},\\-\gamma\zeta_\rho'(W_\alpha), & (x,y,t)\in\Omega^c\times\mathbb{R},\end{cases}\tag{6.8}$$

where $\Omega:=\{(x,y):|y|\leq a\}$, $\Omega^c:=\{(x,y):|y|>a\}$ and $a>0$ is such that (6.5) holds. Since $\zeta_\rho''\leq 0$ on $[\rho,\rho+2]$ and $\zeta_\rho''=0$ otherwise, we derive from the non-negative definiteness of $A=(a^{ij})$ that

$$\zeta_\rho''(W_\alpha)a^{ij}\partial_{y_i}V\partial_{y_j}V\leq 0\quad\text{on}\quad\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}.\tag{6.9}$$

Substituting (6.8) and (6.9) into (6.7) gives

$$\begin{aligned}&-\alpha\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\zeta_\rho'(W_\alpha)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau+\gamma\int_t^{t+T}\iint_{\Omega^c}\zeta_\rho'(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau\\&\quad\leq\max_{\Omega\times\mathbb{R}}|\mathcal{L}V|\int_t^{t+T}\iint_{\Omega}\zeta_\rho'(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau.\end{aligned}$$

In particular,

$$\begin{aligned}-\alpha\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\zeta_\rho'(W_\alpha)\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau&\leq\max_{\Omega\times\mathbb{R}}|\mathcal{L}V|\int_t^{t+T}\iint_{\Omega_\tau}\zeta_\rho'(W_\alpha)\mathrm{d}\mu_\tau\mathrm{d}\tau\\&\leq\max_{\Omega\times\mathbb{R}}|\mathcal{L}V|\times T.\end{aligned}\tag{6.10}$$

Note that $\lim_{\rho\to\infty}\zeta_\rho(t)=t$, which yields $\lim_{\rho\to\infty}\zeta_\rho'(W_\alpha)=1$. Letting $\rho\to\infty$ in (6.10), we find

$$-\alpha\int_t^{t+T}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\mathcal{L}U\mathrm{d}\mu_\tau\mathrm{d}\tau\leq\max_{\Omega\times\mathbb{R}}|\mathcal{L}V|\times T.\tag{6.11}$$

To see $\mathrm{supp}(\mu)\subset\{0_m\}\times\mathbb{R}^n\times\mathbb{R}$, we suppose on the contrary that there exists an closed set $B\subset\mathbb{R}^m$ satisfying $0_m\notin B$ such that

$$\int_t^{t+T}\mu_\tau(\{(x,y):x\in B\})\mathrm{d}\tau>0,\quad\forall t\in\mathbb{R}.$$

Note that $\sup_{B\times\mathbb{R}}\mathcal{L}U<0$ by **(B2)**. Hence,

$$-\alpha\left(\sup_{B\times\mathbb{R}}\mathcal{L}U\right)\int_t^{t+T}\mu_\tau(\{(x,y):x\in B\})\mathrm{d}\tau\leq\max_{\Omega\times\mathbb{R}}|\mathcal{L}V|\times T,$$

which leads to a contradiction when letting $\alpha\to\infty$.

Step 3. We claim that $\mathrm{supp}(\mu) = \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$ and $\mu = (\mu_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (6.4).

Define

$$\mu_t^*(B) := \mu_t(\{0_m\} \times B), \quad \forall B \in \mathcal{B}(\mathbb{R}^n) \text{ and } t \in \mathbb{R}, \qquad \text{and} \qquad \mu^* := (\mu_t^*)_{t\in\mathbb{R}},$$

where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra of $\mathbb{R}^n$. We further define

$$\mathcal{L}_0 := \partial_t + \alpha^{ij}\partial^2_{y_iy_j} + \beta^i\partial_{y_i},$$

where $\alpha^{ij}(y,t) = a^{ij}(0_m, y, t)$ and $\beta^i(y,t) := g^i(0_m, y, t)$ for $(y,t) \in \mathbb{R}^n \times \mathbb{R}$ and $i, j = 1, \dots, n$.

As $\mu = (\mu_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (6.4) and is supported on $\{0_m\} \times \mathbb{R}^n \times \mathbb{R}$, we see that $\mu_t^*(\mathbb{R}^n) = 1$ and $\mu_t^* = \mu_{t+T}^*$ for $t \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathcal{L}_0\phi \mathrm{d}\mu_\tau^* \mathrm{d}\tau = 0, \quad \forall \phi \in C_0^{2,1}(\mathbb{R}^n \times \mathbb{R}).$$

That is, $\mu^* = (\mu_t^*)_{t\in\mathbb{R}}$ is a periodic probability solution of the following FPE

$$\partial_t u = \partial^2_{y_iy_j}(\alpha^{ij}u) - \partial_{y_i}(\beta^i u), \quad (y,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{6.12}$$

By Theorem 2.1, μ^* admits a positive density on $\mathbb{R}^n \times \mathbb{R}$. Hence, $\mathrm{supp}(\mu^*) = \mathbb{R}^n \times \mathbb{R}$, or equivalently, $\mathrm{supp}(\mu) = \{0_m\} \times \mathbb{R}^n \times \mathbb{R}$. Note that V is an Lyapunov function with respect to $\mathcal{L}_0$. Hence, we apply Theorem A to conclude that (6.4) as well as (6.12) admits a unique periodic probability solution. □

When the semi-Lyapunov function in Theorem 6.2 is indeed a strong semi-Lyapunov function, we are able to apply Theorem B to deduce a convergence result.

Theorem 6.3. *Assume* **(B1)**, **(B2)** *and that $\mathcal{L}_{SF}$ admits a strong semi-Lyapunov function. Then, for any global probability solution $\mu = (\mu_t)_{t\in(s,\infty)}$ of the Cauchy problem associated to (6.4) with initial condition $\mu_s = \nu$, where $\nu \in \mathcal{M}_p(\mathbb{R}^m \times \mathbb{R}^n)$ is compactly supported, there holds*

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \iint_{\mathbb{R}^m\times\mathbb{R}^n} \phi \mathrm{d}\mu_{t+kT} = \iint_{\mathbb{R}^m\times\mathbb{R}^n} \phi \mathrm{d}\tilde{\mu}_t, \quad \forall \phi \in C_c^2(\mathbb{R}^m \times \mathbb{R}^n) \text{ and } t \in (s, s+T],$$

where $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ is the unique periodic probability solution of (6.4)

Proof. Let $V \in C_T^{2,1}(\mathbb{R}^n \times \mathbb{R})$ be the strong semi-Lyapunov function with respect to $\mathcal{L}_{SF}$. Arguing as in the proof of Theorem 6.2, we show that the function

$$W(x,y,t) := U(x,t) + V(y,t), \quad \forall (x,y,t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$$

is a strong Lyapunov function with respect to $\mathcal{L}_{SF}$. The conclusion follows from Theorem B and Theorem 6.2. □

6.3. **Convergence of weak solutions of a SDE.** Fix $s \in \mathbb{R}$. Consider the following initial value problem associated to the SDE (1.2):

$$\begin{cases} \mathrm{d}x = V(x,t)\mathrm{d}t + G(x,t)\mathrm{d}W_t, & x \in \mathcal{U}, \\ x_s \sim \nu, \end{cases} \tag{6.13}$$

where ν is a given Borel probability measure on $\mathcal{U}$. We assume V and G are continuous on $\mathcal{U} \times \mathbb{R}$ and T-periodic in t for some $T > 0$.

Recall that a *(globally defined) weak solution* of (6.13) is a triple of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq s}, \mathbb{P})$, an adapted Wiener process $(W_t)_{t\geq s}$ and an adapted stochastic process $(X_t)_{t\geq s}$ such that

$$X_s \sim \nu, \quad X_t = X_s + \int_s^t V(X_\tau, \tau)\mathrm{d}\tau + \int_s^t G(X_\tau, \tau)\mathrm{d}W_\tau, \quad \forall t > s.$$

In the sequel, we simply call $(X_t)_{t\geq s}$ a weak solution of (6.13) without mentioning the underlying probability space and Wiener process.

Let $(a^{ij}) = \frac{1}{2}GG^\top$ and set

$$\mathcal{L} := \partial_t + a^{ij}\partial^2_{ij} + V^i\partial_i.$$

Lemma 6.1. *Let $(X_t)_{t\geq s}$ be a weak solution of (6.13) and μ_t be the distribution of X_t for $t \geq s$. Then, $(\mu_t)_{t\in(s,\infty)}$ is a global probability solution of the Cauchy problem (1.3)-(1.11).*

Proof. It is well-known [24] that under the current assumptions on the coefficients, $(X_t)_{t\geq s}$ induces a solution of the associated martingale problem. Hence, for each $\phi \in C^2_c(\mathcal{U})$, there holds

$$\mathbb{E}\phi(X_t) - \mathbb{E}\phi(X_s) - \int_s^t \mathbb{E}\left[\mathcal{L}\phi(X_\tau)\right]\mathrm{d}\tau = 0, \quad \forall t > s,$$

that is,

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t - \int_{\mathcal{U}} \phi \mathrm{d}\nu - \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau = 0, \quad \forall t > s.$$

The conclusion then follows from Lemma 2.1 (b)(1). □

In the presence of Lemma 6.1, we can apply Theorem B and Theorem C to derive the following convergence results of weak solutions.

Theorem 6.4. *Suppose $\mathcal{L}$ admits a strong Lyapunov function U. Let $(X_t)_{t\geq s}$ be a weak solution of (6.13) with $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$. Then for any sequence of positive integers $\{n_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} n_j = \infty$, there exists a subsequence, still denoted by $\{n_j\}_{j\in\mathbb{N}}$, and a periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ of (1.3) such that*

1. *for each bounded $\phi \in C_T(\mathcal{U}\times\mathbb{R})$, there holds*

$$\lim_{j\to\infty} \frac{1}{n_j T}\int_t^{t+n_j T} \mathbb{E}\phi(X_\tau)\mathrm{d}\tau = \frac{1}{T}\int_0^T \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau, \quad \forall t \geq s, \tag{6.14}$$

2. *for each $\psi \in C^2_c(\mathcal{U})$, there holds*

$$\lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1} \mathbb{E}\psi(X_{t+kT}) = \int_{\mathcal{U}} \psi \mathrm{d}\tilde{\mu}_t, \quad \textit{for a.e. } t > s. \tag{6.15}$$

In particular, if (1.3) admits a unique periodic probability solution $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$, then (6.14) and (6.15) hold for the whole sequence $\mathbb{N}$.

Theorem 6.5. *Assume $GG^\top$ is locally Lipschitz continuous in x and pointwise positive definite. Suppose $\mathcal{L}$ admits an exponentially strong Lyapunov function U. Then, there exist positive constants C_1 and C_2 such that any weak solution $(X_t)_{t\geq s}$ of (6.13) with $\int_{\mathcal{U}} U(\cdot, s)\mathrm{d}\nu < \infty$ satisfies*

$$\left|\mathbb{E}\phi(X_t) - \int_{\mathcal{U}} \phi \mathrm{d}\tilde{\mu}_t\right| \leq C_1 e^{-C_2(t-s)}, \quad \forall t > s$$

for any bounded measurable function ϕ on $\mathcal{U}$, where $\tilde{\mu} = (\tilde{\mu}_t)_{t\in\mathbb{R}}$ is the unique probability solution of (1.3).

Appendix A. Proof of an inequality

In this appendix, we prove the inequality (2.3). Let $\rho_1, \rho_2 \in C(\mathcal{U}\times(s,\infty))$ be respectively a global probability solution and a global sub-probability solution of the Cauchy problem (1.3)-(1.11). Define $w := \frac{\rho_2}{\rho_1}$ and $f_\lambda(t) := e^{\lambda(1-t)} - e^{\lambda}$ for $t \geq 0$, where $\lambda > 0$ is a parameter. Then for any non-negative function $\phi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, there holds

$$\int_{\mathcal{U}} f_\lambda(w)\phi \mathrm{d}\mu_t^1 \leq f_\lambda(1)\int_{\mathcal{U}}\phi \mathrm{d}\nu + \int_s^t\int_{\mathcal{U}} f(w)\mathcal{L}\phi \mathrm{d}\mu_\tau^1 \mathrm{d}\tau, \quad \forall t > s. \tag{A.1}$$

Note that the above inequality is just (2.3).

The rest of this appendix is devoted to the proof of (A.1).

Proof of (A.1). Define

$$\eta(x) = \begin{cases} c_d e^{-\frac{1}{1-|x|^2}}, & \text{if} \quad |x| \leq 1,\\ 0, & \text{if} \quad |x| > 1,\end{cases}$$

where $c_d > 0$ is such that $\int_{\mathbb{R}^d}\eta dx = 1$. It is well-known that $\eta \in C_c^\infty(\mathbb{R}^d)$. Let $\eta_\epsilon(x) := \frac{1}{\epsilon^d}\eta(\frac{x}{\epsilon})$ for $x \in \mathbb{R}^d$ and $0 < \epsilon \ll 1$.

For a measurable function $g : \mathcal{U}\times(s,\infty) \to \mathbb{R}$, we define

$$g_\epsilon(x,t) := \int_{\{y\in\mathbb{R}^d : x-y\in\mathcal{U}\}} g(x-y,t)\eta_\epsilon(y)\mathrm{d}y, \quad (x,t) \in \mathcal{U}\times(s,\infty).$$

In particular, for each $i = 1, 2$,

$$\begin{aligned}\rho_{i,\epsilon}(x,t) &= \int_{\{y\in\mathbb{R}^d : x-y\in\mathcal{U}\}} \rho_i(x-y,t)\eta_\epsilon(y)\mathrm{d}y\\ &= \int_{\mathcal{U}}\rho_i(y,t)\eta_\epsilon(x-y)\mathrm{d}y, \quad (x,t) \in \mathcal{U}\times(s,\infty).\end{aligned}$$

It is not hard to check that

$$\lim_{\epsilon\to 0}\rho_{i,\epsilon} = \rho_i \quad \text{locally uniformly in} \quad \mathcal{U}\times(s,\infty), \tag{A.2}$$

and that for each $0 < \epsilon \ll 1$,

$$\lim_{t\to s}\rho_{i,\epsilon}(x,t) = \nu_\epsilon(x), \quad x \in \mathcal{U}, \tag{A.3}$$

where

$$\nu_\epsilon(x) := \int_{\mathcal{U}}\eta_\epsilon(x-y)\mathrm{d}\nu(y), \quad x \in \mathcal{U}.$$

Note that for each $0 < \epsilon \ll 1$ and $i = 1, 2$, there holds $\rho_{i,\epsilon}(\cdot,t) \leq |\eta_\epsilon|_\infty$ on $\mathcal{U}$ for each $t \in (s,\infty)$, which together with (A.3) and the dominated convergence theorem implies that

$$\lim_{t\to s^+}\rho_{i,\epsilon}(\cdot,t) = \nu_\epsilon \quad \text{in} \quad L^1(\mathcal{U}). \tag{A.4}$$

It is straightforward to check that for each $i = 1, 2$, $\rho_{i,\epsilon}$ satisfies

$$\partial_t\rho_{i,\epsilon} = \partial_{kl}(a^{kl}\rho_{i,\epsilon}) - \partial_k((V^k\rho_i)_\epsilon - R^k_{\rho_i,\epsilon}),$$

where $R^k_{\rho_i,\epsilon} := \partial_l(a^{kl}\rho_i)_\epsilon - \partial_l(a^{kl}\rho_{i,\epsilon})$. Set $w_\epsilon := \frac{\rho_{2,\epsilon}}{\rho_{1,\epsilon}}$. Multiplying by $\phi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$ the equation satisfied by $\rho_{2,\epsilon}$ and integrating by parts, we arrive at

$$\int_{t_1}^{t_2}\int_{\mathcal{U}}\partial_t(w_\epsilon\rho_{1,\epsilon})\phi \mathrm{d}x\mathrm{d}\tau = \int_{t_1}^{t_2}\int_{\mathcal{U}}\left[w_\epsilon\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + \left((V^k\rho_2)_\epsilon - R^k_{\rho_2,\epsilon}\right)\partial_k\phi\right]\mathrm{d}x\mathrm{d}\tau, \quad \forall t_2 > t_1 > s.$$

Setting $\phi = f'_\lambda(w_\epsilon)\psi$ in the above equality, where $\psi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$ is non-negative, we find

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathcal{U}} &\partial_t(w_\epsilon\rho_{1,\epsilon})f'_\lambda(w_\epsilon)\psi \mathrm{d}x\mathrm{d}\tau\\ &=\int_{t_1}^{t_2}\int_{\mathcal{U}}\left[w_\epsilon\rho_{1,\epsilon}a^{kl}\partial_{kl}(f'_\lambda(w_\epsilon)\psi)+\left((V^k\rho_2)_\epsilon - R^k_{\rho_{2,\epsilon}}\right)\partial_k(f'_\lambda(w_\epsilon)\psi)\right]\mathrm{d}x\mathrm{d}\tau,\quad \forall t_2>t_1>s.\end{aligned}\tag{A.5}$$

Note that there holds the equality

$$\partial_t(w_\epsilon\rho_{1,\epsilon})f'_\lambda(w_\epsilon)=\partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon})-(f_\lambda(w_\epsilon)-f'_\lambda(w_\epsilon)w_\epsilon)\,\partial_t\rho_{1,\epsilon}.$$

Inserting the above equality into the left-hand side of (A.5) and then utilizing the equation satisfied by $\rho_{1,\epsilon}$, the following equality follows from straightforward calculations.

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathcal{U}}\partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon})\psi\mathrm{d}x\mathrm{d}\tau &+ \int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f''_\lambda(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\ &=\int_{t_1}^{t_2}\int_{\mathcal{U}}\left[f_\lambda(w_\epsilon)\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_{t_1}^{t_2}\int_{\mathcal{U}}(W_\epsilon\cdot\nabla\psi)f'_\lambda(w_\epsilon)\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_{t_1}^{t_2}\int_{\mathcal{U}}(W_\epsilon\cdot\nabla w_\epsilon)f''_\lambda(w_\epsilon)\psi\mathrm{d}x\mathrm{d}\tau\\ &\quad-\int_{t_1}^{t_2}\int_{\mathcal{U}}f'_\lambda(w_\epsilon)R^k_{\rho_{2,\epsilon}}\partial_k\psi\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_{t_1}^{t_2}\int_{\mathcal{U}}(f_\lambda(w_\epsilon)-w_\epsilon f'_\lambda(w_\epsilon))R^k_{\rho_{1,\epsilon}}\partial_k\psi\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_{t_1}^{t_2}\int_{\mathcal{U}}f''_\lambda(w_\epsilon)(R^k_{\rho_{2,\epsilon}}-w_\epsilon R^k_{\rho_{1,\epsilon}})\partial_k w_\epsilon\psi\mathrm{d}x\mathrm{d}\tau\\ &=\int_{t_1}^{t_2}\int_{\mathcal{U}}\left[f_\lambda(w_\epsilon)\rho_{1,\epsilon}a^{kl}\partial_{kl}\phi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau+\sum_{j=1}^5 I_j,\end{aligned}\tag{A.6}$$

where $W_\epsilon := (V\rho_2)_\epsilon - (V\rho_1)_\epsilon w_\epsilon$.

We estimate the terms I_j, $j=1,\dots,5$. Note that $f'_\lambda(x) = -\lambda e^{\lambda(1-x)}$ and $f''_\lambda(x)=\lambda^2 e^{\lambda(1-x)}$. Obviously,

$$|I_1|\le \lambda e^\lambda \sup_{\mathcal{U}\times[s,t_2]}|\nabla\psi|\int_s^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))}|W_\epsilon|e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau.$$

By Young's inequality, there holds

$$\begin{aligned}|I_2| &\le \delta\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f''_\lambda(w_\epsilon)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau + \frac{1}{4\delta}\sup_{\mathcal{U}\times[s,t_2]}|\psi|\int_{t_1}^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))}\frac{|W_\epsilon|^2}{\rho_{1,\epsilon}}f''_\lambda(w_\epsilon)\mathrm{d}x\mathrm{d}\tau\\ &\le \delta\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f''_\lambda(w_\epsilon)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau + \frac{\lambda^2}{4\delta}e^\lambda\sup_{\mathcal{U}\times[s,t_2]}|\psi|\int_{s}^{t_2}\int_{\mathrm{supp}(\psi(\cdot,\tau))}\frac{|W_\epsilon|^2}{\rho_{1,\epsilon}}e^{-\lambda w_\epsilon}\mathrm{d}x\mathrm{d}\tau,\quad \delta>0.\end{aligned}$$

For I_3 and I_4, we have

$$|I_3| \leq \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |R_{\rho_2,\epsilon}| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau,$$
$$|I_4| \leq \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |R_{\rho_1,\epsilon}|(1+\lambda w_\epsilon) e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau,$$

where $R_{\rho_i,\epsilon} = (R^k_{\rho_i,\epsilon})$, $i=1,2$. For I_5, we find from Young's inequality and $f''_\lambda(x) = \lambda^2 e^{\lambda(1-x)}$ that

$$\begin{aligned}|I_5| \leq\ & 2\delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}} |w_\epsilon|^2 e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau, \quad \delta > 0,\end{aligned}$$

It follows from (A.6) that

$$\begin{aligned}&\int_{t_1}^{t_2} \int_{\mathcal{U}} \partial_t (f_\lambda(w_\epsilon)\rho_{1,\epsilon}) \psi \mathrm{d}x\mathrm{d}\tau + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \psi f''_\lambda(w_\epsilon) a^{kl} \partial_k w_\epsilon \partial_l w_\epsilon \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &\quad \leq \int_{t_1}^{t_2} \int_{\mathcal{U}} [f_\lambda(w_\epsilon)\rho_{1,\epsilon} a^{kl} \partial_{kl}\psi + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon \partial_k \psi] \mathrm{d}x\mathrm{d}\tau \\ &\quad\quad + 3\delta \int_{t_1}^{t_2} \int_{\mathcal{U}} \psi f''_\lambda(w_\epsilon) |\nabla w_\epsilon|^2 \rho_{1,\epsilon} \mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon,\delta), \quad \forall t_2 > t_1 > s,\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}\Omega(\epsilon,\delta) =\ & \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} |W_\epsilon| e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|W_\epsilon|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \lambda e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\nabla\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \left[|R_{\rho_2,\epsilon}| + |R_{\rho_1,\epsilon}|(1+\lambda w_\epsilon)\right] e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_2,\epsilon}|^2}{\rho_{1,\epsilon}} e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau \\ &+ \frac{\lambda^2}{4\delta} e^{\lambda} \sup_{\mathcal{U}\times[s,t_2]} |\psi| \int_s^{t_2} \int_{\mathrm{supp}(\psi(\cdot,\tau))} \frac{|R_{\rho_1,\epsilon}|^2}{\rho_{1,\epsilon}} |w_\epsilon|^2 e^{-\lambda w_\epsilon} \mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Arguing as in the proof of [34, Lemma 3.1 and Lemma 3.2], we find

$$\lim_{\epsilon\to 0} \Omega(\epsilon,\delta) = 0, \quad \forall \delta > 0.$$

It follows from the Newton-Leibniz formula that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathcal{U}}\partial_t(f_\lambda(w_\epsilon)\rho_{1,\epsilon})\psi\mathrm{d}x\mathrm{d}\tau=&\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x\\&-\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x\\&-\int_{t_1}^{t_2}\int_{\mathcal{U}}f_\lambda(w_\epsilon)\rho_{1,\epsilon}\partial_t\psi\mathrm{d}x\mathrm{d}\tau.\end{aligned}\tag{A.8}$$

As $\psi\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, when restricted on $\mathcal{U}\times[s,t_2]$, is compactly supported, and (a^{ij}) is locally uniform positive definite, there is a positive number m such that

$$\left(a^{ij}\partial_i w_\epsilon\partial_j w_\epsilon\right)(x,t)\geq m|\nabla w_\epsilon(x,t)|^2,\quad\forall(x,t)\in\mathrm{supp}(\psi)\cap(\mathcal{U}\times[s,t_2]).$$

This together with $f_\lambda''\geq 0$ and $\psi\geq 0$ yields that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f_\lambda''(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\geq&\frac{3}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f_\lambda''(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\&+\frac{m}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f_\lambda''(w_\epsilon)|\nabla w_\epsilon|^2\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau.\end{aligned}\tag{A.9}$$

Set $\delta=\frac{m}{12}$. We find from (A.7), (A.8) and (A.9) that

$$\begin{aligned}&\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x\\&\quad\leq\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x-\frac{3}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}}\psi f_\lambda''(w_\epsilon)a^{kl}\partial_k w_\epsilon\partial_l w_\epsilon\rho_{1,\epsilon}\mathrm{d}x\mathrm{d}\tau\\&\qquad+\int_{t_1}^{t_2}\int_{\mathcal{U}}\left[\rho_{1,\epsilon}f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\frac{m}{12})\\&\quad\leq\int_{\mathcal{U}}f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x\\&\qquad+\int_{t_1}^{t_2}\int_{\mathcal{U}}\left[\rho_{1,\epsilon}f_\lambda(w_\epsilon)(\partial_t\psi+a^{kl}\partial_{kl}\psi)+f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau+\Omega(\epsilon,\frac{m}{12}).\end{aligned}\tag{A.10}$$

Since $|f_\lambda(t)-f_\lambda(1)|\leq\lambda e^\lambda|t-1|$ holds for all $t\geq 0$, we apply the dominated convergence theorem to find for each $0<\epsilon\ll 1$,

$$\begin{aligned}&\int_{\mathcal{U}}|f_\lambda(w_\epsilon(x,t_1))-f_\lambda(1)|\rho_{1,\epsilon}(x,t_1)\psi(x,t_1)\mathrm{d}x\\&\quad\leq\lambda e^\lambda\int_{\mathcal{U}}|\rho_{2,\epsilon}(x,t_1)-\rho_{1,\epsilon}(x,t_1)|\psi(x,t_1)\mathrm{d}x\\&\quad\to 0\quad\text{as}\quad t_1\to s^+,\end{aligned}$$

By (A.4) and the dominated convergence theorem, we deduce

$$\begin{aligned}
&\int_{\mathcal{U}} |\rho_{1,\epsilon}(x,t_1)\psi(x,t_1) - \nu_\epsilon(x)\psi(x,s)|\mathrm{d}x \\
&\quad \leq \int_{\mathcal{U}} |\rho_{1,\epsilon}(x,t_1) - \nu_\epsilon(x)|\psi(x,t_1)\mathrm{d}x + \int_{\mathcal{U}} \nu_\epsilon(x)|\psi(x,t_1) - \psi(x,s)|\mathrm{d}x \\
&\quad \leq \max_{\mathcal{U}\times[s,t_2]} |\psi| \cdot \|\rho_{1,\epsilon}(\cdot,t_1) - \nu_\epsilon(\cdot)\|_{L^1(\mathcal{U})} + \int_{\mathcal{U}} \nu_\epsilon(x)|\psi(x,t_1) - \psi(x,s)|\mathrm{d}x \\
&\quad \to 0 \quad \text{as} \quad t_1 \to s^+.
\end{aligned}$$

Thus,

$$\int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_1))\rho_{1,\epsilon}(x,t)\psi(x,t_1)\mathrm{d}x \to f_\lambda(1)\int_{\mathcal{U}} \nu_\epsilon(x)\psi(x,s)\mathrm{d}x \quad \text{as} \quad t_1 \to s.$$

Note that $\rho_{1,\epsilon} f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi) + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi$ is integrable on $\mathcal{U}\times[s,t_2)$. It follows that

$$\begin{aligned}
&\lim_{t_1\to s^+} \int_{t_1}^{t_2}\int_{\mathcal{U}} \rho_{1,\epsilon}\left[f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi) + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau \\
&\quad = \int_s^{t_2}\int_{\mathcal{U}} \left[\rho_{1,\epsilon} f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi) + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Passing to the limit $t_1 \to s^+$ in the inequality (A.10) yields

$$\begin{aligned}
&\int_{\mathcal{U}} f_\lambda(w_\epsilon(x,t_2))\rho_{1,\epsilon}(x,t_2)\psi(x,t_2)\mathrm{d}x \\
&\quad \leq f_\lambda(1)\int_{\mathcal{U}} (\nu*\eta_\epsilon)\psi\mathrm{d}x + \int_s^{t_2}\int_{\mathcal{U}} \left[f_\lambda(w_\epsilon)\rho_{1,\epsilon} f_\lambda(w_\epsilon)(\partial_t\psi + a^{kl}\partial_{kl}\psi)\right. \\
&\qquad \left. + f_\lambda(w_\epsilon)(V^k\rho_1)_\epsilon\partial_k\psi\right]\mathrm{d}x\mathrm{d}\tau + \Omega(\epsilon,\frac{m}{12}), \quad \forall t_2 > s.
\end{aligned}$$

As $\lim_{\epsilon\to 0}\Omega(\epsilon,\frac{m}{12}) = 0$, we let $\epsilon \to 0$ in the above inequality to find from (A.2), (A.3) and the dominated convergence theorem that (A.1), with t_2 and ψ replaced by t and ϕ, respectively, holds. This completes the proof. □

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