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NOISE-VANISHING CONCENTRATION AND LIMIT BEHAVIORS OF PERIODIC PROBABILITY SOLUTIONS

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Abstract. The present paper is devoted to the investigation of noisy impacts on the dynamics of periodic ordinary differential equations (ODEs). To do so, we consider a family of stochastic differential equations resulting from a periodic ODE perturbed by small white noises, and study noise-vanishing behaviors of their "steady states" that are characterized by periodic probability solutions of the associated Fokker-Plank equations. By establishing noise-vanishing concentration estimates of periodic probability solutions, we prove that any limit measure of periodic probability solutions must be a periodically invariant measure of the ODE and that the global periodic attractor of a dissipative ODE is stable under general small noise perturbations. For local periodic attractors (resp. local periodic repellers), small noises are constructed to stabilize (resp. de-stabilize) them. Our study provides an elementary step towards the understanding of stochastic stability of periodic ODEs.

1. Introduction

In the traditional mathematical modelling of real systems arising in biology, ecology, engineering, physics, etc., ordinary differential equations

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(ODEs) of the form

$$\dot{x} = V(x,t), \quad x \in \mathcal{U}, \tag{1.1}$$

where $\mathcal{U}$ is an open and connected subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$ and the *vector field* $V : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^d$ is T-periodic in t for some $T > 0$, are often created according to certain biological, chemical or physical laws. The periodic time-dependence in (1.1) is frequently used to model time recurrence or seasonal variations. However, many real systems are subject to small perturbations either from extrinsic environments or intrinsic uncertainties. Such small perturbations could have great impact on the dynamics of these systems. Thus, investigating the stability of (1.1) under small perturbations is a fundamental issue of both realistic and theoretical significance. This problem has been extensively studied in the classical perturbation theory of dynamical systems that particularly deals with systems under small deterministic perturbations (see [18, 41, 10] and references therein).

As for (1.1) under small noise perturbations, its investigation has been attracting a lot of attention in recent years due partly to its power in studying noise-driven phenomena in natural sciences (see e.g. [40, 37, 17, 16, 7, 39]) and modelling complex processes such as chemical reactions (see e.g. [13, 1]), weather-climate systems (see e.g. [19, 29, 28]) and complex fluids (see e.g. [32, 38]). In literature, small noise perturbations are often incorporated into (1.1) as small multiplicative white noises, leading to the following stochastic differential equation (SDE) of Itô type:

$$\mathrm{d}x = V(x,t)\mathrm{d}t + \epsilon G(x,t)\mathrm{d}W_t, \quad x \in \mathcal{U}, \tag{1.2}$$

where the small parameter $\epsilon > 0$ stands for the noise intensity, the *noise matrix* $G : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^{d\times m}$ is T-periodic in t for $m \geq d$ and $W = (W_t)_{t\in\mathbb{R}}$ is a standard m-dimensional Wiener process.

The stability of (1.1) under small noise perturbations can be considered from either a geometric or statistical viewpoint. The former focuses on the stochastic stability of "compact" invariant sets such as *maximal periodic attractors* and *maximal periodic repellers* of (1.1) (see Definition 2.3), while the latter tackles the stochastic stability of *periodically invariant measures* of (1.1) (see Definition 1.3). When $V(x,t) = V(x)$ and $G(x,t) = G(x)$ are time-independent, these problems have been widely studied, especially for gradient systems (see e.g. [36, 23, 31, 3, 16, 21]). For non-gradient systems, the authors recently investigated in [22, 27] the stochastic stability of maximal attractors/repellers. Stochastic stability of SRB measures has been studied for diffeomorphisms on compact manifolds (see e.g. [11, 42]). It is worthwhile to point out that the *large deviation theory* (see e.g. [31, 16])

provides finer results in the case that the unperturbed system admits a finite number of equilibria or limit cycles. In contrast, the study of (1.2) with general periodic coefficients lags behind although theories are being push forward.

In the present paper, we focus on (1.2) with periodic coefficients and study the stability of "compact" invariant sets of (1.1) under small noise perturbations. More precisely, the stability of the global periodic attractor of a dissipative system is investigated under general small noise perturbations, while the stabilization of local periodic attractors and the de-stabilization of local periodic repellers are examined under specific small noise perturbations. Our approach is based on the noise-vanishing concentration of *periodic probability solutions* of the following Fokker-Planck equation (FPE) associated to (1.2):

$$\partial_t u = \epsilon^2 \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u), \quad x \in \mathcal{U}, \tag{1.3}$$

where

$$A := (a^{ij}) = \frac{1}{2}GG^\top$$

is the *diffusion matrix*. Throughout the paper, we denote $\partial_i = \partial_{x_i}$, $\partial^2_{ij} = \partial^2_{x_i x_j}$ for $i, j = 1, \ldots, d$, and adopt the usual summation convention whenever applicable. It is well-known that the distribution of solutions of (1.2) is governed by (1.3) at least when V and G are sufficiently regular.

We recall from [25] the definition of periodic probability solution of (1.3). For each $\epsilon > 0$, let

$$\mathcal{L}_{\epsilon^2 A} := \partial_t + \epsilon^2 a^{ij}\partial^2_{ij} + V^i\partial_i$$

be the differential operator associated to the dual equation of (1.3).

Definition 1.1 (Periodic probability solution). *Let $\epsilon > 0$. A Borel measure μ^ϵ on $\mathcal{U} \times \mathbb{R}$ is called a* periodic probability solution *of* (1.3) *if there is a family of Borel probability measures $\{\mu^\epsilon_t\}_{t\in\mathbb{R}}$ on $\mathcal{U}$ satisfying*

$$\mu^\epsilon_t = \mu^\epsilon_{t+T}, \quad \forall t \in \mathbb{R},$$
$$a^{ij},\, V^i \in L^1_{loc}(\mathcal{U} \times \mathbb{R}, \mathrm{d}\mu^\epsilon_t \mathrm{d}t), \quad \forall i, j \in \{1, \ldots, d\},$$

and

$$\int_{\mathbb{R}}\int_{\mathcal{U}} \mathcal{L}_{\epsilon^2 A}\phi \,\mathrm{d}\mu^\epsilon_t \mathrm{d}t = 0, \quad \forall \phi \in C^{2,1}_0(\mathcal{U} \times \mathbb{R}), \tag{1.4}$$

such that $\mathrm{d}\mu^\epsilon = \mathrm{d}\mu^\epsilon_t \mathrm{d}t$. In this case, we write $\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}$.

The study of SDEs with periodic coefficients dates back to the pioneering work of Khasminskii [30]. Assuming locally Lipschitz continuous coefficients and Lyapunov-type conditions, he was able to show the existence of periodic Markov processes whose distributions are necessarily periodic probability solution of (1.3). Since then, many researchers have looked into the existence issue as well as the global dynamics under different settings (see [35, 24, 43, 14, 9, 8, 15] and references therein). Very recently, the authors of the present paper explored in [25, 26] the existence and uniqueness of periodic probability solutions of (1.3) with irregular coefficients, and the global dynamics of solutions of (1.2) and (1.3) under Lyapunov-type conditions.

To investigate the stochastic stability of maximal periodic attractors/repellers of (1.1), we examine the concentration and limit behaviors of periodic probability solutions of (1.3) as noises vanish. To be more specific, for a family of periodic probability solutions $\{\mu^\epsilon\}_{\epsilon>0}$ of (1.3), we derive concentration estimates of $\{\mu^\epsilon\}_{\epsilon>0}$ near maximal periodic attractors/repellers of (1.1), and investigate possible limit behaviors of μ^ϵ as $\epsilon \to 0$. This gives rise to limit measures of $\{\mu^\epsilon\}_{\epsilon>0}$ defined as follows.

Definition 1.2 (Limit measure). *Let $\{\mu^\epsilon\}_{\epsilon>0}$ be a family of periodic probability solutions of* (1.3)*. A Borel measure μ on $\mathcal{U} \times \mathbb{R}$ is called a* limit measure *of $\{\mu^\epsilon\}_{\epsilon>0}$ if*

(1) *$\mu(\mathcal{U} \times [0,T]) = T$, and*
(2) *there is a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that μ^{ϵ_n} converges to μ under the weak*-topology as $n \to \infty$.*

We refer the reader to Definition 5.1 for the definition of convergence under the weak*-topology. Limit measures of $\{\mu^\epsilon\}_{\epsilon>0}$ are later shown to be *periodically invariant measures* of (1.1) as given in the following definition. Let $(\varphi^{t,s})$ be the two-parameter family generated by the solutions of (1.1).

Definition 1.3 (Periodically invariant measure). *A Borel measure μ on $\mathcal{U} \times \mathbb{R}$ is called a* periodically invariant measure *of* (1.1) *if there is a family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ on $\mathcal{U}$ satisfying:*

(1) *$\mu_t = \varphi_*^{t,s}\mu_s$ for $t > s$, where $\varphi_*^{t,s}\mu_s$ is the usual pushforward measure,*
(2) *μ_0 is an invariant measure of the Poincaré map $P := \varphi^{T,0}$,*

such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$. In this case, we write $\mu = (\mu_t)_{t\in\mathbb{R}}$.

Remark 1.1. The periodicity of (1.1) ensures that periodically invariant measures must be periodic in the sense that if $\mu = (\mu_t)_{t\in\mathbb{R}}$ a periodically invariant measure, then $\mu_t = \mu_{t+T}$ for all $t \in \mathbb{R}$. Therefore, they are periodic generalization of invariant measures and are expected to capture the "steady states" of the system (1.1) from the distributional viewpoint.

We point out that each periodically invariant measure naturally induces an invariant measure of the skew product semi-flow on $\mathcal{U} \times \mathbb{S}_T$ generated by (1.1), where $\mathbb{S}_T := \mathbb{R}/T\mathbb{Z}$ (see Remark 3.1). The skew product semi-flow together with its invariant measures is a classical and widely accepted approach to study the statistical behaviors of (1.1). However, as a Borel measure on the product space $\mathcal{U} \times \mathbb{S}_T$, it is not clear whether an invariant measure of the skew product semi-flow admits t-sections $(\mu_t)_{t\in\mathbb{R}}$ (see Definition 3.1), let alone μ_t being a Borel probability measure for each $t \in \mathbb{R}$. Hence, invariant measures of the skew product semi-flow are not as fine as periodically invariant measures in characterizing the dynamics of (1.1).

We introduce some notations before stating the main results. Note that each set $Z \subset \mathcal{U} \times \mathbb{R}$ can be written as

$$Z = \bigcup_{t\in\mathbb{R}} (Z_t \times \{t\}),$$

where $Z_t = \{x \in \mathcal{U} : (x, t) \in Z\}$ for each $t \in \mathbb{R}$. The sets $\{Z_t\}_{t\in\mathbb{R}}$ are called *t-sections* of Z.

Convention: If a set $Z \subset \mathcal{U}\times\mathbb{R}$ is given, then the notation Z_t always means the t-section of Z, unless otherwise specified.

Definition 1.4.

(1) *A set $Z \subset \mathcal{U} \times \mathbb{R}$ is called* periodic *if $Z_{t+T} = Z_t$ for all $t \in \mathbb{R}$.*

(2) *A periodic set $Z \subset \mathcal{U} \times \mathbb{R}$ is naturally identified with the set $[Z] := \bigcup_{t\in\mathbb{S}_T}(Z_t \times \{t\})$ in the space $\mathcal{U} \times \mathbb{S}_T$.*

(3) *A periodic set $Z \subset \mathcal{U}\times\mathbb{R}$ is* compactly embedded *in an open periodic set $\mathcal{Z} \subset \mathcal{U} \times \mathbb{R}$, if $\overline{[Z]}$ is compact and $\overline{[Z]} \subset [\mathcal{Z}]$. As usual, we write $Z \subset\subset \mathcal{Z}$.*

The first result concerns the concentration of $\{\mu^\epsilon\}_{\epsilon>0}$ in the vicinity of maximal periodic attractors/repellers of (1.1) under general small noise perturbations. The reader is referred to Subsection 2.1 for dynamical aspects related to (1.1). From now on, we start to use some function spaces whose notations are collected in Table 1 located at the end of this section.

Theorem A. *Assume $V \in C^{1,1}_T(\mathcal{U} \times \mathbb{R})$. Let $\mathcal{E}$ be a maximal periodic attractor (resp. maximal periodic repeller) with basin of attraction (resp. basin of expansion) $B(\mathcal{E})$. Let $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ for some $p > d + 2$ satisfy $GG^\top$ being pointwise positive definite and $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ be a family of periodic probability solutions of* (1.3)*. Then, the following hold.*

(1) *For any periodic Borel set* $\mathcal{W} \subset\subset B(\mathcal{E}) \setminus \mathcal{E}$, *there exist constants* $C > 0$ *and* $\epsilon_* > 0$ *such that*

$$\int_t^{t+T} \mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s \leq Te^{C\epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*).$$

(2) *Any limit measure* μ *of* $\{\mu^\epsilon\}_{\epsilon>0}$ *is a periodically invariant measure of* (1.1) *and satisfies* $\mu(B(\mathcal{E}) \setminus \mathcal{E}) = 0$. *In particular, if* $\mathcal{E}$ *is the global periodic attractor of* (1.1), *then any limit measure* μ *of* $\{\mu^\epsilon\}_{\epsilon>0}$ *satisfies* $\mathrm{supp}(\mu) \subset \mathcal{E}$.

Note that Theorem A(2) only asserts some properties of limit measures of $\{\mu^\epsilon\}_{\epsilon>0}$ with no guarantee of their existence. The following corollary provides easily verifiable conditions for the existence of limit measures of periodic probability solutions. We refer the reader to Definition 2.5 for the definition of unbounded uniform Lyapunov functions.

Corollary A. *Assume* $V \in C_T^{1,1}(\mathcal{U} \times \mathbb{R})$. *Let* $G \in C_T(\mathbb{R}, W_{loc}^{1,p}(\mathcal{U}))$ *for some* $p > d+2$ *satisfy* $A := \frac{1}{2}GG^\top$ *being pointwise positive definite. Suppose* $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ *admits an unbounded uniform Lyapunov function in* $\mathcal{U} \times \mathbb{R}$. *Then, there is an* $\epsilon_* > 0$ *such that for each* $\epsilon \in (0, \epsilon_*)$, (1.3) *admits a unique periodic probability solution* μ^ϵ. *Moreover, the family* $\{\mu^\epsilon\}_{\epsilon\in(0,\epsilon_*)}$ *admits limit measures, which are necessarily periodically invariant measures of* (1.1) *and supported in the global periodic attractor of* (1.1).

It is not hard to find an unbounded uniform Lyapunov function in many situations. For instance, when the ODE (1.1) admits an unbounded Lyapunov function U with bounded second-order partial derivatives in x, then U is an unbounded uniform Lyapunov function with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ for any bounded noise matrix $G \in C_T(\mathbb{R}, W_{loc}^{1,p}(\mathcal{U}))$ with $A := \frac{1}{2}GG^\top$. As a result of Corollary A, (1.3) admits a unique periodic probability solution μ^ϵ for each $0 < \epsilon \ll 1$ if in addition A is pointwise positive definite; moreover, μ^ϵ tends to concentrate on the global periodic attractor of (1.1) as $\epsilon \to 0$. In other words, the global periodic attractor of (1.1) is stable under such noise perturbations.

In general, the global periodic attractor of a dissipative system may further contain local periodic attractors/repellers. It is natural to ask whether they can survive from small noise perturbations. The following two theorems provide partial answers to this issue.

Theorem B. *Assume* $V \in C_T^{1,1}(\mathcal{U} \times \mathbb{R})$ *and that* (1.1) *is dissipative. Let* $\mathcal{E}$ *be a local periodic attractor. Then, there exists a noise matrix* $G \in$

$C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ with $GG^\top$ being pointwise positive definite such that the following holds for any family of periodic probability solutions $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ of (1.3): For any periodic Borel set $\mathcal{W} \subset\subset (\mathcal{U}\times\mathbb{R})\setminus\mathcal{E}$, there are $C_1, C_2 > 0$ and $\epsilon_* > 0$ depending on G and $\mathcal{W}$ such that

$$\int_t^{t+T} \mu^\epsilon_s(\mathcal{W}_s)\mathrm{d}s \leq C_1 e^{-C_2\epsilon^{-2}}, \quad \forall t\in\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_*).$$

In particular, each limit measure μ of $\{\mu^\epsilon\}_{\epsilon>0}$ satisfies $\mathrm{supp}(\mu)\subset\mathcal{E}$.

The above result indicates the stabilization of a local periodic attractor by specific noise perturbations, while the next result reveals that a local periodic repeller may not survive from some noise perturbations.

Theorem C. *Assume $V \in C^{1,1}_T(\mathcal{U}\times\mathbb{R})$. Let $\mathcal{E}$ be a local periodic repeller with basin of expansion $B(\mathcal{E})$. Then, there exists a noise matrix $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ with $GG^\top$ being pointwise positive definite such that the following holds for any family of periodic probability solutions $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ of (1.3): For any periodic Borel set $\mathcal{W}\subset\subset B(\mathcal{E})$, there are $C_1, C_2 > 0$ and $\epsilon_* > 0$ depending on G and $\mathcal{W}$ such that*

$$\int_t^{t+T} \mu^\epsilon_s(\mathcal{W}_s)\mathrm{d}s \leq C_1 e^{-C_2\epsilon^{-2}}, \quad \forall t\in\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_*). \tag{1.5}$$

In particular, each limit measure μ of $\{\mu^\epsilon\}_{\epsilon>0}$ satisfies $\mu(B(\mathcal{E})) = 0$.

The proofs of Theorem B and Theorem C rely on delicate analysis comparing the strength of noises in the vicinity of a local periodic attractor/repeller with that away from it. More precisely, in the proof of Theorem B, we find if the strength of the noises near a given local periodic attractor is much weaker than that away from it, then each limit measure of $\{\mu^\epsilon\}_{\epsilon>0}$ is a periodically invariant measure of (1.1) and is supported in this local periodic attractor. Similarly, in the proof of Theorem C, if the strength of the noises near a given local periodic repeller is much stronger than that away from it, then each limit measure of $\{\mu^\epsilon\}_{\epsilon>0}$ is a periodically invariant measure of (1.1) and vanishes on the basin of expansion of this local periodic repeller.

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries including basic dynamical aspects of periodic ODEs, equivalent formalism of periodic probability solutions of (1.3), regularity theory, Lyapunov/anti-Lyapunov functions and Harnack's inequality. In Section 3, an equivalent characterization of periodically invariant measure is given. This provides an easily verifiable condition for limit measures of periodic probability solutions being periodically invariant measures. In Section 4, an integral identity as well as a priori estimates of periodic probability solutions are established assuming the existence of Lyapunov functions/anti-Lyapunov functions. In Section 5, we first construct smooth Lyapunov functions (resp. anti-Lyapunov functions) near maximal periodic attractors (resp. local periodic repellers), which are then used to derive concentration estimates of periodic probability solutions. Finally, we prove Theorem A and Corollary A. Section 6 is devoted to the proof of Theorem B and Theorem C. In Section 7, we study an example to demonstrate the idea of noisy stabilization/destabilization and applications of our results.

For notational simplicity, we use $|\cdot|$ throughout the rest of the paper to denote the absolute value of a number, the norm of a vector, a matrix, the Lebesgue measure of a set, etc.

TABLE 1. Notations

$\nabla_{x,t}U$	$(\partial_1 U, \dots, \partial_d U, \partial_t U)$
$C_c(\mathcal{U})/C_c(\mathbb{R})$	The space of compactly supported continuous functions on $\mathcal{U}/\mathbb{R}$
$C_T(\mathbb{R})$	The space of T-periodic continuous functions on $\mathbb{R}$
$C_c^\infty(\mathcal{U})$	The space of all C^∞ smooth functions in $C_c(\mathcal{U})$
$C_c^2(\mathcal{U})$	The space of all functions in $C_c(\mathcal{U})$ that are twice continuously differentiable
$C_0(\mathcal{U}\times\mathbb{R})$	The space of compactly supported continuous functions on $\mathcal{U}\times\mathbb{R}$
$C_c(\mathcal{U}\times\mathbb{R})$	The space of continuous functions $u:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ such that $u(t,\cdot)\in C_c(\mathcal{U})$ for each $t\in\mathbb{R}$
$C_T(\mathcal{U}\times\mathbb{R})$	The space of T-periodic continuous functions on $\mathcal{U}\times\mathbb{R}$
$C_{c,T}(\mathcal{U}\times\mathbb{R})$	The space of functions in $C_c(\mathcal{U}\times\mathbb{R})$ that is T-periodic in t
$C^{m,n}(\mathcal{U}\times\mathbb{R})$	The space of functions that have continuous derivatives up to m-th order with respect to x and up to n-th order with respect to t
$C_c^{m,n}(\mathcal{U}\times\mathbb{R})$	$C^{m,n}(\mathcal{U}\times\mathbb{R})\cap C_c(\mathcal{U}\times\mathbb{R})$
$C_T^{m,n}(\mathcal{U}\times\mathbb{R})$	$C^{m,n}(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$
$C_{c,T}^{m,n}(\mathcal{U}\times\mathbb{R})$	$C^{m,n}(\mathcal{U}\times\mathbb{R})\cap C_{c,T}(\mathcal{U}\times\mathbb{R})$
$C_0^{m,n}(\mathcal{U}\times\mathbb{R})$	$C^{m,n}(\mathcal{U}\times\mathbb{R})\cap C_0(\mathcal{U}\times\mathbb{R})$
$C_T(\mathbb{R};W_{loc}^{1,p}(\mathcal{U}))$	The space of functions $u\in C_T(\mathcal{U}\times\mathbb{R})$, such that $u(t,\cdot)\in W_{loc}^{1,p}(\mathcal{U})$ for all $t\in\mathbb{R}$ for some $p>d+2$ and for each subdomain $\Omega\subset\subset\mathcal{U}$, the function $t\mapsto\lVert u(t,\cdot)\rVert_{W^{1,p}(\Omega)}$ is continuous

2. PRELIMINARY

In Subsection 2.1, we recall some basic dynamical aspects of periodic ODEs. In Subsection 2.2, we present an equivalent formalism and a regularity result of periodic probability solutions. In Subsection 2.3, Lyapunov/anti-Lyapunov functions of (1.1) and (1.3) are respectively defined in periodic domains to quantify their dissipativity/anti-dissipativity. Moreover, we prove the positive/negative invariance of the sub-level sets of Lyapunov/anti--Lyapunov functions. In Subsection 2.4, we recall the Harnack's inequality for parabolic equations.

2.1. **Periodic ODEs.** In this subsection, we recall some dynamical aspects of the periodic ODE (1.1). We assume $V \in C^{1,1}_T(\mathcal{U} \times \mathbb{R})$.

Denote by $(\varphi^{t,s})$ the local two-parameter family generated by the solutions of (1.1), that is, for $\xi \in \mathcal{U}$, $\varphi^{t,s}(\xi)$ is the local unique solution of (1.1) with initial condition $\varphi^{s,s}(\xi) = \xi$.

Definition 2.1 (Dissipativity)**.** *The system* (1.1) *is said to be* dissipative *if there exists a periodic set $K \subset \mathcal{U} \times \mathbb{R}$ with $[K]$ being compact in $\mathcal{U} \times \mathbb{S}_T$ such that for each $\xi \in \mathcal{U}$, there exists some $t_0 = t_0(\xi) > 0$ such that $\varphi^{s+t,s}(\xi) \in K$ for all $t \geq t_0$ and $s \in \mathbb{R}$.*

Definition 2.2. *A set $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}$ is called*

(1) *positively invariant under $(\varphi^{t,s})$ if $\varphi^{t,s}(\mathcal{V}_s) \subset \mathcal{V}_t$ for all $t \geq s$;*
(2) *negatively invariant under $(\varphi^{t,s})$ if $\varphi^{t,s}(\mathcal{V}_s) \subset \mathcal{V}_t$ for all $t \leq s$;*
(3) *invariant under $(\varphi^{t,s})$ if it is both positively and negatively invariant.*

We suppress the term "under $(\varphi^{t,s})$" in what follows whenever no confusion is caused. Set

$$P^+ := \varphi^{T,0} \quad \text{and} \quad P^- := \varphi^{-T,0}.$$

Definition 2.3. *Let $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}$ be open, connected, periodic and positively invariant* (*resp. negatively invariant*).

(1) *A set $\mathcal{E} \subset \mathcal{V}$ is called a maximal periodic attractor* (*resp. maximal periodic repeller*) *in $\mathcal{V}$ if it is invariant and $\mathcal{E}_0$ is a maximal attractor in $\mathcal{V}_0$ of the discrete dynamical system generated by P^+* (*resp. P^-*).
(2) *Suppose the maximal periodic attractor* (*resp. maximal periodic repeller*) *$\mathcal{E}$ exists in $\mathcal{V}$. If $\mathcal{V} = \mathcal{U} \times \mathbb{R}$, then $\mathcal{E}$ is called the the global periodic attractor* (*resp. global periodic repeller*)*. Otherwise, it is called a local periodic attractor* (*resp. local periodic repeller*).

Remark 2.1. We make some comments on Definition 2.3.

- When $\mathcal{V}$ is positively invariant (resp. negatively invariant), P^+ (resp. P^-) is well-defined and maps $\mathcal{V}_0$ to itself. Therefore, it is reasonable to talk about the maximal attractor in $\mathcal{V}_0$ of the discrete dynamical system generated by P^+ (resp. P^-), which has been extensively studied in literature.
- Maximal periodic attractors (resp. maximal periodic repellers) defined here are essentially the same as maximal attractors (resp. maximal repellers) of the skew-product semi-flow generated by (1.1). The reason we choose maximal periodic attractors and maximal periodic repellers is that they are more compatible with periodically invariant measures given in Definition 1.3.

For a maximal periodic attractor (resp. local periodic repeller) $\mathcal{E}$, its *basin of attraction* (resp. *basin of expansion*) $B(\mathcal{E})$ is defined as follows:

$$B(\mathcal{E}) := \Big\{(x,t) \in \mathcal{U} \times \mathbb{R} : (\varphi^{\tau,t}(x),\tau) \in \mathcal{U} \times \mathbb{R},\ \forall \tau \geq t,\\ \operatorname{dist}(\varphi^{\tau,t}(x), \mathcal{E}_\tau) \to 0,\ \text{as } \tau \to \infty\Big\}$$

respectively,

$$B(\mathcal{E}) := \Big\{(x,t) \in \mathcal{U} \times \mathbb{R} : (\varphi^{\tau,t}(x),\tau) \in \mathcal{U} \times \mathbb{R},\ \forall \tau \leq t,\\ \operatorname{dist}(\varphi^{\tau,t}(x), \mathcal{E}_\tau) \to 0 \text{ as } \tau \to -\infty\Big\}.$$

The proofs of the following results are standard.

Proposition 2.1. *Assume $V \in C^{1,1}_T(\mathcal{U}\times\mathbb{R})$. Let $\mathcal{V}$ be as in the statement of Definition* 2.3. *Suppose there exists a maximal periodic attractor* (*resp. the maximal periodic repeller*) $\mathcal{E}$ *in* $\mathcal{V}$. *Then the following hold.*

(1) $\mathcal{E}$ *is the only maximal periodic attractor* (*resp. local periodic repeller*) *in* $\mathcal{V}$.
(2) $\mathcal{E}$ *is connected, periodic and positively invariant* (*resp. negatively invariant*), *and* $[\mathcal{E}]$ *is compact in* $\mathcal{U} \times \mathbb{S}_T$.
(3) $B(\mathcal{E})$ *is open, connected, periodic and positively invariant* (*resp. negatively invariant*).

Proposition 2.2. *Assume $V \in C^{1,1}_T(\mathcal{U} \times \mathbb{R})$. Let $\mathcal{E}$ be a maximal periodic attractor* (*resp. local periodic repeller*). *If $K \subset \mathcal{U}\times\mathbb{R}$ is periodic and satisfies $K \subset\subset B(\mathcal{E})$, then*

$$\lim_{s\to\infty} \sup_{t\in\mathbb{R}} \operatorname{dist}_H(\varphi^{t+s,t}(K_t), \mathcal{E}_{t+s}) = 0$$

respectively,

$$\lim_{s\to-\infty}\sup_{t\in\mathbb{R}}\operatorname{dist}_H(\varphi^{t+s,t}(K_t),\mathcal{E}_{t+s})=0,$$

where dist_H *is the Hausdorff semi-distance.*

Proposition 2.3. *Assume* $V\in C_T^{1,1}(\mathcal{U}\times\mathbb{R})$. *Let* $\mathcal{E}$ *be a maximal periodic attractor* (*resp. maximal periodic repeller*) *and* $B(\mathcal{E})$ *be its basin of attraction* (*resp. basin of expansion*). *Then, any periodically invariant measure* μ *of* (1.1) *satisfies* $\mu\left(B(\mathcal{E})\setminus\mathcal{E}\right)=0$.

2.2. **Equivalent formalism and regularity.** The following result gives a condition that is equivalent to (1.4).

Lemma 2.1 ([5, 6])**.** *Let* $\epsilon>0$ *and* $\mu^\epsilon=(\mu_t^\epsilon)_{t\in\mathbb{R}}$ *be a Borel measure on* $\mathcal{U}\times\mathbb{R}$ *such that* a^{ij}, $V^i\in L^1_{loc}(\mathcal{U}\times\mathbb{R},\mathrm{d}\mu_t^\epsilon\mathrm{d}t)$ *for each* $i,j\in\{1,\dots,d\}$. *Then,* (1.4) *holds if and only if for each* $\phi\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$, *there exists a subset* $J_\phi\subset\mathbb{R}$ *satisfying* $|\mathbb{R}\setminus J_\phi|=0$ *such that*

$$\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\mu_t^\epsilon=\int_{\mathcal{U}}\phi(\cdot,s)\mathrm{d}\mu_s^\epsilon+\int_s^t\int_{\mathcal{U}}\mathcal{L}_{\epsilon^2A}\phi(\cdot,\tau)\mathrm{d}\mu_\tau^\epsilon\mathrm{d}\tau,\quad\forall s,\,t\in J_\phi.\tag{2.1}$$

Lemma 2.2 ([25, 26])**.** *Fix* $\epsilon>0$. *Let* $\mu^\epsilon=(\mu_t^\epsilon)_{t\in\mathbb{R}}$ *be as in Lemma* 2.1. *If the function*

$$t\mapsto\int_{\mathcal{U}}\phi\mathrm{d}\mu_t^\epsilon$$

is continuous on $\mathbb{R}$ *for each* $\phi\in C_c^2(\mathcal{U})$, *or* μ^ϵ *admits a continuous density, then* (2.1) *hold for all* $s<t$.

We recall the regularity theory of periodic probability solutions of (1.3). Recall $p>d+2$. Let $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathbb{R})$ be the space of measurable functions u on $\mathcal{U}\times\mathbb{R}$ such that $u(\cdot,t)\in W_0^{1,p}(\mathcal{U})$ for a.e. $t\in\mathbb{R}$ and the function $t\mapsto\|u(t,\cdot)\|_{W_0^{1,p}(\mathcal{U})}$ lies in $L^p(\mathbb{R})$. Let $\mathbb{H}^{-1,p'}(\mathcal{U}\times\mathbb{R})$ be the dual space of $\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathbb{R})$, where $p'>1$ is such that $\frac{1}{p}+\frac{1}{p'}=1$.

Let $\mathcal{H}^{1,p}_{loc}(\mathcal{U}\times\mathbb{R})$ be the space of measurable functions u on $\mathcal{U}\times\mathbb{R}$ such that $\eta u\in\mathbb{H}_0^{1,p}(\mathcal{U}\times\mathbb{R})$ and $\partial_t(\eta u)\in\mathbb{H}^{-1,p}(\mathcal{U}\times\mathbb{R})$ for each $\eta\in C_0^{1,1}(\mathcal{U}\times\mathbb{R})$. By [5, Theorem 6.2.2], $\mathcal{H}^{1,p}_{loc}(\mathcal{U}\times\mathbb{R})$ is continuously embedded into some function space consisting of Hölder continuous functions on $\mathcal{U}\times\mathbb{R}$.

Set $\mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R}):=\mathcal{H}^{1,p}_{loc}(\mathcal{U}\times\mathbb{R})\cap C_T(\mathcal{U}\times\mathbb{R})$.

Theorem 2.1 ([4, 5])**.** *Assume* $a^{ij}\in C_T(\mathbb{R};W^{1,p}_{loc}(\mathcal{U}))$ *and* $V^i\in L^p_{loc,T}(\mathcal{U}\times\mathbb{R})$ *for each* $i,j=1,\dots,d$, *where* $p>d+2$. *Suppose* $A=(a^{ij})$ *is pointwise* *positive definite. Fix $\epsilon > 0$ and let μ^ϵ be a periodic probability solution of* (1.3)*. Then, μ^ϵ admits a positive density $u^\epsilon \in \mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R})$.*

2.3. **Lyapunov/Anti-Lyapunov functions.** Following previous works [30, 22, 25], we define Lyapunov/anti-Lyapunov functions of (1.1) and (1.3) in periodically varying domains.

Let $\mathcal{W} \subset \mathcal{U}\times\mathbb{R}$ be open, connected and periodic. We denote $C^{2,1}_T(\mathcal{W})$ as the space of all continuous T-periodic functions in $\mathcal{W}$ that are twice continuously differentiable in x and continuously differentiable in t.

We clarify the meaning of $(x,t) \to \partial\mathcal{W}$. As $\mathcal{W}$ is T-periodic, we can identify it with $[\mathcal{W}] \subset \mathcal{U}\times\mathbb{S}_T$ (see Definition 1.4). Let $\mathbb{E}^d := \mathbb{R}^d \bigcup \partial\mathbb{R}^d$ be the extended Euclidean space, where $\partial\mathbb{R}^d = \{x_*^\infty : x_* \in \mathbb{S}^{d-1}\}$ and x_*^∞ denotes the infinity element of the ray through x_*. Let $\overline{\mathbb{B}^d} = \mathbb{B}^d \bigcup \mathbb{S}^{d-1}$ be the closed unit ball in $\mathbb{R}^d$. Define $h : \mathbb{E}^d \times \mathbb{S}_T \to \overline{\mathbb{B}^d}\times\mathbb{S}_T$ as follows:

$$h(x,t) = \begin{cases} \left(\frac{x}{1+|x|}, t\right), & (x,t) \in \mathbb{R}^d \times \mathbb{S}_T, \\ (x_*, t), & (x,t) = (x_*^\infty, t) \in \partial\mathbb{R}^d \times \mathbb{S}_T. \end{cases}$$

Clearly, h identifies $\mathbb{R}^d\times\mathbb{S}_T$ with $\mathbb{B}^d\times\mathbb{S}_T$ and $\partial\mathbb{R}^d\times\mathbb{S}_T$ with $\mathbb{S}^{d-1}\times\mathbb{S}_T$. If the topology of $\partial\mathbb{R}^d\times\mathbb{S}_T$ is defined as the one inherited from h, then h becomes a homeomorphism.

By virtue of h, the boundary $\partial[\mathcal{W}] \subset \mathbb{E}^d\times\mathbb{S}_T$ of $[\mathcal{W}]$ is defined as the preimage of the boundary of $h([\mathcal{W}])$ in $\overline{\mathbb{B}^d}\times\mathbb{S}_T$, namely, $\partial[\mathcal{W}] = h^{-1}(\partial h([\mathcal{W}]))$. Therefore, $(x,t)\to\partial\mathcal{W}$ if and only if $(x, t \mod T) \to \partial[\mathcal{W}]$, if and only if $h(x, t \mod T) \to \partial h([\mathcal{W}])$.

We start with the definition of compact functions.

Definition 2.4 (Compact function)**.** *A non-negative and continuous function U on $\mathcal{W}$ is called a* compact function *with the* essential upper bound *ρ_M, if*

1. *$U(x,t) < \rho_M$, for $(x,t) \in \mathcal{W}$, and*
2. *$\lim_{(x,t)\to\partial\mathcal{W}} U(x,t) = \rho_M$.*

For a nonnegative and continuous function U on $\mathcal{W}$, we define for each $\rho > 0$ and $t \in \mathbb{R}$

$$\Omega_\rho = \{(x,t) \in \mathcal{W} : U(x,t) < \rho\} \quad \text{ and}$$
$$\Omega_\rho^t = \{x \in \mathcal{W}_t : U(x,t) < \rho\}$$

to be the *ρ-sublevel set* of U and its *t-section*, respectively. Note that the notation for the t-sections of Ω_ρ does NOT follow the **Convention** mentioned right above Definition 1.4.

Definition 2.5 (Lyapunov/Anti-Lyapunov function)**.** *A compact function $U \in C_T^{2,1}(\mathcal{W})$ is called*

(1) *a Lyapunov function* (*resp. an anti-Lyapunov function*) *of* (1.1) *in $\mathcal{W}$, if there exist $\gamma > 0$, called a Lyapunov constant* (*resp. an anti-Lyapunov constant*) *of U, and $\rho_m \geq 0$, called an essential lower bound of U, such that*
$$\mathcal{L}_0 U \leq -\gamma \quad (\textit{resp.} \ \geq \gamma) \quad \textit{in} \quad \mathcal{W} \setminus \overline{\Omega}_{\rho_m},$$
where $\mathcal{L}_0 := \partial_t + V^i \partial_i$ and $\Omega_{\rho_m} := \{(x,t) \in \mathcal{W} : U(x,t) < \rho_m\}$;

(2) *a Lyapunov function* (*resp. an anti-Lyapunov function*) *with respect to $\mathcal{L}_{\epsilon^2 A}$ in $\mathcal{W}$, where $\epsilon > 0$ is fixed, if there exist $\gamma > 0$, called a Lyapunov constant* (*resp. an anti-Lyapunov constant*) *of U, and $\rho_m \geq 0$, called an essential lower bound of U, such that*
$$\mathcal{L}_{\epsilon^2 A} U \leq -\gamma \quad (\textit{resp.} \ \geq \gamma) \quad \textit{in} \quad \mathcal{W} \setminus \overline{\Omega}_{\rho_m}; \tag{2.2}$$

(3) *a uniform Lyapunov function* (*resp. uniform anti-Lyapunov function*) *with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon > 0}$ in $\mathcal{W}$ if there exist positive constants ϵ_*, ρ_m and $\gamma > 0$ such that* (2.2) *holds for all $\epsilon \in (0, \epsilon_*)$.*

Next, we show that the sublevel sets of Lyapunov functions (resp. anti-Lyapunov functions) of (1.1) are positively invariant (resp. negatively invariant).

Proposition 2.4. *Assume $V \in C_T^{1,1}(\mathcal{U} \times \mathbb{R})$. Let U be a Lyapunov function* (*resp. an anti-Lyapunov function*) *of* (1.1) *in $\mathcal{W}$ with essential upper bound ρ_M and essential lower bound ρ_m. Then, Ω_ρ is positively invariant* (*resp. negatively invariant*) *for all $\rho \in [\rho_m, \rho_M]$.*

Proof. We only prove the result when U is a Lyapunov function; the other case can be treated in a similar manner. Let $\gamma > 0$ be the Lyapunov constant of U.

Fix $\rho \in [\rho_m, \rho_M]$ and $(x,t) \in \Omega_\rho$. We need to show that $U(\varphi^{t+\tau,t}(x), t + \tau) < \rho$ for all $\tau \geq 0$. Suppose for contradiction that this fails. Then,
$$\tau_* := \min\left\{\tau > 0 : U(\varphi^{t+\tau,t}(x), t+\tau) = \rho\right\}$$
is well-defined and finite. Moreover, $U(\varphi^{t+\tau_*,t}(x), t+\tau_*) = \rho$. Clearly,
$$U(\varphi^{t+\tau,t}(x), t+\tau) = U(x,t) + \int_0^\tau \mathcal{L}_0 U(\varphi^{t+s,t}(x), t+s)\mathrm{d}s$$

$$\leq U(x,t) - \gamma\tau, \quad \forall \tau \leq \tau_*.$$

It follows that

$$U(\varphi^{t+\tau_*,t}(x), t+\tau_*) \leq U(x,t) - \gamma\tau_* < \rho,$$

which leads to a contradiction. Hence, $(\varphi^{t+\tau,t}(x), t+\tau) \in \Omega_\rho$ for all $\tau \geq 0$. $\square$

2.4. **Harnack's inequality.** Consider the parabolic equation

$$\partial_t u = \partial_i(\alpha^{ij}\partial_j u - \beta^i u), \quad (x,t) \in \mathcal{U} \times \mathbb{R}, \tag{2.3}$$

where α^{ij} and β^i are continuous functions and T-periodic in t for each $i, j = 1, \ldots, d$. Assume (α^{ij}) is pointwise positive definite. Let $\mathcal{W} \subset \mathcal{U} \times \mathbb{R}$ be bounded. Denote by $\lambda_{\mathcal{W}}$ and $\Lambda_{\mathcal{W}}$ respectively the largest and smallest positive constants such that

$$\lambda_{\mathcal{W}}|\xi|^2 \leq \alpha^{ij}(x,t)\xi^i\xi^j \leq \Lambda_{\mathcal{W}}|\xi|^2, \quad \forall (x,t) \in \mathcal{W} \text{ and } \xi \in \mathbb{R}^d.$$

Proposition 2.5 (Harnack's inequality)**.** *Let $u \in \mathcal{H}^{1,p}_{loc}(\mathcal{U}\times\mathbb{R})$, for some $p > d+2$, be a non-negative solution of* (2.3) *in $\mathcal{U}\times\mathbb{R}$. Then for any connected and bounded subdomains $\mathcal{U}_1 \subset\subset \mathcal{U}_2 \subset\subset \mathcal{U}$ and $s < s_1 < t_1 < s_2 < t_2 < t$, there exists $C > 0$, depending only on n, s_1, s_2, t_1, t_2, $\mathcal{U}_1$ and $\mathcal{U}_2$, such that*

$$\sup_{\mathcal{U}_1\times[s_1,t_1]} u \leq C^M \inf_{\mathcal{U}_1\times[s_2,t_2]} u,$$

where

$$M = \Lambda_{\mathcal{U}_2\times[s,t]} + \lambda^{-1}_{\mathcal{U}_2\times[s,t]}\Big(1 + \sum_{i=1}^d \sup_{\mathcal{U}_2\times[s,t]} |\beta^i|^2\Big).$$

Proposition 2.5 is a special case of [2, Theorem 3]. The explicit expression of the constant M follows from the calculations done in [2, 33, 34].

3. Characterization of periodically invariant measures

In this section, we prove an equivalent characterization of periodically invariant measures of (1.1), which provides a convenient way to test that limit measures of periodic probability solutions of (1.3) are periodically invariant measures of (1.1).

We recall from [25] the definition of *t-sections* for σ-finite Borel measures on $\mathcal{U} \times \mathbb{R}$.

Definition 3.1 (t-sections)**.** *A σ-finite Borel measure μ on $\mathcal{U}\times\mathbb{R}$ is said to admit* t-sections*, if there exists a family of σ-finite Borel measures $\{\mu_t\}_{t\in\mathbb{R}}$ on $\mathcal{U}$ such that $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$, namely,*

$$\iint_{\mathcal{U}\times\mathbb{R}} \phi \mathrm{d}\mu = \int_{\mathbb{R}}\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t\mathrm{d}t, \quad \forall \phi \in C_0(\mathcal{U}\times\mathbb{R}).$$

In this case, we denote $\mu = (\mu_t)_{t\in\mathbb{R}}$. Moreover, $(\mu_t)_{t\in\mathbb{R}}$ is called

(1) T-periodic *or simply* periodic*, if $\mu_t = \mu_{t+T}$ for $t\in\mathbb{R}$;*
(2) continuous*, if the function*

$$t \mapsto \int_{\mathcal{U}} g\mathrm{d}\mu_t$$

is continuous on $\mathbb{R}$ for each $g \in C_c^\infty(\mathcal{U})$.

Proposition 3.1. *Assume $V \in C_T^{1,1}(\mathcal{U}\times\mathbb{R})$. Let μ be a Borel measure on $\mathcal{U}\times\mathbb{R}$. If μ satisfies the following conditions:*

(1) *$\mu(\mathcal{U}\times[0,T]) = T$;*
(2) *μ admits continuous and periodic t-sections $(\mu_t)_{t\in\mathbb{R}}$;*
(3) *for each bounded $\phi \in C_T(\mathcal{U}\times\mathbb{R})$, there holds*

$$\int_0^T\int_{\mathcal{U}} \phi(\varphi^{s+t,s}(\cdot), s+t)\mathrm{d}\mu_s\mathrm{d}s = \int_0^T\int_{\mathcal{U}} \phi(\cdot,s)\mathrm{d}\mu_s\mathrm{d}s, \quad \forall t\in[0,\infty),$$

then $\mu = (\mu_t)_{t\in\mathbb{R}}$ is a periodically invariant measure of $(\varphi^{t,s})$.

Proof. By Definition 1.3, it suffices to show that

(a) $\mu_t = \varphi_*^{s+t,s}\mu_s$ for all $s\in\mathbb{R}$ and $t\in[0,\infty)$;
(b) $\mu_0 = \varphi_*^{T,0}\mu_0$;
(c) $\mu_t(\mathcal{U}) = 1$ for all $t\in\mathbb{R}$.

To prove (a), we set $\phi = g\eta$ in (3.1), where $g\in C_c^\infty(\mathcal{U})$ and $\eta\in C_T(\mathbb{R})$, to find

$$\int_0^T \eta(s+t)\int_{\mathcal{U}} g\circ\varphi^{s+t,s}\mathrm{d}\mu_s\mathrm{d}s = \int_0^T \eta(s)\int_{\mathcal{U}} g\mathrm{d}\mu_s\mathrm{d}s, \quad \forall t\in[0,\infty). \quad (3.1)$$

It follows from the periodicity of $\mu = (\mu_t)_{t\in\mathbb{R}}$ and η that

$$\int_0^T \eta(s)\int_{\mathcal{U}} g\mathrm{d}\mu_s\mathrm{d}s = \int_0^T \eta(s+t)\int_{\mathcal{U}} g\mathrm{d}\mu_{s+t}\mathrm{d}s, \quad \forall t\in[0,\infty).$$

Hence,

$$\int_0^T \eta(s+t)\int_{\mathcal{U}} g\circ\varphi^{s+t,s}\mathrm{d}\mu_s\mathrm{d}s = \int_0^T \eta(s+t)\int_{\mathcal{U}} g\mathrm{d}\mu_{s+t}\mathrm{d}s, \quad \forall t\in[0,\infty).$$

Fix $t \in [0, \infty)$ and $g \in C_c^\infty(\mathcal{U})$. Since the functions

$$s \mapsto \int_{\mathcal{U}} g \circ \varphi^{s+t,s} \mathrm{d}\mu_s \quad \text{and} \quad s \mapsto \int_{\mathcal{U}} g \mathrm{d}\mu_{s+t}$$

are continuous on $\mathbb{R}$, and $\eta \in C_T(\mathbb{R})$ is arbitrary, we deduce

$$\int_{\mathcal{U}} g \circ \varphi^{s+t,s} \mathrm{d}\mu_s = \int_{\mathcal{U}} g \mathrm{d}\mu_{s+t}, \quad \forall s \in [0, T].$$

The above identity actually holds for all $s \in \mathbb{R}$ as $\mu = (\mu_t)_{t \in \mathbb{R}}$ is periodic. It then follows from the arbitrariness of $g \in C_c^\infty(\mathcal{U})$ and the density of $C_c^\infty(\mathcal{U})$ in $C_c(\mathcal{U})$ that (a) holds. The properties (b) and (c) follow readily from the periodicity of $(\mu_t)_{t \in \mathbb{R}}$, the property (a) and the condition (1) in the statement. $\square$

Remark 3.1. Note that Proposition 3.1 (3) is equivalent to say that μ, being identified with a Borel measure on $\mathcal{U} \times \mathbb{S}_T$, is invariant under the skew product semi-flow (Φ^τ). Recall that (Φ^τ) is a semi-flow generated by the following system:

$$\begin{cases} x' = V(x, t), \\ t' = 1 \mod T, \end{cases} \qquad (x, t) \in \mathcal{U} \times \mathbb{S}_T,$$

where $x' = \frac{\mathrm{d}x}{\mathrm{d}\tau}$ and $t' = \frac{\mathrm{d}t}{\mathrm{d}\tau}$.

The equivalent characterization is given in the following result. Recall that

$$\mathcal{L}_0 := \partial_t + V^i \partial_i.$$

Proposition 3.2. *Assume $V \in C_T^{1,1}(\mathcal{U} \times \mathbb{R})$. Let μ be a Borel measure on $\mathcal{U} \times \mathbb{R}$ with t-sections $(\mu_t)_{t \in \mathbb{R}}$. Then, μ is a periodically invariant measure of* (1.1) *if and only if the following hold:*

(1) $\mu(\mathcal{U} \times [0, T]) = T$*;*
(2) $(\mu_t)_{t \in \mathbb{R}}$ *is continuous and periodic;*
(3) *there holds*

$$\int_0^T \int_{\mathcal{U}} \mathcal{L}_0 \phi(\cdot, s) \mathrm{d}\mu_s \mathrm{d}s = 0, \quad \forall \phi \in C_{c,T}^{1,1}(\mathcal{U} \times \mathbb{R}).$$

Proof. Necessity. Suppose $\mu = (\mu_t)_{t \in \mathbb{R}}$ is a periodically invariant measure of (1.1). Then, (1) and (2) follow immediately from its definition (see

Definition 1.3) and the continuity of $(\varphi^{t,s})$. It remains to show (3). Note that

$$\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t = \int_{\mathcal{U}} \phi(\varphi^{t,s}(\cdot),t)\mathrm{d}\mu_s, \quad \forall \phi \in C_{c,T}(\mathcal{U}\times\mathbb{R}),\ s\in\mathbb{R} \text{ and } t\geq s.$$

As a result, for each $\phi \in C^{1,1}_{c,T}(\mathcal{U}\times\mathbb{R})$, the function

$$f_\phi : \mathbb{R}\to\mathbb{R}, \quad t\mapsto \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t$$

is T-periodic and continuously differentiable, and satisfies

$$\begin{aligned} f'_\phi(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \phi(\varphi^{t,s}(\cdot),t)\mathrm{d}\mu_s \\ &= \int_{\mathcal{U}} \left(\partial_t\phi + V^i\partial_i\phi\right)(\varphi^{t,s}(\cdot),t)\mathrm{d}\mu_s = \int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot,t)\mathrm{d}\mu_t, \quad \forall t\in(s,\infty). \end{aligned}$$

Integrating the above equality over $[t,t+T]$, we find

$$f(t+T)-f(t) = \int_t^{t+T} f'_\phi \mathrm{d}s = \int_0^T\int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot,s)\mathrm{d}\mu_s\mathrm{d}s, \quad \forall t\in\mathbb{R}.$$

Remark 1.1 and the periodicity of ϕ give

$$f(t+T) = \int_{\mathcal{U}} \phi(\cdot,t+T)\mathrm{d}\mu_{t+T} = \int_{\mathcal{U}} \phi(\cdot,t)\mathrm{d}\mu_t = f(t), \quad \forall t\in\mathbb{R}.$$

Hence,

$$\int_0^T\int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot,s)\mathrm{d}\mu_s\mathrm{d}s = 0$$

and (3) follows.

Sufficiency. By Proposition 3.1, it suffices to show

$$\int_0^T\int_{\mathcal{U}} \phi(\varphi^{s+t,s}(\cdot),s+t)\mathrm{d}\mu_s\mathrm{d}s = \int_0^T\int_{\mathcal{U}} \phi(\cdot,s)\mathrm{d}\mu_s\mathrm{d}s, \tag{3.2}$$

$\forall \phi \in C_{c,T}(\mathcal{U}\times\mathbb{R})$ and $t\in[0,\infty)$. Thanks to the density of $C^{1,1}_{c,T}(\mathcal{U}\times\mathbb{R})$ in $C_{c,T}(\mathcal{U}\times\mathbb{R})$, we only need to show (3.2) for all $\phi\in C^{1,1}_{c,T}(\mathcal{U}\times\mathbb{R})$.

Fix $\phi\in C^{1,1}_{c,T}(\mathcal{U}\times\mathbb{R})$. Obviously, the function

$$t\mapsto \int_0^T\int_{\mathcal{U}} \phi(\varphi^{s+t,s}(\cdot),s+t)\mathrm{d}\mu_s\mathrm{d}s$$

is continuously differentiable on $[0,\infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_0^T\int_{\mathcal{U}}\phi(\varphi^{s+t,s}(\cdot),s+t)\mathrm{d}\mu_s\mathrm{d}s=\int_0^T\int_{\mathcal{U}}\mathcal{L}_0\phi(\cdot,s)\mathrm{d}\mu_s\mathrm{d}s=0, \quad (3.3)$$

where we used the condition (3) in the last equality.

Define

$$\tilde{\phi}_{t'}(x,s):=\phi(\varphi^{s+t',s}(x),s+t'), \quad (x,s)\in\mathcal{U}\times\mathbb{R} \quad \text{and} \quad t'\in[0,\infty).$$

Clearly, $\tilde{\phi}_{t'}\in C^{1,1}_{c,T}(\mathcal{U}\times\mathbb{R})$ for each $t'\in[0,\infty)$. An application of (3.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_0^T\int_{\mathcal{U}}\tilde{\phi}_{t'}(\varphi^{s+t,s}(\cdot),s+t)\mathrm{d}\mu_s\mathrm{d}s=0, \quad \forall t'\in[0,\infty). \quad (3.4)$$

Note that

$$\begin{aligned}&\int_0^T\int_{\mathcal{U}}\tilde{\phi}_{t'}(\varphi^{s+t,s}(\cdot),s+t)\mathrm{d}\mu_s\mathrm{d}s\\&=\int_0^T\int_{\mathcal{U}}\phi(\varphi^{s+t+t',s+t}\circ\varphi^{s+t,s}(\cdot),s+t+t')\mathrm{d}\mu_s\mathrm{d}s\\&=\int_0^T\int_{\mathcal{U}}\phi(\varphi^{s+t+t',s}(\cdot),s+t+t')\mathrm{d}\mu_s\mathrm{d}s, \quad \forall t'\in[0,\infty).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_0^T\int_{\mathcal{U}}\tilde{\phi}_{t'}(\varphi^{s+t,s}(\cdot),t+s)\mathrm{d}\mu_s\mathrm{d}s\\&=\frac{\mathrm{d}}{\mathrm{d}t'}\int_0^T\int_{\mathcal{U}}\phi(\varphi^{s+t',s}(\cdot),s+t')\mathrm{d}\mu_s\mathrm{d}s, \quad \forall t'\in[0,\infty).\end{aligned}$$

This together with (3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t'}\int_0^T\int_{\mathcal{U}}\phi(\varphi^{s+t',s}(\cdot),s+t')\mathrm{d}\mu_s\mathrm{d}s=0, \quad \forall t'\in[0,\infty),$$

which yields (3.2). □

4. INTEGRAL IDENTITY AND MEASURE ESTIMATES

In this section, we first prove an integral identity for periodic probability solutions of (1.3). It is then applied to derive a priori estimates for periodic probability solutions in periodic subdomains of $\mathcal{U}\times\mathbb{R}$ assuming the existence of Lyapunov/anti-Lyapunov functions.

For clarity, we focus on the Fokker-Planck equation (1.3) with $\epsilon = 1$, that is,

$$\partial_t u = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u) \quad \text{in} \quad \mathcal{U} \times \mathbb{R}. \tag{4.1}$$

The corresponding results for (1.3) follows with a^{ij} replaced by $\epsilon^2 a^{ij}$. Denote

$$\mathcal{L}_A := \partial_t + a^{ij}\partial^2_{ij} + \partial_i V^i.$$

Throughout this section, we assume

- $a^{ij} \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ and $V^i \in C^{1,1}_T(\mathcal{U} \times \mathbb{R})$ for all $i, j \in \{1, \ldots, d\}$;
- (a^{ij}) is pointwise positive definite;
- $\mathcal{W}$ is an open and periodic subset of $\mathcal{U} \times \mathbb{R}$.

The integral identity, generalizing the one in [20] for stationary measures, is given in the next result.

Theorem 4.1. *Let $U \in C^{2,1}_T(\mathcal{W})$ be a compact function with essential upper bound $\rho_M > 0$ and satisfy $\nabla_{x,t}U \neq 0$ on $\partial\Omega_\rho$ for some $\rho \in (0, \rho_M)$. If $\mu = (\mu_t)_{t\in\mathbb{R}}$ is a periodic probability solution of (4.1) with a density $u \in \mathcal{H}^{1,p}_{loc,T}(\mathcal{U} \times \mathbb{R})$, then*

$$\int_t^{t+T}\int_{\Omega^s_\rho} \mathcal{L}_A U \mathrm{d}\mu_s \mathrm{d}s = \int_{\partial\Omega_\rho \cap (\mathcal{U}\times[t,t+T])} \frac{a^{ij}\partial_i U \partial_j U}{|\nabla_{x,s}U|} u \mathrm{d}S_{x,s}, \quad \forall t \in \mathbb{R},$$

where $dS_{x,s}$ denotes the Lebesgue measure on $\partial\Omega_\rho \cap (\mathcal{U} \times [t, t+T])$.

Proof. Since $\nabla_{x,t}U \neq 0$ on $\partial\Omega_\rho$, the implicit function theorem ensures that $\partial\Omega_\rho$ is a C^1 hypersurface in $\mathcal{W}$. Define

$$\tilde{U}(x,t) := \begin{cases} U(x,t) - \rho, & (x,t) \in \Omega_\rho, \\ 0, & (x,t) \in \mathcal{W} \setminus \Omega_\rho. \end{cases}$$

Obviously, $\tilde{U} \in C_T(\mathcal{W})$ is supported in $\overline{\Omega}_\rho$ and $\partial_t\tilde{U}, \partial_i\tilde{U}$, $i, j = 1, \ldots, d$ are essentially bounded.

Let $\eta \in C^\infty_c(\mathbb{R}^{d+1})$ be nonnegative and satisfy

$$\iint_{\mathbb{R}^{d+1}} \eta \mathrm{d}x\mathrm{d}t = 1.$$

For $0 < \epsilon \ll 1$, we define

$$\eta_\epsilon(x,t) = \epsilon^{-(d+1)}\eta\big(\frac{x}{\epsilon}, \frac{t}{\epsilon}\big) \ \text{ for } (x,t) \in \mathbb{R}^{d+1}.$$

Set $\tilde{U}_\epsilon = \tilde{U} * \eta_\epsilon$. Then $\tilde{U}_\epsilon$ is a periodic smooth function supported in a neighbourhood Ω_* of $\overline{\Omega}_\rho$ for all $0 < \epsilon \ll 1$, and satisfies

- $\tilde{U}_\epsilon \to \tilde{U}$ in $C_T(\overline{\Omega}_*)$ under the maximum norm as $\epsilon \to 0$;
- $\partial_t\tilde{U}_\epsilon \to \partial_t\tilde{U}$, $\partial_i\tilde{U}_\epsilon \to \partial_i\tilde{U}$ in $L^q(\Omega_*)$ as $\epsilon \to 0$ for any $q > 1$.

We see from Lemma 2.1, Lemma 2.2 and the periodicity of $\tilde{U}_\epsilon$ that for each $0 < \epsilon \ll 1$,

$$\begin{aligned}\int_t^{t+T}\int_{\mathcal{U}}\Big(\partial_t\tilde{U}_\epsilon + a^{ij}\partial^2_{ij}\tilde{U}_\epsilon + V^i\partial_i\tilde{U}_\epsilon\Big)u\mathrm{d}x\mathrm{d}s \\ = \int_{\mathcal{U}}\tilde{U}_\epsilon(\cdot,t+T)u(\cdot,t+T)\mathrm{d}x - \int_{\mathcal{U}}\tilde{U}_\epsilon(\cdot,t)u(\cdot,t)\mathrm{d}x = 0, \quad \forall t\in\mathbb{R}.\end{aligned}\tag{4.2}$$

Since $u \in \mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R})$, we pass to the limit $\epsilon \to 0$ to find that for each $t \in \mathbb{R}$,

$$\begin{aligned}\int_t^{t+T}\int_{\mathcal{U}} a^{ij}\partial^2_{ij}\tilde{U}_\epsilon u\mathrm{d}x\mathrm{d}s &= -\int_t^{t+T}\int_{\mathcal{U}}\partial_j(a^{ij}u)\partial_i\tilde{U}_\epsilon\mathrm{d}x\mathrm{d}s \\ &\to -\int_t^{t+T}\int_{\Omega^s_\rho}\partial_j(a^{ij}u)\partial_i U\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{4.3}$$

Integration by parts yields that

$$\begin{aligned}-\int_t^{t+T}\int_{\Omega^s_\rho}\partial_j(a^{ij}u)\partial_i U\mathrm{d}x\mathrm{d}s \\ = -\int_{\partial(\bigcup_{s\in(t,t+T)}(\Omega^s_\rho\times\{s\}))} a^{ij}\partial_i U\tilde{\nu}_j u\mathrm{d}S_{x,s} + \int_t^{t+T}\int_{\Omega^s_\rho} a^{ij}\partial^2_{ij}U u\mathrm{d}x\mathrm{d}s,\end{aligned}\tag{4.4}$$

where $\mathrm{d}S_{x,s}$ denotes the Lebesgue measure on $\partial(\bigcup_{s\in(t,t+T)}(\Omega^s_\rho\times\{s\}))$ and $\tilde{\nu}_j$ is the j-th component of the unit outward normal vector field $\tilde{\nu}$ along the boundary of $\bigcup_{s\in(t,t+T)}(\Omega^s_\rho\times\{s\})$. Since

$$\partial\Big(\bigcup_{s\in(t,t+T)}(\Omega^s_\rho\times\{s\})\Big) = \Omega^t_\rho\bigcup\Omega^{t+T}_\rho\bigcup\left(\partial\Omega_\rho\cap(\mathcal{U}\times[t,t+T])\right)$$

and

$$\tilde{\nu}_j(x,s) = \begin{cases}0, & \forall (x,s)\in\Omega_t\bigcup\Omega_{t+T},\\ \frac{\partial_j U}{|\nabla_{x,s}U|}, & \forall (x,s)\in\partial\Omega_\rho\cap(\mathcal{U}\times[t,t+T]),\end{cases}$$

there holds

$$-\int_{\partial(\bigcup_{s\in(t,t+T)}(\Omega^s_\rho\times\{s\}))} a^{ij}\partial_i U\tilde{\nu}_j u\mathrm{d}S_{x,s} = -\int_{\partial\Omega_\rho\cap(\mathcal{U}\times[t,t+T])}\frac{a^{ij}\partial_i U\partial_j U}{|\nabla_{x,s}U|}u\mathrm{d}S_{x,s},$$

which together with (4.3) and (4.4) yields

$$\begin{aligned}\int_t^{t+T}\int_{\mathcal{U}} a^{ij}\partial_{ij}^2\tilde{U}_\epsilon u\mathrm{d}x\mathrm{d}s &\to -\int_t^{t+T}\int_{\Omega_\rho^s}\partial_j(a^{ij}u)\partial_i U\mathrm{d}x\mathrm{d}s \\ &= -\int_{\partial\Omega_\rho\cap(\mathcal{U}\times[t,t+T])}\frac{a^{ij}\partial_i U\partial_j U}{|\nabla_{x,s}U|}u\mathrm{d}S_{x,s}+\int_t^{t+T}\int_{\Omega_\rho^s}a^{ij}\partial_{ij}^2 U u\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{4.5}$$

as $\epsilon\to 0$.

For other terms on the left hand side of (4.2), it is easy to see that

$$\int_t^{t+T}\int_{\mathcal{U}}\left(\partial_t\tilde{U}_\epsilon+V^i\partial_i\tilde{U}_\epsilon\right)u\mathrm{d}x\mathrm{d}s\to\int_t^{t+T}\int_{\Omega_\rho^s}\left(\partial_t U+V^i\partial_i U\right)u\mathrm{d}x\mathrm{d}s, \tag{4.6}$$

as $\epsilon\to 0$. The theorem follows from (4.5) and (4.6). □

We apply the integral identity in Theorem 4.1 to derive a priori estimates for periodic probability solutions of (4.1).

Theorem 4.2. *Let $U\in C_T^{2,1}(\mathcal{W})$ be a Lyapunov function (resp. an anti-Lyapunov function) with respect to $\mathcal{L}_A$ in $\mathcal{W}$ with essential upper bound $\rho_M>0$, essential lower bound $\rho_m\geq 0$ and Lyapunov constant (resp. anti-Lyapunov constant) $\gamma>0$. Suppose that $\nabla_{x,t}U\neq 0$ on $\partial\Omega_\rho$ for a.e. $\rho\in(\rho_m,\rho_M)$. The following statements hold for any periodic probability solution $\mu=(\mu_t)_{t\in\mathbb{R}}$ of (4.1) with a density in $\mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R})$.*

(1) *If U is a Lyapunov function, then*

$$\int_t^{t+T}\mu_s(\mathcal{W}_s\setminus\Omega_\rho^s)\mathrm{d}s\leq Te^{-\gamma\int_{\rho_m}^{\rho}\frac{1}{H(s)}\mathrm{d}s},\quad\forall\rho\in(\rho_m,\rho_M)\text{ and }t\in\mathbb{R},$$

where

$$H(\rho):=\sup_{\partial\Omega_\rho}a^{ij}\partial_i U\partial_j U\quad\text{for }\rho\in(\rho_m,\rho_M).$$

(2) *If U is an anti-Lyapunov function, then*

$$\int_t^{t+T}\mu_s(\Omega_{\rho_1}^s\setminus\Omega_{\rho_m}^s)\mathrm{d}s\geq e^{\gamma\int_{\rho_0}^{\rho_1}\frac{1}{H(s)}\mathrm{d}s}\int_t^{t+T}\mu_s(\Omega_{\rho_0}^s\setminus\Omega_{\rho_m}^s)\mathrm{d}s,$$

$\forall\rho_m<\rho_0<\rho_1<\rho_M$ *and* $t\in\mathbb{R}$.

Proof. Let $u\in\mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R})$ be the density of μ. Denote by $\mathcal{I}$ the set of all $\rho\in(\rho_m,\rho_M)$ satisfying $\nabla_{x,t}U\neq 0$ on $\partial\Omega_\rho$. Apparently, $|\mathcal{I}|=\rho_M-\rho_m$ and $\mathcal{I}$ is dense in (ρ_m,ρ_M).

(1) Fix $t \in \mathbb{R}$ and take $\rho, \rho_1 \in \mathcal{I}$ with $\rho < \rho_1$. Applying Theorem 4.1, we find

$$\begin{aligned}
&\int_t^{t+T} \int_{\overline{\Omega}_{\rho_1}^s \setminus \Omega_\rho^s} \mathcal{L}_A U \mathrm{d}\mu_s \mathrm{d}s \\
&= \Big(\int_{\partial\Omega_{\rho_1} \cap (\mathcal{U} \times [t,t+T])} - \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \Big) \frac{a^{ij}\partial_i U \partial_j U}{|\nabla_{x,s} U|} u \mathrm{d}S_{x,s}.
\end{aligned}$$

Since $\mathcal{L}_A U \le -\gamma$ in $\mathcal{W} \setminus \Omega_\rho$ and (a^{ij}) is non-negative definite, we deduce

$$-\gamma \int_t^{t+T} \mu_s(\overline{\Omega}_{\rho_1}^s \setminus \Omega_\rho^s) \mathrm{d}s \ge - \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \frac{a^{ij}\partial_i U \partial_j U}{|\nabla_{x,s} U|} u \mathrm{d}S_{x,s},$$

which leads to

$$\begin{aligned}
\int_t^{t+T} \mu_s(\overline{\Omega}_{\rho_1}^s \setminus \Omega_\rho^s) \mathrm{d}s &\le \frac{1}{\gamma} \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \frac{a^{ij}\partial_i U \partial_j U}{|\nabla_{x,s} U|} u \mathrm{d}S_{x,s} \\
&\le \frac{H(\rho)}{\gamma} \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \frac{u}{|\nabla_{x,s} U|} \mathrm{d}S_{x,s}.
\end{aligned}$$

As $\mathcal{I}$ is dense in (ρ_m, ρ_M), letting $\rho_1 \in \mathcal{I} \to \rho_M$ in the above inequality results in

$$\int_t^{t+T} \mu_s(\mathcal{W}_s \setminus \Omega_\rho^s) \mathrm{d}s \le \frac{H(\rho)}{\gamma} \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \frac{u}{|\nabla_{x,s} U|} \mathrm{d}S_{x,s}, \quad \forall \rho \in \mathcal{I}. \tag{4.7}$$

Set

$$f(\rho) := \int_t^{t+T} \mu_s(\mathcal{W}_s \setminus \Omega_\rho^s) \mathrm{d}s = \int_t^{t+T} \int_{\mathcal{W}_s \setminus \Omega_\rho^s} u \mathrm{d}x \mathrm{d}s, \quad \forall \rho \in (\rho_m, \rho_M).$$

The coarea formula ensures that

$$f'(\rho) = - \int_{\partial\Omega_\rho \cap (\mathcal{U} \times [t,t+T])} \frac{u}{|\nabla_{x,s} U|} \mathrm{d}S_{x,s}, \quad \forall \rho \in \mathcal{I}.$$

It follows from (4.7) that

$$f(\rho) \le -\frac{H(\rho)}{\gamma} f'(\rho), \quad \forall \rho \in \mathcal{I}.$$

Applying the Gronwall's inequality, we conclude from the continuity of f on (ρ_m, ρ_M) that

$$f(\rho) \le f(\rho') e^{-\gamma \int_{\rho'}^{\rho} \frac{1}{H(s)} \mathrm{d}s} \le T e^{-\gamma \int_{\rho'}^{\rho} \frac{1}{H(s)} \mathrm{d}s}, \quad \forall \rho, \rho' \in (\rho_m, \rho_M) \text{ with } \rho' < \rho,$$

which is the same as

$$\int_t^{t+T} \mu_s(\mathcal{W}_s \setminus \Omega_\rho^s)\mathrm{d}s \le Te^{-\gamma\int_{\rho'}^{\rho}\frac{1}{H(s)}\mathrm{d}s}, \quad \forall \rho, \rho' \in (\rho_m, \rho_M) \text{ with } \rho' < \rho.$$

Letting $\rho' \to \rho_m$ concludes the result.

(2) Fix $t \in \mathbb{R}$ and take $\rho, \rho_1 \in \mathcal{I}$ satisfying $\rho_1 > \rho$. We apply Theorem 4.1 to find

$$\begin{aligned}&\int_t^{t+T}\int_{\overline{\Omega}_{\rho_1}^s\setminus\Omega_\rho^s} \mathcal{L}U\mathrm{d}\mu_s\mathrm{d}s\\ &= \Big(\int_{\partial\Omega_{\rho_1}\cap(\mathcal{U}\times[t,t+T])} - \int_{\partial\Omega_{\rho}\cap(\mathcal{U}\times[t,t+T])}\Big)\frac{a^{ij}\partial_i U\partial_j U}{|\nabla_{x,s}U|}u\mathrm{d}S_{x,s}.\end{aligned}$$

Since $\mathcal{L}U \ge \gamma$ in $\mathcal{W}\setminus\Omega_\rho$ and (a^{ij}) is non-negative definite, we have

$$\gamma\int_t^{t+T}\mu_s(\overline{\Omega}_{\rho_1}^s\setminus\Omega_\rho^s)\mathrm{d}s \le \int_{\partial\Omega_{\rho_1}\cap(\mathcal{U}\times[t,t+T])}\frac{a^{ij}\partial_i U\partial_j U}{|\nabla_{x,s}U|}u\mathrm{d}S_{x,s},$$

which yields

$$\int_t^{t+T}\mu_s(\overline{\Omega}_{\rho_1}^s\setminus\Omega_\rho^s)\mathrm{d}s \le \frac{H(\rho_1)}{\gamma}\int_{\partial\Omega_{\rho_1}\cap(\mathcal{U}\times[t,t+T])}\frac{u}{|\nabla_{x,s}U|}\mathrm{d}S_{x,s}. \tag{4.8}$$

Set

$$f(\rho_1) := \int_t^{t+T}\mu_s(\overline{\Omega}_{\rho_1}^s\setminus\Omega_\rho^s)\mathrm{d}s = \int_t^{t+T}\int_{\Omega_{\rho_1}^s\setminus\Omega_\rho^s}u\mathrm{d}x\mathrm{d}s, \quad \forall \rho_1 \in (\rho, \rho_M).$$

Applying the coarea formula, we find

$$f'(\rho_1) = \int_{\partial\Omega_{\rho_1}\cap(\mathcal{U}\times[t,t+T])}\frac{u}{|\nabla_{x,s}U|}\mathrm{d}S_{x,s}, \quad \forall \rho_1 \in \mathcal{I}\cap(\rho,\rho_M).$$

It follows from (4.8) that

$$f(\rho_1) \le \frac{H(\rho_1)}{\gamma}f'(\rho_1), \quad \forall \rho_1 \in \mathcal{I}\cap(\rho,\rho_M).$$

The Gronwall's inequality and the continuity of f in (ρ, ρ_M) ensure

$$f(\rho_1) \ge f(\rho_0)e^{\gamma\int_{\rho_0}^{\rho_1}\frac{\mathrm{d}\rho}{H(\rho)}}, \quad \forall \rho_1 > \rho_0 > \rho,$$

namely,

$$\int_t^{t+T}\mu_s(\Omega_{\rho_1}^s\setminus\Omega_\rho^s)\mathrm{d}s \ge e^{\gamma\int_{\rho_0}^{\rho_1}\frac{\mathrm{d}\rho}{H(\rho)}}\int_t^{t+T}\mu_s(\Omega_{\rho_0}^s\setminus\Omega_\rho^s)\mathrm{d}s, \quad \forall \rho_1 > \rho_0 > \rho.$$

Letting $\rho \to \rho_m$, we arrive at the result. $\square$

5. Quantitative concentration

In Subsection 5.1, we construct uniform Lyapunov functions (resp. uniform anti-Lyapunov functions) with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ in neighbourhoods of maximal periodic attractors (resp. maximal periodic repellers). We then apply Theorem 4.2 to prove Theorem A and Corollary A in Subsection 5.2.

5.1. **Uniform Lyapunov/anti-Lyapunov functions.** Throughout this subsection, let $\mathcal{E}$ be a maximal periodic attractor (resp. maximal periodic repeller) and $B(\mathcal{E})$ be its basin of attraction (resp. basin of expansion).

Theorem 5.1. *Assume $V \in C_T^{1,1}(\mathcal{U}\times\mathbb{R})$. Let $\mathcal{V}^1$ and $\mathcal{V}^2$ be open, connected and periodic subsets of $\mathcal{U}\times\mathbb{R}$ and satisfy $\mathcal{E} \subset\subset \mathcal{V}^1 \subset\subset \mathcal{V}^2 \subset\subset B(\mathcal{E})$. Suppose $\mathcal{V}^2$ is positively invariant (resp. negatively invariant). Then, the following statements hold.*

(1) *There exist an open, connected and positively invariant (resp. negatively invariant) set $\mathcal{W} \subset B(\mathcal{E})$, and a bounded C^∞ Lyapunov function (resp. anti-Lyapunov function) U of (1.1) in $\mathcal{W}$ with essential upper bound $\rho_M > 0$, essential lower bound $\rho_m > 0$ and Lyapunov constant (resp. anti-Lyapunov constant) $\gamma > 0$. Moreover, the following properties hold:*
   - $\mathcal{E} \subset\subset \Omega_{\rho_m} \subset\subset \mathcal{V}^1 \subset\subset \mathcal{V}^2 \subset\subset \mathcal{W}$,
   - $\nabla_{x,t} U \neq 0$ *on* $\partial\Omega_\rho$ *for a.e.* $\rho \in (\rho_m, \rho_M)$, *where*

$$\Omega_\rho := \{(x,t) \in \mathcal{W} : U(x,t) < \rho\}.$$

(2) *Let $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ and $A := \frac{1}{2}GG^\top$. Then, there exists some $\epsilon_* > 0$, depending on γ, $G|_{\mathcal{W}}$ and U, such that U is a uniform Lyapunov function (resp. uniform anti-Lyapunov function) with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ with essential lower bound ρ_m and Lyapunov constant (resp. anti-Lyapunov constant) $\frac{\gamma}{2}$.*

We prove two lemmas before proving the above theorem.

Lemma 5.1. *Assume $V \in C_T^{1,1}(\mathcal{U}\times\mathbb{R})$. Let $\mathcal{V} \subset \mathcal{U}\times\mathbb{R}$ be open and connected, and satisfy $\mathcal{E} \subset\subset \mathcal{V} \subset\subset B(\mathcal{E})$. Then the following hold.*

(1) *For each $(x,t) \in B(\mathcal{E})$, there is a $\tau_* \geq 0$ (resp. $\tau_* \leq 0$) such that $(\varphi^{t+\tau,t}(x), t+\tau) \in \mathcal{V}$ for all $\tau \geq \tau_*$ (resp. $\tau \leq \tau_*$).*
(2) *Let $\mathcal{W} \subset \mathcal{U}\times\mathbb{R}$ be positively invariant (resp. negatively invariant) and satisfy $\mathcal{V} \subset \mathcal{W} \subset B(\mathcal{E})$. Then, $\mathcal{W}$ is connected.*

Proof. (1) It is a simple consequence of Proposition 2.2. (2) For any $(x,t) \in \mathcal{W}$, (1) ensures that the forward orbit (resp. backward orbit) of (1.1) starting at (x,t) enters $\mathcal{V}$ after some finite time. Given the positive invariance (resp. negative invariance) of $\mathcal{W}$, its connectedness follows from that of $\mathcal{V}$. $\square$

Lemma 5.2. *Assume $V \in C^{1,1}_T(\mathcal{U}\times\mathbb{R})$. Let $\mathcal{V}\subset\mathcal{U}\times\mathbb{R}$ be open, connected and periodic, and satisfy $\mathcal{E}\subset\subset\mathcal{V}\subset\subset B(\mathcal{E})$. Then, there exists an open, connected, periodic and positively invariant set* (*resp. negatively invariant set*) *$\mathcal{W}$ satisfying $\mathcal{V}\subset\subset\mathcal{W}\subset\subset B(\mathcal{E})$. Moreover, $\overline{\mathcal{W}}$ is connected and positively invariant* (*resp. negatively invariant*).

Proof. We prove the lemma when $\mathcal{E}$ is a maximal periodic attractor; the other case follows in the same manner. Since $\mathcal{V}\subset\subset B(\mathcal{E})$, there exists an open, connected and periodic set $\mathcal{V}^1$ satisfying $\mathcal{V}\subset\subset\mathcal{V}^1\subset\subset B(\mathcal{E})$. Define

$$\mathcal{W} := \bigcup_{\tau\in[0,\infty)}\bigcup_{t\in\mathbb{R}}\left(\varphi^{t+\tau,t}(\mathcal{V}^1_t)\times\{t+\tau\}\right).$$

Obviously, $\mathcal{V}\subset\subset\mathcal{W}$ and $\mathcal{W}$ is periodic and positively invariant. Hence, the connectedness of $\mathcal{W}$ follows from Lemma 5.1 (2).

To show the openness of $\mathcal{W}$, we consider the skew product semi-flow corresponding to $(\varphi^{t,s})$ (see Remark 3.1):

$$\Phi^\tau((x,t)) := \left(\varphi^{t+\tau,t}(x), (t+\tau) \mod T\right), \quad \forall (x,t)\in\mathcal{U}\times\mathbb{S}_T,\ \tau\ge 0.$$

Then, $\mathcal{W}$ can be identified with $\bigcup_{\tau\ge0}\Phi^\tau(\mathcal{V}^1)$. Applying the semi-flow property of (Φ^τ), we see that for each $\tau\ge0$, $\Phi^\tau(\mathcal{V}^1)$ is open in $\mathcal{U}\times\mathbb{S}_T$. Hence, $\bigcup_{\tau\ge0}\Phi^\tau(\mathcal{V}^1)$ is open in $\mathcal{U}\times\mathbb{S}_T$. The openness of $\mathcal{W}$ in $\mathcal{U}\times\mathbb{R}$ follows.

We show $\mathcal{W}\subset\subset B(\mathcal{E})$. By Proposition 2.2, there holds

$$\lim_{\tau\to\infty}\sup_{t\in\mathbb{R}}\mathrm{dist}_H(\varphi^{t+\tau,t}(\mathcal{V}^1_t),\mathcal{E}_{t+\tau}) = 0,$$

which yields the existence of $\tau_0>0$ such that

$$\mathcal{W}_{\ge\tau_0} := \bigcup_{\tau\in[\tau_0,\infty)}\bigcup_{t\in\mathbb{R}}(\varphi^{t+\tau,t}(\mathcal{V}^1_t)\times\{t+\tau\})\subset\subset B(\mathcal{E}).$$

Since $\mathcal{V}^1\subset\subset B(\mathcal{E})$ and $B(\mathcal{E})$ is positively invariant, we see that

$$\mathcal{W}_{<\tau_0} := \bigcup_{\tau\in[0,\tau_0)}\bigcup_{t\in\mathbb{R}}(\varphi^{t+\tau,t}(\mathcal{V}^1_t)\times\{t+\tau\})\subset\subset B(\mathcal{E}).$$

It follows from $\mathcal{W} = \mathcal{W}_{<\tau_0}\bigcup\mathcal{W}_{\ge\tau_0}$ that $\mathcal{W}\subset\subset B(\mathcal{E})$.

Obviously, $\overline{\mathcal{W}}$ is periodic, compact and positively invariant. The connectedness of $\overline{\mathcal{W}}$ then follows from Lemma 5.1 (2). $\square$

Now, we prove Theorem 5.1.

Proof of Theorem 5.1. We only prove the case when $\mathcal{E}$ is a maximal periodic attractor; the case of a maximal periodic repeller can be treated in the same manner.

We start with (1). By Lemma 5.2, there is an open, connected, periodic and positively invariant set $\mathcal{W}^2$ satisfying $\mathcal{V}^2 \subset\subset \mathcal{W}^2 \subset\subset B(\mathcal{E})$. Moreover, $\overline{\mathcal{W}^2}$ is connected and positively invariant. Applying Proposition 2.2 to $\overline{\mathcal{W}^2}$, we find

$$\lim_{\tau\to\infty}\sup_{t\in\mathbb{R}} \operatorname{dist}_H\left(\varphi^{t+\tau,t}(\overline{\mathcal{W}^2_t}), \mathcal{E}_{t+\tau}\right) = 0.$$

Since $\mathcal{E} \subset\subset \mathcal{V}^1$, there exists $\tau_* > 0$ such that

$$\mathcal{W}^1 := \bigcup_{\tau\geq\tau_*}\bigcup_{t\in\mathbb{R}}(\varphi^{t+\tau,t}(\mathcal{W}^2_t) \times \{t+\tau\}) \subset\subset \mathcal{V}^1.$$

Clearly, $\mathcal{W}^1$ is connected, periodic and positively invariant. The openness of $\mathcal{W}^1$ follows from arguments as in the proof of Lemma 5.2.

We finish the proof of (1) within two steps.

Step 1. We generalize the integral approach in [27] to construct a continuously differentiable function $U_* : B(\mathcal{E}) \to [0,\infty)$ satisfying the following properties:

(a) There holds

$$\sup_{\mathcal{V}^2} U_* < \inf_{\partial\mathcal{W}^2} U_*. \tag{5.1}$$

(b) There exists an open, connected and periodic set $\tilde{\mathcal{W}}^1$ satisfying $\mathcal{W}^1 \subset\subset \tilde{\mathcal{W}}^1 \subset\subset \mathcal{V}^1$ such that

$$\sup_{\tilde{\mathcal{W}}^1} U_* < \inf_{\partial\mathcal{V}^1} U_*. \tag{5.2}$$

(c) There exists some $\gamma > 0$ such that

$$\mathcal{L}_0 U_* \leq -2\gamma \quad \text{in} \quad \overline{\mathcal{W}^2} \setminus \tilde{\mathcal{W}}^1. \tag{5.3}$$

Let $\delta \in (0,1)$ be such that

$$(\partial\mathcal{W}^2)_\delta := \{(x,t) \in \mathcal{U}\times\mathbb{R} : \operatorname{dist}\left((x,t), \partial\mathcal{W}^2\right) < \delta\} \subset\subset B(\mathcal{E})$$

and $(\partial\mathcal{W}^2)_\delta \cap \overline{\mathcal{V}^2} = \emptyset$.

Since $\overline{\mathcal{W}^1}$ and $(\partial\mathcal{W}^2)_\delta$ are periodic, we can find non-negative and T-periodic functions $\zeta,\eta\in C^\infty(\mathcal{U}\times\mathbb{R})$ satisfying

$$\zeta(x,t)\begin{cases}=0, & (x,t)\in\overline{\mathcal{W}^1},\\ >0, & \text{otherwise},\end{cases}\quad \text{and}\quad \eta(x,t)\begin{cases}>0, & (x,t)\in(\partial\mathcal{W}^2)_\delta,\\ =0, & \text{otherwise}.\end{cases}$$

For each $n\in\mathbb{N}$, define

$$U_n(x,t):=\int_0^{\tau_*}(\zeta+n\eta)(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau,\quad \forall(x,t)\in B(\mathcal{E}).$$

Obviously, U_n is well-defined, nonnegative and continuously differentiable.

We claim that there exists some $n_*\in\mathbb{N}$ such that

$$\sup_{\mathcal{V}^2}U_{n_*}<\inf_{\partial\mathcal{W}^2}U_{n_*}.$$

Note that for any $(x,t)\in\partial\mathcal{W}^2$, there is $\tau_{(x,t)}>0$ such that $(\varphi^{t+\tau,t}(x),t+\tau)\in(\partial\mathcal{W}^2)_{\frac{\delta}{2}}$ for all $\tau\in[0,\tau_{(x,t)})$. The continuity of $(\varphi^{t,s})$ and the compactness of $\partial\mathcal{W}^2$ then ensure the existence of some $\tau_0>0$ such that $(\varphi^{t+\tau,t}(x),t+\tau)\in(\partial\mathcal{W}^2)_{\frac{\delta}{2}}$ for all $(x,t)\in\partial\mathcal{W}^2$ and $\tau\in[0,\tau_0)$. As a result, for any $n\in\mathbb{N}$, there holds

$$\begin{aligned}U_n(x,t)&=\int_0^{\tau_*}(\zeta+n\eta)(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau \\ &\geq\int_0^{\tau_0}(\zeta+n\eta)(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau\geq n\tau_0\min_{(\partial\mathcal{W}^2)_{\frac{\delta}{2}}}\eta,\quad\forall(x,t)\in\partial\mathcal{W}^2.\end{aligned}\tag{5.4}$$

As $\mathcal{V}^2$ is positively invariant, we deduce

$$\begin{aligned}U_n(x,t)&=\int_0^{\tau_*}(\zeta+n\eta)(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau \\ &=\int_0^{\tau_*}\zeta(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau\leq\tau_*\sup_{\mathcal{V}^2}\zeta,\quad\forall(x,t)\in\mathcal{V}^2\text{ and }n\in\mathbb{N}.\end{aligned}\tag{5.5}$$

The claim follows readily from (5.4) and (5.5).

Set $U_*:=U_{n_*}$. We show that U_* satisfies the expected properties (a), (b) and (c).

(a) It is trivial.

(b) As $\mathcal{W}^1$ is positively invariant under $(\varphi^{t,s})$, the definitions of ζ, η and U_{n_*} ensure that $U_*\equiv0$ on $\overline{\mathcal{W}^1}$. Obviously, $\inf_{\partial\mathcal{V}^1}U_*(x,t)>0$. Hence, (b) follows from $\mathcal{W}^1\subset\subset\mathcal{V}^1$ and the continuity of $(\varphi^{t,s})$.

(c) Direct calculations give

$$\begin{aligned}\mathcal{L}_0 U_*(x,t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} U_*(\varphi^{t+\tau,t}(x), t+\tau)\Big|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{\tau_*} (\zeta + n_*\eta)(\varphi^{t+\tau+\tau_1,t}(x), t+\tau+\tau_1)\mathrm{d}\tau_1\Big|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_\tau^{\tau_*+\tau} (\zeta + n_*\eta)(\varphi^{t+\tau_1,t}(x), t+\tau_1)\mathrm{d}\tau_1\Big|_{\tau=0}.\end{aligned}$$

As the definition of $\mathcal{W}^1$ ensures

$$(\varphi^{t+\tau,t}(x), t+\tau) \in \overline{\mathcal{W}^1}, \quad \forall (x,t) \in \overline{\mathcal{W}^2} \text{ and } \tau \geq \tau_*,$$

it follows from $\zeta = 0$ and $\eta = 0$ on $\overline{\mathcal{W}^1}$ that

$$\int_\tau^{\tau_*+\tau} (\zeta + n_*\eta)(\varphi^{t+\tau_1,t}(x), t+\tau_1)\mathrm{d}\tau_1 = \int_\tau^{\tau_*} (\zeta + n_*\eta)(\varphi^{t+\tau_1,t}(x), t+\tau_1)\mathrm{d}\tau_1.$$

As a result, we arrive at

$$\begin{aligned}\mathcal{L}_0 U_*(x,t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_\tau^{\tau_*} (\zeta + n_*\eta)(\varphi^{t+\tau_1,t}(x), t+\tau_1)\mathrm{d}\tau_1\Big|_{\tau=0} \\ &= -(\zeta + n_*\eta)(x,t), \quad \forall (x,t) \in \overline{\mathcal{W}^2}.\end{aligned}$$

Setting

$$\gamma := \frac{1}{2} \min_{\overline{\mathcal{W}^2}\setminus\tilde{\mathcal{W}}^1} (\zeta + n_*\eta) > 0$$

in the above equality, we find

$$\mathcal{L}_0 U_* \leq -2\gamma \ \text{ in } \overline{\mathcal{W}^2} \setminus \tilde{\mathcal{W}}^1.$$

This proves (c).

Step 2. We use Morse functions to construct a smooth Lyapunov function of (1.1).

Clearly, we can extend U_* to be a function in $C_T^{1,1}(\mathcal{U} \times \mathbb{R})$. As Morse functions are dense in $C_T^{1,1}(\mathcal{U} \times \mathbb{R}) \approx C^1(\mathcal{U} \times \mathbb{S}_T)$, there exists a sequence of Morse functions

$$\{\tilde{U}_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathcal{U} \times \mathbb{S}_T) \approx C_T^\infty(\mathcal{U} \times \mathbb{R})$$

such that

$$\lim_{n\to\infty} \max_{\overline{\mathcal{W}^2}} \left(|\tilde{U}_n - U_*| + |\partial_t \tilde{U}_n - \partial_t U_*| + \sum_{i=1}^d |\partial_i \tilde{U}_n - \partial_i U_*| \right) = 0.$$

Since

$$\lim_{n\to\infty}\sup_{\mathcal{V}^2}\tilde{U}_n=\sup_{\mathcal{V}^2}U_* \quad\text{and}\quad \lim_{n\to\infty}\inf_{\partial\mathcal{W}^2}\tilde{U}_n=\inf_{\partial\mathcal{W}^2}U_*,$$

we find from (5.1) the existence of some $n_1\in\mathbb{N}$ such that

$$\sup_{\mathcal{V}^2}\tilde{U}_n<\inf_{\partial\mathcal{W}^2}\tilde{U}_n,\quad \forall n\geq n_1. \tag{5.6}$$

Similarly, it follows from (5.2) that there is an $n_2\in\mathbb{N}$ such that

$$\sup_{\tilde{\mathcal{W}}^1}\tilde{U}_n<\inf_{\partial\mathcal{V}^1}\tilde{U}_n,\quad \forall n\geq n_2. \tag{5.7}$$

Since (5.3) and

$$\max_{\overline{\mathcal{W}^2}\setminus\tilde{\mathcal{W}}^1}|\mathcal{L}_0\tilde{U}_n-\mathcal{L}_0U_*|\to 0 \quad\text{as } n\to\infty,$$

we can find some $n_3\in\mathbb{N}$ such that

$$\mathcal{L}_0\tilde{U}_n\leq-\gamma\quad\text{in}\quad \overline{\mathcal{W}^2}\setminus\tilde{\mathcal{W}}^1,\quad \forall n\geq n_3. \tag{5.8}$$

Let $N=\max\{n_1,n_2,n_3\}$ and $U:=\tilde{U}_N$. Then (5.6), (5.7) and (5.8) hold with $\tilde{U}_n$ replaced by U.

Fix $\rho_m\in(\sup_{\tilde{\mathcal{W}}^1}U,\inf_{\partial\mathcal{V}^1}U)$. Clearly,

$$\tilde{\mathcal{W}}^1\subset\subset\Omega_{\rho_m}:=\{(x,t)\in\overline{\mathcal{W}^2}:U(x,t)<\rho_m\}.$$

We show $\Omega_{\rho_m}\subset\subset\mathcal{V}^1$. Suppose on the contrary that there exists $(x,t)\in\overline{\mathcal{W}^2}\setminus\mathcal{V}^1$ such that $U(x,t)\leq\rho_m$. Set

$$\tau':=\inf\left\{\tau\geq 0:(\varphi^{t+\tau,t}(x),t+\tau)\in\mathcal{V}^1\right\}.$$

We apply Lemma 5.1 (with $\mathcal{V}$ replaced by $\mathcal{V}^1$) to find $\tau'<\infty$. The continuity of $(\varphi^{t,s})$ and the pre-compactness of $\mathcal{V}^1$ then lead to $(\varphi^{t+\tau',t}(x),t+\tau')\in\partial\mathcal{V}^1$. It follows that

$$\begin{aligned}\inf_{\partial\mathcal{V}^1}U&\leq U(\varphi^{t+\tau',t}(x),t+\tau')\\&=U(x,t)+\int_0^{\tau'}\mathcal{L}_0U(\varphi^{t+\tau,t}(x),t+\tau)\mathrm{d}\tau\leq\rho_m-\gamma\tau',\end{aligned}$$

which leads to a contradiction.

Clearly, (5.6) yields the existence of some $\rho_M\in(\sup_{\mathcal{V}^2}U,\inf_{\partial\mathcal{W}^2}U)$ such that $\mathcal{V}^2\subset\subset\Omega_{\rho_M}\subset\subset\mathcal{W}^2$. We then see from (5.8) that U is a Lyapunov function of (1.1) in $\mathcal{W}:=\Omega_{\rho_M}$ with essential upper bound ρ_M, essential lower bound $\rho_m>0$ and Lyapunov constant $\gamma>0$.

Lemma 5.1 and Proposition 2.4 ensure that $\mathcal{W}$ is connected and positively invariant. Finally, we see that $\nabla_{x,t}U \neq 0$ in $\mathcal{W}$ except for finite points, which is a property of U being a Morse function on $\mathcal{U} \times \mathbb{S}_T$. This proves (1).

Now, we prove (2). Setting

$$\epsilon_* := \sup\left\{\epsilon > 0 : \epsilon^2 \sup_{\mathcal{W}} |a^{ij}\partial^2_{ij}U| \leq \frac{\gamma}{2}\right\},$$

we find

$$\mathcal{L}_{\epsilon^2 A}U = \partial_t U + V^i\partial_i U + \epsilon^2 a^{ij}\partial^2_{ij}U \leq -\frac{\gamma}{2} \quad (\text{resp.} \geq \frac{\gamma}{2}), \quad \forall \epsilon \in (0, \epsilon_*).$$

Thus, U is a uniform Lyapunov function (resp. uniform anti-Lyapunov function) with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ with essential upper bound ρ_M, essential lower bound ρ_m and Lyapunov constant $\frac{\gamma}{2}$. □

Corollary 5.1. *Assume $V \in C^{1,1}_T(\mathcal{U} \times \mathbb{R})$. Let $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ be such that $A := \frac{1}{2}GG^\top$ is pointwise positive definite, and U, ρ_m, ρ_M, γ and ϵ_* be as in Theorem* 5.1. *The following statements hold for any family of periodic probability solutions $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ of* (1.3).

(1) *If $\mathcal{E}$ is a maximal periodic attractor, then for any $\rho_0, \rho_1, \rho_2 \in (\rho_m, \rho_M]$ with $\rho_0 < \rho_1 < \rho_2$ there holds*

$$\int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_2} \setminus \Omega^s_{\rho_1})\mathrm{d}s \leq Te^{-C\epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*),$$

where

$$C := \frac{\gamma(\rho_0 - \rho_m)}{2\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U} > 0.$$

(2) *If $\mathcal{E}$ is a maximal periodic repeller, then for any $\rho_1, \rho_2, \rho_3 \in (\rho_m, \rho_M]$ with $\rho_1 < \rho_2 < \rho_3$ there holds*

$$\int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_1} \setminus \Omega^s_{\rho_m})\mathrm{d}s \leq e^{-C\epsilon^{-2}} \int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_m})\mathrm{d}s, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*),$$

where

$$C := \frac{\gamma(\rho_3 - \rho_2)}{2\max_{\overline{\Omega}_{\rho_3} \setminus \Omega_{\rho_2}} a^{ij}\partial_i U \partial_j U}.$$

Proof. (1) We apply Lemma 4.2 (1) to find

$$\int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_2} \setminus \Omega^s_{\rho_1})\mathrm{d}s \leq Te^{-\frac{\gamma}{2}\int_{\rho_m}^{\rho_1} \frac{\mathrm{d}s}{\epsilon^2 H(s)}} \leq Te^{-\frac{\gamma}{2}\int_{\rho_m}^{\rho_0} \frac{\mathrm{d}s}{\epsilon^2 H(s)}},$$

$\forall t \in \mathbb{R}$ and $\epsilon \in (0, \epsilon_*)$, where

$$H(\rho) := \sup_{\partial\Omega_\rho} a^{ij}\partial_i U \partial_j U \quad \text{for } \rho \in (\rho_m, \rho_M).$$

Setting

$$C := \frac{\gamma(\rho_0 - \rho_m)}{2\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U\partial_j U} \leq \frac{\gamma}{2}\int_{\rho_m}^{\rho_0} \frac{\mathrm{d}s}{H(s)},$$

we arrive at the result.

(2) Applying Lemma 4.2 (2), we find

$$\int_t^{t+T} \mu_s^\epsilon(\Omega_{\rho_1}^s \setminus \Omega_{\rho_m}^s)\mathrm{d}s \leq e^{-\frac{\gamma}{2}\int_{\rho_1}^{\rho_3}\frac{\mathrm{d}s}{\epsilon^2 H(s)}} \int_t^{t+T} \mu_s^\epsilon(\Omega_{\rho_3}^s \setminus \Omega_{\rho_m}^s)\mathrm{d}s,$$

$\forall t \in \mathbb{R}$ and $\epsilon \in (0, \epsilon_*)$. Setting

$$C := \frac{\gamma(\rho_3 - \rho_2)}{2\max_{(\overline{\Omega}_{\rho_3}\setminus\Omega_{\rho_2})} a^{ij}\partial_i U\partial_j U} \leq \frac{\gamma}{2}\int_{\rho_2}^{\rho_3} \frac{\mathrm{d}s}{H(s)},$$

we derive the result. $\square$

Corollary 5.2. *Assume $V \in C_T^{1,1}(\mathcal{U}\times\mathbb{R})$. Let $G \in C_T(\mathbb{R}, W_{loc}^{1,p}(\mathcal{U}))$ be such that $A := \frac{1}{2}GG^\top$ is pointwise positive definite. If $\{\mu^\epsilon = (\mu_t^\epsilon)_{t\in\mathbb{R}}\}_{\epsilon>0}$ is a family of periodic probability solutions of* (1.3)*, then for any periodic Borel set $\mathcal{W} \subset\subset B(\mathcal{E})\setminus\mathcal{E}$, there are $C>0$ and $0<\epsilon_*\ll 1$ such that*

$$\int_t^{t+T} \mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s \leq Te^{-C\epsilon^{-2}}, \quad \forall t\in\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_*).$$

Proof. Let $\mathcal{W}$ be as in the statement. There exist open, connected and periodic sets $\mathcal{V}^1$ and $\mathcal{V}^2$ in $B(\mathcal{E})$ such that $\mathcal{W}\subset\mathcal{V}^2\setminus\mathcal{V}^1$. By Lemma 5.2, we may assume, without loss of generality, that $\mathcal{V}^2$ is positively invariant.

Set $A := \frac{1}{2}GG^\top$. Theorem 5.1 ensures that $\{\mathcal{L}_{\epsilon^2 A}\}_{\epsilon>0}$ admits a uniform Lyapunov function (resp. uniform anti-Lyapunov function) U in a neighbourhood of $\mathcal{V}^2$ with essential upper bound $\rho_M>0$ and essential lower bound $\rho_m>0$. Moreover, $\Omega_{\rho_m}\subset\subset\mathcal{V}^1$. Clearly, there are $\rho_1,\rho_2\in(\rho_m,\rho_M)$ such that $\Omega_{\rho_1}\subset\subset\mathcal{V}^1\subset\subset\mathcal{V}^2\subset\subset\Omega_{\rho_2}$. Hence, $\mathcal{W}\subset\Omega_{\rho_2}\setminus\Omega_{\rho_1}$.

Corollary 5.1 yields the existence of positive constants C and ϵ_* such that

$$\int_t^{t+T}\mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s \leq \int_t^{t+T}\mu_s^\epsilon(\Omega_{\rho_2}^s\setminus\Omega_{\rho_1}^s)\mathrm{d}s \leq Te^{-C\epsilon^{-2}}, \quad \forall t\in\mathbb{R} \text{ and } \epsilon\in(0,\epsilon_*).$$

This completes the proof. $\square$

5.2. **Concentration of periodic probability solutions.** We prove Theorem A and Corollary A in this subsection.

We recall the convergence under the weak*-topology for a family of Borel measures on $\mathcal{U}\times\mathbb{R}$.

Definition 5.1. *A sequence of Borel measures* $\{\mu^n : n\in\mathbb{N}\}$ *on* $\mathcal{U}\times\mathbb{R}$ *is said to converge to some Borel measure* μ *on* $\mathcal{U}\times\mathbb{R}$ *under the* weak*-topology *as* $n\to\infty$ *if*

$$\lim_{n\to\infty}\iint_{\mathcal{U}\times\mathbb{R}}\phi\mathrm{d}\mu^n=\iint_{\mathcal{U}\times\mathbb{R}}\phi\mathrm{d}\mu,\quad\forall\phi\in C_0(\mathcal{U}\times\mathbb{R}).$$

Remark 5.1. We emphasize that a limit measure μ of a family of periodic probability solutions $\{\mu_\epsilon\}_{\epsilon>0}$ of (1.3) (see Definition 1.2) is not only a limit point of $\{\mu_\epsilon\}_{\epsilon>0}$ under the weak*-topology, but also satisfies the condition $\mu(\mathcal{U}\times[0,T])=T$.

We turn to the proof of Theorem A.

Proof of Theorem A. We only prove the results when $\mathcal{E}$ is a maximal periodic attractor; the case when $\mathcal{E}$ is a local periodic repeller can be treated in the same way.

(1) Fix a periodic Borel set $\mathcal{W}\subset\subset B(\mathcal{E})\setminus\mathcal{E}$. There exist open, connected and periodic sets $\mathcal{V}^1$ and $\mathcal{V}^2$ satisfying $\mathcal{E}\subset\subset\mathcal{V}^1\subset\subset\mathcal{V}^2\subset B(\mathcal{E})$ and $\mathcal{W}\subset\subset\mathcal{V}^2\setminus\mathcal{V}^1$. Lemma 5.2 yields the existence of an open, connected, periodic and positively invariant set $\tilde{\mathcal{V}}$ such that $\mathcal{V}^2\subset\subset\tilde{\mathcal{V}}\subset\subset B(\mathcal{E})$.

Applying Theorem 5.1, we find a uniform Lyapunov function U with respect to $\{\mathcal{L}_{\epsilon^2A}\}_{\epsilon>0}$ in a neighbourhood of $\tilde{\mathcal{V}}$ with essential upper bound $\rho_M>0$, essential lower bound $\rho_m>0$ and Lyapunov constant $\gamma>0$. Moreover, $\Omega_{\rho_m}\subset\subset\mathcal{V}^1$. For $\rho_0,\rho_1\in(\rho_m,\rho_M)$ with $\rho_0<\rho_1$ such that $\Omega_{\rho_1}\subset\mathcal{V}^1$, Corollary 5.1 (1) yields the existence of some $\epsilon_*>0$ such that

$$\int_t^{t+T}\mu_s^\epsilon(\Omega_{\rho_M}^s\setminus\Omega_{\rho_1}^s)\mathrm{d}s\le Te^{-C\epsilon^{-2}},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*),$$

where

$$C:=\frac{\gamma}{2\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU}>0.$$

Due to $\mathcal{W}\subset\mathcal{V}^2\setminus\mathcal{V}^1$, $\Omega_{\rho_1}\subset\mathcal{V}^1$ and $\mathcal{V}^2\subset\Omega_{\rho_M}$, we arrive at

$$\int_t^{t+T}\mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s\le\int_t^{t+T}\mu_s^\epsilon(\Omega_{\rho_M}^s\setminus\Omega_{\rho_1}^s)\mathrm{d}s\le Te^{-C\epsilon^{-2}},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*).$$

(2) Let μ be a limit measure of $\{\mu^\epsilon\}_{\epsilon>0}$. It follows from (1) and the Portmanteau theorem that $\mu(B(\mathcal{E})\setminus\mathcal{E})=0$. It remains to show μ is a periodically invariant measure of (1.1). This is finished within two steps. We may assume, without loss of generality, that $\mu^\epsilon\to\mu$ under the weak*-topology as $\epsilon\to 0$.

Step 1. We prove that μ admits periodic and continuous t-sections $(\mu_t)_{t\in\mathbb{R}}$.

According to Theorem 2.1, μ^ϵ admits a density $u^\epsilon\in\mathcal{H}^{1,p}_{loc,T}(\mathcal{U}\times\mathbb{R})$. Then, Lemma 2.1 and Lemma 2.2 ensures for each $\epsilon>0$,

$$\int_{\mathcal{U}}\phi \mathrm{d}\mu_t^\epsilon=\int_{\mathcal{U}}\phi \mathrm{d}\mu_s^\epsilon+\int_s^t\int_{\mathcal{U}}\mathcal{L}_{\epsilon^2A}\phi \mathrm{d}\mu_\tau^\epsilon \mathrm{d}\tau,\quad \forall\phi\in C_c^2(\mathcal{U}) \text{ and } t>s, \tag{5.9}$$

where $\mathrm{d}\mu_t^\epsilon:=u(x,t)\mathrm{d}x$ for $t\in\mathbb{R}$.

Fix $\phi\in C_c^2(\mathcal{U})$. For each $\epsilon>0$, we define

$$f_\phi^\epsilon(t):=\int_{\mathcal{U}}\phi \mathrm{d}\mu_t^\epsilon,\quad t\in\mathbb{R}.$$

Obviously, f_ϕ^ϵ is T-periodic and satisfies $|f_\phi^\epsilon|_\infty\le|\phi|_\infty$ for all $\epsilon>0$. It follows from (5.9) that for each $\epsilon\in(0,1)$,

$$\begin{aligned}|f_\phi^\epsilon(t)-f_\phi^\epsilon(s)|&=\Big|\int_s^t\int_{\mathcal{U}}\mathcal{L}_{\epsilon^2A}\phi \mathrm{d}\mu_\tau^\epsilon \mathrm{d}\tau\Big|\le\max_{\mathcal{U}\times\mathbb{R}}|\mathcal{L}_{\epsilon^2A}\phi|\times(t-s)\\&\le\Big(\max_{\mathrm{supp}(\phi)\times\mathbb{R}}|a^{ij}\partial_{ij}^2\phi|+\max_{\mathrm{supp}(\phi)\times\mathbb{R}}|V^i\partial_i\phi|\Big)\times(t-s),\quad\forall s<t.\end{aligned}$$

Thus, the family $\{f_\phi^\epsilon\}_{\epsilon\in(0,1)}$ is uniformly bounded and equicontinuous. Applying the Arzelà-Ascoli theorem, we find a subsequence $\{f_\phi^{\epsilon_j}\}_{j\in\mathbb{N}}$, where $\lim_{j\to\infty}\epsilon_j=0$, that uniformly converges to some $f_\phi\in C_T(\mathbb{R})$. Obviously, $|f_\phi|_\infty\le|\phi|_\infty$.

For each $\eta\in C_c(\mathbb{R})$, the dominated convergence theorem yields that

$$\lim_{j\to\infty}\int_{\mathbb{R}}\eta\int_{\mathcal{U}}\phi \mathrm{d}\mu_t^{\epsilon_j}\mathrm{d}t=\lim_{j\to\infty}\int_{\mathbb{R}}\eta f_\phi^{\epsilon_j}\mathrm{d}t=\int_{\mathbb{R}}\eta f_\phi \mathrm{d}t.$$

Since μ^ϵ converges to μ under the weak*-topology as $\epsilon\to0$, there holds

$$\lim_{j\to\infty}\int_{\mathbb{R}}\eta\int_{\mathcal{U}}\phi \mathrm{d}\mu_t^{\epsilon_j}\mathrm{d}t=\iint_{\mathcal{U}\times\mathbb{R}}\eta\phi \mathrm{d}\mu,\quad\forall\eta\in C_c(\mathbb{R}).$$

Hence,

$$\int_{\mathbb{R}}\eta f_\phi \mathrm{d}t=\iint_{\mathcal{U}\times\mathbb{R}}\eta\phi \mathrm{d}\mu,\quad\forall\eta\in C_c(\mathbb{R}). \tag{5.10}$$

Note that f_ϕ must be the unique periodic continuous function on $\mathbb{R}$ satisfying (5.10). Following an argument almost the same as that in the proof of [25, Lemma 4.2], we find a family of σ-finite Borel measures $\{\mu_t\}_{t\in\mathbb{R}}$ such that

$$f_\phi(t) = \int_{\mathcal{U}} \phi \mathrm{d}\mu_t, \quad \forall \phi \in C_c^2(\mathcal{U}) \text{ and } t \in \mathbb{R}.$$

It follows from (5.10) that

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \eta\phi \mathrm{d}\mu_t \mathrm{d}t = \iint_{\mathcal{U}\times\mathbb{R}} \eta\phi \mathrm{d}\mu, \quad \forall \eta \in C_c(\mathbb{R}) \text{ and } \phi \in C_c^2(\mathcal{U}). \tag{5.11}$$

Since the set

$$\Big\{ \sum_{k=1}^n c_k\eta_k\phi_k : n \in \mathbb{N}, \{c_k\}_{k=1}^n \subset \mathbb{R}, \{\eta_k\}_{k=1}^n \subset C_c(\mathbb{R}) \text{ and } \{\phi_k\}_{k=1}^n \subset C_c^2(\mathcal{U}) \Big\}$$

is dense in $C_0(\mathcal{U}\times\mathbb{R})$, (5.11) ensures

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \phi(\cdot, t) \mathrm{d}\mu_t \mathrm{d}t = \iint_{\mathcal{U}\times\mathbb{R}} \phi \mathrm{d}\mu, \quad \forall \phi \in C_0(\mathcal{U}\times\mathbb{R}),$$

that is, $\{\mu_t\}_{t\in\mathbb{R}}$ are t-sections of μ, or $\mu = (\mu_t)_{t\in\mathbb{R}}$.

Moreover, as f_ϕ is T-periodic and continuous for each $\phi \in C_c^2(\mathcal{U})$, the periodicity and continuity of $(\mu_t)_{t\in\mathbb{R}}$ follows.

Step 2. We prove that

$$\int_t^{t+T} \int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot, s) \mathrm{d}\mu_s \mathrm{d}s = 0, \quad \forall t \in \mathbb{R} \text{ and } \phi \in C_{c,T}^{1,1}(\mathcal{U}\times\mathbb{R}). \tag{5.12}$$

Proposition 3.2 then ensures that $\mu = (\mu_t)_{t\in\mathbb{R}}$ is a periodically invariant measure of (1.1).

We claim that

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot, t) \mathrm{d}\mu_t \mathrm{d}t = 0, \quad \forall \phi \in C_0^{2,1}(\mathcal{U}\times\mathbb{R}). \tag{5.13}$$

Fix $\phi \in C_0^{2,1}(\mathcal{U}\times\mathbb{R})$. As $\mu^\epsilon = (\mu_t^\epsilon)_{t\in\mathbb{R}}$ is a periodic probability solution of (1.3), there holds

$$\int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{L}_{\epsilon^2 A}\phi(\cdot, t) \mathrm{d}\mu_t^\epsilon \mathrm{d}t = 0. \tag{5.14}$$

Since μ^ϵ converges to μ under the weak*-topology as $\epsilon \to 0$ and $\mathcal{L}_0\phi \in C_0(\mathcal{U}\times\mathbb{R})$, we find

$$\Big| \int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot, t) \mathrm{d}\mu_t^\epsilon \mathrm{d}t - \int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{L}_0\phi(\cdot, t) \mathrm{d}\mu_t \mathrm{d}t \Big| \to 0 \quad \text{as} \quad \epsilon \to 0.$$

As

$$\int_{\mathbb{R}}\int_{\mathcal{U}}|\mathcal{L}_{\epsilon^2 A}\phi(\cdot,t)-\mathcal{L}_0\phi(\cdot,t)|\,\mathrm{d}\mu_t^\epsilon\mathrm{d}t \le |\mathcal{L}_{\epsilon^2 A}\phi-\mathcal{L}_0\phi|_\infty\times\mu^\epsilon(\mathrm{supp}(\phi))$$
$$\le \epsilon^2\max_{\mathrm{supp}(\phi)}|a^{ij}\partial_{ij}^2\phi|\times\mu^\epsilon(\mathrm{supp}(\phi))\to 0 \quad \text{as} \quad \epsilon\to 0,$$

we derive

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\int_{\mathcal{U}}\mathcal{L}_{\epsilon^2 A}\phi(\cdot,t)\mathrm{d}\mu_t^\epsilon\mathrm{d}t-\int_{\mathbb{R}}\int_{\mathcal{U}}\mathcal{L}_0\phi(\cdot,t)\mathrm{d}\mu_t\mathrm{d}t\Big|\\
&\quad\le\int_{\mathbb{R}}\int_{\mathcal{U}}\Big|\mathcal{L}_{\epsilon^2 A}\phi(\cdot,t)-\mathcal{L}_0\phi(\cdot,t)\Big|\mathrm{d}\mu_t^\epsilon\mathrm{d}t+\Big|\int_{\mathbb{R}}\int_{\mathcal{U}}\mathcal{L}_0\phi(\cdot,t)\mathrm{d}\mu_t^\epsilon\mathrm{d}t\\
&\qquad-\int_{\mathbb{R}}\int_{\mathcal{U}}\mathcal{L}_0\phi(\cdot,t)\mathrm{d}\mu_t\mathrm{d}t\Big|\to 0\quad\text{as}\quad\epsilon\to 0.
\end{aligned}$$

Letting $\epsilon\to 0$ in (5.14), we conclude (5.13).

Since $(\mu_t)_{t\in\mathbb{R}}$ is continuous, Lemma 2.1 and Lemma 2.2 ensure that

$$\int_{\mathcal{U}}\phi(\cdot,t)\mathrm{d}\mu_t=\int_{\mathcal{U}}\phi(\cdot,s)\mathrm{d}\mu_s+\int_s^t\int_{\mathcal{U}}\mathcal{L}_0\phi\mathrm{d}\mu_\tau\mathrm{d}\tau,$$

$\forall\phi\in C_c^{2,1}(\mathcal{U}\times\mathbb{R})$ and $t>s$. It follows from the T-periodicity of $(\mu_t)_{t\in\mathbb{R}}$ that

$$\int_t^{t+T}\int_{\mathcal{U}}\mathcal{L}_0\phi\mathrm{d}\mu_\tau\mathrm{d}\tau=0,\quad\forall t\in\mathbb{R}\text{ and }\phi\in C_{c,T}^{2,1}(\mathcal{U}\times\mathbb{R}).$$

Since $C_{c,T}^{2,1}(\mathcal{U}\times\mathbb{R})$ is dense in $C_{c,T}^{1,1}(\mathcal{U}\times\mathbb{R})$, (5.12) follows. This completes the proof. □

Proof of Corollary A. Let U be the unbounded uniform Lyapunov function and ρ_m be its essential lower bound. We apply [25, Theorem A] and [26, Theorem A] to find an $\epsilon_*>0$ such that (1.3) admits a unique periodic probability solution $\mu^\epsilon=(\mu_t^\epsilon)_{t\in\mathbb{R}}$ for each $\epsilon\in(0,\epsilon_*)$. Theorem 4.2 (1) gives

$$\int_t^{t+T}\mu_s^\epsilon(\mathcal{U}\setminus\Omega_{\rho_m}^s)\mathrm{d}s\le Te^{-\gamma\int_{\rho_m}^\infty\frac{1}{\epsilon^2 H(s)}\mathrm{d}s},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*),\tag{5.15}$$

where

$$H(\rho):=\max_{\partial\Omega_\rho}a^{ij}\partial_i U\partial_j U\quad\text{for }\rho\in(\rho_m,\rho_M).$$

Since

$$\sup_{\epsilon\in(0,\epsilon_*)}\mu^\epsilon(K)<\infty$$

for any compact set $K\subset\mathcal{U}\times\mathbb{R}$, we apply [12, Corollary A2.6.V.] to conclude the existence of a subsequence of $\{\mu^\epsilon\}_{\epsilon\in(0,\epsilon_*)}$, denoted by $\{\mu^{\epsilon_j}\}_{j\in\mathbb{N}}$, that converges to some σ-finite Borel measure μ on $\mathcal{U}\times\mathbb{R}$ under the weak*-topology. Here, $\lim_{j\to\infty}\epsilon_j=0$. Following the arguments as in the proof of Theorem A, we conclude that μ admits periodic and continuous t-sections $(\mu_t)_{t\in\mathbb{R}}$ and satisfies (5.12). By Proposition 3.2, μ is a periodically invariant measure of (1.1) if we can show $\mu(\mathcal{U}\times[0,T])=T$.

The continuity of $(\mu_t)_{t\in\mathbb{R}}$ and the Portmanteau theorem lead to

$$\int_t^{t+T}\mu_s(\mathcal{U})\mathrm{d}s=\mu(\mathcal{U}\times(t,t+T))\le\liminf_{j\to\infty}\mu^{\epsilon_j}(\mathcal{U}\times(t,t+T))=T,\quad\forall t\in\mathbb{R}.$$

As

$$\lim_{\epsilon\to0}\gamma\int_{\rho_m}^{\infty}\frac{1}{\epsilon^2H(s)}\mathrm{d}s=\infty,$$

we find from (5.15) that for any $\delta>0$, there exists $\epsilon_{**}>0$ such that

$$\int_t^{t+T}\mu_s^{\epsilon}(\mathcal{U}\setminus\Omega_{\rho_m}^s)\mathrm{d}s<\delta,\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_{**}).$$

This together with the Portmanteau theorem yields

$$\int_t^{t+T}\mu_s(\overline{\Omega}_{\rho_m}^s)\mathrm{d}s\ge\limsup_{j\to\infty}\int_t^{t+T}\mu_s^{\epsilon_j}(\overline{\Omega}_{\rho_m}^s)\mathrm{d}s>T-\delta,\quad\forall t\in\mathbb{R}.$$

It follows from the arbitrariness of $\delta>0$ that $\mu(\mathcal{U}\times[t,t+T])=T$ for all $t\in\mathbb{R}$. This completes the proof. $\square$

6. NOISY STABILIZATION/DE-STABILIZATION

This section is devoted to the proof of Theorem B and Theorem C. For $G\in C_T(\mathbb{R},W^{1,p}_{loc}(\mathcal{U}))$ with $GG^\top$ being pointwise positive definite, we define

$$\lambda_B:=\inf_{(x,t)\in B}\inf_{\xi\in\mathbb{R}^d\setminus\{0\}}\frac{a^{ij}(x,t)\xi^i\xi^j}{|\xi|^2},\quad\Lambda_B:=\sup_{(x,t)\in B}\sup_{\xi\in\mathbb{R}^d\setminus\{0\}}\frac{a^{ij}(x,t)\xi^i\xi^j}{|\xi|^2},$$

$\forall B\subset\subset\mathcal{U}\times\mathbb{R}$, where $(a^{ij})=\frac12GG^\top$.

Proof of Theorem B. Let $\mathcal{W}\subset\subset(\mathcal{U}\times\mathbb{R})\setminus\mathcal{E}$ be a periodic Borel set. Arguing as in the proof of Theorem A, we could apply Theorem 5.1 to $\mathcal{E}$ to find a Lyapunov function U of (1.1) in some neighbourhood of $\mathcal{E}$ with essential upper bound $\rho_M>0$, essential lower bound $\rho_m>0$ and Lyapunov constant $\gamma>0$. Moreover, $\mathcal{E}\subset\subset\Omega_{\rho_m}$ and $\mathcal{W}\subset\subset(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_m}$, where $\Omega_\rho:=\{(x,t):U(x,t)<\rho\}$ for $\rho\in(0,\rho_M]$. Let ρ_i, $i=0,1,2,3$ be such that $\rho_m<\rho_0<\rho_1<\rho_2<\rho_3<\rho_M$ and $\mathcal{W}\subset(\mathcal{U}\times\mathbb{R})\setminus\Omega_{\rho_2}$.

The proof is finished within two steps. In Step 1, we establish estimates for $\{\mu^\epsilon\}_{\epsilon>0}$ in $\mathcal{W}$ for a general $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ with $A := \frac{1}{2}GG^\top$ being pointwise positive definite, where $\{\mu^\epsilon\}_{\epsilon>0}$ is a family of periodic probability solutions of (1.3). In Step 2, a special noise matrix $G_* \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ is designed according to such estimates to meet the requirements of the theorem.

As (1.1) is assumed to be dissipative, the global periodic attractor $\mathcal{A}$ exists.

Step 1. Let $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ be such that $A := \frac{1}{2}GG^\top$ is pointwise positive definite and $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ be a family of periodic probability solutions of (1.3). Let $\tilde{\mathcal{V}} \subset \mathcal{U} \times \mathbb{R}$ be an open, connected and periodic neighbourhood of $\mathcal{A}$, and satisfy $\Omega_{\rho_M} \subset\subset \tilde{\mathcal{V}}$. Obviously,

$$\mathcal{W} \subset \left(\mathcal{W} \setminus \tilde{\mathcal{V}}\right) \bigcup \left(\tilde{\mathcal{V}} \setminus \Omega_{\rho_3}\right) \bigcup \left(\Omega_{\rho_3} \setminus \Omega_{\rho_2}\right).$$

We derive estimates for $\{\mu_\epsilon\}_{\epsilon>0}$ in the domains $\mathcal{W} \setminus \tilde{\mathcal{V}}$, $\tilde{\mathcal{V}} \setminus \Omega_{\rho_3}$ and $\Omega_{\rho_3} \setminus \Omega_{\rho_2}$.

Since $\mathcal{W} \setminus \tilde{\mathcal{V}} \subset\subset (\mathcal{U} \times \mathbb{R}) \setminus \mathcal{A}$, we apply Corollary 5.2 (with $\mathcal{E}$ and $\mathcal{W}$ replaced by $\mathcal{A}$ and $\mathcal{W} \setminus \tilde{\mathcal{V}}$ respectively) to find positive constants C_1 and ϵ_1 such that

$$\int_t^{t+T} \mu^\epsilon_s(\mathcal{W}_s \setminus \tilde{\mathcal{V}}_s)\mathrm{d}s \le Te^{-C_1\epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_1). \tag{6.1}$$

Corollary 5.1 ensures the existence of an $\epsilon_2 > 0$ such that

$$\int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_2})\mathrm{d}s \le Te^{-C_2\epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_2), \tag{6.2}$$

where

$$C_2 := \frac{\gamma(\rho_0 - \rho_m)}{2\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U}.$$

It remains to estimate $\int_t^{t+T} \mu^\epsilon_s(\tilde{\mathcal{V}}_s \setminus \Omega^s_{\rho_3})\mathrm{d}s$. This is done by a delicate analysis of $\{u^\epsilon\}_{\epsilon>0}$ in a neighbourhood of $\tilde{\mathcal{V}} \setminus \Omega_{\rho_3}$, where u^ϵ is the density of μ^ϵ for each $\epsilon > 0$.

Since $\tilde{\mathcal{V}} \setminus \Omega_{\rho_2}$ needs not be connected, we denote by $\{\mathcal{V}^k : k = 1, \dots, N\}$ the connected components of $\tilde{\mathcal{V}} \setminus \Omega_{\rho_2}$. As $\tilde{\mathcal{V}} \setminus \Omega_{\rho_2}$ is periodic, so is $\mathcal{V}^k$ for each $k \in \{1, \dots, N\}$.

We show that

$$\mathcal{V}^k \cap (\Omega_{\rho_3} \setminus \Omega_{\rho_2}) \neq \emptyset, \quad \forall k \in \{1, \dots, N\}. \tag{6.3}$$

Fix $k \in \{1, \dots, N\}$ and some point $(x,t) \in \mathcal{V}^k$. The connectedness of $\tilde{\mathcal{V}}$ yields the existence of a continuous curve $\Gamma : [0,1] \to \tilde{\mathcal{V}}$ connecting $(x,t) = \Gamma(0)$ and some point $\Gamma(1) \in \Omega_{\rho_2}$. Set

$$s_* := \inf\{s \in (0,1) : \Gamma(s) \in \Omega_{\rho_2}\}.$$

Clearly, $s_* \in (0,1)$, $\Gamma([0,s_*)) \subset \mathcal{V}^k$ and $\Gamma(s_*) \in \partial\Omega_{\rho_2}$. As $\Omega_{\rho_2} \subset\subset \Omega_{\rho_3}$ and Γ is continuous, there exists some $0 < \delta_* \ll 1$ such that

$$\Gamma(s) \in (\Omega_{\rho_3} \setminus \Omega_{\rho_2}) \cap \mathcal{V}^k, \quad \forall s \in (s_* - \delta_*, s_*).$$

This proves (6.3).

Let $\tilde{\mathcal{V}}^1 \subset \mathcal{U} \times \mathbb{R}$ be open, connected and periodic, and satisfy $\tilde{\mathcal{V}} \subset\subset \tilde{\mathcal{V}}^1$. Obviously, $\tilde{\mathcal{V}} \setminus \Omega_{\rho_2} \subset\subset \tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1}$. Set $\epsilon_* := \min\{\epsilon_1, \epsilon_2, 1\}$. For each $k \in \{1, \dots, N\}$ and $0 < \epsilon < \epsilon_*$, Harnack's inequality (see Proposition 2.5) applied to u^ϵ in $\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1}$ gives

$$\sup_{\mathcal{V}^k} u^\epsilon = \sup_{\bigcup_{t\in[0,T]}(\mathcal{V}^k_t \times \{t\})} u^\epsilon \le D_k^{M_\epsilon} \inf_{\bigcup_{t\in[2T,3T]}(\mathcal{V}^k_t \times \{t\})} u^\epsilon = D_k^{M_\epsilon} \inf_{\mathcal{V}^k} u^\epsilon, \tag{6.4}$$

where $D_k = D_k(\mathcal{V}^k, \tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1}, d, T)$ and $M_\epsilon = \epsilon^2 \Lambda_{(\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1})} + \epsilon^{-2} N_\epsilon \lambda^{-1}_{(\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1})}$. In which,

$$N_\epsilon = 1 + \sum_{i=1}^d \sup_{(\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1})} (\epsilon^4 |\partial_j a^{ij}|^2 + |V^i|^2).$$

It follows from $(\tilde{\mathcal{V}} \setminus \Omega_{\rho_2}) = \bigcup_{k=1}^N \mathcal{V}^k$ that

$$\begin{aligned}\int_t^{t+T} \mu_s^\epsilon(\tilde{\mathcal{V}}_s \setminus \Omega^s_{\rho_2}) \mathrm{d}s &= \sum_{k=1}^N \int_t^{t+T} \mu^\epsilon_s(\mathcal{V}^k_s) \mathrm{d}s \le \sum_{k=1}^N \sup_{\mathcal{V}^k} u^\epsilon \Big| \bigcup_{s\in[0,T]} (\mathcal{V}^k_s \times \{s\}) \Big| \\ &\le \sum_{k=1}^N D_k^{M_\epsilon} \times \inf_{\mathcal{V}^k} u^\epsilon \times \Big| \bigcup_{s\in[0,T]} (\mathcal{V}^k_s \times \{s\}) \Big|. \end{aligned} \tag{6.5}$$

Note that (6.2) gives

$$\begin{aligned}\inf_{\mathcal{V}^k} u^\epsilon &\le \inf_{\mathcal{V}^k \cap (\Omega_{\rho_3} \setminus \Omega_{\rho_2})} u^\epsilon \\ &\le \frac{1}{\Big| \bigcup_{s\in[0,T]} [(\mathcal{V}^k_s \cap (\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_2})) \times \{s\}] \Big|} \int_t^{t+T} \mu^\epsilon_s(\mathcal{V}^k_s \cap (\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_2})) \mathrm{d}s \\ &\le \frac{1}{\Big| \bigcup_{s\in[0,T]} [(\mathcal{V}^k_s \cap (\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_2})) \times \{s\}] \Big|} \int_t^{t+T} \mu^\epsilon_s(\Omega^s_{\rho_3} \setminus \Omega^s_{\rho_2}) \mathrm{d}s \end{aligned}$$

$$\leq \frac{Te^{-C_2\epsilon^{-2}}}{\left|\bigcup_{s\in[0,T]}[(\mathcal{V}_s^k\cap(\Omega_{\rho_3}^s\setminus\Omega_{\rho_2}^s))\times\{s\}]\right|},\quad \forall\epsilon\in(0,\epsilon_*),$$

which, together with (6.5), yields

$$\int_t^{t+T}\mu_s^\epsilon(\tilde{\mathcal{V}}_s\setminus\Omega_{\rho_2}^s)\mathrm{d}s \leq Te^{-C_2\epsilon^{-2}}\sum_{k=1}^N\frac{D_k^{M_\epsilon}\left|\bigcup_{s\in[0,T]}(\mathcal{V}_s^k\times\{s\})\right|}{\left|\bigcup_{s\in[0,T]}[(\mathcal{V}_s^k\cap(\Omega_{\rho_3}^s\setminus\Omega_{\rho_2}^s))\times\{s\}]\right|}$$
$$= C_3D_*^{M_\epsilon}e^{-C_2\epsilon^{-2}},\quad \forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*), \tag{6.6}$$

where $D_*=\max\{D_k,k=1,\dots,N\}$ and

$$C_3=T\sum_{k=1}^N\frac{\left|\bigcup_{s\in[0,T]}(\mathcal{V}_s^k\times\{s\})\right|}{\left|\bigcup_{s\in[0,T]}[(\mathcal{V}_s^k\cap(\Omega_{\rho_3}^s\setminus\Omega_{\rho_2}^s))\times\{s\}]\right|}.$$

Step 2. We look for a noise matrix $G_*\in C_T(\mathbb{R};W_{loc}^{1,p}(\mathcal{U}))$ giving

$$\int_t^{t+T}\mu_s^\epsilon(\tilde{\mathcal{V}}_s\setminus\Omega_{\rho_3}^s)\mathrm{d}s\leq e^{C_4-C_5\epsilon^{-2}},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*) \tag{6.7}$$

for some positive constants C_4 and C_5, where $\{\mu^\epsilon:=(\mu_t^\epsilon)_{t\in\mathbb{R}}\}_{\epsilon>0}$ is a family of periodic probability solutions of (1.3) with $(a^{ij}):=\frac{1}{2}G_*G_*^\top$. Consequently, the estimate (6.7), together with (6.1) and (6.2), yields the desired estimate

$$\int_t^{t+T}\mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s\leq\tilde{C}_1e^{-\tilde{C}_2\epsilon^2},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*),$$

where $\tilde{C}_1:=\max\{T,e^{C_4}\}$ and $\tilde{C}_2:=\min\{C_1,C_2,C_5\}$.

To find such a G_*, we take the logarithm on both sides of (6.6) to deduce

$$\ln\int_t^{t+T}\mu_s^\epsilon(\tilde{\mathcal{V}}_s\setminus\Omega_{\rho_2}^s)\mathrm{d}s\leq\ln C_3+M_\epsilon\ln D_*-C_2\epsilon^{-2}$$
$$=\ln C_3+\ln D_*\left(\epsilon^2\Lambda_{(\tilde{\mathcal{V}}^1\setminus\Omega_{\rho_1})}+\epsilon^{-2}N_\epsilon\lambda^{-1}_{(\tilde{\mathcal{V}}^1\setminus\Omega_{\rho_1})}\right)-\frac{\gamma(\rho_0-\rho_m)\epsilon^{-2}}{2\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU}$$
$$\leq C_4-C_5\epsilon^{-2},\quad\forall t\in\mathbb{R}\text{ and }\epsilon\in(0,\epsilon_*),$$

where

$$C_4=|\ln C_3|+|\ln D_*|\Lambda_{(\tilde{\mathcal{V}}^1\setminus\Omega_{\rho_1})}$$

and

$$C_5=-|\ln D_*|N_*\lambda^{-1}_{(\tilde{\mathcal{V}}^1\setminus\Omega_{\rho_1})}+\frac{\gamma(\rho_0-\rho_m)}{2\max_{\overline{\Omega}_{\rho_0}}a^{ij}\partial_iU\partial_jU}.$$

In which,

$$N_* = 1 + \sum_{i=1}^{d} \sup_{(\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1})} (|\partial_j a^{ij}|^2 + |V^i|^2).$$

Hence, $C_5 > 0$ if and only if

$$\max_{\overline{\Omega}_{\rho_0}} a^{ij}\partial_i U \partial_j U < \frac{\gamma(\rho_0 - \rho_m)}{2|\ln D_*| N_*} \lambda_{(\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1})}. \tag{6.8}$$

As $\overline{\Omega}_{\rho_0} \cap (\tilde{\mathcal{V}}^1 \setminus \Omega_{\rho_1}) = \emptyset$, there must exist a $G_* \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ such that (6.8) holds with $(a^{ij}) := \frac{1}{2} G_* G_*^\top$ being pointwise positive definite. It then follows from (6.6) that

$$\int_t^{t+T} \mu_s^\epsilon(\tilde{\mathcal{V}}_s \setminus \Omega^s_{\rho_2}) \mathrm{d}s \le e^{C_4 - C_5 \epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*).$$

This together with $\tilde{V} \setminus \Omega_{\rho_3} \subset \tilde{V} \setminus \Omega_{\rho_2}$ leads to (6.7). □

Proof of Theorem C. Let $\mathcal{W} \subset\subset B(\mathcal{E})$ be a periodic Borel set. Arguing as in the proof of Theorem A, we could apply Theorem 5.1 to find an anti-Lyapunov function U of (1.1) in some neighbourhood of $\mathcal{E}$ with essential upper bound $\rho_M > 0$, essential lower bound $\rho_m > 0$ and anti-Lyapunov constant $\gamma > 0$. Moreover, $\mathcal{E} \subset\subset \Omega_{\rho_m}$ and $\mathcal{W} \subset\subset \Omega_{\rho_M}$, where $\Omega_\rho := \{(x,t) : U(x,t) < \rho\}$ for $\rho \in (0, \rho_M]$. Let ρ_i, $i = 0,1,2,3$ be such that

$$\rho_m < \rho_0 < \rho_1 < \rho_2 < \rho_3 < \rho_M \quad \text{ and } \quad \mathcal{W} \subset \Omega_{\rho_0}.$$

We break the rest of the proof into two steps. In Step 1, we establish estimates for $\{\mu^\epsilon\}_{\epsilon>0}$ in $\mathcal{W}$ for a general noise matrix $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ with $A := \frac{1}{2} GG^\top$ being pointwise positive definite, where $\{\mu^\epsilon\}_{\epsilon>0}$ is a family of periodic probability solutions of (1.3). Step 2 is devoted to the construction of a special noise matrix $G_* \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ based on estimates established in Step 1 to finish the proof.

Step 1. Let $G \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ be such that $A := \frac{1}{2} GG^\top$ is pointwise positive definite, and $\{\mu^\epsilon = (\mu^\epsilon_t)_{t\in\mathbb{R}}\}_{\epsilon>0}$ be a family of periodic probability solutions of (1.3). Denote by $\{u^\epsilon\}_{\epsilon>0}$ the densities of $\{\mu^\epsilon\}_{\epsilon>0}$.

For $\epsilon > 0$, we apply Harnack's inequality (see Proposition 2.5) to u^ϵ in Ω_{ρ_1} to find

$$\sup_{\Omega_{\rho_0}} u^\epsilon = \sup_{\bigcup_{t\in[0,T]}(\Omega^t_{\rho_0}\times\{t\})} u^\epsilon \le C_1^{M_\epsilon} \inf_{\bigcup_{t\in[2T,3T]}(\Omega^t_{\rho_0}\times\{t\})} u^\epsilon = C_1^{M_\epsilon} \inf_{\Omega_{\rho_0}} u^\epsilon,$$

where $C_1 = C_1(\Omega_{\rho_0}, \Omega_{\rho_1}, d, T)$ and $M_\epsilon = \epsilon^2 \Lambda_{\Omega_{\rho_1}} + \epsilon^{-2} N_\epsilon \lambda^{-1}_{\Omega_{\rho_1}}$. In which,

$$N_\epsilon = 1 + \sum_{i=1}^{d} \sup_{\Omega_{\rho_1}} (\epsilon^4 |\partial_j a^{ij}|^2 + |V^i|^2).$$

It follows that

$$\begin{aligned} \int_t^{t+T} \mu_s^\epsilon(\Omega_{\rho_0}) \mathrm{d}s &\le \sup_{\Omega_{\rho_0}} u^\epsilon \times \Big| \bigcup_{s\in[0,T]} (\Omega^s_{\rho_0} \times \{s\}) \Big| \\ &\le C_1^{M_\epsilon} \times \inf_{\Omega_{\rho_0}} u^\epsilon \times \Big| \bigcup_{s\in[0,T]} (\Omega^s_{\rho_0} \times \{s\}) \Big|. \end{aligned} \tag{6.9}$$

Note that Corollary 5.1 ensures the existence of some $\epsilon_* > 0$ such that

$$\int_t^{t+T} \mu_s^\epsilon(\Omega^s_{\rho_0} \setminus \Omega^s_{\rho_m}) \mathrm{d}s \le T e^{-C_2 \epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*), \tag{6.10}$$

where

$$C_2 = \frac{\gamma(\rho_3 - \rho_2)}{2 \max_{(\overline{\Omega}_{\rho_3} \setminus \Omega_{\rho_2})} a^{ij} \partial_i U \partial_j U}.$$

It follows that

$$\begin{aligned} \inf_{\Omega_{\rho_0}} u^\epsilon \le \inf_{\Omega_{\rho_0} \setminus \Omega_{\rho_m}} u^\epsilon &\le \frac{1}{\Big| \bigcup_{s\in[0,T]} [(\Omega^s_{\rho_0} \setminus \Omega^s_{\rho_m}) \times \{s\}] \Big|} \int_0^T \mu_s^\epsilon(\Omega^s_{\rho_0} \setminus \Omega^s_{\rho_m}) \mathrm{d}s \\ &\le \frac{T e^{-C_2 \epsilon^{-2}}}{\Big| \bigcup_{s\in[0,T]} [(\Omega^s_{\rho_0} \setminus \Omega^s_{\rho_m}) \times \{s\}] \Big|}, \quad \forall \epsilon \in (0, \epsilon_*). \end{aligned}$$

This together with (6.9) yields

$$\int_t^{t+T} \mu_s^\epsilon(\Omega_{\rho_0}) \mathrm{d}s \le C_3 C_1^{M_\epsilon} e^{-C_2 \epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*), \tag{6.11}$$

where

$$C_3 := \frac{T \Big| \bigcup_{s\in[0,T]} (\Omega^s_{\rho_0} \times \{s\}) \Big|}{\Big| \bigcup_{s\in[0,T]} [(\Omega^s_{\rho_0} \setminus \Omega^s_{\rho_m}) \times \{s\}] \Big|}.$$

Step 2. We show the existence of some $G_* \in C_T(\mathbb{R}; W^{1,p}_{loc}(\mathcal{U}))$ such that

$$\int_t^{t+T} \mu_s^\epsilon(\mathcal{W}_s)\mathrm{d}s \le (C_3 e^{C_4})e^{C_5\epsilon^{-2}}, \quad \forall t \in \mathbb{R} \text{ and } \epsilon \in (0, \epsilon_*), \tag{6.12}$$

where $\epsilon_* > 0$ is fixed, and $\{\mu^\epsilon := (\mu_t^\epsilon)_{t\in\mathbb{R}}\}_{\epsilon>0}$ is a family of periodic probability solutions of (1.3) with $(a^{ij}) := \frac{1}{2}G_*G_*^\top$.

Taking the logarithm on both sides of (6.11), we deduce

$$\begin{aligned}\ln \int_t^{t+T} \mu_s^\epsilon(\Omega_{\rho_0})\mathrm{d}s &\le \ln C_3 + M_\epsilon \ln C_1 - C_2\epsilon^{-2}\\ &= \ln C_3 + \ln C_1\left(\epsilon^2 \Lambda_{\Omega_{\rho_1}} + \epsilon^{-2} N_\epsilon \lambda^{-1}_{\Omega_{\rho_1}}\right) - C_2\epsilon^{-2} \le C_4 - C_5\epsilon^{-2},\end{aligned}$$

where $C_4 = |\ln C_3| + |\ln C_1|\Lambda_{\Omega_{\rho_1}}$ and

$$C_5 = -|\ln C_1| N_* \lambda^{-1}_{\Omega_{\rho_1}} + \frac{\gamma(\rho_3 - \rho_2)}{2\max_{(\overline{\Omega}_{\rho_3}\setminus\Omega_{\rho_2})} a^{ij}\partial_i U \partial_j U}.$$

In which,

$$N_* = 1 + \sum_{i=1}^d \sup_{\Omega_{\rho_1}} (|\partial_j a^{ij}|^2 + |V^i|^2).$$

Clearly, $C_5 > 0$ if and only if

$$\max_{(\overline{\Omega}_{\rho_3}\setminus\Omega_{\rho_2})} a^{ij}\partial_i U\partial_j U \le \frac{\gamma(\rho_3-\rho_2)}{2|\ln C_1| N_*}\lambda_{\Omega_{\rho_1}}. \tag{6.13}$$

Since $\overline{\Omega}_{\rho_1} \cap (\overline{\Omega}_{\rho_3} \setminus \Omega_{\rho_2}) = \emptyset$, there exists a $G_* \in C_T(\mathbb{R}, W^{1,p}_{loc}(\mathcal{U}))$ such that (6.13) holds, where $(a^{ij}) := \frac{1}{2}G_*G_*^\top$ is pointwise positive definite. It follows from $\mathcal{W} \subset \Omega_{\rho_0}$ and (6.11) that (6.12) holds. □

7. An example

In this section, we demonstrate the noisy stabilization/de-stabilization and applications of concentration estimates of periodic probability solutions by the following stochastic planar system:

$$\begin{cases} \mathrm{d}x_1 = \left\{x_1\left[b(t) - (x_1^2 + x_2^2)\right] - x_2\right\}\mathrm{d}t + \epsilon g^{11}(x_1,x_2,t)\mathrm{d}W_t^1 \\ \qquad\qquad + \epsilon g^{12}(x_1,x_2,t)\mathrm{d}W_t^2, \\ \mathrm{d}x_2 = \left\{x_2\left[b(t) - (x_1^2 + x_2^2)\right] + x_1\right\}\mathrm{d}t + \epsilon g^{21}(x_1,x_2,t)\mathrm{d}W_t^1 \\ \qquad\qquad + \epsilon g^{22}(x_1,x_2,t)\mathrm{d}W_t^2, \end{cases} \tag{7.1}$$

where $0 < \epsilon \ll 1$, $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous, positive and T-periodic function for some $T > 0$, $G := (g^{ij}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ belongs to $C_T(\mathbb{R}, W^{1,p}_{loc}(\mathbb{R}^2))$ for some $p > 4$, is such that $GG^\top$ is pointwise positive definite, and satisfies

$$|G(x_1, x_2, t)| \leq C\left(x_1^2 + x_2^2\right), \quad \forall (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}$$

for some constant $C > 0$, and W^1 and W^2 are independent one-dimensional Wiener processes.

It is not hard to construct an unbounded uniform Lyapunov function for the system (7.1), and therefore, (7.1) admits a unique periodic probability solution for each $0 < \epsilon \ll 1$ (see the proof Theorem 7.1). To study the concentration and limit behaviors of these periodic probability solutions as $\epsilon \to 0$, we need to explore the dynamics of the unperturbed system:

$$\begin{cases} \dot{x}_1 = x_1\left[b(t) - (x_1^2 + x_2^2)\right] - x_2, \\ \dot{x}_2 = x_2\left[b(t) - (x_1^2 + x_2^2)\right] + x_1. \end{cases} \tag{7.2}$$

It is more convenient to study (7.2) in polar coordinates ($x_1 = r\cos\theta$ and $x_2 = r\sin\theta$):

$$\begin{cases} \dot{r} = r\left[b(t) - r^2\right], \quad r \geq 0, \\ \dot{\theta} = 1 \mod 2\pi. \end{cases}$$

Lemma 7.1. *The following hold.*

(1) *The equation* (7.3a) *admits a unique positive T-periodic solution, denoted by $r_*(t)$, that attracts solutions of* (7.3a) *with positive initial data.*

(2) *The system* (7.3a)-(7.3b) *admits a unique local T-periodic repeller $\mathcal{R}^{polar} = \{0\} \times \mathbb{S}_{2\pi} \times \mathbb{R}$ with basin of expansion*

$$B(\mathcal{R}^{polar}) = \bigcup_{t\in\mathbb{R}} \left([0, r_*(t)) \times \mathbb{S}_{2\pi} \times \{t\}\right).$$

(3) *The system* (7.3a)-(7.3b) *admits a unique local T-periodic attractor*

$$\mathcal{J}^{polar} = \bigcup_{t\in\mathbb{R}} \left(\{r^*(t)\} \times \mathbb{S}_{2\pi} \times \{t\}\right)$$

with basin of attraction $B(\mathcal{J}^{polar}) = (0, \infty) \times \mathbb{S}_{2\pi} \times \mathbb{R}$. Moreover,

- *(the non-resonance case) if $\frac{T}{2\pi}$ is irrational, then $\mathcal{J}^{polar}$ consists of quasi-periodic solutions of* (7.3a)-(7.3b) *with the same frequencies; each of them, when identified as an orbit in $\mathcal{J}^{polar}_T$, is dense in $\mathcal{J}^{polar}_T$, where $\mathcal{J}^{polar}_T = \bigcup_{t\in\mathbb{S}_T} \left(\{r_*(t)\} \times \mathbb{S}_{2\pi}\right)$;*

– (*the resonance case*) *if* $\frac{T}{2\pi} = \frac{m}{n}$ *for some relatively prime positive integers* m *and* n, *then* $\mathcal{J}^{polar}$ *consists of* T_{sol}*-periodic solutions of* (7.3a)-(7.3b), *where* $T_{sol} = nT = 2m\pi$.

Proof. Note that the system (7.3a)-(7.3b) is decoupled. Obviously, $r \equiv 0$ is an unstable solution of (7.3a). This together with (7.3b) gives the unique local T-periodic repeller $\mathcal{R}^{polar}$. By the classical comparison principle and contraction arguments, it is not hard to show that (7.3a) admits a unique positive T-periodic solution $r_*(t)$ that attracts solutions of (7.3a) with positive initial data. This together with (7.3b) gives the unique local T-periodic attractor $\mathcal{J}^{polar}$. It is easy to determine the basin of expansion of $\mathcal{R}^{polar}$ and the basin of attraction of $\mathcal{J}^{polar}$.

It remains to explore the structure of $\mathcal{J}^{polar}$. Note that the dynamics of (7.3a)-(7.3b) on $\mathcal{J}^{polar}$ are equivalent to those of the following system:

$$\begin{cases} \dot{r} = 1 \mod T, \\ \dot{\theta} = 1 \mod 2\pi. \end{cases} \tag{7.4}$$

It is well-known that if $\frac{T}{2\pi}$ is an irrational number, then each orbit of (7.4) is quasi-periodic and dense in $\mathbb{S}_T \times \mathbb{S}_{2\pi}$; moreover, these quasi-periodic orbits have the same frequencies. If $\frac{T}{2\pi} = \frac{m}{n}$ is a rational number for some relative prime positive integers m and n, then each orbit of (7.4) is T_{sol}-periodic, where $T_{sol} = nT = 2m\pi$. This completes the proof. $\square$

The next result follows readily from Lemma 7.1.

Proposition 7.1. *Let* $r_*(t)$ *be the unique positive* T*-periodic solution of* (7.3a) *given in Lemma* 7.1. *Then, the following hold.*

(1) *The system* (7.2) *admits a unique local* T*-periodic repeller* $\mathcal{R} = \{(0,0)\} \times \mathbb{R}$ *with basin of expansion*

$$B(\mathcal{R}) = \left\{(x_1, x_2, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : x_1^2 + x_2^2 < r_*(t)^2\right\}.$$

(2) *The system* (7.2) *admits a unique local* T*-periodic attractor*

$$\mathcal{J} = \left\{(x_1, x_2, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : x_1^2 + x_2^2 = r_*(t)^2\right\}$$

with basin of attraction $B(\mathcal{J}) = \left(\mathbb{R}^2 \setminus \{0\}\right) \times \mathbb{R}$. *Moreover,*

– (*the non-resonance case*) *if* $\frac{T}{2\pi}$ *is irrational, then* $\mathcal{J}$ *consists of quasi-periodic solutions of* (7.2) *with the same frequencies; each of them, when identified with an orbit in* $\mathcal{J}_T$, *is dense in* $\mathcal{J}_T$, *where* $\mathcal{J}_T = \left\{(x_1, x_2, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}_T : x_1^2 + x_2^2 = r_*(t)^2\right\}$;

– *(the resonance case) if* $\frac{T}{2\pi} = \frac{m}{n}$ *for some relatively prime positive integers* m *and* n*, then* $\mathcal{J}$ *consists of* T_{sol}*-periodic solutions of* (7.2)*, where* $T_{sol} = nT = 2m\pi$*.*

Now, we turn to the stochastic system (7.1). Denote

$$V = (V^1, V^2)^\top = \begin{pmatrix} x_1 \left[b(t) - (x_1^2 + x_2^2)\right] - x_2 \\ x_2 \left[b(t) - (x_1^2 + x_2^2)\right] + x_1 \end{pmatrix}.$$

The Fokker-Planck equation associated to (7.1) reads

$$\partial_t u = \epsilon^2 \partial_{ij}^2 (a^{ij} u) - \partial_i (V^i u), \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{7.5}$$

where $A = (a^{ij}) := \frac{1}{2} G G^\top$ and $\partial_{ij}^2 = \partial_{x_i x_j}^2$ for $i, j = 1, 2$.

Theorem 7.1. *The following statements hold.*

(1) *For each* $0 < \epsilon \ll 1$*, the stochastic system* (7.5) *admits a unique* T*-periodic probability solution* $\mu^\epsilon = (\mu_t^\epsilon)_{t \in \mathbb{R}}$*. Moreover, the set of limit measures of* $\{\mu^\epsilon\}_{0<\epsilon\ll 1}$*, denoted by* $\mathcal{M}$*, is nonempty, and each element in* $\mathcal{M}$ *is a* T*-periodically invariant measure of* (7.2) *and is supported in* $\mathcal{J}$*.*

(2) *If, in addition,* $\frac{T}{2\pi}$ *is irrational, then* $\mathcal{M} = \{\mu\}$*, where* μ *is the unique* T*-periodically invariant measure of* (7.2) *on* $\mathbb{R}^2 \backslash \{(0,0)\}$ *and satisfies* $\mathrm{supp}(\mu) = \mathcal{J}$*. In particular,* $\mu^\epsilon \to \mu$ *under the weak*-topology as* $\epsilon \to 0$.

Proof. (1) For each $0 < \epsilon \ll 1$, denote

$$\mathcal{L}_{\epsilon^2 A} := \partial_t + \epsilon^2 a^{ij} \partial_{ij}^2 + V^i \partial_i.$$

We construct an unbounded uniform Lyapunov function with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{0<\epsilon\ll 1}$. Define

$$U(x,t) := \frac{|x|^2}{2}, \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

where $x = (x_1, x_2)^\top$ and $|x| := \sqrt{x_1^2 + x_2^2}$. Obviously, U is an unbounded C^2 compact function. We calculate

$$\mathcal{L}_{\epsilon^2 A} U = |x|^2 (b(t) - |x|^2) + \epsilon^2 (a^{11} + a^{22}), \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Since

$$|A| = \frac{1}{2} \left| G G^\top \right| \le \frac{1}{2} C^2 |x|^4,$$

it is not hard to find positive constants ϵ_0, C_1 and R_0 such that for each $\epsilon \in (0, \epsilon_0)$, there holds

$$\mathcal{L}_{\epsilon^2 A} U \leq -C_1 |x|^4, \quad \forall |x| \geq R_0 \text{ and } t \in \mathbb{R}.$$

That is, U is an unbounded uniform Lyapunov function with respect to $\{\mathcal{L}_{\epsilon^2 A}\}_{0<\epsilon\ll 1}$.

It follows from Corollary A that for each $0 < \epsilon \ll 1$, (7.5) admits a unique periodic probability solution $\mu^\epsilon = (\mu_t^\epsilon)_{t\in\mathbb{R}}$. Moreover, $\mathcal{M} \neq \emptyset$ and each element in $\mathcal{M}$ is a T-periodically invariant measure of (7.2) and is supported in the global T-periodic attractor of (7.2).

It remains to show that each $\mu \in \mathcal{M}$ satisfies $\mathrm{supp}(\mu) \subset \mathcal{J}$. Fix an arbitrary $\mu \in \mathcal{M}$ and let $\{\epsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ be such that $\epsilon_j \to 0$ as $j \to \infty$ and $\mu^{\epsilon_j} \to \mu$ under the weak*-topology as $j \to \infty$. Proposition 2.3 yields that $\mathrm{supp}(\mu) \subset \mathcal{R} \cup \mathcal{J}$. Thus, it suffices to prove $\mu(\mathcal{R}) = 0$.

Since $A = (a^{ij})$ is pointwise positive definite and $[B(\mathcal{R})]$ is bounded in $\mathbb{R}^2 \times \mathbb{S}_T$, there exists some $\lambda > 0$ such that

$$a^{ij}(x,t)\xi^i\xi^j \geq \lambda|\xi|^2, \quad \forall (x,t) \in B(\mathcal{R}) \text{ and } \xi \in \mathbb{R}^2.$$

Set

$$D_L := \left\{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : |x| < L\right\},$$

where

$$L := \min_{t\in\mathbb{R}} \min\left\{r^*(t), b(t)\right\}.$$

Obviously, $D_L \subset B(\mathcal{R})$. Direct calculations give

$$\mathcal{L}_{\epsilon^2 A} U(x,t) = |x|^2(b(t) - |x|^2) + \epsilon^2(a^{11} + a^{22})(x,t) \geq 2\epsilon^2\lambda, \quad \forall (x,t) \in D_L.$$

That is, U is an anti-Lyapunov function with respect to $\mathcal{L}_{\epsilon^2 A}$ in D_L with essential upper bound $\frac{1}{2}L^2$, essential lower bound 0 and anti-Lyapunov constant $2\epsilon^2\lambda$.

Recall that Ω_ρ is the ρ-sublevel set of U, and Ω_ρ^t is its t-section. Note that $\Omega_0 = \emptyset$. Hence, for fixed $\rho_1 \in (0, \frac{1}{2}L^2)$, we can apply Lemma 4.2 (2) to find that

$$\begin{aligned}\int_t^{t+T} \mu_s^\epsilon(\Omega_\rho^s)\mathrm{d}s &= \int_t^{t+T} \mu^\epsilon(\Omega_\rho^s \setminus \Omega_0^s)\mathrm{d}s \\ &\leq e^{-2\lambda\int_\rho^{\rho_1}\frac{1}{H(s)}\mathrm{d}s} \int_t^{t+T} \mu^\epsilon(\Omega_{\rho_1}^s \setminus \Omega_0^s)\mathrm{d}s \leq Te^{-2\lambda\int_\rho^{\rho_1}\frac{1}{H(s)}\mathrm{d}s},\end{aligned}$$

$\forall t \in \mathbb{R}$ and $\rho \in (0, \rho_1)$, where

$$H(\rho) := \sup_{\partial\Omega_\rho} a^{ij}\partial_i U \partial_j U.$$

It is easy to verify that

$$H(\rho) \leq \Lambda \sup_{\partial\Omega_\rho} |\nabla U|^2 \leq 2\Lambda\rho, \quad \forall \rho \in [0, \frac{1}{2}L^2],$$

where $\Lambda := \sup_{\overline{D}_L} |A|$. Hence,

$$\int_t^{t+T} \mu_s^\epsilon(\Omega_\rho^s)\mathrm{d}s \leq Te^{-\frac{\lambda}{\Lambda}\int_\rho^{\rho_1} \frac{1}{s}\mathrm{d}s}, \quad \forall t \in \mathbb{R}.$$

Since $\mu^{\epsilon_j} \to \mu$ under the weak*-topology as $j \to \infty$, the Portmanteau theorem yields

$$\mu\Big(\bigcup_{s\in(t,t+T)} (\Omega_\rho^s \times \{s\})\Big) \leq \liminf_{j\to\infty} \int_t^{t+T} \mu_s^{\epsilon_j}(\Omega_\rho^s)\mathrm{d}s \leq Te^{-\frac{\lambda}{\Lambda}\int_\rho^{\rho_1} \frac{1}{s}\mathrm{d}s}, \quad \forall t \in \mathbb{R}.$$

Letting $\rho \to 0$ in the above inequality, we conclude $\mu(\mathcal{R}) = 0$.

(2) If $\frac{T}{2\pi}$ is irrational, then the system (7.4) is uniquely ergodic. This leads to the unique ergodicity of the skew product semi-flow generated by (7.2) on $\mathbb{R}^2 \setminus \{(0,0)\}$. As any element in $\mathcal{M}$, considered as a T-periodically invariant measure of (7.2), can be identified with an invariant measure of the skew product semi-flow, we conclude that $\mathcal{M}$ is a singleton set, that is, $\mathcal{M} = \{\mu\}$, where μ is the unique T-periodically invariant measure of (7.2) on $\mathbb{R}^2 \setminus \{(0,0)\}$. The fact that $\mathrm{supp}(\mu) = \mathcal{J}$ and the convergence of μ^ϵ to μ under the weak*-topology as $\epsilon \to 0$ follow readily. □

Remark 7.1. Theorem 7.1 (1) asserts that the local T-periodic repeller $\mathcal{R}$ and the local T-periodic attractor $\mathcal{J}$ are respectively de-stabilized and stabilized by any noise described at the beginning of this section.

In the case that $\frac{T}{2\pi} = \frac{m}{n}$ is a rational number for some relatively prime positive integers m and n, Lemma 7.1 (3) says that the local T-periodic attractor $\mathcal{J}$ consists of T_{sol}-periodic solutions of (7.2). As a result, the system (7.2) admits many T-periodically invariant measures who are candidates for elements in the set of limit measures $\mathcal{M}$ given in Theorem 7.1 (1). It remains an interesting problem to investigate finer structures of $\mathcal{M}$.

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