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Existence of periodic probability solutions to Fokker-Planck equations with applications ☆

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A B S T R A C T

In the present paper, we consider a Fokker-Planck equation associated to periodic stochastic differential equations with irregular coefficients. We define periodic probability solutions to be periodic analogs of stationary measures for stationary Fokker-Planck equations, and study their existence in both non-degenerate and degenerate cases. In the non-degenerate case, a Lyapunov condition is imposed to ensure the existence of periodic probability solutions to the Fokker-Planck equation with Sobolev coefficients. In the degenerate case with slightly more regular coefficients, the existence is established under the same Lyapunov condition. As applications of our results, we construct periodic probability solutions to Fokker-Planck

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equations associated to stochastic damping Hamiltonian systems and stochastic differential inclusions.

Contents

1. Introduction

Ordinary differential equations (ODEs) of the form

$$\dot{x} = V(x,t), \quad x \in \mathcal{U}, \tag{1.1}$$

where $\mathcal{U} \subset \mathbb{R}^d$ is open and connected, and the vector field V is T-periodic in its second variable, have been widely used to model many processes arising in biology, chemistry, climate, engineering, finance, physics, etc. As real processes are subject to noises and are often lack of mechanisms, the ODE model (1.1) can hardly capture the entire dynamics of these processes. To incorporate such uncertainties into the model (1.1), we consider stochastic perturbations to (1.1) resulting in the following stochastic differential equation (SDE):

$$\mathrm{d}x = V(x,t)dt + G(x,t)\mathrm{d}W_t, \quad x \in \mathcal{U}, \tag{1.2}$$

where the noise intensity $G : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^{d\times m}$ is T-periodic in its second variable and $W := (W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process. We assume $m \geq d$.

One of the central problems concerning the SDE (1.2) is the long-time dynamics of solutions. This problem has been extensively studied when $V(x,t) = V(x)$ and $G(x,t) = G(x)$ are independent of t. Khasminskii [28] initiated the study of the existence and uniqueness of invariant measures and the convergence of solutions to invariant measures when $V(x)$ and $G(x)$ are sufficiently regular. These results are largely generalized and improved in later works (see e.g. [29,41,37,1] and references therein). In this case, the theory of random dynamical system [2] has been developed and applied to study the dynamics with the focus on random attractors, random invariant manifolds, etc.

In modeling complex fluid flows (see e.g. [39]), situations with rough $V(x)$ and $G(x)$ are often the case, and gives rise to challenging mathematical problems. The well-posedness of such equations (with time-dependent coefficients) have attracted a lot of attention in recent years (see e.g. [34,21,44,19] and references therein). Concerning the global dynamics, a large amount of literature has been carried out towards the understanding of the existence and uniqueness of stationary measures of the associated Fokker-Planck equation (or Kolmogorov forward equation) (see e.g. [8,12,6,9,11,7,23] and references therein). The convergence of solutions to Fokker-Planck equations to stationary measures is studied in [26]. We point out that invariant measures of an SDE are necessarily stationary measures of the associated Fokker-Planck equation, while the converse is also true under additional mild conditions (see e.g. [42]).

When $V(x,t)$ and $G(x,t)$ are T-periodic in t, the roles played by invariant measures or stationary measures in the time-independent case are replaced by their periodic analogs, called *periodic solutions*. There exist only a few results on periodic solutions (with different definitions) under rather different conditions. Khasminskii [29] defined periodic solutions in the sense of periodic Markov process and proved the existence under a periodic Lyapunov condition. Chen, Han, Li and Yang [15] studied the existence of classical periodic solutions to the Fokker-Planck equation associated to (1.2) assuming the existence of an unusual Lyapunov function. The coefficients in [29,15] are assumed to be locally Lipschitz. Under the same assumptions on the coefficients, the existence of periodic solutions to semilinear SDEs has been established (see [38,24,16,14] and references therein). Zhao and Zheng [45], and Feng, Zhao and Zhou [20] studied the existence of the so-called random periodic solutions to (1.2) in the framework of random dynamical systems. As random periodic solutions are trajectory based, the study of them does not require the global dissipativity of the system, and therefore, they are in general not expected to control the global dynamics.

The purpose of the present paper is to study periodic solutions to (1.2) with irregular (in particular, non-Lipschitz) coefficients in the sense of distribution. As transition probabilities associated to solutions of (1.2) can hardly be defined in this case, we consider the following Fokker-Planck equation associated to (1.2):

$$\mathcal{L}^* u := -\partial_t u + \partial_{ij}^2 (a^{ij} u) - \partial_i (V^i u) = 0, \quad (x,t) \in \mathcal{U} \times \mathbb{R}, \tag{1.3}$$

where the diffusion matrix $(a^{ij}) := \frac{1}{2} G G^\top$ is pointwise semi-positive definite on $\mathcal{U} \times \mathbb{R}$ and T-periodic in its second variable, the drift field (V^i) is T-periodic in its second variable, $\partial_i = \partial_{x_i}$, $\partial_{ij}^2 = \partial_{x_i x_j}^2$, for all $i, j \in \{1, \ldots, d\}$, and the usual summation convention is used. It is known that the distribution of solutions to (1.2) is governed by (1.3) at least when the coefficients are sufficiently regular, say, locally Lipschitz continuous. Therefore,

Table 1
Notations.

$C_c(\mathcal{U})$	The space of compactly supported continuous functions on $\mathcal{U}$
$C_c^2(\mathcal{U})$	$C_c(\mathcal{U}) \cap C^2(\mathcal{U})$
$C_0(\mathcal{U}\times\mathbb{R})$	The space of compactly supported continuous functions on $\mathcal{U}\times\mathbb{R}$
$C_c(\mathcal{U}\times\mathbb{R})$	The space of continuous functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $u(t,\cdot)\in C_c(\mathcal{U})$ for each $t\in\mathbb{R}$
$C_T(\mathcal{U}\times\mathbb{R})$	The space of T-periodic and continuous functions on $\mathcal{U}\times\mathbb{R}$
$C^{2,1}(\mathcal{U}\times\mathbb{R})$	The space of continuous functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $\partial_i u$, $\partial_{ij}^2 u$ $\partial_t u$ are continuous on $\mathcal{U}\times\mathbb{R}$ for all $i,\, j\in\{1,\dots,d\}$
$C_0^{2,1}(\mathcal{U}\times\mathbb{R})$	$C^{2,1}(\mathcal{U}\times\mathbb{R})\cap C_0(\mathcal{U}\times\mathbb{R})$
$C_c^{2,1}(\mathcal{U}\times\mathbb{R})$	$C^{2,1}(\mathcal{U}\times\mathbb{R})\cap C_c(\mathcal{U}\times\mathbb{R})$
$C_T^{2,1}(\mathcal{U}\times\mathbb{R})$	$C^{2,1}(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$
$C^{\alpha}(\mathbb{R},C^{\gamma}(\mathcal{U}))$	The space of continuous functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $u(t,\cdot)\in C^{\gamma}(\mathcal{U})$ for each $t\in\mathbb{R}$ and the function $t\mapsto \lvert u(t,\cdot)\rvert_{C^{\gamma}(\overline{\Omega})}$ lies in $C^{\alpha}(\mathbb{R})$ for each bounded domain $\Omega\subset\mathcal{U}$
$C_T(\mathbb{R},W^{1,p}_{loc}(\mathcal{U}))$	The space of T-periodic functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $u(t,\cdot)\in W^{1,p}_{loc}(\mathcal{U})$ for each $t\in\mathbb{R}$ and the function $t\mapsto \lVert u(t,\cdot)\rVert_{W^{1,p}(\Omega)}$ is continuous for each bounded domain $\Omega\subset\mathcal{U}$
$L^{\infty}_{loc}(\mathbb{R},W^{1,p}_{loc}(\mathcal{U}))$	The space of measurable functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $u(t,\cdot)\in W^{1,p}_{loc}(\mathcal{U})$ for a.e. $t\in\mathbb{R}$ and the function $t\mapsto \lVert u(t,\cdot)\rVert_{W^{1,p}(\Omega)}$ is locally essentially bounded for each bounded domain $\Omega\subset\mathcal{U}$

the distribution of periodic solutions to (1.2) correspond to periodic solutions to (1.3). While the converse is expected to be true under additional mild assumptions as in the time-independent case mentioned earlier, it remains an interesting open question. As (1.3) with irregular coefficients does not admit classical solutions in general, and we are mainly interested in the distribution of solutions to (1.2) if exist, we look for periodic solutions to (1.3) in the space of Borel probability measures on $\mathcal{U}$. This allows us to deal with much worse coefficients.

From now on, we begin to use some function spaces, which, except the usual ones, are collected in Table 1 in Section 1. For convenience, we denote by

$$\mathcal{L} := \partial_t + a^{ij}\partial^2_{ij} + V^i\partial_i$$

the formal L^2-adjoint of $\mathcal{L}^*$. Motivated by the definition of stationary measures to (1.3) when $V(x,t)=V(x)$ and $G(x,t)=G(x)$ (see e.g. [8]), and measure solutions to (1.3) (see e.g. [4]), we define periodic solutions to (1.3) as follows.

Definition 1.1 *(Periodic probability solution).* A Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is called a *periodic probability solution* to (1.3) if there is a family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ on $\mathcal{U}$ satisfying

$$\mu_t = \mu_{t+T}, \quad \forall t\in\mathbb{R},$$
$$a^{ij}, V^i \in L^1_{loc}(\mathcal{U}\times\mathbb{R}, \mathrm{d}\mu_t\mathrm{d}t), \quad \forall i,\, j\in\{1,\dots,d\}$$

and $\mathcal{L}^*\mu = 0$ in $\mathcal{U}\times\mathbb{R}$ in the sense that

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_t \mathrm{d}t = 0, \quad \forall \phi \in C_0^{2,1}(\mathcal{U} \times \mathbb{R}),$$

such that $\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t$.

Following [29], we use Lyapunov-type functions to quantify the global dissipativity of (1.3), and thus, ensure the existence of periodic probability solutions to (1.3). For a non-negative function $U \in C_T^{2,1}(\mathcal{U} \times \mathbb{R})$ (see Table 1 for the definition), we define for each $\rho > 0$, the ρ-sublevel set

$$\Omega_\rho = \{(x,t) \in \mathcal{U} \times \mathbb{R} : U(x,t) < \rho\}$$

and its t-sections

$$\Omega_\rho^t = \{x \in \mathcal{U} : U(x,t) < \rho\}, \quad \forall t \in \mathbb{R}.$$

Definition 1.2 *(Unbounded Lyapunov function).* A non-negative function $U \in C_T^{2,1}(\mathcal{U} \times \mathbb{R})$ is called an *unbounded Lyapunov function* with respect to $\mathcal{L}$ if there is a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open sets in $\mathcal{U}$ satisfying $\mathcal{U}_n \subset \mathcal{U}_{n+1} \subset\subset \mathcal{U}$ for all $n \in \mathbb{N}$ and $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}_n$ such that

$$\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} U \to \infty \quad \text{as} \quad n \to \infty, \tag{1.4}$$

and there exist a $\rho_m > 0$, called an *essential lower bound* of U, and a constant $\gamma > 0$, called a *Lyapunov constant* of U, such that

$$\mathcal{L}U(x,t) \leq -\gamma, \quad \forall (x,t) \in (\mathcal{U} \times \mathbb{R}) \setminus \overline{\Omega}_{\rho_m}. \tag{1.5}$$

Before stating our main results on the existence of periodic probability solutions to (1.3), we make some assumptions on the coefficients.

(H) Let $p > d+2$. The diffusion matrix $A(x,t) = (a^{ij}(x,t))$ is semi-positive definite for each $(x,t) \in \mathcal{U} \times \mathbb{R}$, and $a^{ij} \in L_{loc}^{\infty}(\mathbb{R}; W_{loc}^{1,p}(\mathcal{U}))$ for each $i,j \in \{1,\dots,d\}$. The drift vector field $V = (V^i)$ satisfies $V^i \in L_{loc}^p(\mathcal{U} \times \mathbb{R})$ for each $i \in \{1,\dots,d\}$.

We point out that only semi-positive definiteness of A is assumed in **(H)**. Our first main result concerning the existence of periodic probability solutions to (1.3) in the non-degenerate case is stated in the following theorem. $A = (a^{ij})$ is called *locally uniformly positive definite* if for each bounded domain $\mathcal{W} \subset\subset \mathcal{U}$, there exist $\lambda_{\mathcal{W}}, \Lambda_{\mathcal{W}} > 0$ such that

$$\lambda_{\mathcal{W}}|\xi|^2 \leq a^{ij}(x,t)\xi^i\xi^j \leq \Lambda_{\mathcal{W}}|\xi|^2, \quad \forall (x,t) \in \mathcal{W} \times \mathbb{R}, \quad \xi \in \mathbb{R}^d.$$

Theorem A. *Assume* **(H)**. *Suppose $A = (a^{ij})$ is locally uniformly positive definite, and there is an unbounded Lyapunov function with respect to $\mathcal{L}$. Then,* (1.3) *admits a periodic* *probability solution with a density in $C^{\alpha}(\mathbb{R}, C^{\gamma}(\mathcal{U}))$ for some $\alpha > 0$ and $\gamma > 0$ depending on d and p.*

In [29, Theorem 3.8] and the footnote on the same page, Khasminskii proved the existence of a special solution, which is a T-periodic Markov process, to the SDE (1.2) with locally Lipschitz coefficients provided the existence of an unbounded Lyapunov function as in Definition 1.2. It is easy to see that the distribution of the special solution is a periodic probability solution to (1.3). Therefore, Theorem A establishes a more general framework.

The existence result in Theorem A is also a periodic counterpart of the existence of stationary measures for stationary Fokker-Planck equations established in [11]. In fact, for time-independent coefficients $(a^{ij}) \in W^{1,p}_{loc}(\mathbb{R}^d)$ being locally uniformly positive definite and $(V^i) \in L^p_{loc}(\mathbb{R}^d)$ for $p > d$, the authors proved in [11] the existence of stationary measures for the stationary Fokker-Planck equation on the whole space $\mathbb{R}^d$ in the presence of a time-independent unbounded Lyapunov function, namely, a function $U \in C^2(\mathbb{R}^d)$ satisfying $U(x) \to \infty$ as $|x| \to \infty$ such that the inequality $a^{ij}\partial^2_{ij}U + V^i\partial_i U \leq -\gamma$ holds in an exterior domain for some $\gamma > 0$. Stationary measures for the stationary Fokker-Planck equations on $\mathbb{R}^d$ and any open and connected domain $\mathcal{U} \subset \mathbb{R}^d$ were later shown to exist in [7, Corollary 2.4.2] and [23], respectively, under quasi-compact or generalized Lyapunov conditions. Similar results for the stationary Fokker-Planck equations on $\mathbb{R}^d$ in the degenerate case have been established in [7, Corollary 2.4.4].

Given Theorem A, we are able to apply perturbation techniques to construct periodic probability solutions to (1.3) in the degenerate case as stated in the next result.

Theorem B. *Assume* **(H)**. *Suppose $A = (a^{ij}) \in C_T(\mathbb{R}; W^{1,p}_{loc}(\mathcal{U}))$ and $V = (V^i) \in C_T(\mathcal{U} \times \mathbb{R})$, and there is an unbounded Lyapunov function with respect to $\mathcal{L}$. Then,* (1.3) *admits a periodic probability solution.*

Our study of (1.3) in the degenerate case is mainly motivated by the following stochastic damping Hamiltonian system:

$$\begin{cases} \mathrm{d}x = y\mathrm{d}t, \\ \mathrm{d}y = -\left[b(x,y)y + \nabla V(x,t)\right]\mathrm{d}t + \sigma(x,y,t)\mathrm{d}W_t, \end{cases} \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{1.6}$$

where $b(x,y)$ is the damping, $V(x,t)$ is the T-periodic potential and $\sigma(x,y,t)$ is the T-periodic noise intensity. The Fokker-Planck equation associated to (1.6) is given by

$$\mathcal{L}^*_H u := -\partial_t u + \partial^2_{y_i y_j}(a^{ij}u) - \partial_{x_i}(y_i u) + \partial_{y_i}\left((b^{ij}y_j + \partial_{x_i}V)u\right) = 0, \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},$$

which is obviously degenerate. Under appropriate assumptions on the coefficients, we are able to construct an unbounded Lyapunov function with respect to $\mathcal{L}_H$, the formal L^2-adjoint of $\mathcal{L}^*_H$, and hence, we can apply Theorem B to find periodic probability solutions to $\mathcal{L}_H^* u = 0$. See Theorem 5.1 for more details. Besides, we use Theorem B to study stochastic differential inclusions in Theorem 5.2.

In the present paper, only the existence of periodic probability solutions to (1.3) is studied. In the forthcoming work [25], we study the uniqueness of periodic probability solutions to (1.3), the global dynamics of (1.3) and the ergodicity of (1.2).

The rest of the paper is organized as follows. In Section 2, we present some preliminaries including the definition of measure solutions to (1.3) in general spatio-temporal domains, the regularity theory of measure solutions to (1.3) and a priori estimates for measure solutions to (1.3). Theorem A and Theorem B are proven in Section 3 and Section 4, respectively. Applications of Theorem B to stochastic damping Hamiltonian systems and stochastic differential inclusions are given in Section 5.

2. Preliminaries

In Subsection 2.1, we define measure solutions to (1.3), present some equivalent formalisms, and recall the regularity theory. In Subsection 2.2, we establish a priori estimates for measure solutions to (1.3).

2.1. Measure solutions and regularity

We first define measure solutions to (1.3). Let Q be an open and connected domain in $\mathcal{U} \times \mathbb{R}$, and $C_0^{2,1}(Q)$ be the space of continuous functions $u : Q \to \mathbb{R}$ such that u is compactly supported and $\partial_i u$, $\partial_{ij}^2 u$ and $\partial_t u$ are continuous on Q for all i, $j \in \{1, \dots, d\}$.

Definition 2.1. A σ-finite Borel measure μ on Q is called a *measure solution* to (1.3) in Q if

$$a^{ij}, V^i \in L^1_{loc}(Q, \mathrm{d}\mu), \quad \forall i, j \in \{1, \dots, d\},$$

and $\mathcal{L}^* \mu = 0$ in Q in the sense that

$$\iint\limits_Q \mathcal{L}\phi \mathrm{d}\mu = 0, \quad \forall \phi \in C_0^{2,1}(Q). \tag{2.1}$$

If, in addition, μ admits a continuous density u in Q, then μ or u is called a *weak solution* to (1.3) in Q.

Arguing as in [7, Proposition 6.1.2] and [10, Lemma 1.1], the following equivalent formalisms hold for (2.1) in the case $Q = \mathcal{U} \times \mathbb{R}$.

Corollary 2.1. *Let μ be a measure solution to (1.3) in $\mathcal{U} \times \mathbb{R}$. Suppose there is a family of σ-finite Borel measures $\{\mu_t : t \in \mathbb{R}\}$ on $\mathcal{U}$ such that $\mathrm{d}\mu = \mathrm{d}\mu_t \mathrm{d}t$. Then, the following conditions are equivalent to (2.1) with $Q = \mathcal{U} \times \mathbb{R}$:*

(1) *for each* $\phi \in C_c^2(\mathcal{U})$, *there exists a subset* $J_\phi \subset \mathbb{R}$ *satisfying* $|\mathbb{R} \setminus J_\phi| = 0$ *such that*

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall s, t \in J_\phi; \tag{2.2}$$

(2) *for each* $\phi \in C_c^{2,1}(\mathcal{U} \times \mathbb{R})$, *there exists a subset* $J_\phi \subset \mathbb{R}$ *satisfying* $|\mathbb{R} \setminus J_\phi| = 0$ *such that*

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall s, t \in J_\phi. \tag{2.3}$$

Remark 2.1. We claim that there exists a subset $J \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus J| = 0$ such that (2.2) (resp. (2.3)) holds for all $\phi \in C_c^2(\mathcal{U})$ (resp. $\phi \in C_c^{2,1}(\mathcal{U} \times \mathbb{R})$) and for all $s, t \in J$. In particular, if μ admits a continuous density in $\mathcal{U} \times \mathbb{R}$, then (2.2) and (2.3) hold for all $s, t \in \mathbb{R}$.

We prove the claim for (2.2); the claim for (2.3) can be proven in the same manner. Let $\mathcal{D}$ be a countable basis for $C_c^2(\mathcal{U})$. For each $\phi \in \mathcal{D}$, there exists $J_\phi \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus J_\phi| = 0$ such that (2.2) holds for all $s, t \in J_\phi$. Set $J := \cap_{\phi \in \mathcal{D}} J_\phi$. Then, $|\mathbb{R} \setminus J| = 0$ and for any $s, t \in J$, there holds

$$\int_{\mathcal{U}} \phi \mathrm{d}\mu_t = \int_{\mathcal{U}} \phi \mathrm{d}\mu_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau, \quad \forall \phi \in \mathcal{D}. \tag{2.4}$$

Now, let $\phi \in C_c^2(\mathcal{U})$. There is a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ such that $\phi_n \to \phi$ in $C_c^2(\mathcal{U})$ as $n \to \infty$ in the sense that

$$\lim_{n \to \infty} \max_{\mathcal{U}} \left(|\phi_n - \phi| + \sum_{i=1}^d |\partial_i \phi_n - \partial_i \phi| + \sum_{i,j=1}^d |\partial_{ij}^2 \phi_n - \partial_{ij}^2 \phi| \right) = 0.$$

It follows that for each $t \in J$, there holds

$$\left| \int_{\mathcal{U}} \phi_n \mathrm{d}\mu_t - \int_{\mathcal{U}} \phi \mathrm{d}\mu_t \right| \leq \max_{\mathcal{U}} |\phi_n - \phi| \to 0 \quad \text{as} \quad n \to \infty.$$

As a^{ij}, $V^i \in L^1_{loc}(\mathcal{U} \times \mathbb{R}, \mathrm{d}\mu_t \mathrm{d}t)$ for all $i, j \in \{1, \dots, d\}$, we see that for $s, t \in J$ with $s < t$

$$\begin{aligned} \lim_{n \to \infty} \int_s^t \int_{\mathcal{U}} |\mathcal{L}\phi_n - \mathcal{L}\phi| \mathrm{d}\mu_t \mathrm{d}t &\leq \lim_{n \to \infty} \int_s^t \int_{\mathcal{U}} \left[\left| a^{ij}(\partial_{ij}^2 \phi_n - \partial_{ij}^2 \phi) \right| + \left| V^i(\partial_i \phi_n - \partial_i \phi) \right| \right] \mathrm{d}\mu_t \mathrm{d}t \\ &= 0. \end{aligned}$$

The claim then follows from setting $\phi = \phi_n$ in (2.4) and letting $n \to \infty$.

For the "In particular" part, let us fix any $\phi \in C_c^2(\mathcal{U})$. As μ admits a density, the functions $t \mapsto \int_{\mathcal{U}} \phi \mathrm{d}\mu_t : \mathbb{R} \to \mathbb{R}$ and $(s,t) \mapsto \int_s^t \int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_\tau \mathrm{d}\tau : \{(s,t) : s < t\} \to \mathbb{R}$ are continuous. Since J is dense in $\mathbb{R}$ and (2.4) holds for all $s,t \in J$, a density argument shows that (2.4) holds for all $s,t \in \mathbb{R}$.

We recall the regularity theory of measure solutions to (1.3) in $\mathcal{U} \times \mathbb{R}$. Recall $p > d+2$. Let $\mathbb{H}_0^{1,p}(\mathcal{U} \times \mathbb{R})$ be the space of measurable functions u on $\mathcal{U} \times \mathbb{R}$ such that $u(\cdot,t) \in W_0^{1,p}(\mathcal{U})$ for a.e. $t \in \mathbb{R}$ and the function $t \mapsto \|u(t,\cdot)\|_{W_0^{1,p}(\mathcal{U})}$ lies in $L^p(\mathbb{R})$. Let $\mathbb{H}^{-1,p'}(\mathcal{U} \times \mathbb{R})$ be the dual space of $\mathbb{H}_0^{1,p}(\mathcal{U} \times \mathbb{R})$, where $p' > 1$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $\mathcal{H}_{loc}^{1,p}(\mathcal{U} \times \mathbb{R})$ be the space of measurable functions u on $\mathcal{U} \times \mathbb{R}$ such that $\eta u \in \mathbb{H}_0^{1,p}(\mathcal{U} \times \mathbb{R})$ and $\partial_t(\eta u) \in \mathbb{H}^{-1,p}(\mathcal{U} \times \mathbb{R})$ for each $\eta \in C_0^\infty(\mathbb{R}^{d+1})$. By [7, Theorem 6.2.2], there exist $\alpha > \frac{1}{p}$ and $\gamma > 0$ (depending only on d and p) such that $\mathcal{H}_{loc}^{1,p}(\mathcal{U} \times \mathbb{R})$ is continuously embedded into $C^{\alpha - \frac{1}{p}}(\mathbb{R}, C^\gamma(\mathcal{U}))$.

Theorem 2.1 *([5,7]). Assume* **(H)**. *Suppose $A = (a^{ij})$ is locally uniformly positive definite. Let $\mu = \{\mu_t\}_{t\in\mathbb{R}}$ be a measure solution to (1.3) in $\mathcal{U} \times \mathbb{R}$. Then, μ admits a positive density $u \in \mathcal{H}_{loc}^{1,p}(\mathcal{U} \times \mathbb{R})$. Moreover, for each closed interval $[t_1,t_2] \subset\subset [\tilde{t}_1,\tilde{t}_2]$ and each open subset $\mathcal{W} \subset\subset \mathcal{W}_1 \subset\subset \mathcal{U}$, there holds*

$$\|u\|_{C^{\alpha-\frac{1}{p}}([t_1,t_2],C^\gamma(\mathcal{W}))} \leq N \int_{\tilde{t}_1}^{\tilde{t}_2} \mu_s(\mathcal{W}_1)\mathrm{d}s \tag{2.5}$$

for some $N > 0$ depending only on d, p, t_1, t_2, $\tilde{t}_1$, $\tilde{t}_2$, $\mathcal{W}$, $\mathcal{W}_1$, $\lambda_{\mathcal{W}_1}$, $\Lambda_{\mathcal{W}_1}$, $\sup_{t\in[\tilde{t}_1,\tilde{t}_2]} \|a^{ij}(\cdot,t)\|_{W^{1,p}(\mathcal{W}_1)}$ and $\|V^i\|_{L^p(\mathcal{W}_1\times[\tilde{t}_1,\tilde{t}_2])}$.

2.2. A priori estimates

We establish measure estimates for measure solutions, with continuous and periodic densities, to (1.3). The proof is inspired by [11, Theorem 2].

Theorem 2.2. *Assume* **(H)**. *Let $U \in C_T^{2,1}(\mathcal{U} \times \mathbb{R})$ be non-negative and satisfy*

$$\mathcal{L}U \leq -\gamma \quad in \quad (\mathcal{U} \times \mathbb{R}) \setminus \overline{\Omega}_{\rho_m} \tag{2.6}$$

for some $\rho_m > 0$ and $\gamma > 0$. Let $\rho_1 > \rho_m$ be such that $\overline{\Omega}_{\rho_1} \subset\subset \mathcal{U} \times \mathbb{R}$. If μ is a measure solution to (1.3) in $\mathcal{U} \times \mathbb{R}$ and admits a density in $C_T(\mathcal{U} \times \mathbb{R})$, then for each $\rho_0 \in (\rho_m, \rho_1)$ there exists $C_* > 0$ (depending only on ρ_m and ρ_0) such that

$$\mu\left(\bigcup_{s\in[t,t+T]} \left((\Omega_{\rho_1}^s \setminus \Omega_{\rho_0}^s) \times \{s\}\right)\right) \leq C\mu\left(\bigcup_{s\in[t,t+T]} \left(\Omega_{\rho_0}^s \times \{s\}\right)\right), \quad \forall t \in \mathbb{R}, \tag{2.7}$$

where $C = \frac{C_*}{\gamma}\left(\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U\right)$.

Proof. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be a family of Borel probability measures on $\mathcal{U}$ such that $\mathrm{d}\mu = \mathrm{d}\mu_t dt$. By the assumptions on μ, Corollary 2.1 and Remark 2.1, there holds for each $\phi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R}) \cap C_T(\mathcal{U}\times\mathbb{R})$

$$\int_t^{t+T}\int_{\mathcal{U}} \mathcal{L}\phi \mathrm{d}\mu_s \mathrm{d}s = \int_{\mathcal{U}} \phi(x,t+T)\mathrm{d}\mu_{t+T} - \int_{\mathcal{U}} \phi(x,t)\mathrm{d}\mu_t = 0, \quad \forall t\in\mathbb{R}. \tag{2.8}$$

Fix $\rho_0 \in (\rho_m, \rho_1)$. Let $\{\zeta_\rho\}_{\rho\in(\rho_0,\rho_1)}$ be a family of smooth non-decreasing functions on $[0,\infty)$ satisfying

$$\zeta_\rho(t) = \begin{cases} 0, & t\in[0,\rho_m], \\ t, & t\in[\rho_0,\rho], \\ \frac{\rho+\rho_1}{2}, & t\in[\rho_1,\infty), \end{cases} \quad \text{and} \quad \zeta_\rho'' \le 0 \text{ on } [\rho,\rho_1].$$

In addition, we let the functions $\{\zeta_\rho\}_{\rho\in(\rho_0,\rho_1)}$ coincide on $[0,\rho_0]$.

Obviously, $\zeta_\rho(U) - \frac{\rho+\rho_1}{2} \in C_c^{2,1}(\mathcal{U}\times\mathbb{R}) \cap C_T(\mathcal{U}\times\mathbb{R})$. Setting $\phi = \zeta_\rho(U) - \frac{\rho+\rho_1}{2}$ in (2.8), we find from

$$\mathcal{L}\left(\zeta_\rho(U) - \frac{\rho+\rho_1}{2}\right) = \zeta_\rho'(U)\mathcal{L}U + \zeta_\rho''(U)a^{ij}\partial_i U\partial_j U$$

that

$$\int_t^{t+T}\int_{\mathcal{U}} \left[\zeta_\rho'(U)\mathcal{L}U + \zeta_\rho''(U)a^{ij}\partial_i U\partial_j U\right] \mathrm{d}\mu_s\mathrm{d}s = 0, \quad \forall t\in\mathbb{R}. \tag{2.9}$$

As $\zeta_\rho' \ge 0$ on $[\rho_m,\rho_1)$, $\zeta_\rho' = 1$ on $[\rho_0,\rho]$ and $\zeta_\rho' = 0$ otherwise, we obtain from (2.6) that

$$\zeta_\rho'(U)\mathcal{L}U \le \begin{cases} -\gamma, & \text{in} \quad \Omega_\rho \setminus \Omega_{\rho_0}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

Since (a^{ij}) is semi-positive definite, $\zeta_\rho'' \not\equiv 0$ on $[\rho_m,\rho_0]$, $\zeta_\rho'' \le 0$ on $[\rho,\rho_1]$ and $\zeta_\rho'' = 0$ otherwise, we find

$$\zeta_\rho''(U)a^{ij}\partial_i U\partial_j U \le \begin{cases} C_*\left(\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U\right), & \text{in} \quad \Omega_{\rho_0}\setminus\overline{\Omega}_{\rho_m}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.11}$$

where $C_* := \max_{t\in[\rho_m,\rho_0]} \zeta_\rho''(t) < \infty$ is independent of $\rho\in(\rho_0,\rho_1)$ due to the construction of $\{\zeta_\rho\}_{\rho\in(\rho_0,\rho_1)}$. Applying (2.10) and (2.11) to (2.9), we find

$$\begin{aligned}
\gamma \int_t^{t+T} \mu_s(\Omega_\rho^s \setminus \Omega_{\rho_0}^s)\mathrm{d}s &\le - \int_t^{t+T} \int_{\mathcal{U}} \zeta_\rho'(U)\mathcal{L}U\mathrm{d}\mu_s\mathrm{d}s \\
&= \int_t^{t+T} \int_{\mathcal{U}} \zeta_\rho''(U)a^{ij}\partial_i U\partial_j U\mathrm{d}\mu_s\mathrm{d}s \\
&\le C_* \left(\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U \right) \int_t^{t+T} \mu_s(\Omega_{\rho_0}^s)\mathrm{d}s, \quad \forall \rho \in (\rho_0, \rho_1).
\end{aligned}$$

Letting $\rho \to \rho_1^-$ in the above inequality, the conclusion follows. $\square$

3. Proof of Theorem A

Subsection 3.1 is devoted to the study of periodic solutions to (1.3) on product spaces of the form $\Omega \times \mathbb{R}$, where $\Omega \subset \mathcal{U}$ is open and bounded. The proof of Theorem A is done in Subsection 3.2.

3.1. Periodic solutions in bounded domains

Let $\Omega \subset\subset \mathcal{U}$ be a bounded domain with smooth boundary. Denote by $C_T(\overline{\Omega} \times \mathbb{R})$ the space of T-periodic and continuous functions on $\overline{\Omega} \times \mathbb{R}$.

Let $\alpha^{ij}, \beta^i \in C_T(\overline{\Omega} \times \mathbb{R})$ be C^3 on $\overline{\Omega} \times \mathbb{R}$, namely, all of their partial derivatives up to the third order are continuous on $\overline{\Omega} \times \mathbb{R}$, for all $i, j \in \{1, \ldots, d\}$. In addition, let (α^{ij}) be uniformly positive definite on $\overline{\Omega} \times \mathbb{R}$. Consider the following eigenvalue problem with the reflecting boundary condition

$$\begin{cases} -\partial_t \phi + \partial_{ij}^2(\alpha^{ij}\phi) - \partial_i(\beta^i\phi) = \lambda\phi & \text{in} \quad \Omega \times \mathbb{R}, \\ \nu_i\left(\partial_j(\alpha^{ij}\phi) - \beta^i\phi\right) = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}, \\ \phi \in C_T(\overline{\Omega} \times \mathbb{R}) \cap C^{2,1}(\Omega \times \mathbb{R}), \end{cases} \tag{3.1}$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit outward normal vector field along $\partial\Omega$, and $C^{2,1}(\Omega \times \mathbb{R})$ is the space of continuous functions $u : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\partial_i u$, $\partial_{ij}^2 u$ $\partial_t u$ are continuous on $\Omega \times \mathbb{R}$ for all i, $j \in \{1, \ldots, d\}$.

Definition 3.1. A number $\lambda \in \mathbb{R}$ is called a *principal eigenvalue* of the eigenvalue problem (3.1), if there is a non-negative and non-zero function $\phi \in C_T(\overline{\Omega} \times \mathbb{R}) \cap C^{2,1}(\Omega \times \mathbb{R})$ such that the pair (λ, ϕ) solves the problem (3.1). In this case, the function ϕ is called a *principal eigenfunction* associated to λ.

Theorem 3.1. 0 *is an algebraically simple and isolated principal eigenvalue of the eigenvalue problem* (3.1)*, and it is the only eigenvalue admitting a non-negative eigenfunction.*

Proof. By the parabolic regularity theory (see e.g. [33,35]), the maximum principle (see e.g. [40]) and the Kreĭn-Rutman theorem (see e.g. [32,22]), the eigenvalue problem admits an eigenvalue λ as in the statement of the theorem. Let ϕ be a non-negative eigenfunction associated to λ.

We show that $\lambda = 0$. Fix $t \in \mathbb{R}$. Integrating the equation satisfied by the pair (λ, ϕ) over $\Omega \times (t, t+T)$, we find

$$\int_t^{t+T}\int_\Omega -\partial_t \phi \mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega \partial^2_{ij}(\alpha^{ij}\phi) - \partial_i(\beta^i \phi)\mathrm{d}x\mathrm{d}s = \lambda \int_t^{t+T}\int_\Omega \phi \mathrm{d}x\mathrm{d}s.$$

Applying the divergence theorem to the second term on the left hand side, we find from the boundary condition satisfied by ϕ and the periodicity of ϕ that

$$\begin{aligned} &-\int_t^{t+T} \frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega \phi \mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega \partial^2_{ij}(\alpha^{ij}\phi) - \partial_i(\beta^i \phi)\mathrm{d}x\mathrm{d}s \\ &\quad = -\int_\Omega \phi(x, t+T)\mathrm{d}x + \int_\Omega \phi(x,t)\mathrm{d}x + \int_t^{t+T}\int_{\partial\Omega} \nu_i \left(\partial_j(\alpha^{ij}\phi) - \beta^i\phi\right)\mathrm{d}S_x\mathrm{d}s = 0. \end{aligned}$$

Hence, $\lambda \int_t^{t+T}\int_{\Omega_\rho} \phi \mathrm{d}x\mathrm{d}s = 0$, which yields $\lambda = 0$. □

We recall the definition of weak solutions to (1.3) in Definition 2.1.

Corollary 3.1. *Assume* **(H)**. *Suppose $A = (a^{ij})$ is locally uniformly positive definite and let U be an unbounded Lyapunov function with respect to $\mathcal{L}$ with an essential lower bound ρ_m and a Lyapunov constant γ. Then, for any $\rho > \rho_m$, there is a weak solution u to (1.3) in Ω_ρ satisfying the following properties:*

- *u is a non-negative, T-periodic and Hölder continuous function on Ω_ρ;*
- *there holds $\int_t^{t+T}\int_{\Omega^s_\rho} u(x,s)\mathrm{d}x\mathrm{d}s = T$ for all $t \in \mathbb{R}$;*
- *for any $\rho_0 \in (\rho_m, \rho)$ there exists $C_* > 0$ depending only on ρ_m and ρ_0 such that*

$$\mu\left(\bigcup_{s\in[t,t+T]} \left((\Omega^s_\rho \setminus \Omega^s_{\rho_0}) \times \{s\}\right)\right) \le C\mu\left(\bigcup_{s\in[t,t+T]} \left(\Omega^s_{\rho_0} \times \{s\}\right)\right), \quad \forall t \in \mathbb{R},$$

where $\mathrm{d}\mu := u\mathrm{d}x\mathrm{d}t$ and $C := \frac{C_*}{\gamma}\left(\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U\right)$.

Proof. To highlight the dependence of $\mathcal{L}$ on the coefficients, we write $\mathcal{L}_{A,V} = \partial_t + a^{ij}\partial^2_{ij} + V^i\partial_i$. Fix $\rho > \rho_m$ and $(x_0, t_0) \in \Omega_{\rho_m}$. Since U satisfies (1.4), there exist a bounded subset

$\Omega \subset \mathcal{U}$ with smooth boundary and a bounded subset $\Omega_* \subset \mathcal{U}$ such that $\Omega \subset\subset \Omega_*$ and $\overline{\Omega}_\rho \subset\subset \Omega \times \mathbb{R}$.

We first construct a sequence of smooth functions $(A_n)_n$ and $(V_n)_n$ to approximate A and V, respectively. To do so, we fix some non-negative function $\eta \in C_0^\infty(\mathbb{R}^{d+1})$ satisfying

$$\eta(x,t) = 0 \text{ for } |(x,t)| \geq 1 \quad \text{and} \quad \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \eta \mathrm{d}x\mathrm{d}t = 1,$$

and define

$$\eta_n(x,t) = n^{d+1}\eta(nx, nt), \quad n \in \mathbb{N}.$$

Define $A_n := (a_n^{ij})$ and $V_n := (V_n^i)$ as follows:

$$a_n^{ij}(x,t) = \int\limits_{\mathbb{R}} \int\limits_{\mathcal{U}} \eta_n(x-y, t-s)\mathbb{1}_{\Omega_*}(y,s)a^{ij}(y,s)\mathrm{d}y\mathrm{d}s, \quad (x,t) \in \Omega_* \times \mathbb{R},$$

$$V_n^i(x,t) = \int\limits_{\mathbb{R}} \int\limits_{\mathcal{U}} \eta_n(x-y, t-s)\mathbb{1}_{\Omega*}(y,s)V^i(y,s)\mathrm{d}y\mathrm{d}s, \quad (x,t) \in \Omega_* \times \mathbb{R}.$$

It is straightforward to check that for all $n \gg 1$, A_n and V_n are smooth and T-periodic in t, and $A_n(\cdot,t)$ and $V_n(\cdot,t)$ are compactly supported in Ω_* for each $t \in \mathbb{R}$. Moreover,

$$\int\limits_s^t \|A_n(\cdot,\tau) - A(\cdot,\tau)\|^q_{W^{1,p}(\Omega)} \mathrm{d}\tau \to 0, \qquad \|V_n - V\|_{L^p(\Omega\times[s,t])} \to 0 \quad \text{as} \quad n \to \infty \tag{3.2}$$

for any $q > 1$ and $t > s$.

For each $n \gg 1$, we apply Theorem 3.1 to find a non-negative function $u_n \in C_T(\overline{\Omega} \times \mathbb{R}) \cap C^{2,1}(\Omega \times \mathbb{R})$ satisfying

$$\partial_t u_n = \partial^2_{ij}(a_n^{ij}u_n) - \partial_i(V_n^i u_n) \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{3.3}$$

and the normalization

$$u_n(x_0,t_0) = 1. \tag{3.4}$$

Since $A = (a^{ij})$ is locally uniformly positive definite, there are positive constants λ and Λ depending only on Ω_* such that

$$\lambda|\xi|^2 \leq a_n^{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \quad \xi \in \mathbb{R}^d, \quad n \gg 1.$$

Moreover, we see from (3.2) that there exists $M > 0$ such that

$$\left\| |\partial_j a_n^{ij}| + |V_n^i| \right\|_{L^p(\Omega\times[t,t+T])} \le M, \quad \forall i \in \{1,\dots,d\}, \quad t\in\mathbb{R}, \quad n \gg 1.$$

We then apply classical results on local Hölder estimates and Harnack's inequality (see e.g. [33,35]) to obtain that for each $\Omega^* \subset\subset \Omega$ and $t \in \mathbb{R}$, there exist constants $\alpha \in (0,1)$ and $C_1, C_2 > 0$ (depending only on d, λ, Λ, M, T, Ω and Ω^*) such that

$$|u_n|_{\alpha,\frac{\alpha}{2};\Omega^*\times[t-T,t]} \le C_1 \sup_{\Omega^*\times[t-T,t]} u_n \le C_2 \inf_{\Omega^*\times[t+T,t+2T]} u_n, \tag{3.5}$$

where $|u|_{\alpha,\frac{\alpha}{2};\Omega^*\times[t-T,t]}$ denotes the sum of the α-Hölder norm in space and the $\frac{\alpha}{2}$-Hölder norm in time in the domain $\Omega^* \times [t-T,t]$.

As u_n is T-periodic and $u_n(x_0,t_0) = 1$ for each $n \gg 1$, we find for each $t \in \mathbb{R}$, $\inf_{\Omega^*\times[t+T,t+2T]} u_n \le 1$ implying $|u_n|_{\alpha,\frac{\alpha}{2};\Omega^*\times[t-T,t]} \le C_2$. Applying the Arzelà-Ascoli theorem and the standard diagonal argument, we find a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, and a non-negative function $u \in C_T(\Omega\times\mathbb{R})$ such that u_n locally uniformly converges to u as $n \to \infty$. By (3.5), u is Hölder continuous in $\Omega\times\mathbb{R}$. Moreover, $u(x_0,t_0) = 1$ due to the normalization (3.4). In particular, u is non-zero.

Now, we show that u is a weak solution to (1.3) in $\Omega\times\mathbb{R}$. Multiplying (3.3) by $\phi \in C_0^{2,1}(\Omega\times\mathbb{R})$ and integrating the resulting equation over $\Omega\times\mathbb{R}$, we conclude from the divergence theorem that

$$\int_{\mathbb{R}}\int_{\Omega} (\mathcal{L}_{A_n,V_n}\phi) u_n \mathrm{d}x\mathrm{d}s = 0. \tag{3.6}$$

Since $\|\mathcal{L}_{A_n,V_n}\phi - \mathcal{L}_{A,V}\phi\|_{L^p(\Omega\times\mathbb{R})} \to 0$ as $n\to\infty$ due to (3.2), and u_n uniformly converges to u on $\mathrm{supp}(\phi)$ as $n\to\infty$, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathcal{U}} |\mathcal{L}_{A_n,V_n}\phi(u_n - u)| \mathrm{d}x\mathrm{d}t \\
&\qquad \le \left(\max_{\mathrm{supp}(\phi)} |u_n - u|\right) |\mathrm{supp}(\phi)|^{1-\frac{1}{p}} \sup_n \|\mathcal{L}_{A_n,V_n}\phi\|_{L^p(\Omega\times\mathbb{R})} \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathcal{U}} |\mathcal{L}_{A_n,V_n}\phi - \mathcal{L}_{A,V}\phi| u(x,t) \mathrm{d}x\mathrm{d}t \\
&\qquad \le \|\mathcal{L}_{A_n,V_n}\phi - \mathcal{L}_{A,V}\phi\|_{L^p(\Omega\times\mathbb{R})} \left(\max_{\mathrm{supp}(\phi)} u\right) |\mathrm{supp}(\phi)|^{1-\frac{1}{p}} \to 0
\end{aligned}$$

as $n\to\infty$. Letting $n\to\infty$ in (3.6), we find $\int_{\mathbb{R}}\int_{\Omega}(\mathcal{L}_{A,V}\phi)u\mathrm{d}x\mathrm{d}s = 0$. Thus, u is a weak solution to (1.3) in $\Omega\times\mathbb{R}$.

Finally, we rescale u to finish the proof. Due to the periodicity of u, $t \mapsto \frac{1}{T}\int_t^{t+T}\int_{\Omega_\rho^s} u(x,s)\mathrm{d}x\mathrm{d}s$ is a constant function on $\mathbb{R}$. Denote this constant by $\tilde{C}$. Set

$$\tilde{u} := \frac{1}{\tilde{C}} u \quad \text{and} \quad \mathrm{d}\tilde{\mu} := \tilde{u}\mathrm{d}x\mathrm{d}t.$$

Clearly, $\tilde{u}$ is a non-negative, T-periodic and Hölder continuous weak solution to (1.3) in $\Omega\times\mathbb{R}$, and hence in particular in Ω_ρ. From Theorem 2.2, we see that for each $\rho_0 \in (\rho_m, \rho)$ there exists C_* depending only on ρ_m and ρ_0 such that

$$\begin{aligned}&\tilde{\mu}\left(\bigcup_{s\in[t,t+T]} \left((\Omega_\rho^s \setminus \Omega_{\rho_0}^s) \times \{s\}\right)\right)\\ &\quad \le \frac{C_*}{\gamma}\left(\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U\right) \tilde{\mu}\left(\bigcup_{s\in[t,t+T]} \left(\Omega_{\rho_0}^s \times \{s\}\right)\right), \quad \forall t \in \mathbb{R}.\end{aligned}$$

This completes the proof. $\square$

3.2. Proof of Theorem A

Let U be an unbounded Lyapunov function with respect to $\mathcal{L}$ with an essential lower bound ρ_m and a Lyapunov constant γ. The proof is done within three steps.

Step 1. We construct a weak solution μ, with a density $u \in C_T(\mathcal{U}\times\mathbb{R}) \cap C^{\alpha-\frac{1}{p}}(\mathbb{R}, C^\gamma(\mathcal{U}))$, to (1.3) in $\mathcal{U}\times\mathbb{R}$.

Let $\{\rho^n\}_{n\in\mathbb{N}} \subset (\rho_m,\infty)$ be increasing and satisfy $\lim_{n\to\infty}\rho^n = \infty$. For each $n \gg 1$, let u_n be the weak solution to (1.3) in Ω_{ρ^n} obtained in Corollary 3.1. Note that for each open set $\mathcal{V} \subset\subset \mathcal{U}$, there exists $n_0 = n_0(\mathcal{V})$ such that $\mathcal{V}\times\mathbb{R} \subset\subset \Omega_{\rho^n}$ for all $n > n_0$. By Theorem 2.1, for each $t \in \mathbb{R}$, there exists $\tilde{N} > 0$ such that

$$|u_n|_{C^{\alpha-\frac{1}{p}}([t,t+T],C^\gamma(\overline{\mathcal{V}}))} \le \tilde{N}, \quad \forall n > n_0. \tag{3.7}$$

Applying the Arzelà-Ascoli theorem and the standard diagonal argument, we may assume, without loss of generality, that u_n converges locally uniformly in $\mathcal{U}\times\mathbb{R}$ to some non-negative function $u \in C_T(\mathcal{U}\times\mathbb{R})$ as $n\to\infty$. For any $t\in\mathbb{R}$ and $\mathcal{V}\subset\subset\mathcal{U}$, it follows easily from (3.7) that $|u|_{C^{\alpha-\frac{1}{p}}([t,t+T],C^\gamma(\overline{\mathcal{V}}))} \le \tilde{N}$ for some $\tilde{N} > 0$. As t and $\mathcal{V}$ are arbitrary in $\mathbb{R}$ and $\mathcal{U}$ respectively, we deduce that $u \in C^{\alpha-\frac{1}{p}}(\mathbb{R}, C^\gamma(\mathcal{U}))$.

Moreover, setting $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t := u\mathrm{d}x\mathrm{d}t$, we conclude from Fatou's lemma that

$$\mu(\mathcal{U}\times[t,t+T]) = \int_t^{t+T}\int_{\mathcal{U}} u(x,s)\mathrm{d}x\mathrm{d}s \le \liminf_{n\to\infty}\int_t^{t+T}\int_{\mathcal{U}} u_n(x,s)\mathbb{1}_{\Omega_{\rho^n}}(x,s)\mathrm{d}x\mathrm{d}s = T$$

for all $t\in\mathbb{R}$. In particular, μ is σ-finite.

Note that for any $\phi \in C_0^{2,1}(\mathcal{U}\times\mathbb{R})$, there exists $n_1\in\mathbb{N}$ such that $\mathrm{supp}(\phi) \subset \Omega_{\rho^n}$ for all $n > n_1$. Since u_n is a weak solution to (1.3) in Ω_{ρ^n}, we find $\iint_{\mathcal{U}\times\mathbb{R}} \mathcal{L}\phi u_n \mathrm{d}x\mathrm{d}t = 0$ for all $n > n_1$. Letting $n\to\infty$, we find

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}\phi u \mathrm{d}x\mathrm{d}t = 0.$$

Since $\phi \in C_0^{2,1}(\mathcal{U}\times\mathbb{R})$ is arbitrary, μ is a weak solution to (1.3) in $\mathcal{U}\times\mathbb{R}$.

Step 2. We claim that u is non-zero on $\mathcal{U}\times\mathbb{R}$. Otherwise, $u\equiv 0$, namely, u_n converges locally uniformly in $\mathcal{U}\times\mathbb{R}$ to 0 as $n\to\infty$. It follows from Corollary 3.1 that there exist positive constants $\rho_0 > \rho_m$ and $C>0$ independent of $n\in\mathbb{N}$ such that

$$\mu^n\left(\bigcup_{s\in[t,t+T]}\left((\Omega^s_{\rho^n}\setminus\Omega^s_{\rho_0})\times\{s\}\right)\right) \le C\mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho_0}\times\{s\}\right)\right),$$

for all $n\gg 1$, where $\mathrm{d}\mu^n = \mathrm{d}\mu^n_t\mathrm{d}t = u_n(x,t)\mathrm{d}x\mathrm{d}t$ in Ω_{ρ^n}. Thus, for each $t\in\mathbb{R}$ there holds

$$\begin{aligned}
T &= \mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho^n}\times\{s\}\right)\right)\\
&= \mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho^n}\setminus\Omega^s_{\rho_0}\right)\times\{s\}\right) + \mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho_0}\times\{s\}\right)\right)\\
&\le C\mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho_0}\times\{s\}\right)\right) + \mu^n\left(\bigcup_{s\in[t,t+T]}\left(\Omega^s_{\rho_0}\times\{s\}\right)\right)\\
&\to 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

which leads to a contradiction.

Step 3. We show that there is $C>0$ such that $\int_{\mathcal{U}} u(x,t)dx = C$ for all $t\in\mathbb{R}$. Thus, the measure $\tilde{\mu}$, defined by $\mathrm{d}\tilde{\mu} := \frac{1}{C}u\mathrm{d}x\mathrm{d}t$, is a periodic probability solution to (1.3).

Let us fix $s\in\mathbb{R}$. Following the arguments in [36, Proposition 2.8], we can find a non-negative function $\tilde{U}\in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ satisfying

(1) $\int_{\mathcal{U}} \tilde{U}(x,s)u(x,s)\mathrm{d}x < \infty$;
(2) $\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}} \tilde{U}\to\infty$ as $n\to\infty$;
(3) $\mathcal{L}\tilde{U}\le 0$ in $(\mathcal{U}\times\mathbb{R})\setminus\tilde{\Omega}_{\tilde{\rho}_m}$ for some $\tilde{\rho}_m>0$, where $\tilde{\Omega}_{\tilde{\rho}_m} = \{(x,t)\in\mathcal{U}\times\mathbb{R} : \tilde{U}(x,t) < \tilde{\rho}_m\}$.

Indeed, suppose for the moment that there is a non-negative function $\theta\in C^2([0,\infty))$ satisfying

$$\theta(0)=0,\quad \lim_{r\to\infty}\theta(r)=\infty,\quad 0\le\theta'(r)\le 1,\quad \theta''(r)\le 0 \quad \text{and}$$
$$\int\limits_{\mathcal{U}} \theta(U(x,s))u(x,s)\mathrm{d}x < \infty.$$

We show $\tilde{U} := \theta(U)$ satisfies required conditions. Properties (1) and (2) follow readily. For (3), we see from $\mathcal{L}U \le -\gamma$ in $(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m}$ that

$$\mathcal{L}\tilde{U} = \theta'(U)\mathcal{L}U + \theta''(U)a^{ij}\partial_i\tilde{U}\partial_j\tilde{U} \le -\gamma\theta'(U) \le 0 \quad \text{in} \quad (\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m},$$

where we recall that $\Omega_{\rho_m} = \{(x,t)\in\mathcal{U}\times\mathbb{R} : U(x,t) < \rho_m\}$. Since $\tilde{U}$ satisfies (2), there must exist some $\tilde{\rho}_m > 0$ such that $\Omega_{\rho_m} \subset\subset \tilde{\Omega}_{\tilde{\rho}_m}$, which yields (3).

It remains to construct the function θ. Note that $\sigma := \mu_s \circ U^{-1}(\cdot, s)$ is a finite Borel measure on $[0,\infty)$, where $\mathrm{d}\mu_s = u(x,s)\mathrm{d}x$. It is not hard to find an increasing sequence of numbers $\{z_k\}_{k\in\mathbb{N}\cup\{0\}}$ satisfying

$$z_0 = 0, \quad z_{k+1} - z_k \ge z_k - z_{k-1} \ge 1 \quad \text{and} \quad \sigma([z_k,\infty)) \le \frac{1}{2^k} \quad \text{for all} \quad k\in\mathbb{N}.$$

Let $\theta_0 : [0,\infty) \to [0,\infty)$ be linear on each interval $[z_k, z_{k+1}]$ and satisfy $\theta_0(z_k) = k$ for all $k \in \mathbb{N}\cup\{0\}$. It is easy to check that θ_0 is an σ-integrable, increasing and concave function on $[0,\infty)$. Let $g \in C^1([0,\infty))$ satisfy

$$g'(z) \le 0 \quad \text{and} \quad g(z) = \theta_0'(z) \text{ if } z \in \left(z_k + \frac{1}{k+3}, z_{k+1} - \frac{1}{k+3}\right) \text{ for all } k\in\mathbb{N}\cup\{0\}.$$

The function $\theta(z) := \int_0^z g(s)\mathrm{d}s$ for $z\in[0,\infty)$ meets the requirements.

Next, we show that $\mu_t(\mathcal{U}) \ge \mu_s(\mathcal{U})$ for all $t \ge s$. We see from **Step 1**, Corollary 2.1 and Remark 2.1 that

$$\int_{\mathcal{U}} \phi(x,t)\mathrm{d}\mu_t = \int_{\mathcal{U}} \phi(x,s)\mathrm{d}\mu_s + \int_s^t\int_{\mathcal{U}} \mathcal{L}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau, \quad \forall\phi\in C_c^{2,1}(\mathcal{U}\times\mathbb{R}), \quad t > s. \tag{3.8}$$

Fix $\rho_0 > \tilde{\rho}_m$ and set $N_0 = [\rho_0] + 1$, where $[\rho_0]$ is the integer part of ρ_0. Let $\{\zeta_N\}_{N\ge N_0}$ be a family of smooth non-decreasing functions on $[0,\infty)$ satisfying

$$\zeta_N(t) = \begin{cases} 0, & t\in[0,\tilde{\rho}_m], \\ t, & t\in[\rho_0, N], \\ N+1, & t\in[N+2,\infty), \end{cases} \qquad \text{and} \quad \zeta_N'' \le 0 \text{ on } [N, N+2].$$

In addition, we let the functions $\{\zeta_N\}_{N\ge N_0}$ coincide on $[0,\rho_0]$.

Obviously, $\zeta_N(\tilde{U}) - (N+1) \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$. Setting $\phi = \zeta_N(\tilde{U}) - (N+1)$ in (3.8), we find

$$\begin{aligned} &\int_{\mathcal{U}} \left(\zeta_N(\tilde{U}) - (N+1)\right)\mathrm{d}\mu_t \\ &= \int_{\mathcal{U}} \left(\zeta_N(\tilde{U}) - (N+1)\right)\mathrm{d}\mu_s + \int_s^t\int_{\mathcal{U}} \left(\zeta_N'(\tilde{U})\mathcal{L}\tilde{U} + \zeta_N''(\tilde{U})a^{ij}\partial_i\tilde{U}\partial_j\tilde{U}\right)\mathrm{d}\mu_\tau\mathrm{d}\tau. \end{aligned} \tag{3.9}$$

Since $\zeta_N' = 0$ on $[0, \tilde{\rho}_m]$, $\zeta_N' \geq 0$ and $\mathcal{L}\tilde{U} \leq 0$ in $(\mathcal{U}\times\mathbb{R})\setminus\tilde{\Omega}_{\tilde{\rho}_m}$, it follows that $\zeta_N'(\tilde{U})\mathcal{L}\tilde{U} \leq 0$ in $\mathcal{U}\times\mathbb{R}$. Similarly, as $\zeta_N'' \not\equiv 0$ on $[\tilde{\rho}_m, \rho_0]$, $\zeta_N'' \leq 0$ on $[N, N+2]$ and $\zeta_N'' = 0$ otherwise, we find from the non-negative definiteness of (a^{ij}) that

$$\zeta_N''(\tilde{U})a^{ij}\partial_i\tilde{U}\partial_j\tilde{U} \leq \begin{cases} C_* \sup_{\tilde{\Omega}_{\rho_0}} a^{ij}\partial_i\tilde{U}\partial_j\tilde{U} =: M, & \text{in} \quad \Omega_{\rho_0}, \\ 0, & \text{otherwise}, \end{cases}$$

where $C_* = \max_{t\in[\tilde{\rho}_m,\rho_0]} \zeta_N''$ is independent of N due to the construction of $\{\zeta_N\}_{N\geq N_0}$ and $\tilde{\Omega}_{\rho_0} = \{(x,t) \in \mathcal{U}\times\mathbb{R} : \tilde{U}(x,t) < \rho_0\}$. We then deduce from (3.9) that

$$\int_{\mathcal{U}} \left(\zeta_N(\tilde{U}) - (N+1)\right) \mathrm{d}\mu_t \leq \int_{\mathcal{U}} \left(\zeta_N(\tilde{U}) - (N+1)\right) \mathrm{d}\mu_s + M(t-s), \quad \forall t > s,$$

which gives

$$0 \leq \int_{\mathcal{U}} \zeta_N(\tilde{U})\mathrm{d}\mu_t \leq \int_{\mathcal{U}} \zeta_N(\tilde{U})\mathrm{d}\mu_s + (N+1)\left[\mu_t(\mathcal{U}) - \mu_s(\mathcal{U})\right] + M(t-s), \quad \forall t > s. \tag{3.10}$$

If $\mu_t(\mathcal{U}) < \mu_s(\mathcal{U})$, then $(N+1)\left[\mu_t(\mathcal{U}) - \mu_s(\mathcal{U})\right] \to -\infty$ as $N \to \infty$, while the construction of $\tilde{U}$ yields $\limsup_{N\to\infty} \int_{\mathcal{U}} \zeta_N(\tilde{U})u(x,s)\mathrm{d}x < \infty$. This leads to a contradiction. Thus, $\mu_t(\mathcal{U}) \geq \mu_s(\mathcal{U})$ for all $t > s$.

Since $s \in \mathbb{R}$ is arbitrary and $(\mu_t)_{t\in\mathbb{R}}$ is T-periodic, there must hold $\mu_t(\mathcal{U}) = \mu_s(\mathcal{U})$ for all $t \geq s$. Let $C = \mu_t(\mathcal{U})$. Then, $C > 0$ thanks to **Step 2**. This completes the proof.

4. Proof of Theorem B

In Subsection 4.1, we introduce the concept of weak periodic probability measures. In Subsection 4.2, we study the limiting properties of weak periodic probability measures under the weak*-topology. The proof of Theorem B is given in Subsection 4.3.

4.1. Weak periodic probability measure

We introduce the following weak version of periodic probability measures.

Definition 4.1 *(Weak periodic probability measure).* A σ-finite Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is called a *weak periodic probability measure (with period T)* if

(1) for each $\eta \in C_c(\mathbb{R})$, there holds $\iint_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu = \int_{\mathbb{R}} \eta\mathrm{d}t$;
(2) for each $\phi \in C_c(\mathcal{U}\times\mathbb{R}) \cap C_T(\mathcal{U}\times\mathbb{R})$, there exists $C_\phi \in \mathbb{R}$ such that

$$\iint_{\mathcal{U}\times[t,t+T)} \phi\mathrm{d}\mu = C_\phi, \quad \forall t \in \mathbb{R}.$$

Let $\mathcal{M}_T(\mathcal{U}\times\mathbb{R})$ be the set of all weak periodic probability measures on $\mathcal{U}\times\mathbb{R}$ as in Definition 4.1. For convenience, we introduce the following notion.

Definition 4.2. A σ-finite Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is said to admit *t-sections*, denoted by $(\mu_t)_{t\in\mathbb{R}}$, if there are σ-finite Borel measures $\{\mu_t, t\in\mathbb{R}\}$ on $\mathcal{U}$ such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$, namely,

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \phi(x,t)\mathrm{d}\mu = \int\limits_{\mathbb{R}}\int\limits_{\mathcal{U}} \phi(x,t)\mathrm{d}\mu_t\mathrm{d}t, \quad \forall \phi\in C_0(\mathcal{U}\times\mathbb{R}).$$

The next result justifies the notion of weak periodic probability measures.

Lemma 4.1. *Let μ be a σ-finite Borel measure on $\mathcal{U}\times\mathbb{R}$. If μ admits t-sections $(\mu_t)_{t\in\mathbb{R}}$, then $\mu\in\mathcal{M}_T(\mathcal{U}\times\mathbb{R})$ if and only if $\mu_{t+T}=\mu_t$ and $\mu_t(\mathcal{U})=1$ for a.e. $t\in\mathbb{R}$.*

Proof. The sufficiency is obvious. We show the necessity. If $\mu=(\mu_t)_{t\in\mathbb{R}}\in\mathcal{M}_T(\mathcal{U}\times\mathbb{R})$, then conditions (1)-(2) in Definition 4.1 read

(1) for each $\eta\in C_c(\mathbb{R})$, $\int_{\mathbb{R}}\eta(t)\mu_t(\mathcal{U})\mathrm{d}t = \int_{\mathbb{R}}\eta\mathrm{d}t$;
(2) for each $\phi\in C_c(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$, there is a constant $C_\phi\in\mathbb{R}$ such that

$$\int\limits_t^{t+T}\int\limits_{\mathcal{U}} \phi(x,s)\mathrm{d}\mu_s\mathrm{d}s = C_\phi, \quad \forall t\in\mathbb{R}.$$

Set $f(t) := \mu_t(\mathcal{U})$ for $t\in\mathbb{R}$. We see from (1) that $f\in L^1_{loc}(\mathbb{R})$ and $h\mapsto\int_{\mathbb{R}}\eta(t+h)f(t)\mathrm{d}t = \int_{\mathbb{R}}\eta\mathrm{d}t$ is a constant function on $\mathbb{R}$, and hence, $\int_{\mathbb{R}}\eta'(t+h)f(t)\mathrm{d}t = 0$ for all $h\in\mathbb{R}$. Since $\eta\in C_c(\mathbb{R})$ is arbitrary, the distributional derivative of f is 0, implying the existence of some $c>0$ such that $f(t)=c$ for a.e. $t\in\mathbb{R}$. Thus, (1) reads $\int_{\mathbb{R}} c\eta(t+h)\mathrm{d}t = \int_{\mathbb{R}}\eta\mathrm{d}t$, which implies $c=1$. That is, $\mu_t(\mathcal{U})=1$ for a.e. $t\in\mathbb{R}$.

Let $\phi\in C_c(\mathcal{U})$, which can be seen as an element in $C_c(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$. Obviously, the function $t\mapsto\int_{\mathcal{U}}\phi(x)\mathrm{d}\mu_t$ is locally integrable. Thus, the functions $t\mapsto\int_0^t\int_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau$ and $t\mapsto\int_{-T}^t\int_{\mathcal{U}}\phi\mathrm{d}\mu_{\tau+T}\mathrm{d}\tau$ are absolutely continuous on $\mathbb{R}$, and hence, there exist subsets $J^1_\phi, J^2_\phi\subset\mathbb{R}$ satisfying $|\mathbb{R}\setminus J^1_\phi|=0$ and $|\mathbb{R}\setminus J^2_\phi|=0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t\int\limits_{\mathcal{U}}\phi\mathrm{d}\mu_\tau\mathrm{d}\tau = \int\limits_{\mathcal{U}}\phi\mathrm{d}\mu_t, \quad \forall t\in J^1_\phi$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-T}^t\int\limits_{\mathcal{U}}\phi\mathrm{d}\mu_{\tau+T}\mathrm{d}\tau = \int\limits_{\mathcal{U}}\phi\mathrm{d}\mu_{t+T}, \quad \forall t\in J^2_\phi.$$

It follows that for each $t \in J_\phi := J_\phi^1 \cap J_\phi^2$ there holds

$$
\begin{aligned}
\int_{\mathcal{U}} \phi \mathrm{d}\mu_{t+T} - \int_{\mathcal{U}} \phi \mathrm{d}\mu_t &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{-T}^{t} \int_{\mathcal{U}} \phi \mathrm{d}\mu_{\tau+T} \mathrm{d}\tau - \int_{0}^{t} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{t+T} \int_{\mathcal{U}} \phi \mathrm{d}\mu_\tau \mathrm{d}\tau \\
&= 0,
\end{aligned}
\tag{4.1}
$$

where we used (2) in the last equality.

Let $\mathcal{D}$ be a countable basis of $C_c(\mathcal{U})$. For each $\phi \in \mathcal{D}$, there is a set $J_\phi \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus J_\phi| = 0$ such that (4.1) holds for all $t \in J_\phi$. Setting $J := \cap_{\phi \in \mathcal{D}} J_\phi$, we find $|\mathbb{R} \setminus J| = 0$ and

$$
\int_{\mathcal{U}} \phi \mathrm{d}\mu_{t+T} = \int_{\mathcal{U}} \phi \mathrm{d}\mu_t, \quad \forall \phi \in \mathcal{D}, \quad t \in J. \tag{4.2}
$$

For each $\phi \in C_c(\mathcal{U})$, there is a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ such that ϕ_n converges to ϕ uniformly in $\mathcal{U}$ as $n \to \infty$. It then follows from (4.2) that $\int_{\mathcal{U}} \phi(x) \mathrm{d}\mu_{t+T} = \int_{\mathcal{U}} \phi(x) \mathrm{d}\mu_t$ for all $t \in J$, which yields $\mu_t = \mu_{t+T}$ for all $t \in J$. This completes the proof. □

The following result is a simple consequence of Lemma 4.1.

Corollary 4.1. *Let μ be a σ-finite Borel measure on $\mathcal{U} \times \mathbb{R}$. If μ admits a density $u \in C(\mathcal{U} \times \mathbb{R})$, then $\mu \in \mathcal{M}_T(\mathcal{U} \times \mathbb{R})$ if and only if $u \in C_T(\mathcal{U} \times \mathbb{R})$ and satisfies $\int_{\mathcal{U}} u(x,t) \mathrm{d}x = 1$ for a.e. $t \in \mathbb{R}$.*

The next result shows that any σ-finite Borel measure on $\mathcal{U} \times \mathbb{R}$ satisfying an additional condition admits t-sections.

Lemma 4.2. *Let μ be a σ-finite Borel measure on $\mathcal{U} \times \mathbb{R}$ satisfying*

$$
\iint_{\mathcal{U} \times \mathbb{R}} \eta \mathrm{d}\mu = \int_{\mathbb{R}} \eta \mathrm{d}t, \quad \forall \eta \in C_c(\mathbb{R}).
$$

Then, μ admits t-sections $(\mu_t)_{t \in \mathbb{R}}$. Moreover, $\mu_t(\mathcal{U}) = 1$ for a.e. $t \in \mathbb{R}$.

Proof. For $\phi \in C_c(\mathcal{U})$, let L_ϕ be the functional on $C_c(\mathbb{R})$ defined by

$$
L_\phi \eta = \iint_{\mathcal{U} \times \mathbb{R}} \phi(x) \eta(t) \mathrm{d}\mu, \quad \forall \eta \in C_c(\mathbb{R}).
$$

Obviously, L_ϕ is continuous and linear. The Riesz representation theorem yields the existence of a signed Borel measure ν_ϕ on $\mathbb{R}$ such that

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \phi(x)\eta(t)\mathrm{d}\mu = \int\limits_{\mathbb{R}} \eta \mathrm{d}\nu_\phi, \quad \eta \in C_c(\mathbb{R}). \tag{4.3}$$

Moreover, ν_ϕ is a Borel measure if and only if ϕ is non-negative. We see from (4.3) that

$$\left|\int\limits_{\mathbb{R}} \eta(t)\mathrm{d}\nu_{|\phi|}\right| \le |\phi|_\infty \iint\limits_{\mathcal{U}\times\mathbb{R}} |\eta|\mathrm{d}\mu = |\phi|_\infty \int\limits_{\mathbb{R}} |\eta|\mathrm{d}t, \quad \forall \eta \in C_c(\mathbb{R}), \tag{4.4}$$

which implies that both $\nu_{|\phi|}$ and ν_ϕ have no atom. By the Radon-Nikodym theorem, there exists a unique $f_\phi \in L^1_{loc}(\mathbb{R})$ such that $\mathrm{d}\nu_\phi = f_\phi(t)\mathrm{d}t$. Clearly, f_ϕ is non-negative if and only if ϕ is non-negative. It follows from (4.4) that $|f_\phi|_\infty \le |\phi|_\infty$, namely, $f_\phi \in L^\infty(\mathbb{R})$. Thus, there is a subset $J_\phi \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus J_\phi| = 0$ such that $|f_\phi(t)| \le |\phi|_\infty$ for all $t \in J_\phi$.

Set $C_c^+(\mathcal{U}) := \{\phi \in C_c(\mathcal{U}) : \phi \ge 0\}$. Let $C_*(\mathcal{U})$ be the completion of $C_c(\mathcal{U})$ under the supremum norm. It is well known that $C_*(\mathcal{U})$ is a separable metric space. As subspaces of $C_*(\mathcal{U})$, both $C_c^+(\mathcal{U})$ and $C_c(\mathcal{U})$ are also separable.

Let $\mathcal{D}^+$ be a countable basis of $C_c^+(\mathcal{U})$. We extend $\mathcal{D}^+$ to be a countable basis, denoted by $\mathcal{D}$, of $C_c(\mathcal{U})$. For each $\phi \in \mathcal{D}$, there exists a subset J_ϕ satisfying $|\mathbb{R} \setminus J_\phi| = 0$ such that $|f_\phi(t)| \le |\phi|_\infty$ for all $t \in J_\phi$. Setting $J := \cap_{\phi\in\mathcal{D}} J_\phi$, it follows that $|\mathbb{R} \setminus J| = 0$ and $|f_\phi(t)| \le |\phi|_\infty$ for all $\phi \in \mathcal{D}$ and $t \in J$.

Fix $t \in J$ and define the functional K_t on $\mathcal{D}$ by setting

$$K_t\phi = f_\phi(t), \quad \forall \phi \in \mathcal{D}.$$

Since K_t is linear and $|K_t\phi| \le |f_\phi(t)| \le |\phi|_\infty$ for all $\phi \in \mathcal{D}$, K_t can be extended to be a continuous and linear functional $\overline{K}_t$ on $C_c(\mathcal{U})$. Moreover, $\overline{K}_t$ is positive. To see this, let $\phi \in C_c^+(\mathcal{U})$. As $\mathcal{D}^+$ is dense in $C_c^+(\mathcal{U})$, there exists a sequence $\{\phi_n\}_n \subset \mathcal{D}^+$ such that ϕ_n uniformly converges to ϕ in $\mathcal{U}$ as $n \to \infty$. Thus,

$$\overline{K}_t(\phi) = \lim_{n\to\infty} K_t(\phi_n) = \lim_{n\to\infty} f_{\phi_n}(t) \ge 0.$$

We then apply Riesz representation theorem to find a Borel measure μ_t on $\mathcal{U}$ such that

$$\overline{K}_t\phi = \int\limits_{\mathcal{U}} \phi \mathrm{d}\mu_t, \quad \forall \phi \in C_c(\mathcal{U}).$$

For $t_* \in \mathbb{R} \setminus J$, we define μ_{t_*} to be the zero measure on $\mathcal{U}$. We claim that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$, that is, μ admits t-sections. Indeed, for any $\phi \in \mathcal{D}$, we see from the definition of $\{\mu_t, t \in \mathbb{R}\}$ that

$$\int_{\mathcal{U}} \phi(x)\mathrm{d}\mu_t = \begin{cases} f_\phi(t), & t \in J, \\ 0, & t \in \mathbb{R} \setminus J. \end{cases}$$

As $|\mathbb{R} \setminus J| = 0$, we find for each $\eta \in C_c(\mathbb{R})$

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \phi(x)\eta(t)\mathrm{d}\mu_t\mathrm{d}t = \int_{\mathbb{R}} \eta(t) \int_{\mathcal{U}} \phi(x)\mathrm{d}\mu_t\mathrm{d}t = \int_{\mathbb{R}} \eta(t) f_\phi(t)\mathrm{d}t,$$

which together with the definition of f_ϕ imply that

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \phi(x)\eta(t)\mathrm{d}\mu_t\mathrm{d}t = \iint_{\mathcal{U}\times\mathbb{R}} \phi(x)\eta(t)\mathrm{d}\mu. \tag{4.5}$$

Let

$$\mathcal{F} = \left\{ \sum_{k=1}^{n} c_k\phi_k\eta_k : n \in \mathbb{N},\ \{\phi_k\}_{k=1}^n \subset \mathcal{D},\ \{\eta_k\}_{k=1}^n \subset C_0(\mathbb{R}) \text{ and } \{c_k\}_{k=1}^n \subset \mathbb{R} \right\}.$$

As $\mathcal{D}$ is dense in $C_c(\mathcal{U})$, $\mathcal{F}$ is dense in $C_0(\mathcal{U} \times \mathbb{R})$. We conclude from (4.5) that

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \psi(x,t)\mathrm{d}\mu_t\mathrm{d}t = \iint_{\mathcal{U}\times\mathbb{R}} \psi(x,t)\mathrm{d}\mu, \quad \forall \psi \in C_0(\mathcal{U} \times \mathbb{R}),$$

that is, $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$. Thus,

$$\int_{\mathbb{R}} \eta\mathrm{d}t = \iint_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu = \int_{\mathbb{R}} \eta\mu_t(\mathcal{U})\mathrm{d}t, \quad \forall \eta \in C_c(\mathbb{R}),$$

which implies that $\mu_t(\mathcal{U}) = 1$ for a.e. $t \in \mathbb{R}$. This completes the proof. □

As a simple consequence of Lemma 4.1 and Lemma 4.2, we have the following result.

Corollary 4.2. *Let $\mu \in \mathcal{M}_T(\mathcal{U} \times \mathbb{R})$. Then μ admits t-sections $(\mu_t)_{t\in\mathbb{R}}$ and there exists a subset $J \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus J| = 0$ such that $\mu_{t+T} = \mu_t$ and $\mu_t(\mathcal{U}) = 1$ for all $t \in J$.*

4.2. Limiting properties of weak periodic probability measures

We recall the weak*-topology for Borel measures on $\mathcal{U} \times \mathbb{R}$.

Definition 4.3. A sequence of σ-finite Borel measures $\{\mu^n, n \in \mathbb{N}\}$ on $\mathcal{U} \times \mathbb{R}$ is said to converge to a σ-finite Borel measure μ on $\mathcal{U} \times \mathbb{R}$ under the *weak*-topology* as $n \to \infty$ if

$$\lim_{n\to\infty}\iint\limits_{\mathcal{U}\times\mathbb{R}}\phi\mathrm{d}\mu^n=\iint\limits_{\mathcal{U}\times\mathbb{R}}\phi\mathrm{d}\mu,\quad\forall\phi\in C_0(\mathcal{U}\times\mathbb{R}).$$

Lemma 4.3. *Let $\{\mu^n, n\in\mathbb{N}\}$ and μ be σ-finite Borel measures on $\mathcal{U}\times\mathbb{R}$. Assume that μ^n converges to μ under the weak*-topology as $n\to\infty$. Then for each $\phi\in C_c(\mathcal{U}\times\mathbb{R})$, there is a set $J_\phi\subset\mathbb{R}$ such that $\mathbb{R}\setminus J_\phi$ is at most countable, in particular $|\mathbb{R}\setminus J_\phi|=0$, and*

$$\iint\limits_{\mathcal{U}\times[s,t)}\phi\mathrm{d}\mu=\lim_{n\to\infty}\iint\limits_{\mathcal{U}\times[s,t)}\phi\mathrm{d}\mu^n,\quad\forall s,t\in J_\phi,\ s<t.$$

Proof. Let us fix $\phi\in C_c(\mathcal{U}\times\mathbb{R})$ and consider the measures ν and $\{\nu_n, n\in\mathbb{N}\}$ on $\mathbb{R}$ defined by

$$\nu(I)=\iint\limits_{\mathcal{U}\times I}\phi\mathrm{d}\mu,\quad\nu_n(I)=\iint\limits_{\mathcal{U}\times I}\phi\mathrm{d}\mu^n,\quad I\in\mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. As μ^n converges to μ under the weak*-topology as $n\to\infty$, we deduce that ν_n converges to ν under the weak*-topology as $n\to\infty$, that is, $\lim_{n\to\infty}\int_{\mathbb{R}}\eta\mathrm{d}\nu_n=\int_{\mathbb{R}}\eta\mathrm{d}\nu$ for all $\eta\in C_c(\mathbb{R})$. As a measure on $\mathbb{R}$, ν admits at most countably many atoms. Let $\mathcal{S}_\phi$ be the set of atoms of ν. We see that if s, $t\notin\mathcal{S}_\phi$, namely, $[s,t)$ is a continuous set of ν, then the Portmanteau theorem implies that $\lim_{n\to\infty}\nu_n([s,t))=\nu([s,t))$. Setting $J_\phi:=\mathbb{R}\setminus\mathcal{S}_\phi$, the conclusion follows. □

In the next result, we show that any limiting measure under the weak*-topology of a sequence of measures in $\mathcal{M}_T(\mathcal{U}\times\mathbb{R})$ is periodic in the sense of Definition 4.1 (2).

Lemma 4.4. *Let $\{\mu^n, n\in\mathbb{N}\}\subset\mathcal{M}_T(\mathcal{U}\times\mathbb{R})$ and μ be a σ-finite Borel measure on $\mathcal{U}\times\mathbb{R}$. If μ^n converges to μ under the weak*-topology as $n\to\infty$, then for each bounded $\phi\in C_T(\mathcal{U}\times\mathbb{R})$, there exists a constant $C_\phi\in\mathbb{R}$ such that*

$$\iint\limits_{\mathcal{U}\times[t,t+T)}\phi\mathrm{d}\mu=C_\phi,\quad\forall t\in\mathbb{R}.\tag{4.6}$$

In particular, $\mu(\mathcal{U}\times(a,b))<\infty$ for all $-\infty<a<b<\infty$.

Proof. Clearly, it suffices to prove (4.6) for non-negative and bounded functions $\phi\in C_T(\mathcal{U}\times\mathbb{R})$.

We first prove (4.6) for non-negative functions $\phi\in C_c(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$. By Definition 4.1 (2), for each $n\in\mathbb{N}$ and $\phi\in C_c(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$ there is $C_{n,\phi}\in\mathbb{R}$ such that

$$\iint\limits_{\mathcal{U}\times[t,t+T)}\phi\mathrm{d}\mu^n=C_{n,\phi},\quad\forall t\in\mathbb{R}.$$

Let us fix a non-negative function $\phi \in C_c(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$ and let J_ϕ be as in Lemma 4.3. Let $\tilde{J}_\phi := J_\phi \cap (J_\phi - T)$. Then $|\mathbb{R} \setminus \tilde{J}_\phi| = 0$ and for any $t \in \tilde{J}_\phi$, we have $t \in J_\phi$ and $t + T \in J_\phi$. Applying Lemma 4.3, we find some $C_\phi \geq 0$ such that

$$\iint\limits_{\mathcal{U}\times[t,t+T)} \phi \mathrm{d}\mu = \lim_{n\to\infty} \iint\limits_{\mathcal{U}\times[t,t+T)} \phi \mathrm{d}\mu^n = \lim_{n\to\infty} C_{n,\phi} =: C_\phi, \tag{4.7}$$

holds for all $t \in \tilde{J}_\phi$.

Next, we show that (4.7) holds for all $t \in \mathbb{R}$. Let $t_* \in \mathbb{R} \setminus \tilde{J}_\phi$. Since $\tilde{J}_\phi$ is dense in $\mathbb{R}$, there is an increasing sequence $\{t_n\}_{n\in\mathbb{N}} \subset \tilde{J}_\phi$ such that $t_n \to t_*$ as $n \to \infty$. Define the Borel measure ν_ϕ on $\mathbb{R}$ by setting

$$\nu_\phi(I) = \iint\limits_{\mathcal{U}\times I} \phi \mathrm{d}\mu, \quad \forall I \in \mathcal{B}(\mathbb{R}).$$

Clearly, ν_ϕ is σ-finite. Applying the dominated convergence theorem to the sequence $t \mapsto \mathbb{1}_{[t_n,t_n+T)}(t)$, we deduce from (4.7) that

$$\nu_\phi([t_*, t_* + T)) = \lim_{n\to\infty} \nu_\phi([t_n, t_n + T)) = C_\phi.$$

Hence, (4.7) holds for all $t \in \mathbb{R}$. That is, (4.6) holds for all non-negative $\phi \in C_c(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$.

Finally, for any non-negative and bounded $\phi \in C_T(\mathcal{U} \times \mathbb{R})$, there is a non-decreasing sequence of non-negative functions $\{\phi_n\}_n \subset C_c(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$ such that ϕ_n converges locally uniformly in $\mathcal{U} \times \mathbb{R}$ to ϕ as $n \to \infty$. As $\iint_{\mathcal{U}\times[t,t+T)} \phi \mathrm{d}\mu$ is finite, we apply the monotone convergence theorem to find

$$\iint\limits_{\mathcal{U}\times[t,t+T)} \phi \mathrm{d}\mu = \lim_{n\to\infty} \iint\limits_{\mathcal{U}\times[t,t+T)} \phi_n \mathrm{d}\mu = \lim_{n\to\infty} C_{\phi_n}, \quad \forall t \in \mathbb{R}.$$

This completes the proof. □

4.3. Proof of Theorem B

Let U be an unbounded Lyapunov function with respect to $\mathcal{L}$ with an essential lower bound ρ_m and a Lyapunov constant γ. The proof is done within six steps. To highlight the dependence of $\mathcal{L}$ on A and V, we write $\mathcal{L}_{A,V}$ for $\mathcal{L}$.

Step 1. We construct a candidate measure on $\mathcal{U} \times \mathbb{R}$.

Since $A = (a^{ij})$ is semi-positive definite, the matrix $A + \epsilon I$ is locally uniformly positive definite for any $\epsilon > 0$, where I is the $d \times d$ identity matrix.

We identify $C_T^\infty(\mathcal{U} \times \mathbb{R})$ with $C^\infty(\mathcal{U} \times \mathbb{S}_T)$ and write $C_T^\infty(\mathcal{U} \times \mathbb{R}) \approx C^\infty(\mathcal{U} \times \mathbb{S}_T)$, where $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$. By a partition of unity (see e.g. [27]), there exist a locally finite open cover $(\mathcal{V}_\beta)_{\beta\in\mathcal{B}}$ of $\mathcal{U} \times \mathbb{S}_T$ and functions $(f_\beta)_{\beta\in\mathcal{B}} \subset C_c^\infty(\mathcal{U} \times \mathbb{S}_T)$ such that

(1) $\mathrm{supp}(f_\beta) \subset \mathcal{V}_\beta$ for all $\beta \in \mathcal{B}$;
(2) $0 \le f_\beta(x,t) \le 1$ for all $(x,t) \in \mathcal{U} \times \mathbb{S}_T$ and $\beta \in \mathcal{B}$;
(3) $\sum_{\beta\in\mathcal{B}} f_\beta(x,t) = 1$ for all $(x,t) \in \mathcal{U} \times \mathbb{S}_T$.

Set $C_\beta := \frac{\gamma}{2(1+\max_{\overline{\mathcal{V}}_\beta} |D^2U|)}$, where D^2U denotes the Hessian of U. For $n \in \mathbb{N}$, let

$$\epsilon_n(x,t) := \frac{1}{n} \sum_{\beta\in\mathcal{B}} f_\beta(x,t) C_\beta, \quad (x,t) \in \mathcal{U} \times \mathbb{S}_T.$$

Clearly, $\epsilon_n \in C^\infty(\mathcal{U} \times \mathbb{S}_T) \approx C_T^\infty(\mathcal{U} \times \mathbb{R})$ for each n. Moreover, ϵ_n converges locally uniformly in $\mathcal{U} \times \mathbb{R}$ to 0 as $n \to \infty$ and $\epsilon_n \sum_{i=1}^d \partial_{ii}^2 U \le \frac{\gamma}{2}$ in $\mathcal{U} \times \mathbb{R}$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, writing $A_n = (a_n^{ij}) := A + \epsilon_n I$, we have

$$\mathcal{L}_{A_n,V} U \le -\frac{\gamma}{2} \quad \text{in} \quad (\mathcal{U} \times \mathbb{R}) \setminus \overline{\Omega}_{\rho_m}. \tag{4.8}$$

That is, U is an unbounded Lyapunov function with respect to $\mathcal{L}_{A_n,V}$ for each $n \in \mathbb{N}$ with a uniform essential lower bound ρ_m and a uniform Lyapunov constant $\frac{\gamma}{2}$.

Applying Theorem A, we find that for each $n \in \mathbb{N}$, there exists a periodic probability solution μ^n to (1.3) with A replaced by A_n. We see that $\sup_n \mu^n(K) < \infty$ for any compact set $K \subset \mathcal{U} \times \mathbb{R}$. Then, we apply [17, Corollary A2.6.V.] to conclude the existence of a subsequence, still denoted by $\{\mu^n\}_n$, such that μ^n converges to some σ-finite Borel measure μ on $\mathcal{U} \times \mathbb{R}$ under the weak*-topology as $n \to \infty$. The measure μ is the candidate.

We apply Lemma 4.4 to conclude that for each $\phi \in C_c(\mathcal{U} \times \mathbb{R}) \cap C_T(\mathcal{U} \times \mathbb{R})$, there exists $C_\phi \in \mathbb{R}$ such that

$$\iint\limits_{\mathcal{U}\times[t,t+T)} \phi \mathrm{d}\mu = C_\phi, \quad \forall t \in \mathbb{R}. \tag{4.9}$$

Step 2. We show that μ is a measure solution to (1.3) in $\mathcal{U} \times \mathbb{R}$. Since μ^n is a periodic probability solution to (1.3) with A replaced by A_n, there holds

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}_{A_n,V}\phi \mathrm{d}\mu^n = 0, \quad \forall \phi \in C_0^{2,1}(\mathcal{U} \times \mathbb{R}).$$

Fix $\phi \in C_0^{2,1}(\mathcal{U} \times \mathbb{R})$. As $\max_{\mathrm{supp}(\phi)} |\mathcal{L}_{A_n,V}\phi - \mathcal{L}_{A,V}\phi| \to 0$ as $n \to \infty$, we find

$$\left| \iint\limits_{\mathcal{U}\times\mathbb{R}} (\mathcal{L}_{A_n,V}\phi - \mathcal{L}_{A,V}\phi)\mathrm{d}\mu^n \right| \le |\mathcal{L}_{A_n,V}\phi - \mathcal{L}_{A,V}\phi|_\infty \times \sup_n \mu^n(\mathrm{supp}(\phi)) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{L}_{A,V}\phi \in C_0(\mathcal{U} \times \mathbb{R})$ and μ^n converges to μ under the weak*-topology as $n \to \infty$, we find

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}_{A,V}\phi \mathrm{d}\mu = \lim_{n\to\infty} \iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}_{A,V}\phi \mathrm{d}\mu^n.$$

Thus,

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}_{A,V}\phi \mathrm{d}\mu = \lim_{n\to\infty} \iint\limits_{\mathcal{U}\times\mathbb{R}} \mathcal{L}_{A_n,V}\phi \mathrm{d}\mu^n = 0.$$

Since $\phi \in C_0^{2,1}(\mathcal{U}\times\mathbb{R})$ is arbitrary, we conclude that μ is a measure solution to (1.3) in $\mathcal{U}\times\mathbb{R}$.

Step 3. We show that μ is non-trivial. Suppose on the contrary that μ is the zero measure so that $\lim_{n\to\infty}\mu^n(K) = 0$ for each compact set $K \subset \mathcal{U}\times\mathbb{R}$.

Fix $\rho_0 > \rho_m$. Due to (4.8), we can apply Theorem 2.2 to each μ^n on Ω_{ρ_1} for $\rho_1 \in (\rho_0,\infty)$ and then take $\rho_1 \to \infty$ to find

$$\begin{aligned}
&\mu^n\left(\bigcup_{s\in[t,t+T]} ((\mathcal{U}\setminus\Omega^s_{\rho_0})\times\{s\})\right)\\
&\quad\le \frac{C_*}{\gamma}\left(\max_{\overline{\Omega}_{\rho_0}} a_n^{ij}\partial_i U\partial_j U\right)\mu^n\left(\bigcup_{s\in[t,t+T]}(\Omega^s_{\rho_0}\times\{s\})\right), \quad \forall t\in\mathbb{R},
\end{aligned}$$

where $C_* > 0$ depends only on ρ_m and ρ_0.

As A_n converges locally uniformly in $\mathcal{U}\times\mathbb{R}$ to A as $n\to\infty$, there holds

$$\max_{\overline{\Omega}_{\rho_0}} a_n^{ij}\partial_i U\partial_j U \to \max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U \quad \text{as} \quad n\to\infty.$$

Thus, there is $C > 0$ independent of n such that

$$\mu^n\left(\bigcup_{s\in[t,t+T]} ((\mathcal{U}\setminus\Omega^s_{\rho_0})\times\{s\})\right) \le C\mu^n\left(\bigcup_{s\in[t,t+T]}(\Omega^s_{\rho_0}\times\{s\})\right), \quad \forall t\in\mathbb{R}.$$

Then, for each $t\in\mathbb{R}$,

$$\begin{aligned}
T &= \mu^n\left(\bigcup_{s\in[t,t+T]}(\mathcal{U}\times\{s\})\right)\\
&= \mu^n\left(\bigcup_{s\in[t,t+T]}(\mathcal{U}\setminus\Omega^s_{\rho_0})\times\{s\}\right) + \mu^n\left(\bigcup_{s\in[t,t+T]}(\Omega^s_{\rho_0}\times\{s\})\right)\\
&\le (1+C)\mu^n\left(\bigcup_{s\in[t,t+T]}(\Omega^s_{\rho_0}\times\{s\})\right)
\end{aligned}$$

$$\to 0 \quad \text{as} \quad n \to \infty,$$

which leads to a contradiction.

Step 4. We show that for any $\eta_1, \eta_2 \in C_c(\mathbb{R})$ with $\max \operatorname{supp}(\eta_1) < \min \operatorname{supp}(\eta_2)$, there holds

$$\begin{aligned}
&\iint\limits_{\mathcal{U}\times\mathbb{R}} \phi(x,t_2)\eta_2(t_2)\mathrm{d}\mu \int\limits_{\mathbb{R}} \eta_1(t_1)\mathrm{d}t_1 \\
&\quad = \iint\limits_{\mathcal{U}\times\mathbb{R}} \phi(x,t_1)\eta_1(t_1)\mathrm{d}\mu \int\limits_{\mathbb{R}} \eta_2(t_2)\mathrm{d}t_2 + \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left[\iint\limits_{\mathcal{U}\times[t_1,t_2)} \mathcal{L}_{A,V}\phi\mathrm{d}\mu\right] \eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2.
\end{aligned} \tag{4.10}$$

As μ^n, with a continuous density, is a periodic probability solution to (1.3) with A replaced by A_n, we see from Corollary 2.1 and Remark 2.1 that for each $\phi \in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, there holds

$$\int\limits_{\mathcal{U}} \phi(x,t_2)\mathrm{d}\mu^n_{t_2} = \int\limits_{\mathcal{U}} \phi(x,t_1)\mathrm{d}\mu^n_{t_1} + \int\limits_{t_1}^{t_2}\int\limits_{\mathcal{U}} \mathcal{L}_{A_n,V}\phi\mathrm{d}\mu^n_\tau\mathrm{d}\tau, \quad \forall t_1 < t_2.$$

Multiplying the above equality by $\eta_1(t_1)$ and then integrating the resulting equality with respect to t_1 over $\mathbb{R}$, we arrive at

$$\begin{aligned}
&\int\limits_{\mathbb{R}} \left[\int\limits_{\mathcal{U}} \phi(x,t_2)\mathrm{d}\mu^n_{t_2}\right] \eta_1(t_1)\mathrm{d}t_1 \\
&\quad = \int\limits_{\mathbb{R}} \left[\int\limits_{\mathcal{U}} \phi(x,t_1)\mathrm{d}\mu^n_{t_1}\right] \eta_1(t_1)\mathrm{d}t_1 \\
&\qquad + \int\limits_{\mathbb{R}} \left[\int\limits_{t_1}^{t_2}\int\limits_{\mathcal{U}} \mathcal{L}_{A_n,V}\phi\mathrm{d}\mu^n_\tau\mathrm{d}\tau\right] \eta_1(t_1)\mathrm{d}t_1, \quad \forall t_2 > \max \operatorname{supp}(\eta_1).
\end{aligned}$$

Multiplying the above equality by $\eta_2(t_2)$ and then integrating the resulting equality with respect to t_2 over $\mathbb{R}$, we deduce

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left[\int\limits_{\mathcal{U}} \phi(x,t_2)\mathrm{d}\mu^n_{t_2}\right] \eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2 \\
&\quad = \int\limits_{\mathbb{R}} \left[\int\limits_{\mathcal{U}} \phi(x,t_1)\mathrm{d}\mu^n_{t_1}\right] \eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2
\end{aligned}$$

$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{t_1}^{t_2}\int_{\mathcal{U}}\mathcal{L}_{A_n,V}\phi\mathrm{d}\mu_\tau^n\mathrm{d}\tau\right]\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2,$$

that is,

$$\begin{aligned}&\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_2)\eta_2(t_2)\mathrm{d}\mu^n\int_{\mathbb{R}}\eta_1(t_1)\mathrm{d}t_1\\&\quad=\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_1)\eta_1(t_1)\mathrm{d}\mu^n\int_{\mathbb{R}}\eta_2(t_2)\mathrm{d}t_2\\&\qquad+\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}(\mathcal{L}_{A_n,V}\phi-\mathcal{L}_{A,V}\phi)\mathrm{d}\mu^n\right]\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2\\&\qquad+\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu^n\right]\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2.\end{aligned}\tag{4.11}$$

Since μ^n converges to μ under the weak*-topology as $n\to\infty$, we find

$$\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_2)\eta_2(t_2)\mathrm{d}\mu^n\to\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_2)\eta_2(t_2)\mathrm{d}\mu$$

and

$$\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_1)\eta_1(t_1)\mathrm{d}\mu^n\to\iint_{\mathcal{U}\times\mathbb{R}}\phi(x,t_1)\eta_1(t_1)\mathrm{d}\mu,$$

as $n\to\infty$. Since

$$\iint_{\mathcal{U}\times[t_1,t_2)}|\mathcal{L}_{A_n,V}\phi-\mathcal{L}_{A,V}\phi|\,\mathrm{d}\mu^n\le|\mathcal{L}_{A_n,V}\phi-\mathcal{L}_{A,V}\phi|_\infty\times(t_2-t_1),$$

we find

$$\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\iint_{\mathcal{U}\times[t_1,t_2)}|\mathcal{L}_{A_n,V}\phi-\mathcal{L}_{A,V}\phi|\,\mathrm{d}\mu^n\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2=0.$$

As $\mathcal{L}_{A,V}\phi\in C_c(\mathcal{U}\times\mathbb{R})$, we apply Lemma 4.3 to find a subset set $J\subset\mathbb{R}$ (depending on $\mathcal{L}_{A,V}\phi$) satisfying $|\mathbb{R}\setminus J|=0$ such that

$$\lim_{n\to\infty}\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu^n=\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu,\quad\forall t_1,\,t_2\in J\text{ with }t_1<t_2,$$

which enables us to use the dominated convergence theorem to obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu^n\right]\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2\\
&\quad=\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu^n\right]\mathbb{1}_{\{(s,t)\in\mathbb{R}\times\mathbb{R}:s<t\}}(t_1,t_2)\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2\\
&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu\right]\mathbb{1}_{\{(s,t)\in\mathbb{R}\times\mathbb{R}:s<t\}}(t_1,t_2)\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2\\
&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\iint_{\mathcal{U}\times[t_1,t_2)}\mathcal{L}_{A,V}\phi\mathrm{d}\mu\right]\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2.
\end{aligned}
$$

Thus, letting $n\to\infty$ in (4.11), we find (4.10).

Step 5. We show the existence of some $C>0$ such that

$$
\iint_{\mathcal{U}\times\mathbb{R}}\eta\mathrm{d}\mu=C\int_{\mathbb{R}}\eta\mathrm{d}t,\quad\forall\eta\in C_c(\mathbb{R}).\tag{4.12}
$$

Clearly, it suffices to show that (4.12) holds for any $\eta\in C_c^+(\mathbb{R})$, where $C_c^+(\mathbb{R})$ is the set of non-negative functions in $C_c(\mathbb{R})$.

We first claim that for any $\eta\in C_c^+(\mathbb{R})$ satisfying $|\mathrm{supp}(\eta)|<T$, there holds

$$
\iint_{\mathcal{U}\times\mathbb{R}}\eta\mathrm{d}\mu\le\iint_{\mathcal{U}\times\mathbb{R}}\eta(\cdot-\delta)\mathrm{d}\mu,\quad\forall\delta>T.\tag{4.13}
$$

To see this, let us fix $s\in\mathbb{R}$ and $h\in(0,T)$. Arguing as in **Step 3** in the proof of Theorem A, we find a non-negative function $\tilde{U}\in C_T^{2,1}(\mathcal{U}\times\mathbb{R})$ satisfying

(1) $\iint_{\mathcal{U}\times[s,s+h]}\tilde{U}\mathrm{d}\mu<\infty$;
(2) $\inf_{(\mathcal{U}\setminus\mathcal{U}_n)\times\mathbb{R}}\tilde{U}\to\infty$ as $n\to\infty$;
(3) $\mathcal{L}_{A,V}\tilde{U}\le 0$ in $(\mathcal{U}\times\mathbb{R})\setminus\tilde{\Omega}_{\tilde{\rho}_m}$ for some $\tilde{\rho}_m>0$, where $\tilde{\Omega}_{\tilde{\rho}_m}=\big\{(x,t)\in\mathcal{U}\times\mathbb{R}:\tilde{U}(x,t)<\tilde{\rho}_m\big\}$.

Fix $\rho_0>\tilde{\rho}_m$ and set $N_0=[\rho_0]+1$, where $[\rho_0]$ is the integer part of ρ_0. Let $\{\zeta_N\}_{N\ge N_0}$ be a family of smooth and non-decreasing functions on $[0,\infty)$ satisfying

$$\zeta_N(t)=\begin{cases}0, & t\in[0,\tilde\rho_m],\\ t, & t\in[\rho_0,N],\\ N+1, & t\in[N+2,\infty),\end{cases}\qquad \text{and}\quad \zeta_N''\le 0 \text{ on } [N,N+2].$$

In addition, we let the functions $\{\zeta_N\}_{N\ge N_0}$ coincide on $[0,\rho_0]$.

Obviously, $\zeta_N(\tilde U)-(N+1)\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$. Let $\eta\in C_c(\mathbb{R})$ be non-negative, non-zero and satisfy $\mathrm{supp}(\eta)\subset[s,s+h]$. Setting $\phi=\zeta_N(\tilde U)-(N+1)$ in (4.10) with $\eta_1=\eta$ and $\eta_2=\eta(\cdot-\delta)$ for $\delta>T$, we find

$$\begin{aligned}&\iint\limits_{\mathcal{U}\times\mathbb{R}}\left[\zeta_N(\tilde U)-(N+1)\right]\eta_2\mathrm{d}\mu\int\limits_{\mathbb{R}}\eta_1(t_1)\mathrm{d}t_1\\ &\quad=\iint\limits_{\mathcal{U}\times\mathbb{R}}\left[\zeta_N(\tilde U)-(N+1)\right]\eta_1\mathrm{d}\mu\int\limits_{\mathbb{R}}\eta_2(t_2)\mathrm{d}t_2 \\ &\qquad+\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\left(\iint\limits_{\mathcal{U}\times[t_1,t_2)}\left[\zeta_N'(\tilde U)\mathcal{L}_{A,V}\tilde U+\zeta_N''(\tilde U)a^{ij}\partial_i\tilde U\partial_j\tilde U\right]\mathrm{d}\mu\right)\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2.\end{aligned}\tag{4.14}$$

Since $\zeta_N'=0$ on $[0,\tilde\rho_m]$, $\zeta_N'\ge 0$ and $\mathcal{L}\tilde U\le 0$ in $(\mathcal{U}\times\mathbb{R})\setminus\tilde\Omega_{\tilde\rho_m}$, we have $\zeta_N'(\tilde U)\mathcal{L}\tilde U\le 0$ in $\mathcal{U}\times\mathbb{R}$. As $\zeta_N''\not\equiv 0$ on $[\tilde\rho_m,\rho_0]$, $\zeta_N''\le 0$ on $[N,N+2]$ and $\zeta_N''=0$ otherwise, we find from the non-negative definiteness of (a^{ij}) that

$$\zeta_N''(\tilde U)a^{ij}\partial_i\tilde U\partial_j\tilde U\le\begin{cases}C_*\sup_{\tilde\Omega_{\rho_0}}a^{ij}\partial_i\tilde U\partial_j\tilde U=:M, & \text{in}\quad \Omega_{\rho_0},\\ 0, & \text{otherwise},\end{cases}$$

where $C_*=\max_{t\in[\tilde\rho_m,\rho_0]}\zeta_N''$ is independent of N due to the construction of $\{\zeta_N\}_{N\ge N_0}$ and $\tilde\Omega_{\rho_0}=\left\{(x,t)\in\mathcal{U}\times\mathbb{R}:\tilde U(x,t)<\rho_0\right\}$. As η_1 and η_2 are compactly supported, we see from Lemma 4.4 that there is $\tilde C>0$ such that $\mu(\mathcal{U}\times[t_1,t_2))\le\tilde C$ for all $t_1\in\mathrm{supp}(\eta_1)$ and $t_2\in\mathrm{supp}(\eta_2)$. We then deduce from (4.14) that

$$\begin{aligned}&\iint\limits_{\mathcal{U}\times\mathbb{R}}\left[\zeta_N(\tilde U)-(N+1)\right]\eta_2\mathrm{d}\mu\int\limits_{\mathbb{R}}\eta_1(t_1)\mathrm{d}t_1\\ &\quad\le\iint\limits_{\mathcal{U}\times\mathbb{R}}\left[\zeta_N(\tilde U)-(N+1)\right]\eta_1\mathrm{d}\mu\int\limits_{\mathbb{R}}\eta_2(t_2)\mathrm{d}t_2+M\tilde C\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2,\end{aligned}$$

which gives

$$\iint\limits_{\mathcal{U}\times\mathbb{R}}\zeta_N(\tilde U)\eta_2\mathrm{d}\mu\int\limits_{\mathbb{R}}\eta_1(t_1)\mathrm{d}t_1$$

$$\leq \iint\limits_{\mathcal{U}\times\mathbb{R}} \zeta_N(\tilde{U})\eta_1 \mathrm{d}\mu \int\limits_{\mathbb{R}} \eta_2(t_2)\mathrm{d}t_2 + (N+1)\left(\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_2 \mathrm{d}\mu \int\limits_{\mathbb{R}} \eta_1 \mathrm{d}t - \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_1 \mathrm{d}\mu \int\limits_{\mathbb{R}} \eta_2 \mathrm{d}t\right)$$
$$+ M\tilde{C}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \eta_1(t_1)\mathrm{d}t_1\eta_2(t_2)\mathrm{d}t_2.$$

Since $\int_{\mathbb{R}} \eta_1 \mathrm{d}t = \int_{\mathbb{R}} \eta_2 \mathrm{d}t = \int_{\mathbb{R}} \eta \mathrm{d}t > 0$, we find

$$\begin{aligned} 0 \leq \iint\limits_{\mathcal{U}\times\mathbb{R}} \zeta_N(\tilde{U})\eta_2 \mathrm{d}\mu \leq \iint\limits_{\mathcal{U}\times\mathbb{R}} \zeta_N(\tilde{U})\eta_1 \mathrm{d}\mu + (N+1)\left(\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_2 \mathrm{d}\mu - \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_1 \mathrm{d}\mu\right) \\ + M\tilde{C}\int\limits_{\mathbb{R}} \eta \mathrm{d}t \end{aligned} \tag{4.15}$$

As $\tilde{U}$ is integrable with respect to μ over $\mathcal{U} \times [s, s+h]$, there holds $\limsup_{N\to\infty} \iint_{\mathcal{U}\times\mathbb{R}} \zeta_N(\tilde{U})\eta_1 \mathrm{d}\mu < \infty$. If $\iint_{\mathcal{U}\times\mathbb{R}} \eta_1 \mathrm{d}\mu > \iint_{\mathcal{U}\times\mathbb{R}} \eta_2 \mathrm{d}\mu$, a contradiction is easily derived by letting $N \to \infty$ in (4.15). Thus, $\iint_{\mathcal{U}\times\mathbb{R}} \eta_1 \mathrm{d}\mu \leq \iint_{\mathcal{U}\times\mathbb{R}} \eta_2 \mathrm{d}\mu$.

As $s \in \mathbb{R}$ and $h \in (0, T)$ are arbitrary, we see that for any $\eta \in C_c^+(\mathbb{R})$ satisfying $|\mathrm{supp}(\eta)| < T$, there holds (4.13).

Next we claim that for any $\eta \in C_c^+(\mathbb{R})$ satisfying $|\mathrm{supp}(\eta)| < T$, there holds

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta \mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\mu, \quad \forall \delta > T. \tag{4.16}$$

Let $\eta \in C_c^+(\mathbb{R})$ satisfy $|\mathrm{supp}(\eta)| < T$ and $\delta > T$. Let k_0 be a positive integer such that $k_0T - \delta > T$. Applying (4.13) to η and $\eta(\cdot - \delta)$ respectively, we find

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta \mathrm{d}\mu \leq \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\mu \leq \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - k_0T)\mathrm{d}\mu.$$

The claim follows if we can show

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta \mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - k_0T)\mathrm{d}\mu. \tag{4.17}$$

We show (4.17). Let $[a, b]$ satisfy $b - a < T$ and $\mathrm{supp}(\eta) \subset [a, b]$. Define $\tilde{\eta} \in C_T(\mathbb{R})$ by setting

$$\tilde{\eta}(t) = \begin{cases} \eta(t - kT), & t \in [a + kT, b + kT] \text{ and } k \in \mathbb{Z} \\ 0, & t \in \mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}}[a + kT, b + kT]. \end{cases}$$

It is straightforward to check that

$$\iint\limits_{\mathcal{U}\times[a,a+T)} \tilde{\eta}\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu \quad \text{and} \quad \iint\limits_{\mathcal{U}\times[a+k_0T,a+(k_0+1)T)} \tilde{\eta}\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - k_0T)\mathrm{d}\mu$$

By Lemma 4.4, there holds

$$\iint\limits_{\mathcal{U}\times[a,a+T)} \tilde{\eta}\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times[a+k_0T,a+(k_0+1)T)} \tilde{\eta}\mathrm{d}\mu,$$

which indicates $\iint_{\mathcal{U}\times\mathbb{R}} \eta(t)\mathrm{d}\mu = \iint_{\mathcal{U}\times\mathbb{R}} \eta(t-k_0T)\mathrm{d}\mu$. The claim thus follows.

Now we show that (4.16) actually holds for all $\eta \in C_c^+(\mathbb{R})$ and $\delta \in \mathbb{R}$, namely,

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\mu, \quad \forall \eta \in C_c^+(\mathbb{R}), \quad \delta \in \mathbb{R}. \tag{4.18}$$

Let $\eta \in C_c^+(\mathbb{R})$. By a partition of unity (see e.g. [27]), there exists a locally finite open cover $\{\mathcal{I}_\beta\}_\beta$ on $\mathbb{R}$ and functions $\{f_\beta\}_\beta$ satisfying

- for each β, $\mathcal{I}_\beta$ is an open interval satisfying $|\mathcal{I}_\beta| < T$;
- $\mathrm{supp}(f_\beta) \subset \mathcal{I}_\beta$ and $\sum_\beta f_\beta(t) = 1$ for $t \in \mathbb{R}$

Define $\eta_\beta := \eta f_\beta$. Then $\eta_\beta \in C_c^+(\mathbb{R})$ and $|\mathrm{supp}(\eta_\beta)| < T$. Applying (4.16) to each η_β and then summarizing the resulting equalities, we find

$$\sum_\beta \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_\beta\mathrm{d}\mu = \sum_\beta \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta_\beta(\cdot - \delta)\mathrm{d}\mu, \quad \forall \delta > T.$$

Applying Fubini's theorem, we find

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\mu, \quad \forall \delta > T. \tag{4.19}$$

For $\delta \le T$, let $\delta_1 > T$. We then apply (4.19) to find

$$\iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta(\cdot - \delta - \delta_1)\mathrm{d}\mu = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu.$$

Thus, (4.18) holds.

Finally, we show the existence of some $C > 0$ such that (4.12) holds. Let us consider the functional L on $C_c(\mathbb{R})$ defined by

$$L\eta = \iint\limits_{\mathcal{U}\times\mathbb{R}} \eta\mathrm{d}\mu, \quad \eta \in C_c(\mathbb{R}).$$

Clearly, L is linear, continuous and positive. Applying Riesz representation theorem, there exists a σ-finite Borel measure ν on $\mathbb{R}$ such that $\int_{\mathbb{R}} \eta \mathrm{d}\nu = \iint_{\mathcal{U}\times\mathbb{R}} \eta \mathrm{d}\mu$ for all $\eta \in C_c(\mathbb{R})$. It follows from (4.18) that

$$\int_{\mathbb{R}} \eta \mathrm{d}\nu = \int_{\mathbb{R}} \eta(\cdot - \delta)\mathrm{d}\nu, \quad \forall \eta \in C_c(\mathbb{R}), \quad \delta \in \mathbb{R},$$

which implies that ν is translation-invariant. By [18, Theorem 0.1], there is a constant $C \geq 0$ such that $\mathrm{d}\nu = C\mathrm{d}t$ leading to $\iint_{\mathcal{U}\times\mathbb{R}} \eta \mathrm{d}\mu = C\int_{\mathbb{R}} \eta \mathrm{d}t$ for all $\eta \in C_c(\mathbb{R})$. As μ is non-trivial, it follows that $C > 0$.

Step 6. Let $C > 0$ be as in (4.12). Define $\tilde{\mu} := \frac{1}{C}\mu$. It follows from (4.12) and (4.9) that $\tilde{\mu} \in \mathcal{M}_T(\mathcal{U} \times \mathbb{R})$. We show that $\tilde{\mu}$ admits t-sections $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ satisfying $\tilde{\mu}_t = \tilde{\mu}_{t+T}$ and $\tilde{\mu}_t(\mathcal{U}) = 1$ for all $t \in \mathbb{R}$.

Applying Corollary 4.2, we find $\tilde{\mu}$ admits t-sections $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ satisfying $\tilde{\mu}_t = \tilde{\mu}_{t+T}$ and $\tilde{\mu}_t(\mathcal{U}) = 1$ for all $t \in J_0$, where $J_0 \subset \mathbb{R}$ satisfies $|\mathbb{R} \setminus J_0| = 0$.

By **Step 2**, $\tilde{\mu}$ (or equivalently μ) is a measure solution to (1.3) in $\mathcal{U}\times\mathbb{R}$. By Corollary 2.1 and Remark 2.1, we find a subset $\tilde{J} \subset \mathbb{R}$ satisfying $|\mathbb{R} \setminus \tilde{J}| = 0$ such that

$$\int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_t = \int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_s + \int_s^t \int_{\mathcal{U}} \mathcal{L}_{A,V}\phi \mathrm{d}\tilde{\mu}_\tau \mathrm{d}\tau, \quad \forall \phi \in C_c^2(\mathcal{U}) \text{ and } s,\ t \in \tilde{J}. \tag{4.20}$$

Since $\mathcal{L}_{A,V}\phi \in C_c(\mathcal{U})$ is bounded, we see from (4.20) that

$$\left| \int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_t - \int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_s \right| \leq |\mathcal{L}_{A,V}\phi|_\infty |t - s|, \quad \forall s,\ t \in \tilde{J}.$$

Thus, the function $t \mapsto \int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_t : \tilde{J} \to \mathbb{R}$ has a unique periodic and continuous extension to $\mathbb{R}$. We denote this extension by F_ϕ.

For each $t_* \in \mathbb{R} \setminus \tilde{J}$, let us consider the functional K_{t_*} on $C_c(\mathcal{U})$ defined by

$$K_{t_*}\phi = F_\phi(t_*), \quad \phi \in C_c(\mathcal{U}).$$

As $|F_\phi(t_*)| = |\lim_{\tilde{J}\ni t\to t_*} \int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_t| \leq |\phi|_\infty$, we see that K_{t_*} is linear, continuous and positive. Applying Riesz representation theorem, we find a σ-finite Borel measure $\tilde{\nu}_{t_*}$ satisfying

$$\int_{\mathcal{U}} \phi \mathrm{d}\tilde{\nu}_{t_*} = K_{t_*}(\phi), \quad \forall \phi \in C_c(\mathcal{U}).$$

Redefine $\tilde{\mu}_{t_*} := \nu_{t_*}$. We see from $\int_{\mathcal{U}} \phi(x)\mathrm{d}\tilde{\mu}_{t_*} = F_\phi(t_*) = \lim_{\tilde{J}\ni t\to t_*} \int_{\mathcal{U}} \phi(x)\mathrm{d}\mu_t$ that (4.20) holds for all $s, t \in \mathbb{R}$. Clearly $\tilde{\mu}_t = \tilde{\mu}_{t+T}$ for all $t \in \mathbb{R}$ and $\tilde{\mu}_t(\mathcal{U}) = 1$ for $t \in J_0$.

Note that the arguments in **Step 3** in the proof of Theorem A do not require (a^{ij}) to be non-degenerate. Thus, we can follow the lines to argue that $\tilde{\mu}_t(\mathcal{U}) = 1$ for all $t \in \mathbb{R}$.

This completes the proof.

5. Applications

In this section, we apply our results to study stochastic damping Hamiltonian systems and stochastic differential inclusions in Subsection 5.1 and Subsection 5.2, respectively.

5.1. Stochastic damping Hamiltonian system

Consider the following stochastic damping Hamiltonian system:

$$\begin{cases} \mathrm{d}x = y\mathrm{d}t, \\ \mathrm{d}y = -\left[b(x,y)y + \nabla V(x,t)\right]\mathrm{d}t + \sigma(x,y,t)\mathrm{d}W_t, \end{cases} \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{5.1}$$

where the damping $b = (b^{ij}) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ is continuous, the potential $V : \mathbb{R}^d \times \mathbb{R} \mapsto (0,\infty)$ is twice continuously differentiable in its first variable and continuously differentiable and T-periodic in its second variable, the noise intensity $\sigma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^{d\times m}$ belongs to $C(\mathbb{R}, W^{1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^d))$ and is T-periodic in its third variable, where $p > d + 2$ and $m \geq d$ are fixed, and $(W_t)_{t\in\mathbb{R}}$ is the standard m-dimensional Wiener process.

The Fokker-Planck equation associated to (5.1) reads

$$\begin{aligned} \mathcal{L}^*_H u :=& -\partial_t u + \partial^2_{y_i y_j}(a^{ij}u) - \partial_{x_i}(y_i u)_j \\ &+ \partial_{y_i}\left((b^{ij}y + \partial_{x_i}V)u\right) = 0, \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \end{aligned} \tag{5.2}$$

where the diffusion matrix $(a^{ij}) := \frac{\sigma\sigma^\top}{2}$ is semi-positive definite. Let

$$\mathcal{L}_H := \partial_t + a^{ij}\partial^2_{y_i y_j} + y^i\partial_{x_i} - (b^{ij}y_j + \partial_{x_i}V)\partial_{y_i}$$

be the L^2-formal adjoint of $\mathcal{L}^*_H$.

We make the following assumptions on the coefficients.

- **(H1)** There is $b_0 > 0$ such that $b^{ij}y_i y_j \geq b_0|y|^2$ for all $y \in \mathbb{R}^d$.
- **(H2)** The functions σ and $\partial_t V$ are uniformly bounded on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ and $\mathbb{R}^d \times \mathbb{R}$, respectively.
- **(H3)** There exists a lower bounded function $\Phi \in C^2(\mathbb{R}^d)$ such that

$$\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} \sum_{i,j=1}^d \left| -b^{ji}(x,y)\frac{x_j}{|x|} + \partial_{x_i}\Phi(x) \right| < \infty.$$

(H4) $\nabla_x V \cdot \frac{x}{|x|} \to \infty$ as $|x| \to \infty$.

We remark that **(H1)** says that the system (5.1) is actually damped. When $b(x, y)$ is bounded, the function Φ in **(H3)** can be taken to be 0.

Theorem 5.1. *Suppose* **(H1)-(H4)**. *Then,* (5.2) *admits a periodic probability solution.*

Proof. We first follow [43,13] to construct an unbounded Lyapunov function with respect to $\mathcal{L}_H$. Define

$$\begin{aligned} E(x,y,t) &= \frac{|y|^2}{2} + V(x,t), \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \\ G(x,y) &= \eta(|x|)\frac{x \cdot y}{|x|}, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \end{aligned}$$

where $\eta \in C^\infty([0,\infty))$ satisfies

$$\eta(t) = \begin{cases} 0, & t \le \frac{1}{2}, \\ 1, & t > 1. \end{cases}$$

Let α, $\beta > 0$ (to be chosen) and define

$$U(x,y,t) = \exp\left\{\alpha E(x,y,t) + \beta\left(G(x,y) + \Phi(x)\right)\right\}, \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},$$

where Φ is as in **(H3)**. Clearly, $U \in C^{2,1}_T(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$ is positive and satisfies $\sup_{t\in\mathbb{R}} U(x,y,t) \to \infty$ as $|x|+|y| \to \infty$. For the last property of U, the lower boundedness of Φ is used. In particular, U satisfies (1.4).

We claim the existence of some $\gamma > 0$ such that

$$\sup_{t\in\mathbb{R}} \mathcal{L}_H U(x,y,t) \le -\gamma, \quad \forall |x| + |y| \gg 1, \tag{5.3}$$

which implies that U is an unbounded Lyapunov function with respect to $\mathcal{L}$. We compute

$$\begin{aligned} \frac{\mathcal{L}_H U}{U} &= \alpha\mathcal{L}_H E + \beta\mathcal{L}_H(G+\Phi) + a^{ij}\left(\alpha\partial_{y_i}E + \beta\partial_{y_i}G\right)\left(\alpha\partial_{y_j}E + \beta\partial_{y_j}G\right) \\ &= \alpha\mathcal{L}_H E + \beta\mathcal{L}_H(G+\Phi) + a^{ij}\left(\alpha y_i + \beta\frac{x_i}{|x|}\right)\left(\alpha y_j + \beta\frac{x_j}{|x|}\right) \\ &= \alpha\mathcal{L}_H E + \beta\mathcal{L}_H(G+\Phi) + \alpha^2 a^{ij} y_i y_j + 2\alpha\beta a^{ij}\frac{x_i y_j}{|x|} + \beta^2 a^{ij}\frac{x_i x_j}{|x|^2}, \\ &\qquad \forall |x| > 1 \text{ and } y \in \mathbb{R}^d. \end{aligned}$$

Direct calculations show that

$$\mathcal{L}_H E = \partial_t V - b^{ij} y_i y_j + \sum_{i=1}^{d} a^{ii},$$

$$\mathcal{L}_H \Phi = y_i \partial_{x_i} \Phi,$$

$$\mathcal{L}_H G = \mathcal{L}_H \left(\frac{x \cdot y}{|x|} \right) = -\left(b^{ij} y_j + \partial_{x_i} V \right) \cdot \frac{x_i}{|x|} + \frac{|y|^2}{|x|} - \frac{x_i x_j y_i y_j}{|x|^3}, \quad \forall |x| > 1 \text{ and } y \in \mathbb{R}^d.$$

As $\sum_{ij} x_i x_j y_i y_j = \left[\sum_i (x_i y_i)\right]^2 \geq 0$, we see that

$$\mathcal{L}_H G \leq -\left(b^{ij} y_j + \partial_{x_i} V \right) \cdot \frac{x_i}{|x|} + |y|^2, \quad \forall |x| > 1 \text{ and } y \in \mathbb{R}^d.$$

Thus,

$$\begin{aligned} \frac{\mathcal{L}_H U}{U} &\leq \alpha \left(\partial_t V - b^{ij} y_i y_j + \sum_{i=1}^{d} a^{ii} \right) \\ &\quad + \beta \left[-\left(b^{ij} y_j + \partial_{x_i} V \right) \cdot \frac{x_i}{|x|} + |y|^2 + y_i \partial_{x_i} \Phi \right] \\ &\quad + \alpha^2 a^{ij} y_i y_j + 2\alpha\beta a^{ij} \frac{x_i y_j}{|x|} + \beta^2 a^{ij} \frac{x_i x_j}{|x|^2}, \quad \forall |x| > 1 \text{ and } y \in \mathbb{R}^d. \end{aligned}$$

Setting

$$(\mathrm{I}) := -\alpha b^{ij} y_i y_j + \beta \left(-b^{ji} \frac{x_j}{|x|} + \partial_{x_i} \Phi \right) y_i + \beta |y|^2 + \alpha^2 a^{ij} y_i y_j + 2\alpha\beta a^{ij} \frac{x_i y_j}{|x|},$$

$$(\mathrm{II}) := \alpha \partial_t V + \alpha \sum_{i=1}^{d} a^{ii} - \beta \partial_{x_i} V \frac{x_i}{|x|} + \beta^2 a^{ij} \frac{x_i x_j}{|x|^2},$$

we find $\frac{\mathcal{L}_H U}{U} \leq (\mathrm{I}) + (\mathrm{II})$.

Set

$$M_1 := \sup_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}} \frac{|\sigma\sigma^\top|}{2}, \quad M_2 := \sup_{\mathbb{R}^d \times \mathbb{R}} |\partial_t V| \quad \text{and}$$

$$M_3 := \sup_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i,j=1}^{d} \left| -b^{ji}(x,y) \frac{x_j}{|x|} + \partial_{x_i} \Phi(x) \right|.$$

Due to **(H2)** and **(H4)**, M_1, M_2 and M_3 are finite. For (I), we see from **(H1)** and the definitions of M_1, M_2 and M_3 that

$$\begin{aligned}(\mathrm{I}) &\leq -\alpha b_0|y|^2 + \beta M_3|y| + \beta|y|^2 + \alpha^2 M_1|y|^2 + 2\alpha\beta M_1|y| \\ &\leq (-\alpha b_0 + \alpha^2 M_1 + \beta)|y|^2 + \beta M_3|y| + 2\alpha\beta M_1|y|.\end{aligned}$$

Let us fix $0 < \alpha < \frac{b_0}{M_1}$ and then choose $\beta > 0$ so small that $-\alpha b_0 + \alpha^2 M_1 + \beta < 0$. It is clear that there exists $\delta_1 > 0$ such that (I) ≤ -1 for all $|y| \geq \delta_1$. Similarly, we find

$$(\mathrm{II}) \leq \alpha M_2 + \alpha\sqrt{d}M_1 + \beta^2 M_1 - \beta\partial_{x_i}V\frac{x_i}{|x|}.$$

It is easy to see from **(H4)** that there is $\delta_2 > 1$ such that (II) ≤ -1 for $|x| \geq \delta_2$. Thus,

$$\frac{\mathcal{L}_H U}{U} \leq -2, \quad \forall (x,y,t) \in \{(x,y,t) : |x| \geq \delta_1,\ |y| \geq \delta_2\} =: \mathcal{U}_*,$$

which implies that

$$\mathcal{L}_H U \leq -\gamma \quad \text{on} \quad \mathcal{U}_*,$$

where $\gamma = 2\inf_{\mathcal{U}_*} U > 0$. This proves the claim (5.3).

Thus, we apply Theorem B to find a periodic probability solution to (5.2). □

5.2. Stochastic differential inclusion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $Cl(\mathbb{R}^d)$ and $Cl(\mathbb{R}^{d\times m})$ denote the class of all closed subsets in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively.

We consider the following stochastic differential inclusion (SDI)

$$\mathrm{d}x \in B(x,t)\mathrm{d}t + \Sigma(x,t)\mathrm{d}W_t, \quad (x,t) \in \mathcal{U} \times \mathbb{R}, \tag{5.4}$$

where $B : \mathcal{U} \times \mathbb{R} \to Cl(\mathbb{R}^d)$ and $\Sigma : \mathcal{U} \times \mathbb{R} \to Cl(\mathbb{R}^{d\times m})$ are set-valued functions and $(W_t)_{t\in\mathbb{R}}$ is the standard m-dimensional Wiener process. We assume both B and Σ are T-periodic in their second variables.

A stochastic process $x = (x_t)_{t\in\mathbb{R}}$ adapted to $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is called a solution to (5.4) if for any $s < t$, there holds

$$x_t - x_s \in \int_s^t B(x_\tau, \tau)\mathrm{d}\tau + \int_s^t \Sigma(x_\tau, \tau)\mathrm{d}W_\tau, \quad \mathbb{P}\text{-a.s.}. \tag{5.5}$$

We refer the reader to [30,31] for the definition of the integrals on the right hand side of (5.5). It is worthwhile to point out that if there are continuous selections b and σ from B and Σ, respectively, then a solution $x = (x_t)_{t\in\mathbb{R}}$ to the SDE

$$\mathrm{d}x = b(x,t)\mathrm{d}t + \sigma(x,t)\mathrm{d}W_t, \quad x \in \mathcal{U}$$

must be a solution to (5.4). According to this, we can study the long-time behavior of the distribution of $x = (x_t)_{t\in\mathbb{R}}$. Given this, we define periodic probability solutions to (5.4) as follows.

Definition 5.1. A σ-finite Borel measure μ is called a *periodic probability solution* to (5.4) if there are T-periodic selections $b = (b^i) \in B$ and $\sigma \in \Sigma$ such that μ is a *periodic probability solution* to the Fokker-Planck equation

$$\partial_t u = \partial^2_{ij}(a^{ij}u) - \partial_i(b^i u), \quad (x,t) \in \mathcal{U} \times \mathbb{R},$$

where $A = (a^{ij}) = \frac{\sigma\sigma^\top}{2}$.

Definition 5.2. A non-negative function $U \in C^{2,1}_T(\mathcal{U} \times \mathbb{R})$ is called an unbounded Lyapunov function for (5.4) if U is an unbounded Lyapunov function with respect to $\mathcal{L}_{A,b}$ for all T-periodic selections σ and b from Σ and B, respectively, where $A = \frac{\sigma\sigma^\top}{2}$.

We recall the following definitions from [3].

Definition 5.3. Let X be a Polish space, and $F : \mathcal{U} \times \mathbb{R} \to 2^X$ be a set-valued function.

(1) F is called *lower semi-continuous* if for each fixed $(x_0, t_0) \in \mathcal{U} \times \mathbb{R}$, the following holds: for any $y_0 \in F(x_0, t_0)$ and any neighborhood N_{y_0} of y_0 in X, there is a neighborhood $N_{(x_0,t_0)}$ of (x_0, t_0) such that $F(x,t) \cap N_{y_0} \neq \emptyset$ for all $(x,t) \in N_{x_0,t_0}$.
(2) F is called *Lipschitz continuous* in $Q \subset \mathcal{U} \times \mathbb{R}$ if there is $L_Q > 0$ such that

$$\sup_{y_i \in F(x_i,t_i), i=1,2} |y_1 - y_2| \le L_Q |(x_1,t_1) - (x_2,t_2)|, \quad \forall (x_i,t_i) \in Q,\ i = 1,2.$$

F is called *locally Lipschitz continuous* in $\mathcal{U} \times \mathbb{R}$ if for any bounded subdomain $Q \subset\subset \mathcal{U} \times \mathbb{R}$, F is Lipschitz continuous in Q.

Lemma 5.1. *Let B be a lower semi-continuous function from $\mathcal{U} \times \mathbb{R}$ to the class of closed convex subsets of $\mathbb{R}^d$ and be T-periodic in its second variable. Then, a function $b \in C_T(\mathcal{U} \times \mathbb{R})$ can be selected from B.*

Proof. Arguing as in the proof of the Michael's selection theorem (e.g., [3, Section 1.11, Theorem 1]), we can construct a sequence of continuous functions $\{b_n\}_n$ on $\mathcal{U} \times \mathbb{R}$ such that b_n converges locally uniformly to some $b \in C(\mathcal{U} \times \mathbb{R})$ as $n \to \infty$, and $\sup_{(x,t)\in\mathcal{U}\times\mathbb{R}} d(b_n(x,t), B(x,t)) \le 2^{-n}$ for all $n \gg 1$. Since B is periodic, we can certainly construct b_n to be T-periodic in its second viable for each n. As a result, b is T-periodic in its second variable. $\square$

Lemma 5.2. *Let Σ be a locally Lipschitz continuous function from $\mathcal{U} \times \mathbb{R}$ to the class of compact convex subsets of $\mathbb{R}^{d\times m}$ and be T-periodic in its second variable. Then, a locally*

Lipschitz continuous function σ on $\mathcal{U} \times \mathbb{R}$ that is T-periodic in its second variable can be selected from Σ.

Proof. Let $\tilde{\mathcal{U}} \subset \mathcal{U}$ be a bounded subdomain. We claim that if $\tilde{\Sigma}$ is a Lipschitz continuous function from $\tilde{\mathcal{U}} \times \mathbb{R}$ to the class of compact convex subsets of $\mathbb{R}^{d\times m}$, and is T-periodic in its second variable, then a Lipschitz continuous function $\tilde{\sigma}$ on $\tilde{\mathcal{U}} \times \mathbb{R}$ that is T-periodic in its second variable can be selected from $\tilde{\Sigma}$. Indeed, as in the proof of the barycentric selection theorem (e.g., [3, Theorem 1, Section 1.9]), the function

$$\tilde{\sigma}(x,t) := \frac{1}{\lambda^{d\times m}(\tilde{\Sigma}(x,t)+B_0(1))} \iint\limits_{\tilde{\Sigma}(x,t)+B_0(1)} y \mathrm{d}\lambda^{d\times m}(y), \quad (x,t) \in \tilde{\mathcal{U}} \times \mathbb{R}$$

is a Lipschitz continuous selection from $\tilde{\Sigma}$, where $\lambda^{d\times m}$ is the Lebesgue measure on $\mathbb{R}^{d\times m}$ and $B_0(1)$ is the unit ball in $\mathbb{R}^{d\times m}$ centered at the origin. It is clear that $\tilde{\sigma}$ is T-periodic in the second variable.

Now, let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be an increasing sequence of bounded domains in $\mathcal{U}$ satisfying $\mathcal{U} := \cup_{n=1}^{\infty}\mathcal{U}_n$. For each $n \in \mathbb{N}$, define $\Sigma_n(x,t) = \Sigma(x,t)$ for all $(x,t) \in \mathcal{U}_n \times \mathbb{R}$. Then Σ_n is Lipschitz continuous in $\mathcal{U}_n \times \mathbb{R}$ due to the T-periodicity in its second variable. Applying the claim, we see that there is a T-periodic Lipschitz continuous selection σ_n from Σ_n. It is clear that $\sigma_m = \sigma_n$ on $\mathcal{U}_n \times \mathbb{R}$ for $m > n$. Then, $\sigma(x,t) := \sigma_n(x,t)$ for $(x,t) \in \mathcal{U}_n \times \mathbb{R}$ and $n \in \mathbb{N}$, is a T-periodic locally Lipschitz continuous selection from Σ as required. $\square$

Theorem 5.2. *Let B be a lower semi-continuous function from $\mathcal{U} \times \mathbb{R}$ to the class of closed convex subsets in $\mathbb{R}^d$, and Σ be a locally Lipschitz continuous function from $\mathcal{U} \times \mathbb{R}$ to the class of compact convex subsets of $\mathbb{R}^{d\times m}$. Suppose B and Σ are T-periodic in their second variables. If (5.4) admits an unbounded Lyapunov function U, then (5.4) admits a periodic probability solution.*

Proof. Applying Lemma 5.1 and Lemma 5.2, respectively, we see that there exist a T-periodic continuous selection b from B and a T-periodic locally Lipschitz continuous selection σ from Σ.

Clearly, $A = (a^{ij}) = \frac{\sigma\sigma^\top}{2}$ is semi-positive definite and locally Lipschitz continuous. Thus, we apply Theorem B to conclude the existence of a periodic probability solution to (5.4). $\square$

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