1. Which of the sets below are compact?

(a) \( \{ x \in \mathbb{R}^N : r \leq \|x\| \leq R \} \) with \( 0 < r < R \);
(b) \( \{ x \in \mathbb{R}^N : r < \|x\| \leq R \} \) with \( 0 < r < R \);
(c) \( \{(t, \sin \frac{1}{t}) : t \in (0, 2018]\} \);
(d) \( \{ \frac{1}{n} : n \in \mathbb{N} \} \);
(e) \( \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\} \).

Justify your answers.

Solution: In each case, let the set under consideration be denoted by \( K \).

(a) As \( K = B_R[0] \cap B_r(0)^c \subset B_r[0] \) is closed and bounded, it is compact by the Heine–Borel Theorem.

(b) As \( \{B_\rho(0) : r < \rho \} \) is an open cover for \( K \) without a finite subcover, \( K \) cannot be compact.

(c) As \( \{(t, \sin \frac{1}{t}) : t \in (0, 2018]\} \) is clearly bounded, so is its closure \( K \), which is therefore compact by Heine–Borel.

(d) Assume that \( K \) is compact. The \( K \) is, in particular, closed, i.e., \( K^c \) is open. As \( 0 \in K^c \), there is \( \epsilon > 0 \) such that \( (-\epsilon, \epsilon) \subset K^c \). Choose \( n \in \mathbb{N} \) so large that \( \frac{1}{n} < \epsilon \). It follows that \( \frac{1}{n} \in K \cap K^c \), which is impossible.

(e) Let \( \{U_i : i \in I\} \) be an open cover for \( K \). Choose \( i_0 \in I \) such that \( 0 \in U_{i_0} \). Let \( \epsilon > 0 \) be such that \( (-\epsilon, \epsilon) \subset U_{i_0} \), and choose \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0} < \epsilon \). It follows that \( \frac{1}{n} \in (-\epsilon, \epsilon) \subset U_{i_0} \) for \( n \geq n_0 \). For \( k = 1, \ldots, n_0 - 1 \), chose \( i_k \in I \) such that \( \frac{1}{k} \in U_i \). It follows that
\[ K \subset U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{n_0-1}}. \]

Hence, \( K \) is compact. (This is a special case of Problem 4.)

2. A set \( S \subset \mathbb{R}^N \) is called star shaped if there is \( x_0 \in S \) such that \( tx_0 + (1-t)x \in S \) for all \( x \in S \) and \( t \in [0, 1] \). Show that every star shaped set is connected, and give an example of a star shaped set that fails to be convex.

Solution: Let \( S \) be star shaped, and let \( x_0 \in S \) be as in the definition. Assume that there is a disconnection \( \{U, V\} \) of \( S \). Without loss of generality suppose that \( x_0 \in U \). Let \( x \in V \cap S \), and set
\[ \tilde{U} := \{ t \in \mathbb{R} : tx_0 + (1-t)x \in U \} \quad \text{and} \quad \tilde{V} := \{ t \in \mathbb{R} : tx_0 + (1-t)x \in V \}. \]
As in the proof for the connectedness of convex sets, one sees that \( \{\tilde{U}, \tilde{V}\} \) is a disconnection for \([0, 1]\), which is impossible.

Set, for instance,

\[
S := \{(x, y) \in \mathbb{R}^2 : y \leq |x|\}.
\]

For \((x, y) \in S\), i.e., such that \(y \leq |x|\), and \(t \in [0, 1]\), we have \((1-t)y \leq |(1-t)x|\), so that \(((1-t)x, (1-t)y) = t(0,0) + (1-t)(x,y) \in S\). Hence, \(S\) is star shaped.

Clearly, \((1,1), (-1,1) \in S\) whereas

\[
(0,1) = \frac{1}{2}(1,1) + \frac{1}{2}(-1,1) \notin S.
\]

Hence, \(S\) is not convex.

3. Let \(C \subset \mathbb{R}^N\) be connected. Show that \(\overline{C}\) is also connected.

**Solution:** Assume that there is a disconnection \(\{U, V\}\) for \(\overline{C}\). It is then obvious that \((C \cap U) \cap (C \cap V) = \emptyset\) and \((C \cap U) \cup (C \cup V) = C\). Assume that \(C \cap U = \emptyset\), i.e., \(C \subset U^c\). As \(U\) is open, \(U^c\) is closed, so that \(\overline{C} \subset U^c\) as well, i.e., \(\overline{C} \cap U = \emptyset\). But this is impossible because \(\{U, V\}\) is a disconnection for \(\overline{C}\). Similarly, one sees that \(C \cap V \neq \emptyset\).

All in all, \(\{U, V\}\) is a disconnection for \(C\), which is impossible because \(C\) is connected.

4. Let \(S \subset \mathbb{R}^N\), and let \(x \in \mathbb{R}^N\). Show that \(x \in \overline{S}\) if and only if there is a sequence \((x_n)_{n=1}^{\infty}\) in \(S\) such that \(x = \lim_{n \to \infty} x_n\).

**Solution:** Suppose that there is a sequence \((x_n)_{n=1}^{\infty}\) in \(S\) such that \(x = \lim_{n \to \infty} x_n\).

As \((x_n)_{n=1}^{\infty}\) is also contained in \(\overline{S}\) and since \(\overline{S}\) is closed, it follows that \(x \in \overline{S}\).

Conversely, let \(x \in \overline{S}\). If \(x \in S\), there certainly is a sequence \((x_n)_{n=1}^{\infty}\) converging to \(x\): just set \(x_n := x\) for \(n \in \mathbb{N}\). If \(x \notin S\), then \(x\) must be a cluster point of \(S\) by the definition of \(\overline{S}\), i.e., for each \(n \in \mathbb{N}\), there is \(x_n \in B_{\frac{1}{n}}(x) \cap S\), so that \(x_n \to x\).

5. Let \((x_n)_{n=1}^{\infty}\) be a convergent sequence in \(\mathbb{R}^N\) with limit \(x\). Show that \(\{x_n : n \in \mathbb{N}\} \cup \{x\}\) is compact.

**Solution:** Let \(\{U_i : i \in \mathbb{I}\}\) be an open cover for \(K := \{x_n : n \in \mathbb{N}\} \cup \{x\}\). Choose \(i_0 \in \mathbb{I}\) such that \(x \in U_{i_0}\). Since \(U_{i_0}\) is open, it is a neighborhood of \(x\). Hence, there is \(n_0 \in \mathbb{N}\) such that \(x_n \in U_{i_0}\) for all \(n \geq n_0\). For \(j = 1, \ldots, n_0 - 1\), choose \(i_j \in \mathbb{I}\) such that \(x_j \in U_{i_j}\). It follows that

\[
K \subset U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{n_0-1}},
\]

so that \(K\) is compact as claimed.
6*. Show that \( \mathbb{R}^N \setminus \{0\} \) is disconnected if and only if \( N = 1 \).

**Solution**: If \( N = 1 \), then \( \{(-\infty, 0), (0, \infty)\} \) is a disconnection for \( S := \{x \in \mathbb{R}^N : x \neq 0\} \).

Let \( N \geq 2 \) and assume that there is a disconnection \( \{U, V\} \) for \( S \). Fix \( x \in U \cap S \) and \( y \in V \cap S \).

Suppose first that \( x + t(y - x) \neq 0 \) for all \( t \in \mathbb{R} \). Define
\[
\tilde{U} := \{t \in \mathbb{R} : x + t(y - x) \in U \cap S\}
\]
and
\[
\tilde{V} := \{t \in \mathbb{R} : x + t(y - x) \in V \cap S\}.
\]
As in the proof of the connectedness of convex sets, one sees that \( \{\tilde{U}, \tilde{V}\} \) is a disconnection for \( \mathbb{R} \), which is not possible.

Suppose now that there is \( t_0 \in \mathbb{R}^N \) such that \( x + t_0(y - x) = 0 \). Since \( y \neq 0 \), we have \( t_0 \neq 1 \) and thus \( x = -\frac{t_0}{1-t_0}y \). Let \( j \in \{1, \ldots, N\} \) be such that \( y_j \neq 0 \); then we have \( -\frac{t_0}{1-t_0} = \frac{x_j}{y_j} \) and thus \( x = \frac{x_j}{y_j}y \). Let \( \epsilon > 0 \) be such that \( B_\epsilon(x) \subset U \cap S \). Fix \( k \in \{1, \ldots, N\} \setminus \{j\} \), and define \( \tilde{x} \in \mathbb{R}^N \) by letting
\[
\tilde{x}_l := \begin{cases} x_l, & l \neq k, \\ x_k + \epsilon, & k = l, \end{cases}
\]
for \( l = 1, \ldots, N \). It follows that \( \tilde{x} \in B_\epsilon(x) \subset U \cap S \). Assume that there is \( \tilde{t}_0 \in \mathbb{R} \) such that \( \tilde{x} + \tilde{t}_0(y - \tilde{x}) = 0 \). Then—as before—it follows that
\[
\tilde{x} = \frac{x_j}{y_j}y = \frac{x_j}{y_j}y = x,
\]
which is impossible by the definition of \( \tilde{x} \). Hence, \( \tilde{x} + t(y - \tilde{x}) \neq 0 \) must hold for all \( t \in \mathbb{R} \), which is impossible as we just saw.