1. (a) Let \((x_n)_{n=1}^{\infty}\) be a sequence in \(\mathbb{R}^N\) such that there is \(\theta \in (0, 1)\) with
\[
\|x_{n+2} - x_{n+1}\| \leq \theta \|x_{n+1} - x_n\|
\]
for \(n \in \mathbb{N}\). Show that \((x_n)_{n=1}^{\infty}\) converges.

(Hint: Show first that \(\|x_{n+1} - x_n\| \leq \theta^n \|x_2 - x_1\|\) for \(n \in \mathbb{N}\), and then use this and the fact that \(\sum_{n=0}^{\infty} \theta^n\) converges to show that \((x_n)_{n=1}^{\infty}\) is a Cauchy sequence.)

(b) (Banach’s Fixed Point Theorem.) Let \(\emptyset \neq F \subset \mathbb{R}^N\) be closed, and let \(f: F \to \mathbb{R}^N\) be such that \(f(F) \subset F\) and that there is \(\theta \in (0, 1)\) with
\[
\|f(x) - f(y)\| \leq \theta \|x - y\|
\]
for \(x, y \in F\). Show that there is a unique \(x_0 \in F\) such that \(f(x_0) = x_0\).

Solution:

(a) We use induction to prove that
\[
\|x_{n+1} - x_n\| \leq \theta^n \|x_2 - x_1\|
\]
for \(n \in \mathbb{N}\). The claim is trivially true for \(n = 1\). Suppose now that the claim has been proven for a particular \(n \in \mathbb{N}\). Then
\[
\|x_{n+2} - x_{n+1}\| \leq \theta \|x_{n+1} - x_n\| \leq \theta \theta^{n-1} \|x_2 - x_1\| = \theta^n \|x_2 - x_1\|
\]
holds, which proves the claim for \(n + 1\).

Let \(m > n \geq 2\). We obtain:
\[
\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\|
\]
\[
= \sum_{k=n}^{m-1} \|x_{k+1} - x_k\|
\]
\[
\leq \sum_{k=n}^{m-1} \theta^{k-1} \|x_2 - x_1\|
\]
\[
= \sum_{k=n-1}^{m-2} \theta^k \|x_2 - x_1\|
\]
\[
= \|x_2 - x_1\| \left(\sum_{k=0}^{m-2} \theta^k - \sum_{k=0}^{n-2} \theta^k\right)
\]
Let $\epsilon > 0$. Since $\sum_{n=0}^{\infty} \theta^n$ converges, $(\sum_{k=0}^{n} \theta^k)_{n=1}^{\infty}$ is a Cauchy sequence. Hence, there is $n_\epsilon \in \mathbb{N}$ such that
\[
\left| \sum_{k=0}^{n-2} \theta^k - \sum_{k=0}^{m-2} \theta^k \right| < \frac{\epsilon}{\|x_2 - x_1\| + 1}
\]
for $n, m \geq n_\epsilon$.

Let $n, m \geq n_\epsilon$. If $n = m$, we have $\|x_m - x_n\| = 0 < \epsilon$. If $n > m$, note that $\|x_m - x_n\| = \|x_n - x_m\|$ and switch the roles of $n$ and $m$. Hence, we may suppose that $m > n$. We thus have
\[
\|x_m - x_n\| \leq \|x_2 - x_1\| \left( \sum_{k=0}^{n-2} \theta^k - \sum_{k=0}^{m-2} \theta^k \right) < \frac{\epsilon \|x_2 - x_1\|}{\|x_2 - x_1\| + 1} < \epsilon.
\]
Hence, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and therefore converges.

(b) Define $(x_n)_{n=1}^{\infty}$ inductively as follows. Let $x_1 \in F$ be arbitrary, and for $n \in \mathbb{N}$, set $x_{n+1} := f(x_n)$. It follows that
\[
\|x_{n+2} - x_{n+1}\| = \|f(x_{n+1}) - f(x_n)\| \leq \theta \|x_{n+1} - x_n\|
\]
for $n \in \mathbb{N}$. By Problem 4 on Assignment #4, $(x_n)_{n=1}^{\infty}$ converges to some $x_0 \in \mathbb{R}^N$, and as $F$ is closed we have $x \in F$. Since $f$ is continuous, we have
\[
f(x_0) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = x_0.
\]
This proves the existence of $x_0$.

To see that $x_0$ is unique, let $\tilde{x}_0 \in F$ be such that $f(\tilde{x}_0) = \tilde{x}_0$. It follows that
\[
\|x_0 - \tilde{x}_0\| = \|f(x_0) - f(\tilde{x}_0)\| \leq \theta \|x_0 - \tilde{x}_0\|.
\]
As $\theta \in (0, 1)$, this means that $\|x_0 - \tilde{x}_0\| = 0$ and thus $x_0 = \tilde{x}_0$.

2. Let $D := \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and let
\[
f : D \to \mathbb{R}, \quad (x, y) \mapsto \frac{x^2}{y}
\]
Show that:

(a) $\lim_{t \to 0^+} f(tx_0, ty_0) = 0$ for all $(x_0, y_0) \in D$;

(b) $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

Solution:
(a) Let \((x_0, y_0) \in D\). For \(t \in \mathbb{R} \setminus \{0\}\), we then have that \((tx_0, ty_0) \in D\) as well such that

\[
f(tx_0, ty_0) = \frac{t^2 x_0^2}{ty_0} = \frac{t x_0^2}{y_0}
\]

It follows that \(\lim_{t \to 0} f(tx_0, ty_0) = 0\).

(b) For \(n \in \mathbb{N}\), set \((x_n, y_n) := \left(\frac{1}{n}, \frac{1}{n^2}\right)\), so that

\[
f(x_n, y_n) = \frac{1}{n^2} = 1.
\]

It follows that \(\lim_{n \to \infty} f(x_n, y_n) = 1\). Since by (a), \(\lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = 0\), we conclude that that \(\lim_{(x,y) \to (0,0)} f(x, y)\) does not exist.

3. Let \(\emptyset \neq D \subset \mathbb{R}^N\) have the property that every continuous function \(f : D \to \mathbb{R}\) is bounded. Show that \(D\) is compact.

Solution: Assume that \(D\) is not compact. By Heine–Borel, there are two possibilities.

Case 1: \(D\) is unbounded. Then

\[
f : D \to \mathbb{R}, \quad x \mapsto \|x\|
\]

is an unbounded continuous function.

Case 2: \(D\) is not closed, i.e., there is \(x_0 \in \overline{D} \setminus D\). Then

\[
f : D \to \mathbb{R}, \quad x \mapsto \frac{1}{\|x - x_0\|}
\]

is an unbounded continuous function.

Both cases lead to contractions, so that \(D\) must be both closed and bounded, i.e., compact.

4. Let \(\emptyset \neq D \subset \mathbb{R}^N\). A function \(f : D \to \mathbb{R}^M\) is called Lipschitz continuous if there is \(C \geq 0\) such that

\[
\|f(x) - f(y)\| \leq C\|x - y\|
\]

for all \(x, y \in D\).

Show that:

(a) each Lipschitz continuous function is uniformly continuous;

(b) if \(f : [a, b] \to \mathbb{R}\) is continuous such that \(f\) is differentiable on \((a, b)\) with \(f'\) bounded on \((a, b)\), then \(f\) is Lipschitz continuous;

(c) the function

\[
f : [0, 1] \to \mathbb{R}, \quad x \mapsto \sqrt{x}
\]

is uniformly continuous, but not Lipschitz continuous.
Solution:

(a) Suppose that, for \( f : D \to \mathbb{R}^M \), there is \( C \geq 0 \) such that

\[
\| f(x) - f(y) \| \leq C \| x - y \|
\]

for all \( x, y \in D \). Let \( \epsilon > 0 \), and choose \( \delta := \frac{\epsilon}{C+1} \). For \( x, y \in D \) with \( \| x - y \| < \delta \), it follows that

\[
\| f(x) - f(y) \| \leq C \| x - y \| < \frac{\epsilon}{C+1} < \epsilon.
\]

Hence, \( f \) is uniformly continuous.

(b) Set \( C := \sup_{\xi \in (a,b)} |f'(\xi)| \). Let \( x, y \in [a,b] \), and suppose without loss of generality that \( x < y \). By the Mean Value Theorem, there is \( \xi \in (x,y) \) such that

\[
f'(\xi) = \frac{f(y) - f(x)}{y - x},
\]

so that

\[
|f(x) - f(y)| = |f'(\xi)||x - y| \leq C|x - y|.
\]

(c) As \( f \) is continuous and as \([0,1]\) is compact, it follows that \( f \) is uniformly continuous. Assume that there is \( C \geq 0 \) as in the definition of Lipschitz continuity. It then follows that

\[
\frac{1}{2\sqrt{x}} = f'(x) \leq C
\]

for \( x \in (0,1] \), which is impossible.

5. Let \( C \subset \mathbb{R}^N \). We say that \( x_0, x_1 \in C \) can be joined by a path if there is a continuous function \( \gamma : [0,1] \to \mathbb{R}^N \) with \( \gamma([0,1]) \subset C \), \( \gamma(0) = x_0 \), and \( \gamma(1) = x_1 \). We call \( C \) path connected if any two points in \( C \) can be joined by a path.

Show that any path connected set is connected.

**Solution:** Assume that \( C \) is not connected, i.e., there is a disconnection \( \{U,V\} \) for \( C \). Choose \( x_0 \in U \cap C \) and \( x_1 \in V \cap C \). Since \( C \) is path connected, there is a continuous function \( \gamma : [0,1] \to \mathbb{R}^N \) with \( \gamma([0,1]) \subset C \), \( \gamma(0) = x_0 \), and \( \gamma(1) = x_1 \). Since \( \gamma \) is continuous, there are open sets \( \tilde{U}, \tilde{V} \subset \mathbb{R} \) such that

\[
\tilde{U} \cap [0,1] = \gamma^{-1}(U) \quad \text{and} \quad \tilde{V} \cap [0,1] = \gamma^{-1}(V).
\]

It is easy to see that \( \{\tilde{U}, \tilde{V}\} \) is a disconnection for \([0,1]\), which is impossible.

6*. Let

\[
C := \left\{ \left(x, \sin \left( \frac{1}{x} \right) \middle| x > 0 \right) \right\} \subset \mathbb{R}^2.
\]
Show that $C$ is connected, but not path connected. *(Hint: Show that $\{0\} \times [-1,1] \in C$ and that any point in $\{0\} \times [-1,1]$ cannot be joined by a path with any point of the form $(x, \sin \left(\frac{1}{x}\right))$ with $x > 0$.)

**Solution:** The map

$$(0, \infty) \rightarrow \mathbb{R}^2, \quad t \mapsto \left(t, \sin \left(\frac{1}{t}\right)\right)$$

is continuous and has $C$ as its range. As $(0, \infty)$ is connected, $C$ is connected as is $C$ by Solution 3 to Assignment #4.

Let $y \in [-1,1]$, and let $x_y > 0$ be such that $\sin x_y = y$. For $n \in \mathbb{N}$, let $x_n := \frac{1}{2n\pi + x_y}$. It follows that

$$\left(x_n, \sin \left(\frac{1}{x_n}\right)\right) = (x_n, \sin x_y) = (x_n, y) \rightarrow (0, y),$$

so that $(0, y) \in \overline{C}$.

Let $y \in [-1,1]$, let $t_0 > 0$, and suppose that there is a continuous function $\gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow \overline{C}$ such that $\gamma(0) = (0, y)$ and $\gamma(1) = \left(t_0, \sin \left(\frac{1}{t_0}\right)\right)$. Let $a := \sup\{t \in [0,1] : \gamma_1(t) = 0\}$. It follows that $\gamma_1(a) = 0$, $a \in [0,1)$, and $\gamma_2(t) = \sin \left(\frac{1}{\gamma_1(t)}\right)$ for $t \in (a,1]$. Consider

$$\tau : [0,1] \rightarrow [a,1], \quad t \mapsto a + t(1-a)$$

Then $\gamma \circ \tau$ is a path joining $(0, \gamma_2(a))$ with $\left(t_0, \sin \left(\frac{1}{t_0}\right)\right)$. Replacing $\gamma$ by $\gamma \circ \tau$, we can thus suppose without loss of generality that $\gamma_1(t) > 0$ for all $t \in (0,1]$.

Let $n \in \mathbb{N}$, and note that $\lim_{t \rightarrow 0} \gamma_1(t) = 0 < \gamma_1 \left(\frac{1}{n}\right)$. Choose $m_n \in \mathbb{N}$ such that:

- if $n$ is even, then so is $m_n$, and if $n$ is odd, so is $m_n$;
- $\frac{1}{m_n\pi + \frac{\pi}{2}} \leq \gamma_1 \left(\frac{1}{n}\right)$.

Then use the Intermediate Value Theorem to find $t_n \in (0, \frac{1}{n}]$ such that $\gamma_1(t_n) = \frac{1}{m_n\pi + \frac{\pi}{2}}$.

It follows that $t_n \rightarrow 0$, so that $\gamma(t_n) \rightarrow (0, y)$. However, we have

$$\gamma_2(t_n) = \sin \left(m_n\pi + \frac{\pi}{2}\right) = (-1)^n$$

for $n \in \mathbb{N}$, which does not converge as $n \rightarrow \infty$. 