1. Let $U := \mathbb{R}^2 \setminus \{(0, 0)\}$, and let $f: U \to \mathbb{R}^2$, $(x, y) \mapsto \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right)$.

(a) Calculate $\det J_f(x, y)$ for all $(x, y) \in U$.

(b) Determine $f(U)$. Does it contain a non-empty open subset?

Solution:

(a) Let $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Since

$$\frac{\partial}{\partial x} \frac{x}{\sqrt{x^2 + y^2}} = \frac{1}{x^2 + y^2} \left(\frac{x^2}{\sqrt{x^2 + y^2}} - \frac{x^2}{x^2 + y^2}\right),$$

$$\frac{\partial}{\partial y} \frac{y}{\sqrt{x^2 + y^2}} = \frac{1}{x^2 + y^2} \left(\frac{y^2}{\sqrt{x^2 + y^2}} - \frac{y^2}{x^2 + y^2}\right),$$

and

$$\frac{\partial}{\partial x} \frac{y}{\sqrt{x^2 + y^2}} = \frac{\partial}{\partial y} \frac{x}{\sqrt{x^2 + y^2}} = \frac{1}{x^2 + y^2} \left(-\frac{xy}{\sqrt{x^2 + y^2}}\right),$$

we obtain that

$$\det J_f(x, y) = \frac{1}{(x^2 + y^2)^2} \left(x^2 + y^2 - x^2 - y^2 + \frac{x^2 y^2}{x^2 + y^2} - \frac{x^2 y^2}{x^2 + y^2}\right) = 0.$$

(b) Clearly, $f(U)$ is the circle of radius 1 centered at $(0, 0)$. This set does not have interior points and thus contains no non-empty open subset.

2. Is the following “theorem” true or not?

Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, let $x_0 \in U$, and let $f \in C^1(U, \mathbb{R}^N)$ be such that $f(V)$ is open for each open neighborhood $V \subset U$ of $x_0$. Then $\det J_f(x_0) \neq 0$.

Give a proof or provide a counterexample.

Solution: Let

$$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3,$$

and let $x_0 = 0$. For any $x \in \mathbb{R}$ and $\epsilon > 0$, we have $f((x-\epsilon, x+\epsilon)) = ((x-\epsilon)^3, (x+\epsilon)^3)$, so that $f(V)$ is open for each open subset of $V$. On the other hand, $f'(0) = 0$. 

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3. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, and let $f \in C^1(U, \mathbb{R}^N)$ be such that $\det J_f(x) \neq 0$ for all $x \in U$.

(a) Show that
\[ (*) \quad U \to \mathbb{R}, \quad x \mapsto \|f(x)\| \]
has no local maximum.

(b) Suppose that $U$ is bounded (so that $\overline{U}$ is compact) and that $f$ has a continuous extension $\tilde{f}: \overline{U} \to \mathbb{R}^N$. Show that the continuous map
\[ (**) \quad \overline{U} \to \mathbb{R}, \quad x \mapsto \|\tilde{f}(x)\| \]
attains its maximum on $\partial U$.

Solution:

(a) Assume that $(*)$ attains a local maximum at $x_0 \in U$, i.e., there is an open neighborhood $V \subset U$ of $x_0$ such that
\[ (***) \quad \|f(x)\| \leq \|f(x_0)\| \]
holds for $x \in V$. Since $\det J_f(x) \neq 0$ for all $x \in U$, it follows that $f(U)$ is open. Hence, there is $\epsilon > 0$ such that $B_r(f(x_0)) \subset f(U)$. This, however, contradicts $(***)$.

(b) Since $\overline{U}$ is compact and $\tilde{f}$ is continuous, $(**)$ attains its maximum at a point $x_0 \in \overline{U}$. By (a), $x_0 \notin U$ must hold, so that $x_0 \in \partial U$.

4. Let
\[ f: \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2. \]
Show that, there is $\epsilon > 0$ and a $C^1$-function $\phi: (-\epsilon, \epsilon) \to \mathbb{R}$ with $\phi(0) = 1$ such that $y = \phi(x)$ solves the equation $f(x, y) = 1$ for all $x \in \mathbb{R}$ with $|x| < \epsilon$. Show without explicitly determining $\phi$ that
\[ \phi'(x) = -\frac{x}{\phi(x)} \quad (x \in (-\epsilon, \epsilon)). \]

Solution: Let
\[ g: \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2 - 1, \]
so that $f(x, y) = 1$ if and only if $g(x, y) = 0$. Clearly,
\[ g(0, 1) = 0 \quad \text{and} \quad \frac{\partial g}{\partial y}(0, 1) = 2. \]
The existence of $\epsilon > 0$ and $\phi$ is then an immediate consequence of the Implicit Function Theorem.

Also by the Implicit Function Theorem, we have

$$\phi'(x) = -\left(\frac{\partial g(x, \phi(x))}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}(x, \phi(x)) = -\frac{x}{\phi(x)}.$$ 

5. Show that there are $\epsilon > 0$, and $u, v, w \in C^1(B_{\epsilon}((1,1)), \mathbb{R})$ such that $u(1,1) = 1$, $v(1,1) = 1$, and $w(1,1) = -1$, and

$$u(x,y)^5 + x v(x,y)^2 - y + w(x,y) = 0,$$

$$v(x,y)^5 + y u(x,y) - x + w(x,y) = 0,$$

and

$$w(x,y)^4 + y^5 - x^4 = 1$$

for $(x,y) \in B_{\epsilon}((1,1))$.

Solution: Define

$$f : \mathbb{R}^5 \to \mathbb{R}^3,$$

$$(x,y,u,v,w) \mapsto (u^5 + xv^2 - y + w, v^5 + yu - x + w, w^4 + y^5 - x^4 - 1).$$

and note that $f(1,1,1,1,-1) = 0$. Also, we have

$$\begin{bmatrix}
\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\
\frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\
\frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w}
\end{bmatrix} = \begin{bmatrix}
5u^5 & 2v & 1 \\
y & 5v^4 & 1 \\
0 & 0 & 4w^3
\end{bmatrix};$$

evaluation at $(1,1,1,1,-1)$ yields the matrix

$$\begin{bmatrix}
5 & 2 & 1 \\
1 & 5 & 1 \\
0 & 0 & -4
\end{bmatrix},$$

the determinant of which is computed as $-92$. The existence of $\epsilon > 0$ and $u, v, w \in C^1(B_{\epsilon}((1,1)), \mathbb{R})$ as required then follows from the Implicit Function Theorem.

6. Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ be such that

$$f^{-1}(\{0\}) = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1 \text{ or } (x+1)^2 + y^2 = 1\}.$$

(a) Sketch $f^{-1}(\{0\}).$
(b) Show that
\[ \frac{\partial f}{\partial y}(0,0) = \frac{\partial f}{\partial x}(0,0) = 0. \]

Solution:

(a) The zero set of \( f \) is the union of two circles of radius one, where one is centered at \((-1,0)\) and the other one at \((1,0)\):

(b) In each neighborhood \( V \) of 0 and each \( x \in V \), there are two \( y \in \mathbb{R} \) with \( f(x, y) = 0 \). By the implicit function theorem, this is possible only if \( \frac{\partial f}{\partial y}(0,0) = 0 \). Exchanging the roles of \( x \) and \( y \), yields \( \frac{\partial f}{\partial x}(0,0) = 0 \) as well.