1. Let $I$ be a compact interval, and let $f = (f_1, \ldots, f_M) : I \to \mathbb{R}^M$. Show that $f$ is Riemann integrable if and only if $f_j : I \to \mathbb{R}$ is Riemann integrable for each $j = 1, \ldots, M$ and that, in this case,

$$\int_I f = \left( \int_I f_1, \ldots, \int_I f_M \right)$$

holds.

**Solution:** Suppose that $f$ is Riemann integrable. Fix $k \in \{1, \ldots, M\}$, and let $y = (y_1, \ldots, y_M)$ be the Riemann integral of $f$ over $I$. Let $\epsilon > 0$. Then there is a partition $\mathcal{P}_\epsilon$ of $I$ such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each associated Riemann sum $S(f, \mathcal{P})$, we have

$$|S(f_k, \mathcal{P}) - y_k| \leq \|S(f, \mathcal{P}) - y\| < \epsilon.$$

This means that $f_k$ is Riemann integrable with $\int_I f_k = y_k$.

Conversely, suppose that $f_j$ is Riemann integrable with integral $y_j$ for $j = 1, \ldots, M$. Set $y := (y_1, \ldots, y_M)$. Let $\epsilon > 0$. For each $j = 1, \ldots, M$, there is a partition $\mathcal{P}_j$ of $I$ such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_j$, we have

$$|S(f_j, \mathcal{P}) - y_j| < \frac{\epsilon}{\sqrt{M}}$$

for each Riemann sum $S(f_j, \mathcal{P})$. Let $\mathcal{P}_\epsilon$ be a common refinement of $\mathcal{P}_1, \ldots, \mathcal{P}_M$. Then for every refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each Riemann sum $S(f, \mathcal{P})$, we obtain

$$\|S(f, \mathcal{P}) - y\| \leq \sqrt{M} \max_{j=1,\ldots,M} |S(f_j, \mathcal{P}) - y_j| < \sqrt{M} \frac{\epsilon}{\sqrt{M}} = \epsilon.$$

Consequently, $f$ is Riemann integrable with $\int_I f = y$.

2. Let $I \subset \mathbb{R}^N$ be a compact interval, and let $f : I \to \mathbb{R}^M$ be Riemann integrable. Show that $f$ is bounded.

**Solution:** Assume towards a contradiction that $f$ is not bounded.

Let $\mathcal{P}$ be a partition of $I$—with corresponding subdivision $(I_\nu)_\nu$ of $I$—such that

$$\left\|S(f, \mathcal{P}) - \int_I f\right\| < 1$$

for each Riemann sum $S(f, \mathcal{P})$ of $f$ corresponding to $\mathcal{P}$. In particular, this means that

$$\|S(f, \mathcal{P})\| \leq 1 + \left\|\int_I f\right\| =: C$$
for each such Riemann sum $S(f, P)$. Since $f$ is assumed to be unbounded and since $I = \bigcup_\nu I_\nu$, there is at least one $\nu_0$ such that $f$ is unbounded on $I_{\nu_0}$. Choose $x_{\nu_0} \in I_{\nu_0}$ such that
\[
\|f(x_{\nu_0})\| > \frac{1}{\mu(I_{\nu_0})} \left( C + \left\| \sum_{\nu \neq \nu_0} f(x_{\nu}) \mu(I_{\nu}) \right\| \right).
\]
For the Riemann sum
\[
S_0(f, P) := \sum_{\nu} f(x_{\nu}) \mu(I_{\nu}),
\]
we thus obtain
\[
\|S_0(f, P)\| = \left\| \sum_{\nu} f(x_{\nu}) \mu(I_{\nu}) \right\|
\geq \|f(x_{\nu_0})\| \mu(I_{\nu_0}) - \left\| \sum_{\nu \neq \nu_0} f(x_{\nu}) \mu(I_{\nu}) \right\|
= \|f(x_{\nu_0})\| \mu(I_{\nu_0}) - \left\| \sum_{\nu \neq \nu_0} f(x_{\nu}) \mu(I_{\nu}) \right\|
> C.
\]
which is impossible.

3. Let $\emptyset \neq D \subset \mathbb{R}^N$ be bounded, and let $f, g : D \to \mathbb{R}$ be Riemann-integrable. Show that $fg : D \to \mathbb{R}$ is Riemann-integrable.

Do we necessarily have
\[
\int_D fg = \left( \int_D f \right) \left( \int_D g \right) ?
\]

(Hint: First, treat the case where $f = g$ and then the general case by observing that $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2).$

Solution: Without loss of generality suppose that $D$ is a compact interval $I$.

Let $C \geq 0$ such that $|f(x)| \leq C$ for $x \in I$. Let $\epsilon > 0$ and let $P_\epsilon$ be a partition of $I$ such that
\[
|S_1(f, P_\epsilon) - S_2(f, P_\epsilon)| < \frac{\epsilon}{2(C + 1)}
\]
for all Riemann sums $S_1(f, P_\epsilon)$ and $S_2(f, P_\epsilon)$ corresponding to $P_\epsilon$. Let $(I_\nu)_\nu$ the subdivision of $I$ induced by $P_\epsilon$, and let $x_\nu, y_\nu \in I_\nu$ be support points. As in the proof of Proposition 4.2.12(iii), one sees that
\[
\sum_{\nu} |f(x_\nu) - f(y_\nu)| \mu(I_\nu) < \frac{\epsilon}{2(C + 1)}.
\]
It follows that
\[
\sum_{\nu} |f(x_{\nu})^2 - f(y_{\nu})^2| \mu(I_{\nu}) = \sum_{\nu} |f(x_{\nu}) + f(y_{\nu})||f(x_{\nu}) - f(y_{\nu})| \mu(I_{\nu}) \\
\leq \sum_{\nu} 2C|f(x_{\nu}) - f(y_{\nu})| \mu(I_{\nu}) \\
< 2C \frac{\epsilon}{2(C+1)} \\
< \epsilon.
\]

Hence, \( f^2 \) is Riemann-integrable by Corollary 4.2.6.

For Riemann-integrable \( f, g : I \to \mathbb{R} \), we have
\[
fg = \frac{1}{2}((f + g)^2 - f^2 - g^2),
\]
so that \( fg \) is also Riemann-integrable.

However, we have, for instance,
\[
\int_0^1 x^2 \, dx = \frac{1}{3} \neq \frac{1}{4} = \left( \int_0^1 x \, dx \right)^2.
\]

4. Let \( \emptyset \neq D \subset \mathbb{R}^N \) have content zero, and let \( f : D \to \mathbb{R}^M \) be bounded. Show that \( f \) is Riemann-integrable on \( D \) such that
\[
\int_D f = 0.
\]

**Solution:** Let \( C \geq 0 \) be such that \( \|f(x)\| \leq C \) for \( x \in D \).

Let \( I \subset \mathbb{R}^N \) be a compact interval such that \( D \subset I \), and extend \( f \) to \( \tilde{f} : I \to \mathbb{R}^M \) as pointed out in class. Let \( \epsilon > 0 \), and choose a partition \( P \) of \( I \) with corresponding subdivision \( (I_{\nu})_{\nu} \) of \( I \) such that
\[
\sum_{I_{\nu} \cap D \neq \emptyset} \mu(I_{\nu}) < \frac{\epsilon}{C+1}.
\]

Let \( Q \) be a refinement of \( P \) with corresponding subdivision \( (J_{\lambda})_{\lambda} \). It follows that
\[
\sum_{J_{\lambda} \cap D \neq \emptyset} \mu(J_{\lambda}) < \frac{\epsilon}{C+1}.
\]

For each \( \lambda \), pick a support point \( y_{\lambda} \in J_{\lambda} \). Then we have
\[
\left\| \sum_{\lambda} \tilde{f}(y_{\lambda}) \mu(J_{\lambda}) \right\| = \left\| \sum_{J_{\lambda} \cap D \neq \emptyset} f(y_{\lambda}) \mu(J_{\lambda}) \right\| \leq C \sum_{J_{\lambda} \cap D \neq \emptyset} \mu(J_{\lambda}) < \epsilon.
\]

It follows that \( \int_D f = 0 \).
5. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open with content, and let $f : U \to [0, \infty)$ be bounded and continuous such that $\int_U f = 0$. Show that $f \equiv 0$ on $U$.

_Solution:_ Assume that there is $x_0 \in U$ such that $f(x_0) \neq 0$, i.e., $f(x_0) > 0$. By the continuity of $f$, there is $\delta > 0$, such that $B_\delta(x_0) \subset U$ and $f(x) > \frac{f(x_0)}{2}$ for all $x \in B_\delta(x_0)$. Let

$$J := \left[ x_{0,1} - \frac{\delta}{3\sqrt{N}}, x_{0,1} + \frac{\delta}{3\sqrt{N}} \right] \times \cdots \times \left[ x_{0,N} - \frac{\delta}{3\sqrt{N}}, x_{0,N} + \frac{\delta}{3\sqrt{N}} \right],$$

so that $J \subset B_\delta(x_0)$. We thus obtain

$$\int_I f \geq \int_I f \chi_J = \int_J f = \frac{f(x_0)}{2} \mu(J) > 0,$$

which is a contradiction.

6*. The function

$$f : [0, 1] \times [0, 1] \to \mathbb{R}, \quad (x, y) \mapsto xy$$

is continuous and thus Riemann integrable. Evaluate $\int_{[0,1] \times [0,1]} f$ using only the definition of the Riemann integral, i.e., in particular, without use of Fubini’s Theorem.

_Solution:_ For $n \in \mathbb{N}$, let

$$\mathcal{P}_n := \left\{ \frac{j}{n} : j = 0, \ldots, n \right\} \times \left\{ \frac{k}{n} : k = 0, \ldots, n \right\}.$$ 

For $(j, k) \in \{0, \ldots, n\}$, let $x_{j,k} := \left( \frac{j}{n}, \frac{k}{n} \right)$. The corresponding Riemann sum is then

$$S_n(f, \mathcal{P}_n) = \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{jk}{n^2} \frac{1}{n^2} = \frac{1}{n^4} \left( \sum_{j=1}^{n} j \right) \left( \sum_{k=1}^{n} k \right) = \frac{1}{n^4} \frac{n^2(n+1)^2}{4} \to \frac{1}{4}.$$ 

We claim that $\int_{[0,1]^2} f = \frac{1}{4}$.

Let $\epsilon > 0$, and choose $\delta > 0$ such that $|(f(x, y) - f(x', y')| < \frac{\epsilon}{3}$ for all $(x, y), (x', y') \in [0,1]^2$ such that $\|(x, y) - (x', y')\| < \delta$. Choose a partition $\mathcal{P}_0$ of $I$ such that the following are true for the corresponding subdivision $(I_\nu)_\nu$ of $[0,1]^2$:

- if $(x, y), (x', y') \in I_\nu$ for some $\nu$, then $\|(x, y) - (x', y')\| < \delta$;
• if \( P \) is any refinement of \( P_0 \), then \( |S(f, P) - \int f| < \frac{\epsilon}{3} \) for any Riemann sum \( S(f, P) \) corresponding to \( P \).

Choose \( n_0 \in \mathbb{N} \) be such that the following are true for the corresponding subdivision \( (J_\mu)_\mu \) of \([0, 1]^2\):

• if \((x, y), (x', y') \in J_\mu \) for some \( \mu \), then \( \|(x, y) - (x', y')\| < \delta \);
• for any \( n \geq n_0 \), we have \( \left| \frac{1}{4} - S_n(f, P_n) \right| < \frac{\epsilon}{3} \).

Let \( Q \) be any common refinement of \( P_0 \) and \( P_{n_0} \), and let \((K_\lambda)_\lambda \) be the corresponding partition of \([0, 1]^2\), and let \( S(f, Q) \) be a corresponding Riemann sum. Then we have

\[
\left| \frac{1}{4} - \int_{[0,1]^2} f \right| \leq \left| \frac{1}{4} - S_{n_0}(f, P_{n_0}) \right| - \left| S_{n_0}(f, P_{n_0}) - S(f, Q) \right| + \left| S(f, Q) - \int_{[0,1]^2} f \right| < \frac{\epsilon}{3} + \left| S_{n_0}(f, P_{n_0}) - S(f, Q) \right|
\]

Let \( S(f, Q) = \sum_\lambda f(x_\lambda)\mu(K_\lambda) \) with \( x_\lambda \in K_\lambda \), and \( S_{n_0}(f, P_{n_0}) = \sum_\nu f(y_\nu)\mu(I_\nu) \). It follows that

\[
\left| S_{n_0}(f, P_{n_0}) - S(f, Q) \right| = \left| \sum_\nu f(y_\nu)\mu(I_\nu) - \sum_\lambda f(x_\lambda)\mu(K_\lambda) \right| \\
\leq \sum_\nu \sum_{K_\lambda \subset I_\nu} |f(y_\nu) - f(x_\lambda)| \mu(K_\lambda) \\
< \frac{\epsilon}{3},
\]

so that, all in all, \( \left| \frac{1}{4} - \int_{[0,1]^2} f \right| < \epsilon \). As \( \epsilon > 0 \) was arbitrary, this means that \( \int_{[0,1]^2} f = \frac{1}{4} \) as claimed.