1. Calculate $\int_{\gamma} f \cdot dx$, where $f(x, y) = (xy, ye^x)$, and $\gamma$ is the polygonal path connecting $(0, 0)$, $(2, 0)$, $(2, 1)$, $(0, 1)$, and $(0, 0)$ (in this order).

Solution: First note that

$$\int_{\gamma} f \cdot dx = \int_{[(0,0),(2,0)]} f \cdot dx + \int_{[(2,0),(2,1)]} f \cdot dx + \int_{[(2,1),(0,1)]} f \cdot dx + \int_{[(0,1),(0,0)]} f \cdot dx.$$

Since

$$\int_{[(0,0),(2,0)]} f \cdot dx = \int_0^2 (0, 0) \cdot (1, 0) \, dt = 0,$$

$$\int_{[(2,0),(2,1)]} f \cdot dx = \int_0^1 (2t, te^2) \cdot (0, 1) \, dt = \frac{e^2}{2},$$

$$\int_{[(2,1),(0,1)]} f \cdot dx = \int_0^2 (2 - t, e^{(2-t)}) \cdot (-1, 0) \, dt = -2,$$

and

$$\int_{[(0,1),(0,0)]} f \cdot dx = \int_0^1 (0, 1 - t) \cdot (0, -1) \, dt = \int_0^1 -1 + t \, dt = -\frac{1}{2},$$

we obtain that

$$\int_{\gamma} f \cdot dx = \frac{1}{2}(e^2 - 5).$$

2. Let $f = (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$P(x, y, z) = e^{yz}, \quad Q(x, y, z) = xze^{yz}, \quad \text{and} \quad R(x, y, z) = xy e^{yz}$$

for $x, y, z \in \mathbb{R}$, and let

$$\gamma : [0, 6\pi] \to \mathbb{R}, \quad t \mapsto (\cos t, \sin t, 666t).$$

Evaluate the curve integral

$$\int_{\gamma} P \, dx + Q \, dy + R \, dz.$$

Solution: A routine calculation shows that $\text{curl} \, f = 0$. Hence, by a result presented in class, $f$ is conservative, i.e., a $(C^1)$-gradient field. Consequently, there is a function $F : \mathbb{R}^3 \to \mathbb{R}$ such that $f = \nabla F$. 

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As \( P = \frac{\partial F}{\partial x} \), there is \( G : \mathbb{R}^2 \to \mathbb{R} \) such that

\[
F(x, y, z) = xe^{yz} + G(y, z)
\]

for \( x, y, z \in \mathbb{R} \). Fix \( x \) and differentiate with respect to \( y \); we obtain

\[
\frac{\partial G}{\partial y}(y, z) = \frac{\partial F}{\partial y}(y) - xze^{yz} = 0.
\]

for \( y, z \in \mathbb{R} \). Consequently, there is \( H : \mathbb{R} \to \mathbb{R} \) such that

\[
G(y, z) = H(z).
\]

Fixing \( y \) and differentiation with respect to \( z \), we get

\[
H'(z) = \frac{\partial G}{\partial z}(x, y) = \frac{\partial F}{\partial z}(x, y, z) - xye^{yz} = 0
\]

for \( z \in H \), which implies that \( H \) is constant. It follows that

\[
F(x, y, z) = xe^{yz} + C
\]

for \( x, y, z \in \mathbb{R} \) for some constant \( C \). From the Fundamental Theorem for Curve Integrals, we get that

\[
\int_{\gamma} P \, dx + Q \, dy + R \, dz = F(1, 0, 3996\pi) - F(1, 0, 0) = e - e = 0.
\]

3. Show that

\[
K := \{(x, y) \in \mathbb{R}^2 : x, y, x + y \in [0, 1]\}
\]

is a normal domain (with respect to both coordinate axes) and use Green’s Theorem to compute

\[
\int_{\partial K} ye^x \, dx + xe^y \, dy.
\]

Solution: Note that

\[
K = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1 - x\},
\]

and thus, if

\[
\phi_1(x) := 0 \quad \text{and} \quad \phi_2(x) := 1 - x.
\]

for \( x \in [0, 1] \) is a normal domain with respect to the \( x \)-axis. Similarly, one sees that \( K \) is a normal domain with respect to the \( y \) axis.
By Green’s Theorem, we thus have
\[ \int_{\partial K} ye^x \, dx + xe^y \, dy = \int_K e^y - e^x = \int_0^1 \left( \int_0^{1-x} e^y - e^x \, dy \right) \, dx. \]
Since
\[ \int_0^{1-x} e^y - e^x \, dy = e^{1-x} - 1 - (1 - x)e^x, \]
this leaves us with
\[ \int_{\partial K} ye^x \, dx + xe^y \, dy = \int_0^1 e^{1-x} - 1 - (1 - x)e^x \, dx \]
\[ = -e^{1-x}\big|_{x=1}^{x=0} - 1 - e^x\big|_{x=0}^{x=1} + \int_0^1 xe^x \, dx \]
\[ = -1 + e - 1 - e + 1 + \int_0^1 xe^x \, dx \]
\[ = -1 + \int_0^1 xe^x \, dx. \]
Integration by parts yields
\[ \int_0^1 xe^x \, dx = xe^x\big|_0^1 - \int_0^1 e^x \, dx = e - e + 1 = 1, \]
so that
\[ \int_{\partial K} ye^x \, dx + xe^y \, dy = 0. \]

4. Let \( \Phi \) be a surface in \( \mathbb{R}^3 \) with parameter domain \( K \subset \mathbb{R}^2 \), let \( \gamma : [a, b] \to K \) be a \( C^1 \)-curve, and let \( \alpha := \Phi \circ \gamma \). Show that \( \alpha'(t) \) is orthogonal to \( N(\gamma(t)) \) for each \( t \in [a, b] \).

Interpret this in geometric terms.

**Solution:** The chain rule in several variables yields that
\[ \alpha'(t) = \gamma_1'(t) \frac{\partial \Phi}{\partial s}(\gamma(t)) + \gamma_2'(t) \frac{\partial \Phi}{\partial t}(\gamma(t)) \]
for \( t \in [a, b] \). As \( N(\gamma(t)) = \frac{\partial \Phi}{\partial s}(\gamma(t)) \times \frac{\partial \Phi}{\partial t}(\gamma(t)) \) is orthogonal to \( \frac{\partial \Phi}{\partial s}(\gamma(t)) \) and \( \frac{\partial \Phi}{\partial t}(\gamma(t)) \), we obtain
\[ \alpha'(t) \cdot N(\gamma(t)) = \gamma_1'(t) \frac{\partial \Phi}{\partial s}(\gamma(t)) \cdot N(\gamma(t)) + \gamma_2'(t) \frac{\partial \Phi}{\partial t}(\gamma(t)) \cdot N(\gamma(t)) = 0 \]
for \( t \in [a, b] \).

In geometric terms, this means that, for any point \( x_0 \in K \), the vectors \( N(x_0) \) stands perpendicularly on each tangential vector to the surface \( \Phi \) passing through \( \Phi(x_0) \).
5. Let $a < b$, and let $f \in C^1([a, b], \mathbb{R})$ such that $f \geq 0$. Viewing the graph of $f$ as a subset of the $xy$-plane in $\mathbb{R}^3$ and rotating it about the $x$-axis generates a surface in $\mathbb{R}^3$, a so called rotation surface. Show that the area of this surface is

$$2\pi \int_a^b f(t) \sqrt{1 + f'(t)^2} \, dt.$$ 

What is the area of the outer hull of a cone with height $h > 0$ of which the basis is a circle of radius $r$?

Solution: Let $r > 0$, and define

$$g: [a - r, b + r] \to \mathbb{R}, \quad t \mapsto \begin{cases} f'(a), & t \in [a - r, a], \\ f'(t), & t \in [a, b], \\ f'(b), & t \in [b, b + r] \end{cases}$$

Then $g$ is continuous and extends $f'$. Define

$$\tilde{f}: (a - r, b + r) \to \mathbb{R}, \quad t \mapsto f(a) + \int_a^t g(s) \, ds,$$

so that $\tilde{f}$ is a $C^1$-function extending $f$.

The rotation surface is parametrized as

$$\Phi(s, t) := s \mathbf{i} + \tilde{f}(s) \cos t \mathbf{j} + \tilde{f}(s) \sin t \mathbf{k}$$

for $(s, t) \in \mathbb{R} \times (a - r, b + r)$ with parameter domain $[a, b] \times [0, 2\pi]$, so that

$$\frac{\partial \Phi}{\partial s}(s, t) := \mathbf{i} + f'(s) \cos t \mathbf{j} + f'(s) \sin t \mathbf{k}$$

and

$$\frac{\partial \Phi}{\partial t}(s, t) := -f(s) \sin t \mathbf{j} + f(s) \cos t \mathbf{k}$$

for $s \in [a, b]$ and $t \in [0, 2\pi]$. It follows that

$$N(s, t) = \frac{\partial \Phi}{\partial s}(s, t) \times \frac{\partial \Phi}{\partial t}(s, t)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(s) \cos t & f'(s) \sin t \\ 0 & -f(s) \sin t & f(s) \cos t \end{vmatrix}$$

$$= f'(s) f(s) \mathbf{i} - f(s) \cos t \mathbf{j} - f(s) \sin t \mathbf{k}$$

for $s \in [a, b]$ and $t \in [0, 2\pi]$ and hence—$f$ is non-negative—

$$\|N(s, t)\| = \sqrt{f(s)^2 f'(s)^2 + f(s)^2 (\sin t)^2 + f(s)^2 (\cos t)^2} = f(s) \sqrt{1 + f'(s)^2}$$
for \( s \in [a, b] \) and \( t \in [0, 2\pi] \). The definition of surface area and Fubini’s Theorem yield that

\[
\text{surface area} = \int_{[a,b] \times [0,2\pi]} f(s) \sqrt{1 + f'(s)^2} \\
= \int_0^{2\pi} \left( \int_{a}^{b} f(s) \sqrt{1 + f'(s)^2} \, ds \right) \, dt \\
= 2\pi \int_{a}^{b} f(s) \sqrt{1 + f'(s)^2} \, ds.
\]

The outer hull of a cone is the rotation surface obtained from the function

\[ f : [0, h] \to \mathbb{R}, \quad t \mapsto \frac{r}{h} t. \]

We thus obtain

\[
\text{surface area} = 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \int_{0}^{h} t \, dt \\
= \pi rh \sqrt{1 + \frac{r^2}{h^2}} \\
= \pi r \sqrt{r^2 + h^2}.
\]

6*. Let \( R > 0 \). Determine the area of the part of the sphere

\[ \{(x, y, z) \in \mathbb{R}^3 : z \geq 0, \ x^2 + y^2 + z^2 = R^2\} \]

that lies inside the cylinder

\[ \left\{ (x, y, z) \in \mathbb{R}^3 : \left( x - \frac{R}{2} \right)^2 + y^2 \leq \frac{R^2}{4} \right\}. \]

**Solution:** First note that

\[ \left( x - \frac{R}{2} \right)^2 + y^2 \leq \frac{R^2}{4} \iff x^2 - Rx + y^2 \leq 0. \]

Let \( x = r \cos \theta \) and \( y = r \sin \theta \) with \( r > 0 \). Then we have

\[ x^2 - Rx + y^2 \leq 0 \iff r^2 \leq Rr \cos \theta \iff r \leq R \cos \theta. \]

Hence we can parametrize the part of sphere whose area we wish to determine by

\[ \Phi(\theta, r) := r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \sqrt{R^2 - r^2} \mathbf{k} \]

with parameter domain

\[ K := \left\{ (r, \theta) : \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], r \in [0, R \cos \theta] \right\}. \]
It follows that
\[ \frac{\partial \Phi}{\partial r}(r, \theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} - \frac{r}{\sqrt{R^2 - r^2}} \mathbf{k} \]
and
\[ \frac{\partial \Phi}{\partial \theta}(r, \theta) = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \]
for \((r, \theta) \in K\). The normal vector \(N(r, \theta)\) is then calculated as
\[
N(r, \theta) = \frac{\partial \Phi}{\partial r}(r, \theta) \times \frac{\partial \Phi}{\partial \theta}(r, \theta)
= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & -\frac{r}{\sqrt{R^2 - r^2}} \\
-r \sin \theta & r \cos \theta & 0
\end{vmatrix}
= \frac{r^2}{\sqrt{R^2 - r^2}} \cos \theta \mathbf{i} - \frac{r^2}{\sqrt{R^2 - r^2}} \sin \theta \mathbf{j} + r \mathbf{k},
\]
so that
\[
\|N(r, \theta)\| = \sqrt{\frac{r^4}{R^2 - r^2} + r^2} = r \sqrt{\frac{r^2}{R^2 - r^2} + 1} = r \frac{R}{\sqrt{R^2 - r^2}}
\]
for \((r, \theta) \in K\). The surface area is then calculated as follows:
\[
\text{surface area} = \int_{K} r \frac{R}{\sqrt{R^2 - r^2}}
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_{0}^{R \cos \theta} \frac{r}{\sqrt{R^2 - r^2}} \, dr \right) \, d\theta
= -R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_{0}^{R \cos \theta} \frac{-2r}{2\sqrt{R^2 - r^2}} \, dr \right) \, d\theta
= -R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{R^2 - r^2} \bigg|_{r=0}^{r=R \cos \theta} \, d\theta
= -R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (R |\sin \theta| - R) \, d\theta
= -R^2 \left( 2 \int_{0}^{\frac{\pi}{2}} |\sin \theta| \, d\theta - \pi \right)
= -R^2 (2 - \pi)
= R^2 (\pi - 2).
\]