1. Let \( r \in \mathbb{N} \), and let \((x_0, x_1, \ldots, x_r) \in \mathbb{R}^N\). Show that

\[
\{[x_0, x_1, \ldots, x_r]\} = \left\{ s_0x_0 + s_1x_1 + \cdots + s_rx_r : s_0, s_1, \ldots, s_r \geq 0, \sum_{j=0}^r s_j = 1 \right\}.
\]

**Solution:** We have

\[
\{[x_0, x_1, \ldots, x_r]\} = \{x_0 + t_1(x_1 - x_0) + \cdots + t_r(x_r - x_0) : (t_1, \ldots, t_r) \in \Sigma_r\}
\]

\[
= \left\{ x_0 + t_1(x_1 - x_0) + \cdots + t_r(x_r - x_0) : t_1, \ldots, t_r \geq 0, \sum_{j=1}^r t_j \leq 1 \right\}
\]

\[
= \left\{ \left(1 - \sum_{j=1}^r t_j \right) x_0 + t_1x_1 + \cdots + t_r x_r : t_1, \ldots, t_r \geq 0, \sum_{j=1}^r t_j \leq 1 \right\}
\]

\[
= \left\{ s_0x_0 + s_1x_1 + \cdots + s_rx_r : s_0, s_1, \ldots, s_r \geq 0, \sum_{j=0}^r s_j = 1 \right\},
\]

which proves the claim.

2. Prove or give a counterexample to the following generalization of the Alternating Series Test:

Let \((a_n)_{n=1}^\infty\) be a sequence of non-negative reals such that \(\lim_{n \to \infty} a_n = 0\).

Then \(\sum_{n=1}^\infty (-1)^{n-1}a_n\) converges.

*(Hint: Try \(a_n := \left| \frac{1}{n} - \frac{(-1)^n}{\sqrt{n}} \right|\)).*

**Solution:** For \(n \in \mathbb{N}\), let

\[
a_n := \left| \frac{1}{n} - \frac{(-1)^n}{\sqrt{n}} \right| \geq 0,
\]

so that \(\lim_{n \to \infty} a_n\). Since \(\sqrt{n} \leq n\) for \(n \in \mathbb{N}\), it is immediate that

\[
(-1)^{n-1}a_n = \frac{1}{n} - \frac{(-1)^n}{\sqrt{n}}
\]

for \(n \in \mathbb{N}\). Assume towards a contradiction that \(\sum_{n=1}^\infty (-1)^{n-1}a_n\) converges. Since the series \(\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}\) converges by the Alternating Series Test, this implies that

\[
\sum_{n=1}^\infty (-1)^{n-1}a_n + \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n}
\]

also converges, which is clearly false.
3. Let \((a_n)_{n=1}^{\infty}\) be a decreasing sequence of non-negative real numbers. Show that \(\sum_{n=1}^{\infty} a_n\) converges if and only if \(\sum_{n=1}^{\infty} 2^n a_{2^n}\) converges.

What can you conclude about the convergence of \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) for \(p \in \mathbb{R}\)?

**Solution:** For \(n \in \mathbb{N}\), let

\[ s_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad S_n := \sum_{k=1}^{n} 2^k a_{2^k} \]

Suppose that \(\sum_{n=1}^{\infty} a_n\) converges to \(s\). For \(n \in \mathbb{N}\), we see that

\[ s \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n}) \]
\[ \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n}, \]

and thus

\[ S_n = 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n} \leq 2(s - a_1). \]

Hence, the increasing sequence \((S_n)_{n=1}^{\infty}\) is bounded and therefore converges.

Suppose, on the other hand, that \(\sum_{n=1}^{\infty} 2^n a_{2^n}\) converges to \(S\). Then, we see that, for \(n \in \mathbb{N}\),

\[ a_1 + a_2 + \cdots + a_{2^n-1} = a_1 + (a_2 + a_3) + \cdots + (a_{2^n-1} + \cdots + a_{2^n-1}) \]
\[ \leq a_1 + 2a_2 + \cdots + 2^{n-1} a_{2^n-1} \]
\[ \leq S + a_1. \]

so that the increasing sequence \((s_n)_{n=1}^{\infty}\) is bounded and therfore converges.

To investigate the convergence of \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) for arbitrary \(p \in \mathbb{R}\), first note that \((\frac{1}{n^p})_{n=1}^{\infty}\) does not tend to zero if \(p \leq 0\), so that \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) has to diverge.

Let \(p > 0\). Then \((\frac{1}{n^p})_{n=1}^{\infty}\) decreases, so that the criterion just proved is applicable. We thus have:

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \iff \quad \sum_{n=1}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=1}^{\infty} \left(\frac{2}{2^p}\right)^n < \infty \]
\[ \iff \quad \frac{2}{2^p} < 1 \]
\[ \iff \quad p > 1. \]

4. Test the following series for convergence and absolute convergence:

(a) \(\sum_{n=1}^{\infty} \left(\frac{2n}{n}\right)^{-1}\);

(b) \(\sum_{\nu=42}^{\infty} \frac{7\nu^2 \cos(2021\nu^7)}{\sqrt{\nu^4 - 1}}\);

(c) \(\sum_{k=4}^{\infty} \frac{(-1)^k^3}{\log(\sqrt{k + \log k})}\);
(d) \( \sum_{m=2}^{\infty} \frac{1}{(\log m)^p} \) where \( p > 0 \).

**Solution:**

(a) For \( n \in \mathbb{N} \), set \( a_n := \left( \frac{2n}{n} \right)^{-1} = \frac{(n!)^2}{(2n)!} \). It follows that

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)!^2 (2n)!}{(2n+2)! (n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to 4.
\]

Hence, there series converges (absolutely) by the limit ratio test.

(b) For \( \nu \in \mathbb{N} \), set \( a_\nu := \frac{7\nu^4 \cos(2021\nu^7)}{\sqrt{\nu} (\nu^4 - 1)} \) and \( b_\nu := \frac{1}{\sqrt{\nu}} \). Then

\[
\frac{|a_\nu|}{b_\nu} = \frac{7\nu^4 |\cos(2021\nu^7)|}{\sqrt{\nu} (\nu^4 - 1)} = \frac{7|\cos(2021\nu^7)|}{\sqrt{\nu} - \frac{1}{\sqrt{\nu}}} \to 0
\]

holds, so that \( \sum_{n=1}^{\infty} a_n \) converges absolutely by the limit comparison test.

(c) For \( k \geq 4 \), set \( a_k := \frac{1}{\log(\sqrt{k} + \log k)} \). Then \( (a_k)_{k=4}^{\infty} \) is a decreasing sequence converging to zero. Since \((-1)^k = (-1)^k\) for all \( k \in \mathbb{N} \). The convergence of the series follows from the alternating series test. First, note that \( \log k \leq k \) for \( k \) large enough. For such \( k \) we then have

\[
a_k \geq \frac{1}{\log 2k} \geq \frac{1}{2k}.
\]

Since \( \sum_{k=1}^{\infty} \frac{1}{k} = \infty \) diverges, so does \( \sum_{k=4}^{\infty} a_k \) by the comparison test. Hence, the series in this problem does not converge absolutely.

(d) For \( m \geq 2 \), set \( a_m := \frac{1}{(\log m)^p} \). Since \( p > 0 \), \( (a_m)_{m=2}^{\infty} \) is a decreasing sequence of non-negative reals. By Problem 2, \( \sum_{m=2}^{\infty} a_m \) therefore converges if and only if \( \sum_{m=1}^{\infty} 2^m a_{2m} \) does. For \( m \in \mathbb{N} \), we have

\[
2^m a_{2m} = \frac{2^m}{(\log 2^m)^p} = \frac{2^m}{m^p(\log 2)^{2p}}.
\]

Since \( \frac{2^m}{m^p} \not\to 0 \) for \( p > 0 \), it follows that \( \sum_{m=1}^{\infty} 2^m a_m \) and thus \( \sum_{m=2}^{\infty} \frac{1}{(\log m)^p} \)

diverges.

5. Let \( p \) be a polynomial, and let \( \theta \in (-1, 1) \). Show that the series \( \sum_{n=1}^{\infty} p(n)\theta^n \) converges absolutely.

**Solution:** Let \( \nu \) be the degree of \( p \), i.e.,

\[
p(x) = a_\nu x^\nu + q(x),
\]

where \( a_\nu \neq 0 \), and \( q \) is a polynomial whose degree is strictly less than \( \nu \). We then have

\[
p(x + 1) = a_\nu x^\nu + \sum_{k=1}^{\nu} a_\nu \binom{\nu}{k} x^{\nu-k} + q(x + 1).
\]

of degree strictly less than \( \nu \) in \( x \)
Hence, we obtain
\[
\frac{p(n + 1)}{p(n)} = \frac{a_\nu n^\nu + \sum_{k=1}^{\nu} a_\nu (\nu) n^{\nu-k} + q(n + 1)}{a_\nu n^\nu + q(n)}
= \frac{a_\nu + \frac{1}{n^\nu} \sum_{k=1}^{\nu} a_\nu (\nu) n^{\nu-k} + \frac{q(n + 1)}{n^\nu}}{a_\nu + \frac{q(n)}{n^\nu}} \to 1.
\]
(We may take the quotient safely, since polynomials only have a finite number of zeros.)

Let \(a_n := p(n)\theta^n\). By the foregoing, we obtain that
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{p(n + 1)}{p(n)} \right| |\theta| \to |\theta| < 1.
\]
Hence, the series converges by the limit ratio test.

6*. Let \(r > 0\), let \(x_0 := [0 0]^T\), \(x_1 := [2\pi 0]^T\), \(x_2 := [2\pi \frac{\pi}{2}]^T\), and \(x_3 := [0 \frac{\pi}{2}]^T\), and let \(K\) be the rectangle in \(\mathbb{R}^2\) with the vertices \(x_0, x_1, x_2, \) and \(x_3\), and let
\[
\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (s, t) \mapsto r(\cos s)(\cos t) \hat{i} + r(\sin s)(\cos t) \hat{j} + r \sin t \hat{k}
\]
be the surface in \(\mathbb{R}^3\) with parameter domain \(K\). Set \(\Sigma_1 := [x_0, x_1, x_2]\) and \(\Sigma_2 := [x_0, x_2, x_3]\), let \(\Phi_k := \Phi \circ \Sigma_k\) for \(k = 1, 2\), and let \(\Psi := \Phi_1 \oplus \Phi_2\). Determine \(\{\Psi\}\) in geometric terms.

Solution: From the given data, it is clear that
\[
\{\Psi\} = \{\Phi_1\} \cup \{\Phi_2\} = \{\Phi(s, t) : (s, t) \in K\}
\]
i.e., the upper half of the sphere in \(\mathbb{R}^3\) with radius \(r\) and centered at the origin.