Solutions #6

1. Let \( r \in \mathbb{N} \), and let \( \sigma \in S_r \). Show that there are \( n \leq r - 1 \) and transpositions \( \tau_1, \ldots, \tau_n \in S_r \) such that \( \sigma = \tau_1 \circ \cdots \circ \tau_n \). (Hint: Induction on \( r \).)

Solution: The case \( r = 1 \) is clear: in this case, the identity is the only element of \( S_r \), which is a product of zero transpositions.

Suppose that the claim is true for all elements of \( S_r \), and let \( \sigma \in S_{r+1} \).

Case 1: \( \sigma(r + 1) = r + 1 \).

Then \( \sigma(\{1, \ldots, r\}) \subset \{1, \ldots, r\} \), so that \( \sigma|_{\{1, \ldots, r\}} \in S_r \). By the induction hypothesis, there are \( n \leq r - 1 \) and \( \tau_1, \ldots, \tau_n \in S_r \) such that \( \sigma|_{\{1, \ldots, r\}} = \tau_1 \circ \cdots \circ \tau_n \). Define transpositions \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \in S_{r+1} \) by letting \( \tilde{\tau}_j|_{\{1, \ldots, r\}} = \tau_j \) for \( j = 1, \ldots, n \). It is then clear that \( \sigma = \tilde{\tau}_1 \circ \cdots \circ \tilde{\tau}_n \).

Case 2: \( \sigma(r + 1) \neq r + 1 \).

Let \( \tau \in S_{r+1} \) be such that \( \tau(\sigma(r + 1)) = r + 1 \), \( \tau(r + 1) = \sigma(r + 1) \), and \( \tau(k) = k \) for \( k \in \{1, \ldots, r, r + 1\} \setminus \{r + 1, \sigma(r + 1)\} \). It follows that \( \tau \circ \sigma \in S_{r+1} \) with \( (\tau \circ \sigma)(r + 1) = r + 1 \). By Case 1, this means that there are \( n \leq r - 1 \) and transpositions \( \tau_1, \ldots, \tau_n \in S_{r+1} \) with \( \tau \circ \sigma = \tau_1 \circ \cdots \circ \tau_n \), so that \( \sigma = \tau^{-1} \circ \tau_1 \circ \cdots \circ \tau_n \), i.e., \( \sigma \) is a product of at most \( n + 1 \leq r \) transpositions.

2. Let

\[
\omega := \sin(xy)\,dx + e^{xz}\,dy - zx\,dz \quad \text{and} \quad \phi := xyz\,dx \wedge dz + \cos z\,dx \wedge dy.
\]

Compute \( \omega \wedge \phi \) and \( d(\omega \wedge \phi) \), expressing both differential forms in their respective standard representation.

Solution: Using the properties of \( \wedge \), we obtain

\[
\omega \wedge \phi = xyz\sin(xy)\left(\frac{dx \wedge dx \wedge dz + \sin(xy)(\cos z)\,dx \wedge dx \wedge dy}{=0}\right) \\
+ xyz e^{xz}\,dy \wedge dx \wedge dz + e^{xz}(\cos z)\,dy \wedge dx \wedge dy \\
- xyz^2\,dz \wedge dx \wedge dz - z\cos z\,dz \wedge dx \wedge dy \\
= -(xyz e^{xz} + z\cos z)\,dx \wedge dy \wedge dz.
\]

With \( \omega \wedge \phi \in \Lambda^3(\mathcal{C}^1(\mathbb{R}^3)) \), we have \( d(\omega \wedge \phi) \in \Lambda^4(\mathcal{C}^0(\mathbb{R}^3)) = \{0\} \), so that \( d(\omega \wedge \phi) = 0 \).
3. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, and let $r \in \mathbb{N}_0$. Then $\omega \in \Lambda^r(\mathcal{C}^1(U))$ is called closed if $d\omega = 0$. Show that:

(a) each $\omega \in \Lambda^N(\mathcal{C}^1(U))$ is closed;
(b) $\omega = \sum_{j=1}^N f_j \, dx_j \in \Lambda^1(\mathcal{C}^1(U))$ is closed if and only if

$$\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}$$
on $U$ for all $j, k = 1, \ldots, N$.

**Solution:**

(a) As $\omega \in \Lambda^N(\mathcal{C}^1(U))$, then $d\omega \in \Lambda^{N+1}(\mathcal{C}^0(U)) = \{0\}$.

(b) Let $\omega = \sum_{j=1}^N f_j \, dx_j \in \Lambda^1(\mathcal{C}^1(U))$. Then

$$d\omega = \sum_{j=1}^N \sum_{k=1}^N \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j$$

$$= \sum_{j,k=1}^N \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j$$

$$= \sum_{1 \leq j < k \leq N} \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j + \sum_{1 \leq k < j \leq N} \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j$$

$$= \sum_{1 \leq j < k \leq N} \left( \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} \right) \, dx_k \wedge dx_j,$$

so that $d\omega = 0$ if and only if $\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}$ for all $j, k = 1, \ldots, N$.

4. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, and let $r \in \mathbb{N}$. Then $\omega \in \Lambda^r(\mathcal{C}^0(U))$ is called exact if there is $\phi \in \Lambda^{r-1}(\mathcal{C}^1(U))$ such that $\omega = d\phi$. Show that:

(a) if $\omega \in \Lambda^r(\mathcal{C}^1(U))$ is exact, then it is closed;
(b) if $U$ is star shaped and $\omega \in \Lambda^1(\mathcal{C}^1(U))$ is closed, then $\omega$ is exact.

**Solution:**

(a) Let $\phi \in \Lambda^{r-1}(\mathcal{C}^1(U))$ such that $\omega = d\phi$. Then $d\omega = d(d\phi) = 0$.

(b) Suppose that $\omega = \sum_{j=1}^N f_j \, dx_j \in \Lambda^1(\mathcal{C}^1(U))$ is closed. By the previous problem, this means that $\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}$ for all $j, k = 1, \ldots, N$. By Theorem 6.2.15 from the
notes, this means that there is \( \phi \in C^2(U) = \Lambda^0(C^2(U)) \) such that \( \frac{\partial \phi}{\partial x_j} = f_j \) for \( j = 1, \ldots, N \), i.e.,

\[
d\phi = \sum_{j=1}^{N} \frac{\partial \phi}{\partial x_j} \, dx_j = \omega.
\]

5. Let \( \emptyset \neq U \subset \mathbb{R}^N \) be open, and let \( f_1, \ldots, f_n \in C^2(U) \). Show that \( df_1 \wedge \cdots \wedge df_n \) is closed. \((\text{Hint: Induction on } n. )\)

Solution: For \( n = 1 \), the clear is clear by Theorem 7.2.6 of the notes.

For the induction step note that, by Proposition 7.2.5(iii),

\[
d(d(f_1 \wedge \cdots \wedge df_n \wedge df_{n+1}) = d(d f_1 \wedge \cdots \wedge df_n) \wedge df_{n+1} + (-1)^n (d f_1 \wedge \cdots \wedge df_n) \wedge d(df_{n+1}) = 0,
\]

which yields the claim.

6*. Let \( \emptyset \neq U \subset \mathbb{R}^N \) be open, and let \( f = (f_1, \ldots, f_N) \in C^1(U, \mathbb{R}^N) \). Show that

\[
df_1 \wedge \cdots \wedge df_N = \det J_f \, dx_1 \wedge \cdots \wedge dx_N.
\]

Solution: As

\[
df_j = \sum_{k=1}^{N} \frac{\partial f_j}{\partial x_k} \, dx_k
\]

for \( j = 1, \ldots, N \), the properties of \( \wedge \) yield

\[
df_1 \wedge \cdots \wedge df_N = \sum_{k_1=1}^{N} \cdots \sum_{k_N=1}^{N} \frac{\partial f_1}{\partial x_{k_1}} \cdots \frac{\partial f_N}{\partial x_{k_N}} \, dx_{k_1} \wedge \cdots \wedge dx_{k_N}
\]

\[
= \sum_{k_1, \ldots, k_N=1}^{N} \frac{\partial f_1}{\partial x_{k_1}} \cdots \frac{\partial f_N}{\partial x_{k_N}} \, dx_{k_1} \wedge \cdots \wedge dx_{k_N}
\]

\[
= \sum_{\sigma \in S_N} \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_N}{\partial x_{\sigma(N)}} \, dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(N)}
\]

\[
= \sum_{\sigma \in S_N} (\text{sgn } \sigma) \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_N}{\partial x_{\sigma(N)}} \, dx_1 \wedge \cdots \wedge dx_N
\]

which proves the claim.