1. Let $\Phi$ be a surface in $\mathbb{R}^3$ with parameter domain $K \subset \mathbb{R}^2$, let $\gamma : [a, b] \to K$ be a $C^1$-curve, and let $\alpha := \Phi \circ \gamma$. Show that $\alpha'(t)$ is orthogonal to $N(\gamma(t))$ for each $t \in [a, b]$.

Interpret this in geometric terms.

*Solution:* The chain rule in several variables yields that

$$\alpha'(t) = \gamma_1'(t) \frac{\partial \Phi}{\partial s}(\gamma(t)) + \gamma_2'(t) \frac{\partial \Phi}{\partial t}(\gamma(t))$$

for $t \in [a, b]$. As $N(\gamma(t)) = \frac{\partial \Phi}{\partial s}(\gamma(t)) \times \frac{\partial \Phi}{\partial t}(\gamma(t))$ is orthogonal to $\frac{\partial \Phi}{\partial s}(\gamma(t))$ and $\frac{\partial \Phi}{\partial t}(\gamma(t))$, we obtain

$$\alpha'(t) \cdot N(\gamma(t)) = \gamma_1'(t) \left( \frac{\partial \Phi}{\partial s}(\gamma(t)) \cdot N(\gamma(t)) \right) + \gamma_2'(t) \left( \frac{\partial \Phi}{\partial t}(\gamma(t)) \cdot N(\gamma(t)) \right) = 0$$

for $t \in [a, b]$.

In geometric terms, this means that, for any point $x_0 \in K$, the vector $N(\Phi(x_0))$ stands perpendicularly on each tangential vector to the surface $\Phi$ passing through $\Phi(x_0)$.

2. Let $\Phi$ and $\Psi$ be $C^2$-surfaces with parameter domain $K$, which is a normal region, such that $\Phi|_{\partial K} = \Psi|_{\partial K}$, and let $f : V \to \mathbb{R}^3$ be continuously differentiable where $V \subset \mathbb{R}^3$ is open and contains $\{\Phi\} \cup \{\Psi\}$. Show that

$$\int_{\Phi} \text{curl } f \cdot n \, d\sigma = \int_{\Psi} \text{curl } f \cdot n \, d\sigma.$$

*Solution:* Let $\gamma$ be a piecewise $C^1$-curve that parametrizes the positively oriented boundary of $K$. Then $\Phi|_{\partial K} = \Psi|_{\partial K}$ implies that $\Phi \circ \gamma = \Psi \circ \gamma$. Stokes’ Theorem then yields

$$\int_{\Phi} \text{curl } f \cdot n \, d\sigma = \int_{\Phi \circ \gamma} f \cdot d(x,y,z) = \int_{\Psi \circ \gamma} f \cdot d(x,y,z) = \int_{\Psi} \text{curl } f \cdot n \, d\sigma.$$

3. Let $S$ be the surface of the ball centered at $(0,0,0)$ with radius $r > 0$. Compute

$$\int_S x^3 \, dy \wedge dz + y^3 \, dz \wedge dx + z^3 \, dx \wedge dy.$$
Solution: Let $V$ the the closed ball centered at $(0, 0, 0)$ with radius $r > 0$. By Gauß’ Theorem, we have

$$
\int_S x^3 \, dy \wedge dz + y^3 \, dz \wedge dx + z^3 \, dx \wedge dy
$$

$$
= 3 \int_V x^2 + y^2 + z^2
$$

$$
= 3 \int_{[0,r] \times [0,2\pi] \times [-\pi,\pi]} \rho^4 \cos \sigma,
$$

through passage to spherical coordinates,

$$
= 6\pi \left( \int_0^r \rho^4 \, d\rho \right) \left( \int_{-\pi}^{\pi} \cos \sigma \, d\sigma \right)
$$

$$
= 6\pi \cdot \frac{r^5}{5} \cdot 2
$$

$$
= \frac{12}{5} \pi r^5.
$$

4. Let $V$ be a normal domain with boundary $S$ such that $N \neq 0$ on $S$ throughout, and let $f$ and $g$ be $\mathbb{R}$-valued $C^2$-functions on an open set containing $V$.

(a) Prove Green’s First Formula:

$$
\int_V (\nabla f) \cdot (\nabla g) + \int_V f \Delta g = \int_S f D_n g \, d\sigma.
$$

(b) Prove Green’s Second Formula:

$$
\int_V (f \Delta g - g \Delta f) = \int_S (f D_n g - g D_n f) \, d\sigma.
$$

(Hint for (a): Apply Gauß’ Theorem to the vector field $f \nabla g$.)

Solution:

(a) Let $w := f \nabla g$. Since $D_n g = (\nabla g) \cdot n$, we obtain that

$$
\int_S f D_n g \, d\sigma = \int_S f (\nabla g) \cdot n \, d\sigma = \int_S w \cdot n \, d\sigma.
$$

By the rules of differentiation, we obtain that

$$
\text{div} \ w = (\nabla f) \cdot (\nabla g) + f \Delta g.
$$

With Gauß’ Theorem, we finally get

$$
\int_S f D_n g \, d\sigma = \int_S w \cdot n \, d\sigma
$$

$$
= \int_V \text{div} \ w
$$

$$
= \int_V (\nabla f) \cdot (\nabla g) + \int_V f \Delta g
$$

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(b) By Green’s First Formula,
\[ \int_V (\nabla f) \cdot (\nabla g) + \int_V f \Delta g = \int_S f D_n g \, d\sigma \]
holds, and with \( f \) and \( g \) interchanged, we obtain
\[ \int_V (\nabla f) \cdot (\nabla g) + \int_V g \Delta f = \int_S g D_n f \, d\sigma. \]
Subtracting the second equation from the first one yields the claim.

5. Let \( \emptyset \neq U \subset \mathbb{R}^3 \) be open, and suppose that \( f \in C^2(U, \mathbb{R}) \) is harmonic, i.e., satisfies \( \Delta f = 0 \). Let \( V \subset U \), \( S \) and \( n \) be as in the previous problem. Show that
\[ \int_S D_n f \, d\sigma = 0 \quad \text{and} \quad \int_S f D_n f \, d\sigma = \int_V \|\nabla f\|^2. \]

Solution: For the first identity, apply Green’s First Formula (with \( f \) and \( g \) interchanged) with \( g \equiv 1 \) (so that \( \nabla g \equiv 0 \)). We obtain
\[ \int_S D_n f \, d\sigma = \int_V (\nabla f) \cdot (\nabla g) + \int_V g \Delta f \]
\[ = \int_V g \Delta f, \quad \text{since} \ \nabla g \equiv 0, \]
\[ = 0, \quad \text{since} \ \Delta f \equiv 0. \]

For the second identity, apply Green’s First Formula with \( f = g \), so that
\[ \int_S f D_n f \, d\sigma = \int_V (\nabla f) \cdot (\nabla f) + \int_V f \Delta f \]
\[ = \int_V (\nabla f) \cdot (\nabla f) \]
\[ = \int_V \|\nabla f\|^2. \]

6. Let \( a, b > 0 \). Use Green’s Theorem to compute the area of the ellipse
\[ E := \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}. \]

Solution: Clearly, \( E \) is a normal domain. By Corollary 6.3.7, this means that
\[ \mu(E) = \frac{1}{2} \int_{\partial E} x \, dy - y \, dx. \]
Let \( \gamma : [0, 2\pi] \to \mathbb{R}^2, \) \( t \mapsto (a \cos t, b \sin t). \)
Then $\gamma$ is a $C^1$-curve parametrizing $\partial E$ in counterclockwise orientation. It follows that

$$
\mu(E) = \frac{1}{2} \int_{\partial E} x \, dy - y \, dx \\
= \frac{1}{2} \int_0^{2\pi} (-b \sin t, a \cos t) \cdot (-a \sin t, b \cos t) \, dt \\
= \frac{1}{2} \int_0^{2\pi} ab \left( (\sin t)^2 + (\cos t)^2 \right) \, dt \\
= \pi ab.
$$