1. Determine the maximum and the minimum on

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto (x-1)^2 + y^2$$

on

$$K := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\},$$

as well as the points at which they are attained.

**Solution:** Since $K$ is compact and $f$ is continuous, $f$ attains both a minimum and a maximum on $K$.

First, compute the gradient of $f$:

$$\frac{\partial f}{\partial x}(x,y) = 2(x-1) \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 2y.$$ 

Consequently, $\nabla f$ has only one zero, namely $(1,0)$, which lies in the interior of $K$. Hence, $f$ has only one stationary point in the interior of $K$. It is clear that $f(1,0) = 0$, whereas $f(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$ with $(x,y) \neq (1,0)$. It follows that $f$ attains its minimum on $K$—namely 0—at $(1,0)$.

To search for the maximum of $f$ on $K$, we need to investigate the boundary: if $f$ attains its maximum on $K$, it must be on the boundary, i.e., we have a local maximum under the constraints $\phi = 0$, where $\phi(x,y) = x^2 + y^2 - 4$. Suppose that $f$ attains its maximum at $(x,y)$. The Lagrange multiplier theorem yields $\lambda \in \mathbb{R}$ such that $\nabla f(x,y) = \lambda \nabla \phi(x,y)$. We thus obtain the equations

$$2x - 2 = 2\lambda x \quad \text{(1)}$$

and

$$2y = 2\lambda y \quad \text{(2)}$$

Furthermore, the constraint $\phi = 0$ yields

$$x^2 + y^2 = 4 \quad \text{(3)}$$

Suppose first that $y \neq 0$. Then division by $y$ in (2) yields $\lambda = 1$. Plugging into (1), we get $2x - 2 = 2x$, which is impossible. Hence, this case cannot occur, i.e., we have $y = 0$. Due to (3), we have $x = \pm 2$, so that $(2,0)$ and $(-2,0)$ are the only candidates for a local extremum under the constraint $\phi = 0$ to be attained at. Since

$$f(-2,0) = 9 \quad \text{and} \quad f(2,0) = 1,$$

we conclude that $f$ attains on $K$ the maximum 9 at $(-2,0)$. 

2. Determine the maximum and minimum of

\[ f: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto xy + z^2 \]

on

\[ B := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 2\} \]

as well as all the points at which they are attained.

Solution: First, note that

\[ (\nabla f)(x, y, z) = (y, x, 2z) \]

for \((x, y, z) \in \mathbb{R}^3\). Hence, \((0, 0, 0)\) is the only stationary point of \(f\) in the interior of \(B\), at which \(f\) attains the value 0.

Let

\[ \phi: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto x^2 + y^2 + z^2 - 2, \]

so that \(\partial B = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = 0\}\). Clearly,

\[ (\nabla \phi)(x, y, z) = (2x, 2y, 2z) \]

holds for \((x, y, z) \in \mathbb{R}^3\). We thus need to simultaneously solve the equations

\begin{align*}
(4) & \quad y = \lambda 2x, \\
(5) & \quad x = \lambda 2y, \\
(6) & \quad 2z = \lambda 2z,
\end{align*}

and

\[ (7) \quad x^2 + y^2 + z^2 = 2. \]

If \(x = 0\), then \(y = 0\) by (4), so that \(z = \pm \sqrt{2}\) by (7). Evaluating \(f\) at \((0, 0, \pm \sqrt{2})\), we obtain the value 2 at both points.

Suppose that \(x \neq 0\). Plugging (4) into (5), we obtain \(x = 4\lambda^2 x\) and thus \(\lambda = \pm \frac{1}{2}\).

From (6), we conclude that \(2z = \pm z\) and thus \(z = 0\). By (4), \(y = \pm x\) holds. Plugging into (7) yields \(2x^2 = 2\), i.e., \(x = \pm 1\). As \(y = \pm x\), we thus have to test \(f\) at the points \((1, 1, 0), (-1, -1, 0), (1, -1, 0), \) and \((-1, 1, 0)\); the value of \(f\) at the first two of those points is 1, and it is \(-1\) and the two last ones.

Hence, the maximum value is 2, attained at \((0, 0, \pm \sqrt{2})\), and the minimum value is \(-1\), attained at \((\pm 1, \mp 1, 0)\).
Determine the maximum and the minimum of
\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2 + xy \]
on \[ K := \{(x, y) \in \mathbb{R}^2 : 2 \leq x^2 + y^2 \leq 8\}. \]

Also, find all points in \( K \), where the maximum and the minimum, respectively, are attained.

**Solution:** First determine all stationary points of \( f \) in the interior of \( K \).

Calculate the first partial derivatives of \( f \) and obtain
\[ \frac{\partial f}{\partial x} = 2x + y \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y + x. \]

Suppose that
\[ 2x + y = 2y + x; \]
then subtracting both \( x \) and \( y \) from both sides of this equation yields \( x = y \). Hence, if \( 2x + y = 0 = 2y + x \) is possible only if \( x = 0 = y \). Consequently, \( f \) has no stationary points in the interior of \( K \).

Consequently, \( f \) must attain its maximum and minimum on
\[ \partial K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 8\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\} \]

Let
\[ \phi_1 : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2 - 8 \]
so that \( K_1 = \{(x, y) \in \mathbb{R}^2 : \phi_1(x, y) = 0\} \). We have:
\[ \frac{\partial \phi_1}{\partial x} = 2x \quad \text{and} \quad \frac{\partial \phi_1}{\partial y} = 2y. \]

If \( f \) attains its maximum or minimum at \((x, y) \in K_1\), it thus has a local extremum there under the constraint \( \phi_1 = 0 \). The Lagrange multiplier theorem thus yields \( \lambda \in \mathbb{R} \) such that
\[ 2x + y = 2\lambda x \]
and
\[ 2y + x = 2\lambda y. \]

The constraint yields a third equation, namely
\[ x^2 + y^2 = 8. \]
Subtract (9) from (8) and obtain

\[(y - x) = 2\lambda(x - y)\]  

**Case 1:** \(x \neq y\).

Then division of (11) by \(x - y\) yields \(\lambda = -\frac{1}{2}\). Plugging into (1) yields \(x = -y\). From (10), we conclude that \(2x^2 = 8\), or equivalently, that \(x = \pm 2\). It follows that \((2, -2)\) and \((-2, 2)\) are possible candidates for a local extremum of \(f\) under the constraint \(\phi_1 = 0\) to be attained at. We have \(f(2, -2) = f(-2, 2) = 4\).

**Case 2:** \(x = y\). Then (3) yields \((2, 2)\) and \((-2, -2)\) as possible candidates with \(f(2, 2) = f(-2, -2) = 12\).

Let \(\phi_2 : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2 - 2\) so that \(K_2 = \{(x, y) \in \mathbb{R}^2 : \phi_2(x, y) = 0\}\) and

\[
\frac{\partial \phi_2}{\partial x} = 2x \quad \text{and} \quad \frac{\partial \phi_2}{\partial y} = 2y.
\]

If \(f\) attains its maximum or minimum at \((x, y) \in K_2\), it thus has a local extremum there under the constraint \(\phi_2 = 0\), so that—for some \(\lambda \in \mathbb{R}\)—the equations

\[
\begin{align*}
2x + y &= 2\lambda x, \\
2y + x &= 2\lambda y,
\end{align*}
\]

and

\[
x^2 + y^2 = 2
\]

hold. As for \(\phi_1\), we obtain \((y - x) = 2\lambda(x - y)\).

**Case 1:** \(x \neq y\). As for \(\phi_1\), \(\lambda = -\frac{1}{2}\) holds, and plugging into (12) yields \(x = -y\). From (14), we obtain \(2x^2 = 2\), i.e., \(x = \pm 1\). Hence, \((1, -1)\) and \((-1, 1)\) are possible candidates for a local extremum of \(f\) under the constraint \(\phi_2 = 0\) to be attained at with \(f(1, -1) = f(-1, 1) = 1\).

**Case 2:** \(x = y\). Then (12) yields \((1, 1)\) and \((-1, -1)\) as possible candidates with \(f(1, 1) = f(-1, -1) = 3\).

All in all, the maximum of \(f\) on \(K\) is 12—attained at \((2, 2)\) and \((-2, -2)\)—and its minimum on \(K\) is 1—attained at \((1, -1)\) and \((-1, 1)\).

4. Of course, the definition of a local extremum under a constraint, can also be formulated with respect to a vector-valued function \(\phi\). Prove the following generalization of the Lagrange multiplier theorem from class:
Let \( \emptyset \neq U \subset \mathbb{R}^N \) be open, let \( \phi \in C^1(U, \mathbb{R}^M) \) with \( M < N \), let \( f \in C^1(U, \mathbb{R}) \) have a local extremum at \( x_0 \in U \) under the constraint \( \phi(x) = 0 \), and suppose that \( J_\phi(x_0) \) has rank \( M \). Then there are \( \lambda_1, \ldots, \lambda_M \in \mathbb{R} \) such that

\[
(\nabla f)(x_0) = \sum_{j=1}^{M} \lambda_j (\nabla \phi_j)(x_0).
\]

**Solution:** We denote the first \( N - M \) coordinates in \( \mathbb{R}^N \) by \( x_1, \ldots, x_{N-M} \) and the remaining \( N - M \) ones by \( y_1, \ldots, y_M \). By rearranging the variables, we can suppose without loss of generality that \( \partial \phi / \partial y \big|_{x_0} \) is invertible.

The implicit function theorem, then yields open neighborhoods \( V \subset \mathbb{R}^{N-M} \) of \((x_0, 1, \ldots, x_{0,N-M})\) and \( W \subset \mathbb{R}^M \) of \((x_{0,(N-M)+1}, \ldots, x_{0,N})\) along with a function \( \psi \in C^1(V, \mathbb{R}^M) \) such that \( \psi(x_0, 1, \ldots, x_{0,N-M}) = (x_{0,(N-M)+1}, \ldots, x_{0,N}) \) and

\[
\{(x, y) \in V \times W : \phi(x, y) = 0 \} = \{(x, \psi(x)) : x \in V \}.
\]

In particular, we have \( \phi(x, \psi(x)) = 0 \) for all \( x \in V \). The chain rule in several variables yields

\[
0 = \frac{\partial \phi}{\partial x}(x, \psi(x)) + \frac{\partial \phi}{\partial y}(x, \psi(x)) J_\psi(x) \quad (x \in V)
\]

and thus, in particular,

\[
0 = \frac{\partial \phi}{\partial x}(x_0) + \frac{\partial \phi}{\partial y}(x_0) J_\psi(x_0, 1, \ldots, x_{0,N-M}).
\]

Consider

\[
g : V \to \mathbb{R}, \quad x \mapsto f(x, \psi(x)).
\]

Since \( f \) has a local extremum at \( x_0 \) under the constraint \( \phi(x, y) = 0 \), the function \( g \) has a local extremum at \((x_{0,1}, \ldots, x_{0,N-M})\), so that

\[
0 = (\nabla g)(x_{0,1}, \ldots, x_{0,N-M}) = \frac{\partial f}{\partial x}(x_0) + \frac{\partial f}{\partial y}(x_0) J_\psi(x_{0,1}, \ldots, x_{0,N-M}).
\]

Let

\[
[y_1, \ldots, y_M] := \frac{\partial f}{\partial y}(x_0) \left( \frac{\partial \phi}{\partial y}(x_0) \right)^{-1}
\]

For this definition, it is immediate that

\[
\frac{\partial f}{\partial y}(x_0) = \frac{\partial f}{\partial y}(x_0) \left( \frac{\partial \phi}{\partial y}(x_0) \right)^{-1} \frac{\partial \phi}{\partial y}(x_0)
\]

\[
= [y_1, \ldots, y_M] \frac{\partial \phi}{\partial y}(x_0)
\]

\[
= y_1 \frac{\partial \phi_1}{\partial y}(x_0) + \cdots + y_M \frac{\partial \phi_M}{\partial y}(x_0)
\]
and thus
\begin{equation}
\frac{\partial f}{\partial x_j}(x_0) = \lambda_1 \frac{\partial \phi_1}{\partial x_j}(x_0) + \cdots + \lambda_M \frac{\partial \phi_M}{\partial x_j}(x_0)
\end{equation}
for \( j = N - M + 1, \ldots, N \). On the other hand, we have by (16) and (15) that
\begin{align*}
\frac{\partial f}{\partial x}(x_0) &= -\frac{\partial f}{\partial y}(x_0) J_\psi(x_0, 1, \ldots, x_0, N - M) \\
&= -\lambda_1 \cdots \lambda_M \frac{\partial \phi}{\partial y}(x_0) J_\psi(x_0, 1, \ldots, x_0, N - M) \\
&= [\lambda_1, \ldots, \lambda_M] \frac{\partial \phi}{\partial x}(x_0),
\end{align*}
i.e.,
\begin{equation}
\frac{\partial f}{\partial x_j}(x_0) = \lambda_1 \frac{\partial \phi_1}{\partial x_j}(x_0) + \cdots + \lambda_M \frac{\partial \phi_M}{\partial x_j}(x_0)
\end{equation}
for \( j = 1, \ldots, N - M \) as well. Together, (17) and (18) yield the claim.

5. An \( N \)-dimensional cube is a subset \( C \) of \( \mathbb{R}^N \) such that
\[ C = [x_1 - r, x_1 + r] \times \cdots \times [x_N - r, x_N + r] \]
with \( x_1, \ldots, x_N \in \mathbb{R} \) and \( r > 0 \).

Let \( \emptyset \neq U \subset \mathbb{R}^N \) be open and let \( Z \subset U \) be compact with content zero. Show that, for each \( \epsilon > 0 \), there are cubes \( C_1, \ldots, C_n \subset U \) with
\[ Z \subset C_1 \cup \cdots \cup C_n \quad \text{and} \quad \sum_{j=1}^{n} \mu(C_j) < \epsilon. \]
(This was stated in class, but not proven. Your task is to prove it here.)

Solution: Clearly, we can find closed intervals \( I_1, \ldots, I_n \subset U \) such that \( Z \subset \bigcup_{j=1}^{n} I_j \) and \( \sum_{j=1}^{n} \mu(I_j) < \epsilon \). Making each \( I_j \) just a little bit larger, we can achieve that
\[ I_j = [a_{j,1}, b_{j,1}] \times \cdots \times [a_{j,N}, b_{j,N}] \]
with \( a_{j,1}, b_{j,1}, \ldots, a_{j,N}, b_{j,N} \in \mathbb{Q} \). Let \( d \in \mathbb{N} \) be such that \( \gamma_k := d(b_{j,k} - a_{j,k}) \in \mathbb{N} \) for \( k = 1, \ldots, N \). Let \( \nu = (\nu_1, \ldots, \nu_k) \) be such that \( \nu_k \in \{1, \ldots, \gamma_k\} \), and let
\[ C_{\nu,j} := \left[ a_{j,1} + (\nu_1 - 1) \frac{1}{d}, a_{j,1} + \nu_1 \frac{1}{d} \right] \times \cdots \times \left[ a_{j,1} + (\nu_N - 1) \frac{1}{d}, a_{j,1} + \nu_N \frac{1}{d} \right]. \]
Then \( C_{\nu,j} \) is a cube with sidelength \( \frac{1}{d} \). By construction, it is clear that
\[ I_j = \bigcup_{\nu} C_{\nu,j}. \]
Moreover, since $C_{\nu,j}$ and $C_{\mu,j}$ have at most boundary points in common if $\nu \neq \mu$, we also obtain that

$$\sum_{\nu} \mu(C_{j,\nu}) = \mu(I_j).$$

6*. Show that a slice of pizza of radius $r > 0$ and with angle $\alpha$ has the area $\frac{1}{2}r^2\alpha$.

**Solution:** Use polar coordinates, and obtain

$$\text{area of the slice} = \int_{\{(\rho,\theta) : \rho \in [0,r], \theta \in [0,\alpha]\}} \rho \, d\rho \, d\theta$$

$$= \int_0^r \left( \int_0^\alpha \rho \, d\theta \right) \, d\rho$$

$$= \alpha \int_0^r \rho \, d\rho$$

$$= \frac{r^2}{2} \alpha.$$